A quantum-probabilistic paradigm: non-consequential reasoning and state dependence in investment choice

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Abstract

Seminal findings involving payoffs (Shafir and Tversky, 1992; Tversky and Shafir, 1992; Shafir, 1994) showed that individuals exhibit state-dependent behaviour in different informational contexts. In particular, in the condition of ambiguity as well as risk, individuals tend to exhibit ambiguity aversion. The core principle of rational (consequential) behaviour conceived by Savage (1954), that is the 'Savage Sure Thing' principle, has been shown to be violated. In mathematical language, this violation is equivalent to the violation of the "Law of total probability", (Kolmogorov, 1933). Given the importance of original findings in the call for a generalization of classical expected utility, we perform in this paper a set of experiments related to expressing investment preferences: i) under objective risk, ii) after a preceding gain, or loss. In accordance with previous findings we detected state dependence in human judgement (previous gain or loss changed the preference state of the participants) as well as violation of consequential reasoning under risk. We propose a quantum probabilistic model of agents' preferences, where non-consequentialism and state dependence can be well explained via interference of complex probability amplitudes. A geometric depiction of the experimental findings with a *state reconstruction* procedure from statistical data via the inverse

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Born's rule (1926), allows for an accurate representation of agents' preference formation in risky investment choice.

Keywords: Decision theory; non-consequential reasoning; investment choice; state dependence; quantum probability; generalized observables.

1 Introduction

Modern economic theory has naturally been preoccupied with the formalization of decision making. Theories of rational thought, such as the expected utility (EUT) model by von Neumann and Morgenstern (1944), its generalization under subjective uncertainty, the subjective expected utility (SEUT) by Savage (1954), as well as a hybrid paradigm by Anscombe-Aumann (1963) became to varying degrees 'workhorse models' in many applied and theoretical economic models.

Given the enormous difficulty of proposing an axiomatic framework, which can distil the essence of such a complicated topic as human decision making, it is no surprise that the axioms and logic in the above models were proven to be violated by experimental-based decision making. Various paradoxes plagued the expected utility research community, e.g., the Ellsberg paradox (1961), the Allais paradox (1953), and recently the Rabin and Thaler (2001) and Machina (2009) paradoxes, which pointed to non-classical processing of information (not conforming to the cannons of the classical Kolmogorov probability theory) and hence, falsifying some of the core axioms of these theories.[1] Violations of the independence axiom (also known as the 'Sure Thing Principle' (STP), Savage (1954)), which postulates consequential reasoning via Bayes' conditioning as the appropriate operator for updating knowledge) were also revealed by Shafir and Tversky (1992), Tversky and Shafir (1992), Shafir (1994), Croson (1999). The above findings showed that agents avoid carrying out probabilistic assessment of consequences, given by their acts in the different states and possess event non-separability in their probabilistic assessment of states, cf. examination in Gilboa (1987), Bastardi and Shafir (1998), Machina (1987; 2005), Karni (2014) and Marinacci (2015).[2]

[1]Machina (2005) establishes a division of the types of violations into three main categories: event-separability violation (aka violation of irrelevant alternatives) shown in Allais paradox; state-dependence violations that question agents probabilistic sophistication where acts only depend on the subjective probability measure assigned to consequences (only utility of consequence plays a role and the state in which it realises does not matter) and finally, ambiguity aversion shown in the classical Ellsberg setting.

[2]Interestingly, the behaviour that is not consistent with the STP was also ascribed to finance market agents by Shafir and Tversky (1992), as manifested in low trading activity

Broad-based research in mathematical psychology and economics addressed the above paradoxes, with a particular focus on the origins of the violation of *independence* axiom for risky and ambiguous situations (see discussions in Kahneman and Tversky, 1979; Machina, 1982; Holt, 1986; Gilboa, 1987 and others). Without the aim of being exhaustive, we can mention some well-known contributions that were aimed at overcoming the linearity restriction of the subjective probabilistic beliefs: max-min expected utility by Gilboa and Schmeidler (1989), in which individuals can possess multiple probabilistic priors; seminal 'Choquet expected utility' by Schmeidler (1989) and the subjective probability version by Gilboa (1989) that introduces non-additive probabilistic capacities, to relax the linearity constraints of probabilities, given by the independence axiom. Klibanoff et al. (2005) further investigated the agents' non-additive subjective beliefs that are revealed in Ellsberg type paradoxes, by axiomatising a formulation with a 'second layer of uncertainty', via the transformation of the subjective belief function (depending on model uncertainty) over the objective probabilistic measures. Another prominent contribution to tackle the paradoxes was conceived by Kahneman and Tversky (1979) known as 'Prospect Theory' (PT) and its rank-dependent modification by Tversky and Kahneman (1992) known as Cumulative Prospect Theory (CPT). A more general exposition by Karni et al. (1983), coined the 'state-dependent' SEUT, relaxes the notion of probabilistic sophistication, whereby agents may not evaluate the consequences separately from states (i.e. the utility of the consequence can be state dependent). The above generalizations gained wide recognition in economics and were successfully implemented in finance to describe 'anomalous' phenomena, such as the 'equity premium puzzle', by Mehra and Prescott (1985) and state dependence in investment preferences (Shefrin and Statman, 1985; Benartzi and Thaler, 1995; and Odean, 1998). At the same time, some difficulties with the application of the above frameworks were identified in the economic literature. Takemura (2014), Thaler and Johnson (1990) discuss the problem of empirically establishing the form of personal value function in PT that stems from the difficulty in detecting a unique personal reference point, given the editing rules that different decision makers (DMs) can apply.[3] Machina (2009) and Baillon et al (2011) also challenged the assumptions of rank dependent probabilities applied in EUT generalizations by Schmeidler (1989)

before the 1998 Presidential elections showing market players' unwillingness to implement any form of mixed strategy.

[3]More precisely, Thaler and Johnson (1990), Barkan and Busemeyer (2003) point out that a DM can implement different editing rules of the risky and uncertain prospects (lotteries) e.g. by coding the prior gains and losses separately from the current DM task, or alternatively incorporating them within the initial DM state, i.e. the reference point.

and Tversky and Kahneman (1992) due to the violation of 'tail separability'. Given the variety of EUT and SEUT generalizations, Kahneman (2003), drew attention to the existence of DM contextuality and the non-static nature of human preferences. More recently, Dzhafarov and Kujala (2013), Dzhafarov et al. (2017) carried out an extensive analysis of various types of contextual influences, and devised a special framework to analyse contextual influences on systems of random variables in psychology and decision theory.

In the search for more general and unified probabilistic theories, to model decision making processes and belief updates in economics and finance (and of course in all other domains of social science), theorists and practitioners turned their attention to the quantum probabilistic paradigm. The calculus and logic of quantum theory is by now widely applied interdisciplinarily in decision theory and cognition with a growing number of contributions to quantum probabilistic models of decision making in economics, neuroeconomics, game theory and finance (Khrennikov and Haven (2009), Pothos and Busemeyer (2009; 2013), Brandenburger (2010), Danilov and Lambert-Mogiliansky (2010), Bagarello (2012), Bagarello and Haven (2014), Busemeyer and Bruza (2012), Hawkins and Frieden (2012), Haven and Khrennikov (2013), Aerts et al (2014; 2016), Khrennikov (2015), Favre et al. (2016), Haven and Sozzo (2016), Khrennikova and Haven (2016; 2017), Takahashi (2017), and Khrennikova (2017)). The above contributions utilize the mathematical framework of quantum theory, which is based on a quantum probability that is a measure on subspaces of a multidimensional state space (the Hilbert state space), cf. Von Neumann (1932). Since the axiomatics of logic on subspaces is different from classical Boolean logic, the projection valued measures (that allow to reproduce probabilistic measures) do not obey some operations of classical Kolmogorov set theory, such as commutativity and distributivity. Decision makers' beliefs and preference states are represented as complex vectors and can describe well 'ambiguity aversion' as the process of forming prior probabilistic beliefs about states of nature and the conditional probabilities, as well as indeterminacy in the process of preference formation. As formalised by Pothos and Busemeyer (2013), p. 255.

> "In QP [Quantum probability] theory, probabilistic assessment is often strongly context- and order-dependent, individual states can be superposition states (that are impossible to associate with specific values), [and] our thesis is that they provide a more accurate and powerful account of certain cognitive processes."

In light of the above exposition, our paper's contribution is twofold. Firstly, we seek to examine individual behaviour in a financial investment setting by exploring through a controlled experiment, whether investment decisions under risk adhere to the postulates of consequential reasoning and event separability in STP. Another envisaged aim is to explore the existence of state non-separability in investment choices, with respect to the previously realized gains or losses. The paper is structured as follows. We briefly review the classical decision theories under risk and uncertainty (section 2), followed by an introduction of the core principle of consequential reasoning, STP. We illustrate empirical evidence that poses a challenge to the consequential paradigm. In section (3) we present the basics of the quantum probabilistic approach to human belief and preference formation. In sections (4) and (5) we exemplify the descriptive features of quantum probability via collected experimental findings. We further suggest in section (5) a QP framework that can well accommodate the experimental statistics, and we conclude in section (6).

2 SEUT and consequential thinking

The core aim of SEUT (Subjective Expected Utility) synthesized by Savage (1954)[4] is to operationally render an individual's preference relation between *acts* based on the perceived subjective expected utility of their *consequences* in different *states* of the world, given by some subjective *probability* estimates over states. The choice space p of DM (decision maker) is a set of all *consequence functions* from the space of acts to the space of consequences, following Kreps (1988).[5] Hence, the SEUT decision form is defined by a quartet of variables: $\{S, C, F, p\}$, given by the set of states (S), set of consequences (C), set of acts (F) and the consequence function (p). DM establishes her preference formation by forming some subjective probability estimates $\pi(s)$ over events (different states of the world), where each $s \in S$. The latter corresponds to the whole set of all available mutually exclusive states, where only one state will realize.[6] Since SEUT formalises a decision rule under uncertainty, the beliefs about states of the world are given by the *classical probability measure* formalised by Kolmogorov (1933).[7] The measure, π, is

[4]In classical EUT under objective risk by von Neumann and Morgenstern (1944) the function of states into the consequences is already specified externally, hence the DM has only to care about the utility of each consequence (payoff) and associated objective probability.

[5]See also Gilboa (1987), Machina (2005), Marinacci (2015) for a detailed presentation of the theoretical foundations.

[6]Realisation of states can be understood as a realization of values of some random variables, such as states of the economy, or asset price values.

[7]For further comparison with the quantum probability model of (random) observables, we emphasize the *functional representation of observables* in classical theory.

countably (and, in particular, *finitely*) additive: for disjoint subsets (events) of the sample space, Ω; $E_1, E_2, E_3 ... E_n ... \in E, E_i \cap E_j = \emptyset, i \neq j$,

$$\pi(E_1 \cup E_2 \cup E_3 ... \cup E_n) = \pi(E_1) + \pi(E_2) + \pi(E_3) + ... + \pi(E_n) \quad (2.1)$$

In particular, if disjoint sets form the partition of the whole sample space, Ω, i.e. $\cup_n E_n = \Omega$, we have:

$$\pi(E_1 \cup E_2 \cup E_3 ... \cup E_n) = 1. \quad (2.2)$$

Hence, in SEUT the probability measure over all states is additive to unity, $\sum_{s \in S} \pi(s) = 1$.[8]

In the DM process, states are mapped into corresponding consequences $c \in C$, where technically, each consequence is specified as a function p given by the acts, $f \in F$ conditioned on the state that occurs; $p : F \times S \to C$, where $c = p(f, s)$. This functional representation allows to derive a decision rule that is based solely on *consequences*, e.g., an indifference relation $f_1 \sim f_2$ between two acts holds, *iff* $p(f_1, s) = p(f_2, s)$ for $\forall s \in S$. The ranking of consequences is established via a real valued function $u : C \to R$. The $u(.)$ with higher numerical value is always preferred to lower numerical value, specifying the *subjective utility* of a DM. The function u associates consequences in C with some real numbers, where its expectation value is given by: $\sum_{i=1}^{n} u(c_i)\pi(s_i)$. As such, a weakly transitive binary relation on a set of acts F can be established (e.g. $f_1 \succeq f_2$), *iff* a person possesses a subjective utility function and the expectation value of the functional $V(f_1)$ is higher than (or equal to) the expectation of $V(f_2)$, formally: $V(f_1) \geq V(f_2)$, i.e., $\sum_{s \in S} \pi(s) u(p(f_1, s)) \geq \sum_{s \in S} \pi(s) u(p(f_2, s))$.

2.1 Sure thing principle (STP)

Consequentialism lies at the core in the STP formulation of SEUT (it is equivalent to the independence axiom in the von Neumann and Morgenstern (1944) EUT formulation, with risky lotteries), cf. Savage (1954). This principle assumes that only consequences are important, and their utility does not depend on any particular state of the world, s_i. The principle (also known as Postulate 2 of SEUT) can be formulated as: if a person prefers act f_1 to f_2 either knowing that state s_1 occurred, or state s_2 ($s_1, s_2 \in S$) occurred then he prefers f_1 to f_2, and her preferences over acts are independent from the actual state realization. This also implies that $V(f_1) \succ V(f_2)$,

[8]For simplicity in this and the following formulation we assume a finite number of states of the world, each associated with a probabilistic measure.

meaning that the expected utility of possible consequences of act f_1 is higher in both states of the world. This principle was reinstated in probabilistic terms in Shafir and Tversky (1992), Khrennikov and Haven (2009), Pothos and Busemeyer (2009) and others, showing that the violation of STP for a population of decision makers is equal to the violation of additivity of the probability disjunctions in the formula of total probability (henceforth FTP) in the Kolmogorovian set theory. The formula is obtained if two conditions are satisfied: i) the additivity of measures, and ii) the subjective probabilistic beliefs can be undated via Bayes' formula of conditional probability. For the Savage example with two acts and two states of the world the formula can be stated in a simple manner:

$$\pi_T(f_1) = \pi(f_1 \cap s_1) + \pi(f_1 \cap s_2). \quad (2.3)$$

The formula can be expanded by replacing the joint probability of acts in different states of the world via Bayes' conditional probability:

$$\pi_T(f_1) = \pi(f_1 \mid s_1)\pi(s_1) + \pi(f_1 \mid s_2)\pi(s_2). \quad (2.4)$$

where, $s_1 \cup s_2 = S; \pi(s_1) = 1 - \pi(s_2)$ and $f_1 \cup f_2 = F$, $\pi(f_1) = 1 - \pi(f_2)$.

With the aid of (2.4) one can express the total probability (π_T) of realization of act f_1 (respective f_2), given the conditional $\pi(f_1 \mid s_1), \pi(f_1 \mid s_2)$ and prior probabilities $\pi(s_1), \pi(s_2)$. Hence, FTP is representing the baseline probability of an event, given different disjoint paths of its realisation. In case the total probability of an act is equal to one, a DM knows for sure that in all states the act f_1 will be chosen i.e. $\pi(f_1 \succ f_2) = 1$. The total probability can only be obtained if the DM possesses a joint probability distribution (she can combine the acts and states in the same probability state space).

Evidence on violation of STP was collected for both objective and subjective probability distributions; cf. Allais (1953), Ellsberg (1961), Tversky and Shafir (1992), Shafir (1994), Croson (1999), Pothos and Busemeyer (2009), Machina (2009) and others. Non-consequential reasoning as a form of non-Bayesian processing of information in the 'agree to disagree' paradox was also explored in Khrennikov (2015).

2.2 State dependence

The classical generalizations of SEUT, approach the probabilistic violations exhibited by individuals in the process of their evaluation of consequences. Yet, state dependence can also be shown, whereby the form of the individual utility function can be state dependent, i.e., $u(c_i|s)$ cf. Karni et al. (1983). Hence, an individual can possess different utility functions in different states and show *preference reversals* over acts. Specific attention is paid in the literature to realizations of states that yield positive, or negative monetary consequences (known as previous gains and losses). Some more general examples can be: states of health of the decision maker, states of the financial market, etc.

Thaler and Johnson (1990), Tversky et al (1990), Tversky and Kahneman (1991) and Shafir (1994) showed that the existence of the previous gains and losses affects the subsequent preferences under risk and uncertainty.[9] This phenomenon was coined as 'reference dependence' by Kahneman and Tversky (1979) and Tversky and Kahneman (1992). The devised PT and CPT addressed the effect of the prior outcomes upon the change in preferences, by proposing so called 'editing rules' that a DM can employ. When editing the risky or uncertain prospects, the prior certain outcomes are incorporated into the reference point and hence, different value functions can exist for a DM, depending on the cumulative perception of the monetary consequences. CPT is characterized by two specific (loss and gain) value functions that have a different curvature, showing that the sensitivity of a DM to a possible loss is almost double the sensitivity to a possible gain, based on the experimental evidence (Tversky and Kahneman, 1992; Rabin and Thaler, 2001; Kahneman, 2003).

3 Quantum probability theory of preferences

Quantum probability (QP) is a complete probabilistic framework that can be well applied, as a descriptive decision making model under risk and uncertainty.[10] In general QP builds on two assumptions: i) human beliefs can be ambiguous, and no exact probabilistic distribution can be specified, ii) state dependence of preference formation, where preferences over consequences can differ in different states.[11] We proceed with a complete representation

[9]Preference reversals are at variance with the SEUT presuming that only the integration of the possible monetary consequences with the total existing wealth can take place.

[10]We remark that different notations, such as 'quantum-like', cf., Haven and Khrennikov (2013), 'Quantum probability theory' by Busemeyer and Bruza (2012), and 'Quantum Decision Theory' by Favre et al. (2016) are in use, to denote the application of quantum mechanical calculus to macroscopic phenomena in cognition and decision theory. The umbrella of 'quantum-like' models also includes frameworks beyond the 'standard quantum formalism', cf. Khrennikov (2010), Aerts et al. (2016). We also remark that the framework is widely used to describe various probabilistic fallacies and preference reversals in riskless choice, such as order effects (Busemeyer and Bruza, 2012) and voting preference reversals, (Khrennikova and Haven, 2016).

[11]Applications of the QP formalism does not imply the necessity of an existence of a classical utility function, instead preference formation can be modeled via decision oper-

of both beliefs and preferences, given the evidence on state dependence of preferences, section (2.2). We briefly sketch the axiomatic representation of human beliefs and preferences by means of QP and the geometric properties of Hilbert space:

- The assembly of beliefs about events[12] in a DM task correspond to unit length vectors (the so called basis vectors), ψ, that are one dimensional subspaces of the Hilbert space, H. The Hilbert space is a complex linear space, endowed with a scalar product, denoted as $\langle\psi_1|\psi_2\rangle$ and is complete with respect to the metric determined by the norm defined as:
$$\|\psi\| = \sqrt{\langle\psi|\psi\rangle}.$$
  The norm defines the metric (distance) on $H : d(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|$.

  In applications of the quantum formalism to cognitive phenomena and decision theory, a *finite dimensional Hilbert space* is usually applied, in order to simplify the complexity of the models. The state space is derived empirically, where one can represent all the observables in two-dimensions or use a maximum state space size to correspond to all the elementary event-act combinations, cf. analysis in Haven and Khrennikov (2009), Pothos and Busemeyer (2009), Busemeyer and Bruza (2012), Khrennikova and Haven (2016).[13]

- The uncertainty of a DM, associated with the beliefs about state realisation and preferences, is encoded in the *superposition* of the various *belief states*, or *DM states* (cf. monographs Busemeyer and Bruza

ators that can combine payoff utility with some other cognitive factors, cf. Pothos and Busemeyer (2009). Yet, some contributions use QP as a tool to model only the violations of type (i), ambiguity of human beliefs, cf. Haven and Sozzo (2016). In the former approach of quantum probabilistic modeling, the obtained probability of choosing a specific option is associated with the *preference* of the DM rather than only with the quantification of her *degree of belief*, as in standard utility based economic models.

[12]Events can denote both states of the world and preferences over acts, in the words of SEUT.

[13]The derivation of an appropriate state space still remains an unsolved problem. Two-dimensional state space allows for a simple representation of information processing and preference formation, (while even a four dimensional state space is already characterized by a large number of free parameters), yet suffers from the existence of 'hidden parameters' that mathematically corresponds to the impossibility of the usage of conventional Hermitian projectors that have to obey normalization with respect to identity. This problem was addressed in Khrennikova and Haven (2017), who derived a generalized operator that allows to represent any number of observables with dichotomous values in a two dimensional plane.

(2012); Bagarello (2012); Haven and Khrennikov (2013) for an extensive introduction to QP and quantum dynamics). We remark that the distribution of beliefs does not obey the probability measure by Kolmogorov (1933), based on a σ-algebra of events, and hence the commutativity and distributivity of events are relaxed. Moreover, the prognosis of preferences over acts is also obtained in a form of probabilistic distribution, rather than a deterministic relationship.

We can represent events by fixing in H an orthonormal basis (e_j), i.e., $\langle e_i|e_j\rangle = \delta_{ij}$. Vectors can be represented through their coordinates:

$$\psi_1 = (k_1, ..., k_n), \psi_2 = (b_1, ..., b_n).$$

In the above coordinate representation the inner product of the vectors has the form:

$$\langle \psi_1|\psi_2\rangle = \sum_j \bar{k}_j b_j,$$

where $\bar{k}$ denotes the complex conjugate. The superposition state of the decision making state is depicted through normalized vectors in H, i.e., ψ such that $\langle\psi|\psi\rangle = 1$. Such normalized vector determines a pure state up to the phase factor $e^{i\theta}$, $\theta \in [0, 2\pi)$, i.e., two vectors ψ_1 and $\psi_2 = e^{i\theta}\psi_1$ would describe the same decision making state.

- States of the world that are given by random variables in classical probability theory, are given in QP by so called *observables*. The operator projectors, e.g. E_i act upon the belief state ψ and update it, in respect to the basis states e_i corresponding to the possible states of the world s_i.[14] In a similar mode, the preference question (or a lottery) is given by another set of observables with respect to the same state ψ (we allude to it as a 'DM state', if a preference observable acts upon it). The projector operator F_i acts upon ψ and transforms it into one of the basis states, f_j corresponding to concrete preferences over acts. Observables in a conventional quantum framework are represented by Hermitian operators, e.g. $A = \sum_i a_i E_i$, where a_i are eigenvalues of operator A. Eigenvalues label the outcomes, e.g., number of possible acts, or states of the world. E_i are orthogonal projectors onto the corresponding subspaces. One can assign another operator B to depict the preference formation, $B = \sum_i b_i F_i$. The b_i are eigenvalues corresponding to the possible preference realizations (acts), and F_i are projectors onto the 'preference subspaces'. The above representation is valid for operators

[14]We remark that the states of the world, and preferences (acts), are represented by different sets of observables.

with non-degenerate spectra, where each eigenvalue corresponds to a one-dimensional eigenstate.

- Loosely speaking a particular preference outcome (corresponding to the eigenvalue of an observable in quantum jargon) is obtained by projection of the unit length vector ψ (that we call the belief, or DM vector), onto one of the bases (which can be a one dimensional ray, or a multidimensional subspace, depending on the complexity of the model) of the decision making space (the complex Hilbert space). The squared projectors correspond to the probability of observing a particular act, or belief about the realization of an event. This information processing algorithm is borrowed from the quantum measurement scheme, given by the so called Born rule, Born (1926). It can be expressed as: $\pi(A = a_i) = \langle E_i\psi|\psi\rangle = \|E_i\psi\|^2$. The latter expression means that e.g., the belief about the probability of a state of the world is given by a squared length of the projected vector onto the subspace that denotes this event (before the actual realization of the event, but also before the DM obtains a belief about the certainty of the occurrence of the event). When sequential measurements are used, DM performs a *state update*, after the E_i projective measurement took place, and the new normalized state is given by: $\psi_{a_i} = E_i\psi/\|E_i\psi\|$. This is the canonical version of the projection postulate in quantum formalism, von Neumann (1932). Hence, the conditional probability for the sequence of A, B measurements will be given as: $\pi(B = b_i|A = a_i) = \langle F_i\psi_{a_i}|\psi_{a_i}\rangle = \|F_i\psi_{a_i}\|^2$. See a visualisation in fig. (1) for a case of a two dimensional state space.

- By representing the DM state in respect to the observables that the DM state of the agent confronts, one can decompose it in respect to the eigenvectors of the corresponding observable A with the corresponding eigenvectors $e_1, ..., e_n$ that form an *orthonormal* (obeying unit-length and orthogonality of the basis vectors) basis in the decision making state denoted as H. The decision making state can be represented in terms of the *complex coordinates* $c_i \in C$. Such a combination of pure states e_i is called the *superposition representation*: $\psi = c_1e_1 + ... + c_ne_n$. This form of linear representation of DM states allows to restate the Born rule[15] for the probabilistic distribution of the post-measurement states associated with respective events $\pi_{ai} = |c_i|^2$.

[15] Again the latter expression reads as: probability of observing the eigenvalue of observable A (associated with a specific event), is given by the squared complex amplitude associated with the basis state, e_i. We continue to denote probability by the letter π.

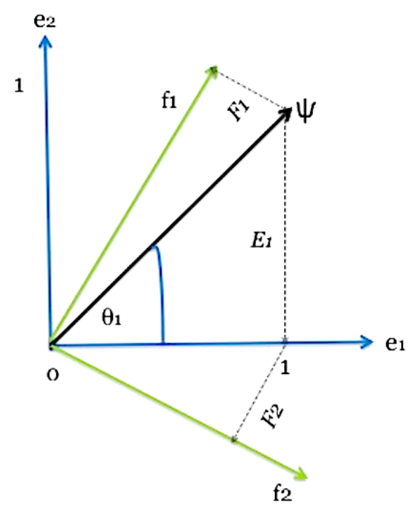

Figure 1: Graphical representation of sequential state transition onto the eigenbases under projective measurement scheme.

- Observables that can not be measured on the DM state ψ simultaneously are represented by *non-commuting* Hermitian operators. Observables which can be measured simultaneously, i.e., represented by *commuting* Hermitian operators, share the basis consisting of common eigenvectors. When the observables cannot be processed simultaneously by the DM state, one observes a violation of FTP, that indicates the lack of a joint distribution of random variables, hence the total probability associated with some act f_i cannot be assessed by the DM. The order of preference formation depends on an ensemble of factors, to mention a few: a) the order in which question measurement about preferences takes place; b) the personal choice of answering the decision making tasks (questions) that can, in particular, depend on the representativeness of the events; c) time that is given for the decision making task, and other internal and external factors, cf. Kahneman (2003), Busemeyer and Bruza (2012).

- QP is a non-deterministic framework, where the functional approaches of utility theory and its generalizations is replaced by DM state and projectors acting upon it. The beliefs in respect to pursuing particular acts are partly based on a personal value, associated with the corresponding consequences (e.g. value of payoffs), but also created in the process of interaction of the DM state with other observables. Hence, in the spirit of Karni et al. (1983), the realization of a particular state can have a direct impact on the individual evaluation of consequences. In QP models, this effect is coined 'contextuality', cf., Bruza and Busemeyer (2012), Haven and Khrennikov (2013), Dzhafarov et al. (2017).

4 Experimental Data on Investment Preferences

In order to further explore the existence of; i) the disjunction effect in the probabilistic update in risky investment indicating STP violation; ii) the state dependence of investment behaviour, in the light of previous gains and losses, we carried out a series of so called 'Portfolio game' experiments, cf. Khrennikova (2016) for an extended presentation and data analysis. These experiments were designed to extend the widely cited 'Two-stage gambling' task into the hypothetical setting of a financial market.[16] The contribution of this paper can be considered as a first attempt to generalize the experimental setup of a 'casino' into a financial environment. At this stage, we used the same payoff-probability combinations as in Tversky and Shafir (1992), yet the more subjective nature of risk was present, due to the probability being based on market forecasts, rather than on the frequency of a spin of a roulette wheel. We ran a total of three experiments that we labeled 'Pilot experiment', 'Main experiment' and 'Belief elucidation' experiment. The description of the experiments is presented in the next section, (4.1).

4.1 Experimental design

Both in the initial 'Pilot experiment' and the 'Main experiment', the investment task was presented to the participants as a *portfolio game* with a reinvestment opportunity for the second investment. Hence, the investment in the portfolio consisted of two periods, where the participation in the first investment period was presented as given. The participants had to decide for the participation (in the form of yes/no) in the second investment period in three experimental conditions of the portfolio game. The initial information in all three settings was as follows:

[16] The original two-stage gambling experiment from Tversky and Shafir (1992) was replicated in the same financial setting by Kühberger et al (2001) and Lambdin and Burdsal (2007).

"Imagine that you are an investor on the financial market. You have borrowed £1000 to invest in a portfolio" and will have to return it at the end of the portfolio game. Please neglect interest rates on the £1000. You can own the positive return the portfolio might make. Equally, you could obtain a negative return on your portfolio investment. Assume the time is now 9:00 am. Consider two future times: 10:00 am and 11:00 am. The portfolio is predicted to have a strictly 50% *chance of obtaining a* +20% *profit and* 50% *chance to generate a loss of* −10% *at 10:00 am. Equally, the portfolio has a* 50% *chance of obtaining a* +20% *profit and* 50% *chance of having a loss of* −10% *at 11:00 am. Consequently, you can either gain £200 or lose £100 at 10:00 am and 11:00 am. You can only acquire information about the realized portfolio return from a portfolio manager. This means you cannot obtain information about the portfolio's price change from any other source (internet, newspapers, etc.). At the same time, the portfolio manager has a purely informative role and cannot influence the price of the portfolio".*

The above description was followed in each condition by specific information and a dichotomous choice question supported by a graphical illustration exemplified in the Appendix, (7).

1. **No Information (NYK)**: "Imagine that at 10:00 am no information was released by the portfolio manager. This means you do NOT know whether you have a profit of £200 or a loss of £100. Would you continue playing and owning (or dis-owning) the returns of the portfolio between 10:00 am and 11:00 am, or would you prefer to quit the game now?"

2. **Won**: "At 10:00 am the portfolio manager releases information that the portfolio had a positive return and you made a profit of £200 on your portfolio investment. Would you continue playing and owning (or dis-owning) the returns of the portfolio for the second round between 10:00 am and 11:00 am, or would you prefer to quit the game now?"

3. **Lost**: "At 10:00 am the portfolio manager releases information that the portfolio had a negative return and you lost £100 of your portfolio investment. Would you continue playing and owning (or dis-owning) the returns of the portfolio for the second round between 10:00 am and 11:00 am, or would you prefer to quit the game now?"

The above experimental design was aimed to ascertain, whether the portfolio game participation frequency in the second period would differ in different experimental conditions. Furthermore, we aimed to get additional evidence related to STP violation found in previous studies by analysing, whether the NYK playing frequency is below the weighted average playing frequency after a loss or gain, cf. summary of frequencies from previous studies in Appendix, table (A1).

In the third experiment, the so called 'belief elucidation', the three informational settings were juxtaposed next to each other on the same page: *"Imagine that at 10:00 am NO information was released from the portfolio manager. This means you do NOT know for sure whether you have a profit of £200 or a loss of £100. If you believe that you have obtained a profit of £200, would you continue to play the portfolio game for the next round between 10:00 am and 11:00 am, or would you prefer to quit the game now?"* In a similar vein, a question is asked about playing the next round, if you believe that you lost. Finally, the 'No information' question was given to the participants: *"Would you play the portfolio game for the second round between 10:00 am and 11:00 am before knowing the outcome of the first round of the portfolio game?"*

Those experiments aimed to elucidate beliefs of the participants about their winning or losing of the portfolio game, to form conditional preferences in the NYK setting. In a sense, this experiment allowed the participants to form a 'mental decision tree' and hence, avoid non-consequential reasoning. A similar approach was applied for different disjunction effect experiments (testing violation of STP) in Shafir and Tversky (1992), Tversky and Shafir (1992), Croson (1999), and Busemeyer and Bruza (2012). Another research objective that was not explicitly followed in previous STP experiments was to observe, whether individuals exhibit state dependence in preferences, after a gain and after a loss, as noted by Thaler and Johnson (1990).[17]

Additionally, in the 'main experiment' and 'belief elucidation' experiment we devised a risk attitude question to measure participants' risk preferences and realize whether the size of the negative payoff might be too high for them to accept.[18] Finally, some personal questions were asked, such as gender, age, country of origin, annual income range, presence of trading experience of securities on the financial market.

[17]The null hypothesis about the absence of the disjunction effect in the previous experiments implies that the individuals ought to play both after a gain, and after a loss. Yet, the relative frequency of playing (quitting) in the respective settings is not explicitly discussed in the original setup. See however Kühberger et al (2001), seeking to provide an interpretation of the low playing frequency after a loss in their experiments.

[18]The levels of risk were acceptable, where 46% of students indicated that they would accept a 50/50 chance investment, with an expected payoff of £50 and above. In line with the findings of Shafir and Tversky (1992), this frequency was comparable with accepting the portfolio game in the NYK setting.

4.2 Procedure

For all three portfolio game experiments, the students were sampled from various Postgraduate and Undergraduate Programs at the School of Business, University of Leicester.[19] Firstly, a 'pilot study' was carried out, where we utilized between-group design with N=118, consisting of 71% female and 29% male students from various postgraduate programs. We allocated the students to three experimental conditions, by randomly assigning each seminar group that we approached to an experimental condition, to obtain approximately the same number of participants for each condition. To overcome the possible biases that can be associated with between group design we also run a within group replication of the same experiment, that we called the 'main experiment'. In the main experiment N= 60 students, 60% females and 40% males, took place in all three conditions with a time interval of two-three weeks between the conditions, to eliminate the memory effect. Finally, for the 'belief elucidation' experiment we obtained N= 29 (by design of the questions the experiment was within-group) answers, with 45% females and 55% males.

4.3 Results

- **Pilot experiment**: The results for the pilot experiment were as following: 67% of students were willing to participate in the second investment round after a previous loss of £100, yet, only 40.5% of students were willing to play after a sure gain of £200 and finally 52% of students were willing to play for the second period in the NYK setting. The difference between the Won and Lost conditions was significant, $X^2(1) = 13,982$, $p < 0.01$. The difference between NYK and respective Lost and Won behaviour was not significant. We could conclude that disjunction effect was negligible for this sample of participants, yet preference reversals in playing after a gain and after a loss were present. No significant relationship was detected in terms of gender and playing/quitting behaviour.

- **Main experiment**: In this experiment the same participants were participating in all these settings that allowed to add additional evi-

[19]We remark that the participants took some courses in statistics and finance. Students were not sampled from the first and second years of study, so as to make sure that they possess a minimal knowledge of the probabilistic calculus and finance terminology. The experimental studies were carried out in accordance with the 'University of Leicester Code of Practice and Research Code of Conduct' and ethical approval (ref: pk198-d0eb) was obtained from the Research Ethics Committee of the School of Business.

dence to the study on disjunction affect and preference reversals. In this setting, the highest playing frequency of 66.7% was once again observed for the Lost condition, followed by 65% of participants playing the second period, when they knew that they gained and finally, 48.3% playing the second period in the NYK setting. To analyse further the differences in statistics across pilot and main experiment, we computed the average playing frequency across all conditions for the pilot study, which was 53.3% and for the main experiment it was 60%. We should also note that gender and program of study composition in the samples were different, where males in general were more willing to play across all settings. We ran a set of significance tests; Cochrans Q test, ($p < 0.046$), followed by McNemars test to find out the specific differences between conditions. The results of McNemars test are: significant difference in choices between NYK and Lost conditions ($p < 0.035$) and no significant difference between other conditions. No significant relationship between gender and investment choices was detected. Hence, the findings indicated that the disjunction effect existed, yet the preference reversals after a previous gain, respective loss were minimal.[20] We ran a Chi-Square test for goodness of fit to test for the existence of the disjunction effect, where NYK playing frequency, 48%, was compared with the benchmark playing frequency of 65.85% that we computed via (2.4), with $X^2(1) = 8.187, p < 0.04$, showing that the disjunction effect was present.

- **Belief elucidation experiment**: After considering a hypothetical loss 55.2% of participants would invest again, given a hypothetical gain 48% of participants would invest again and in NYK 55.2% of participants would invest. Cochran's Q test did not show any significant differences ($p = 0.670$) between the frequencies, related to participants' hypothetical preferences in the three settings. The results support previous findings (Tversky and Shafir, 1992; Croson, 1999) whereby the framing of the decision making task externally forced the participants to evaluate the consequences of their actions in the two states of the world and form their evaluation of the preferences. On an aggregate level, preference reversals after a gain and loss were also minimal showing state independent risk-attitude and, hence preference ranking.

[20] As noted by Lambdin and Burdsal (2007) it is also important to take into account unspecified percentage comparisons, and seek to analyse the behavioural pattern of each participant in order to detect the exact direction of preference reversals. Such a detailed analysis of the collected statistics is performed in Khrennikova (2016). We do not report it in the present study due to its limited scope.

We summarized our results together with the results of 'Two stage gambling tasks' for a comparative analysis in Appendix (7), table, (A1).

4.4 Discussion

We would like to recall that there are two components of preference formation that are revealed in our study and in previous studies. Firstly, contextuality (that we can allude to as 'state dependence') of preferences related to personal risk attitudes (i.e. the same payoff can be preferred in one setting, but rejected in another setting). Such changes in preferences are at variance with EU theories, where the absolute values of payoffs matter for the DM, but not her earlier gains/losses (and more complex contextual circumstances). Another component is related to personal probabilistic assessment and information update in respect to some random variables that can affect the payoffs. The DM can exhibit ambiguity aversion and hence, not follow the canons of consequential preference formation.

4.4.1 Choice in the presence of prior losses and gains

The obtained findings in section(4.3), indicate that preference reversals occur for many participants after a sure preceding gain/loss. The main difference between the findings of Tversky and Shafir (1992), Kühberger et al. (2001), Lambdin and Burdsal (2007), and our findings (which persisted in both the 'Pilot study' and the 'Main experiment') is that, after a sure loss, the participants are most willing to play for the second period. The acceptance of risky investments in this setting is explained initially in Kahneman and Tversky (1979) as 'loss aversion'. According to Thaler and Johnson (1990) a DM will be risk seeking for complex losses, by integrating the previous losses with her subsequent investment choice. This is due to the need to break even and recover the previous losses. Loss aversion is also widely observed among investors in the financial market, known as the disposition effect, cf. Shefrin and Statman (1985), Odean (1998).

4.4.2 No information (NYK)

As we outlined in section, (2.1), empirical evidence shows that individuals tend not to accept any subsequent gamble, both under objective uncertainty (risk) and subjective uncertainty (ambiguity), if they do not know any certain outcome. The ambiguity avoidance situations have been well explained in the studies exploring variants of the Ellsberg Paradox, Ellsberg (1961), Gilboa and Schmeidler (1989), Shafir and Tversky (1992), Shafir (1994), Klibanoff et al (2005), Busemeyer and Bruza (2012) and others. The main findings of these studies, as well as our experiment is that participants are not able to (or prefer to avoid) form classical probabilistic subjective beliefs about the states of the world and hence consider the consequences in a SEUT manner. We note that the original, two step gambling experiments always involved objective risks, of a very simple nature, giving an equal chance to realize again and face a loss.[21] By inferring the decisions in the state of a gain, as well as in the state of a loss, the classical probabilistic paradigm would imply that the DM can form a joint distribution of their conditional beliefs about her acts in a risky setting. Since the risk is objective, it means that the participants have no reason for ambiguity avoidance. Yet, we can observe that in our study and in the previous studies, (A1), the consequential reasoning approach does not explain the observed variance in preference frequencies. Hence, following the explanation of Shafir and Tversky (1992), Shafir (1994), Bastardi and Shafir (1998) and Croson (1999), we suppose that in the two-stage risky choice we deal with an emergence of 'disjunction effect', where the DM cannot carry out a hypothetical evaluation of consequences of the different states of the world (the 'good economy' state accompanied by a sure gain and the 'bad economy' accompanied by a loss, in our simple set-up). The assumption is further confirmed by the control experiment (elucidation experiment), where disjunction effect and dependence of the preferences on a realized state (gain, loss) was absent.

5 QP framework of investment preferences

The aim of this theoretical analysis is to assess the *classicality* of participants' probabilistic assessment of upcoming information, based on the evaluation of prior probabilities and the usage of the Bayesian updating scheme. As a next step we aim to devise a QP description of preference formation for a representative agent. We use a DM preference representation via a so called DM state, that can be obtained through the usage of a generalized Born rule. We use a generalization of Born's rule in order to be able to apply non-Hermitian positive valued projectors, cf. Khrennikova and Haven (2017).

[21] The set-up can be of course generalized to genuinely uncertain situations, such as introducing some real financial assets and their subsequent payoff realization. This would be one of the future directions of our research.

5.1 Interference effects and DM state reconstruction

We denote the set of acts $f_1, f_2 \in F$ corresponding to 'play' respective 'quit'. The states of the world are given by $s_1, s_2 \in S$ and correspond to 'Won', or 'Lost' settings in the portfolio game. A representative agent would hence prefer the act that has a higher probability of its realization (another explanation is that the QP can provide a probabilistic prognosis for a group of DMs. Yet, this interpretation would require to complicate the model, by introducing a *mixed* DM state representation, to encode the individual differences in the initial DM states.) According to (2.4), the marginal probability of a person in a NYK setting to choose some option f_j should be equal to the sum of disjunctions of f_j conditioned upon the events s_1, s_2 (in our setting, a loss, or a gain in a previous period of the portfolio game). By embedding the averaged frequencies for playing in the different settings[22], from the pilot and main experiments, cf. table (A1), into the equation (2.4), we obtain:

$$\pi(f_1) \neq \pi(f_1|s_1) \cdot \pi(s_1) + \pi(f_1|s_2) \cdot \pi(s_2) = \pi_T(f_1); \quad (5.1)$$

$$0.5 \neq 0.549 \cdot 0.5 + 0.67 \cdot 0.5 = 0.6095 \quad (5.2)$$

Based on the results we can observe super-additivity of disjunctions with a probabilistic difference of -0.1095 between the marginal probability of f_1 and the total probability of its realization ($\pi_T(f_1) = 0.6095$). The above discrepancy suggests that the violation of the classical probabilistic assessment of information takes place, and STP is not followed by some individuals. Along with the earlier studies utilizing QP for representing reasoning and decision making, we use a quantum generalization of FTP, the so called quantum formula of total probability (QFTP), due to von Neumann (1932), to reconstruct the initial DM state from the psychological data (see introduction on QP in section 3). For two dichotomous variables the formula has the form:

$$\pi(f_1) = \pi(f_1 \mid s_1) \cdot \pi(s_1) + \pi(f_1 \mid s_2) \cdot \pi(s_2)+ \quad (5.3)$$

$$2\cos\theta_1 \sqrt{\pi(s_1) \cdot \pi(f_1 \mid s_1) \cdot \pi(s_2) \cdot \pi(f_1 \mid s_2)} \quad (5.4)$$

with the data:

$$0.5 = 0.549 \cdot 0.5 + 0.67 \cdot 0.5 + 2\cos\theta_1 \sqrt{0.5 \cdot 0.549 \cdot 0.67 \cdot 0.5} \quad (5.5)$$

[22]Total probability for f_2 (quit the second round) can be also computed with a sub-additivity in probability $\pi = 0.1095$, i.e. the sum of the probabilistic violations for the outcomes f_1 and f_2 is equal to zero.

We aimed to compute the so called interference angle in (5.5), also known as the *phase* between the complex coordinates (that represents the initial DM state in respect to a given observable). We obtain $cos\theta_1 = -0.180544$, and the interference angle $\theta_1 = 1.7523$ rad. We recall that the negative value of $cos\theta$ signifies a destructive interference of the probability amplitudes related to preference formation in respect to f_1. The probabilistic interference related f_2 (quit) equals to $cos\theta_2 = 0.283838$, $\theta_2 = 1.283$ rad.[23] We can interpret the destructive interference (where the θ corresponds to phases between basis vectors in the superposition DM state) of probability waves as leading to lower probability of playing preference, when only one preference observable acts upon the DM state ψ, in the absence of interaction of the DM state with the observable related to belief formation on s_1, s_2. Hence, the DM state transits into the eigenstates corresponding to eigenvalues f_1, or f_2, yet remains in a superposition state in respect to the observable with eigenvalues corresponding to s_1, s_2.

By knowing the interference angle from (5.5), it is possible to reconstruct the DM state ψ, that is the initial superposition state with respect to the preference operator that we denoted as P. We reconstruct the DM state via the inverse Born rule formulated by Born (1926).[24] The DM state vector (ψ) is defined through a linear combination of complex coordinates (c_1, c_2), $|\psi\rangle = c_1|e_1^P\rangle + c_2|e_2^P\rangle$, where $|e_1^P\rangle, |e_2^P\rangle$ is a basis of ψ with respect to the operator P. The square of the complex coordinate, c_1 gives the unconditional probability for f_1 preference. Hence, the determination of quantum probabilities from probability amplitudes is possible and vice versa. The coordinate can be represented as:

$$c_1 = \sqrt{\pi(s_1) \cdot \pi(f_1|s_1)} + e^{i\theta_1}\sqrt{\pi(s_2) \cdot \pi(f_1|s_2)} \tag{5.6}$$

In the same vein, the complex coordinate c_2, that gives the probability of

[23]Since the interference terms are less than one, the statistical data can be accommodated in a Hilbert space. Higher magnitudes of interference can also be observed in psychological data, cf. Khrennikov (2010) and Khrennikov and Haven (2013). For comparison, we also analysed the statistics from the previous gambling experiments (A1), and obtained probabilistic super-additivity of -0.204 for f_1, with a negative interference $cos\theta_1 = -0.34466$ of a higher magnitude, with an angle $\theta_1 = 1.9227$ rad.

[24]The inverse Born rule, is the essential tool for reconstruction of the superposition state of a quantum, or psychological system from the experimental data. Its application enables to obtain the agents' generalized initial DM state ψ, with the aid of the matrix of transition probabilities. In a psychological context, transition probabilities denote the conditional preferences that are firm preferences, obtained after a question/information measurement is carried out on the DM state. Another promising approach for a belief state reconstruction from the preferences is the application of "quantum tomography". Quantum tomography allows to measure unknown belief states from the known (final observed) preference, or belief states.

'quit' preference, f_2 could be obtained from the statistics of the experiment:

$$c_2 = \sqrt{\pi(s_1)\cdot\pi(f_2|s_1)} + e^{i\theta_2}\sqrt{\pi(s_2)\cdot\pi(f_2|s_2)} \quad (5.7)$$

We recall that by Euler's formula: $e^{i\theta} = cos\theta + isin\theta$. This allows to decompose $e^{i\theta}$, to express the coordinates via complex numbers:

$$c_1 = \sqrt{0.5\cdot 0.549} + (-0.18055 + 0.9836i)\cdot\sqrt{0.5\cdot 0.67} = 0.4194 + 0.5693i \quad (5.8)$$

We note that the complex amplitude gives us the probability of finding the (initial) DM state in the eigenvalue corresponding to preference f_1, $\pi(f_1) = |c_1|^2 = |0.4194 + 0.5693i|^2 = 0.5$. For f_2 we get:

$$c_2 = \sqrt{0.5\cdot 0.451} + (0.2838 + 0.95887i)\cdot\sqrt{0.5\cdot 0.33} = 0.59016 + 0.3895i \quad (5.9)$$

Hence, $\pi(f_2) = |c_2|^2 = |0.59016 + 0.3895i|^2 = 0.5$ that is the probability of preference f_2 to take place. The above computations enable to faithfully represent the initial DM state in a complex two dimensional Hilbert space, from the obtained statistics on preference distribution: $|\psi\rangle = (0.4194 + 0.5693i)|e_1^P\rangle + (0.59016 + 0.3895i)|e_2^P\rangle$.

To represent the other operator V transforming the initial DM state (or better to say, the *belief state*) with respect to the eigenvalues corresponding to s_1, s_2, one needs to introduce a class of more general operators. A simple form of orthogonal Hermitian operators cannot be applied, to describe the belief state with respect to the V observable, due to the matrix of transition probabilities not satisfying double stochasticity (it satisfies left stochasticity through):

$$\begin{bmatrix}\pi(f_1|s_1) & \pi(f_1|s_2)\\ \pi(f_2|s_1) & \pi(f_2|s_2)\end{bmatrix}; \begin{bmatrix}0.549 & 0.67\\ 0.451 & 0.33\end{bmatrix}$$

We can see that: $\pi(f_1|s_1) + \pi(f_1|s_2) \neq \pi(f_2|s_1) + \pi(f_2|s_2) \neq 1$. This means that the basis vectors $|e_1^V\rangle, |e_2^V\rangle$, denoting the DM's belief state with respect to V are *non-orthogonal*. One would need to introduce projectors (unless a state space increase, or degenerate spectra is considered) that *do not obey orthogonality*, imposed on classical Hermitian projectors. In quantum physics one solves this representation problem by considering positive operator valued measures (POVMs).

Definition: *A POVM is a family of linear operators $A = (V_j)$ such that each V_j is Hermitian and positive semidefinite, obeying the normalisation condition, where I is the identity operator:*

$$V \equiv \sum_j V_j = I. \quad (5.11)$$

Although POVMs serve well to describe an important class of phenomena in quantum physics, in application to decision theory, it is convenient to proceed with an even wider class of operator valued measures, relaxing the normalization constraint, i.e. $\sum_j V_j \neq I$, where V_j are generalised projectors, cf., Khrennikova and Haven (2017). In this contribution we also adopt the formalism of such *non-orthogonal* generalized POVMs, to reconstruct the DM state, ψ, with respect to the V observable, related to states of the world that the agent has to consider. The initial DM state vector is in a similar manner represented through the eigenbasis, corresponding to the V projective measurement: $|\psi\rangle = k_1|e_1^V\rangle + k_2|e_2^V\rangle$, where k_1, k_2 are the corresponding complex coordinates. The probability for s_1, respective s_2, is given by the squared complex amplitudes, i.e. $\pi(s_1) = |k_1|^2$, $\pi(s_2) = |k_2|^2$, and $|k_1|^2 + |k_2|^2 = 1$. The basis $|e_1^V\rangle, |e_2^V\rangle$ can also be expressed via a system of complex coordinates, with respect to the second state transition of a DM giving the conditional probabilities $\pi(f_j|s_j)$, $j = 1, 2$.

$$|\psi\rangle = \sqrt{\pi(s_1)}|e_1^V\rangle + \sqrt{\pi(s_2)}|e_2^V\rangle \tag{5.12}$$

where the basis of the generalized POVM can be expressed via an orthogonal basis (e_1^P, e_2^P) with respect to P given by conventional orthogonal projectors:

$$|e_1^V\rangle = \sqrt{\pi(f_1|s_1)}\ |e_1^P\rangle + \sqrt{\pi(f_2|s_1)}\ |e_2^P\rangle \tag{5.13}$$

$$|e_2^V\rangle = e^{i\theta_1}\sqrt{\pi(f_1|s_2)}\ |e_1^P\rangle + e^{i\theta_2}\sqrt{\pi(f_2|s_2)}\ |e_2^P\rangle \tag{5.14}$$

Applying the projectors V_1 and V_2[25] onto the initial DM state ψ, allows to obtain the probability distribution of s_1 and s_2. The projectors have the matrix representation, cf. Khrennikova and Haven (2017) for detailed formulation.

$$\begin{gathered} V_1 = \frac{1}{\sqrt{\pi(f_1|s_1)\pi(f_2|s_2)} - \sqrt{\pi(f_2|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}}} \times \\ \begin{bmatrix} \sqrt{\pi(f_1|s_1)\pi(f_2|s_2)} & -\sqrt{\pi(f_1|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}} \\ \sqrt{\pi(f_2|s_1)\pi(f_2|s_2)} & -\sqrt{\pi(f_2|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}} \end{bmatrix} \end{gathered} \tag{5.15}$$

where the difference between the phases related to complex coordinates c_1, c_2 is $\Delta_{12} = (\theta_1 - \theta_2) = 1.7523 - 1.283 = 0.4693$ rad.

[25] Since the matrix of transition probabilities is not doubly stochastic, the vectors e_1^V, e_2^V are non-orthogonal, hence the corresponding projectors V_1 and V_2 are neither orthogonal.

$$V_2 = \frac{1}{\sqrt{\pi(f_1|s_1)\pi(f_2|s_2)} - \sqrt{\pi(f_2|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}}} \times \begin{bmatrix} -\sqrt{\pi(f_2|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}} & \sqrt{\pi(f_1|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}} \\ -\sqrt{\pi(f_2|s_1)\pi(f_2|s_2)} & \sqrt{\pi(f_1|s_1)\pi(f_2|s_2)} \end{bmatrix} \tag{5.16}$$

With respect to the above defined projectors, eq.(5.15)- (5.16), one can build the components of the generalized POVM (denoted as Q_j, $j = 1, 2$), corresponding to the basis e_1^V, e_2^V, by taking: $Q_j = V_j^\star V_j$.

$$Q_1 = \frac{1}{\mathcal{K}} \begin{bmatrix} \pi(f_2|s_2) & -\sqrt{\pi(f_1|s_2)\pi(f_2|s_2)}\, e^{i\Delta_{12}} \\ -\sqrt{\pi(f_1|s_2)\pi(f_2|s_2)}\, e^{-i\Delta_{12}} & \pi(f_1|s_2) \end{bmatrix} \tag{5.17}$$

where:

$$\mathcal{K} = |\sqrt{\pi(f_1|s_1)\pi(f_2|s_2)} - \sqrt{\pi(f_2|s_1)\pi(f_1|s_2)}e^{i\Delta_{12}}|^2 = \\ \pi(f_1|s_1)\pi(f_2|s_2) + \pi(f_2|s_1)\pi(f_1|s_2) - 2\, \sqrt{\pi(f_1|s_1)\pi(f_2|s_2)\pi(f_2|s_1)\pi(f_1|s_2)} \cos\Delta_{12} \tag{5.18}$$

In the same way:

$$Q_2 = \frac{1}{\mathcal{K}} \begin{bmatrix} \pi(f_2|s_1) & -\sqrt{\pi(f_1|s_1)\pi(f_1|s_1)} \\ -\sqrt{\pi(f_1|s_1)\pi(f_2|s_1)} & \pi(f_1|s_1) \end{bmatrix} \tag{5.19}$$

By calculating the scalar product of the components ($Q_j, j = 1, 2$) with the initial DM state, i.e. $\langle Q_j\psi, \psi\rangle$, we can obtain the respective marginal probabilities, $\pi(s_1), \pi(s_2)$. Hence, we can express the DM state with respect to, V, as a generalised POVM: $|\psi\rangle = 0.707|e_1^V\rangle + 0.707|e_2^V\rangle$.

5.2 Preference formation algorithm in QP scheme

Throughout the two-stage portfolio game, participants are assumed to be prepared in an initial DM state ψ, upon which the two observables P and V act in different settings. The P observable relates to the question on playing the second round of the portfolio game, and the V observable expresses the impact of the information on a monetary gain, respective loss. The two observables are represented in a two-dimensional Hilbert space with non-degenerate spectra, hence the bases are simply one dimensional rays. For the P measurement, the eigenvalues corresponding to the f_1, f_2 preference outcome are $p_j, j = 1, 2$, in the basis, $(|e_1^P\rangle, |e_2^P\rangle)$. The basis is orthonormal, i.e. $\langle e_1^P|e_2^P\rangle = 0$ and $e_1^P = (1, 0), e_2^P = (0, 1)$. The belief state (we also allude to it as DM state), ψ, can be expressed in the eigenbasis: $|\psi\rangle = c_1|e_1^P\rangle + c_2|e_2^P\rangle$

with $|c_1|^2 + |c_2|^2 = 1$ by the normalization condition. We are reminded that $|c_1|^2 = \pi(f_1) = 0.5$ and $|c_2|^2 = \pi(f_2) = 0.5$.

Another observable (V) corresponds to the delivery of information related to winning, or losing the first period of the portfolio game that we denoted as states of the world to occur. As mentioned, the observable $V = (Q_1, Q_2)$ is composed of non-orthogonal projectors, V_1, V_2. The projectors act onto the basis, e_1^V, e_2^V, with respect to ψ. The possible realizations of the first round of the portfolio game states s_1, s_2 correspond to the eigenvalues $v_j, j = 1, 2$. In the context of NYK, the belief state of the DM is only affected by the operator P. The DM is still in a state of superposition (indeterminacy) with respect to possible outcomes of the first round of the portfolio game, given by V. Hence, a direct state transition $\psi \longrightarrow \psi_{p_j}$ onto one of the eigenvectors $(|e_1^P\rangle, |e_2^P\rangle)$ takes place. The squared complex amplitudes of the projectors onto these eigenvectors give us the probability of this state transition. We remind that the order of the two projective measurements (direct state transition, respective two consecutive state transitions) is important in creating the violation of FTP.

When participants are given information about the outcome of the first round of the portfolio game, a generalized POVM V acts upon the initial DM state ψ. It is updated with respect to the basis vectors $|e_1^V\rangle$, or $|e_2^V\rangle$. A new updated DM state, $|\psi_{v_j}\rangle = V_j|\psi\rangle / \|V_j|\psi\rangle\|$ emerges. In this state, beliefs about states, s_j are given with $\pi = 1$. Next, another projective measurement takes place as observable P acts upon the updated state ψ_{v_j}. We get a state transition $\psi_{v_j} \longrightarrow \psi_{p_j}$ with probability $|\langle\psi_{v_j} e_j^P\rangle|^2$, which denotes the conditional probability, $\pi(P = p_j | V = v_j)$. Depending on the observables that act upon the DM state, two different state transition schemes can take place, $[\psi \longrightarrow \psi_{p_j}]$ and $[\psi \longrightarrow \psi_{v_j} \longrightarrow \psi_{p_j}]$, which are characterized by different final probability distributions. The measurements are state dependent (i.e. the path, through which the final preference state is reached, can alter the probability distribution of preferences). The phase between the bases is a way of measuring the degree of state dependence (non-commutativity of the operators P and V, in QP terminology). The state dependence (contextuality) of measurements implies that the probabilities from the first (unconditional) preference question and the conditional preferences of DM cannot be coupled through FTP:

$$\pi(P = p_j) \neq \pi(V = v_2) \cdot \pi(P = p_j | V = v_2) + \pi(V = v_1) \cdot \pi(P = p_j | V = v_1), \tag{5.20}$$

since:

$$|\langle\psi|e_j^P\rangle|^2 = |\langle\psi|e_1^V\rangle \cdot \langle e_1^V|e_j^P\rangle + \langle\psi|e_2^V\rangle \cdot \langle e_2^V|e_j^P\rangle|^2 = |\langle\psi|e_1^V\rangle \cdot \langle e_1^V|e_j^P\rangle|^2 + |\langle\psi|e_2^V\rangle \cdot \langle e_2^V|e_j^P\rangle|^2 + 2cos\theta|\langle\psi|e_1^V\rangle \cdot \langle e_1^V|e_j^P\rangle| \cdot |\langle\psi|e_2^V\rangle \cdot \langle e_2^V|e_j^P\rangle| \neq |\langle\psi|e_2^V\rangle|^2 \cdot |\langle e_2^V|e_j^P\rangle|^2 + |\langle\psi|e_1^V\rangle|^2 \cdot |\langle e_1^V|e_j^P\rangle|^2. \tag{5.21}$$

The QP scheme in section (3) explains the non-additivity of the probability disjunctions, based on *probability interference* incorporated in the interference term and hence, it relaxes the constraints on the additivity of probability measures posed by the distributive axiom.

6 Final Remarks

By analysing experimental findings on investment preferences under risk and comparing them with investment preferences after a gain or loss we aimed to devise a framework for depicting preference reversals that yield violations of event separability postulated in STP and also indicate fluctuations in risk attitude, given a particular DM state. The proposed quantum representation of belief state transition in the process of preference formation is updated by the rules of a quantum projective measurement, where interference of probability amplitudes captures the mode of non-consequential reasoning. The phase relates the informational content of the DM-operators that can capture: i) ambiguity in the process of belief formation about possible states of the world, and corresponding consequences of different acts leading to non-consequentialism; ii) state dependence of preferences, as the actualisation of a state of the world (represented as a DM belief state update in QP) can change the probabilistic distribution of preferences (e.g., risk attitude towards some risky payoffs can change, depending on which state of the world the DM finds herself in).

In future works, by collecting a broader range of evidence, we aim to devise a more complete axiomatization of projective measurements, to describe investment preferences, given different subjective and objective risks.

7 Appendix

We present a summarising table (A1) with the results of the previous experiments and our own results as well as the computed weighted average results (separately the previous studies and own experiments).

Table 1: Summary of the acceptance rate of the second gamble across all studies

Playing frequency across conditions	NYK	Won	Lost	Total sample
1 Original	36%	69%	59%	N=98 (within group)
1 Original	38%	69%	57%	N=213 (between gr)
1 Belief Elucidation version	84%	71%	56%	N=87 (within group)
2 Replication of original	46.8%	60%	47%	N= 177 (between gr.
2 Replication of original	42.9%	80%	37.1%	N=35 (within gr.)
2 Payoffs $(4; -2)	61.9%	82.8%	69.8%	N= 184 (between gr.)
2 Real payoffs $ (4;-2)	37.5%	67.6%	32.1%	N=97 (within gr.)
3 Replication of original	36.8%	63%	45.6%	N=57 (within gr.)
3 Three Card Monte	24%	70%	38%	N= 57 (within gr.)
3 Reversed Three Card Monte	60%	73%	49%	N= 57 (within gr.)
4 Pilot study	52.8%	40.5%	67.5%	N= 118 (between gr.)
4 Main study	48.3%	65%	66.7%	N=60 (within gr.)
4 Belief elucidation	55%	48.2%	55%	N=29 (within gr.)
Mean: Previous replications	39.3%[a]	67.5%	51.9%	Sum of all subsamples
Mean: Pilot+ Main study	50%	54.9%	67%	

1-study by Tversky and Shafir (1992);
2-study by Kühberger et al. (2001);
3-study by Lambdin and Burdsal (2007);
4-present study.
a) We computed weighted averages to account for differences in the sample sizes across the different studies. We omitted replications with different payoffs and probability distributions.

Graphical illustration of the possible monetary payoffs in the different experimental conditions

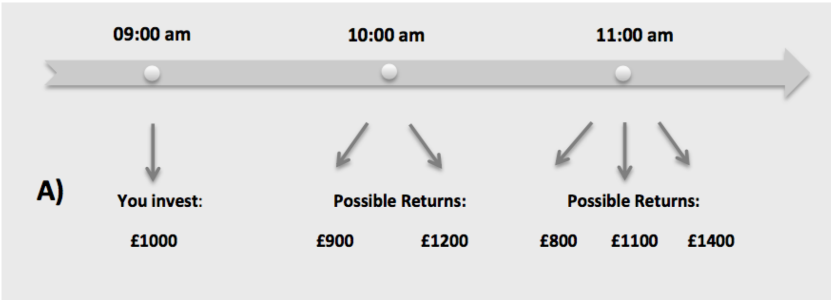

Figure 2: Graphical representation of possible monetary payoffs in NYK condition in case of option A (play) is selected. Similar representations are shown to participants for option B (quit).

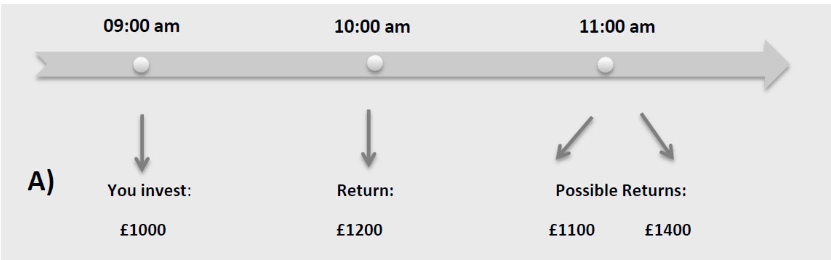

Figure 3: Graphical representation of possible monetary payoffs in the Won condition in case of option A (play) is selected. Similar representations are shown to participants for option B (quit).

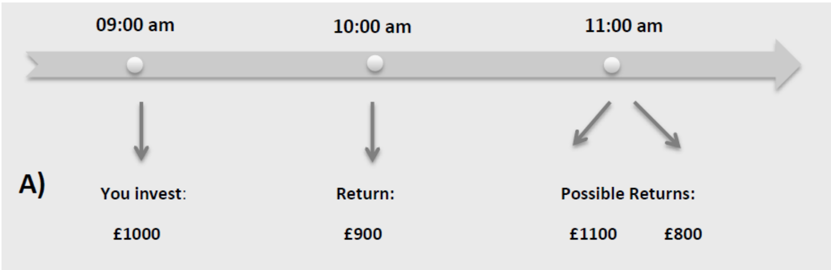

Figure 4: Graphical representation of possible monetary payoffs in the Lost condition in case of option A (play) is selected. Similar representations are shown to participants for option B (quit).

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