Asset trading under non-classical ambiguity and heterogeneous beliefs

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Abstract

We propose discrete time asset trading framework based on quantum probability formalism that represents well the ambiguity of agents in respect to the fundamental values and price states of the traded assets. Divergence of beliefs alike classical finance frameworks (e.g. works by Harrison and Kreps, 1978 [24]; Scheinkman and Xiong, 2003 [50]) produces different expectations of agents about the future price distribution of the traded risky asset. The model accounts for the emergence of heterogeneous beliefs from agents' ambiguity about both the future asset price states and the fundamentals, as opposed to the strands that attribute heterogeneous beliefs to asymmetric information and different, yet firm prior beliefs about stochastic processes over fundamentals. The introduced quantum probability paradigm allows to depict a genuine ambiguity of agents in respect to the future realization of payoff relevant variables and prices. There are two sources of ambiguity: i) the imperfect market knowledge of agents, manifest in a divergence of ambiguous priors, ii) uncertainty about the probability distribution of price states and dividends in the next trading period. Agents update their beliefs via Born rule (instead of Bayesian update) when observing the realised price outcomes and dividend signals. An important feature relates to individual traders' not possessing a joint probability distribution over the payoff relevant variables and price outcomes that brings up attraction, respective aversion to ambiguity in their interpretation of public signals. On the level of the composite model of stock exchange, formed by the expectations of two ensembles of agents, an *interference term* can serve as a quantitative testable

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31 prediction in respect to the excess volatility of asset prices created by
32 traders' optimistic and pessimistic beliefs.

33 **keywords:** asset trading; speculative asset prices; heterogeneous beliefs;
34 ambiguity; state dependence; quantum probability.

35 1 Introduction

36 There is a long standing debate in the standard neoclassical finance litera-
37 ture on the impact of uncertainty on financial asset prices. The best known
38 models going back to Miller, [44] have held that uncertainty is represented
39 by some measure of divergence of investors' opinions, grounded in the pres-
40 ence of information asymmetries in incomplete markets. Miller conceived
41 that "*the very concept of uncertainty implies that reasonable men differ in*
42 *their forecasts*", [44], p.1151. Such divergence of states of beliefs causes the
43 deviation of asset prices from the so called fundamental value as predicted
44 by the asset pricing models such as CAPM.¹ Since the central premise of
45 rational expectations (RE) models is homogeneity of investors' beliefs, along
46 with some restricted perfect market conditions, arbitrage is not possible in
47 the long run and asset prices do not deviate from what the fundamentals
48 predict. This is also an important implication of the seminal EMH (Efficient
49 Market Hypothesis) by [23] that builds upon the notion of common knowledge
50 of market actors about all the available information as well as singularity in
51 mapping from the observed fundamentals to asset prices. As a consequence,
52 in the absence of new information all agents agree on the fair price of a
53 risky asset. When new information about payoff relevant variables arrives,
54 the agents react to the price relevant news following a classical Bayesian up-
55 date scheme.² The assumption of equal priors and homogeneous posterior
56 beliefs, as a result of common knowledge that is central to the above finance
57 frameworks, was shown to be a rather idealized postulation, in particular,
58 when one deals with such a complex system as finance market containing
59 multiple sources of noisy information, as opposed to an observation of some
60 simple events, such as a coin toss. For decades finance literature was pre-
61 occupied with the identification of the sources of heterogeneous beliefs and
62 their impact upon asset prices, associated with periods of high volatility and

¹Capital asset pricing model establishes the 'right' rate of return for holding a risky asset that together with expected cash flow projections, or all expected future dividends allows to assess a current 'fair' price of that asset.

²Under asymmetric information with non-biased private signals, rational agents would ideally be able to infer other traders' information from the observed asset prices and hence a fully revealing rational expectations equilibrium is attained, [21].

63 ‘bubbles’, [11] [24], [32], [45], [50], as well as recently, [1], [9], [38], [19], [13].
 64 The main causes of heterogeneous beliefs are divergent priors that can steam
 65 from overconfidence as well as optimism and pessimism. Asymmetric infor-
 66 mation and incomplete markets is also a widely identified factor that triggers
 67 uncertainty about the fair asset value among the less informed ensembles of
 68 agents. The markets can also be incomplete for all types of agents, giving
 69 raise to divergence of opinions, [1]. As a consequence the agents’ externally
 70 irrational beliefs about the fundamentals are not a common knowledge. The
 71 above works espouse that information processing of agents follows Bayesian
 72 scheme with some noisy estimates that distort a singularity of opinions based
 73 on the above mentioned cognitive factors. The ‘noisy’ forecasts are captured
 74 via classical Markov processes and related stochastic equations, with some
 75 adjustments to the chosen parameters, to capture a disagreement about the
 76 fundamentals, see, e.g., [9], [13], [38], [50]. Works by [1] and [32] explore dif-
 77 ferences in interpretation of signals and their mapping to prices values under
 78 incomplete and fully available public information respectively.

79 The present paper aims to serve as contribution to the exiting theoretic-
 80 al frameworks on asset trading under heterogeneous beliefs and ambiguity.
 81 Agents can hold ambiguous beliefs about the next period asset price distribu-
 82 tion of a financial asset in an informationally incomplete market. We model
 83 agents ambiguity via a ‘projective’ probability calculus, based on quantum
 84 probability (QP). This formalism aims to go beyond SEUT (Subjective Ex-
 85 pected Utility Theory) formalization of preferences that builds upon classical
 86 probability theory of [39]. The main premise of our framework is that agents
 87 can be non-Bayesian and find themselves in a deeper state of ambiguity, given
 88 by indeterminate superposition states of opinions. Following earlier works on
 89 speculative bubbles by [24] and [50], we assume a basic setup with infinitely
 90 lived risk neutral investors, short sales constrains, no liquidity constraints
 91 and frictionless markets.³ Agents maximize discounted expected value in
 92 a classical SEUT mode. They hold non-classical beliefs that are updated
 93 via the Born rule. Our model assumptions carry some similarities with the
 94 framework of [1], characterized by informational incompleteness of the mar-
 95 ket that triggers agents’ divergence in opinion about the liquidation value of
 96 the risky asset and the noisy dividend signal in the next trading period. In
 97 their model the agents have different likelihood functions mapping the ex-
 98 pectations over fundamentals onto the price states. We seek to modify their
 99 framework by introducing a quantum probability (QP) based scheme of in-

³Risk attitudes do not affect risk premium and the impossibility of short sales does not allow pessimists to short the asset, thereby giving rise to bubbly and, in other trading periods, as belief states switch, to deflationary pressures.

100 formation processing and belief formation under informational ambiguity of
101 agents about the dividend signals and prices in the next trading period. It
102 is important to note that agents form separate beliefs about the price evolu-
103 tion and dividend signal distribution, contrary to the assumptions of classical
104 DDM (Dividend Discount Models) in which only the future dividend streams
105 matter for the current asset price. The state space of the agents, the so called
106 Hilbert space, consists of the subspaces related to the outcomes of price and
107 dividend observables, which act upon the belief state vector (ψ) of each agent.
108 QP calculus relaxes the assumption of a joint probability distribution over
109 prices and fundamentals that is axiomatized in classical probability theory via
110 the commutativity rule of probability distributions.⁴ Agents can be affected
111 by non-commuting observables (given by random variables in classical prob-
112 ability theory), associated with dividends and prices. A quantum formula
113 of total probability introduced in eq. (2) contains an additional interference
114 term that mathematically depicts the interference effects in agents' beliefs.
115 Positive interference of probability amplitudes pertains to overweighting of
116 probability of price value. Negative interference gives raise to pessimistic be-
117 liefs, manifest in under-weighting of the probabilistic prognosis in respect to
118 the realizations of future increase in price and dividends. Non-classical am-
119 biguous beliefs produce trading preferences associated with upwards volatility
120 in respect to the fundamental valuation based on RE. As beliefs of agents
121 oscillate between the trading periods, a state transition to pessimistic beliefs
122 creates trading preferences that contribute to deflationary pressures.

123 Given the importance of the non-singularity of beliefs in affecting the
124 asset price volatility we operate with two ensembles of agents. Their het-
125 erogeneous ambiguous beliefs are caused by optimism or pessimism affect.⁵
126 We also conceive that the cognitive states of pessimism and optimism trigger
127 ambiguity attraction respective aversion among agents, when evaluating the
128 asset price distributions in the next trading period, which is in line with find-
129 ings in [49]. In the absence of dividend signals, agents' ambiguity attraction
130 in creates beliefs that the asset price will go up in the coming trading period.
131 Upward price volatility emerges and the risky asset price raises above its
132 fundamental value. The coefficients of interference allow to quantify the de-
133 gree of overvaluation respective undervaluation by the agents. Divergence of

⁴Non-commutativity brings up a non-satisfaction of the core rule of classical probability theory, the formula of total probability, [39]. A deviation from classicality in information processing, given by a violation of the independence axiom of SEUT has been detected in a large body of studies in economics, starting with the seminal Ellsberg paradox.

⁵In the proposed model uncertainty about divided distribution is assumed to be symmetrical, and the heterogeneity in the limiting probability distribution is solely due to divergence in the prior ambiguous beliefs.

agents' beliefs is mathematically represented via a weighted sum of different pure states, ψ , producing a composite mixed state of ambiguous beliefs. The difference in the phase between the belief states of the two ensembles of agents allows in the similar manner to [50] to quantify the difference in ambiguous beliefs. The two types of agents interact, as one agent type trades based on their optimistic beliefs about the prices in the next trading period. The optimistic agents can switch their beliefs to pessimistic in the next trading period t_1 , and deflationary pressures on prices can emerge. This behaviour is in line with [46], showing that short-sale constraints, combined with pessimistic mood during crises can lead to undervaluation of assets. A switching of beliefs occurs, as the observed price states feed back into ambiguous belief states of the agents. Since the operators of prices and dividends do not commute, a different order of price and signal observations can create different limiting probability distributions of beliefs known in the literature as 'order effect', [55].⁶

QP is by now widely applied in economics and decision theory, in particular, to formalise information processing under ambiguity. Probabilistic measures given by quantum probability amplitudes can be interpreted as classical objective, or subjective probability. Just to mention few, the works [47], [12], [26], [55], [2], [27], [35], [10], and [15] formalize the applicability of QP as a probabilistic framework in decision theory under uncertainty and risk. Dynamical models are also widely applied in information modeling in aggregate financial, economic and social systems, see for instance [4], [5], [6], [7], [8] [25], [28], [29], [26], [34].⁷ Monographs by [4]-[5] generalize the applicability of quantum mechanics and quantum field theory to modeling of the dynamics of financial instruments on the capital market. An agent-based model of asset trading via the introduction of raising and lowering operators, affecting the share holdings and prices as agents interact is devised in [6]-[7]. The information dynamics under the existence of arbitrage is modeled via wave function in [25]. Similarly, the work by [28] introduces Schrödinger equation to derive states of equilibrium and dis-equilibrium in an economic system. Quantum Markovian dynamics is applied in [34], to derive long term equilibrium states of asset prices.

⁶In Appendix 8 we espouse a multi-period belief evolution setup, where the internal belief evolution can create differences in information update, following the observation of a sequence of dividend and price outcomes.

⁷We focus here on quantum dynamical models. A survey of the achievements of models borrowed from physics in the fields of economics and finance can be found in the monograph by [41] as well as recent works by [31] and [3]. The latter work examines the potential of physical frameworks to serve as alternative financial models to model asset price formation beyond EMH.

167 To sum up, our setup is aiming to enrich the field of above reviewed
168 contributions of QP models to capture asset pricing under two deviations
169 from RE given by, i) non-classical information processing and heterogene-
170 ity in opinions, ii) informationally incomplete markets. To the best of our
171 knowledge, there are no contributions to speculative asset pricing that focus
172 on these two phenomena by representing agents' opinion update by QP prob-
173 abilistic measures. The paper is organized as follows: in the next Section 2
174 we sketch the mathematical differences between the classical and quantum
175 probability information processing schemes. We motivate the usage of quan-
176 tum probability framework as a descriptive DM model for agents' decision
177 making under uncertainty. In Section 3 we provide a mathematical frame-
178 work underpinning the all-important distinction of quantum versus classical
179 modeling of heterogeneous beliefs under uncertainty. In Section 4 we illus-
180 trate the geometric properties of agents belief state evolution, and introduce
181 belief and price behaviour operators. In Section 5 we define the Born rule of
182 information update in the QP measurement scheme. In Section 6 we sum-
183 marize the possible empirical predictions of our framework, and in Section 7
184 we conclude.

185 1.1 Related Literature

186 It has been shown in a large body of finance studies that heterogeneous and
187 irrational beliefs can generate speculative pressures on capital markets mani-
188 fest in asset price bubbles, which may sustain for long periods, see accounts in
189 [51]-citeShiller2. A survey of bubble emergence emphasising the role of diver-
190 gent beliefs on agents' speculative behaviour can be found in [56]. The impact
191 of heterogeneous beliefs is not the only cause of the existence of inflationary-
192 deflationary pressures on asset prices. Excess volatility can also take place
193 under rational expectations, caused for instance by dynamic inconsistency
194 of agents and credit constraints, [43]. Shiller documents: "*speculative bubble*
195 *(is) a natural consequence of the principles of social psychology coupled with*
196 *imperfect news media and information channels.*" [52], p.1487.

197 The 'agree to disagree' phenomenon and its effects on asset trading,
198 whereby speculative behaviour can emerge was firstly formalized in the asset
199 trading framework by [24], followed by a dynamical representation in [50].
200 The existence of disperse beliefs is reflected in deviations from the RE equilib-
201 rium price that is based on homogenous expectations of agents.⁸ In partially

⁸Seminal 'no-trade theorem' due to [53] postulates the impossibility of the emergence of bubbles under RE. Under the existence of symmetric information and classical information processing scheme, traders are aware of the true probability distribution of future returns and hence, the fundamental value of assets is a common knowledge. Any time any bubbly

202 revealing equilibriums of asset trading, the agents can be aware of other
 203 agents' possessing divergent beliefs, due to different private information sets,
 204 or due to overestimation of the informativeness of some public signals. This
 205 awareness can trigger trading behaviour directed to benefit from perceived
 206 overvaluation of assets by other less rational agents, see asset trading under
 207 'beauty contest' by [11]. A similar assumption about the lack of rationality
 208 of other agents is made in [50]. Among other, the models by [24], [50], [11],
 209 assume symmetric upcoming information, yet the agents can make different
 210 forecasts due to optimism and different priors. In particular the work by [11]
 211 addresses higher order beliefs (i.e. the beliefs of investors about the beliefs of
 212 others) under symmetric information and different priors, expanding on the
 213 seminal work of Harsanyi.

214 Optimism as a cognitive feature of the decision makers also contributes
 215 to a divergence in prior belief states (i.e. the degree of optimism will cause
 216 different, yet firm prior beliefs among ensembles of agents under symmetric
 217 information) as postulated in [24]. The heterogeneity in beliefs is an im-
 218 portant trigger of speculative trade, as beliefs about asset valuation switch
 219 between agents over the trading periods. Frameworks by [45] and [11] also
 220 formalize asset trading with divergent prior opinions. Learning among agents
 221 can occurs over time, as agents observe a sequence of dividends and prices
 222 converge to the fundamentals over long term, [1], [19], [45]. At the same
 223 time, agents can update separately their price expectations and dividend ex-
 224 pectations, based on the observed market outcomes, [1]. Finally, managerial
 225 decision making under different opinions is explored in [19]. In their work
 226 the role of asymmetric information and optimism, characterising the diver-
 227 gence of beliefs in principal-agent relationships can create under-investment.
 228 The stream of literature that explores emergence of adverse selection due to
 229 asymmetric information is also broad based with a focus on identifying the
 230 degree of adverse selection impact on the finance market performance.⁹

231 Divergence of opinions can be coupled with adverse selection, where some
 232 traders lack the private information possessed by other traders. A widely used
 233 measure of the existence of adverse selection is the bid-ask spread, that quan-
 234 tifies agents' attempts to minimise their possible losses, due to the lack of
 235 complete information, [20]. Contribution by [9] addresses the impact of ad-
 236 verse selection by devising a continuous dynamical asset pricing model for

trends are due to emerge, an agent can infer information from prices by possessing the same likelihood function as other agents.

⁹Adverse selection problem would naturally not exist in efficient markets, since less informed investors would follow the more informed ones, since the rationality of all agents is common knowledge, [21].

237 rational investors under incomplete, but symmetrical information.¹⁰ In addition to the above reviewed causes, heterogeneous beliefs can also emerge as a
 238 result of non Bayesian information processing. As mentioned, the divergence
 239 in beliefs can result from different ‘biases’ that cause the non-classical processing of information where ‘noise’ is present often categorised as optimism
 240 and pessimism. The ‘noise’ in the estimation of the dividend rate can be captured via a Markov process following [38], with a coefficient π that serves to
 241 create optimism ($\pi > 0$), or pessimism ($\pi < 0$) in respect to the variance of the drift factor μ . Another cause behind the violation of Bayesian update and
 242 other axioms of classical probability can be due to agents’ employing a fundamentally different mechanism of information updating under uncertainty,
 243 rather than information processing rationality implied by the neo-classical normative decision theories. In real finance setting, the agents often cannot
 244 reach resolution from uncertainty about the realization of future states of the world. The agents can trade, while being ambiguous about the future probabilistic
 245 distribution of asset price returns, or have ambiguous expectations about the informativeness of private signals. Belief formation and update under ambiguity and ambiguous information is already well researched in asset
 246 trading theories, mainly via the usage of ‘max-min expected utility’ (MEU) and a dynamical modification thereof, see [13], [16], and references herein.
 247 Agents can exhibit ambiguity aversion and ambiguity attraction that affect their preference formations as espoused in ([18],[36], [17]). In recent contributions, [14] also seek to examine the effect of an interaction of public and
 248 private information upon asset prices via an introduction of two ensembles of agents. The informed agents exhibit ambiguity aversion and hence bias the full revelation of prices for the other (less informed) ensemble of agents.
 249 Further, the effect of short sale constraint upon return volatility is formalized by [46], showing that as the private information becomes more ambiguous, negative effects on asset price dynamics emerge when coupled with short sale
 250 bans. Finally, work by [30] extends the analysis of bubbles to include over-time regime shifts in the fundamentals to provide the necessary conditions for bubble emergence in derivative markets. To round up, classical probabilistic
 251 heterogeneous beliefs in the sense of [39] are likely to be attributed to: i) divergence of prior beliefs due to some cognitive differences of agents; ii) lack of common knowledge caused by asymmetric, or uncertain information, and
 252 iii) non classical mode of information processing, such that Bayesian reasoning is not employed due to some ‘biases’ or ‘noise’. The last case demands a
 253

¹⁰In this context, it is important to observe how the effect of uncertainty on prices differs from the standard risk attitude divergence. Different preferences about the required risk premium can yield divergence in required discount rate among investors and hence, their affect their valuation of the fair price as shown in the early work by [42].

274 relaxation of some of the classical probabilistic axioms, such as distributiv-
275 ity and commutativity. Under uncertainty and information asymmetry, the
276 beliefs of agents can be also ambiguous and contextual factors can give raise
277 to ambiguity aversion that affect their preference formation, [13], [17], [18],
278 [36]. Ambiguity attraction is less researched in ambiguity based asset trading
279 works, yet can be closely related to optimistic behaviour under uncertainty,
280 [2], [49]. We note that the reviewed frameworks also make opposite pre-
281 dictions in respect to the emergence of speculative trading; while ambiguity
282 aversion implies under-pricing, heterogeneous beliefs and optimism can lead
283 to overpricing, thus creating bubbles. The present framework aims to unify
284 these predictions, by relating instances of overpricing and under-pricing, via
285 the transformation of agents' heterogeneous (beliefs) in different time periods
286 that follow the rules of projective measurements of quantum probability.

287 Ambiguity impact on decision preferences, as well as information process-
288 ing under ambiguity has been well addressed in recent studies in economics
289 and decision theory via the usage of QP, rather than classical theory of prob-
290 ability and stochastic processes, for instance, [47], [12], [26], [33] address
291 the emergence beliefs and preferences under non-classical ambiguity that de-
292 scribe well the violation of classical Bayesian updating scheme in 'Savage
293 Sure Thing principle' problems and the 'agree to disagree' paradox. In [27]
294 additional empirical evidence on non-consequential preferences in investment
295 choices is collected and accommodated in QP framework. A QP model for
296 order effects is formalized in [55] that accounts for state dependence in infor-
297 mation processing. Ellsberg and Machina paradox behaviour from ambigu-
298 ous beliefs is formalised in [2] with aid of QP calculus. The work by [35]
299 proposes decision making scheme via the usage of creation and annihilation
300 operators from the quantum information theory. The existence of 'zero prior
301 paradox' that challenges Bayesian updating from uninformative priors is at-
302 tested and solved with the aid of projective scheme of information update in
303 work by [10]. Finally, [15] apply the QP formalism of information update un-
304 der the study of persuasion in investment and consumption choice, showing
305 that the non- satisfaction of the recursive dynamic consistency of choices can
306 be mathematically depicted through incompatible information observables.
307 The usage of QP as an alternative descriptive (and potentially normative)
308 framework of preference formation under uncertainty can be justified given
309 the body of empirical evidence in the above mentioned and related studies
310 in economics and psychology. The findings affirm an existence of different
311 attitudes among decision makers towards ambiguity and risk that are not in
312 accord with the SEUT or non-linear probabilistic transformations thereof.
313 The main advantage of QP is that it is a complete probabilistic framework
314 that allows to accurately depict indeterminacy of agents and its overtime

315 dynamics. The axiomatic of quantum probability is based on a different
 316 mathematics, where QP is by definition a non set-theoretic probability the-
 317 ory relaxing the distributivity axiom (exclusivity of events and their additiv-
 318 ity) and commutativity (context independent joint probability distribution).
 319 The prior ambiguous beliefs are modeled in QP framework as superposition of
 320 agents' belief states. Superposition state allows to reproduce the ambiguity
 321 of agents associated with the probability distributions of future asset prices
 322 and dividends. The initial belief state is an indefinite, 'superposition' state
 323 of various probability distributions, or preferences and interference effects
 324 can be present. We would like to emphasize that superposition representa-
 325 tion is fundamental to differentiate between the classical and quantum belief
 326 state description. Our proposal, along with the existing studies using QP
 327 based decision theory, is that the superposition representation of probability
 328 amplitudes captures better the ambiguous beliefs than a classical ensemble
 329 description. The quantum probability is obtained from quantum wave func-
 330 tion (probability amplitudes) that can also vacillate over time.¹¹ The random
 331 variables in QP are given by observables and the events by subspaces of a
 332 Hilbert space, rather than by the sigma algebra on the probability sample
 333 space. To round up, the QP approach to information processing is consid-
 334 ered in the literature as a viable mathematical framework of information
 335 processing that can also be applied to agents' information processing on the
 336 financial market. In proposed model we aim to develop the belief formation
 337 scheme about fundamentals and price in a discrete time setup and describe
 338 the emergent non-classical asset price equilibrium today. In the next sections
 339 we continue with the formulation of the quantum probabilistic setup, to de-
 340 vise a model of belief state formation and update under uncertainty in the
 341 context of asset trading.

342 **2 Model of trading under uncertainty in quan-** 343 **tum probability (QP) framework**

344 The aim of the framework is to elucidate how non-classical heterogeneous be-
 345 liefs of market participants create interference effects that amplify the trading
 346 optimism and in other periods trading pessimism. The state of the market
 347 participants changes after an informational context and the interference af-
 348 fects can create amplification of buying respective selling, even under no
 349 upcoming information,[51]-[52]. The states of participants update (e.g. af-

¹¹For an extended mathematical analysis cf. monographs by [12], [26] and survey by [48].

350 ter observation of previous trading dynamics) and trading bids and asks can
351 create sudden price moves for no apparent reason.

352 In our model we capture the amplification mechanism via non-classical
353 belief indeterminacy, where the ambiguous beliefs about the price states can
354 interfere with the ambiguous beliefs on fundamentals' distribution.

355 Before we present information processing under QP under asset trading,
356 we review in a simple dichotomous scheme the information update under
357 uncertainty that lies at the heart of SEUT.

358 **2.1 Agents' decision making: classical probability based** 359 **information processing**

360 In the classical probabilistic information update, each infinitely lived agent
361 from the population N is endowed with an initial pure belief state with
362 respect to one risky asset value I in question. Every agent operates in belief
363 space for dividends, D and a belief space, K for asset price realizations, upon
364 which she makes the hold and sell decisions. The composite state space that
365 includes all possible asset price and dividend realisations up to the decision
366 time t is Ω^t , where $t = 0, 1, 2, \dots$ are some discrete points in time. Hence,
367 every agent is a composite system of D and K .¹² To simplify the exposition
368 of probabilistic update of the model at this stage we consider two types of
369 beliefs about the discrete asset price movements, P_+ and P_- , that correspond
370 to agents' decisions to buy or sell the asset now. There is a dividend signal
371 that we denote in dichotomous form as positive or negative (S_+ or S_-) related
372 to the asset valuation. The arrival of such a signal in classical finance theory
373 changes the initial belief state of the agents in a Bayesian updating scheme.
374 What is important, in classical probabilistic framework the random variables
375 corresponding to the asset prices, and informational signals are partitions of
376 the same sample space, i.e. the agents have a joint probability distribution of
377 signals and corresponding price outcomes, due to Kolmogorov [39] probability
378 theory. The agents can also form a joint probability distribution of all asset
379 price states given the dividend realisations in a given decision making context.

380 The key rules of classical information processing scheme (due to SEUT
381 and its modifications based on classical probability theory) imply that the
382 agents make a joint probabilistic evaluation of all possible signal-events and
383 the corresponding consequences. These probabilities are corresponding to

¹²On the finance market only the actions and decisions of actors are visible through the changes in asset prices. Hence, the beliefs about asset prices are given by the expectations of the agents about the asset price given the future dividends. Each realization of the dividend at time $t = 1$ allows to assume a specific value of the asset price.

384 firm beliefs and no indeterminacy in agent's expectations is present at each
 385 time point. If the probability of the price increase is higher than of a price
 386 decrease, the risk neutral decision maker will have a higher expectation value
 387 from holding the asset between $t = 0$ to $t = 1$. For now, we assume that
 388 the price increase and decrease size in next trading period is of the same
 389 magnitude. We also assume that in the pricing of the asset the agent would
 390 use some risk neutral discount factor r , i.e. $P_{t=0} = e^{-r\Delta t} E_{t=1}(P_{t=1})$. The
 391 expectation value for price at t_1 is obtained by agents via analysing the
 392 probabilistic distribution of fundamentals (approximated by dividends). Af-
 393 ter the dividend outcome news are observed, each agent is able to evaluate in
 394 a Bayesian mode the conditional price state distributions. We remark that
 395 $S_{\pm}$ denotes the news about the dividend increase or decrease in the next
 396 trading period. We note that alike the classical finance frameworks the ac-
 397 tual dividend distribution causes a proportional decrease in the price value
 398 on the ex-dividend date. In incomplete markets some signals are more dif-
 399 ficult to verify and noisy assessments can take place. This brings agents to
 400 have different evaluation of the new equilibrium price that corresponds to a
 401 divergence of (classical) probabilistic beliefs about payoff relevant signals. If
 402 the probability of the price increase and decrease is the same, then the price
 403 is in a short term equilibrium, until new informational signals reach the fi-
 404 nance market, [23]. For the composite finance market one can observe the
 405 frequency of trading after the informational signal, e.g. if S_+ , the company
 406 will pay the dividend with certainty. In an ideal case, if all the agents buy
 407 the asset (they hold singular beliefs and information update) the price goes
 408 up to the new equilibrium price. In this setting, the commutativity axiom
 409 is also satisfied, and no context effects (related to the order of information
 410 processing) are present, $p(P_+|S_1 \cap S_2) = p(P_+|S_2 \cap S_1)$, where p denotes prob-
 411 ability and S_1, S_2 some dividend signals. Next, under uncertainty the agents
 412 are able to evaluate instantaneously the past and present information and
 413 form subjective probabilities, associated with future signals and conditional
 414 price realizations. This mode of information processing can be formalized
 415 with the aid of the formula of total probability (FTP) which lies at the heart
 416 of classical probability theory.

In the general case the P_i corresponds to the realization of a concrete
 price value, the S_k corresponds some informational signal and p is associated

probability measure.¹³

$$p(P_i) = \sum_{k=1}^{\infty} p(P_i \cap S_k) = \sum_{k=1}^{\infty} p(S_k) \times \frac{p(P_i \cap S_k)}{p(S_k)} \quad (1)$$

417 We remind that the disjoint subsets S_k and P_i belong to the same probability
 418 sample space Ω . A simple decision tree represents information processing of
 419 agents in classical probability framework.

Figure 1: The chance nodes are given by circles and the belief/decision nodes are given by squares.

420 We can see from figure 2.1 that when a positive or negative signal reaches
 421 the market the belief states of agents update via Bayes rule, giving the condi-
 422 tional probability for $(P_+|S_+)$, $(P_+|S_-)$, $(P_-|S_+)$, $(P_-|S_-)$. The conditional
 423 probabilities can be interpreted in Bayesian fashion as each agent's subjective
 424 beliefs about a price increase, given that $S_{\pm}$ is true. Naturally, an econom-
 425 ically rational agent would always assume a P_+ realisation, if the signal is
 426 positive and vice versa.¹⁴ Under a divergence of beliefs, given an objective-
 427 frequency interpretation, one can observe that some populations of agents

¹³We operate with discrete probability measures to follow the formulation of classical neo-economic decision theories. We also restrict the formulation to dichotomous outcomes of variables, $P_i = \pm$ and $S_k = \pm$.

¹⁴We discuss in more detail the objective versus subjective interpretation of classical and quantum probability in Section 3).

believe in e.g. $(P_+|S_+)$ and less frequently in $(P_-|S_+)$. From here one can derive a divergence of beliefs based on classical probability that yields trading of the risky asset among these ensembles agents. If all agents hold the same beliefs about the fundamental asset price, given the set of informational signals, one would observe $p(P_\pm)$ to be equal to unity. The same mechanism applies, when the agents update their belief states in respect to the negative signal S_- .¹⁵

With dichotomous signals and price realizations we get:

$$p(P_+) = p(S_+) \times p(P_+|S_+) + p(S_-) \times p(P_+|S_-)$$

for the price realisation P_+ . The probability (p) on the left-handside provides a probabilistic prognosis of the asset price increase (to a specific value) assuming a representative agent information processing under a set of different informational scenarios. Here we also assume that agents act upon their beliefs by maximizing the expected utility. In the same way, a total probability of (P_-) expectation can be expressed. Under the frequency interpretation, one can observe the frequency of agents that would hold or buy the unit of the asset for the next trading period, based on their firm beliefs.

2.2 Possibility of a violation of the classical mode of information processing: quantum probability representation

Extensive evidence on decision making under uncertainty and risk shows that agents often do not process information in classical probabilistic mode and do not employ Bayesian updating scheme. There are contexts, in which agents can process information in classical probabilistic mode, and contexts in which agents are not able, or prefer not to process information classically. Setting of uncertainty pertains to the non classical mode of information processing supported by empirical findings from psychology and behavioural economics, see a theoretical analysis and discussion in [2], [12], [26], [27], [47] and [48]. The above list of QP motivated works to information processing fallacies and preference reversals is far from being exhaustive. As documented by [51], and references herein, a real setting of the finance market is characterised by a vast level of information complexity and ambiguity, and hence agents'

¹⁵In a more general setup, a random variable S can have multiple realizations (dividend values), upon which the agents condition the price outcomes. In classical finance models with continuous probability distribution of dividends, agents would possess a price function that maps each value of the dividend into a real value of a price at any time, t .

458 decision making may not follow the cannons of classical probability based
 459 information processing.

460 QP has been shown to be able to describe the divergence in agents' pos-
 461 terior beliefs that are at variance with Bayesian inference and linearity of
 462 probability measures. The interference effects and the order of information
 463 processing can affect agents' limiting distribution of beliefs as shown in [47],
 464 [55], [33], and [10]. Ambiguous beliefs are well captured via probability inter-
 465 ference that can create amplification of optimistic, or pessimistic expectations
 466 under ambiguity via probability wave interference.

467 In the quantum probability framework instead of a probability sample
 468 space, the price and fundamentals observables are represented in the com-
 469 plex Hilbert state space, and the events are given by subspaces. In the
 470 simplest two dimensional model, these are one-dimensional rays. A QP for-
 471 mulation relaxes the distributive axiom, where a joint distribution of price
 472 probabilities and dividend signals may not be accessible to the decision mak-
 473 ers. Non-satisfaction of the commutativity condition of classical probability
 474 theory, as the order of information processing affects the final distribution of
 475 agents' beliefs, is given via incompatible observables in quantum probability
 476 setup. When price and dividend observables are not measured, the agent's
 477 belief state is an ambiguous state, in which different beliefs about signals
 478 and prices may coexist. It is only when the agents trade the asset based on
 479 their expectations, the measurements becomes classical Von Neuman-Lüders
 480 measurement, where the belief state collapses either into P_+ or P_- .¹⁶ Given
 481 that each agent is endowed at $t = 0$ with $1/N$ units of the risky asset I and
 482 no liquidity constraints, the beliefs about the price states bring the acts to
 483 hold and buy, respective to sell. When the information is present on the
 484 finance market and it is verifiable, an ideal case from the viewpoint of EMH
 485 is that all agents buy an asset, until particular price threshold is reached, es-
 486 tablishing a new unique REE. In that way no overreaction or under-reaction
 487 takes place by any ensembles of agents. Under divergence of beliefs, agents
 488 can be in different initial (quantum) belief states. The divergence in initial
 489 belief states can be caused by optimism, respective pessimism. Under the
 490 impossibility of short selling, the agents can: i) buy the asset, even if not able
 491 to resolve the uncertainty about the informational signals and price expecta-
 492 tions in the next trading period, ii) overreact to the dividend signals based
 493 on their prior optimistic belief states about the asset price. More precisely,
 494 a classical FTP is replaced with a more general quantum formula of total

¹⁶On the real finance market the measurements can also be 'unsharp'. For instance, an agent is almost confident that she will trade an asset, given her assessment of the future price realization, yet some ambiguity is still present in her preference formation.

495 probability (QFTP) that has an additional *interference term* that we denote
 496 by λ . A representation for a dichotomous outcome case:

$$p(P_+) = p(P_+|S_+)p(S_+) + p(P_+|S_-)p(S_-) + 2\cos\theta\sqrt{p(P_+|S_+)p(S_+)p(P_+|S_-)p(S_-)} \quad (2)$$

$$\lambda = p(P_+) - p(P_+|S_+)p(S_+) - p(P_+|S_-)p(S_-) \quad (3)$$

497 The parameter $\cos\theta \neq 0$ makes the whole term λ either negative or pos-
 498 itive. The quantum probabilistic formulation allows for the interference of
 499 agents' beliefs about signals and price realizations, giving raise to inflation-
 500 ary (the $\lambda > 0$), or deflationary pressures (the $\lambda < 0$) on the on asset prices
 501 under ambiguity. In mathematical language, we expect sub-additivity effects
 502 respective super-additivity effects of agents' beliefs about asset price states
 503 in the next trading period $t = 1$. If $\lambda = 0$ then QFTP collapses into FTP,
 504 and agents update the information in a classical mode under ambiguity. This
 505 means that no deviation from the fundamental value is observed. In several
 506 recent studies in economics and finance, the above mentioned interference
 507 angle (θ) was quantified experimentally and decision making contexts asso-
 508 ciated with its observed values were explored, cf.[47], [26], [2], [27]. The
 509 last work specifically explores preference formation in an investment context
 510 under risk, called a 'Portfolio Game'.

511 When agents are ambiguous but their initial (prior) belief states are iden-
 512 tical, the total frequency of their trading under uncertainty can be approx-
 513 imated by the probability of price increase in the next trading period. The
 514 preference for buying under uncertainty is based on agents' belief interfer-
 515 ence of probability amplitudes related to $S_{\pm}$, and $P_{\pm}$. One can obtain the
 516 total probability of P_+ , given by the left hand side of (2). If all agents
 517 hold time separable rational preferences, then the discounted expected value
 518 $(E) \sum (P_{t=1\pm})$, gives a equilibrium price $P_{t=0}$. When the decision makers are
 519 in non-classical ambiguous belief states, $P_{t=0}$ can be different from the price
 520 based on classical information processing, (1). The *interference term* denoted
 521 in eq. (3) as λ , quantifies this difference. The variable θ denotes the angle
 522 of the belief state wave function of the agents. Constructive and destructive
 523 interference terms in the quantum probability framework explain the rela-
 524 tionship between trading under uncertainty and the inflationary, respective
 525 deflationary pressures on the asset price. Interference in belief formation is
 526 shaped by incompatible decision making contexts, encountered by financial
 527 agents under uncertainty.¹⁷ We represent such a state update in Section, 4

¹⁷The theoretical propositions that are derived from the model can be tested empirically

528 and extend the formulation in Appendix, 8. The above formalization assumes
529 a representative agent (there is no divergence in probability interference an-
530 gle and the initial belief states) and hence, a *pure state* representation can
531 be used. When the agents are in different pure belief states they will have
532 different probability interference magnitudes under ambiguity that are mod-
533 eled as a composite (mixed) state of given by weighted sum of pure states,
534 please see Section (3).

535 3 Uncertainty versus diversity

536 In this section we analyse in more detail the quantum versus classical repre-
537 sentations of uncertainty and diversity of probabilistic beliefs. The difference
538 can be perceived as subtle, yet we aim to draw a distinction between different
539 subjective beliefs of agents as opposed to different ambiguous beliefs of agents
540 on the possible values of the signal and the price. The below mathematical
541 analysis is just the first step towards the understanding of this important
542 distinction in asset trading context.

543 In the simplest quantum probability based model, market's state is based
544 on the two dimensional qubit state space H . Consider in H the orthonormal
545 basis $|+\rangle, |-\rangle$, where $|\pm\rangle$ are the states representing the beliefs that the
546 price of some infinitely lived financial asset will go up and down in the next
547 trading period. Since uncertainty is present in respect to the outcome of the
548 future distribution of dividends and prices, the agents hold ambiguous beliefs
549 about the asset price states and hence, their trading under ambiguous beliefs
550 generates deviations from the fundamental price value of the stock I .

Consider the state of the market *before* the arrival of the information
about the value of some fundamentals. In a state of maximal ambiguity, the
pure state representation of the beliefs of a representative agent are given as
the uniform superposition of the price states, respective dividend states.

$$|\psi\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}. \quad (4)$$

This is the state of a maximal uncertainty of market's agents about the
future dividend raise/fall and the consequent price raise/fall. If the agents
possess some prior information (e.g. observed realization of previous dividend
and price outcomes), they can make stronger expectations about the price
would go up or down, and hence the superposition state would be not

by the measurement of interference terms from the limiting probabilities and belief state
reconstruction via the Born rule.

uniform, where the squared modulus of the complex state coordinates c, d would sum to unity and each provide a classical probabilistic outcome on the price state ($\pm$).

$$|\psi\rangle = c|+\rangle + d|-\rangle, \quad c, d \in \mathbf{C}, \quad |c|^2 + |d|^2 = 1. \quad (5)$$

Such a skewed superposition would also exist, if the agents exhibit e.g. over-optimism, or pessimism under ambiguity.

The above representation gives the state of indeterminacy of beliefs, but not the state of diversity, since all agents are assumed to have the same pure state of beliefs, as in the notion of a ‘representative agent’, see eq.(4). In this model diversity of beliefs is generated through transition from a pure state (represented by a normalized vector ψ) to a mixed state (represented by a density operator ρ). In the latter, agents can hold heterogeneous ambiguous beliefs. The important mathematical distinction is that each of such belief-components of the mixture ρ is also a state (pure) of ambiguity. Thus, we can speak about probabilistic *diversity of uncertainties*, in contrast to *diversity of certainties* in the classical measure-theoretic models.¹⁸ Let us present briefly the latter for an ensemble of financial agents. Let Ω be the ensemble of all agents of the market. It is endowed with the classical probabilistic structure: a σ -algebra $\mathcal{F}$ and a probability measure P . Let $\xi : \Omega \rightarrow \{\pm\}$ be a random variable representing expectations of agents A, B about behavior of the price of an asset.¹⁹ Set $\Omega_{\pm} = \{\omega \in \Omega : \xi(\omega) = \pm\}$. Then each agent $\omega_A \in \Omega_+$ believes that the price would go up, and each agent $\omega_B \in \Omega_-$ believes that the price would go down. The probability distribution $p(\pm) = P(\Omega_{\pm})$ gives the measure of diversity of price behaviour beliefs. In this set-up, each agent of each agent type has a definite expectation of price behavior.

Now let us present belief formation in QP framework. Let ρ be a density operator and $|e_1\rangle, |e_2\rangle$ be its basis of eigenvectors with the eigenvalues q_1, q_2 which are non-negative and sum up to one:

$$\rho = q_1|e_1\rangle\langle e_1| + q_2|e_2\rangle\langle e_2|. \quad (6)$$

Let us re-expand the vectors $|e_1\rangle, |e_2\rangle$ with respect to the price-expectation basis $|+\rangle, |-\rangle$: $|e_j\rangle = c_j|+\rangle + d_j|-\rangle$. Thus with the weights q_1 and q_2 the

¹⁸This is a fundamental remark that separates the proposed model from the information asymmetry based models, where diversity of expectations is encoded in different prior and posteriors probability distributions that agents possess over the asset valuation.

¹⁹Future venues of research can focus on a generalization of the model to a market portfolio of assets, where the agents form expectations about the price behaviour of the composite finance market dynamics, see discussion in [51] and theoretical contribution in [34].

574 ensemble of agents Ω is split into two sub-ensembles Ω_A and Ω_B characterizing
 575 that agents belonging to each of them have the same state of uncertainty
 576 about possible behavior of the price.

In quantum probability random variables are given by observables, hence we introduce *the price expectations observable*, P , for the asset I represented by the operator having the vectors $|\pm\rangle$ as eigenvectors with eigenvalues ± 1 :

$$P = |+\rangle\langle+| - |-\rangle\langle-| \equiv P_+ - P_- . \quad (7)$$

Before forming a concrete preference on holding, or selling the unit of an asset, an agent has to resolve her ambiguity about the possible behavior of the stock price realization. For the simplicity of the model, the preferences are given classically, based on the discounted expected future value of stock price in the next trading period. To resolve the (non-classical) ambiguity the agent performs a (self-)measurement of the corresponding price expectation observable represented by the operator P . This operator, jointly with agent's belief-state ψ that is in a superposition of the different informational signals given as in eq. (5), encodes the probabilities that the price of this asset will go up or down.²⁰ They are given by Born's rule (one of the core postulates of quantum mechanics):

$$p(\pm) = |\langle\pm|\psi\rangle|^2 = \|P_{\pm}\psi\|^2. \quad (8)$$

577 For a pure state Born rule normalizes the quantum measurement scheme
 578 on an observable. More specifically, it identifies the probability rule for ob-
 579 serving probability of a realization of an eigenstate (price value) after the
 580 measurement of the price behaviour observable P .

For a mixed state with a density matrix ρ , Born rule can be expressed via a trace formalism:

$$p(\pm) = \text{Tr} \rho P_{\pm} \quad (9)$$

581 In this setup the limiting distribution of obtaining a concrete price output
 582 $\pm$ for asset I in the next trading period is given by taking the trace of the
 583 action of a projector $P_{\pm}$ upon the mixed state ρ .

584 One can measure the heterogeneity of beliefs about the asset liquidation
 585 value at $t = 1$, by considering, beliefs of the ensembles of agents $A, B \in \Omega$
 586 separately, in a similar mode as in [50].²¹ If one cohort of agents is in
 587 the pure state ψ_A and the other is in the pure state ψ_B , the difference in

²⁰To exemplify such a state transition, for the moment, we employ pure states.

²¹The dynamics of the value of belief heterogeneity denoted by k can be in their model crucial for creating upwards volatility in asset prices, if $k > y\sigma^2 Q$, where Q denotes the total supply of the asset and y is a standard measure of risk aversion.

ambiguous beliefs is obtained by measuring the angle (phase) between the state vectors ψ_A and ψ_B , such that $\theta_{A-B} = \Delta_{AB}$. If the ensemble A holds exactly the opposite ambiguous beliefs to the ensemble B , then $\Delta_{AB} = \pi$.

3.1 Interpretation of agents' beliefs: subjective and objective models

In quantum probabilistic models to decision making there are explored two basic models for completion of the process of decision making by an agent. These models are based on the two core interpretations of probabilities, *objectivist* and *subjectivist*.²²

- *Objective probability*: an agent actually performs a measurement on one of the outcomes, ± 1 , and depending on this outcome she makes her decision. The probabilities given by eq.(8) are objective (frequency) probabilities. They do not have any meaning for an individual agent. They can be found by collecting statistics for a large ensemble of agents. Such probability is interpreted as statistical frequency of agents that expect the asset to up or down.
- *Subjective probability*: an agent assigns subjective probabilities to possible outcomes given by eq. (8) and then she proceeds as in the classical subjective decision making framework by calculating the odds and making her choice by comparing the odds with 1 (certainty). Subjective probability approach is also known in economic literature as the 'degree of belief'. In this paradigm one treats the probability as individual agents' beliefs about the realisations of dividend and price outcomes.

The second interpretation seems to be closer to the classical subjective probability models of decision making, such as SEUT and its non-linear generalisations. Moreover, the behavior of such an agent can be considered to be more rational from the viewpoint of neo-classical economic theory (see also analysis in section 2.1.) For instance, if the probability of a price decrease is unlikely, $p(P-) \ll 1$, then it is rational to make the decision that the price is to go up. Thus, an agent with the belief-state encoding very low probability that the price is to go down *never* expects that the price will go down and acts accordingly. This interpretation of expectation formation also allows to operate with the notion of a 'representative agent', assuming that

²²A discussion on the interpretation of subjective and objective probability in experiments is provided in an early work by [54], for applications of QP interpretation in decision theory see [26].

all agents have similar subjective beliefs and form identical preferences. In contrary, an objective probability, or frequency interpretation implies that single agents have different trading preferences. Of course, the probability (the frequency of agents, who have some certain beliefs and trade upon them) of such a decision is very low, but not zero, as in the subjective interpretation model. It might be that the validity of these two models can be tested experimentally. In fact, such an experimental comparison is not just a ‘quantum probability theoretic problem’. This is a problem of application of objective versus subjective probabilities in models of decision making for populations of agents.

3.2 State dependence in asset trading and feedback reaction

How do the observed price states feed back into the ambiguous belief states of the agents about the asset price in the next trading period?

By making the decision $\alpha_t = \pm 1$ for the asset I at time t (which is given by $t = 0$ in the first trading period), an agent’s initial state ψ is projected onto the eigenvector $|\alpha_t\rangle$ that corresponds to an eigenstate for a particular value of price realization for that asset in the current trading period.²³ After the price realization up to time t of the asset is observed, the agent has to make a decision about the possible price behavior of the asset at time $t + 1$, and she performs a measurement of the corresponding expectation observable, for the updated belief-state $|+_{t+1}\rangle$. The index $t + 1$ denotes agent’s ambiguous beliefs about the dividends-prices in the subsequent trading period. The eigenvalues $\alpha_t = \pm 1$ of the price behaviour observable P_t , are given with the probability:

$$p_{t \rightarrow (t+1)}(\alpha_t \rightarrow \alpha_{t+1}) = |\langle \alpha_t | \alpha_{t+1} \rangle|^2. \quad (10)$$

The above mathematical exposition of state transition provides *quantum transition probability*. They have also an objective meaning. Consider an ensemble of agents in the same state ψ who made the decision α_t with respect to the price’s behavior of the asset. In the next step the agents form preferences about the subsequent period asset price realizations and consider only those whose decision is α_{t+1} . In this way it is possible to find the frequency-probability $p_{t \rightarrow (t+1)}(\alpha_t \rightarrow \alpha_{t+1})$. Following a classical tradition, we can consider the above output as the quantum analogue of the

²³In a simple setup with two types of price movements we fix only two eigenvectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$, corresponding to the eigenvalues $a = \pm 1$. These price outcomes are observed by the agent on the finance market when trading takes place.

conditional probabilities, $p_{t \rightarrow (t+1)}(\alpha_t \rightarrow \alpha_{t+1}) \equiv p_{t+1|t}(\alpha_{t+1}|\alpha_t)$. We remark that the trading in this setup takes place under informational ambiguity in respect to the next trading period, when the agents are still in an indeterminacy given as a superposition in respect to next coming dividend signals, $p(S\pm)$, $\psi_t = \beta_1|+\rangle + \beta_2|-\rangle$, $|\beta_1|^2 + |\beta_2|^2 = 1$. Hence, in each of the subsequent updated belief states about price behaviour the agents are in superposition in respect to the fundamentals that can change the price and interference effects as in eq.(2) exist for each agent's pure belief state (that can be approximated by a type of a representative agent). By using the probabilities (8)-(10) we can define the quantum joint probability distribution for price expectation about the price of the asset I in both trading periods, t and $t + 1$.

$$p_{t,(t+1)}(\alpha_t, \alpha_{t+1}) = p_t(\alpha_t)p_{(t+1)|t}(\alpha_{t+1}|\alpha_t). \quad (11)$$

This joint probability respects the *order structure* of beliefs, where the observed price outcome at time t changes the beliefs about the asset price distribution at $t + 1$.²⁴ In general:

$$p_{t,(t+1)}(\alpha_t, \alpha_{t+1}) \neq p_{(t+1),t}(\alpha_{t+1}, \alpha_t). \quad (12)$$

Equation, (12) is an exhibition of the *order effect* that is not in accord with Bayesian probability updating scheme, see theoretical analysis in [48], [55]. Order effects bring a non-satisfaction of the joint probability distribution and give raise to violation of the commutativity principle of classical probability. Order effects in a state update under ambiguity can exist for: i) preference formation related to a sequence of asset price observation as depicted above; ii) information processing related to the order of the sequences of state updates from observed dividend signal realizations. Non-commuting observables allow to depict agents' state dependence in belief formation that affects their trading preferences and hence the RE equilibrium price departures.²⁵ If state dependence is absent, the observable operators are commuting and the agent possesses a joint probability distribution for the infinite sequence of decision variables, given by some element, $\omega \in \Omega^t, \omega = \{P_t, S_t\}_{t=0}^\infty$.

It is important to remark that in the general quantum probability setup the operators for stock price behaviour at different time points do not commute, i.e., $[P_t, P_{t+1}] \neq 0$. This means that the price (and dividend signal)

²⁴The same state update scheme takes place in respect to the informational signals, i.e. the state update in respect to the price implies that the bases associated with the dividend realizations have a different phase in respect to this new updated state at ψ_t . The limiting probability distribution of the asset prices, when current prices and dividends at time t are observed is determined by agents' *order* of measurement of the corresponding observables.

²⁵In the formalization above we focus on dependence of the belief states upon the realised price states.

662 observables overtime, are complementary and agents cannot form a joint
 663 probability space of these random variables in the process of information
 664 processing. The order of price and dividend observations creates the pes-
 665 simistic, respective optimistic belief states of the agents that deviate from
 666 the classical joint evaluation of past and future price and dividend realiza-
 667 tions. The important consequence is that it is impossible to define a family
 668 of random variables for dividends and prices denoted as $\xi_i : \Omega \rightarrow \{\pm 1\}$ on
 669 the same classical probability space, $(\Omega, \mathcal{F}; P)$, which would reproduce the
 670 quantum probabilities $p(\pm 1) = |\langle \pm | \psi \rangle|^2$ as $P(\xi_i = \pm)$ and quantum tran-
 671 sition probabilities $p_{t \rightarrow (t+1)}(\alpha_t \rightarrow \alpha_{t+1}) = |\langle \alpha_t | \alpha_{t+1} \rangle|^2$, $\alpha_t, \alpha_{t+1} = \pm$, as the
 672 classical conditional probabilities $P(\xi_{t+1} = \alpha_{t+1} | \xi_t = \alpha_t)$.

673 In QP model the agents do not form definite expectations about the price
 674 behavior, and the observed price realization can change their future expect-
 675 ations about the asset price. This type of state dependence in beliefs is not
 676 in accord with classical RE pricing models that imply a current price depen-
 677 dence only on the future discounted payoff relevant variables. Furthermore,
 678 the agents exhibit ambiguity in respect to the probabilistic composition of
 679 future dividend signals and impact on price value, whereby the interference
 680 of these beliefs gives raise to a deviation of the belief distribution from the
 681 classically modeled rational beliefs. Given a price observation, agents can
 682 form conditional expectations only *sequentially* and not jointly. In the next
 683 section we present an operational depiction of the heterogeneous asset price
 684 belief evolution under informational ambiguity.

685 4 Creation and annihilation operators

686 We present a belief state space construct based on two ensembles of repre-
 687 sentative agents in different initial belief states and describe the operators
 688 that create their beliefs about the price of the risky asset. Consider a type
 689 A agent, and her belief-state space K , about the price behaviour let it be
 690 a two dimensional qubit state space.²⁶ We define an orthonormal basis in
 691 K , denoted as $|0\rangle, |1\rangle$. The states are interpreted as follows: they represent,
 692 respectively, agent A 's beliefs that the price of the stock will decrease or
 693 increase under ambiguity. In general A is in a state of a superposition of
 694 these 'core beliefs' representing her ambiguity in respect to the asset price

²⁶As noted above, the preference states of agents are visible from the market data, where the belief states about price behaviour are playing a key role, since the agents trade upon their beliefs, when there are no liquidity constraints. We introduce two operators A and B that describe the interaction of A 's beliefs about the price behaviour and the actual finance market price behaviour.

695 configuration at $t + 1$:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \quad (13)$$

696 where $c_j \in \mathbf{C}$ and $|c_0|^2 + |c_1|^2 = 1$.

697 The initial state ψ encodes all price and dividend realization histories
698 and encodes the prior ambiguous state of an agent. Two ensembles of agents
699 that we introduce later, can have different states ψ that also change over the
700 run of trading periods. Mathematically, weighted combinations of the pure
701 states, mixed states, are employed.

702 Following [35] we introduce the so called creation and annihilation oper-
703 ators a^* , a , having the following role in the setting of asset trading.²⁷

The operator a^* ‘creates’ a belief that the price would go up, $a^*|0\rangle = |1\rangle$,
and the operator a ‘annihilates’ the belief that the price would go up, $a|1\rangle =$
 $|0\rangle$. It can be interpreted as the operator of the creation of a belief that the
the price is to go down. Hence, these operators provide a mathematical tool
for the generation of firm beliefs about the price change. These operators
satisfy canonical anti-commutation relations:

$$\{a, a^*\} = I, \{a, a\} = 0, \{a^*, a^*\} = 0, \quad (14)$$

where I is the unit operator and $\{A; B\} = AB + BA$ denotes anti-commutator
of two operators A, B . In the basis $|0\rangle, |1\rangle$ the operators can be represented
by 2×2 matrices:

$$a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (15)$$

704 We now introduce the *operator of the price behavior* $B = a^*a$. We remark
705 that $B|1\rangle = |1\rangle$ and $B|0\rangle = 0$. Thus, in the basis of sharp beliefs about the
706 price behavior B has the diagonal form $B = \text{diag}(0, 1)$. This operator, in fact,
707 coincides with the orthogonal projector onto the vector $|1\rangle$, i.e., $B = |1\rangle\langle 1|$.
708 This operator represents the observable of the actual price behavior on the
709 finance market. Agents can make self-inspections of their beliefs about the

²⁷The paper by [6] apply raising a lowering operators to describe the process of cre-
ation and reduction of traders’ stock holdings. See also the work by [7] that uses this
formulation to describe trading between two agents, or a system of ‘n agents in a general
trading game. The operators can be applied to describe non classical dynamics in more
complex macroscopic systems, [8]. An interpretation of raising and lowering operators in
our framework is given by their operational role in modeling the price changes in financial
markets, while belief update of agents takes place.

710 possible price changes throughout the investment process.²⁸ By applying the
 711 inverse Born's rule one can reconstruct the subjective probability distribution
 712 of the dividend signals from agents' initial belief states coupled with the
 713 observed price states, given that $P_t = e^{-r\Delta t}(|c_1|^2 P_+ + |c_2|^2 P_-)$. The squared
 714 complex coordinates c_1, c_2 denote the total probability of the price increase
 715 or decrease given by eq. (2).

716 In this set-up an agent that is isolated from the surrounding informational
 717 environment, can observe her trading preferences and the uncertainty that
 718 can be updated via her observations of the asset price behaviour. We remark
 719 that the observed asset prices can further enhance the agent's A optimism
 720 in respect to the realization of future positive signals and price outcomes, as
 721 depicted in Section, 3.2.

722 Now we consider the system of two types of agents (A,B) with belief-
 723 state spaces $K_i, i = 1, 2$, with bases $|0\rangle_i, |1\rangle_i$. The belief-state space of this
 724 system is given by the tensor product $K = K_1 \otimes K_2$ and it has the basis
 725 $|00\rangle, |10\rangle, |01\rangle, |11\rangle$.²⁹ The basis states $|\alpha\beta\rangle$ are the states of sharp beliefs,
 726 e.g., in the state $|00\rangle$ both agents believe with certainty that the price of the
 727 financial asset will go down.

The individual ambiguity of the agents is encoded in superposition of the
 form in eq.(13). The joint belief-state of two agents is given by the factor
 product:

$$|\psi_1\rangle \otimes |\psi_2\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes (k_0|0\rangle + k_1|1\rangle) \quad (16)$$

The most general belief-state of these two agents has the form of superposi-
 tion:

$$|\psi\rangle = c_{00}|00\rangle + c_{10}|10\rangle + c_{01}|01\rangle + c_{11}|11\rangle \quad (17)$$

728 where $c_{ij} \in \mathbf{C}$ and $|c_{00}|^2 + \dots + |c_{11}|^2 = 1$.

The creation and annihilation operators of agents are lifted to the belief-
 state space K and we denote them by bold symbols, e.g., $\mathbf{a}_1 = a_1 \otimes I$. These
 operators satisfy so called *qubit commutation relations*. For the fixed i , such
 operators satisfy the canonical commutation relations, see eq.(14) for the one
 dimensional fermionic system, but for different i, j they commute:

$$[\mathbf{a}_i, \mathbf{a}_j^*] = [\mathbf{a}_i, \mathbf{a}_j] = [\mathbf{a}_i^*, \mathbf{a}_j^*] = 0, \quad (18)$$

729 where $[A, B] = AB - BA$ is the usual commutator.

²⁸In the work by [14] a price function of a form $p(s)$ is introduced, which maps the
 signals into asset prices, and the agents can infer the probability distributions from the
 price function by taking its inverse. We do not directly associate multiple signals (a set of
 different signals) with the observed price, since the beliefs about the signals are ambiguous.

²⁹Here we simplified notation, $|\alpha\rangle_i \equiv |\alpha\rangle$ and $|\alpha\rangle_1 \otimes |\beta\rangle_2 \equiv |\alpha\beta\rangle, \alpha, \beta = 0, 1$.

730 For the composite state of the two agent types the introduced operators
 731 generate agents' joint belief dynamics and hence, give the price expectations
 732 associated with the composite set of agents that trade the risky asset.

733 5 The Born rule of information update

734 After formulating the belief state evolution process with the aid of the price-
 735 creation and annihilation operators, the next stage is to explain how the
 736 ambiguity of agents' beliefs is resolved to classical probabilistic distribution
 737 of belief states after arrival of signals. By this we mean how: a) each agent's
 738 belief state and, b) a mixture of agents' belief states giving the composite fi-
 739 nance market will update once a signal about asset prices reaches the market.
 740 The probability of state realization and conditional probabilities of signals
 741 are given by Born rule (for mathematical details cf. monographs [12], [26]).³⁰
 742 Born rule specifies probability to obtain a particular result of measurement
 743 (eigenvalue) after a measurement on a pure state ψ , or mixed state given by
 744 a density matrix ρ . The formulation of Born rule differs for pure and mixed
 745 states, yet in both cases it specifies the classical limiting probability distri-
 746 bution associated with each eigenvalue realization. We specify the dividend
 747 signal observable λ with a corresponding operator S that has dichotomous
 748 eigenvalues ± 1 , with $|\pm_k\rangle$ as its eigenvectors.

As specified in eq.(8), we can observe the probability of arrival of some new dividend signal for the pure initial belief state ψ as:

$$p(S_{\pm}) = \|S_{\pm}\psi\|^2. \quad (19)$$

749 We can in the similar way introduce μ as the price movement observable
 750 with corresponding operator, P that is now measured after the arrival after
 751 dividend outcome $\pm$.

The operator has eigenvectors $|\pm\rangle$. Conditional probability from eq. 20 specifying the probability of obtaining a price value $\pm$, under the condition that signal $\pm$ was observed is given in a similar way as in eq.(10). We denote here the signal and price operator eigenvectors via $\lambda_{\pm}$ respective $\mu_{\pm}$ to elucidate the state transition scheme from one (normalized) eigenvector to another:

$$p(P_{\pm}|S_{\pm}) = |\langle\lambda_{\pm}|\mu_{\pm}\rangle|^2 \quad (20)$$

752 The conditional probability in eq.(20) contains the information about the op-
 753 erator S that updates the price behaviour and as a result trading preferences,
 754 after an information arrival, such as some signal. Moreover, the conditional

³⁰Probabilities can be subjective or objective, as discussed in Section 3.

probability will contain the information regarding interaction between the observables μ and λ acting upon the initial ambiguous belief state of financial agents.³¹ In a real finance market setting the agents can also have an internal dynamics of their ambiguous beliefs in addition to the updates of the S , P as well as the ambiguity in respect to these observables may not be resolved concurrently. In the Appendix, 8 we devise a more detailed mathematical representation of agents information processing and measurement scheme about the price behaviour.

The order of information (or the order, in which the agent chooses to process the information) alike the choice on asset trading exposed in Section 3 can affect her updated belief state. Order effects can be modeled in QP via different eigenvectors associated with the observables and hence, the effect of phase between the eigenvectors upon the final conditional probabilities under different sequences of measurement schemes surfaces. We recap that the impact of measurement sequence upon conditional probabilities is due to incompatible observables in this setup. The order effect is important to understand the state update under different orderings of information that can create posterior biased belief state of overoptimism, respective pessimism. The effect of such state updates via Born rule (the sum of which is given by the composite market mixed state) can give raise to sudden price behaviour changes, due to interference effects of information and action states. Such a process can be captured via dynamical quantum probabilistic models, see [47], [28], [12], [26], [34], [40] applying the Schroedinger equation and its extensions to model the belief and information dynamics.

6 Discussion: empirical predictions?

Our model so far has described the quantum probabilistic formulation of the uncertainty, or divergence of belief states of agents that creates asset price volatility in the capital market. The central contribution of this work is due to formalization of the of divergence of beliefs in classical versus quantum probabilistic framework given ambiguity about both dividend and price states shaped by agents internal states and informationally incomplete markets. The motivation to apply QP is to capture agents' trading under a deeper, endogenous uncertainty.

However, for empirical prediction of the model we need to consider how the measurements of belief states performed by the signal and price behaviour

³¹The QP update algorithm allows also to depict information update from uninformative and close to zero priors that cannot be captured by classical Bayesian update, see the work by [10].

790 operators on the initial belief states of the agents, or the market belief state
 791 as a whole, impacts the probabilities of movements of asset prices. Certainly,
 792 according to QP formulation the probabilities are obtained from the familiar
 793 trace formulation, as shown in the above model, which contains the interfer-
 794 ence terms for different ensembles of agents. Since the size of the interference
 795 term indicates the magnitude of probabilistic interference, we can surely talk
 796 about e.g. sub-additivity of baseline probability for P_+ , if it is above the sum
 797 of the conditional probabilistic disjunctions given the different signals (the
 798 so called ‘disjunction effect’ is present). On the aggregate finance market,
 799 we can interpret it as a bubble in the condition of uncertainty, when the ac-
 800 tual signals are not measured by market participants in a Bayesian fashion.
 801 When interference effect is positive, an overweighting of probability takes
 802 place, and the $p(P_+)$ is above the total probability of P_+ , given the different
 803 sets of information as specified in FTP.

804 As noted, when $\cos\theta = 0$ no interference is present and QFTP collapses
 805 into its classical analogue the FTP so that agents have a classical probabilis-
 806 tic distribution of the asset price expectations, given the different signals. In
 807 this case all agents agree on the fundamental value, given that there is no
 808 dispersion in their information processing. A positive, respective negative
 809 magnitude of the interference term also depends on the belief evolution dy-
 810 namics under uncertainty, and periods of ‘optimistic’, respective ‘pessimistic’
 811 trading cycles can emerge, where bubbles can burst suddenly, without any
 812 warning signals, [56]. We remark that in contrary to the modifications of
 813 classical probability calculus that are introduced to describe volatility cy-
 814 cles form agents’ beliefs in the classical finance literature, the endeavour of
 815 QP based framework of asset trading is to consider a different probability
 816 calculus that is complete in terms of its axiomatics.

817 Since the two ensembles of agents can have different belief states about
 818 the price ups and downs, the whole market can be given mathematically a
 819 system of these ensembles of agents in different pure belief states, denoted
 820 via a mixed state. This representation allows to depict divergence of the
 821 uncertain (pure) belief states of agents, where under short selling constraints
 822 agent ensembles with positive λ create inflationary pressures. The model
 823 generalized to markets with no short selling constraints, makes the pessimists
 824 and also speculators act by creating periods of asset undervaluation, whereby
 825 the evolution of the mixed state in the QP model allows to observe the net
 826 effect on the limiting probability distribution of the price states, given in
 827 eq.(9). The proposed theoretical model can be advanced further, by replacing
 828 the notion of risk neutral agent and introducing a discount factor based on
 829 the risk aversion of the different ensembles of agents, see [1].

830 Finally, the QP based model of beliefs has the potential to provide man-

831 agers and practitioners with an insight into the possible response patterns
 832 of investors to new share issuance as well as valuation of the traded risky
 833 assets on the secondary markets. The studies can be based on the empiri-
 834 cal investigations of interference terms and of dispersion of opinions data as
 835 mentioned above. Such studies would bring more insights about the future
 836 expectations of primary and secondary markets of risky assets. The proba-
 837 bilistic prognosis of agents' price valuation under the subjectivist probability
 838 interpretation can allow for better understanding of cognitive processes of
 839 individual investors as attested experimentally in, e.g. [27], [37]. The 'ob-
 840 jectivist' interpretation of the quantum probabilities would correspond to
 841 the prognosis of the frequency of agents buying and selling the asset under
 842 ambiguity and hence, provide an indication of the composite finance market
 843 trading volume.

844 **7 Conclusion**

845 We have suggested a QP based model of asset trading behaviour under un-
 846 certainty. Our model fundamentally differentiates from the standard neoclas-
 847 sical finance models of price behaviour under heterogeneous beliefs, based on
 848 classical probability theory and non-additive modifications. The main mo-
 849 tivation for adopting an alternative modeling is that there is a high degree
 850 of divergence in predictions in the standard literature regarding the price
 851 behavior under uncertainty, for example, whether diversity of beliefs leads to
 852 adverse selection problem, or overpricing respective under-pricing following
 853 speculative trading. There is also a lack of a unifying framework describing
 854 the mechanism of speculative bubble formation but also trading that can lead
 855 to market crashes. Our model aims to offer an alternative foundation to spec-
 856 ulative asset pricing under ambiguity, based on QP of belief representation.
 857 First of all, the description of uncertainty in the model is based on a super-
 858 position of belief states, and not on classical probability distributions. We
 859 also deploy a novel technique of annihilation-creation operators, to describe
 860 observables that measure the "belief state" of the market. The interference of
 861 probability amplitudes related to price states and fundamentals gives a mea-
 862 sure of over-pricing and in other trading periods under-pricing, following the
 863 state dependence on observed asset price states. The proposed framework in
 864 the subjectivist probability interpretation, is providing a quantifiable testable
 865 prediction on price volatility in respect to the from fundamental value, un-
 866 der belief ambiguity. The framework can be further tested in experimental
 867 asset trading markets, where one can reckon the degree of inflationary, or
 868 deflationary pressures created by the ensembles of traders in different be-

869 lief states under ambiguity. The agents' asset trading preferences can be
870 revealed under uncertainty and contrasted with their preferences after some
871 informational signals reached them to attest the classicality of their belief
872 update in investment, as performed in a similar setting in [27]. Order effects
873 belief evolution, given by the order of the observed price states and payoff
874 relevant variables can be also further tested with a similar setup as recently
875 proposed in [40]. We hope that this simple theoretical model will bring up
876 new experimental studies in the area of investment preferences of agents for
877 different types of financial instruments and financial markets, coupled with
878 the impact of stock market news upon the evolution of agents' expectations
879 that can be potentially modeled in a QP based decision theoretic framework.

880 8 Appendix

881 We espouse a more detailed mathematical representation of the existence of
882 internal dynamics of the belief states of the agent types, given by their diver-
883 gent mode of information processing, beyond the measurements of dividend
884 and price signals at the specific points in time $(t_0 - t)$.

885 Without loss of generality, let us consider a Hilbert state space H of
886 an arbitrary dimension, in which two operators S and P with respective
887 eigenvalues ± 1 and eigen-subspaces $H_{\pm}^S$ and $H_{\pm}^P$, act upon the preference
888 state of the agents. The corresponding projectors are denoted by $S_{\pm}$ and
889 $P_{\pm}$, i.e., $S = S_+ - S_-$ and $P = P_+ - P_-$.

890 The measurements of S (or P) for the state ψ with the outcomes ± 1
891 projects ψ onto the state $\psi_{\pm}^S = S_{\pm}\psi$ (or $\psi_{\pm}^P = P_{\pm}\psi$). The belief state of an
892 agent can be modified (updated) through the measurements of S or P . This is
893 basics of the (belief) measurement scheme due to the 'Von Neuman-Lüders'
894 projection postulate of quantum theory.

895 It is useful to extend exposition of quantum probabilistic belief update
896 to include the *internal dynamics* of the belief state $t \rightarrow \psi_t$ in order to better
897 approximate the information processing to real finance market environment.
898 This is a belief dynamics of agents in the absence of the updates, given by
899 the measurements of S and P observables. Such dynamics is mathematically
900 described by a family of unitary operators U_t that transform the ambiguous
901 belief distribution overtime, where $\psi_t = U(t - t_0)\psi_0$.

902 Suppose that at instances of time $t_1, \dots, t_n, \dots$ the agent performs a mea-
903 surements of S , or in other words she gets a signal that dividends will raise
904 or fall, and at instances of time $s_1, \dots, s_n, \dots$ she measures P , so she updates
905 her expectations about the asset price based on the observed price dynamics.
906 It natural to assume that $t_1 < s_1 < t_2 < \dots < t_n < s_n \dots$ since the agents

trade under informational ambiguity.

Hence a complete dynamics of an agent's belief state can be represented as a series of projections coupled with the (agent specific) unitary evolution:

$$\psi_{t_0} \rightarrow \psi_{t_1} = \frac{S_{\lambda_1} U_{t_1-t_0} \psi_{t_0}}{\|S_{\lambda_1} U_{t_1-t_0} \psi_{t_0}\|} \rightarrow \psi_{s_1} = \frac{P_{\mu_{s_1}} U_{s_1-t_1} \psi_{t_1}}{\|P_{\mu_{s_1}} U_{s_1-t_1} \psi_{t_1}\|} \rightarrow \dots,$$

where $\lambda_{t_j} = \pm 1$ and $m_{s_i} = \pm 1$ are the outcomes of the measurements of S and P , respectively.

For any t , the probabilities of the possible measurements outcomes S and P are given by the Born rule: $p(S = \pm 1) = \|S_{\pm} \psi_t\|^2$ and $p(P = \pm 1) = \|P_{\pm} \psi_t\|^2$. In particular, for an instance of time t_j , we obtain $p(S = \lambda_{t_j}) = 1$, for the next instance of time s_i , we obtain $p(P = m_{s_i}) = 1$.

The model also specifies *transition probabilities*, e.g., the probability of transition from belief λ_{t_j} about the dividend distribution at $t = t_j$ to a belief μ_{s_j} about the price distribution: $p(\lambda_{t_j} \rightarrow \mu_{s_j}) = |\langle \psi_{s_j} | \psi_{t_j} \rangle|^2$.

In the simplified model that we proposed earlier, we omit the impact of the internal evolution of the belief state, i.e., to set $U_t \equiv I$. Here the state dynamics is reduced to a series of state updates resulting from the measurements of the core variables S and P :

$$\psi_{t_0} \rightarrow \psi_{t_1} = \frac{S_{\lambda_{t_1}} \psi_{t_0}}{\|S_{\lambda_{t_1}} \psi_{t_0}\|} \rightarrow \psi_{s_1} = \frac{P_{\mu_{s_1}} \psi_{t_1}}{\|P_{\mu_{s_1}} \psi_{t_1}\|} \rightarrow \dots,$$

Thus, for $t_j \leq t < s_j$, we get $\psi_t = S_{\lambda_{t_j}} \dots P_{\mu_{s_1}} S_{\lambda_{t_1}} \psi_{t_0} / \|S_{\lambda_{t_j}} \dots P_{\mu_{s_1}} S_{\lambda_{t_1}} \psi_{t_0}\|$, and for $s_j \leq t < t_{j+1}$,

$$\psi_t = P_{\mu_{s_j}} S_{\lambda_{t_j}} \dots P_{\mu_{s_1}} S_{\lambda_{t_1}} \psi_{t_0} / \|P_{\mu_{s_j}} S_{\lambda_{t_j}} \dots P_{\mu_{s_1}} S_{\lambda_{t_1}} \psi_{t_0}\|.$$

The sources of internal dynamics can be manifold and are given by agent specific variables. Some salient triggers of agent specific belief evolution detected experimentally [37], [40] due to individual differences in learning from own gains and losses as well as decision outcomes, coupled with agent's cognitive capacity and risk attitude.

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