# Quantum-like model of subjective expected utility

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**Abstract**

We present a very general quantum-like model of lottery selection based on representation of beliefs of an agent by pure quantum states. Subjective probabilities are mathematically realized in the framework of quantum probability (QP). Utility functions are borrowed from the classical decision theory. But in the model they are represented not only by their values. Heuristically one can say that each value $u_i = u(x_i)$ is surrounded by a cloud of information related to the event $(A, x_i)$. An agent processes this information by using the rules of quantum information and QP. This process is very complex; it combines counterfactual reasoning for comparison between preferences for different outcomes of lotteries which are in general compelementary. These comparisons induce interference type effects (constructive or destructive). The decision process is mathematically represented by the comparison operator and the outcome of this process is determined by the sign of the value of corresponding quadratic form on the belief state. This operational process can be decomposed into a few subprocesses. Each of them can be formally treated as a comparison of subjective expected utilities and interference factors (the latter express, in particular, risks related to lottery selection). The main aim of this paper is to analyze the mathematical structure of these processes in the most general situation: representation of lotteries by noncommuting operators.

**Keywords:** Lottery selection, subjective expected utility, quantum-like model, belief state, decision operator, interference effects

# 1 Introduction

Recently quantum probability (QP) has begun to be widely used to model the processes of decision making (DM) in cognitive psychology, behavioral economics and finance. [1] From the outset we stress that, in fact, QP is not a calculus of probabilities, but of quantum states, "complex probability amplitudes". Of course, probabilities can be generated from this calculus - via the Born rule, see section 4, formula (8). However, intrinsically the process of DM is represented in terms of the amplitudes and not probabilities.[2]

One may consider the appeal to QP to model DM instead of the usage of the conventional probabilistic measures too exotic. Yet, we recall that as early as the 1970s, Tversky, Kahenman and other researchers in psychology and economics following the seminal paradoxes by Allais (1953), Ellsberg (1961) have been demonstrating cases where classical probability (CP) prescription and actual human thinking persistently diverge, seeking to explain these deviations away from the normative DM frameworks (Kaheneman and Tversky 1972; Tversky and Kahneman 1974, Shafir 1994, Kahneman, 2003; Kahneman and Thaler, 2006).

The main inquiry of the experimental studies was often focused

[1]See, e.g., monographs Khrennikov (2004a, 2010), Busemeyer and Bruza (2012) , Bagarello (2012), Haven and Khrennikov (2013), Asano et al. (2015), reviews Brandenburger (2010), Pothos and Busemeyer (2013), Khrennikov (2015a), Haven and Sozzo (2016), Takahashi et al. (2017), the recent handbook edited by Haven and Khrennikov (2017a), and the first textbook for students (Haven et al. 2017); see also closely related study by Narens (2015, 2016) who used the general formalism based on lattices. The first steps in this direction were done in 1990th, see, e.g., Khrennikov (1999). We also mention a few recent papers on applications to finances and economics (Hawkins and Frieden, 2012, 2016; Khrennikova, 2016, 2017; Haven and Sozzo, 2016).

[2]Typically this fundamental issue was not stressed in works on the QP-based models of DM, since the basic effects, such as, e.g., violation of the Sure Thing Principle, were finally expressed via the Born rule in terms of probability. However, this issue is very important for our model and we shall turn to it later.

on human evaluation and revision of probabilities in uncertain DM situations. Following questions naturally emerged:

*Do people obey the rules of classical probability theory, and if not, in which circumstances? Are there any other laws that can be applied to formalize human judgments and preferences?*

After demonstrating first comprehensive evidence on deviation of human preference formation from the postulates of von Neumann and Morgenstern (1944) theories (VNM), Ellsberg (1961, p.646) proposed that: "there would be simply no way to infer meaningful probabilities for those events from their [participants] choices, and theories which purported to describe their uncertainty in terms of probabilities would be quite inapplicable in that area (unless quite different operations for measuring probability were devised)."

A wide array of elegant generalizations of classical probabilistic formulations of rational decision theories (VNM) were devised following the emerging empirical evidence. (We allude to von Neumann and Morgenstern's (1944) expected utility formulation under objective risk, as well as Savage's (1954) subjective expected utility over consequences in uncertain states of the world.) Generalized utility theories focused particularly on eliminating the linearity in probabilistic measures, and seeking to relax the assumption of agents' possessing firm and state-independent probabilistic estimates.[3]

As articulated by Machina (1989), the main appeal of the devised generalizations of VNM formulations was to reach three goals; the empirical (fit to the experimental data), the theoretical (allow to use the formulation in the most general settings, from trading on the financial markets, to insurance and gambling) and finally, the normative status (logical implications such as rationality of the assumptions in VNM).

Another stream of case by case explanations, based on heuristics and biases pioneered by Kahneman and Tversky[4], also gained wide recognition in behavioral economics, that mainly emerged due to extensive collection of empirical evidence on human preference formation and judgment.

Yet, these convincing explanations of each such bias and its effect on DM formation were placed under critique largely due to their lack of theoretical coherence and normative appeal (e.g., Wolford, 1991; Gigenzenger, 1996). We stress that prospect theory by Kahneman and Tversky (1979), and the advanced version, cumulative prospect

[3]See (Kahneman and Tversky, 1979; Machina, 1982, 1989; 2005, Gilboa and Schmeidler, 1989, 1994; Schmeidler, 1989; Tversky and Kaheneman, 1992; Klibanoff et al., 2005).

[4]See (Kahneman and Tversky, 1972, Tversky and Kahneman 1974); see also an earlier excellent survey of the behavioral factors that ought to falsify the postulates of modern decision theories under uncertainty and risk by Simon (1959).

theory by Tversky and Kahneman (1992) can be considered as great accomplishments in the pursuit for a generalized and structured DM framework, encompassing some of the effects of human heuristics and biases.

Today, we still find ourselves at the theoretical crossroads, with considerable divisions across conflicting, entrenched theoretical positions that revolve around the following dilemmas:

- Should we continue to rely on CP as the basis for descriptive and normative predictions in decision making and perhaps ascribe inconsistencies to methodological idiosyncrasies or intrinsic noise, cf. (Costello and Watts, 2014)?
- Should we abandon probability theory completely and instead pursue explanations based on heuristics and biases, as proposed by Tversky and Kahneman?

Yet, a need for a DM framework with theoretical foundations that can be utilized in economics, finance and other domains persists. Hence, by generalizing or replacing classical VNM framework, one is compelled to maintain the theoretical foundations of the alternative decision theory. Probabilistic and statistical methods are undeniably the cornerstones of modern scientific methodology in all spheres of social science. Thus, although the heuristic approach to decision making cannot be discarded completely and serves as an important tool to research the nature of human reasoning, it appears that it is more natural to approach novel probabilistic models to formalize preference formation. Hence, in the present contribution we proceed with the slogan:

*QP instead of heuristics and biases!*

Application of the laws of QP (treated as the calculus of probability amplitudes), instead of CP, can resolve some paradoxes of classical DM theory, see section 2. The number of different 'paradoxes' generated by the classical DM theory is startling. The authors of a recent review (Ert and Erev 2015) identified 35 basic paradoxes. The history of decision theory, can be characterized by advancement of the theoretical frameworks via creation and resolution of paradoxes through modifications of the theory. As an example, von Neumann-Morgenstern (VNM) expected utility theory was generalized to prospect theory after numerous empirical studies (cf. Tversky and Kahenman, 1992; Shafir 1994). However, any modification suffered from new paradoxes.

It seems that the use of QP can resolve such paradoxes (including Allais (1953), Ellsberg (1961) and Machina (2009) paradoxes), at least this is claimed in the recent paper of Asano et al. (2017), see this paper for the detailed mathematical modeling of these paradoxes. In this paper the authors develop a model of selection of lotteries under uncertainty based on the quantum formalism: representation of belief-states as quantum states and introduction of *comparison operator* $D$ which is based on a classical utility function $u(x)$. This operator is the basic mathematical object of the theory and it provides the operational representation of the process of comparison of utilities of two lotteries. This quantum-like model can be coupled to expected utility approach (endowed with either objective or subjective interpretations of utility functions and probabilities). In particular, this model reproduces VNM expected utility theory (in this case probabilities can be interpreted objectively) and prospect theory (including its representation with cumulative probability weighting function, Tversky and Kahenman, 1992). Moreover, the quantum-like model of lottery selection recreates *one special form of the probability weighting function used in prospect theory.* We recall that in prospect theory, the probability weighting function is the important concept to explain the violation of independence axiom in VNM theory. Actually, from phenomenological discussions, various weighting functions have been proposed (Prelec, 1998; Rieger and Wang, 2006; Tversky and Kahneman,1992; Gonzales and Wu, 1999; Wu and Gonzales,1996). Especially, we note the form of the two-parameter weighting function,

$$w_{\lambda,\delta}(x) = \frac{\delta x^{\lambda}}{\delta x^{\lambda} + (1-x)^{\lambda}}, \tag{1}$$

which was discussed in (Gonzalez and Wu, 1999). The parameters $\lambda$ and $\delta$ control the curvature and elevation of the function, respectively. Such a phenomenological function with $\lambda = 1/2$ corresponds to the subjective probabilities derived from the usage of the QP-framework, see (Asano et al. 2017) for a detailed discussion. It would be interesting to find a purely psychological motivation for this specific shape of the weighting function corresponding to $\lambda = 1/2$.

The quantum-like model of Asano et al. (2017) not only reproduces the output of prospect theory (for the aforementioned special choice of the weighting function), but depending on the belief-state of an agent can lead to new decision rules, including the existence of new parameters besides the subjective probabilities. These parameters are given by relative phases expressing correlations between different outcomes of lotteries $A$ and $B$, within a single lottery or between two lotteries. The presence of phases induces the effects of constructive and destructive interference.

In this paper we advance the model proposed in (Asano et al., 2017) by representing Alice's beliefs about the lotteries' outcomes by two arbitrary orthonormal bases.[5] In the operator terms this corresponds to representation of beliefs by two in general noncommuting operators $A$ and $B$, see (2). In (Asano et al., 2017) beliefs about the lotteries were represented by commuting operators.[6] In the case of commuting $A$ and $B$ the interference effects were generated solely by comparison operator $D$. In the general model with noncommuting lottery operators there are two sources of interference between beliefs: a) compelmentarity of lotteries, b) reflections generated by operator $D$.

In the probabilistic terms, in the present model the DM-process is split into four sub-processes, see section 7. The previous model (Asano et al., 2017) handled only one of these processes (Process 1, see section 7). The new counterparts of the DM-process model describe mathematically Alice's reflections in respect to the selection of lotteries. These reflections are modeled with the aid of *quantum transition probabilities.* For complementary lotteries (represented by noncommuting operators), these probabilities are nontrivial and their presence generates complex reflections of a decision maker, cf. (Asano et al., 2017). The transition probabilities are involved in the creation of more complex subjective probabilities for lotteries' outcomes than the probabilities of Process 1 only, see Asano et al. (2017). The devised model is quite complicated from the viewpoint of mathematical computations. We reproduce the detailed model derivation in the special appendix (appendix 1).

The structure of the model is very rich. To demonstrate at least some of its distinguishing features, we analyze in great detail the example of lotteries with two outcomes, see section 8 and appendix 2. This simple example shows that, in fact, a quantum-like agent uses the probability amplitudes (and not the squares of their absolute values) as weights for averaging of utility function. So, the agent evaluates expected utility via usage of square roots (probability amplitudes). Of course, the straightforward probability interpretation of this construction is impossible (since amplitudes need not be non-negative real numbers). For the probabilistic interpretation, one has to proceed with four counterparts of the process of decision making considered in section 7. At the same time modeling based on amplitudes is at-

[5]Since the present model generalizes the model from (Asano et al., 2017), it resolves the paradoxes of the classical DM theory, see (Asano et al., 2017) for detailed presentation (with corresponding numerical simulation).

[6]It is worth noting that both representations, by commutative and noncommutative operators, have been popular in cognitive modeling, see (Pothos and Busemeyer, 2009; Wang and Busemeyer, 2011; Trueblood and Busemeyer, 2011, Broekaert et al., 2017) and (Khrennikov, 2003, 2004a,b, 2006, 2010, 2015c; Busemeyer et al., 2011; Khrennikov and Basieva, 2014; White et al., 2014; Asano et al., 2015).

tractive by its simplicity. One can proceed in this direction by using signed and complex "probabilities" which are widely used in quantum mechanics and recently started to be applied to decision making, see de Barros et al. (2016, 2017).

Now we turn to the interpretation of QP as the calculus of probability amplitudes. In the models in (Asano et al., 2017) and this paper, the decision rule is represented in the form $\langle D\Psi|\Psi\rangle \geq 0$, where $\Psi$ is the belief state of an agent. Although, as was emphasized, this decision rule can represented in the form of comparison of averages of the utility function $u(x)$ with respect to subjective probabilities (or weighting function of them), the probabilistic representation is not intrinsic. Of course, its derivation is important to couple the quantum-like model with the existing DM-models of the VNM and prospect theory type. However, such a probability representation shadowed the fundamental feature of our model which *does not pre-assume that agents really compare lotteries by through calculation of averages (with respect to probabilities or their weighting).* An agent uses the objective or subjective probabilities $(P_i)$ and $(Q_j)$ corresponding to the outcomes of lotteries $A$ and $B$ to create her belief states about these lotteries, $|\Psi_A\rangle$ and $|\Psi_B\rangle$. Then she superposes these states to create state $|\Psi\rangle$ of her beliefs about two lotteries. This state is substituted in the quadratic form of comparison operator $D$ and the value of this form in state of superposed beliefs $\Psi$ determines agent's concrete decision. This is the individual decision, not statistical. However, it can be coupled to statistics through the probabilistic decomposition of the quadratic form $\langle D\Psi|\Psi\rangle$, see section 7. This decomposition expresses potentiality of realization of the concrete decisions encoded in the belied state $\Psi$. The comparison operator $D$ is a cognitive feature of an agent. In this model the process of decision making is algorithmic and it is reduced to calculation of the value of the comparison-form on the belief-state.

We remark that, as well as prospect theory or regret theory, our quantum-like model of DM is of the descriptive type. As the creators of other descriptive theories, we try to explain paradoxes and decision problems. Our explanation is derived from the quantum theory endowed with the information interpretation of states. An agent whose information processing can be described by the quantum formalism uses the belief-weighting procedure based on coefficients in state's expansion with respect to a basis corresponding to a set of beliefs. Such beliefs-weighting explains the basic paradoxes of DM theory. For the moment, we do not have a neurophysiological mechanism of such weighting, see (Khrennikov, 2011, de Barros, 2012, Busemeyer et al., 2017) for some attempts to model it.

# 2 From the von Neumann-Morgenstern expected utility theory to quantum(-like) modeling of subjective expected utility

In their book von Neumann and Morgenstern (1944) introduced an expected utility function over lotteries, or gambles. The type of uncertainty which was embedded in their expected utility[7] approach was objective uncertainty (i.e. an uncertainty which is formalizable by using objective probabilities). A key theorem in the VNM theory establishes the so called expected utility representation, which in essence requires that preferences over lotteries satisfy a specified number of axioms[8]. As economic history has shown, some of their axioms were not as 'natural' as expected and Allais' paradox (see Allais, 1953) showed a violation of the so called substitution axiom[9]. Whilst VNM developed an axiomatic choice framework along objective uncertainty, it surely is the case that real life decisions can revolve around subjective choice situations. The purpose of Savage's theory is to consider choice under subjective uncertainty (see Savage, 1954). In words, the Savage model can be summarized as follows (Kreps (1988) (p. 195-196): " Savage models ...hold that you should assess probabilities for the subjectively uncertain events, probabilities that add up to one, and then choose whichever gamble gives the highest subjective expected utility."

Savage (1954) formulated the famous *Sure Thing Principle* which is an essential axiom (amongst seven other axioms) which allows for the existence of an equivalence relation between a preference over acts[10] and an ordering of expected utilities.

We remark that in purely probabilistic terms, this principle is equivalent to the validity of the law of total probability (see Khrennikov, 2010). Hence, a violation of this law for our quantum-like model of DM will be equivalent to a violation of Savage's Sure Thing Principle (see Busemeyer et al., 2006). We note that the well known Ellsberg paradox (see Ellsberg, 1961) specifically refers to a violation of the sure-thing principle[11].

[7] We note that in economics there has been a long standing discussion between expected value and expected utility. This debate relates to the so called St. Petersburg paradox (see Blavatskyy , 2005 and Samuelson, 1977).

[8] In essence, those preferences need to satisfy the 'substitution' and 'continuity' axioms.

[9] This axiom says that for three lotteries, $a$,$b$ and $c$ the preference of lottery $a$ over lottery $b$ will imply that the weighted average of lotteries $a$ and $c$ is preferred over the weighted average over lotteries $b$ and $c$.

[10] An act, as per Kreps (1988), p. 128, is a function from a set of states to a set of prizes.

[11] The sure thing principle can also be found back in von Neumann-Morgenstern's theory but only if one considers probabilities without finite support (see Krep, 1988, p. 59 and

In the expected utility representation of the Savage approach (as well as in the VNM approach), the utility function will be bounded and unique (up to an affine transformation). The linearity[12] of the preference function is tightly connected to the substitution axiom which forms part of the VNM theory. Violation of this axiom, was shown to occur via the Allais paradox which we already mentioned above. More paradoxes exist. The Machina paradox (see Machina, 2009) will challenge a whole variety of expected utility approaches, such as the max-min expected utility (see Gilboa and Schmeidler, 1989) and the Choquet expected utility (see Gilboa and Schmeidler, 1994). See also Haven and Sozzo (2016) for more of a discussion on why non-classical probability can be an answer in the presence of such paradox (see also Machina, 1983, 1987 and Erev et al., 2016).

All these DM-theories are mathematically formalized with the aid of *classical probability* (CP), the axiomatics of Kolmogorov (1933). Those axiomatics are based on the set-representation of events and the measure-representation of probabilities. Let us make the following point. The constraints posed on a DM-model by the CP-calculus can have fundamental consequences. The most important set of CP-constraints is related to the set-representation of events. In fact, this is the special representation of *classical Boolean logic.* Thus, practically all probabilistic utility models (not only the expected utility ones) are based on the implicit assumption that all agents use the special calculus of propositions known as the *Boolean algebra.*[13] This assumption precedes, e.g. the axioms about the rationality of agents. To even formulate such axioms, one has to appeal to Boolean logic. We also remark that *expected* utility theory uses mathematical expectation which corresponds to the CP-model.

It would be interesting to investigate what consequences will emerge if we consider a relaxation of some of the axioms of CP for DM under uncertainty. However, this general project has a very high complexity: one can create a huge variety of novel 'non-Kolmogorovean models' and to analyze all possible consequences for DM is really impossible. In particular, in mathematical applications we can find a variety of generalized averaging procedures leading to nonclassical notions of mathematical expectation. Therefore, it would be natural to try to go beyond CP (and Boolean logic) on the basis of some concrete and

following).

[12]Mark Machina did note in Machina (1982) (p. 295) that ....“from the fact that differentiable functions are 'locally linear', and that for preference functionals over probabililty distributions, linearity is equivalent to expected utility maximization.”

[13]There are a few exceptions. For example, Narens (2015, 2016) used intuitionistic logic to provide a framework for DM that is not Boolean.

well developed non-CP model which has already demonstrated its applicability to the solution of non-trivial problems in not only natural science but also in economics, psychology and other areas of social science.

Such a non-CP model is now well known: this is the probabilistic counterpart of the mathematical apparatus of quantum mechanics - QP, cf. (Khrennikov et al., 2014; Boyer-Kassem et al., 2015). It is essential we make a remark about possible interpretations of quantum mechanics. The variety of interpretations of quantum mechanics is huge and we have no possibility to present even the most important ones (see Khrennikov, 2010).

Let us point to the recently developed subjective probability interpretation, known as *Quantum Bayesianism* (QBism) (see e.g. Fuchs and Schack, 2011). By this interpretation, QP is the machinery used by agents to update *subjective probabilities* for outcomes of experiments. *QBism stresses the private agent perspective in quantum theory.* The only shortcoming of QBism (from our perspective) is that it handles only decisions about outcomes of observations done for quantum physical systems. This shortcoming of QBism was discussed in the papers by Khrennikov (2016a) and Haven and Khrennikov (2017b), where we extended the applications of QBism to areas outside of quantum physics. Development of QBism is very important for justification of applications of the quantum formalism to modeling the process of decision making based on subjective probability - QBism is the only interpretation of QP based on subjective probability.

Thus, QP is treated as formalizing the DM-process by an agent who follows the rules of quantum logic. The latter relaxes some basis rules of classical logic. Here, in particular, an agent can violate the law of distributivity between conjunction and disjunction. The QBism interpretation of the quantum formalism is very supporting for its applications to decision making, since it justifies the use of subjective probability.

In this paper we shall present a concrete QP-based model of the DM-process under uncertainty which is generated by a complex information environment, including internal representations of lotteries by a decision maker. Here lotteries should not be reduced to mechanical devices such as roulettes. These are generators of events with complex inter-relation, inside each of the lotteries as well as between lotteries.

# 3 Quantum-like model of selection of lotteries

There are two lots, say $A = (x_i, P_i)$ and $B = (y_i, Q_i)$, where $(x_i)$ and $(y_i)$ are outcomes and $(P_i)$ and $(Q_i)$ are probabilities of these otcomes. All of the outcomes are different from each other. Which lot do you select?

## 3.1 Classical probability modeling

An agent, say Alice, can simulate the experience that she draws the lot $A$ (or $B$) and gets the outcome $x_i$ (or $y_i$). Let us represent such an event by $(A, x_i)$ (or $(B, x_i)$). As usual, Alice assigns the utilities $u(x_i)$ and $y(x_i)$ of $(A, x_i)$ and $(B, y_i)$, respectively. Here, $u(x)$ is a utility function of outcome $x$. [14] By using the utility function the agent evaluates various comparisons for making the preference $A \succeq B$ or $B \succeq A$.

The first mathematically consistent theory of decision making was VNM expected utility theory based on VNM axioms (Completeness, Transitivity, Independence, Continuity). VNM axioms are given for the relation of utilities like $u \succeq v$ and the operation using probability like $pu+(1-p)v$. This motivates an agent to operate with the expected utilities, $E_A = \sum u(x_i)P_i$ and $E_B = \sum u(y_i)Q_i$, and to use their difference as the criterion for making the preference.

However, the VNM decision theory is not free of paradoxes as we observed in section 2. This problem is fundamentally coupled to the interpretation of probability used in the VNM theory. VNM used the frequency (statistical) interpretation of probability. Therefore it is natural to test models of decision making based on other interpretations. The most powerful alternative to the frequency probability is the *subjective probability.* The subjective probability is not the frequency of an event obtained on the basis a large number of trial experiments, rather it is the measure of belief about whether a specific outcome is likely to occur.

[14] In the classical model the utility of an event depends on only its outcome. However, in the quantum-like model utility has the complex Hilbert space representation encoding all circumstances of realization of outcomes.

## 3.2 Representation of lotteries by orthonormal bases in belief-state space

In our model of decision making we describe subjective probability by using the framework of quantum theory, cf. with the classical modeling with the aid of subjective probabilities, e.g. (Gilboa, 2009). We emphasize that the quantum formalism operates with a state before measurement. In quantum-like models of cognition and decision making a quantum state is treated as the belief state of an agent. Such a state formalizes the mental representation for uncertain (unmeasured) events.

Consider the space of belief states of an agent. In accordance with the quantum-like modeling of cognition belief-states are represented by normalized vectors of a complex Hilbert space $H$. These are so-called pure states. More generally, belief-states are represented by density operators encoding classical probabilistic mixtures of pure states. But in this paper we restrict our consideration to pure belief-states (although it will be convenient to represent them by operators.)

Lotteries $A$ and $B$ are mathematically realized as two orthonormal bases in $H : (|i_a\rangle)$ and $(|j_b\rangle)$. [15] Any vector $|i_a\rangle$ represents the event $(A, x_i)$ - "selecting of the $A$-lottery which generates the outcome $x_i$.″ The same can be said about vectors of the $B$-basis. These events are not real, but imaginable. Alice plays with potential outcomes of the lotteries and compares them.[16]

We can also represent lotteries by Hermitian operators, the lotteries operators:

$$A = \sum_i x_i |i_a\rangle \ , B = \sum_j y_j |j_b\rangle. \tag{2}$$

As in the classical theory, each outcome $x_i$ has some utility $u_i = u(x_i)$ (say amount of money). Thus our model is based on a mapping from eigenstates of the "lottery-operators" to utilities (amounts of money), $|i_a\rangle \to u(x_i)$, $|j_b\rangle \to u(y_j)$. Thus starting with two lotteries

[15] In the Dirac notations (see, e.g., (Dirac, 1995)) a vector is denoted by a rather complex symbol, $|\phi\rangle$. However, this symbolism is convenient to compose the scalar product. Each vector $|\phi\rangle$ determines the continuous linear functional on $H$ which is denoted as $\langle\phi|$. Then action of the functional $\langle\phi|$ on the vector $|w\rangle$ is represented by the formal multiplication of these two symbols: $\langle\phi||w\rangle \equiv \langle\phi|w\rangle$, the scalar product of these two vectors, see, e.g., (Dirac, 1995). This algebra will be heavily used in this paper, see especially appendix 1. We also remark that any pure state $|\phi\rangle$ determines the orthogonal projector; in the Dirac notations, $\sigma_\phi = |\phi\rangle\langle\phi|$. This is the density operator representation of a pure state.

[16] This is a kind of counterfactual reasoning. From this viewpoint, we treat the quantum formalism as a mathematical device for counterfactual reasoning. Of course, we well aware that this not the only possible representation for such a reasoning; in future other models of counterfactual reasoning can be in the use.

$A$ and $B$ with outcomes $(x_i)$ and $(y_j)$ with corresponding utilities $u_i = u(x_i)$ and $v_j = u(y_j)$, Alice represents these utilities (coupled to their lotteries) by two orthonormal bases in the belief-state space $H$ :

$$u_i \to |i_a\rangle, \; v_j \to |j_b\rangle. \tag{3}$$

We emphasize coupling of utilities to lotteries. Utility (derived from some monetary amount) has not only the value, but also so to say the "color" determined by circumstances surrounding the corresponding lottery - *lottery's context.* Therefore even the same outcome $z = x_i = y_j$ of two lotteries (having the same utility value $u = u(z)$) may be represented by two different vectors: $|i_a\rangle \neq |j_b\rangle$. Moreover, outcomes inside each lottery are also coupled through selection of the fixed orthonormal basis. Finally, we remark that mapping (3) encodes the correlations between outcomes of lotteries (and their utilities). These correlations are mathematically expressed through quantum transition probabilities. We shall define them in the next section, see, e.g., (Haven et al, 2017) for details.

The lotteries operators can be noncommuting, i.e., $[A, B] \neq 0$. In quantum theory noncommuting Hermitian operators represent complementary (or incompatible) observables: they cannot be measured jointly. In our quantum-like model of lottery selection, we can speak about *complementary lotteries* which are represented by noncommuting operators. In the DM-process for complementary lotteries, Alice does not create the the joint image of outcomes of both of them. In mathematical terms the latter means the impossibility to determine the joint probability distribution for the pairs of outcomes $(x_i, y_j)$. Thus, instead of weighting probabilistically the pairs of outcomes, Alice analyzes the possibility of realization of an outcome say $x_i$ of the $A$-lottery and she accounts its utility $u(x_i)$. Then under the assumption of such realization she imagines possible realizations $(y_j)$ of the $B$-lottery and compares the utilities $u(y_j)$ and $u(x_i)$. "Suppose I have selected the $A$-lottery and its outcome $x_i$ was realized. What would be my earning (lost) if (instead) I were selected the $B$-lottery and its outcome $y_j$ were realized?" This kind of counterfactual reflections is mathematically described by the Hilbert space formalism and transition from the $A$-basis to the $B$-basis. Outputs of these comparisons are weighted through accounting Hilbert space coordinates, see the discussion on QP as a calculus of probability amplitudes at the very end of introduction. This accounting is described by the special comparison operator $D$. Since Alice cannot handle both lotteries simultaneously, she starts with imaging one of them say $A$, as in the above consideration. Then she performs similar counterfactual reasoning starting with the $B$-lottery. The comparison operator $D$ has two counterparts representing the processes $A \to B$ and $B \to A$ comparisons, see section 6, formula (20). In the operator terms transitions from one basis to another are represented by transition operators $E_{i_a \to j_b}, E_{j_b \to i_a}$, see (19). And the comparison operator $D$ is compounded of these operators.

### 3.3 Quantum transition probabilities

We now consider the notion of the quantum transition (conditional) probability. For our applications, it is sufficient to consider transitions between the states $(|i_a\rangle)$ and $(|m_b\rangle)$. We have

$$\langle m_b|\ i_a\rangle = \sqrt{p(m_b|i_a)}e^{i\theta_{i_a \to m_b}}, \tag{4}$$

where $p(m_b|i_a) = p(i_a \to m_b)$ is the probability of transition from the state $i_a$ to the state $m_b$. Thus

$$p(m_b|i_a) = |\langle m_b|\ i_a\rangle|^2 \tag{5}$$

This is the Born rule of quantum theory. Symmetry of a scalar product implies that

$$p(i_a \to m_b) = p(m_b \to i_a), \text{i.e.}, p(m_b|i_a) = p(i_a|m_b).$$

We also remark that the corresponding transition phases are related as $\theta_{i_a \to m_b} = -\theta_{m_b \to i_a}$.

## 4 Probabilities and phases

Here we shall discuss the meaning of coefficients in the expansion of a quantum state $|\psi\rangle$ with respect to an orthonormal basis. For simplicity, we consider the two dimensional state space. Here we represent some dichotomous observable by Hermitian operator $A$ with the eigenvalues $(x_1, x_2)$ and eigenvectors $|1\rangle, |2\rangle$. Any state $|\psi\rangle$ can be expanded with respect to this basis:

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle, \tag{6}$$

where $c_1, c_2$ are complex numbers and

$$|c_1|^2 + |c_1|^2 = 1. \tag{7}$$

By using the quantum terminology the state $|\psi\rangle$ is *superposition* of the (eigen)states $|1\rangle, |2\rangle$. We remark that the use of the linear space representation is very common in a variety of cognitive and psychological models, see, e.g., (Shepard, 1987; Nosofsky, 1988; Wills and Pothos, 2012). Thus one might think that the only uncommon feature of the model is the use of complex numbers. However, since each complex number $z$ can be represented as $z = u + iv$, where $u, v$ are real numbers, any complex linear model of dimension $n$ can be treated as the real model of dimension $2n$.

*The main distinguishing feature of the quantum model is that the coefficients have a probabilistic meaning given by the famous Born's rule.* For the state $|\psi\rangle$ of the form (6), the numbers

$$p_j = |c_j|^2 \tag{8}$$

are interpreted as the probabilities of the outcomes $x_j$ of the observable $A$ having the basis of eingenvectors $|1\rangle, |2\rangle$. Thus the absolute values of the coefficients in the expansion (6) have the clear meaning: these are square roots of probabilities, $|c_j| = \sqrt{p_j}$. However, any complex number has also the phase: $c_j = |c_j|e^{i\theta_j}, j = 1, 2$. The interpretation of phases is more complicated. Why do we need phases at all? Why is it not sufficient to work with states with real coefficients? From the viewpoint of the Born rule, it seems that it would be sufficient to proceed with superpositions of the form:

$$|\psi\rangle = \sqrt{p_1}|1\rangle + \sqrt{p_2}|2\rangle. \tag{9}$$

One of the possibilities to provide a consistent interpretation to the phases is to consider the *dynamical model of states generation.* The crucial point is that to have the law of conservation of probability, see (7), we have to consider the *unitary dynamics.* And a unitary dynamics can generate nontrivial phases starting with superpositions of the form (9). The dynamics of the quantum state is described by the Schrödinger equation:

$$i\frac{\partial|\psi\rangle}{\partial t}(t) = \mathcal{H}|\psi\rangle(t),\ |\psi\rangle(0) = |\psi_0\rangle, \tag{10}$$

where $\mathcal{H}$ is the generator of quantum dynamics, a Hermitian positively definite operator. It has the dimension of frequency, i.e., 1/time. [17] Therefore $\mathcal{H}$ can be called the *oscillation operator*. To understand better its meaning, let us consider its eigenvalues $\omega_1, \omega_2$ and corresponding eigenvectors $|e_1\rangle, |e_2\rangle$. In this basis the Schrödinger equation

[17] In physics the left-hand side of the equation contains also the Planck constant having the dimension of action= time× energy. Therefore the generator has the dimension of energy. It is called Hamiltonian and has the meaning of the energy observable. In applications outside of physics we treat it formally as dynamics' generator. In financial modeling (Haven and Khrennikov, 2013) $\mathcal{H}$ was interpreted as a kind of financial energy; in social modeling (Khrennikov, 2016b) it was interpreted as a kind of social energy. However, such interpretations suffer of the absence of measurement methodology.

is the system of two ordinary differential equations for the complex coordinates $z_j(t), j = 1, 2$,

$$i\frac{dz_j}{dt}(t) = \omega_j z_j(t), \tag{11}$$

Its solution has the form

$$z_j(t) = e^{-i\omega_j t} z_{0j}. \tag{12}$$

These are two oscillatory processes. Their combination gives the complete state-oscillations:

$$|\psi\rangle(t) = e^{-i\omega_1 t} z_{01}|e_1\rangle + e^{-i\omega_2 t} z_{02}|e_2\rangle. \tag{13}$$

Thus even if, for the initial state, the coefficients $z_{0j} \in \mathbf{R}$, the dynamics generates nontrivial phases and complex coefficients .

Following the model of dynamical decision making (Khrennikov, 2004a, 2004b, 2006; Pothos and Busemeyer, 2009, Asano et al., 2015), agent's state evolves driven by the Schrödinger equation until the moment of decision making $T = T_{\rm dm}$. The simplest problem of decision making can be represented as a measurement of some observable, say dichotomous, represented by a Hermitian operator $A$ with eigenvectors $|1\rangle, |2\rangle$. By expanding the state $|\psi\rangle(t)$ with respect to this basis we get the representation:

$$|\psi\rangle(t) = c_1(t)|1\rangle + c_2(t)|2\rangle, \tag{14}$$

where $c_j(t) = e^{-i\gamma_j t}\sqrt{p_j(t)}$. Now at the instant of the self-measurement of the mental observable $A$ an agent uses the state

$$|\psi\rangle = e^{-i\theta_1}\sqrt{p_1}|1\rangle + e^{-i\theta_2}\sqrt{p_2}|2\rangle, \tag{15}$$

where $\theta_j = \gamma_j T$ and $p_j = p_j(T)$. If the operator $A$ coincides with $\mathcal{H}$, then $\theta_j = \omega_j T$ (but this is the very special situation).

For this state, an agent makes the $A$-observation and she obtains the output $a_j$ with the probability $p_j$. This is *the objective probability model of decision making*: for a large ensemble of agents the probability-frequency of the output $x_j$ equals to $p_j$ (Khrennikov, 2016a; Haven and Khrennikov, 2017b). Another model is based on *the subjective interpretation of probability* (Khrennikov, 2016a; Haven and Khrennikov, 2017b). An agent assigns subjective probabilities of the outputs by extracting them from the state (15), then she computes odds $O(1/2) = \frac{p_1}{p_2}$ and she makes her choice depending on the value of odds.

In this paper we shall study a more complex problem of comparison of two lotteries which cannot be reduced to quantum-like modeling of a single observable. The process of comparison involves two in general incompatible observables $A$ and $B$. We shall proceed with the subjective interpretation of probabilities. However, the main feature of the quantum-like process of decision making, namely, reduction of this process to elementary oscillations, will be crucial even in the coming model of comparison of lotteries, see section 7.

# 5 Belief-state

The state of Alice's beliefs about the lottery $A$ can be represented as superposition

$$|\Psi_A\rangle = \sum_i \sqrt{P_i} e^{i\theta_{ai}} |i_a\rangle.$$

The probability of realization of the event $(A, x_i)$ is given by the Born rule and equals to $P_i = |\langle i_a|\Psi_A\rangle|^2$. In the same way the state of beliefs about the lottery $B$ can be represented as superposition

$$|\Psi_B\rangle = \sum_i \sqrt{Q_i} e^{i\theta_{bi}} |i_b\rangle.$$

To point that an index serves to describe the lottery $A$ (lottery $B$), we shall label it by its own index, say $i_a$ or $j_b$. And we omit these labels, $a, b$, when the meaning of indexes be clear or their coupling to $A$ and $B$ would not be important.

Alice superposes her belief-states about the lotteries and her total belief-state is created via superposition of her beliefs about the $A$-lottery and the $B$-lottery. Thus the overall $\psi$ is the superposition of the $\psi$'s s for two individual lotteries.

$$|\Psi\rangle = |\Psi_A\rangle + |\Psi_B\rangle. \tag{16}$$

However, since in general the states representing Alice's beliefs about the lotteries are not orthogonal[18], i..e., in general $\langle\Psi_A|\Psi_B\rangle \neq 0$, the vector $|\Psi\rangle$ is not normalized and the state of combined beliefs is obtained via normalization: $|\Phi\rangle = |\Psi\rangle/\||\Psi\rangle\|$. To make presentation simpler, we shall proceed with the vector $|\Psi\rangle$, i.e., without normalization.[19] In future we shall call such vectors *unnormalized states or simply states* (if this would not generate misunderstanding).

[18]Nonorthogonality of the states $|\Psi_A\rangle$ and $|\Psi_B\rangle$ means that the beliefs about two lotteries are not complementary. There is an "overlap" between them. The presence of such overlap plays the important role in the process of decision making, see section 8.

[19]Since the normalization factor is positive and the decision rule is based on inequality $\langle D\Psi|\Psi\rangle \geq 0$, this factor does not play any role in the process of comparison of lotteries, see section 6.

In further calculations it is useful to use the operator representation of $|\Psi\rangle$ :

$$\sigma \equiv \sigma_\Psi = |\Psi\rangle\langle\Psi| = \sigma_A + \sigma_B + \sigma_{B\to A} + \sigma_{A\to B}, \tag{17}$$

where

$$\sigma_A = |\Psi_A\rangle\langle\Psi_A| = \sum_{i,j} \sqrt{P_i P_j} e^{i(\theta_{ai}-\theta_{aj})} |i_a\rangle\langle j_a|$$

$$\sigma_B = |\Psi_B\rangle\langle\Psi_B| = \sum_{i,j} \sqrt{Q_i Q_j} e^{i(\theta_{bi}-\theta_{bj})} |i_b\rangle\langle j_b|$$

$$\sigma_{B\to A} = |\Psi_A\rangle\langle\Psi_B| = \sum_{i,j} \sqrt{P_i Q_j} e^{i(\theta_{ai}-\theta_{bj})} |i_a\rangle\langle j_b|$$

$$\sigma_{A\to B} = |\Psi_B\rangle\langle\Psi_A| = \sum_{i,j} \sqrt{P_i Q_j} e^{-i(\theta_{ai}-\theta_{bj})} |j_b\rangle\langle i_a|.$$

We remark that, since $|\Psi\rangle$ is not normalized, $\mathrm{tr}\sigma \neq 1$. But we repeat that this is not important in the process of lotteries selection described in section 6.

If we consider the normalized state $|\Phi\rangle$, then corresponding operator $\sigma_\Phi = |\Phi\rangle\langle\Phi|$ is a density operator - the operator representation of pure state $|\Phi\rangle$. But we stress that the state of beliefs about two lotteries is a pure state.

The ability to create superposition of states is the main distinguishing feature of an agent using the rules of quantum logic. However, besides creation of superpositions, quantum-like Alice can also create mixtures of belief-states:

$$\sigma = \frac{1}{2}[\sigma_A + \sigma_B]. \tag{18}$$

By operating with such mixtures Alice can reproduce decision making corresponding to VNM expected utility model.

# 6 Comparison operator

In the classical expected utility theory Alice calculates the averages of the utility function. In the quantum-like model Asano et. al (2017) the utility function determines the *comparison operator*. Invention of such an operator is based on mappings from eigenstates of the "lottery-operators" to utilities (amounts of money), $|i_a\rangle \to u(x_i)$ and $|j_b\rangle \to u(y_j)$, see section 3.2.

We remark that we simply borrow the utility function from classical (objective or subjective) utility theory. Thus we do not contribute, e.g., in analysis of possible shapes of utility functions. Then we use QP to model subjective probabilities. And this modeling is encoded in quantum representation of the belief-state. However, this representation is only the first step towards the quantum world of decision making. The crucial step is the quantum-like operational description of the process of comparison of lotteries with the aid of quantum states transitions which are encoded in the comparison operator. This process can be structured as combination of comparison of a few SEUs and the interference type factors of the $\cos\theta$-form, where $\theta$ represents the combination of phases of a few processes of preferring of outcomes of the lotteries. These interference factors represent additional features of the subjective images of lotteries in the mind of an agent, in particular, *risks*, see (Asano et al., 2017), for discussion.

Let us introduce the *transition operators*

$$E_{i_a\to j_b} = |j_b\rangle\langle i_a|,\ E_{j_b\to i_a} = |i_a\ \rangle\langle j_b|. \tag{19}$$

We have, e.g., $E_{i_a\to j_b}|i_a\rangle = |j_b\rangle$. This operator describes the process of transition from preferring the state $|i_a\rangle$ to preferring the state $|j_b\rangle$. The operator $E_{j_b\to i_a} = |i_a\rangle\langle j_b|$ describes transition in the opposite direction. We stress that these are transitions between the belief-states of Alice. We remark that $E_{j_b\to i_a} = E^\star_{i_a\to j_b}$, i.e., elementary transitions in opposite directions are represented by adjoint operators.

Now we introduce the two comparison operators:

$$D_{B\to A} = \sum_{n,m}(u(x_n)-u(y_m))e^{i\gamma_{m_b\to n_a}}E_{m_b\to n_a} = \sum_{n,m}(u(x_n)-u(y_m))e^{i\gamma_{m_b\to n_a}}|n_a\rangle\langle m_b|.$$

$$D_{A\to B} = \sum_{n,m}(u(y_m)-u(x_n))e^{i\gamma_{n_a\to m_b}}E_{n_a\to m_b} = \sum_{n,m}(u(y_m)-u(x_n))e^{i\gamma_{n_a\to m_b}}|m_b\rangle\langle n_a|.$$

The operator $D_{B\to A}$ represents the utility of selection of the lottery $A$ relatively to the utility of selection of the lottery $B$. We can say that by transition from the potential outcome $(B, y_m)$ to the potential outcome $(A, x_n)$ Alice earns utility $u(x_n)$ and at the same time she loses utility $u(y_m)$. (If $u(x) = x$ and $x$ has the meaning of cash amounts (say USD), then by such a transition Alice (potentially) earns $x_n - y_m$ USD.)

In the same way we interpret the transition operator $D_{A\to B}$. This operator represents the utility of selection of the lottery $B$ relatively to the utility of selection of the lottery $A$. These operators represent the process of Alice's reflections in the process of decision making. Her mind fluctuates between preferring outcomes of the $A$-lottery to outcomes of the $B$-lottery (formally represented by the operator $D_{B\to A}$) and inverse preferring (formally represented by the operator $D_{A\to B}$).

Finally, she has to compare how much she can earn (in average) by preferring $A$ to $B$ comparing with preferring $B$ to $A$. This process is formally described by *the complete comparison operator:*[20]

$$D = D_{B\to A} - D_{A\to B}. \tag{20}$$

This operator has the form:

$$\begin{aligned} D &= \sum_{n,m} (u(x_n) - u(y_m)) e^{i\gamma_{m_b \to n_a}} |n_a\rangle\langle m_b| - \sum_{n,m} (u(y_m) - u(x_n)) e^{i\gamma_{n_a \to m_b}} |m_b\rangle\langle n_a| \\ &= \sum_{n,m} u_{nm} (e^{i\gamma_{m_b \to n_a}} |n_a\rangle\langle m_b| + e^{i\gamma_{n_a \to m_b}} |m_b\rangle\langle n_a|), \end{aligned} \tag{21}$$

where

$$u_{nm} = u(x_n) - u(y_m).$$

Since all quantum observables are represented by Hermitian operators, the phases should be related as follows:

$$\gamma_{n_a \to m_b} = -\gamma_{m_b \to n_a}. \tag{22}$$

The comparison operator $D$ gives us the integral judgment. Only heuristically can we treat the $D$-based judgment as the result of comparison of two relative utilities represented by the operators $D_{B\to A}$ and $D_{A\to B}$. We remark that the operators $D_{B\to A}$ and $D_{A\to B}$ are not Hermitian. Hence, they cannot be treated as observables. We have that $D^{\star}_{A\to B} = -D_{B\to A}$ and $D = D_{B\to A} + D^{\star}_{B\to A}$.

The quantum analog of (subjective) expected utility theory is based on the natural decision rule:

**Decision rule.** *If the average of the comparison operator $D$ is non-negative, i.e., $\langle D\rangle = \mathrm{tr} D\sigma = \langle D\Psi|\Psi\rangle \geq 0$, then $A \succeq B$.*

Using Eqs. (17) and (20) the trace can be written as the sum of four components:

$$\mathrm{tr} D\sigma = \frac{1}{2}\mathrm{tr} D\sigma_A + \frac{1}{2}\mathrm{tr} D\sigma_B + \Delta_1 + \Delta_2,$$

where

$$\Delta_1 = \frac{1}{2}\mathrm{tr}(D_{B\to A}\sigma_{A\to B} - \mathrm{tr} D_{A\to B}\sigma_{B\to A}),$$

$$\Delta_2 = \frac{1}{2}\mathrm{tr}(D_{B\to A}\sigma_{B\to A} - \mathrm{tr} D_{A\to B}\sigma_{A\to B}).$$

[20]The operators $D_{B\to A}$ and $D_{A\to B}$ are basic for comparison of lotteries. As everywhere in quantum theory, the order of preferring plays the crucial role. The first operator encodes preferring of $A$ over $B$ and the second one encodes preferring of $B$ over $A$. Then Alice compares their outputs. This last comparison is encoded in the operator $D$.

Which, together with the formal expressions for each components derived in Appendix 1, facilitates an operational understanding of the decision-making process as we shall see in the next section.

From calculations in appendix 1, we obtain that

$$\frac{1}{2}\mathrm{tr}D\sigma_A = \sum_{i,j,m} u_{jm}\sqrt{p(m_b|i_a)P_iP_j}\cos\Theta^A_{ij;m}, \tag{23}$$

where $\Theta^A_{ij;m} = \theta_{i_a\to m_b} - \gamma_{i_a\to m_b} + \theta_{ai} - \theta_{aj}$.

$$\frac{1}{2}\mathrm{tr}D\sigma_B = \sum_{i,j,n} u_{nj}\sqrt{p(n_a|i_b)Q_iQ_j}\cos\Theta^B_{ij;n} \tag{24}$$

where $\Theta^B_{ij;n} = \theta_{i_b\to n_a} - \gamma_{i_b\to n_a} + \theta_{bi} - \theta_{bj}$.

Calculations in appendix 1 show that it is natural to consider the following combinations of traces for comparison operators and "transition states", see (42), (43):

$$\Delta_1 = \frac{1}{2}\mathrm{tr}(D_{B\to A}\sigma_{A\to B} - D_{A\to B}\sigma_{B\to A}) = \sum_{i,j} u_{ij}\sqrt{P_iQ_j}\cos\Theta_{ij}. \tag{25}$$

where $\Theta_{ij} = \theta_{bj} - \theta_{ai} + \gamma_{j_b\to i_a}$;

$$\Delta_2 = \frac{1}{2}\mathrm{tr}(D_{B\to A}\sigma_{B\to A} - D_{A\to B}\sigma_{A\to B}) = \sum_{i,j,n,m} u_{nm}\sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j}\cos\Gamma_{ij,nm}, \tag{26}$$

where $\Gamma_{ij,nm} = \theta_{j_b\to n_a} - \theta_{i_a\to m_b} + \gamma_{n_a\to m_b} + \theta_{bj} - \theta_{ai}$.

# 7 Analysis of the basic counterparts of the process of comparison of lotteries

As we have seen, the average of the comparison operator is naturally decomposed into four counterparts representing special subprocesses of the process of decision making. We start with the simplest expression.

**Process 1:** Its output is represented by the quantity $\Delta_1$, see (25). To simplify considerations, let us assume that all phases $\Theta_{ij}$ in the sum are equal[21], i.e., $\Theta_{ij} \equiv \Theta$. Thus

$$\Delta_1 = \Big[\sum_i u(x_i)\sqrt{P_i}\sum_j\sqrt{Q_j} - \sum_j u(y_j)\sqrt{Q_j}\sum_i\sqrt{P_i}\Big]\cos\Theta.$$

[21] As was mentioned, the model has very rich structure. It can generate very complex behavioral patterns. We consider only the simplest of them.

Following Asano et al. (2017), consider the normalized difference

$$\frac{\Delta_1}{\sum_{i,j}\sqrt{P_i Q_j}} = \Big[\sum_i u(x_i)\tilde{P}_i - \sum_j u(y_j)\tilde{Q}_j\Big]\cos\Theta, \tag{27}$$

where the quantities

$$\tilde{P}_i = \frac{\sqrt{P_i}}{\sum_i \sqrt{P_i}},\ \tilde{Q}_j = \frac{\sqrt{Q_j}}{\sum_j \sqrt{Q_j}} \tag{28}$$

can be interpreted as subjective probabilities.[22] Alice assigns these probabilities to outcomes of the lotteries in the process of comparison of the relative utility of the $B \to A$ transition (based on her beliefs encoded in the $\sigma_{B\to A}$ counterpart of her belief state) with the relative utility of the $A \to B$ transition (based on her beliefs encoded in the $\sigma_{A\to B}$ counterpart of her belief state). We remark that the quantities

$$\langle u\rangle_{\tilde{P}} = \sum_i u(x_i)\tilde{P}_i \text{ and } \langle u\rangle_{\tilde{Q}} = \sum_j u(y_j)\tilde{Q}_j$$

are *expected utilities* for the lotteries with respect to these subjective probabilities. Thus, for this part of the process of comparison of two lotteries, Alice assigns subjective probabilities $(\tilde{P}_i)$ and $(\tilde{Q}_j)$ given by the square root transformation of the original probabilities $(P_i)$ and $(Q_j)$.[23] Then she calculates subjective expected utilities and compare them. The final step of comparison is taking into account the sign of the factor $\cos\Theta$. This is really nontrivial quantum counterpart of the decision process.

We remark that the square root transformation of probabilities can be directly coupled to selection of weighting functions in *prospect theory,* see (1) in introduction.

**Process 2:** Now we analyze the counterpart of the decision making process based on the comparison average given by the quantity

$$\frac{1}{2}\text{tr } D\sigma_A = \sum_{i,j,m} u_{jm}\sqrt{p(m|i)P_i P_j}\cos\Theta_{ij;m},$$

see (23). In this subprocess Alice uses only the part of her belief-state (given by $\Psi_A$) representing her beliefs about the $A$-lottery. She

[22] We remark that $\sum_i \tilde{P}_i = 1$ and $\sum_j \tilde{Q}_j = 1$.

[23] The latter can be treated as objective statistical probabilities. As was pointed out in introduction, the quantum formalism specifies the square root transformation as the map determining subjective probabilities. This is the very special form of the weighting function, see (1), used in the classical SEU-theory. As was already pointed out, it is important to find purely psychological justification for fixing $\lambda = 1/2$ in (1).

compares these $A$-beliefs with possible transitions to the $B$-states. Such transitions are expressed through the transition probabilities $p(m|i) = p(i_a \to m_b)$ and the phases $\theta_{i_a \to m_b}$ and $\gamma_{i_a \to m_b}$. These phases represent correlations between the events $(A, x_i)$ and $(B, x_m)$. These quantities are of the subjective nature. Alice tries to treat two lotteries separately, but she has a variety of correlations between them coming from the analysis of the situation and the previous experience. The transition probabilities $p(m|i)$ are also subjective quantities.

To simplify analysis, we again assume that all trasition phases $\Theta_{ij;m}$ are equal, i.e., $\Theta_A \equiv \Theta_{ij;m}$. Besides the subjective probabilities $\tilde{P}_i$, we consider subjective probabilities of transition from the lottery $A$ to the lottery $B$ given by

$$\tilde{Q}_{m;A} = \frac{\sum_i \sqrt{p(m|i) P_i}}{\sum_{i,m} \sqrt{p(m|i) P_i}}. \tag{29}$$

(We have $\sum_m \tilde{Q}_{m;A} = 1$.)[24] Consider now the normalized quantity

$$\frac{\mathrm{tr} D \sigma_A}{2 \sum_{i,j,m} \sqrt{p(m|i) P_i P_j}} = \Big[ \sum_j u(x_j) \tilde{P}_j - \sum_m u(y_m) \tilde{Q}_{m;A} \Big] \cos \Theta_A$$

$$= [\langle u \rangle_{\tilde{P}} - \langle u \rangle_{\tilde{Q}_A}] \cos \Theta_A,$$

where the quantities in the brackets are expected utilities with respect to the corresponding subjective probability distributions. Thus in this decision subprocess Alice assigns subjective probabilities to the $A$-events (by using the square root transformation). Then she assigns subjective probabilities $p(m|i)$ for transitions $i_a \to m_b$. They can be interpreted in the following way. Alice assumed that the event $(A, x_i)$ would happen (with the probability $\tilde{P}_i$, but with the probability $p(m|i)$ she changes her mind and assumes that the event $(B, y_m)$ would happen. The output of such mental fluctuations is assignment of subjective probability to the events $(B, y_m)$ conditioned on the beliefs about the $A$-states, the probability $\tilde{Q}_{m;A}$. Then Alice computes the difference between expected utilities. If $cos \Theta_A \geq 0$, then she uses this difference to order the lotteries as $A \succeq B$. However, if $\cos \Theta_A \leq 0$, then $B \succeq A$. Thus the ordering is opposite to SEU-ordering. (We recall that this is not the final ordering of the lotteries, but just ordering generated by the subprocess under consideration.) From the viewpoint of classical SEU-theory, this situation seems to be counterintuitive. The role of phases in quantum-like modeling of lottery

[24]If Alice were acting on the basis of the classical (Kolmogorov) probability and if she were not using the square root weighting of probabilities, then the quantity $\tilde{Q}_{m;A}$ would be equal to the original probability $Q_m = \sum_i p(m|i) P_i$.

selection is a complex question. It was discussed in (Asano et al., 2017), where the $\cos\theta$-factor was associated with risks.

**Process 3:**

The counterpart of the decision process based on comparison corresponding Alice's beliefs about the $B$-lottery is analyzed similarly - again under the assumption of coincidence of all phases: in (24) $\Theta_B \equiv \Theta_{ij;n}$. Consider subjective probabilities of transition from the lottery $B$ to the lottery $A$ given by

$$\tilde{P}_{n;B} = \frac{\sum_i \sqrt{p(n|i)Q_i}}{\sum_{i,n} \sqrt{p(n|i)Q_i}}. \tag{30}$$

(We have $\sum_m \tilde{P}_{n;B} = 1$.) Then we have

$$\frac{\mathrm{tr} D\sigma_B}{2\sum_{i,n}\sqrt{p(n|i)Q_i Q_j}} = \Big[\sum_n u(x_n)\tilde{P}_{n;B} - \sum_j u(y_j)\tilde{Q}_j\Big]\cos\Theta_B \tag{31}$$

$$= [\langle u\rangle_{\tilde{P}_B} - \langle u\rangle_{\tilde{P}}]\cos\Theta_B.$$

Thus the output of this counterpart of the decision process is based on comparison of subjective expected utilities and relative phases. One of the expected utilities is based on the subjective account of probability of the transition (in Alice's mind) from the beliefs about the $B$-lottery to the $A$-lottery and another is so to say straightforward subjective probability based on the square root transform of the initial probabilities $(Q_j)$.

**Process 4:**

Finally, we analyze the most complicated counterpart of the process of decision making corresponding to the comparison term $\Delta_2$. This term compares the utilities of transitions $B \to A$ and $A \to B$ when Alice appeals to the counterparts of her state representing beliefs about these transitions. This process is characterized by ambiguity and intensive fluctuations of Alice's mind in both directions; the intensity of these fluctuations is given by the transition probabilities $p(n_a|j_b) = p(j_b \to n_a)$ and $p(m_b|i_a) = p(i_a \to m_b)$. Correlations between outcomes of lotteries are encoded in the phases $\theta_{j_b\to n_a}, \theta_{i_a\to m_b} = -\theta_{m_b\to i_a}$. Since the general form of dependence of $\Delta_2$ on the phases is very complex, we again assume that

$$\Delta_2 = \sum_{i,j,n,m} u_{nm}\sqrt{p(m_b|i_a)p(n_a|j_b)P_i Q_j}\cos\Gamma, \tag{32}$$

where $\Gamma = \theta_{j_b\to n_a} - \theta_{i_a\to m_b} + \gamma_{n_a\to m_b} + \theta_{bj} - \theta_{ai}$. Now, as in the previous processes, we can represent this comparison term as difference of two expected utilities with respect to the subjective probabilities $(\tilde{P}_{i;B})$ and $(\tilde{Q}_{j;A})$ :

$$\frac{\Delta_2}{\sum_{i,j,n,m}\sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j}}\cos\Gamma$$

$$=\Big[\sum_n u(x_n)\frac{\sum_i\sqrt{p(n|j)Q_j}}{\sum_{j,n}\sqrt{p(n|j)Q_j}}-\sum_m u(y_m)\frac{\sum_i\sqrt{p(m|i)P_i}}{\sum_{i,m}\sqrt{p(m|i)P_i}}\Big]\cos\Gamma=$$

$$[\langle u\rangle_{\tilde{Q}_A}-\langle u\rangle_{\tilde{P}_B}]\cos\Gamma.$$

To finalize this the most complex counterpart of the process of comparison of the lotteries, Alice takes into account the signs of difference between subjective expected utilities and of the interference cos-term.

**Complete process of lottery selection.** Under the simplification assumption about phases, we can write the average of the comparison operator as

$$\langle D\rangle_\Psi=c_1[\langle u\rangle_{\tilde{P}}-\langle u\rangle_{\tilde{Q}}]\cos\Theta+c_2[\langle u\rangle_{\tilde{P}}-\langle u\rangle_{\tilde{Q}_A}]\cos\Theta_A \tag{33}$$

$$+c_3[\langle u\rangle_{\tilde{P}_B}-\langle u\rangle_{\tilde{P}}]\cos\Theta_B+c_4[\langle u\rangle_{\tilde{Q}_A}-\langle u\rangle_{\tilde{P}_B}]\cos\Gamma.$$

where the weights $c_j>0, j=1,2,3,4$, can be found in the above considerations for the subprocesses 1-4.

In the accordance with the decision rule, if $\langle D\rangle\geq 0$, then $A\succeq B$.

# 8 Example: lotteries with two outcomes

Consider two lotteries, $A$ and $B$, having two outcomes $(x_1,x_2)$ and $(y_1,y_2)$ and the utilities, $u_i=u(x_i)$ and $v_j=u(y_j)$. The probabilities $P_i,Q_j$ will be specified later. We start with calculation of the matrix of the comparison operator $D$ in the basis $|1_a\rangle,|2_a\rangle$. By definition we have

$$D=(u_1-v_1)[|1_a\rangle\langle 1_b|+|1_b\rangle\langle 1_a|]+(u_1-v_2)[|1_a\rangle\langle 2_b|+|2_b\rangle\langle 1_a|]$$

$$+(u_2-v_1)[|2_a\rangle\langle 1_b|+|1_b\rangle\langle 2_a|]+(u_2-v_2)[|2_a\rangle\langle 2_b|+|2_b\rangle\langle 2_a|].$$

Consider in the qubit space two bases

$$|1_a\rangle\ =\left(\begin{array}{c}1\\0\end{array}\right),\qquad |2_a\rangle\ =\left(\begin{array}{c}0\\1\end{array}\right). \tag{34}$$

and

$$|1_b\rangle\ =(|1_a\rangle+|2_a\rangle)/\sqrt{2}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\1\end{array}\right),|2_b\rangle\ =(|1_a\rangle-|2_a\rangle)/\sqrt{2}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\-1\end{array}\right). \tag{35}$$

We can find the matrix of the comparison operator $D$ in the basis $|1_a\rangle, |2_a\rangle$ :

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 4u_1 - 2v_1 - 2v_2 & 2u_2 - 2v_1 \\ 2u_2 - 2v_1 & 2v_2 - 2v_1 \end{pmatrix}$$

## 8.1 Starting with the uniform probability distributions

Let now $P_1 = P_2 = Q_1 = Q_2 = 1/2$, i.e.,

$$|\psi_A\rangle = = (|1_a\rangle + |2_a\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |\psi_B\rangle \ = (|1_b\rangle + |2_b\rangle)/\sqrt{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{36}$$

$$|\psi\rangle \ = |\psi_A\rangle + |\psi_B\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}. \tag{37}$$

We obtain the following inequality determining selection of lotteries: $\langle\psi|D|\psi\rangle$

$$= C\Big((1+\sqrt{2})^2 u_1 + (1+\sqrt{2}+1)u_2 - ((1+\sqrt{2})^2 + 2(1+\sqrt{2}) + 1)v_1 - ((1+\sqrt{2})^2 - 1)v_2\Big) =$$

$$= C'\Big([(1+\sqrt{2})u_1 + u_2] - [(1+\sqrt{2})v_1 + v_2]\Big) \geq 0, \tag{38}$$

where $C, C'$ are positive factors.

The main point is that the weights of utilities for both lotteries coincide with the coefficients (up to normalization) in the expansions of the (unnormalized) state $|\psi\rangle$ (which combines Alice's beliefs about the lotteries) with respect to the corresponding bases. We can rewrite the comparison inequality as follows

$$(u_1 \tilde{P}_1 + u_2 \tilde{P}_2) \geq (v_1 \tilde{Q}_1 + v_2 \tilde{Q}_2), \tag{39}$$

where the subjective probabilities are given by the expressions:

$$\tilde{P}_1 = \tilde{Q}_1 = \frac{(1+\sqrt{2})}{2+\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\tilde{P}_2 = \tilde{Q}_2 = \frac{1}{2+\sqrt{2}}.$$

Thus by using the state expansion with respect to the $A$-basis, see (34), and by *treating amplitudes, the coefficients with respect to the $A$-basis, as subjective probabilities*, Alice calculates SEU for the $A$-lottery. We stress that she uses the subjective "probabilities-amplitudes" encoded in the state $|\psi\rangle$ combining beliefs about both lotteries and not probabilities (objective or subjective) encoded in the state $|\psi_A\rangle$ representing solely beliefs about the $A$-lottery. Now we remark that the complete belief state $|\psi\rangle$ can be represented as well in the form:

$$|\psi\rangle = (1+\sqrt{2})|1_b\rangle + |2_b\rangle. \tag{40}$$

Alice uses this representation to calculate SEU for the $B$-lottery.

How can one explain, e.g., the increase of probability of the outcome $x_1$ comparing with the outcome $x_2$? In the state $|\psi_A\rangle$ both these outcomes are equally possible. Now Alice started to combine (by using the rules of quantum logic)[25] the beliefs about the two lotteries and she found that some beliefs about the $B$-lottery encoded in the state $|\psi_B\rangle$ can be interpreted as the beliefs in favor of the outcomes $x_1$ and $x_2$, but the additional weight assigned to $x_1$ is higher than the weight assigned additionally to $x_2$. (In our example the latter is zero.) This is a kind of *constructive and destructive interference* for probabilities assigned to beliefs in favor of the outcomes $x_1$ and $x_2$, respectively. We recall that

$$|\psi_B\rangle \ = (|1_b\rangle + |2_b\rangle)/\sqrt{2} = \frac{1}{2}[(|1_a\rangle + |2_a\rangle) + (|1_a\rangle - |2_a\rangle)]. \tag{41}$$

The beliefs in favor of $x_1$ which are present in the states $|1_b\rangle, |2_b\rangle$ interfere constructively and beliefs in favor of $x_2$ interfere destructively.

In this simple two dimensional example we considered the quantum process of lottery selection straightforwardly as weighting of the utility function with respect to the coefficients in the expansion of the complete belief-state with respect to lotteries' bases and comparison of such weighted sums - 'averages". 'The example is good for the illustrative purpose, since here the coefficients are positive. This gives the possibility to interpret them straightforwardly as probabilities and the weighted sums as averages of SEU. In general such direct coupling to subjective probability is impossible, see appendix 2, where we consider the case of lotteries with arbitrary objective probabilities $P_1, P_2$ and $Q_1, Q_2$. As was shown in section 7, in general such coupling can be established through decomposition of the process of decision making into four subprocess. Each of them can be described in terms SEU with QP-realization of subjective probabilities. The example of this section can (but need not) be considered as a sign that it may be that decision makers do not use the language of probability (neither objective nor subjective) at all, but they proceed with signed or even complex weighting of utility, see appendix 2 for continuation of this discussion.

[25] These rules are formally represented by linear algebra in complex Hilbert space. So, Alice expands $|\psi_B\rangle$ with respect to the $A$-basis.

# 9 Concluding remarks

In this paper we present a novel quantum-like model of lottery selection based on representation of beliefs of an agent by pure quantum states. Subjective probabilities are mathematically realized in the framework of QP. Utility functions are borrowed from the classical theory, but they are represented not only by their values. In the quantum representation each event of realization of an outcome of a lottery and its utility are "blurred". Heuristically one can say that each value $u_i = u(x_i)$ is surrounded by a cloud of information related to the event $(A, x_i)$. An agent process this information by using the rules of quantum information and QP. In general this process is very complex; it combines fluctuations between preferences for different outcomes of lotteries. These fluctuations induce interference type effects (constructive or destructive). The decision process which is formally represented by the comparison operator can be decomposed into a few subprocesses. Each of them can be formally treated as comparison of SEUs and interference factors (the latter express, in particular, risks related to lottery selection).

Model's structure is very rich and further analysis of this structure will demand essential mathematical and interpretational efforts and we plan to continue research in this direction.

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# Appendix 1: calculations of quantum averages

We start with calculation of average $\mathrm{tr} D_{B\to A}\sigma_A$.

$$\mathrm{tr} D_{B\to A}\sigma_A = \mathrm{tr}\Big(\sum_{n,m} u_{nm} e^{i\gamma_{m_b\to n_a}}|n_a\rangle\langle m_b|\Big)\Big(\sum_{i,j}\sqrt{P_i P_j}e^{i(\theta_{ai}-\theta_{aj})}|i_a\rangle\langle j_a|\Big)$$

$$= \sum_{i,j,n,m} u_{nm}\sqrt{P_i P_j}e^{i(\theta_{ai}-\theta_{aj}+\gamma_{m_b\to n_a})}\langle j_a|n_a\rangle\langle m_b|i_a\rangle$$

By using the definition of quantum transition (conditional) probabilities, see (4), (5), we obtain:

$$\mathrm{tr} D_{B\to A}\sigma_A = \sum_{i,j,m} u_{jm}\sqrt{p(m_b|i_a)P_iP_j}e^{i(\theta_{i_a\to m_b}+\gamma_{m_b\to j_a}+\theta_{ai}-\theta_{aj})}$$

In the same way we obtain

$$\mathrm{tr} D_{A\to B}\sigma_A = -\mathrm{tr}\Big(\sum_{n,m} u_{nm}e^{i\gamma_{n_a\to m_b}}|m_b\rangle\langle n_a|\Big)\Big(\sum_{i,j}\sqrt{P_iP_j}e^{i(\theta_{ai}-\theta_{aj})}|i_a\rangle\langle j_a|\Big)$$

$$= -\sum_{i,j,n,m} u_{nm}\sqrt{P_iP_j}e^{i(\theta_{ai}-\theta_{aj}+\gamma_{n_a\to m_b})}\langle n_a|i_a\rangle\langle j_a|m_b\rangle$$

$$= -\sum_{i,j,m} u_{im}\sqrt{p(j_a|m_b)P_iP_j}e^{i(\theta_{m_b\to i_a}+\gamma_{i_a\to m_b}+\theta_{ai}-\theta_{aj})}$$

This gives us expression (23) for $\mathrm{tr} D\sigma_A$. The calculations leading to expression (24) for $\mathrm{tr} D\sigma_B$ just repeat the previous ones.

Now we find traces for comparison operators and "transition states":

$$\mathrm{tr} D_{B\to A}\sigma_{B\to A} = \mathrm{tr}\Big(\sum_{n,m} u_{nm}e^{i\gamma_{m_b\to n_a}}|n_a\rangle\langle m_b|\Big)\Big(\sum_{i,j}\sqrt{P_iQ_j}e^{i(\theta_{ai}-\theta_{bj})}|i_a\rangle\langle j_b|\Big)$$

$$\sum_{i,j,n,m} u_{nm}\sqrt{P_iQ_j}e^{i(\theta_{ai}-\theta_{bj}+\gamma_{n_a\to m_b})}\langle j_b|n_a\rangle\langle m_b|i_a\rangle$$

$$= \sum_{i,j,n,m} u_{nm}\sqrt{p(n_a|j_b)p(m_b|i_a)P_iQ_j}e^{i(\theta_{i_a\to m_b}-\theta_{j_b\to n_a}+\gamma_{n_a\to m_b}+\theta_{ai}-\theta_{bj})}.$$

$$\mathrm{tr} D_{A\to B}\sigma_{B\to A} = -\mathrm{tr}\Big(\sum_{n,m} u_{nm}e^{i\gamma_{n_a\to m_b}}|m_b\rangle\langle n_a|\Big)\Big(\sum_{i,j}\sqrt{P_iQ_j}e^{i(\theta_{ai}-\theta_{bj})}|i_a\rangle\langle j_b|\Big)$$

$$= -\sum_{i,j,n,m} u_{nm}\sqrt{P_iQ_j}e^{i(\theta_{ai}-\theta_{bj}+\gamma_{n_a\to m_b})}\langle j_b|m_b\rangle\langle n_a|i_a\rangle$$

$$= -\sum_{i,j} u_{ij}\sqrt{P_iQ_j}e^{i(\theta_{ai}-\theta_{bj}+\gamma_{i_a\to j_b})}.$$

In the same way

$$\mathrm{tr} D_{A\to B}\sigma_{A\to B} = -\sum_{i,j,n,m} u_{nm}\sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j}e^{i(\theta_{j_b\to n_a}-\theta_{i_a\to m_b}+\gamma_{m_b\to n_a}+\theta_{bj}-\theta_{ai})}$$

and

$$\mathrm{tr} D_{B\to A}\sigma_{A\to B} = \mathrm{tr}\Big(\sum_{n,m} u_{nm}e^{i\gamma_{m_b\to n_a}}|n_a\rangle\langle m_b|\Big)\Big(\sum_{i,j}\sqrt{P_iQ_j}e^{i(\theta_{bj}-\theta_{ai})}|j_b\rangle\langle i_a|\Big) =$$

$$\sum_{i,j} u_{ij}\sqrt{P_iQ_j}e^{i(\theta_{bj}-\theta_{ai}+\gamma_{j_b\to i_a})}.$$

Thus

$$\Delta_1 = \frac{1}{2}\mathrm{tr}(D_{B\to A}\sigma_{A\to B} - D_{A\to B}\sigma_{B\to A}) = \sum_{i,j} u_{ij}\sqrt{P_iQ_j}\cos(\theta_{bj}-\theta_{ai}+\gamma_{j_b\to i_a}). \tag{42}$$

Then

$$\Delta_2 = \frac{1}{2}\mathrm{tr}(D_{B\to A}\sigma_{B\to A} - D_{A\to B}\sigma_{A\to B}) =$$

$$\frac{1}{2}\sum_{i,j,n,m} u_{nm}\sqrt{p(n_a|j_b)p(m_b|i_a)P_iQ_j}e^{i(\theta_{i_a\to m_b}-\theta_{j_b\to n_a}+\gamma_{n_a\to m_b}+\theta_{ai}-\theta_{bj})}$$

$$-\frac{1}{2}\sum_{i,j,n,m} u_{nm}\sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j}e^{i(\theta_{j_b\to n_a}-\theta_{i_a\to m_b}-\gamma_{n_a\to m_b}+\theta_{bj}-\theta_{ai})}.$$

Finally, we get

$$\Delta_2 = \sum_{i,j,n,m} u_{nm}\sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j}\cos(\theta_{j_b\to n_a}-\theta_{i_a\to m_b}+\gamma_{n_a\to m_b}+\theta_{bj}-\theta_{ai}). \tag{43}$$

# Appendix 2: lotteries with two outcomes and arbitrary probabilities

We consider the same bases for the lotteries as in section 8, see (34), (35), but now the probabilities $P_1, P_2$ and $Q_1, Q_2$ are arbitrary.

$$|\psi_A\rangle \ = \sqrt{P_1}|1_a\rangle + \sqrt{P_2}|2_a\rangle,\ |\psi_B\rangle \ = \sqrt{Q_1}|1_b\rangle + \sqrt{Q_2}|2_b\rangle. \tag{44}$$

The complete belief-state can be written as

$$|\psi\rangle \ = c_1|1_a\rangle + c_2|2_a\rangle, \tag{45}$$

where

$$c_1 = \sqrt{P_1} + \frac{\sqrt{Q_1}+\sqrt{Q_2}}{\sqrt{2}},\ c_2 = \sqrt{P_2} + \frac{\sqrt{Q_1}-\sqrt{Q_2}}{\sqrt{2}}. \tag{46}$$

Hence, the decision inequality has the form:

$$\langle\psi|D|\psi\rangle \sim \Big([2c_1^2u_1 + 2c_1c_2u_2] - [(c_1+c_2)^2v_1 + (c_1^2-c_2^2)v_2]\Big) \geq 0. \tag{47}$$

It can be rewritten as

$$\langle\psi|D|\psi\rangle \sim 2c_1[c_1u_1 + c_2u_2] - (c_1+c_2)[(c_1+c_2)v_1 + (c_1-c_2)v_2]\Big) \geq 0. \tag{48}$$

Thus Alice assigns to the outcomes of the lotteries the weights $c_1, c_2$ and $d_1 = c_1 + c_2, d_2 = c_1 - c_2$. As we have seen in (45), the weights $c_1, c_2$ correspond to the coefficients in the expansion of the complete state $|\psi\rangle$ with respect to the $A$-basis. It is easy to see that

$$|\psi\rangle = \frac{d_1|1_b\rangle + d_2|2_b\rangle}{\|d_1|1_b\rangle + d_2|2_b\rangle\|}. \tag{49}$$

The comparison inequality (48) can be written as comparison of two subjective utilities with respect to the probabilities:

$$\tilde{P}_1 = \frac{c_1}{c_1 + c_2} = \frac{\sqrt{P_1} + \frac{\sqrt{Q_1}+\sqrt{Q_2}}{\sqrt{2}}}{\sqrt{P_1} + \sqrt{P_2} + \sqrt{2Q_1}}, \tag{50}$$

$$\tilde{Q}_1 = \frac{c_1 + c_2}{2c_1} = \frac{\sqrt{P_1} + \sqrt{P_2} + \sqrt{2Q_1}}{2[\sqrt{P_1} + \frac{\sqrt{Q_1}+\sqrt{Q_2}}{\sqrt{2}}]}, \tag{51}$$

and $\tilde{P}_2 = 1 - \tilde{P}_1, \tilde{Q}_2 = 1 - \tilde{Q}_1$. Thus

$$\langle\psi|D|\psi\rangle \sim \Big((c_1 + c_2)[u_1\tilde{P}_1 + u_2\tilde{P}_2] - 2c_1[v_1\tilde{Q}_1 + v_2\tilde{Q}_2]\Big) \geq 0. \tag{52}$$

However, as was emphasized in section 8, generally the quantities $\tilde{P}_1, \tilde{P}_2$ and $\tilde{Q}_1, \tilde{Q}_2$ cannot be interpreted as probabilities since $\tilde{P}_2$ and $\tilde{Q}_2$ can become negative (and, hence, $\tilde{P}_1$ and $\tilde{Q}_1$ can become larger than 1). Therefore to keep to the probabilistic reasoning, we have to split the process of comparison of the lotteries in the four components (see section 7); each of this component can be interpreted as comparison of two subjective utilities.

At the same time appearance of negative and even complex "probabilities" is rather common in quantum theory, starting with works of Dirac and Wigner, see Khrennikov (2009) for detailed review. One can proceed formally with such signed or complex distributions and develop sufficiently advanced mathematical formalism, including analogs of the central limit theorem and the law of large numbers (Khrennikov, 2009) . Recently signed "probabilities" were actively used to model the proces of decision making (de Barros et al., 2016, 2017). One can consider the calculus of signed (or even complex) distributions as an alternative to the QP-calculus.

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