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On the consistency of the reaction-telegraph process within finite domains

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Corresponding Author:	Sergei Petrovskii University of Leicester Leicester, Leicestershire UNITED KINGDOM	
Corresponding Author Secondary Information:		
Corresponding Author's Institution:	University of Leicester	
Corresponding Author's Secondary Institution:		
First Author:	Paulo F.C. Tilles	
First Author Secondary Information:		
Order of Authors:	Paulo F.C. Tilles Sergei Petrovskii	
Order of Authors Secondary Information:		
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Abstract:	<p>Reaction-telegraph equation (RTE) is a mathematical model that has often been used to describe natural phenomena, with specific applications ranging from physics to social sciences. In particular, in the context of ecology, it is believed to be a more realistic model to describe animal movement than the more traditional approach based on the reaction-diffusion equations. Indeed, the reaction-telegraph equation arises from more realistic microscopic assumptions about individual animal movement (the correlated random walk) and hence could be expected to be more relevant than the diffusion-type models that assume the simple, unbiased Brownian motion. However, the RTE has one significant drawback as its solutions are not positively defined. It is not clear at which stage of the RTE derivation the realism of the microscopic description is lost and/or whether the RTE can somehow be 'improved' to guarantee the solutions positivity. Here we show that the origin of the problem is twofold. Firstly, the RTE is not fully equivalent to the Cattaneo system from which it is obtained; the equivalence can only be achieved in a certain parameter range and only for the initial conditions containing a finite number of Fourier modes. Secondly, the Dirichlet type boundary conditions routinely used for reaction-diffusion equations appear to be meaningless if used for the RTE resulting in solutions with unrealistic properties. We conclude that, for the positivity to be regained, one has to use the Cattaneo system with boundary conditions of Robin type or Neumann type, and we show how relevant classes of solutions can be obtained.</p>	
Response to Reviewers:	<p>Reply to Reviewer 1</p> <p>All changes made in the manuscript are marked in blue colour.</p> <p>1) The referee is right: there are restrictions on what current fields can be realised within a given density field, and the observed 'population appears from nowhere' feature indeed comes from the imposition of a positive rate of change of density on the</p>	

initial condition without taking into account its dependence to the density. However, the RTE is often considered as a standalone, consistent model just by itself, hence disregarding the 'intermediate system' like Cattaneo's. If one accepts this as a working hypothesis, than almost any initial condition for $\partial_t u(z,0)$ should be acceptable, as it only describes the initial rate of change of the density. We added a brief discussion of this issue at the bottom of page 5 and top of page 6.

2) In the same spirit, from a purely mathematical point of view any boundary condition should be acceptable. But when we are considering the RTE to describe population dynamics, we want to keep both the density hypothesis and the correlated random walk hypothesis. In principle, it should be possible to force the solutions to become positive with just a redefinition of the variables/parameters, but we cannot guarantee that this 'new system' describes a microscopic correlated random walk. In order to ensure both hypothesis (within the scope of a finite domain), then one must look at the microscopic dynamics that takes place at the boundaries, which essentially means that we cannot consider the RTE alone: we have to think about the dynamics in terms of the original process (Cattaneo system).

3) The typo is corrected.

Reply to Reviewer 2

All changes made in the manuscript are marked in blue colour.

1) We are thankful to the Reviewer for pointing out at highly relevant literature on run-and-tumble processes that we indeed were not aware about. We added the recommended references to the revised text (with proper comments); see page 2. We do hope that this will make our work interesting for a broader audience.

2) Reference to Malakar et al. (2018) is added along with a brief comment; see the top of page 19.

3) The typo is corrected.

4) In order to ensure that the mathematical problem for the reaction-telegraph equation is well-posed, i.e. possesses a unique solution, it must be complemented by the boundary and the initial conditions. The general theory of differential equations states that the number of the required initial conditions has to be equal to the order of the highest time-derivative contained by the equation. Hence, there must be two initial conditions. A somewhat more tricky issue is the choice of the specific conditions. We added an explanation and a brief discussion of this issue to the revised manuscript; see the bottom of page 5 and the top of page 6.

On the consistency of the reaction-telegraph process within finite domains

Paulo F. C. Tilles^{a,b} & Sergei V. Petrovskii^{a,c1}

^a Department of Mathematics, University of Leicester, Leicester, LE1 7RH, UK

^b Departamento de Matemática, Universidade Federal de Santa Maria, Santa Maria RS,
Brazil

^c Peoples Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya St,
Moscow 117198, Russian Federation

Abstract

Reaction-telegraph equation (RTE) is a mathematical model that has often been used to describe natural phenomena, with specific applications ranging from physics to social sciences. In particular, in the context of ecology, it is believed to be a more realistic model to describe animal movement than the more traditional approach based on the reaction-diffusion equations. Indeed, the reaction-telegraph equation arises from more realistic microscopic assumptions about individual animal movement (the correlated random walk) and hence could be expected to be more relevant than the diffusion-type models that assume the simple, unbiased Brownian motion. However, the RTE has one significant drawback as its solutions are not positively defined. It is not clear at which stage of the RTE derivation the realism of the microscopic description is lost and/or whether the RTE can somehow be ‘improved’ to guarantee the solutions positivity. Here we show that the origin of the problem is twofold. Firstly, the RTE is not fully equivalent to the Cattaneo system from which it is obtained; the equivalence can only be achieved in a certain parameter range and only for the initial conditions containing a finite number of Fourier modes. Secondly, the Dirichlet type boundary conditions routinely used for reaction-diffusion equations appear to be meaningless if used for the RTE resulting in solutions with unrealistic properties. We conclude that, for the positivity to be regained, one has to use the Cattaneo system with boundary conditions of Robin type or Neumann type, and we show how relevant classes of solutions can be obtained.

Keywords: reaction-telegraph equation, reaction-Cattaneo system, nonnegativity, Robin boundary conditions

¹Corresponding author. Email: sp237@le.ac.uk

1 Introduction

Mathematical modelling is a powerful and widely used toolbox of theoretical ecology [1, 2, 3, 4, 5]. There are numerous mathematical frameworks, techniques and approaches that can be used to understand the properties of the dynamics of populations, communities and ecosystems. Curiously, sometimes a fruitful mathematical framework can be based on apparently unrealistic assumptions and/or has some properties that may seem infeasible. One example is diffusion where a dispersing² animal or seed (or, more generally, a particle) is assumed to experience a change in the movement direction between any two infinitesimally close moments. Strictly speaking, this assumption is irrelevant as at small spatial/temporal scales the movement velocities are correlated [6]. Remarkably, the diffusion equation was shown to be a good quantitative model of animal movement when the movement is considered at a spatial scale much larger than the animal size both in a natural environment [7] and under controlled laboratory conditions [8]. On a multi-generational timescale, the diffusion equation becomes the reaction-diffusion equation (a system of reaction-diffusion equations if several interacting species are explicitly taken into account) and the reaction-diffusion framework has been used to successfully describe a variety of ecological phenomena including biological invasions [9, 10, 11] and ecological pattern formation [12, 13, 14].

However, the question remains as to whether the reaction-diffusion framework can perhaps be further improved if the underlining assumption of the infinitely frequent turns is lifted or changed to something more realistic. Correlated walk is thought to be a more appropriate alternative stochastic process as it assumes that the animals can move along for some time before turning, the time between subsequent turns being a random variable (e.g. a Poisson process) [15, 16]. [In the physics literature, this type of correlated random walk was first studied under the name of ‘persistent random walk’ \[21, 22\]. More recently, it has emerged and was further studied under the new name of ‘run and tumble processes’ \[23, 24, 25, 26, 27\], because it describes the motion of some bacteria like *E. coli* \[28\].](#) A macroscopic counterpart of the correlated random walk is the telegraph equation [17, 18], which can be derived from different arguments (i.e. not related to random walk) such as inertia or finite relaxation time [19] and hence it can be regarded as a general model applicable to a certain class of phenomena occurring in systems of various origin [20].

Once multiplication (population growth) is taken into account, it turns into the reaction-telegraph equation (RTE) [29, 30]. As well as the telegraph equation, the RTE can be derived in different ways, e.g. from a microscopic stochastic process [31, 32] as well as from the energy and/or mass conservation combined with a modified Fick law (known as the Cattaneo equation); for an overview and further references see [20]. We also mention here that, beyond the ecological applications, the telegraph equation and the reaction-telegraph equation were previously used, generally successfully, in many areas of natural sciences [33, 34, 35, 36] and remain to be a focus of attention [37, 38, 39, 40, 41, 42].

Given that it is obtained from more realistic assumptions, arguably, the RTE can be

²Here we are considering "dispersal" in a generic sense: movement along a certain path.

regarded as a more adequate model than the reaction-diffusion equation and it was indeed used successfully in several studies to understand the dynamics of real world populations and/or to sensibly interpret field data [30, 43, 44, 45]. As it happens, however, the RTE has some unrealistic properties too, as its solutions are not positively defined, i.e. can become negative at some locations in space and/or at some moments of time [29, 46, 47]. This non-positivity of the solution is in fact rather paradoxical. Indeed, in any derivation of the telegraph or reaction-telegraph equation, the next-to-final stage is given by the Cattaneo system which is essentially a system of mass-balance equations. In case this system is considered in an unbounded space, its solutions are positively defined [30]. The following question therefore arises: how the positivity disappears when the Cattaneo system is transformed, using only apparently identical mathematical transformations, into the reaction-telegraph equation? We mention here that, in spite of considerable literature on the RTE (including the seminal paper by Hillen [46] where the solution's existence and uniqueness are studied comprehensively), the loss of positivity has received relatively little attention and is still not well understood.

In this paper, we consider the linear RTE with the purpose to reveal the origin of the non-positivity. We show that there are two possible reasons for that. Firstly, we show that the RTE is in fact not equivalent to the Cattaneo system from which it is usually derived. We show that, generally speaking, the solution of the RTE only satisfies the Cattaneo system if additional compatibility conditions are imposed. Secondly and more importantly, we show that the non-positivity can result from inadequately imposed boundary conditions. In particular, we show that the standard 'zero-function' Dirichlet boundary conditions that are routinely used for the diffusion and reaction-diffusion equations in fact have no sense for the RTE. We then show that positively defined solutions can be obtained when meaningful boundary conditions are used (e.g. Neumann or Robin type).

2 Cattaneo system, the telegraph equation and the RTE

Let us consider an animal performing a correlated random walk on a line with a constant speed v . A Poisson process of rate λ determines the rate at which the direction of motion may be altered with time (where the change in the direction happens with a probability $1/2$). Let $u_{\pm}(x, t)$ be the probability densities that describe the situation where the animal is going to the right (+) or to the left (-). We then define functions u and j as follows:

$$u(x, t) = u_+(x, t) + u_-(x, t), \quad j(x, t) = u_+(x, t) - u_-(x, t), \quad (1)$$

where $u(x, t)$ is the probability density of finding the animal around the position x at time t , and the function $j(x, t)$ is proportional to the flux (the actual flux should be written as vj in order to have the correct dimension); since v is a constant parameter, we refer to j simply as the flux.

By performing a few standard steps to count the transitions between the animals moving to the left and to the right as a result of turning (for details see [17, 18, 30]), the following system of equations can be readily obtained:

$$\partial_t u_+(x, t) + v \partial_x u_+(x, t) = \frac{\lambda}{2} [u_-(x, t) - u_+(x, t)], \quad (2a)$$

$$\partial_t u_-(x, t) - v \partial_x u_-(x, t) = \frac{\lambda}{2} [u_+(x, t) - u_-(x, t)], \quad (2b)$$

where ∂_t and ∂_x denote the partial derivatives with respect to t and x , respectively. For convenience, we may introduce the characteristic time τ (i.e. the mean time between turnings) and the diffusion coefficient D as

$$\tau = \frac{1}{\lambda}, \quad D = \frac{v^2}{\lambda}. \quad (3)$$

Equations (2) then take the following form:

$$\partial_t u_+(x, t) + \sqrt{\frac{D}{\tau}} \partial_x u_+(x, t) = \frac{1}{2\tau} [u_-(x, t) - u_+(x, t)], \quad (4a)$$

$$\partial_t u_-(x, t) - \sqrt{\frac{D}{\tau}} \partial_x u_-(x, t) = \frac{1}{2\tau} [u_+(x, t) - u_-(x, t)]. \quad (4b)$$

In order to obtain a set of equations for the density and the flux, as defined in Eq. (1), we only need to take linear combinations of Eqs. (4), thus resulting in

$$\partial_t u(x, t) + \sqrt{\frac{D}{\tau}} \partial_x j(x, t) = 0, \quad (5a)$$

$$\partial_t j(x, t) + \sqrt{\frac{D}{\tau}} \partial_x u(x, t) = -\frac{1}{\tau} j(x, t), \quad (5b)$$

which is known as the Cattaneo system [19, 46, 47]. Here Eq. (5a) is the standard continuity equation (or the mass conservation equation). Equation (5b), also known as the Cattaneo equation, can be regarded as a generalized Fick law that takes into account the final relaxation time τ [19, 20]; it turns into the usual Fick law in the limit $\tau \rightarrow 0$.

Differentiating Eq. (5a) with respect t and Eq. (5b) with respect to x and considering a linear combination of them, we obtain the so called telegraph equation:

$$\tau \partial_t^2 u(x, t) + \partial_t u(x, t) = D \partial_x^2 u(x, t). \quad (6)$$

According to our original setup (see the beginning of this section), the solution $u(x, t)$ of Eq. (6) is the probability density to find the animal at given location x and given time t . In case we have a population of N identical animals moving independently, each of them following the same stochastic process, and N is sufficiently large, then $\tilde{u}(x, t) = Nu(x, t)$ is the population density. Since Eq. (6) is linear, having multiplied it by N , it becomes the equation for the population density. Similarly, $\tilde{u}_+(x, t)$ and $\tilde{u}_-(x, t)$ are the population densities of animals moving to the right and to the left, respectively. In the rest of the

paper, we therefore interpret the solutions of Eqs. (4), (5) and (6) as the corresponding population densities but omitting tildes for the sake of notation's simplicity.

Once the mass conservation equation (5a) is considered in terms of the population density, it is straightforward to take into account multiplication by adding the corresponding term to the right-hand side. Here we are only going to consider a linear growth characterized by the growth rate α . Since we are concerned with analyzing the properties of the system on a finite domain, we consider it for $0 < x < L$.

If we define the following dimensionless variables and parameters,

$$z = \frac{x}{L}, \quad \zeta = \frac{t}{\tau}, \quad \mu = \alpha\tau \quad \beta^2 = \frac{D\tau}{L^2}. \quad (7)$$

we arrive at the dimensionless reaction-Cattaneo system:

$$\partial_\zeta u(z, \zeta) + \beta \partial_z j(z, \zeta) = \mu u(z, \zeta), \quad (8a)$$

$$\partial_\zeta j(z, \zeta) + \beta \partial_z u(z, \zeta) = -j(z, \zeta). \quad (8b)$$

After using the same procedure that lead to equation (6), we obtain the dimensionless RTE

$$\partial_\zeta^2 u(z, \zeta) + (1 - \mu) \partial_\zeta u(z, \zeta) = \beta^2 \partial_z^2 u(z, \zeta) + \mu u(z, \zeta). \quad (9)$$

Note that the dimensionless form (9) of the RTE cannot be obtained from the telegraph equation (6) just by adding the growth term to the right-hand side (as is done in case of diffusion) as it would miss the new factor $(1 - \mu)$ in the left-hand side, cf. [30].

3 Dirichlet boundary conditions

Firstly, we are going to analyze the reaction-telegraph equation (9) subject to arguably the simplest, 'intuitive' boundary conditions requiring the population density to be zero at the domain boundaries:

$$u(0, \zeta) = u(1, \zeta) = 0. \quad (10)$$

The standard ecological interpretation of conditions (10) is that the population density is zero outside of the given domain, presumably as a result of very harsh environment. Using the analogy with the diffusion equation, we will refer to (10) as the Dirichlet boundary conditions.

In order to ensure that the mathematical problem (9–10) is well-posed, i.e. possesses a unique solution, it must be complemented by the initial conditions. Note that the general theory of differential equations states that the number of the required initial conditions has to be equal to the order of the highest time-derivative contained by the equation, e.g. see [48]. Correspondingly, for the RTE (9) there must be two initial conditions, which we consider in the following general form:

$$u(z, 0) = u_0(z), \quad (11a)$$

$$\partial_t u(z, 0) = v_0(z). \quad (11b)$$

We mention here that, whilst the biological meaning of condition (11a) is straightforward, the biological meaning of condition (11b) is somewhat obscure. Moreover, if one carefully follows the derivation of Eq. (9) from the very beginning, then it becomes clear that the initial condition for the time derivative of the density is not entirely independent but may be related, according to Eq. (8a), to both the density and the flux. However, at the moment we are considering the hypothesis that the RTE is consistent by itself, as there is considerable literature that deals directly with the telegraph equation or the RTE, hence completely disregarding the ‘intermediate system’ (as given by the Cattaneo system or by the corresponding microscopic process). If true, it means that almost any initial condition for $\partial_t u(z, 0)$ should be acceptable to describe the initial rate of change of the population density.

We endeavour to obtain the solution of the problem (9–11) by applying the method of the separation of variables, i.e. by looking for a solution in the following form:

$$u(z, \zeta) = U(z) \psi(\zeta). \quad (12)$$

Having substituted Eq. (12) into Eq. (9), we arrive at

$$\frac{\psi''(\zeta)}{\psi(\zeta)} + (1 - \mu) \frac{\psi'(\zeta)}{\psi(\zeta)} - \mu = \beta^2 \frac{U''(z)}{U(z)} = -\kappa^2, \quad (13)$$

where constant κ should be determined in order to make the equations consistent with the boundary conditions.

The general solution for the spatial part is readily written as

$$U(z) = C_1 \cos\left(\frac{\kappa z}{\beta}\right) + C_2 \sin\left(\frac{\kappa z}{\beta}\right). \quad (14)$$

In order for the boundary conditions to be satisfied we must set $C_1 = 0$ while the constant κ has to assume any of the values

$$\kappa_n = n\pi\beta, \quad n = 1, 2, \dots \quad (15)$$

Hence, Eq.(14) becomes a countable set of functions, i.e., the eigenfunctions of the classical Sturm-Liouville problem:

$$U_n(z) = \sin(n\pi z), \quad n = 1, 2, \dots \quad (16)$$

Since Eq. (9) is linear, its general solution satisfying boundary conditions (10) is given by the Fourier series over eigenfunctions (16):

$$u(z, \xi) = \sum_n \psi_n(\zeta) \sin(n\pi z), \quad (17)$$

where ψ_n is the solution of the following ordinary differential equation:

$$\psi_n''(\zeta) + (1 - \mu) \psi_n'(\zeta) = (\mu - n^2 \pi^2 \beta^2) \psi_n(\zeta), \quad (18)$$

subject to the initial conditions (11), so that

$$\psi_{0,n} = 2 \int_0^1 dz u_0(z) \sin(n\pi z), \quad \psi'_{0,n} = 2 \int_0^1 dz v_0(z) \sin(n\pi z). \quad (19)$$

The solution of (18–19) can be written as

$$\psi_n(\zeta) = e^{\frac{\mu-1}{2}\zeta} \left[\psi_{0,n} \cosh\left(\frac{\Lambda_n \zeta}{2}\right) + \frac{2\psi'_{0,n} + (1-\mu)\psi_{0,n}}{\Lambda_n} \sinh\left(\frac{\Lambda_n \zeta}{2}\right) \right], \quad (20)$$

where we have used the notation

$$\Lambda_n = \sqrt{(1+\mu)^2 - 4n^2\pi^2\beta^2}. \quad (21)$$

Here $\sinh(\cdot)$ and $\cosh(\cdot)$ are hyperbolic functions if Λ_n is real and they turn into the corresponding trigonometric functions if Λ_n is imaginary. If we define the critical value of β_n as

$$\beta_n^* = \frac{1+\mu}{2\pi n}, \quad (22)$$

then for $\beta > \beta_n^*$ the function $\psi_n(\zeta)$ is trigonometric for a given n . Since $\{\beta_n^*\}$ is a monotonously decreasing sequence, in case $\beta > \beta_1^*$ all functions $\psi_n(\zeta)$ for any $n = 1, 2, \dots$, are trigonometric. From Eqs. (17) and (20) we obtain the solution for the RTE that satisfies the given boundary and initial conditions:

$$\begin{aligned} u(z, \zeta) = & e^{\frac{\mu-1}{2}\zeta} \sum_{n=1}^{\infty} \left[\psi_{0,n} \cosh\left(\frac{\Lambda_n \zeta}{2}\right) \right. \\ & \left. + \frac{2\psi'_{0,n} + (1-\mu)\psi_{0,n}}{\Lambda_n} \sinh\left(\frac{\Lambda_n \zeta}{2}\right) \right] \sin(n\pi z). \end{aligned} \quad (23)$$

A closer look at Eq. (23) immediately reveals that the RTE considered on a bounded domain possesses some unrealistic properties. Let us consider a trivial case where the initial population density is zero everywhere, i.e. $u_0 \equiv 0$. Correspondingly, $\psi_{0,n} = 0$ for all $n = 1, 2, \dots$. Since the biological meaning of the second initial condition is unclear, function $v_0(z)$ can be positive as well as negative. Let us consider a special case $v_0(z) = C \sin(\pi z)$ where $C \neq 0$ is a constant, then $\psi'_{0,1} = C$ and $\psi'_{0,n} = 0$ for $n \geq 2$. Solution (23) then takes the following form:

$$u(z, \zeta) = \left(\frac{2C}{\Lambda_1} \right) \sin(\pi z) e^{\frac{\mu-1}{2}\zeta} \sinh\left(\frac{\Lambda_1 \zeta}{2}\right). \quad (24)$$

Since $\sinh(\cdot)$ is a monotonously growing function, $u(z, \zeta)$ given by Eq. (24) will become non-zero at any $t > 0$: the population apparently appears from nowhere! Moreover, if $C < 0$ then the population density will become negative, which is similarly a pure nonsense. Only in the special case of $v_0(z) \equiv 0$ the solution (24) will remain zero at any $t > 0$. However, as we will show below, in a general case the condition $v_0(z) \equiv 0$ alone does not yet guarantee that the solution will have sensible properties.

In a more general case of nontrivial $u_0(z) \geq 0$, the solution can still become negative; in fact, even grow towards $-\infty$. Consider the case of $\mu > 1$. Then, because of the pre-series exponential factor in Eq. (23), every term in the series exhibits an overall growth tendency. If we choose $\beta < \beta_1^*$, the first term contains only hyperbolic functions and hence is not oscillating, but even this special case poses problems. For instance, if we consider a initial condition containing only the first mode, i.e. $u_0(z) = \psi_{0,1} \sin(\pi z)$, then in the large- ζ limit Eq. (23) becomes

$$u(z, \zeta) \simeq \sin(\pi z) \psi_{0,1} e^{\frac{\mu-1+\Lambda_1}{2}\zeta} \left[1 + \frac{2\chi_1 + 1 - \mu}{\sqrt{(1+\mu)^2 - 4\pi^2\beta^2}} \right], \quad (25)$$

where $\chi_1 = \psi'_{0,1}/\psi_{0,1}$. Since $\psi_{0,1} > 0$ due to its meaning, the positivity of this solution is only guaranteed if the term inside brackets is positive. Therefore, for the initial conditions that satisfy $\chi_1 > (1 - \mu)/2$, any value of $\beta < \beta_1^*$ guarantees that $u(z, \zeta) \rightarrow +\infty$ for $\zeta \rightarrow \infty$ for any $z \in (0, 1)$, but if $-1 < \chi_1 < (1 - \mu)/2$, this asymptotic positivity is only guaranteed if

$$\beta < \frac{\sqrt{(1 + \chi_1)(\mu - \chi_1)}}{2\pi}. \quad (26)$$

Even more drastically, in the case of the initial conditions satisfying $\chi_1 < -1$, $\tilde{u}(z, \zeta) \rightarrow -\infty$ in the large- ζ limit regardless of the value of β .

Since the reaction-telegraph equation (9) is obtained from the Cattaneo system (8) after some additional manipulations [20, 30], a solution of the RTE is not necessarily a solution of the Cattaneo system. Correspondingly, now we are going to check whether the general solution (23) is consistent with the Cattaneo system in the case of the boundary conditions (10).

3.1 Consistency of the solution with the Cattaneo system

In order to check the consistency, we must assume that equation (23) correspond to the solution for the density and use the system of equations (8) to obtain either a solution for the flux function $j(z, \zeta)$ or a consistency condition that must be satisfied. Note that we do not have explicit boundary conditions for the flux.

If we first consider equation (8b), then we obtain a general solution for the flux function that may be written as

$$j(z, \zeta) = g_{0,0} e^{-\zeta} + e^{\frac{\mu-1}{2}\zeta} \sum_{n=1}^{\infty} \frac{\cos(n\pi z)}{\Lambda_n^2 - (1 + \mu)^2} g_n(\zeta), \quad (27)$$

where the temporal functions $g_n(\zeta)$ are defined as

$$\begin{aligned} g_n(\zeta) = & e^{-\frac{1+\mu}{2}\zeta} \{ g_{0,n} [\Lambda_n^2 - (1 + \mu)^2] + 4n\pi\beta (\psi'_{0,n} - \mu\psi_{0,n}) \} \\ & + 4n\pi\beta (\mu\psi_{0,n} - \mu\psi'_{0,n}) \cosh\left(\frac{\Lambda_n\zeta}{2}\right) \\ & + \frac{2n\pi\beta [2(1 + \mu)\psi'_{0,n} + (1 - \mu^2 - \Lambda_n^2)\psi_{0,n}]}{\Lambda_n} \sinh\left(\frac{\Lambda_n\zeta}{2}\right). \end{aligned} \quad (28)$$

The first consistency condition arises because of the term proportional to $e^{-\frac{1+\mu}{2}\zeta}$ (the first term in the right-hand side of Eq. (28)) as such a term does not appear in the density solution (23), and thus cannot be obtained from the continuity equation (8a). If we impose

$$g_{0,n} = 4n\pi\beta \frac{\mu\psi_{0,n} - \psi'_{0,n}}{\Lambda_n^2 - (1+\mu)^2}, \quad (29)$$

then the exponential term disappears, and we also avoid the inconsistency of having 3 initial conditions - $g_{0,n}$, $\psi_{0,n}$ and $\psi'_{0,n}$ - for a second order system.

The second consistency condition appears when we substitute both solutions (23) and (27) back on the continuity equation, which is expressed as

$$\frac{(2n\pi\beta)^2}{(1+\mu)^2 - |(1+\mu)^2 - (2n\pi\beta)^2|} = 1, \quad n = 1, 2, \dots \quad (30)$$

Equation (30) is actually an infinite set of conditions, as it must be valid for every integer n , and the only value of β capable of satisfying all of them is $\beta = 0$, which is achieved in the limit ($L \rightarrow \infty$), i.e. when the domain size becomes infinite. This result explains why the RTE is normally well-posed if considered in an infinite space. In a finite domain, the only way for a solution of the RTE to satisfy the Cattaneo system is for a specific initial condition that contains only a finite number of modes in its Fourier series expansion, and only for specific ranges of the parameter β . For instance, if we assume initial conditions defined in terms of only the first mode, i.e.,

$$u(z, 0) = u_0 \sin(\pi z), \quad \partial_t u(z, 0) = v_0 \sin(\pi z), \quad (31)$$

then it will only be a solution for the system if $\beta < \beta_1^*$. If the initial conditions are expanded into the first N modes of the series expansion, i.e.,

$$u(z, 0) = \sum_{n=1}^N u_n \sin(n\pi z), \quad \partial_t u(z, 0) = \sum_{n=1}^N v_n \sin(n\pi z), \quad (32)$$

then it requires $\beta < \beta_N^*$ for the solution to satisfy the Cattaneo system. Note that even in this case it does not guarantee the positivity of the solution for all $z \in (0, 1)$ and $\zeta > 0$.

3.2 General solution for the Cattaneo system

As a possible source of the inconsistencies could be that we are obtaining a solution for the RTE and trying to make it describe the solutions of the Cattaneo system, we could look for the solutions of the latter one from the beginning. Just as before, the solution may be written as the following series expansion:

$$u(z, \zeta) = \sum_{n=1}^{\infty} \sin(n\pi z) \psi_n(\zeta), \quad (33a)$$

$$j(z, \zeta) = \phi_{0,0} e^{-\zeta} + \sum_{n=1}^{\infty} \cos(n\pi z) \phi_n(\zeta), \quad (33b)$$

which satisfy both the required boundary conditions and spatial part of the differential equation, while the temporal part is described by the system of equations

$$\psi'_n(\zeta) = \mu\psi_n(\zeta) + n\pi\beta\phi_n(\zeta), \quad \phi'_n(\zeta) = -\phi_n(\zeta) - n\pi\beta\psi_n(\zeta). \quad (34)$$

For general initial conditions $\psi_{0,n}$ and $\phi_{0,n}$, the solution therefore can be written as

$$\begin{aligned} \psi_n(\zeta) = & e^{\frac{\mu-1}{2}\zeta} \psi_{0,n} \cosh\left(\frac{\Lambda_n\zeta}{2}\right) \\ & + e^{\frac{\mu-1}{2}\zeta} \frac{(1+\mu)\psi_{0,n} + 2n\pi\beta\phi_{0,n}}{\Lambda_n} \sinh\left(\frac{\Lambda_n\zeta}{2}\right), \end{aligned} \quad (35a)$$

$$\begin{aligned} \phi_n(\zeta) = & e^{\frac{\mu-1}{2}\zeta} \phi_{0,n} \cosh\left(\frac{\Lambda_n\zeta}{2}\right) \\ & - e^{\frac{\mu-1}{2}\zeta} \frac{(1+\mu)\phi_{0,n} + 2n\pi\beta\psi_{0,n}}{\Lambda_n} \sinh\left(\frac{\Lambda_n\zeta}{2}\right). \end{aligned} \quad (35b)$$

It is readily seen that solutions that asymptotically grow towards infinitely large negative values tend to disappear, essentially because the parameter μ appears with a positive sign on the second term in equation (35a). But we still cannot guarantee that the solutions are positively defined, which is due to the fact that the hyperbolic functions become trigonometric ones whenever Λ_n becomes imaginary. To illustrate a feasible example in which it happens, let us consider a situation where the $j(z, 0) = 0$ (density of particles going right and left at the beginning is exactly the same), and the initial condition for the density is just the first mode of the series expansion, $u(z, 0) = u_0 \sin(\pi z)$. If we set $\beta > \beta_1^*$, then these conditions lead to the density function

$$u(z, \zeta) = u_0 \sin(\pi z) e^{\frac{\mu-1}{2}\zeta} \left[\cos\left(\frac{\Lambda_1^*\zeta}{2}\right) + \frac{1+\mu}{\Lambda_1^*} \sin\left(\frac{\Lambda_1^*\zeta}{2}\right) \right], \quad (36)$$

where $\Lambda_1^* = \sqrt{4\pi^2\beta^2 - (1+\mu)^2} > 0$. It is straightforward to see that, if we take ζ at any of the points $\zeta_m = 2\pi(2m+1)/\Lambda_1^*$, with $m = 0, 1, 2, \dots$, the density becomes negative, as

$$u(z, \zeta_m) = -u_0 \sin(\pi z) e^{\frac{\mu-1}{2}\zeta_m} < 0 \quad (37)$$

for all $z \in (0, 1)$.

4 Robin boundary conditions

In the previous section, we have shown that, when the reaction-telegraph equation or the corresponding Cattaneo system are considered in a bounded domain, the intuitive Dirichlet-type boundary conditions (implying zero population density at the domain's boundaries) typically result in a solution that attain negative values. This solution's property may be acceptable when the equations are considered in the context of other natural sciences or engineering, e.g. see [33, 38, 41, 39, 40], but is meaningless in the context of population dynamics.

A possible origin of these negative solutions may lie on the choice of the boundary conditions, as in principle they may not correspond to the physical constraint we are trying to impose on the system. When we were considering the Dirichlet boundary conditions, our objective was to describe a system where the solutions are null everywhere outside the domain, but this condition is microscopically equivalent to either the case where the particles are killed at the boundaries (just as an absorbing barrier), or to the case where the particles going outside the domain are not allowed to return. The difference between this two cases lies specifically on the boundary conditions, as the first one requires the density to be zero at the borders, while the second one does not. These boundary conditions can be made to correspond to the Dirichlet type when one is looking at the equations for $u_{\pm}(z, \zeta)$, but on the density-flux representation they are implemented via Robin boundary conditions.

To start, let us apply the method of separation of variables to write a solution on the form

$$u(z, \zeta) = U(z) \Psi(\zeta), \quad j(z, \zeta) = J(z) \Phi(\zeta). \quad (38)$$

If we insert it back on the Cattaneo system, after some manipulations we may write

$$\left[\frac{\Psi'(\zeta)}{\Psi(\zeta)} - \mu \right] \frac{\Psi(\zeta)}{\Phi(\zeta)} = -\beta \frac{J'(z)}{U(z)} \equiv \kappa_A^2, \quad (39a)$$

$$\left[\frac{\Phi'(\zeta)}{\Phi(\zeta)} + 1 \right] \frac{\Phi(\zeta)}{\Psi(\zeta)} = -\beta \frac{U'(z)}{J(z)} \equiv -\kappa_B^2, \quad (39b)$$

where κ_A and κ_B are constants that should be determined in order to make the equations consistent. A general solution of equations (39) may be written as

$$U(z) = C_1 \cos\left(\frac{\kappa_A \kappa_B}{\beta} z\right) + C_2 \frac{\kappa_B}{\kappa_A} \sin\left(\frac{\kappa_A \kappa_B}{\beta} z\right), \quad (40a)$$

$$J(z) = C_2 \cos\left(\frac{\kappa_A \kappa_B}{\beta} z\right) - C_1 \frac{\kappa_A}{\kappa_B} \sin\left(\frac{\kappa_A \kappa_B}{\beta} z\right), \quad (40b)$$

for the spatial part, and for the temporal part,

$$\Psi(\zeta) = e^{\frac{\mu-1}{2}\zeta} \left[C_3 \cosh\left(\frac{\Delta}{2}\zeta\right) + \frac{(1+\mu)C_3 + 2\kappa_A^2 C_4}{\Delta} \sinh\left(\frac{\Delta}{2}\zeta\right) \right], \quad (41a)$$

$$\Phi(\zeta) = e^{\frac{\mu-1}{2}\zeta} \left[C_4 \cosh\left(\frac{\Delta}{2}\zeta\right) - \frac{(1+\mu)C_4 + 2\kappa_B^2 C_3}{\Delta} \sinh\left(\frac{\Delta}{2}\zeta\right) \right], \quad (41b)$$

where we have defined $\Delta = \sqrt{(1+\mu)^2 - (2\kappa_A \kappa_B)^2}$. Note that the system have six parameters that must be determined by the proper initial values and boundary conditions.

In order to implement the boundary conditions, we are only going to consider the physical situation where the particles are not allowed to come back once they have left the domain³. This physical constraint is satisfied if there is no incoming density at the borders,

³The counterpart situation where the particles are destroyed at the boundaries is described by the condition of no outgoing density at the borders, which is satisfied if $u_-(0, \zeta) = u_+(1, \zeta) = 0$. The Robin set of conditions $u(0, \zeta) = j(0, \zeta)$ and $u(1, \zeta) = -j(1, \zeta)$ is only marginally different from the one used in the text, thus generating similar results.

which means $u_+(0, \zeta) = 0$ and $u_-(1, \zeta) = 0$. A Robin set of conditions $u(0, \zeta) = -j(0, \zeta)$ and $u(1, \zeta) = j(1, \zeta)$ guarantee the two constraints are satisfied, which lead to a set of equations for the free parameters that may be written as

$$C_1 C_3 = -C_2 C_4, \quad (42a)$$

$$(C_2 C_4 - C_1 C_3)(1 + \mu) = 2\kappa_A \kappa_B \left(\frac{C_1 C_4 \kappa_A}{\kappa_B} - \frac{C_2 C_3 \kappa_B}{\kappa_A} \right), \quad (42b)$$

$$(C_2 C_4 - C_1 C_3) \cos \left(\frac{\kappa_A \kappa_B}{\beta} \right) = \left(\frac{C_1 C_4 \kappa_A}{\kappa_B} + \frac{C_2 C_3 \kappa_B}{\kappa_A} \right) \sin \left(\frac{\kappa_A \kappa_B}{\beta} \right), \quad (42c)$$

$$\begin{aligned} & \left[(1 + \mu) \left(\frac{C_1 C_4 \kappa_A}{\kappa_B} - \frac{C_2 C_3 \kappa_B}{\kappa_A} \right) + 2(C_1 C_3 - C_2 C_4) \kappa_A \kappa_B \right] \sin \left(\frac{\kappa_A \kappa_B}{\beta} \right) = \\ & \left[(C_1 C_3 + C_2 C_4)(1 + \mu) + 2\kappa_A \kappa_B \left(\frac{C_1 C_4 \kappa_A}{\kappa_B} + \frac{C_2 C_3 \kappa_B}{\kappa_A} \right) \right] \cos \left(\frac{\kappa_A \kappa_B}{\beta} \right). \end{aligned} \quad (42d)$$

After substituting (42a) on the rest of the equations (42), we may determine the constants

$$C_1 = C_2 \frac{\kappa_B}{\kappa_A} \left[\cot \left(\frac{\kappa_A \kappa_B}{\beta} \right) + \frac{1 + \mu}{2\kappa_A \kappa_B} \right], \quad (43a)$$

$$C_3 = C_4 \frac{\kappa_A}{\kappa_B} \left[\cot \left(\frac{\kappa_A \kappa_B}{\beta} \right) - \frac{1 + \mu}{2\kappa_A \kappa_B} \right], \quad (43b)$$

and obtain a transcendental equation that defines the multiplicity of solutions:

$$\tan^2 \left(\frac{\kappa_A \kappa_B}{\beta} \right) = \frac{4\kappa_A^2 \kappa_B^2}{(1 + \mu)^2 - 4\kappa_A^2 \kappa_B^2}. \quad (44)$$

But here we are faced with a problem. In order for the spatial solutions (40) to be written as trigonometric functions, both parameters κ_A and κ_B must be real. But this restriction lead to only a finite number of possible values for $\kappa_A \kappa_B$, as any solution for equation (44) requires the constraint $4\kappa_A^2 \kappa_B^2 < (1 + \mu)^2$ to be satisfied. Consequently it is not possible to determine a basis for expanding a general initial condition. Nevertheless, we may still look at the solutions we are able to obtain to address the question of whether the density is positively defined or not.

If we recall the temporal part of the solutions in equation (41), it is easy to conclude that presence of hyperbolic functions in the time evolution may account for a quite complicated positivity analysis, specially because they may become trigonometric functions for some range of the parameters. But we still have one degree of freedom left to work with, since the implementation of the boundary conditions only determine the values of $\kappa_A \kappa_B$ and not each one individually. A suitable way to eliminate the unused degree of freedom is to set the condition⁴

$$\kappa_B^2 = 1 + \mu - \kappa_A^2, \quad (45)$$

⁴The appearance of equation (45) can also be understood as a result of the boundary condition at $z = 0$ being implemented for the spatial and temporal parts independently: if we impose that $U(0) = J(0)$, we obtain $C_1 = C_2$ and $C_4 = -C_3$; now the temporal parts must satisfy $\Psi(\zeta) = -\Phi(\zeta)$, which generates equation (45).

which requires $C_1 = C_2$ and $C_4 = -C_3$ in order for it to be consistent with equations (43) and (44). If we define $C = C_1 C_3 > 0$ for the multiplying constant, then with the aid of the rescalings

$$\bar{\kappa} = \frac{\kappa_A}{\sqrt{1+\mu}}, \quad \bar{\beta} = \frac{\beta}{1+\mu}, \quad (46)$$

we are able to write a more suitable representation for the density and the flux, i.e.,

$$u_n(z, \zeta) = C e^{(1+\mu)(1-2\bar{\kappa}_n^2)\zeta} \left[\frac{g_n}{\bar{\kappa}_n} \sin\left(\frac{\bar{\kappa}_n g_n}{\bar{\beta}} z\right) + \cos\left(\frac{\bar{\kappa}_n g_n}{\bar{\beta}} z\right) \right], \quad (47a)$$

$$j_n(z, \zeta) = C e^{(1+\mu)(1-2\bar{\kappa}_n^2)\zeta} \left[\frac{\bar{\kappa}_n}{g_n} \sin\left(\frac{\bar{\kappa}_n g_n}{\bar{\beta}} z\right) - \cos\left(\frac{\bar{\kappa}_n g_n}{\bar{\beta}} z\right) \right], \quad (47b)$$

where we have defined $g_n = \sqrt{1 - \bar{\kappa}_n^2}$ and $\bar{\kappa}_n$ is the n^{th} root of the modified transcendental equation

$$\tan\left(\frac{\bar{\kappa}\sqrt{1-\bar{\kappa}^2}}{\bar{\beta}}\right) = \frac{2\bar{\kappa}\sqrt{1-\bar{\kappa}^2}}{2\bar{\kappa}^2 - 1}, \quad (48)$$

which may only be defined if the solution exists and satisfies the constraint $0 < \bar{\kappa}_n < 1$. The exponential term in equation (47a) guarantees that the density at each spatial point either increases (in absolute value) or decreases towards zero, so the positivity of the solution is guaranteed if the spatial part

$$U_n(z) = \frac{g_n}{\bar{\kappa}_n} \sin\left(\frac{\bar{\kappa}_n g_n}{\bar{\beta}} z\right) + \cos\left(\frac{\bar{\kappa}_n g_n}{\bar{\beta}} z\right) \quad (49)$$

is non-negative at each point, which means that any solution represented by equation (47a) whose initial condition is non-negative, remains non-negative for all $\zeta > 0$.

The numerical evaluation of the roots of equation (48) shows that the solutions of the reaction-Cattaneo system in the form of equations (47) only exist for $\bar{\beta} \leq 0.5$ where the first root is born, as is shown in Fig. 1 (left figure). As $\bar{\beta}$ is decreased other roots are generated in pairs at branching points, where we have labeled the roots with "+" for those that satisfy $\bar{\kappa}_{n+} \rightarrow 1$ as $\bar{\beta} \rightarrow 0$, and with "-" for those that satisfy $\bar{\kappa}_{n-} \rightarrow 0$ in the same limit. The solutions corresponding to the first few roots of Eq. (48) are illustrated in Fig. 1 (right figure) and their properties indicate that at least the first solution $U_1(z)$ is positively defined. However, in order to show it more rigorously, we must first determine some general properties of solutions.

Mathematical analysis shows that $U_n(1-z) = \pm U_n(z)$, which means that the solutions are either symmetric or anti-symmetric with respect to $z = 1/2$, thus satisfying $U_n(0) = 1$ and $U_n(1) = \pm U_n(0)$. For small values of $\bar{\beta}$ all roots $\bar{\kappa}_{n-}$ tend to zero proportional to $\bar{\beta}$, and from a series expansion of equation (48) it is possible to show that

$$\bar{\kappa}_{n-} = n\pi\bar{\beta} + O(\bar{\beta}^2), \quad n = 1, 2, \dots, \quad (50)$$

where $\bar{\kappa}_{1-} = \bar{\kappa}_1$.

Figure 1: (Color online) Left figure: first seven roots of the transcendental equation (48); the first root $\bar{\kappa}_1$ appears at $\bar{\beta} = 0.5$, while the other roots $\bar{\kappa}_n$ for $n > 1$ are born in pairs at the branching points $\bar{\beta} \approx 0.109$ ($\bar{\kappa}_2$), $\bar{\beta} \approx 0.064$ ($\bar{\kappa}_3$) and $\bar{\beta} \approx 0.046$ ($\bar{\kappa}_4$); the roots satisfy $\bar{\kappa}_{n+} \rightarrow 1$ and $\bar{\kappa}_{n-} \rightarrow 0$ in the limit of $\bar{\beta} \rightarrow 0$. Right figure: first five analytical solutions for the spatial part $U_n(z)$ in equation (49) with $\beta = 0.05$; the curves represent $\bar{\kappa}_1 \approx 0.144$ (black curve), $\bar{\kappa}_{2-} \approx 0.297$ (blue curve), $\bar{\kappa}_{2+} \approx 0.984$ (orange curve), $\bar{\kappa}_{3-} \approx 0.48$ (red curve) and $\bar{\kappa}_{3+} \approx 0.924$ (green curve).

4.1 Sufficient condition of the solution positivity

For the reasons that will become clear later, now we focus on the solution $U_1(z)$. For convenience, we use the following notations:

$$A(\bar{\beta}) = \frac{g_1}{\bar{\kappa}_1}, \quad \omega(\bar{\beta}) = \frac{\bar{\kappa}_1 g_1}{\bar{\beta}}, \quad (51)$$

so that

$$U_1(z) = A(\bar{\beta}) \sin[\omega(\bar{\beta})z] + \cos[\omega(\bar{\beta})z]. \quad (52)$$

Here the auxiliary parameters A and ω are the amplitude coefficient of the sine term and the frequency coefficient, respectively. The numerical evaluation of these coefficients are shown in Fig. 2. It follows from equation (50) that $0 < \omega < \pi$, which implies that $A \sin(\omega z) \geq 0$ and has at most one maximum value inside the domain of z . In order to show that $U_1(z) > 0$, it will be required an analysis in three different domains of the parameter $\bar{\beta}$, delimited by the points $\bar{\beta}_+ = 1/\pi$ and $\bar{\beta}_- = \sqrt{2}/(3\pi)$, where

$$\begin{aligned} A_+ &= A(\bar{\beta}_+) = 1, & \omega_+ &= \omega(\bar{\beta}_+) = \pi/2, \\ A_- &= A(\bar{\beta}_-) = \sqrt{3 + 2\sqrt{2}}, & \omega_- &= \omega(\bar{\beta}_-) = 3\pi/4. \end{aligned}$$

For $\bar{\beta} > \bar{\beta}_+$ the cosine term is always positive because $\omega < \pi/2$, so it follows immediately that $U_1(z) > 0$. In the second region where $\bar{\beta}_- < \bar{\beta} < \bar{\beta}_+$, the fact that $\pi/2 < \omega < 3\pi/4$ implies that $\sin(\omega z) + \cos(\omega z) \geq 0$, thus $A \sin(\omega z) + \cos(\omega z) > 0$ because $A > 1$. For last, whenever $0 < \bar{\beta} < \bar{\beta}_-$ we have $3\pi/4 < \omega < \pi$, which means that the minimum value of the function $\sin(\omega z) + \cos(\omega z)$ occurs at $z = 1$. Since $A > 1$, the minimum value of

Figure 2: The amplitude A and the frequency ω (see Eqs. (51–52)) shown as functions of $\bar{\beta}$.

$U_1(z)$ will either be at $z = 0$, where $U_1(0) = 1$, or at $z = 1$, but as

$$U_1(1) = \sin(\omega) \left[A + \frac{2\bar{\kappa}_1^2 - 1}{2\bar{\kappa}_1 g_1} \right] = \sin(\omega) \frac{A^2 + 1}{2A} > 0,$$

then the minimum value of $U_1(z)$ is positive. Hence we are able to conclude that $U_1(z) > 0$ for all $z \in [0, 1]$ and consequently $u_1(z, \zeta)$ is positively defined.

With the above result in mind, we are now able to identify the sufficient condition that ensures solution nonnegativity. Specifically, we are going to show that any initial condition that satisfies a certain criteria will be positively defined for all $\zeta > 0$. As a starting point, we have to consider $\bar{\beta} < 0.5$ so that at least one solution exists. Let $N(\bar{\beta})$ be the number of existing solutions in equation (47). Considering a linear combination of partial solutions (47a) and redefining the constants, we arrive at the following solution of the reaction-Cattaneo system:

$$\begin{aligned} \tilde{u}(z, \zeta) &= a_1 u_1(z, \zeta) + \sum_{n=2\pm}^{N(\bar{\beta})} a_{n\pm} u_{n\pm}(z, \zeta) \\ &= e^{(1+\mu)\zeta} \left[a_1 e^{-2(1+\mu)2\bar{\kappa}_1^2\zeta} U_1(z) + \sum_{n=2\pm}^{N(\bar{\beta})} a_{n\pm} e^{-2(1+\mu)2\bar{\kappa}_{n\pm}^2\zeta} U_{n\pm}(z) \right], \end{aligned} \quad (53)$$

which obviously corresponds to the following initial condition:

$$\tilde{u}(z, 0) = a_1 U_1(z) + \sum_{n=2\pm}^{N(\bar{\beta})} a_{n\pm} U_{n\pm}(z). \quad (54)$$

For the initial condition to be meaningful, it must be nonnegative:

$$\tilde{u}(z, 0) \geq 0, \quad (55)$$

Figure 3: Density $u(z, \zeta)$ obtained as a solution of the reaction-Cattaneo system (8) with Robin-type boundary conditions ($u(0, \zeta) = -j(0, \zeta)$ and $u(1, \zeta) = j(1, \zeta)$) shown at different moments (for the curves in the middle of the domain, top to bottom, $\zeta = 0, 0.2, 0.3, 0.4, 0.5, 0.7, 1, 2$). Parameters are $\mu = 0.1$ and $\beta = 0.9$, and the initial conditions are $u(z, 0) = 2 \sin(\pi z)$ and $j(z, 0) = 0$.

for all $z \in [0, 1]$. Since we have proved that $U_1(z)$ is nonnegative, a sufficient condition of the nonnegativity of the initial condition is given by

$$a_1 U_1(z) \geq \sum_{n=2\pm}^{N(\bar{\beta})} a_{n\pm} U_{n\pm}(z). \quad (56)$$

Now we take into account that the first eigenvalue κ_1 is the smallest one (see Fig. 1), i.e. $\bar{\kappa}_n > \bar{\kappa}_1$ whenever $n > 1$. Therefore, the terms with $n \geq 2$ inside brackets of solution (53) decay with time faster than the first term. Therefore, if condition (56) holds, then the first term in (53) will remain larger than the rest of the series for any $\zeta > 0$ and hence solution (53) will be nonnegative for any $\zeta > 0$.

As an example, Fig. 3 shows the solution of the reaction-Cattaneo system obtained for the Robin boundary conditions. Apparently, the solution remains positive at all time. Note that the same condition used for the reaction-telegraph equation results in a solution that is not nonnegative.

Thus, we have proved the solution's nonnegativity for a class of the initial conditions that contains only a finite number of terms in the Fourier series, i.e. in case high-frequency terms are absent. This implies that the initial condition must be described by a sufficiently smooth function, because a good approximation of a non-smooth or discontinuous function by the Fourier series in the vicinity of a jump requires an infinite number of terms. For an initial condition of the reaction-Cattaneo system given by a discontinuous function, the question of its positivity, strictly speaking, remains open. Interestingly, having solved the reaction-Cattaneo system with the Robin boundary conditions numerically, we observe that the solution is positive even for a discontinuous initial condition; see Fig. 4. That indicates that condition (56) is sufficient but not necessary, and a nonnegative initial

Figure 4: (Color online) Numerical solution for the density $u(z, \zeta)$ obtained from the reaction-Cattaneo system (8) with the Robin-type boundary conditions, $u(0, \zeta) = -j(0, \zeta)$ and $u(1, \zeta) = j(1, \zeta)$, and the piecewise-constant initial conditions, $u(z, 0) = H(z - 1/4) - H(z - 3/4)$ and $j(z, 0) = 0$, where $H(\cdot)$ is the Heaviside step-function. Left figure: system's dynamics at early stages; the discontinuous initial condition (dashed curve) generates two sets of traveling fronts propagating away from the position of the discontinuity ($\zeta = 0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$). Right figure: system's dynamics at later stages after the traveling fronts propagating towards each other collide in the middle of the domain (at $z = 1/2$, from top to bottom, $\zeta = 0.4, 0.6, 0.8, 1, 1.2, 1.5, 2.5$).

condition is likely to remain nonnegative. An analytical proof of this statement however remains a challenge.

5 Zero flux boundary condition

For last, let us consider the physical condition in which the total density is constrained to be inside the domain for all times, which should be implemented as a zero flux boundary condition $j(0, \zeta) = j(1, \zeta) = 0$. If we look at equation (8b), a no-flux condition implies that the spatial derivative of the density at the boundaries should be zero, which is just a Neumann boundary condition if we consider the reaction-telegraph equation (9). From the separation of variables method described in section 3, we can write the spatial solution that satisfies the boundary conditions as

$$U_n(z) = \cos(n\pi z), \quad n = 0, 1, \dots, \quad (57)$$

where the constants κ_n are just the same as in equation (15). Since the differential equation for the temporal part is just the same as in equation (18), if the initial conditions are defined as

$$\psi_{0,n} = 2 \int_0^1 dz u_0(z) \cos(n\pi z), \quad \psi'_{0,n} = 2 \int_0^1 dz v_0(z) \cos(n\pi z), \quad (58)$$

the general solution of the system may be written as

$$\begin{aligned}
u(z, \zeta) = & \frac{\psi_{0,0} + \psi'_{0,0}}{1 + \mu} e^{\mu\zeta} + \frac{\mu\psi_{0,0} - \psi'_{0,0}}{1 + \mu} e^{-\zeta} \\
& + e^{\frac{\mu-1}{2}\zeta} \sum_{n=1}^{\infty} \psi_{0,n} \cos(n\pi z) \cosh\left(\frac{\Lambda_n \zeta}{2}\right) \\
& + e^{\frac{\mu-1}{2}\zeta} \sum_{n=1}^{\infty} \frac{2\psi'_{0,n} + (1 - \mu)\psi_{0,n}}{\Lambda_n} \cos(n\pi z) \sinh\left(\frac{\Lambda_n \zeta}{2}\right), \tag{59}
\end{aligned}$$

where Λ_n was defined in equation (21). In order for this solution to be physically consistent, we must compare it to the solution of the Cattaneo system, which may be obtained by the series expansions

$$u_c(z, \zeta) = \psi_{0,0} e^{\mu\zeta} + \sum_{n=1}^{\infty} \cos(n\pi z) \psi_n(\zeta), \tag{60a}$$

$$j_c(z, \zeta) = \sum_{n=1}^{\infty} \sin(n\pi z) \phi_n(\zeta), \tag{60b}$$

where the temporal parts satisfy the coupled equations

$$\psi'_n(\zeta) = \mu\psi_n(\zeta) - n\pi\beta\phi_n(\zeta), \quad \phi'_n(\zeta) = n\pi\beta\psi_n(\zeta) - \phi_n(\zeta). \tag{61}$$

When we write the general solution for the temporal part in the form of

$$\begin{aligned}
\psi_n(\zeta) = & e^{\frac{\mu-1}{2}\zeta} \psi_{0,n} \cosh\left(\frac{\Lambda_n \zeta}{2}\right) \\
& + e^{\frac{\mu-1}{2}\zeta} \frac{(1 + \mu)\psi_{0,n} - 2n\pi\beta\phi_{0,n}}{\Lambda_n} \sinh\left(\frac{\Lambda_n \zeta}{2}\right), \tag{62a}
\end{aligned}$$

$$\begin{aligned}
\phi_n(\zeta) = & e^{\frac{\mu-1}{2}\zeta} \phi_{0,n} \cosh\left(\frac{\Lambda_n \zeta}{2}\right) \\
& - e^{\frac{\mu-1}{2}\zeta} \frac{(1 + \mu)\phi_{0,n} - 2n\pi\beta\psi_{0,n}}{\Lambda_n} \sinh\left(\frac{\Lambda_n \zeta}{2}\right), \tag{62b}
\end{aligned}$$

where the initial conditions $\psi_{0,n}$ must be the same as in equation (58), we obtain that $u(z, \zeta) = u_c(z, \zeta)$ if the initial conditions for the time derivative $\psi'_{0,n}$ satisfy

$$\psi'_{0,n} = \mu\psi_{0,n} - n\pi\beta\phi_{0,n}. \tag{63}$$

This condition implies a solution of the reaction-telegraph equation will be consistent for general initial conditions (11) as long as the constraint $\psi'_{0,0} = \mu\psi_{0,0}$ is satisfied. If this relation does not hold, we may change the initial condition for the time derivative by adding a constant,

$$v_0(z) \rightarrow v_0(z) + \frac{\mu}{2}\psi_{0,0} - \int_0^1 dz v_0(z), \tag{64}$$

which guarantees the consistency of the solution.

In terms of the positivity of the solutions, the exponential term $\psi_{0,0}e^{\mu\zeta}$ in Eq. (60a) guarantees that any oscillation of the higher order terms do not drive the solution towards negative values, so that the reaction-telegraph equation thus present positively defined solutions for the case of no flux boundary conditions.

Interestingly, the same conclusion about the positivity of the solution in the case of the zero-flux boundary conditions was made in a recent paper by Malakar et al. [24] for a slightly different model (which included both the run-and-tumble process and usual Brownian diffusion but does not include the growth term). Arguably, it suggests the generality of this result.

6 Discussion and conclusions

The main purpose of this study is to reveal the reasons why the solutions of the reaction-telegraph equation are not positively defined [29], especially when considered on a bounded domain, and whether the positivity can somehow be imposed or restored. Note that the non-positivity of those solutions is a puzzling fact and can probably be regarded as a paradox. Indeed, the reaction-telegraph equation is obtained by some apparently identical mathematical transformations from the reaction-Cattaneo system where the latter arise as a result of counting animals moving in different directions and hence is positively defined by definition. The non-positivity is therefore a result of the problem becoming ill-posed, but it remains unclear where the ill-posedness comes from.

Having performed an analytical study of both the reaction-telegraph equation and the reaction-Cattaneo system with a linear growth, we arrive at the following results:

- We have shown that, generally speaking, the reaction-telegraph equation is not equivalent to the reaction-Cattaneo system. A solution of the reaction-telegraph equation is only a solution of the reaction-Cattaneo system when certain consistency conditions hold. That leads to a conclusion that a well-defined model is the reaction-Cattaneo system, not the reaction-telegraph equation.
- The simplest, ‘intuitive’ Dirichlet-type boundary conditions that are routinely used for the diffusion and reaction-diffusion equations in ecological applications, e.g. $u(0, t) = u(1, t) = 0$, are meaningless and makes the problem ill-posed when used for the reaction-telegraph equation or reaction-Cattaneo system, hence resulting in solutions non-positivity. For the boundary conditions to describe a physically sensible situation and preserve positivity, they need to be either of Neumann type or Robin type; the problem then becomes well-posed.

Note that in this paper we have only considered a linear problem. Analytical investigation of a corresponding nonlinear problem where the population growth is a nonlinear function of the population density (e.g. the logistic growth of the growth with the Allee effect) is mathematically much more challenging. Meanwhile, the properties of the nonlinear problem may differ significantly from the linear one, both for the reaction-Cattaneo

system and for the reaction-telegraph equation, in particular enhancing solutions positivity. Intuitively, one can expect that the existence of a stable steady state of the nonspatial system (as given by the nonspatial version of Eq. (8a)) should enhance the existence of a stable nonnegative stationary solution of system (8) or equation (9) subject to Dirichlet conditions (10). In its turn, such attracting nonnegative stationary solution can make the emergence of non-positive solutions less likely. The enhanced nonnegativity of solutions of the nonlinear reaction-telegraph equation was indeed demonstrated numerically in other studies [49, 50].

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References

- [1] Hastings A, An ecological theory journal at last, *Theoretical Ecology*, 1, 1-4 (2008).
- [2] Hastings A, *Population Biology: Concepts and Models*. Springer-Verlag, New York (1997).
- [3] Kot M, *Elements of Mathematical Ecology*. Cambridge University Press, Cambridge UK (2001).
- [4] Mangel M, *The Theoretical Biologist’s Toolbox: Quantitative Methods for Ecology and Evolutionary Biology*. Cambridge University Press, Cambridge UK (2006).
- [5] Smith JM, *Models in Ecology*. Cambridge University Press, Cambridge UK (1974).
- [6] Pyke GH, Understanding movements of organisms: it’s time to abandon the Levy foraging hypothesis, *Methods in Ecology and Evolution*, 6, 1-16 (2015).
- [7] Kareiva PM, Local movement in herbivorous insecta: applying a passive diffusion model to mark-recapture field experiments, *Oecologia*, 57, 322-327 (1983).
- [8] Bearup D, Benefer CM, Petrovskii SV, Blackshaw R, Revisiting Brownian motion as a description of animal movement: a comparison to experimental movement data, *Methods in Ecology and Evolution*, 7, 1525-1537 (2016).
- [9] Hastings A, Cuddington K, Davies KF, Dugaw CJ, Elmendorf S, Freestone A, Harrison S, Holland M, Lambrinos J, Malvadkar U, Melbourne BA, Moore K, Taylor C, Thomson D, The spatial spread of invasions: new developments in theory and evidence, *Ecology Letters*, 8, 91-101 (2005).

- [10] Shigesada N, Kawasaki K, Biological Invasions: Theory and Practice. Oxford University Press, Oxford (1997).
- [11] Skellam JG, Random dispersal in theoretical populations, *Biometrika*, 38, 196-218 (1951).
- [12] Hastings A, Harisson S, McCann K, Unexpected spatial patterns in an insect outbreak match a predator diffusion model, *Proceedings of the Royal Society of London B*, 264, 1837-1840 (1997).
- [13] Malchow H, Petrovskii SV, Venturino E, *Spatiotemporal Patterns in Ecology and Epidemiology: Theory, Models, Simulations*. CRC Press, Boca Raton (2008).
- [14] Meron E, *Nonlinear Physics of Ecosystems*. CRC Press, Boca Raton (2015).
- [15] Kareiva PM, Shigesada N, Analyzing insect movement as a correlated random walk, *Oecologia*, 56, 234-238 (1983).
- [16] Othmer HG, Dunbar SR, Alt W, Models of dispersal in biological systems, *Journal of Mathematical Biology*, 26, 263-298 (1988).
- [17] Goldstein S, On diffusion by discontinuous movements, and on the telegraph equation, *The Quarterly Journal of Mechanics and Applied Mathematics*, 4, 129-156 (1951).
- [18] Kac M, A stochastic model related to the telegraph's equation, *Rocky Mountain Journal of Mathematics*, 4, 497-509 (1956).
- [19] Joseph DD, Preziosi L, Heat waves, *Reviews of Modern Physics*, 61, 41-73 (1989).
- [20] Mendez V, Fedotov S, Horsthemke W, *Reaction-Transport Systems: Mesoscopic Foundations, Fronts, and Spatial Instabilities*. Springer, Berlin (2010).
- [21] Weiss GH, Some applications of persistent random walks and the telegrapher's equation. *Physica A*, 311, 381-410 (2002).
- [22] Masoliver J, Lindenberg K, Continuous time persistent random walk: a review and some generalizations. *European Physical Journal B*, 90, 107 (2017).
- [23] Angelani L, Run-and-tumble particles, telegrapher's equation and absorption problems with partially reflecting boundaries. *Journal of Physics A*, 48, 495003 (2015).
- [24] Malakar K, Jemseena V, Kundu A, Kumar KV, Sabhapandit S, Majumdar SN, Redner S, Dhar A, Steady state, relaxation and first-passage properties of a run-and-tumble particle in one-dimension. *Journal of Statistical Mechanics*, 043215 (2018).
- [25] Evans MR, Majumdar SN, Run and tumble particle under resetting: a renewal approach. *Journal of Physics A*, 51, 475003 (2018).

- [26] Dhar A, Kundu A, Majumdar SN, Sabhapandit S, Schehr G, Run-and-tumble particle in one-dimensional confining potential: steady state, relaxation and first passage properties. *Physical Review E*, 99, 032132 (2019).
- [27] Le doussal P, Majumdar SN, Schehr G, Non-crossing run and tumble particles on a line. *Physical Review E*, 100, 012113 (2019).
- [28] Berg HC, *E. coli in Motion*. Springer, Berlin (2014).
- [29] Haderer KP, Reaction transport systems in biological modelling. In *Mathematics Inspired by Biology*, 95-150. Springer, Berlin (1999).
- [30] Holmes EE, Are diffusion models too simple? A comparison with telegraph models of invasion, *The American Naturalist*, 142, 779-795 (1993).
- [31] Dunbar SR, Othmer HG, On a nonlinear hyperbolic equation describing transmission lines, cell movement, and branching random walks. In *Nonlinear Oscillations in Biology and Chemistry*, 274-289. Springer, Berlin (1986).
- [32] Dunbar SR, A branching random evolution and a nonlinear hyperbolic equation, *SIAM Journal on Applied Mathematics*, 48, 1510-1526 (1988).
- [33] Mainardi F, Signal velocity for transient waves in linear dissipative media, *Wave Motion*, 5, 33-41 (1983).
- [34] Sobolev SL, Transport processes and travelling waves in systems with local nonequilibrium, *Sov. Phys. Usp.*, 34(3), 217-229 (1991).
- [35] Lakestani M, Saray BN, Numerical solution of telegraph equation using interpolating scaling functions, *Comput. Math. Appl.*, 60, 1964-1972 (2010).
- [36] Gilding BH, Kersner R, Wavefront solutions of a nonlinear telegraph equation, *J. Diff. Eqs.*, 254, 599-636 (2013).
- [37] Buono P-L, Eftimie R, Symmetries and pattern formation in hyperbolic versus parabolic models of self-organised aggregation, *J. Math. Biol.*, 71, 847-881 (2015).
- [38] Artale Harris P, Garra R, Nonlinear heat conduction equations with memory: Physical meaning and analytical results, *J. Math. Phys.*, 58, 063501 (2017).
- [39] Di Crescenzo A, Martinucci B, Zacks S, Telegraph process with elastic boundary at the origin, *Methodology and Computing in Applied Probability*, 20, 333-352 (2018).
- [40] Giusti A, Dispersion relations for the time-fractional Cattaneo-Maxwell heat equation, *J. Math. Phys.*, 59, 013506 (2018).
- [41] Sobolev SL, On hyperbolic heat-mass transfer equation, *Int. J. Heat & Mass Transfer*, 122, 629-630 (2018).

- [42] Hajipour, M., Jajarmi, A., Malek, A., Baleanu, D., Positivity-preserving sixth-order implicit finite difference weighted essentially non-oscillatory scheme for the nonlinear heat equation, *Appl. Math. Comput.*, 325, 146-158 (2018).
- [43] Niwa H, Migration dynamics of fish schools in heterothermal environments, *Journal of Theoretical Biology*, 193, 215-231 (1998).
- [44] Murray AG, O’Callaghan M, Jones B, Simple models of massive epidemics of herpesvirus in Australian (and New Zealand) pilchards, *Environment International*, 27, 243-248 (2001).
- [45] Ortega-Cejas V, Fort J, Mendez V, The role of the delay time in the modeling of biological range expansions, *Ecology*, 85, 258-264 (2004).
- [46] Hillen T, Existence theory for correlated random walks on bounded domains, *Canadian Applied Mathematics Quarterly*, 18, 1-40 (2010).
- [47] Hillen T, Swan A, The diffusion limit of transport equations in biology. In *Mathematical models and methods for living systems*, 73-129. Springer, Berlin (2014).
- [48] Garabedian PR, *Partial Differential Equations*. AMS Chelsea Publishing, Providence RI (1998).
- [49] Alharbi WG, Petrovskii SV, Critical domain problem for the reaction-telegraph equation model of population dynamics, *Mathematics*, 6, 59 (2018).
- [50] Cirilo E, Petrovskii SV, Romeiro N, Natti P, Investigation into the critical domain problem for the reaction-telegraph equation using advanced numerical algorithms, *International Journal of Applied and Computational Mathematics*, 5, 54 (2019).