# Quantum Dynamical Modeling of Competition and Cooperation between Political Parties: the Coalition and Non-coalition Equilibrium Model

Polina Khrennikova,

University of Leicester, University Road, Leicester LE1 7RH.
*pk198@le.ac.uk*

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**Abstract**

We propose a model of parties' dynamical decision-making related to becoming a member of a coalition or pursuing a competitive strategy. Our approach is based on the mathematical formalism of quantum information theory. The devised model has no direct relation to quantum physics, only its mathematical apparatus and methodology are applied, in particular the quantum probability and the theory of open quantum systems. The latter describes the most general form of adaptive dynamics of a system interacting with an environment. In our model the environment is composed of the electorate, or more specifically the informational bath generated by the parties' electorate, which is a key part of the socio-economic context surrounding the political party as an decision-making entity. The key feature of the quantum model is the ability to capture the strong interrelation of the parties' decision making states, through the notion of entanglement. The preferences of different parties evolve simultaneously and non-separably in the joint information space. We model the approaching of the state of political equilibrium by using the Markov approximation of the quantum master equation. Illustrative examples of numerical simulations are presented to specify, how the model works operationally.

**Keywords**: Political theory; Game theory; Political Coalitions; Non- separability; Quantum like models; Quantum probability; Entanglement; Quantum Information Theory; Quantum Master Equation.

# 1 Introduction

Mathematical modeling of creation of coalitions between political parties and, more generally, of establishing cooperation with respect to the special political and economic issues is by now a well researched field. Generally speaking, the choices of political parties depend on a number of psychological and institutional parameters. Different models consider different parameters as being more salient to the parties' decisions. If one would search to construct a classical stochastic model with multiple loading factors that would also change over the time dynamics, one would obtain an extremely complex model. In this contribution, we propose a model that is based on the formalism of quantum information theory (quantum Markovian dynamics). The advantage of the devised model is that it reduces essentially the complexity of the classical stochastic models. The model can be potentially adapted to a variety of political issues, where the parties are uncertain in respect to cooperation/non-cooperation with other parties on some political matters. However, as was pinpointed by one of the reviewers of this paper, the topic of (non)cooperation would require a more scrutinized analysis, where the party can often cooperate only to a certain degree, involving several issues on which the party has to decide. In the case of a coalition formation the party has formally only two choices in the form of yes/no. In this piece of work we are proceeding on a formal level, by presenting a model that describes an equilibrium state of the parties that operate in a country with multiparty political system. The core decisions that these parties have to make are simplified to the set of two choices *to enter a political coalition (alliance) or to abstain from entering a coalition (alliance)”*[1].

In this paper we do not have a possibility to review in detail the "classical methods" for the investigation of the domain of cooperation and competition, including the context of coalition formation in politics, see, e.g., monographs by Davies, Hinich&Ordeshook (1970) and Dhillon (2005) for extended treat-

[1]In the spirit of information theory we encode each single party's decision state by the so called quantum bits. Each quantum bit is encoding a probabilistic superposition of obtaining some binary (customarily denoted in quantum information theory as zeros or ones) outcomes. This allows us to represent decision states with the choice outcomes in the form of ”yes” and ”no” with the aid of qubits. This approach, as will be shown, simplifies the model construction essentially.

ments. For our purpose, it is important to point out that one of the main aims of "classical mathematical modeling" is to study the overall existence and the process of approaching to the states of an *equilibrium* of preferences for (non)cooperation between parties[2].

Certainly, game theory plays a crucial role in this setting, since coalition formation is a strategic process that embraces a complexity of factors for each partaking party. Each party has to consider the preferences and aims of the other parties, in order to establish its best strategy and ultimately achieve an optimal equilibrium for all political players involved. For a treatment from a game theoretic perspective on the coalition formation, consult Greenberg (1994) and Riker (1962). In his landmark work on political coalitions Riker (1962) puts forward a well-known theory on political bargaining, stating that the main aim of each separate party is not to win the support of the largest amount of voters, but to form a "minimal winning coalition". According to the theory, such type of party's behavior enables it to save its energy and resources that would be spent in an extensive election campaign. In contrary, more recent works by Greenberg (1994)and Brams&Fishburn (1992) show evidence on the electorate playing a central role in the formation of party's cooperative/non- cooperative strategy. In the later study, Brams&Fishburn (1992) articulate that voters are active complements in terms of shaping the strategy of the parties in multi-party political systems. In this respect, the ultimate aim for the political parties is to form such coalitions that would satisfy the voters, by bringing a convergence of their political interests and ideology. For instance, Meffert&Geschwend (2010) carried out a study on voters in Austria and found out that the voting behaviour of Austrian electorate displays "non- separability". The collected statistics showed that Austrian voters are considering all the election outcomes simultaneously, including the potential coalition possibilities of the parties. The complex mode of voters' information processing can establish voting preferences for some political party, given that it will become a member of a particular coalition. The victory of the political parties depends on the "message" that the existing coalition or the potential coalition members convey to the electorate[3]. The

[2]We highlight that in this work we operate with the words "cooperation" and "non-cooperation", as conceived in the classical game theory. In the proposed model these terms more precisely denote the acts of"entering a coalition/ alliance" or "not entering a coalition/ alliance".

[3]An interesting example of the complex interplay of voters' expectations and the strategies of the political parties is the success of an intricate multi-party coalition, termed "Alliance". This coalition came to power in Sweden in the 2006 parliamentary elections after 4 center-right parties merged together to oppose the leading party, the Social Democrats. The process of coalition formation was accompanied by various disagreements. Finally, the "Alliance" was able to formulate a joint political program, the so called "Manifesto". The

parties that solely focus on the preferences of voters, the so called "vote maximizers", are highly dependent on the voting behavior of the electorate, see the seminal work by Downs (1957). At the same time, a political party may place more value on sustaining its ideology, the so called policy seeking behaviour. The third factor that may determine the strategy of the party is its aspiration for power, fulfilled by the means of increasing the number of its cabinet seats. Strom (1990) explored the above factors' impact on parties' behaviour, and formalized a "three factor" theory of coalition formation. Despite the orthogonal representation of the three key factors; policy seeking, cabinet seats seeking and voter support seeking in this spatial model, the author acknowledges that these factors are often not mutually exclusive but interconnected i.e., they are non-separably coupled in the process of party's decision making. Naturally, the ideology of the party is reflecting the aspirations of the voters as well as its desire for cabinet sits. Consequently, it becomes not possible for a party to fulfil its goals without the voters support, in a multi-party democracy. Moreover, the support of voters is vital for the very existence of the party on the political arena, where the most multi-party political systems have a requirement of passing an election threshold.

We also remark the importance of the timing of the coalition formation, as often discussed in political literature. A pre-election alliance emerges, when the parties participate in the elections process as a joint "team". Similarly, after the elections, the power distribution cannot be altered by other means than by creating a coalition with other parties, in order to form a minority/majority government. Notable cases of alliances[4] that emerged before the elections were held are the "Alliance" in Sweden and"Syriza" in Greece (Widfeldt, 2007; Syriza Party Homepage, 2013). The type of alliance-seeking behavior can be characterized by the parties' need to gain the support of voters as a result of the created image by the alliance members. Conventionally, in the process of alliance formation, the involved parties search to keep close their ideological ties on the so called left-right policy axis. In such contexts, the parties are dependent on the beliefs of voters about their success as an alliance. As a consequence, the parties search to be perceived by the voters as a strong and reliable political entity, see a discussion in the Electoral Knowledge Network (2012). The impact of voters is even more imperative for the post-election coalition emergence. In some cases the parties are left with no other options, but to establish a coalition agreement to stay in power. Many alliances and coalitions, such as the "Grand Coalitions" in Germany,

success of this multi-party coalition was attributed to the transparency of the conveyed information about their coalition plans and the subsequent supportive voting behaviour of the Swedish electorate. For detailed statistics consult the study by Widfeldt (2007).

[4] Pre-election coalitions are often termed alliances.

Italy and the Netherlands, as well as the coalition in the UK, were created in order to secure cabinet sits for the party members. This strategy enabled the parties to form a Government with majority sits, see mass-media coverage in Financial Times (2012), BBC News (2010), (2013), Spiegel Online International (2013).

The coalition formation is a complex process and an optimal equilibrium has to be established for the whole arrangement of participants. The voters definitely have a great impact on the strategic planing of their representative political parties. The voters are effectively shaping the strategy of these parties through their voting behavior on the election day. However, the parties that enter a coalition also keep in mind that the voters' support can swing in favour of an another political party, if their interests become neglected. The party's success in the subsequent elections can be easily jeopardized.

As Downs (1957), p.35, formulated in his milestone work: *..."the main goal of every party is the winning of elections. Thus, all its actions are aimed at maximizing votes."*

In the proposed model, the timing of the coalition formation can be tuned with the aid of appropriate Hamiltonian and Lindblad operators that incorporate the internal and external state fluctuations of the parties decision-making states. At this stage, we will primarily focus on the second type of coalition formation, the post election coalitions, where the voters' behavior greatly shapes the final choices of the political parties. In fact, the ultimate decisions of the parties can be very distinct from their initial preferences[5] Similarly to classical game-theoretic models, the proposed modeling, based on the mathematical tools of quantum physics, captures the approaching of a stable state of a decision equilibrium.

## 1.1 A Note on Non-separability of Political Decisions

Non-separability or strong interrelation of political decisions has been explored in more recent political studies and spatial representations of such preferences where devised (Lacy &Niou, 2000; Lacy, 2001; Finke, 2009; Finke &Fleig, 2013). These studies show that preferences of voters and also Governments are often not evolving in isolation; the issues and their outcomes are not unconditioned and unconstrained, but irreducibly connected with each

[5] As we can see, the above mentioned examples of coalitions are in a sense "exotic" in terms of the very polar ideological position of the coalition members. Despite of the initially rival political behavior, these parties can arrive to an equilibrium state of political cooperation, at least in a short term perspective. We show how this behavior can be captured mathematically in a simulation, see the Figures, 1-2 in section (4.3).

other by the decision-making states of the subjects. We briefly outline the characteristics of non-separability as defined in political science. We adopt a classical definition from Lacy (2001). Non-separability can be also characterized by the different "degrees" of its strength as well as different directions of its appearance.

Let $\mathbf{J} = \{1, ..., J\}$ be a set of issues. Let $\mathbf{o} = (o_1, ..., o_j)$ be a $\mathbf{J}$-tuple of outcomes across all $J$ issues. Define $x$ and $y$ as mutually exclusive and exhaustive non empty subsets of the $\mathbf{o}$. $x'$ is an outcome that differs from $x$ on at least one issue, and $y'$ differs from $y$ on at least one issue. Now suppose individual $i$ has a reflexive and transitive weak preference relation[6], $\succeq_i$, ordering all $\mathbf{J}$-tuples of policy outcomes. Then $i's$ preferences are:

- separable iff for all $x, y, y'$, $(x, y) \succeq_i (x', y)$ and $(x, y') \succeq_i (x', y')$.
- completely non-separable iff for all $x$ there exists a $y$ and $y'$ such that $(x, y) \succeq_i (x', y)$ and $(x', y') \succeq_i (x, y')$. (Lacy 2001, p. 240)

Non- separability reveals a more complex nature of human preferences, where in a political context the outcomes of one political issue in sense generate preferences for the outcomes of other issues. In contrast to what is often assumed in traditional political science studies, preferences are not fixed over time and isolated from other decision-making contexts. In political literature, this phenomenon has been mainly studied among voters (due to the possibility to obtain detailed statistics through surveys and opinion/exit polls). Non-separability of governmental and party decisions has not been so widely explored at this stage. However, Finke (2009) and Finke &Fleig (2013) present statistics on the existence of EU member states' non-separable behavior related to several political issues.

As mentioned above, we propose for a quantum formulation of the non-separability of parties' decision- making states in the context of coalition construction, where the preferences of different parties can strongly interrelate with each other. The motivation for this development stems from the findings elaborated by Zorn &Smith (2011), Khrennikova, Haven &Khrennikov (2014), Khrennikova &Haven (2016) and Khrennikova(2015). These studies provide broad argumentation, including empirical evidence on the non-classical origins of non-separability, i.e. it is not just about the probabilistic conditioning of decision outcomes in a Bayesian fashion.

[6]To establish for a formal representation of a preference relation, economic axioms are serving as building blocks that allow to establish an ordering of preferences. Reflexivity steams from the axiom of preference completeness and pertains to a preference equivalence, where e.g., for outcomes $x$ and $y$, $x \sim y$, iff $x = y$.

It is worth to mention that the probabilistic features of the quantum formalism are closely linked to the state space representation. When we talk about the multifariousness of the representation of preference states and the classical spatial models of voting resting on the Euclidean (weighted) linear space, characterized by metric distances between the preference points, we witness that due to the geometric properties of the Euclidean space this representation suffers from the constraints of detecting the various specific features of non-separability, such as its direction. When the direction of measurement matters we are faced with the violation of the principle of commutativity. With other words, the direction of non separability should not play any role, in spatial models, for instance, an outcome $A$ followed by an outcome $B$ would not differ from a different ordering of their realization. Likewise, the Euclidean space representation of preferences is not coupled to the probabilistic nature of the outcomes, which lies at the heart of the quantum representation of the observables. A more comprehensive account on the similarities and differences of spatial models based on the Euclidean state space as opposed to the Hilbert space can be found in Khrennikova &Haven(2016).

## 1.2 Applicability of Quantum Formalism to Decision processes

Recently, the mathematical formalism of quantum theory and its methodology found a variety of applications beyond physical phenomena: in cognition, psychology, psychophysics, economics, finance, and most recently in politics. Since the number of publications in this novel field of research increases rapidly and the diversity of applications is vast, we refer only to the monographs by Busemeyer &Bruza (2012), Haven &Khrennikov (2013) and references in them. Modeling of decision-making processes in a quantum framework is becoming an established interdisciplinary field. Some notable contributions are by Busemeyer, Wang and Townsend (2006), Pothos &Busemeyer (2009) and Lambert-Mogiliansky &Busemeyer (2012). This paper can be considered as a part of this development, namely decision-making processes in politics.

Inside the quantum-like field, essential efforts were made in the foundational studies, in particular, *on the justification of the applicability of the methods of quantum theory to cognition, psychology and decision-making.* The main motivation for such applications lies in the complex probabilistic structure of human decision-making and judgement. Since more than half a century, when the the foundational "rational economic decision theories" were firstly formalized, psychologists collected experimental statistical data that exhibited features paradoxical from the viewpoint of classical decision theories, which rest upon the classical probability theory, see for example, the seminal experiments carried out by Tversky and Shafir (1992) and Shafir & Tversky (1992). Various fallacies of human reasoning were discovered, e.g., conjunction and disjunction effects, order effects and framing effects. Essentially, one can treat these fallacies as an exhibition of contextuality of human behaviour, where human judgements and choices are intrinsically context-dependent. The features of the experimental data can be mathematically formalized as violations of the laws of classical probability theory. More specifically, the formula of total probability is violated, as well as the Bayesian updating scheme. As a consequence, Bell's inequality (which derivation is based on the possibility to represent statistical data, by using a single classical probability space) is also violated. It is well-known that statistical data, collected in quantum physical experiments, contravene the laws of classical probability theory. For example, the basic quantum effect, interference, demonstrated in the two slit experiment, is probabilistically equivalent to the violation of the formula of total probability (Feyman and Hibbs, 1965). Due to the very similar features of psychological and quantum experimental data, it became natural for the researchers in this field to apply the formalism of quantum theory interdisciplinary. A particular focus is placed on geometric properties and probability theory of QM, to model cognitive processes.

As a result of the endeavours by the constantly growing "Quantum Cognition" community members, the statistical data collected in cognitive psychology, sociology, and politics was successfully modeled, including the description of aforementioned psychological effects, see e.g., Pothos and Busemeyer (2009), Asano et al. (2012), Lambert-Mogiliansky and Busemeyer (2012), Bysemeyer and Bruza (2012) , Haven and Khrennikov (2013), Busemeyer et al.(2006), Wang and Busemeyer (2013) and Khrennikova (2014a),(2014b). Nevertheless, by borrowing the mathematical apparatus of quantum physics one confronts a following foundational problem, namely: *Can one guarantee that the quantum probabilistic formalism would completely capture the decision making processes of individuals?* It is fair to say that this was the implicit assumption of the modern decision theories under risk and uncertainty that utilized the classical Kolmogorovian probability theory as a complete mathematical apparatus for dealing with the involved uncertainties. Similarly, at this stage of development, one cannot guarantee that some of the surfacing psychological effects or their combinations will be in accord with the principles of quantum theory. However, one should stress that even in the quantum physical community, nobody can guarantee that in the future developments, the present quantum formalism will not be modified e.g., to correspond with the Einstein's general relativity.

Another advantage of the quantum formalism is that this is a complete theory i.e., it is not an ad hoc modification of the classical probability theory[7]. Quantum probability is a theory that is composed of valid and complete set of rules, such as Born rule. By applying this formalism to cognition, psychology, political studies one doesn't need to construct complicated models, taking into account all the impact factors. Another complication of the model construct is related to the impossibility to determine some of the decision-making factors empirically, at least with a good precision. By representing the cognitive phenomena with the aid of the quantum formalism one can talk about the minimization of the complexity that classical theories would carry, even if the construction of a classical probabilistic model is formally attainable (i.e.,"hidden variables may exist").

On the conceptual level, the notions used in quantum theory deeply resonate with the heuristics of cognitive modeling. For example, consider the notion of superposition of states: the majority of psychologists would accept that human mind can be in a superposition of a few mental states, i.e., the preferences on some matter are not fixed, but vacillate as time passes. However, such a qualitative explanation would be merely a heuristic statement. In contrary, the quantum formalism provides a mathematical justification for the above mentioned effect. One can allude to formal models, describing the psychological phenomena (cf. with the Euclidean models used in political studies, section 1.1) Last but not least, we point to the non-Boolean structure of quantum logic, which can be mathematically confirmed, e.g., in the violation of the distributivity and commutativity axioms. The former occurs for a variety of human judgements, e.g., when the statistical data cannot be expressed by the means of the formula of total probability. The violation of commutativity is manifest in an "order effect", whose investigation plays an important role in psychology. In quantum models this effect is represented by the means of non-commutative observables.

### 1.2.1 Evolution of the field of "quantum political studies"

Political decision making is a special sphere where humans have to make many decisions with far- reaching implications for the individuals and the society in general. This can involve ballot casting in different types of elections from local to national. On the party level the involved parties as political

[7] In a nutshell, the quantum probabilistic framework is able to accommodate statistical data that a classical probability model can. At the same time, the quantum probability is a more general theory than the classical probability theory, which can also contain non-classical phenomena.

entities have the responsibilities to strategically plan their political actions, by taking into consideration all the possible consequences. Of course, the decisions made in politics are context specific, i.e., it can be difficult to associate the political decisions with some concrete payoffs and risks as formalized by modern decision theories in economics. Nevertheless, traditional political theory is highly inspired by the modern economic schools and it is naturally assumed that as the political decisions are taken, the individuals act in a rational way, by exhibiting a consistency of their preferences (at least in a short term perspective). If these assumptions of the economic theories hold, a well defined ranking of political preferences can be established. One should note that until the more recent contributions, political preferences were represented as separable in the spatial models in politics. To be more specific, each political preference would exist on its own, independently (separably) from other preferences, as conceived in the foundational work by Enelow and Hinich (1984). At the same time, new pieces of information cause changes in the existing preferences and degrees of beliefs. As postulated by the normative choice theories, when the information is uncertain, the decision makers update their preferences and beliefs in a Bayesian fashion.

Decision making in politics, as well as in other social spheres is of a complex nature and multiple pieces of information have to be considered. In various decision-making contexts, preferences on different issues are strongly interrelated (non-separable) as well as the information is not processed in a classical mode. One may talk about irrationality of human reasoning. At the same time, one could argue that the traditional models of human reasoning may have limitations, whereas the quantum formalism provides a worthy illumination of the observed probabilistic fallacies as well as other paradoxes of human reasoning.

In the "quantum political studies" we highlight pioneering articles by Zorn and Smith (2011), Khrennikova et al. (2014), Khrennikova (2015), Bagarello (2015b) and Khrennikova &Haven (2016). The study by Zorn and Smith (2011) is exploring US electorate's voting behaviour in the Congress and Presidential elections, with the focus on the bipartisan tactics of some voters. This part of the electorate prefers to "put the eggs in different baskets" (called "ticket splitting" in political literature) by voting, e.g., for Democrats in the Congress election and at the same time basting ballots for a Republican President. Zorn and Smith (2011) are interconnecting politics and quantum formalism, by pointing to the role of quantum entanglement as a powerful tool for modeling statistical non-separability of voters' preferences. This idea was reinforced in Khrennikova (2015) and Khrennikova &Haven (2016). The authors showed with the aid of statistics on voters preferences that non-separability cannot be attributed to a simple Bayesian conditioning and accommodated in a classical probabilistic framework (see the above discussion on violation of the laws of classical probability). The features of non-separability emergence indicated that the quantum representation of observables could serve as a noteworthy alternative. In Khrennikova et al. (2014) this approach was combined with the theory of open quantum systems, to capture the time dynamics of voters' preferences as well. This theory gives the most general mathematical model of the state's adaptive dynamics of a system interacting with an environment.

In a series of papers (Asano , Tanaka, Basieva & Khrennikov, 2011; Asano, Ohya, Tanaka, Basieva &Khrennikov, 2012; Asano, Basieva, Khrennikov, Ohya and Yamato, 2013) the theory of open quantum systems, and more generally quantum adaptive dynamics were applied to model decision making. The applications ranged from modeling irrational behavior in games of the Prisoners Dilemma type to recognition of ambiguous figures (for the latter work, consult Asano, Khrennikov , Ohya, Tanaka and Yamato, 2014). These works introduce the concept of a psychological "bath" to describe the dynamics and stabilization of a mental state to a classical decision-state. Khrennikova et al. (2014) explored the application of the theory of open quantum systems to model the bipartisan behaviour of the American electorate, by extending the quantum -like treatment of voters' preference states, proposed in Zorn and Smith (2011). The first quantum model of creation of coalitions between political parties, taking into account the impact of the voters' behavior, was elaborated by Bagarello (2015b). The author applied the mathematical formalism of quantum field theory to derive dynamical equations for evolution of parties' preferences for creation of political alliances.

In the present study, following the treatment of this subject in Bagarello(2015b), we propose a model of coalition formation between political parties, by exploring the quantum entanglement of preferences and aspirations of party leaders and their electorates[8]. The model could be potentially applied to more general areas cooperation establishment, with respect to special political and economic issues.

The "environment" that impacts the behavior of the party as a system is a complex combination of factors, where the key role is played by the electorate and their preferences. The role of electorate can be more or less crucial for the formation of the final decision equilibrium depending, on the timing of the coalition formation. The motivation for deriving these methodological assumptions comes from the previous findings in this game-theoretic area

[8]For an extended elaboration on the existence of "pseudo-classical non separability" in decision-making tasks that is mathematically and conceptually reflected in the quantum notion of entanglement, consult Zorn and Smith (2011)

of competition and coalition formation, discussed in the Introduction part. In accordance with the proposed quantum model, the states of preferences for (non)cooperation of a group of political parties $\mathcal{P}_1, ..., \mathcal{P}_n$ are represented in a complex Hilbert space. The key point is that these states are strongly interconnected, i.e., entangled. The preferences of different parties evolve simultaneously and non-separably in the joint information space. We model the approaching to the state of political equilibrium by using the Markov approximation of the quantum master equation. Since the multi-parties' state is represented in the tensor product of the state spaces for each individual party, the dimension of the state space increases exponentially, with the growth of the number of parties.

Coming back to the classical modeling of creation of political coalitions, we can add that our approach extends the classical Markov dynamics of approaching the equilibrium state characterizing a yes/no decision of each party, with respect to a political coalition establishment. We use a more general Markov dynamics given by the quantum master equation. In some sense this approach provides a possibility to represent a deeper state of uncertainty on the political arena, namely the uncertainty expressed by a superposition of alternatives. As was already stressed, another distinguishing feature of the model is a possibility to represent a deeper non-separability between preferences of different parties, non-separability in the form of entanglement. One can also rise the issue of contextuality of decisions on the political arena. Correlations corresponding to entangled states related to the decisions, irreducibly depend on the political contexts. Coming back to coalition modeling, one can say that in such states, political parties do not have their own, intrinsic and fixed preferences for (non)cooperation. Their preferences are characterized by a contextual complexity, with respect to the preferences of other parties.[9]

The ideas and methods elaborated in this paper, can be considered as first steps towards the application of the theory of open quantum systems for mathematical modeling of political decision processes. We hope that the model and methods developed in this paper, will be applicable to a variety of problems in decision-making and more specifically, cooperation and competition cases. The quantum Markov equation that we apply, is a most widely used approximation of quantum master equation, describing quantum adaptive dynamics. It also provides the most general scheme of a

[9]Contextuality is one of the bridges between the standard quantum theory for physical systems and cognition and psychology, see works by De Barros &Suppes, (2009), Dzhafarov &Kujala (2012), Dzhafarov &Kujala (2013) and Asano et al. (2014).

quantum measurement (Zurek, 2003). Hence, it can be considered as an apt candidate for the decision-making processes. Undeniably, to implement this mathematical model to collective decision making (where the political party is considered as a system), one has to justify its applicability. It is well known that its derivation is based on a set of assumptions on interaction of a system and an environment. Therefore, by applying this equation one has to make sure that the assumptions are satisfied. This is especially important to do this procedure, when applying the quantum Markov equation outside its original domain of application. Such an analysis was performed by Khrennikova et al. (2014), in the section "Matching of the Assumptions of Applicability", devising the principles of model's applicability to decision making process in a social environment. In principle, this procedure could be repeated in the context of the present work. At this stage, due to the limited scope, we refer to the aforementioned paper, since exists an essential similarity between the models for adaptive political decision-making of voters in Khrennikova et al. (2014) and the present paper.

By formulating a novel mathematical model, based on the formalism of quantum mechanics and quantum information theory, we are aware that the reader may be not familiar with the mathematical and conceptual formalism of quantum mechanics (QM) that is used throughout this paper. We briefly introduce some of the core notions of QM in the next section, 2. The reader can consult books by Jaeger (2007) and Busch, Grabowki & Lahti (1995) for an in depth mathematical treatment of the notions introduced in the following section.

# 2 Brief introduction to quantum formalism

The state space of QM is based on a complex Hilbert space $H$, i.e., a complex linear space, endowed with a scalar product, denoted as $\langle\psi_1|\psi_2\rangle$ which is complete with respect to the norm: $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. Normalized vectors of $H$, i.e., $\psi$ such that $\langle\psi|\psi\rangle = 1$, represent a special class of states of quantum systems, namely, the *pure states.* A normalized vector determines a pure state up to the phase factor $e^{i\theta}$, i.e., two vectors $\psi_1$ and $\psi_2 = e^{i\theta}\psi_1$ determine the same pure state.

To study open quantum systems, i.e., quantum systems interacting with environment, we also have to consider the so-called *mixed states.* They are represented by *density operators*, i.e., operators which are Hermitian, positive semi-definite and trace one. We recall that a linear operator $\rho$ is Hermitian if, for any pair of vectors $\phi_1, \phi_2$, $\langle\rho\phi_1|\phi_2\rangle = \langle\phi_1|\rho\phi_2\rangle$; it is positive semi-definite if, for any vector $\phi$, $\langle\rho\phi|\phi\rangle \geq 0$.

We remark that a pure state $\psi$ also can be represented by the density operator – the orthogonal projector onto the vector $\psi$. Denote it $\rho_\psi$. Any density operator $\rho$ can be represented as a weighted sum of such orthogonal projectors:

$$\rho = \sum_i q_i \rho_{\psi_i}, \tag{1}$$

where $q_i \in [0,1], \sum_i q_i = 1$, and $(\psi_i)$ are pure states. This expansion leads to the interpretation of the mixed state $\rho$ as representing an ensemble composed of quantum systems in pure states $(\psi_i)$. The weight $q_i$ gives the probability to pick up a system in the state $\psi_i$ from this ensemble.

In the quantum formalism *observables* are represented by Hermitian operators. Consider a state represented by the density operator $\rho$ and an observable represented by the Hermitian operator $A = \sum_i a_i P_{a_i}$, where $(a_i)$ are its eigenvalues and $(P_{a_i})$ are projectors onto the corresponding eigensubspaces [10]. The probability to obtain the concrete value $a_i$ as the result of a measurement is given by the Born's rule, formulated by Born (1926). $p_\rho(a_i) \equiv p_\rho(P_{a_i}) = \mathrm{Tr}\rho\mathrm{P}_{\mathrm{a_i}}$. In particular, if $\rho_\psi$ is a pure state, then $p_{\rho_\psi}(a_i) = \langle P_{a_i}\psi|\psi\rangle = \|P_{a_i}\psi\|^2$.

The so called "Dirac's notations" are widely used in quantum information theory. Vectors of $H$(the Hilbert state space) are called *ket-vectors*, they are denoted as $|\psi\rangle$. Let us restrict our consideration to the case of a finite dimensional $H$ and consider an observable $A$. As such, the normalized eigenvectors $e_i$ of $A$ form an orthonormal basis in $H$. Let $Ae_i = a_i e_i$. In Dirac's notation $e_i$ is written as $|a_i\rangle$ and, hence, any pure state can be written as $|\psi\rangle = \sum_i c_i |a_i\rangle, \ \sum_i |c_i|^2 = 1$.

Qubit states are represented with the aid of some observables with nondegenerate spectra having the eigenvalues $0, 1$. Denote the corresponding eigenvectors as $|i\rangle, i = 0, 1$. Then $|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \ |c_0|^2 + |c_1|^2 = 1$. Naturally, each qubit space is two dimensional.

A pair of qubits is represented in the tensor product of single qubit spaces, here pure states can be represented as superpositions of four eigenstates:

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|00\rangle, \tag{2}$$

where $\sum_{ij} |c_{ij}|^2 = 1$. In the same way the $n$-qubit state is represented in the tensor product of $n$ one qubit state spaces (it has the dimension $2^n$) : $|\psi\rangle = \sum_{x_j=0,1} c_{x_1 \ldots x_n} |x_1 \ldots x_n\rangle$, where $\sum_{x_j=0,1} |c_{x_1 \ldots x_n}|^2 = 1$. We remark that the dimension of the $n$ qubit state space grows exponentially with the growth of $n$.

[10] In the finite dimensional case any Hermitian operator can be represented in this form.

Consider the tensor product $H = H_1 \otimes H_2 \otimes ... \otimes H_n$ of Hilbert spaces $H_k, k = 1, 2, ..., n$. The states of the space $H$ can be *separable and non-separable*. Non-separable states would be the so called *entangled states*. Let us start with representing mathematically the non- separable and separable pure states. The states from the first class, i.e., separable pure states, can be represented in the form:

$$|\psi\rangle = \otimes_{k=1}^{n} |\psi_k\rangle = |\psi_1 ... \psi_n\rangle, \tag{3}$$

where $|\psi_k\rangle \in H_k$. The states which cannot be represented in this way are called non-separable, entangled. Essentially, the mathematical representation of entanglement is very simple, it means an impossibility of tensor product factorization.

For example, we consider the tensor product of two qubit spaces. In each of them we select an orthonormal basis, denoted as $|0\rangle, |1\rangle$. The corresponding orthonormal basis in the tensor product has the form $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Then so called Bell's states (Bell, 1987):

$$|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}; \ \ |\Phi^-\rangle = (|00\rangle - |11\rangle)/\sqrt{2}; \tag{4}$$

$$|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}; \ \ |\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \tag{5}$$

are entangled.

Now consider a quantum state given by a density operator in $H$. This state is called separable, if it can be factorized in the product of density operators in spaces $H_k$ :

$$\rho = \otimes_{k=1}^{n} \rho_k, \tag{6}$$

otherwise the state $/rho$ is called entangled. We remark that an interpretation of entanglement for mixed states is even more intricate than for the pure states.

Although the notion of entanglement is mathematically straightforward, its physical interpretation is one of the main challenges of modern quantum foundations. In this paper we have no possibility to discuss the problem of interpretations of entanglement in quantum physics versus cognition and psychology. We proceed operationally and use entanglement as a mathematical tool for representation of correlations in *a multi-contextual framework*, see, e.g., De Barros and Suppes (2009) for a foundational discussion.

# 3 State space of the (non)coalition creation model

## 3.1 One party preference state space

On the political arena each party $\mathcal{P}_j$ can either prefer to cooperate or not with other parties, $\mathcal{P}_i, i \neq j$. The preference space of $\mathcal{P}_j$ for cooperation with the fixed party $\mathcal{P}_i$ can be mathematically represented (by applying the notations of QM) as one qubit space $H$ with the basis $(|0\rangle, |1\rangle)$ encoding preferences for (non)cooperation. The dichotomous nature of the outcomes, in the form of yes/no stems from the requirements dictated by the election procedure in a multi-party political system. We remind that we treat the cooperation or non-cooperation in the setting of this model as a decision of some party to form (not to form) a coalition with some other party(ies). Clearly, a creation of a coalition does not guarantee that all policies of a party are supported by other members of the coalition. The process of establishing a coalition can be very fragile, i.e., if you give up some of your policies you lose the potential and existing voters as a party. We do not have a possibility to analyze the whole life cycle of the coalition, thus we treat the agreement (disagreement) of entering a coalition as a "final destination" of a party's decision-making process.

One of the main rationales for the quantum-like information description is that $\mathcal{P}_j$'s preferences can be in the *superposition of non-cooperation and cooperation*. Such superpositions are naturally represented in the quantum formalism as:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \tag{7}$$

where $c_0$ and $c_1$ are complex numbers, $|c_0|^2+|c_1|^2 = 1$. Here $|c_0|^2$, $|c_1|^2$ give the probabilities $p_0, p_1$ that $\mathcal{P}_j$ will make the decisions to (non)cooperate. We remark that complex numbers have not only amplitudes, but also phases, $c_k = |c_k|e^{i\theta_k}$. In the quantum formalism the phases, more precisely the relative phase $\theta_2-\theta_1$, also play an important role. The presence of relative phases contribute nontrivially to the state dynamics, either the Schrödinger dynamics describing the evolution of the preference state of a party in the isolation from a "social environment" or the dynamics based on the quantum master equation taking into account interaction with a "social environment". Here the situation differs crucially from say classical Markovian state dynamics which takes into account only the probabilities $p_0, p_1$.

For the fixed political party $\mathcal{P}_j$, the complete state space for preferences for (non)cooperation, $H_j$ is represented (in accordance with quantum information theory) as the tensor product state space corresponding to preferences for other political parties $\mathcal{P}_i, i \neq j$. By denoting the latter as $H_{ji}$ we write

$$H_j = H_{j1} \otimes ... \otimes H_{j(j-1)} \otimes H_{j(j+1)} \otimes ... \otimes H_{jn}. \quad (8)$$

The dimension of this space is equal to $d = 2^{n-1}$. Here $n$ is the total number of political parties under consideration.

Such a state space, i.e., reflecting only the preferences of one fixed political party for other parties, represents another purely quantum information effect, namely, *entanglement*: entanglement of the (non)cooperation preferences of $\mathcal{P}_j$ for other parties. The states of the space $H_{ji}$ can be *separable and non-separable* (entangled).

From the interpretational viewpoint, the notion of entanglement is one of the most complicated notions of QM. One of the features of entanglement (in the framework of our modeling) is that the party $\mathcal{P}_j$ cannot treat its preferences for (non)cooperation with the parties $\mathcal{P}_i, i \neq j$, separately. The party $\mathcal{P}_j$ cannot split its preference state $|\psi\rangle \in H_j$ into the preference states related to individual $\mathcal{P}_i$. To proceed to a decision on (non)cooperation with the fixed $\mathcal{P}_i$, $\mathcal{P}_j$ takes into account its possibilities of (non)cooperation with all $\mathcal{P}_k, k \neq j, i$.

The presence of entanglement (non-separability effect) is even stronger in the multi-parties preference state space, see section 3.2.

Finally, we remark that even separable preference states carry an essential degree of quantumness, related to the superposition effect. Suppose that each qubit state $|\psi_i\rangle$ in (7) is superposition of the preferences for non-cooperation and cooperation, see (7). Then this state has the form of superposition

$$|\psi\rangle = \sum_X c_X |X\rangle, \quad (9)$$

where $|X\rangle = |x_1...x_{j-1}x_{j+1}...x_n\rangle, x_j = 0, 1$ and $\sum_X |c_X|^2 = 1$, the numbers $|c_X|^2$ give the probabilities $p_X$ of $\mathcal{P}_j$'s decisions on (non)-cooperation with other parties. An arbitrary (pure) state $|\psi\rangle$ of preferences of the political party $\mathcal{P}_j$ for (non)cooperation with the political parties $\mathcal{P}_i, i \neq j$, can be represented in the form (9). This superposition state also encodes the quantum interference effect.

## 3.2 Multi-parties preference state space

In the light of the previous considerations, we can say that the preferences of each party $\mathcal{P}_j$ for (non)cooperation with other parties can be mathematically represented by the tensor product of one qubit state spaces, corresponding to the party's the preferences for (non)cooperation with other individual parties. This space was previously denoted as $H_j$. The real coalition formation perspective involves the preferences of all parties for each other. The complete preference state space for all parties involved, is mathematically represented as the tensor product $H = \otimes_j H_j$. In the qubit representation its vectors have the form:

$$|\Psi\rangle = \sum_{\mathcal{X}} C_{\mathcal{X}} |\mathcal{X}\rangle, \tag{10}$$

where $\mathcal{X} = X_1...X_n, X_j = x_1...x_{j-1}x_{j+1}...x_n, x_j = 0, 1$ and $\sum_{\mathcal{X}} |C_{\mathcal{X}}|^2 = 1$.

The dimension of this space is equal to $D_n = 2^{n(n-1)}$. We remark that $D_n$ increases considerably with the increase of the number of parties on the political arena. The appearance of one additional party (of the size and political influence, such that this party is taken into account by other parties) increases essentially the dimension of the state space and hence, the complexity of the process of decision making. For example, $D_2 = 4$, but already $D_3 = 64$, and the appearance of the fourth party would lead to a state space of a very large dimension, $D_4 = 4096$ (correspondingly, the emergence of a fifth party on the political arena, implies a drastic complication of the political situation, $D_5 = 1048576$). In the political reality, the state space is a proper subspace of $H$, because some types of cooperation would be in principle impossible.

Coming back to the example of coalition creation at the Swedish political arena, theoretically, the left parties, such as the Left-party (Vänster partiet) and the Social-Democratic party cannot reach to a decision of cooperation with the nationalist party- the Swedish democrats and vice versa. This constrain reduces eightfold the dimension of the state space. Further, the two leftist parties are typically cooperating with each other on the Swedish political arena. In principle, they can be treated as a single party - the state space dimension shrinks by a factor 4. Effectively, simply as the result of the principle of disagreement between the leftist and nationalist parties, the dimension is reduced by a factor 32. In the case of the existence of five major parties this leads to the state space of the dimension $D'_5 = 32768 << D_5 = 1048576$. There can be other political constraints minimizing the state space dimension[11]. Nevertheless, even with all these constraints, due to the elevated

[11]The Swedish Green Party (Miljöpartiet) cooperates actively with the Social-Democratic party, but, for many questions, its cooperation with the Left-party is impossible; at the same time the Social-Democratic party demonstrates ( that is relatively new party to be parliamentary represented) the wish to cooperate with both, the Swedish Green Party and the Left-party. The Swedish Moderate Party (Moderaterna) can in principle cooperate with the Swedish Democrats, but the cooperation with the Left Party is completely excluded. See Widfeldt (2014) for a detailed discussion. Thus, even the

dimension of the state space, the task of modeling of the process of approaching a consensus between parties (even if they are few of them ) can become a complex multi-dimensional mathematical problem.

In the preference space $H$ we again obtain both quantum effects, namely, superposition and entanglement. As a result of entanglement, the political parties in a sense "lose their individual control over decisions on (non)cooperation with other parties." The decisions of each political party $\mathcal{P}_j$ are irreducibly connected with the possible decisions of other parties.

Mathematically, a preference state is separable if it can be represented in the form:

$$|\Psi\rangle = \otimes_{j=1}^{n} |\Psi_j\rangle = |\Psi_1 ... \Psi_n\rangle, \tag{11}$$

where $\Psi_j \in H_j$. An entangled state cannot be represented in this way.

For $n \geq 3$, there exists an another kind of entanglement- the multipartite entanglement, that has new features, absent in the case of bipartite entanglement. Its interpretation is even a more complicated task than of the bipartite entanglement.

# 4 Decision making and state's dynamics

Following the tradition of quantum-like modeling of the dynamical processes of decision making, c.f. Busemeyer et al. (2006), Asano et al. (2011), Asano et al. (2012), Busemeyer and Bruza (2012), Bagarello (2012), Haven and Khrennikov (2013), Pothos and Busemeyer (2013), Khrennikova et al. (2014), Bagarello (2015a), Bagarello (2015b) and Khrennikova &Haven (2016) we represent the process of establishing of cooperation between political parties as a quantum state dynamics. The simplest quantum state evolution is described by the Schrödinger's equation. It models an evolution of the state of a quantum system, which can be treated (at least with some degree of approximation) as isolated from the outer informational surrounding. If the influence of the environment cannot be neglected, then the state evolution is modeled by the quantum master equation. The latter is typically very complicated, this is why its (quantum) Markovian approximation is very popular in many applications.

political arena of such a small country as Sweden is characterized by a high complexity of constraints on the information state space. There are of course different factors that can always shift these constrains, for instance, in a situation when totally opposite parties in terms of their policy and ideology come to a cooperation agreement. These types of coalitions can emerge as a result of a strong mutual aspiration for power and the particular timing of coalition formation. Further examples can be found in section 5.

We point out that decision-making models based on Schrödinger's equation and the quantum master equation (which describes the nontrivial influence of an environment) differ substantially.

## 4.1 Decision making process by Schrödinger's equation

We start with a brief mathematical remark to delineate the core features of such a construct of a system's quantum dynamics. Solutions of Schrödinger's equation different from stationary ones are represented as linear combinations of imaginary exponents (combinations of sines and cosines). Such linear combinations fluctuate as functions of time and no limit exists for $t \to \infty$. They cannot approach a concrete state with the time increasing, i.e., $\lim_{t\to\infty} \psi(t)$ does not exist.

Therefore, in applications to the dynamics of cognitive systems, to make a decision, decision makers in the process of coalition creation, e.g., the leaders of a political party, would have to intervene into the dynamics of the state in an "authoritarian way" leading to a type of "collapse of the state". It is important to discern that such a collapse would be produced by a decision of any political party, if their preference states are entangled with the preference state of other parties. Decisions of such a type can of course be possible and even quite common for parties with very strong leaders or internal party spirit. In such a context, decisions (related to establishing a coalition with other parties) would be made in isolation, without the adjustment to the aspirations of the electorate, as well as of the society in whole - the so called"common social environment".

Besides of the fluctuating behaviour of the solutions, another problematic feature of Schrödinger's dynamics for preference states is that, as was already pointed out, it preserves the stationary states of Hamiltonians forever. Suppose that there are two parties $\mathcal{P}_1, \mathcal{P}_2$, then each state space is just a qubit space, i.e., $H = H_1 \otimes H_2$ is the four dimensional state space. If the joint Hamiltonian of the pair of parties $\mathcal{H}$ has, e.g., the state $\Psi_0 = |00\rangle$ as an eigenstate, i.e., $\mathcal{H}\Psi_0 = \lambda_0\Psi_0$, then the preference state $\Psi(t)$ will have the form $\Psi(t) = e^{-it\lambda_0/\gamma}\Psi_0$, where $\gamma$ is a factor determining the time scale of the dynamics (if the $\mathcal{H}$ is chosen as a dimensionless quantity). As a consequence, this kind of dynamics in principle cannot lead to establishing a cooperation between these two political parties, i.e., to the state $\Psi = |11\rangle$.

**Remark 1.** (Interpreting Hamiltonian) In QM $\mathcal{H}$ has the dimension of energy and here $\gamma = \hbar$ is the reduced Planck constant $\hbar = h/2\pi$. It has the dimension of action= energy$\times$time. One may search to proceed in the same way by inventing a notion of "political energy" (or "social energy") which is heuristically quite natural. However, in such an approach, the main challenge is the development of a measurement methodology for such kind of "mental energy." This is a complicated problem that would require further analysis of empirical data and we postpone a discussion on it to future publications. In this paper we proceed operationally, by devising the overall structure of the quantum dynamics applied to party's decision making. At this stage we are preliminarily considering some possible components that could constitute the social analogue of the Hamiltonian operator. Due to the novelty of the application of quantum-like models to political science and decision processes, the Hamiltonian is treated in the developed model merely as the generator of a state dynamics. It is palpable that a mental state (individual or collective) can evolve over time. In the quantum-like model states (pure) are represented in the complex linear space and the dynamics is also assumed to be linear. In the case of an isolated cognitive (or social, or political) system the state-evolution is described as unitary dynamics. Hamiltonian is the generator of this unitary dynamics. Thus, in our setting the Hamiltonian is in a sense a phenomenological entity. Nevertheless, the question of construction of a concrete Hamiltonians has to be addressed. In physics there are two basic procedures of constructing Hamiltonians. The most known and widely used is the one based on the Schrödinger quantization procedure. One borrows from the classical physics (presented in the Hamiltonian formalism) the Hamiltonian function combined from kinetic and potential energies, $\mathcal{H}(q,p) = \frac{p^2}{2m} + V(x)$, and then quantizes it by utilizing instead of classical coordinate and momentum variables, $x, p$, the corresponding quantum operators, obtained by the rules postulated by Schrödinger: $x$ is mapped into the operator of multiplication by $x$ and $p$ is mapped into operator proportional to the derivative. We recall that the state space of the Schrödinger representation is given by the space $H = L_2(\mathbf{R}^3)$ of square integrable functions. Unfortunately, this approach is problematic, if possible at all, to generalize to the applications in cognition and decision-making in social and political studies. We do not have a classical Hamiltonian theory of social phenomena. Roughly speaking, there is nothing to quantize. However, there is another way to obtain Hamiltonians, which is more promising for our applications. In quantum field theory, Hamiltonians are often constructed in the Fock space, with the aid of operators of creation and annihilation of quanta (excitations of a quantum field). Processes of creation and annihilation are meaningful not only for quanta of physical fields, but even for quanta of information fields having, e.g., mental, or social, or political interpretation. As the starting point, one can introduce operators of creation and annihilation $a$ and $a^\star$, an introduction can be found in e.g. Miller (2008).

We now come back to the unitary evolution of the state of political preferences. The important case of such conservation of the initial preference is the case of the absence of the direct interaction between the parties $\mathcal{P}_1, \mathcal{P}_2$. This situation is described by the Hamiltonian of the form:

$$\mathcal{H}_0 = \mathcal{H}_{01} \otimes I + I \otimes \mathcal{H}_{02}, \tag{12}$$

where $\mathcal{H}_{0j} : H_j \to H_j, j = 1, 2$, are Hamiltonians generating the preference dynamics of parties, which do not try to negotiate or send other signals to each other in favour or against a political cooperation. For example, the leaders of $\mathcal{P}_1$ can have a meeting to discuss their own preferences to non/cooperate with $\mathcal{P}_2$ (this sort of decision- making activity contributes to $\mathcal{H}_{01}$). Of course, we understand well that this would be an idealization of the real political situation. On a real political arena, the process of "signaling" between parties and electorates cannot be ignored in principle. However, even in physics the notion of an isolated system is an idealization of the real physical situation, since the vacuum is as well contributing into the system's dynamics. Nevertheless, such a separation of internally and externally generated dynamics is a useful approach that can be used both in physics and for social phenomena.

These free Hamiltonians can be defined with the aid of "number operators:"

$$N_j|i\rangle = i|i\rangle, i = 0, 1. \tag{13}$$

In the matrix form we have

$$N_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

Then $\mathcal{H}_{0j} = \omega_j N_j$, where the parameters $\omega_j$ determine the frequencies of oscillations.

We can observe that a combination of such non-interactive dynamics with the impact of a social environment can transfer the non-cooperation state $\Psi_0 = |00\rangle$ into the cooperation-state $\Psi = |11\rangle$. Roughly speaking, even if parties do not interact directly, the social environment, in particular, their own electorate may turn parties' preferences to cooperation, formalized in a joint coalition preference.

**Remark 2.** (Determination of the initial state) In QM to reconstruct a state, it suffices to know its coefficients in a proper basis. The absolute values of coefficients are given by probabilities (more specifically, by their square roots). The problem of the phase determination is more complex. In general, one has to use the powerful machinery of the quantum tomography. However, in the simplest case of a qubit-state, i.e., in the case of the two dimensional state space, the phase can be easily determined by using probabilities for measurements of two complementary observables. In decision-making such observables are given by two complementary questions, i.e., questions which exhibit non-commutative effects. Such effects have been well researched in cognitive psychology (the so called "order effects"). Busemeyer et al. (2006), Wang&Busemeyer (2013) and Khrennikova(2014b) actively explore these effects and search to model such questions-observables in a quantum framework, with the aid of experimental data from different decision making contexts. Khrennikova (2014a) performs a so called state reconstruction with the aid of obtained statistics, to examine the applicability of Born's Rule for psychological data. It should be noted that a real experimental realization of studies on political non/cooperation and more specifically coalition formation (in the form of opinion polls) can be considered as a non-trivial problem.

## 4.2 Markovian quantum master equation in decision making

One of the main distinguishing features of solutions of the Markovian quantum master equation is that here a non-stationary solution $\rho(t)$ can stabilize to a stationary solution $\rho_d$ representing the collective decision of all parties on (non)cooperation. Opposite to Schrödinger's equation, the quantum master equation can transform pure states into mixed states. This is a dynamical equation in the space of density operators. Therefore the limiting strategy determining the decision on cooperation can be a mixed state even if the initial joint state of parties' preferences was a pure state. Thus in general it determines only the probabilities of various pure strategies. For example, if there are two parties $\mathcal{P}_1, \mathcal{P}_2$ then each state space is just qubit space. If, e.g.,

$$\rho_d = P|00\rangle\langle 00| + Q|11\rangle\langle 11|, P + Q = 1, P, Q \geq 0, \tag{15}$$

then the probability that both parties will prefer non-cooperation (cooperation) equals to $P$ (to $Q$). If, e.g.,

$$\rho_d = P|01\rangle\langle 01| + Q|10\rangle\langle 10|, P + Q = 1, P, Q \geq 0, \tag{16}$$

then the probability that $\mathcal{P}_1$ ($\mathcal{P}_2$) will prefer non-cooperation and $\mathcal{P}_2$ ($\mathcal{P}_1$) will prefer cooperation equals to $P$ (to $Q$). We remark that the decision-states, e.g., (15), (16) are in some sense classical states. Superposition indeterminacy which can be present in the initial state, say

$$\rho_0 = |\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|], \tag{17}$$

or

$$\rho_0 = |\Psi^+\rangle\langle\Psi^+| = \frac{1}{2}[|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|] \tag{18}$$

disappears in the process of approaching the stationary state (here the Bell states $|\Phi^+\rangle, |\Psi^+\rangle$ were defined in (4), (5)). This is the most typical scenarios of the evolution driven by quantum Markov master equation. However, for some classes of equations the decision state $\rho_d$ can be a pure state as well, e.g., $\rho_d = |00\rangle\langle 00|$ (the definite non-cooperation preference of both parties) or $\rho_d = |11\rangle\langle 11|$ (the definite cooperation preference of both parties). Moreover, the limiting stationary states can have non-zero off-diagonal elements. In such a case the quantum(-like) indeterminacy is not resolved completely, see section 4.3 for examples.

In the general case of $n$ parties the social environment contributing to parties' preference dynamics can be split, into three sub-environments, in a similar way as performed in Bagarello (2015b): $\mathcal{R}_j, j = 1, ..., n$, represents the preferences of the stable part of the electorate of the party $\mathcal{P}_j$; $\mathcal{R}$ represents the preferences of the "unstable electorate", people who either have no definite political preferences or even having some preferences can easily change them. It is natural that $\mathcal{R}_j$ acts only onto the preferences of $\mathcal{P}_j$, i.e., it is represented by a dynamical generator $R_j$ in the state space $H_j$. The preferences of the unstable electorate $\mathcal{R}$ have to be taken into account by all political parties; in general these preferences are represented by a dynamical generator $R$ acting in the state space $H$.

The important special case is that the operator of unstable electorate $\mathcal{R}$ acts separately, but in the same way, to the preference state of each party. Here we can invent an operator, say $S$, acting in the $2^{(n-1)}$ dimensional Hilbert space (all $H_j$ are isomorphic to it). Then the impact of the social environment to the preference state of $\mathcal{P}_j$ is generated by the sum of operators $R_j + S$. One can say that, although the political parties do not try to negotiate directly, they preferences are inter-related through the impact of the unstable electorate.

We now write the Markovian approximation of the quantum master equation as expounded in Ohya and Volovich (2011):

$$\frac{d\rho}{dt}(t) = -\frac{i}{\gamma}[\mathcal{H}, \rho(t)] + L(\rho(t)), \tag{19}$$

where $\mathcal{H}$ is a Hermitian operator acting in $H$ and $L$ is a linear operator acting in the space of linear operators $B(H)$ in $H$ (such maps are often called super-operators). Typically, the operator $\mathcal{H}$ represents the state dynamics in the absence of environment. However, in general $\mathcal{H}$ can also contain contribution of the impact of the environment. The super-operator $L$ has to map density operators into density operators, i.e., it has to preserve Hermiticity, positive definiteness, and the trace. These conditions constraint essentially the class of possible generators $L$. By adding some additional condition the so called complete positive definiteness, we obtain the possibility to describe the class of generators precisely (see the book by Ohya and Volovich (2011) for technical details.) They have the form:

$$L\rho = \sum_k \alpha_k [C_k \rho C_k^\star - (C_k^\star C_k \rho + \rho C_k^\star C_k)/2] = \sum_k \alpha_k [C_k \rho C_k^\star - \frac{1}{2}\{C_k^\star C_k, \rho\}], \tag{20}$$

where the symbol $C^\star$ is used to denote the adjoint operator of $C$. Hence, the operators $R_j$ are of the form (20), where the operators $C_k$ acts in the $(n-1)$-qubit space, and the operator $R$ is also of this form with the operators $C_k$ acting in the $n(n-1)$-qubit space. Operators $C_k$ encode the special features of a social environment.

## 4.3 Numerical simulation

In the case when only two parties are dominating a political arena, the preference dynamics is represented in a four dimensional Hilbert space. The density matrix has 16 elements and besides the problem of numerical simulation, the more technical problem of visualization arises. At this stage, we search to make the present pilot modeling of the political preference dynamics with the aid of the quantum Markov equation not too complex. For illustrative purpose, we proceed as in Khrennikova et al. (2014), by reducing the dimension of the state space to two.

In other words, we consider the two dimensional sub-model of the general four dimensional model presented earlier, corresponding to the political context, in which two parties can either both agree to cooperate or non-cooperate in respect to a coalition formation. We reduce the modeling task to the subspace with the basis $e_1 = |00\rangle, e_2 = |11\rangle$. It is assumed that at the beginning (i.e., before interaction with the "electorate environment") the two parties were in superposition of the basic states:

$$|\psi\rangle = c_1|00\rangle + c_2|11\rangle,\ |c_1|^2 + |c_2|^2 = 1. \tag{21}$$

We also assume that in the absence of interaction with the "electorate bath" the state of preferences fluctuates driven by the Schrödinger's dynamics with the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{22}$$

where $\lambda > 0$ is the parameter describing the intensity of flipping from $|00\rangle$ to $|11\rangle$ and vice versa. The simplest perturbation of such Schrödinger equation is given by the Lindblad term of the form:: $C\rho C^\star - (C^\star C\rho + \rho C^\star C)/2 = C\rho C^\star - \frac{1}{2}\{C^\star C, \rho\}$. We select the operator $C$ by using its matrix in the basis $e_1, e_2 : C = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$, hence, $C^\star = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$, where the parameter $\lambda$ which is responsible for the interaction between preferences of the two political parties and the electorate is selected the same as in the Hamiltonian (22), just for simplicity of illustration.

We comment briefly on the choice of the operator $C$. This operator combines the preferences of the two parties. Hence, it represents the unstable part of the electorate, which demands are contributing to the decisions of both parties. In this model, for simplicity, we did not take into account the "separate electorates" of these parties.

Thus, we proceed with the quantum master equation:

$$\frac{d\rho}{dt}(t) = -i[\mathcal{H}, \rho(t)] + C\rho(t)C^\star - \frac{1}{2}\{C^\star C, \rho(t)\}. \tag{23}$$

We present dynamics corresponding to symmetric superposition,

$$c_1 = c_2 = \frac{1}{\sqrt{2}}, \tag{24}$$

see Fig. 1, and to a strongly asymmetric superposition

$$c_1 = \sqrt{0.9}, c_2 = \sqrt{0.1}, \tag{25}$$

see Fig. 2.

In this dynamics of parties' preferences for establishing or not-establishing a particular political coalition (formally, entering a coalition agreement), the interaction with the "unstable electorate environment" plays a pivotal role. Strong oscillations of the state dynamics that persist in the absence of an interaction with the "electorate bath", are speedily damped under the influence of "electorate bath" and the matrix elements $\rho_{11} \equiv \rho_{00,00}, \rho_{22} \equiv \rho_{11,11}, \rho_{12} \equiv \rho_{00,11}$, and $\bar{\rho}_{12} = \rho_{21} \equiv \rho_{11,00}$ stabilize to some definite values. This entails that the preferences of the parties, which were in a fluctuating superposition of choices stabilize under the impact of the "electorate bath."

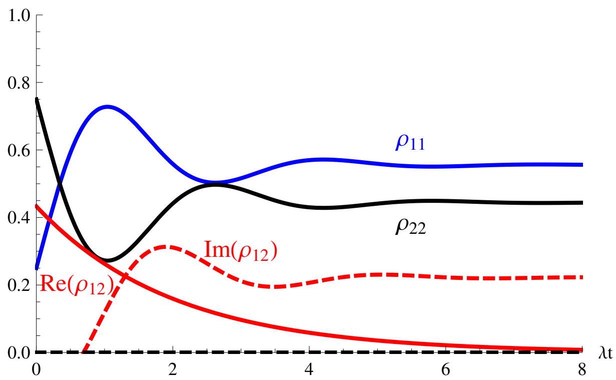


Figure 1: Stabilization of the matrix elements of the density operator; the initial state is symmetric superposition of state $|00\rangle$ and $|11\rangle$.

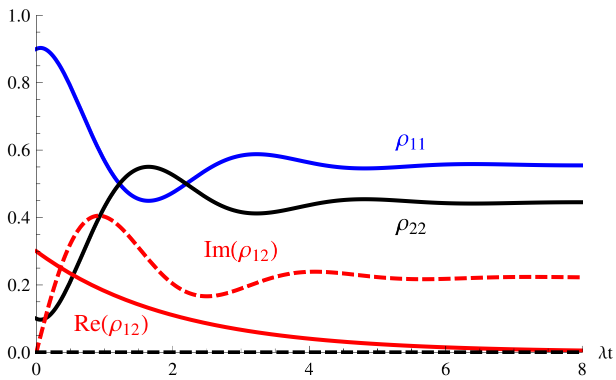


Figure 2: Stabilization of the matrix elements of the density operator; the initial state is stronly asymmetric superposition of states $|00\rangle$ and $|11\rangle$.

In the $\rho_{\text{lim}}$ the elements $\rho_{00,00} \approx 0.6, \rho_{11,11} \approx 0.4$ determine the corresponding probabilities for the particular choices of the parties, e.g. $p(00) \approx 0.6, p(11) = 0.4$. on the Fig.4.3. For illustrative purposes, we selected an interaction of political parties with the electorate bath, such that both initial states, (24) and (25), generate the same limiting distribution of preferences (in fact, this state can be generated from any initial state). Under the pressure of the social environment, the parties started with a superposition (24) increase the 00-preference and the parties that started with a superposition (25) decrease this preference. The resulting distribution of choices is the same for both political contexts (with the initial state (24) and with the initial state (25)).

The results presented in this section were obtained with the aid of a numerical simulation by using the standard package of "Matematica" software.

# 5 Concluding remarks

This paper extends the methods of quantum cognitive psychology and decision-making to the field of political science. The presented model takes into account the preferences and aspirations of the political parties, their members and their electorate. The author hopes that this contribution will further strengthen the area of research related to interdisciplinary applications of models borrowed quantum formalism and information theory to decision making processes on the political arena.

On the real political arena the state of preferences of a group of political parties is a cocktail of power seeking, policy seeking as well as additional economic and financial factors, non-separably coupled to their decision-making outcomes. In this note, following the methodological approach elaborated in Zorn and Smith (2011), quantum information entanglement is used, to represent the non-separability of all aforementioned factors and their intrinsic multi-parties coupling.

In line with the models used by Asano et al. (2011) and Asano et al. (2012), we adopted the open quantum system theory, to model the process of establishing the so called political behavioral equilibrium , the final decision state of (non)cooperation. In this equilibrium state, the parties either firmly decide to establish a political coalition, or continue to pursue a political opposition. In the setting of this exposition, the electorate bath, surrounding the party, plays a pivotal role. An important cluster, constituting the electorate bath, is the so called unstable part of the electorate. These undecided voters can be categorised by an absence of a definite political ideology and hence, make their decisions "irrationally" from the viewpoint of the neoclassical economics theories. Nevertheless, the support of this group of voters can play an imperative role with far-reaching ramifications for the goals and actions of political parties.

One of the main findings of the introduced theory of open quantum systems, is that the obtained equilibrium state exists for a wide class of quantum Markov dynamics. At the same time, when applying the presented model, one should bear in mind that as always is the case in quantum theory, this equilibrium is of a stochastic nature. As such, the quantum approach is not functioning as a deterministic model, for predicting a particular decision outcome. Another important characteristic of the model is related to the possibility to describe a class of quantum Markov dynamics, where the final decision state does not depend on the state of initial preferences, see section 4.3 for some concrete examples. At the first sight, this feature of the proposed dynamical model might be considered as unrealistic. One may doubt that such a class of dynamical systems, capturing a dramatic departure from the initial decision-making states of the parties to their final actions, would correspond to the real world process of political coalition formation. Surprisingly, on the real political arena in many countries, one can find numerous examples of (non)cooperation decisions that match well with class of decision-making dynamics. We highlighted briefly some cases of grand coalitions, discussed in mass-media (BBC News 2010; BBC News, 2013; Spiegel Online International, 2013; Financial Times, 2012). Coalitions between ideologically distant parties also periodically occurred in European politics. A notable example is the sudden coalition between the opposition parties Fine Gael and Labour in Ireland covered in The Guardian (2011), as well as more recently, the coalition between Liberal Democrats and the Conservatives in the UK (BBC, 2010) and finally, a very recent case of a Right Wing party joining Syriza in Greece discussed in the Wall Street Journal (2015). This list could be extended with other examples, including the cases of ideologically connected parties that did not manage to establish a coalition agreement.

In this work we were primarily interested in a class of quantum dynamical systems, producing a unique steady state, independent of the initial conditions. In general, a quantum dynamical system can have a manifold of steady states, corresponding to different initial conditions.

Future studies on the dynamics of coalitions and alliances between political parties will be based on a more extensive analysis of coalition (non)formation cases, in order to establish the concrete parameters for the devised dynamical operators. The cases of "Grand Coalitions" and other exotic coalitions would benefit from further studies, to refine the proposed model. At last, the choice to form a coalition is not the end of the story. The "collapse" of the coalitions, including the success or failure of post-coalition members would benefit from a more in-depth exploration, to capture this process by a suitable dynamical model.

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