Which Random Walk is Faster? Methods to Compare Different Step Length Distributions in Individual Animal Movement

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Abstract. Good understanding of individual animal movement is needed in the context of epidemiology in order to predict the rate of spread of infectious diseases. It is also required for problems arising in nature conservation, biological invasion, pest monitoring, etc. A question that often appears in the centre of the movement studies is which movement pattern is ‘faster’ or more efficient. For instance, it is widely believed that the pattern quantified by a power law distribution of movement steps is faster than the Brownian motion. Here we show that the answer to this question may be not so straightforward and depends on the way how different step length distributions are compared.

Keywords and phrases: random walks, probability distributions, individual animal movement

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1. Introduction

Recent decades have seen outbreaks of several dangerous infectious diseases across the world such as Swine influenza [1,2], Ebola fever [3], dengue fever [4,5], Zika fever [6] and others. Once a disease outbreak occurs, one immediate task is to localize it in space, i.e. to block its spatial spread or, in case blocking is not possible, at least to predict the direction and rate of the spread. The latter is important as it provides an estimate of the time that is available to local authorities and medical units to mobilize their resources. On a global scale, it also contributes to creating an efficient framework for epidemics forecast and control [7]. Therefore, there is a need for a good understanding of the factors controlling the rate of disease spatial spread, which is a timely problem of high practical importance. Factors controlling the disease spread depend on the modes of its transmission. In the case of the human-to-human transmission, peculiarities of human travel are most important [8], including the properties of the relevant transport networks [9]. Alternatively, many diseases are vector-borne, e.g. transmitted by insects; examples are given by malaria, dengue fever and Zika virus. Good understanding of the rates and patterns of insect movement is therefore necessary, and this is one of the reasons why individual animal movement (including insects) has been a focus of intense research and controversy over the last few decades [10–13]. A question that has often

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appeared in the focus of the movement studies is which movement pattern is ‘faster’ - or, ultimately, more efficient, e.g. that allows a foraging animal to find its food item faster or more food items over the same time. For instance, it has been a consensus that Lévy walks are ‘faster’ than Brownian motion. In this paper, however, we revisit this problem and show that the answer in fact is not obvious and depends on the way how different movement patterns are compared.

Mathematical modeling of the individual movement of ‘particles’, whether they are animals, seeds, pollen or bacteria, has long been a major focus in ecology and biology using a variety of stochastic (e.g. random walk) and deterministic (e.g. diffusion equation) theoretical approaches [10,12,14–20]. The choice of the probability density function for the distribution of step length (also referred to as the dispersal kernel) is usually a central part of the model. In the deterministic framework, it affects the type of the equation that describes the evolution of the probability density function to find the moving ‘particle’ at a given location in space, so that the standard diffusion equation may be replaced by a fractional diffusion model [17]. For the explicitly stochastic, individual-based modeling [20?–22], which is the framework that we consider in this paper, the choice of the step length distribution equates to the assumption of the movement pattern such as given, for instance, by the Brownian motion for thin-tailed (an exponentially-bounded) dispersal kernel or Lévy walks [24] for a fat-tailed one (i.e. decaying as a power law with the exponent between 1 and 3) [11]. The latter is thought to be the movement strategy actually employed by many animal species [25–27] as it is known to, under certain conditions, optimize their foraging [28].

Thus, the main theoretical argument behind the choice of a fat-tailed dispersal kernel is that the corresponding dispersing animal moves ‘faster’, i.e. explore the space more efficiently, whilst a thin-tailed kernel is thought to correspond to a slower, less efficient movement [28]. In this paper, however, we argue that this is not necessarily true. We consider several different methods to compare different step length distributions and show that different methods can give different results.

A generic idea behind the comparison of different step length distribution is to equate the corresponding characteristic movement length. Such a characteristic length is usually thought of as the square root of the variance of the dispersal kernel. The processes described by a fat-tailed kernel where the variance does not exist (because of the slow decay at the kernel tail) are often called scale-free [29,30]. However, this terminology is rather misleading as there are several different ways how a characteristic length can be constructed from a given kernel [31]. Moreover, the approach based on equating the variance is not really illuminating as it leaves out of scope the most interesting cases of fat-tailed kernels, instead embracing only the well-known and well-understood movement patterns that are equivalent to the Brownian motion. Therefore, an alternative and less restrictive approach is required allowing us to equate dispersal kernels with different properties. However, it is not at all clear how different ways to compare the step length distribution may affect the conclusion as to which dispersal kernel corresponds to the fastest dispersal.

The distributions we have chosen to compare for this paper are a few variations on the exponential/normal distribution, i.e. the exponential family [34–36], as well as power-law based distributions. The variants of the exponential/normal distribution are similar to e^{-ax^μ} where μ affects the shape and tail of the distribution, how thick or thin the tail is, and a relates to the variance, the normal and exponential distributions are examples of these. There are two types of power-law distributions that we will be looking at; these are distributions based on $(x+h)^{-\mu}$ and $(x^\mu+h^\mu)^{-1}$ where μ again affects the tail of the distribution and h relates to the variance.

2. Method

The idea of our approach is as follows. We will be comparing the distributions by altering the different parameters, σ , λ , a or h , to make a certain characteristic value or quantity, such as the variance, equal for all distributions. Once we have found the parameters that equalize the distributions (using different characteristic quantities) we will then find the mean, the square mean and the variance. These will then be used to calculate the mean squared displacement (MSD), $E(R_n^2)$, either theoretically by using the

Distributions	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$	Variance
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sqrt{\frac{2}{\pi}}\sigma$	σ^2	$\frac{\pi-2}{\pi}\sigma^2$
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$\frac{2\Gamma(\frac{4}{3})}{3\Gamma(\frac{5}{3})}a^{-\frac{2}{3}}$	$\frac{2}{3\Gamma(\frac{5}{3})}a^{-\frac{4}{3}}$	$\frac{6\Gamma(\frac{5}{3})-2(\Gamma(\frac{4}{3}))^2}{9(\Gamma(\frac{5}{3}))^2}a^{-\frac{4}{3}}$
$\lambda e^{-\lambda x}$ or Exponential	λ^{-1}	$2\lambda^{-2}$	λ^{-2}
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$6a^{-2}$	$120a^{-4}$	$84a^{-4}$
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$60a^{-3}$	$20160a^{-6}$	$16560a^{-6}$
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law	$\frac{h}{\mu-2}$	$\frac{2h^2}{(\mu-2)(\mu-3)}$	$\frac{(\mu-1)h^2}{(\mu-2)^2(\mu-3)}$
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$\frac{h}{\sqrt{3}}$	$\frac{h^2}{2}$	$\frac{h^2}{6}$
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$\frac{h}{\sqrt{2}}$	h^2	$\frac{h^2}{2}$
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	h	N/A	N/A
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	N/A	N/A	N/A

TABLE 1. All the distributions with the ways to calculate their mean distance, mean squared distance and variance.

following equation:

$$E(R_n^2) = nE(L^2) + 2E(L)^2 \frac{c}{1-c} \left(n - \frac{1-c^n}{1-c} \right), \quad (2.1)$$

or numerically (using individual based modelling with the chosen step length distribution) if either the variance or the mean do not exist and Eq. 2.1 is irrelevant. Here $E(L^2)$ is the mean square step length, $E(L)$ is the mean step length, c is related to the possible turning angles of the particle and n is the number of steps along the movement path; see [13] for details.

We will be looking at the following distributions. For the distributions based around e^{-ax^μ} we will focus on when $\mu = 2$, the half-normal distribution, $\mu = 1$, the exponential distribution, $\mu = 1.5$, $\mu = 0.5$ and $\mu = \frac{1}{3}$, the last three are a spread of distributions from the exponential family. The $(x+h)^{-\mu}$ based distributions have a general solution which is seen in table 1. As the distributions based on $(x^\mu + h^\mu)^{-1}$ are similar to the exponential family and cannot be generalized, we will be focusing our attention on four different values for μ : $\mu = 6$, $\mu = 4$, $\mu = 3$ and $\mu = 2$, the last two are fat-tailed distributions and thus will need to be simulated to compare them to the other distributions.

The first two ways in which we will be comparing distributions will be ‘standard’, i.e. by changing the different parameters, σ , λ , a or h , to give them equal square mean or variance. In order to include fatter-tailed distribution where the variance does not exist, we also consider a third way to compare distributions which will be based on equating the mean. Finally, aiming to further extend the list to include the fat-tailed distributions where the mean does not exist either, we consider a fourth way where we will be changing the parameters to equate the characteristic distance y defined by the following equation:

$$\int_0^y P(x, a) dx = C, \quad (2.2)$$

where parameter C is therefore the probability to find the dispersing animal within the given distance y from its original location (i.e. before dispersal), cf. [32, 33]. In the context of a biological system - say a natural reserve or nature protection area - C can be defined by an acceptable risk that the dispersing animal will leave the safe area of the size y .

FIGURE 1. Plots of the frequency of all the distributions when they all have variance equal to one. Part a and b show the distributions based on e^{-x^μ} . Part c shows the distributions based on $(x+h)^{-\mu}$. Part d shows the distributions based on $(x^\mu + h^\mu)^{-1}$.

Equation 2.2 gives the formula for working out the parameters of the compared step length distributions when using this method. $P(x, a)$ is the distribution being compared, a is the parameter of the distribution, x is the step length and y is the limit to the step length. This calculates the frequency of the step lengths between 0 and y and makes it equal to constant, C . Obviously for some distributions the square mean and variance do not exist, i.e. is ∞ , for $\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ where $\mu \leq 3$, $\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$ and $\frac{2h}{\pi}(x^2+h^2)^{-1}$, and even the mean does not exist for $\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ where $\mu \leq 2$ and $\frac{2h}{\pi}(x^2+h^2)^{-1}$. So the fourth method is already looking like a better way for comparing these distributions.

Distributions	Parameter	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$	Variance	Fastest
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = \sqrt{\frac{\pi}{\pi-2}}$	$\sqrt{\frac{2}{\pi-2}} \approx 1.324$	$\frac{\pi}{\pi-2} \approx 2.752$	1	2nd
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = \left(\frac{6\Gamma(\frac{5}{3})-4(\Gamma(\frac{4}{3}))^2}{9(\Gamma(\frac{5}{3}))^2}\right)^{\frac{3}{4}}$	$\frac{2\Gamma(\frac{4}{3})}{\sqrt{6\Gamma(\frac{5}{3})-4(\Gamma(\frac{4}{3}))^2}} \approx 1.197$	$\frac{6\Gamma(\frac{5}{3})}{6\Gamma(\frac{5}{3})-4(\Gamma(\frac{4}{3}))^2} \approx 2.432$	1	3rd
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = 1$	1	2	1	4th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = \sqrt[4]{84}$	$\sqrt{\frac{3}{7}} \approx 0.6546$	$\frac{10}{7} \approx 1.429$	1	6th
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a = \sqrt[6]{16560}$	$\sqrt{\frac{5}{23}} \approx 0.4663$	$\frac{28}{25} = 1.12$	1	7th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$h = (\mu-2) \cdot \sqrt{(\mu-3)(\mu-1)}$	$\sqrt{\frac{\mu-3}{\mu-1}} < 1$	$1 < \frac{2\mu-4}{\mu-1} < 2$	1	Between 4th and 8th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu \leq 3$	N/A	N/A	N/A	N/A	N/A
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h = \sqrt{6}$	$\sqrt{2} \approx 1.414$	3	1	1st
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$h = \sqrt{2}$	1	2	1	4th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	N/A	N/A	N/A	N/A	N/A
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	N/A	N/A	N/A	N/A	N/A

TABLE 2. The mean distance, mean squared distance and variance for various distributions based on e^{-ax^μ} , $(x+h)^{-\mu}$ and $(x^\mu+h^\mu)^{-1}$ with variance equal to one. The rightmost column gives the fastest distributions under the equal variance condition

Distributions	Parameter	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$	Variance	Fastest
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = \sqrt{\frac{\pi}{2}}$	1	$\frac{\pi}{2} \approx 1.571$	$\frac{\pi}{2} - 1^2 \approx 0.5708$	7th
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = \left(\frac{2\Gamma(\frac{4}{3})}{3\Gamma(\frac{5}{3})}\right)^{\frac{3}{2}}$	1	$\frac{3\Gamma(\frac{5}{3})}{2(\Gamma(\frac{4}{3}))^2} \approx 1.698$	$\frac{3\Gamma(\frac{5}{3})}{2(\Gamma(\frac{4}{3}))^2} - 1^2 \approx 0.6981$	6th
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = 1$	1	2	1	4th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = \sqrt{6}$	1	$\frac{10}{3} \approx 3.333$	$\frac{7}{3} \approx 2.333$	3rd
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a = \sqrt[3]{60}$	1	$\frac{28}{5} = 5.6$	$\frac{23}{5} = 4.6$	2nd
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu \geq 3$	$1 \leq h = \mu - 2$	1	$2 < \frac{2\mu-4}{\mu-3}$	$1 < \frac{\mu-1}{\mu-3}$	Between 1st and 4th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu < 3$	N/A	N/A	N/A	N/A	N/A
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h = \sqrt{3}$	1	$\frac{3}{2} = 1.5$	0.5	8th
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$h = \sqrt{2}$	1	2	1	4th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	$h = 1$	1	∞	∞	1st
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	N/A	N/A	N/A	N/A	N/A

TABLE 3. The mean distance, mean squared distance and variance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions and with mean distance equal to one. The rightmost column gives the fastest distributions under the equal mean condition.

Distributions	Parameter	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$	Variance	Fastest
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = 1$	$\sqrt{\frac{2}{\pi}} \approx 0.7979$	1	$1 - \frac{2}{\pi} \approx 0.3634$	2nd
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = \left(\frac{2}{3\Gamma(\frac{5}{3})}\right)^{\frac{3}{4}}$	$\frac{2\Gamma(\frac{4}{3})}{\sqrt{6\Gamma(\frac{5}{3})}} \approx 0.7674$	1	$1 - \left(\frac{2\Gamma(\frac{4}{3})}{\sqrt{6\Gamma(\frac{5}{3})}}\right)^2 \approx 0.4111$	3rd
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \sqrt{2}$	$\frac{1}{\sqrt{2}} \approx 0.7071$	1	$1 - \frac{1}{2} = 0.5$	4th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = \sqrt[4]{120}$	$\sqrt{\frac{3}{10}} \approx 0.5477$	1	$1 - \frac{3}{10} = 0.7$	6th
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a = \sqrt[6]{20160}$	$\sqrt{\frac{5}{28}} \approx 0.4226$	1	$1 - \frac{5}{28} \approx 0.8214$	7th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$\frac{0 < h}{\sqrt{\frac{(\mu-2)(\mu-3)}{2}}}$	$\frac{\sqrt{\frac{\mu-3}{2(\mu-2)}}}{< \frac{1}{\sqrt{2}}} \approx 0.7071$	1	$\frac{1}{2} < 1 - \frac{\mu-3}{2(\mu-2)} = \frac{\mu-1}{2(\mu-2)} < 1$	Between 4th and 7th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu \leq 3$	N/A	N/A	N/A	N/A	N/A
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h = \sqrt{2}$	$\sqrt{\frac{2}{3}} \approx 0.8165$	1	$1 - \frac{2}{3} = 0.3$	1st
$\frac{2\frac{3}{2}h^3}{\pi}(x^4+h^4)^{-1}$	$h = 1$	$\frac{1}{\sqrt{2}} \approx 0.7071$	1	$1 - \frac{1}{2} = 0.5$	4th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	N/A	N/A	N/A	N/A	N/A
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	N/A	N/A	N/A	N/A	N/A

TABLE 4. The mean distance, mean squared distance and variance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions and with mean squared distance equal to 1. The rightmost column gives the fastest distributions under the mean distance condition. So distance will only be different under persistent movement and all distributions will be the same if all angles around the circle have equal chance of being chosen.

FIGURE 2. Plots of the frequency of all the distributions when they all have mean equal to one. Part a and b show the distributions based on e^{-x^μ} . Part c shows the distributions based on $(x+h)^{-\mu}$. Part d shows the distributions based on $(x^\mu + h^\mu)^{-1}$.

3. Results

Tables 2, 3 and 4 show the mean, square mean and variance for when the variance, mean distance and mean square distance are equal respectively. Figures 1, 2 and 3 show the step length distributions in the cases where they were equated using the same variance (Fig.1), the same mean (Fig.2) and the same square mean (Fig.3); see tables 2, 3 and 4 respectively.

Table 5 shows the parameter, mean and square mean when the integral is equal to one half and Table 6 shows the parameter and variance when the integral is equal to one half. This parameter is then used to find the mean distance, mean squared distance and variance so that the distributions can be compared easily. Figures 4, 5 and 6 show how the distributions look when the parameters are set to different values of C , which are shown in the tables 5, .2 and .3 respectively.

Distributions	$y = 1$ and $C = \frac{1}{2}$	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = (\sqrt{2}erf^{-1}(\frac{1}{2}))^{-1}$	$\frac{1}{\sqrt{\pi}erf^{-1}(\frac{1}{2})} \approx 1.183$	$\frac{1}{2(erf^{-1}(\frac{1}{2}))^2} \approx 2.198$
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = (\Gamma^{-1}(\frac{2}{3}, \frac{1}{2}(\Gamma(\frac{2}{3}))))$	$\frac{2\Gamma(\frac{4}{3})}{3(\Gamma^{-1}(\frac{2}{3}, \frac{1}{2}(\Gamma(\frac{2}{3}))))^{\frac{2}{3}}\Gamma(\frac{5}{3})} \approx 1.266$	$\frac{2}{3(\Gamma^{-1}(\frac{2}{3}, \frac{1}{2}(\Gamma(\frac{2}{3}))))^{\frac{4}{3}}\Gamma(\frac{5}{3})} \approx 2.716$
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \ln(2)$	$\frac{1}{\ln(2)} \approx 1.443$	$\frac{2}{\ln(4)} \approx 4.163$
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = (W(\frac{-1}{2e}) + 1)$	$\frac{6}{(W(\frac{-1}{2e}) + 1)^2} \approx 2.130$	$\frac{120}{(W(\frac{-1}{2e}) + 1)^4} \approx 15.12$
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a \approx 2.674$	≈ 3.138	≈ 55.15
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$(\sqrt{2}-1)^{-1} < h$ $((\frac{1}{2})^{\frac{1}{1-\mu}} - 1)^{-1}$	$\frac{1}{(\mu-2)((\frac{1}{2})^{\frac{1}{1-\mu}} - 1)} < 2.414$	$\frac{2}{(\mu-3)(\mu-2)((\frac{1}{2})^{\frac{1}{1-\mu}} - 1)^2}$
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$h = (\sqrt{2}-1)^{-1}$	$\frac{1}{(\sqrt{2}-1)} \approx 2.414$	N/A
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$h = 1$	N/A	N/A
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h \approx 1.904$	≈ 1.099	≈ 1.813
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$h \approx 1.766$	≈ 1.248	≈ 3.117
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	$h \approx 1.559$	≈ 1.559	N/A
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	$h = 1$	N/A	N/A

TABLE 5. The case when the integration between 0 and y is equal to a constant, $C = \frac{1}{2}$. We have the values of the parameter, mean distance and square mean distance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

Distributions	$y = 1$ and $C = \frac{1}{2}$	Variance	Fastest
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = (\sqrt{2}erf^{-1}(\frac{1}{2}))^{-1}$	$\frac{\pi-2}{2\pi(erf^{-1}(\frac{1}{2}))^2} \approx 0.7987$	11th
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = (\Gamma^{-1}(\frac{2}{3}, \frac{1}{2}(\Gamma(\frac{2}{3}))))$	$\frac{6\Gamma(\frac{5}{3})-2\Gamma(\frac{4}{3})}{9(\Gamma^{-1}(\frac{2}{3}, \frac{1}{2}(\Gamma(\frac{2}{3}))))^{\frac{4}{3}}(\Gamma(\frac{5}{3}))^2} \approx 1.117$	10th
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \ln(2)$	$\frac{1}{\ln(4)} \approx 2.081$	8th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = (W(\frac{-1}{2e}) + 1)$	$\frac{84}{(W(\frac{-1}{2e})+1)^4} \approx 10.59$	6th
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a \approx 2.674$	≈ 45.30	5th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$(\sqrt{2}-1)^{-1} < h$ $((\frac{1}{2})^{\frac{1}{1-\mu}} - 1)^{-1}$	$\frac{\mu-1}{(\mu-3)(\mu-2)^2((\frac{1}{2})^{\frac{1}{1-\mu}} - 1)^2}$	Between 4th and 13th
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$h = (\sqrt{2}-1)^{-1}$	N/A	3rd
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$h = 1$	N/A	1st
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h \approx 1.904$	≈ 0.6044	12th
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$h \approx 1.766$	≈ 1.559	9th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	$h \approx 1.559$	N/A	7th
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	$h = 1$	N/A	2nd

TABLE 6. The case when the integration between 0 and y is equal to a constant, $C = \frac{1}{2}$. We have the values of the parameter and variance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

FIGURE 3. Plots of the frequency of all the distributions when they all have square mean equal to one. Part a and b show the distributions based on e^{-x^μ} . Part c shows the distributions based on $(x+h)^{-\mu}$. Part d shows the distributions based on $(x^\mu + h^\mu)^{-1}$.

Figure 7 shows the displacement of various distributions from the exponential family on a log scale for time and displacement. It has no consistencies in the direction and so the lines are straight which is to be expected, due to the mean squared displacement being equal to a constant multiplied by step number as $c = 0$ for equation 2.1, [13]. When the direction of movement has consistency, i.e. figure 8, the lines are more interesting and no longer straight.

Figure 9 and 10 shows the displacement of various distributions, based on $(x+h)^{-\mu}$ and $(x^\mu + h^\mu)^{-1}$, on a log scale for time and displacement. It has no consistencies in the direction and so the lines are straight which is to be expected as before, for figure 7. The erratic nature of the lines for the distribution based on $(x+h)^{-2}$ and $(x^2 + h^2)^{-1}$ is due to these distributions having very fat tails, which means very

FIGURE 4. Plots of the frequency of all the distributions when they all have $y = 1$ and $C = \frac{1}{2}$. Part a and b show the distributions based on e^{-x^μ} . Part c shows the distributions based on $(x + h)^{-\mu}$. Part d shows the distributions based on $(x^\mu + h^\mu)^{-1}$.

high step lengths are relatively common. Thus vast numbers of simulations have to be run to smooth out the line.

4. Discussion and conclusions

In this paper, we considered several step length distributions (dispersal kernels) that are often used to describe individual movement of a ‘particle’ (such as animal, seed or pollen). We used several somewhat different ways to make the kernels equivalent by equating a chosen characteristic quantity or characteristic length. For each of those ways, we then calculated the MSD as a function of time and identified the dispersal kernel that corresponds to the fastest growing MSD. Interestingly and rather counter-intuitively,

FIGURE 5. Plots of the frequency of all the distributions when they all have $y = 1$ and $C = \frac{9}{10}$. Part a and b show the distributions based on e^{-x^μ} . Part c shows the distributions based on $(x + h)^{-\mu}$. Part d shows the distributions based on $(x^\mu + h^\mu)^{-1}$.

we found that the kernel corresponding to the fastest growing MSD, and hence to the fastest dispersal, depends on how the kernels are equated. Hence the conclusion as to what is the ‘most efficient’ movement pattern appears to be relative rather than absolute. It can depend on the ecological context and/or the details of a specific habitat and species traits.

In particular, we mention here that, although the half-normal distribution is usually considered to be slower than other distributions, such as the exponential or standard power-laws, because it has a much faster rate of decay at large distances, if we equate the variance, table 2, of such distributions then the half-normal is faster than other common distributions. This is interesting to note, as it is contrary to initial thoughts on distributions. However, if any other method of comparison, shown here, is used then the normal distribution is not the fastest.

FIGURE 6. Plots of the frequency of all the distributions when they all have $y = 1$ and $C = \frac{1}{10}$. Part a and b show the distributions based on e^{-x^μ} . Part c shows the distributions based on $(x + h)^{-\mu}$. Part d shows the distributions based on $(x^\mu + h^\mu)^{-1}$.

The new method detailed in this paper for comparing distributions, (see Eq. 2.2 and the comments after the equation), is more applicable than other methods such as equating the mean or variance, as it can be applied to more distributions. However, the distributions this allows to be compared are fat-tailed and often questioned, [37–39], as to their relevance to step lengths. So this increase in the range of distributions able to be compared does not seem to be important. Despite this, there are other advantages to this method; such as no longer needing to calculate the expected step length which skews the earlier methods of comparison.

An interesting observation, if we look back at tables 2, 3 and 4, is that we see the exponential distribution, $\lambda e^{-\lambda x}$, and the $\frac{2\frac{3}{2}h^3}{\pi} (x^4 + h^4)^{-1}$ distribution have the same mean, square mean and variance when compared in these ways. The power-law distribution, $\frac{\mu-1}{h^{1-\mu}} (x + h)^{-\mu}$, also has the same mean, square

FIGURE 7. The predicted mean squared displacement, [13], for various distributions based on e^{-ax^μ} all with $y = 1$ and $C = \frac{1}{2}$. There is no consistency to the direction of movement

FIGURE 8. The predicted mean squared displacement, [13], for various distributions based on e^{-ax^μ} all with $y = 1$ and $C = \frac{1}{2}$. Here there is quite high consistency to the direction of movement

mean and variance when compared in these ways when μ tends to infinity. Figures 1, 2 and 3 show that the frequency of step lengths is very different, so while these distributions are very similar when choosing a distribution to base the step length on, they are crucially different and picking the ‘wrong’ one would impact the information that can be gained from such simulations. However, they are different when the integral between 0 and y is equal, as seen in tables 5, .2 and .3.

Another thing of interest is that the order, fastest to slowest, of all the distributions is reversed between when they have equal variance and when they have equal mean. Also the order from when they have equal mean is kept for the square mean and mostly for when the integral between 0 and y is equal.

FIGURE 9. The predicted and estimated mean squared displacement, [13], for various distributions based on $(x+h)^{-\mu}$ all with $y = 1$ and $C = \frac{1}{2}$. There is no consistency to the direction of movement. The lines for the lower powers, i.e. $(x+h)^{-3}$ and $(x+h)^{-2}$, are estimated by averaging a large number of randomly generated paths, the other lines are produced using the MSD described by Eq. (2.1).

FIGURE 10. The predicted and estimated mean squared displacement, [13], for various distributions based on $(x^\mu + h^\mu)^{-1}$ all with $y = 1$ and $C = \frac{1}{2}$. There is no consistency to the direction of movement. The lines for the lower powers, i.e. $(x^3 + h^3)^{-1}$ and $(x^2 + h^2)^{-1}$, are estimated by averaging a large number of randomly generated paths, the other lines are produced using the MSD described by Eq. (2.1).

Interestingly for the case of $C = \frac{9}{10}$ the half-normal distribution is faster than the distribution based on $e^{-ax^{1.5}}$, which is unexpected because the half-normal distribution has a larger power. This is due to the shape of the distributions near the origin and is only the case with value of C close to 1.

For ease of comparison table 7 shows the fastest distributions for each method of comparison and for three values of C . *We can clearly see that different methods for comparing distributions result in different orders of magnitudes of displacement.* Interestingly the fat-tailed distributions, distributions that lack a variance, are not always the fastest when C has different values. For example when $C = \frac{1}{10}$ the distribution $\frac{a^3}{6}e^{-a\sqrt[3]{x}}$ is faster than $2h^2(x+h)^{-3}$ and $\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$ both of which lack variances.

Table .1 shows the value of the integral between 0 and y for the general case and the three cases we are focusing on.

Tables .2 and .3 show the parameter, mean distance and square mean distance when the integral is equal to $\frac{9}{10}$ and $\frac{1}{10}$ respectively as in table 5. While tables .4 and .5 show the variance and order of fastest when the integral is equal to $\frac{9}{10}$ and $\frac{1}{10}$ respectively as in table 6.

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Distributions	Equal Variance	Equal Mean	Equal Square Mean	If $C = \frac{1}{2}$	If $C = \frac{9}{10}$	If $C = \frac{1}{10}$
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	2nd	8th	2nd	11th	11th	11th
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	3rd	7th	3rd	10th	12th	9th
$\lambda e^{-\lambda x}$ or Exponential	4th	5th	4th	8th	10th	8th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	6th	4th	6th	6th	7th	6th
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	7th	3rd	7th	5th	6th	3rd
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	Between 4th and 8th	Between 2nd and 5th	Between 4th and 8th	Between 4th and 13th	Between 5th and 13th	Between 5th and 13th
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	N/A	1st	N/A	3rd	4th	4th
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	N/A	N/A	N/A	1st	1st	1st
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	1st	9th	1st	12th	9th	12th
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	4th	5th	4th	9th	8th	10th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	N/A	1st	N/A	7th	3rd	7th
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	N/A	N/A	N/A	2nd	2nd	2nd

TABLE 7. Looking at which is the fastest for different methods of comparison between all the distributions.

Distributions	General Case	If $C = \frac{1}{2}$	If $C = \frac{9}{10}$	If $C = \frac{1}{10}$
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$y = \sqrt{2}\sigma \cdot \text{erf}^{-1}(C)$	$y \approx 0.6745\sigma$	$y \approx 1.645\sigma$	$y \approx 0.1257\sigma$
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$y = \left(\frac{1}{a}\Gamma^{-1}\left(\frac{2}{3}, (1-C)\left(\Gamma\left(\frac{2}{3}\right)\right)\right)\right)^{\frac{2}{3}}$	$y \approx 0.5215a^{-\frac{2}{3}}$	$y \approx 1.420a^{-\frac{2}{3}}$	$y \approx 0.0913a^{-\frac{2}{3}}$
$\lambda e^{-\lambda x}$ or Exponential	$y = \frac{-1}{\lambda} \ln(1-C)$	$y \approx 0.6931\lambda^{-1}$	$y \approx 2.303\lambda^{-1}$	$y \approx 0.1054\lambda^{-1}$
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$y = \frac{(W(\frac{C-1}{a^2})+1)^2}{a^2}$	$y \approx 2.817a^{-2}$	$y \approx 15.13a^{-2}$	$y \approx 0.2828a^{-2}$
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	Solved numerically	$y \approx 19.12a^{-3}$	$y \approx 150.7a^{-3}$	$y \approx 1.338a^{-3}$
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$y = h \left((1-C)^{\frac{1}{1-\mu}} - 1 \right)$	$0 < y \approx 0.4142h$	$0 < y \approx 2.162h$	$0 < y \approx 0.0541h$
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$y = h \left((1-C)^{-\frac{1}{2}} - 1 \right)$	$y \approx 0.4142h$	$y \approx 2.162h$	$y \approx 0.0541h$
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$y = h \left((1-C)^{-1} - 1 \right)$	$y = h$	$y = 9h$	$y \approx 0.1111h$
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	Solved numerically	$y \approx 0.5251h$	$y \approx 1.089h$	$y \approx 0.1047h$
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	Solved numerically	$y \approx 0.5664h$	$y \approx 1.394h$	$y \approx 0.1111h$
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	Solved numerically	$y \approx 0.6415h$	$y \approx 1.985h$	$y \approx 0.1210h$
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	$y = h \cdot \tan\left(\frac{\pi C}{2}\right)$	$y = h$	$y \approx 6.314h$	$y \approx 0.1584h$

TABLE .1. The general case of the integration between 0 and y being equal to a constant C and the cases where $C = \frac{1}{2}$, $\frac{9}{10}$ and $\frac{1}{10}$ for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

Distributions	$y = 1$ and $C = \frac{9}{10}$	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = (\sqrt{2}\text{erf}^{-1}(\frac{9}{10}))^{-1}$	$\frac{1}{\sqrt{\pi}\text{erf}^{-1}(\frac{9}{10})} \approx 0.4851$	$\frac{1}{2(\text{erf}^{-1}(\frac{9}{10}))^2} \approx 0.3696$
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = (\Gamma^{-1}(\frac{2}{3}, \frac{1}{10}(\Gamma(\frac{2}{3}))))$	$\frac{2\Gamma(\frac{4}{3})}{3(\Gamma^{-1}(\frac{2}{3}, \frac{1}{10}(\Gamma(\frac{2}{3}))))^{\frac{2}{3}}\Gamma(\frac{5}{3})} \approx 0.4643$	$\frac{2}{3(\Gamma^{-1}(\frac{2}{3}, \frac{1}{10}(\Gamma(\frac{2}{3}))))^{\frac{4}{3}}\Gamma(\frac{5}{3})} \approx 0.3661$
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \ln(10)$	$\frac{1}{\ln(10)} \approx 0.4343$	$\frac{2}{(\ln(10))^2} \approx 0.3772$
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = (W(\frac{-1}{10e}) + 1)$	$\frac{6}{(W(\frac{-1}{10e}) + 1)^2} \approx 0.3966$	$\frac{120}{(W(\frac{-1}{10e}) + 1)^4} \approx 0.5242$
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a \approx 5.322$	≈ 0.3980	≈ 0.8872
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$(\sqrt{10}-1)^{-1} < h$ $= ((\frac{1}{10})^{\frac{1}{1-\mu}} - 1)^{-1}$	$\frac{1}{(\mu-2)((\frac{1}{10})^{\frac{1}{1-\mu}} - 1)} < 0.4625$	$\frac{2}{(\mu-3)(\mu-2)((\frac{1}{10})^{\frac{1}{1-\mu}} - 1)^2}$
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$h = (\sqrt{10}-1)^{-1}$	$\frac{1}{(\sqrt{10}-1)} \approx 0.4625$	N/A
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$h = \frac{1}{9}$	N/A	N/A
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h \approx 0.9185$	≈ 0.5303	≈ 0.4218
$\frac{2^{\frac{2}{3}}h^3}{\pi}(x^4+h^4)^{-1}$	$h \approx 0.7174$	≈ 0.5073	≈ 0.5146
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	$h \approx 0.5039$	≈ 0.5039	N/A
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	$h = 0.1584$	N/A	N/A

TABLE .2. The case when the integration between 0 and y is equal to a constant, $C = \frac{9}{10}$. We have the values of the parameter, mean distance and square mean distance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

Distributions	$y = 1$ and $C = \frac{1}{10}$	Mean Distance - $E(x)$	Mean Squared Distance - $E(x^2)$
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = (\sqrt{2}erf^{-1}(\frac{1}{10}))^{-1}$	$\frac{1}{\sqrt{\pi}erf^{-1}(\frac{1}{10})} \approx 6.350$	$\frac{1}{2(erf^{-1}(\frac{1}{10}))^2} \approx 63.33$
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = (\Gamma^{-1}(\frac{2}{3}, \frac{9}{10}(\Gamma(\frac{2}{3}))))$	$\frac{2\Gamma(\frac{4}{3})}{3(\Gamma^{-1}(\frac{2}{3}, \frac{9}{10}(\Gamma(\frac{2}{3}))))^{\frac{2}{3}}\Gamma(\frac{5}{3})} \approx 7.225$	$\frac{2}{3(\Gamma^{-1}(\frac{2}{3}, \frac{9}{10}(\Gamma(\frac{2}{3}))))^{\frac{4}{3}}\Gamma(\frac{5}{3})} \approx 88.65$
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \ln(\frac{10}{9})$	$\frac{1}{\ln(\frac{10}{9})} \approx 9.491$	$\frac{2}{(\ln(\frac{10}{9}))^2} \approx 180.2$
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = (W(\frac{-9}{10e}) + 1)$	$\frac{6}{(W(\frac{-9}{10e}) + 1)^2} \approx 21.21$	$\frac{120}{(W(\frac{-9}{10e}) + 1)^4} \approx 1500$
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a \approx 1.102$	≈ 44.83	≈ 11260
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$\left(\sqrt{\frac{10}{9}} - 1\right)^{-1} < h$ $= \left(\left(\frac{9}{10}\right)^{\frac{1}{1-\mu}} - 1\right)^{-1}$	$\frac{1}{(\mu-2)\left(\left(\frac{9}{10}\right)^{\frac{1}{1-\mu}} - 1\right)} < 18.49$	$\frac{2}{(\mu-3)(\mu-2)\left(\left(\frac{9}{10}\right)^{\frac{1}{1-\mu}} - 1\right)^2}$
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$h = \left(\sqrt{\frac{10}{9}} - 1\right)^{-1}$	$\frac{1}{\left(\sqrt{\frac{10}{9}} - 1\right)} \approx 18.49$	N/A
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$h = 9$	N/A	N/A
$\frac{3h^5}{\pi}(x^6 + h^6)^{-1}$	$h \approx 9.550$	≈ 5.514	≈ 45.60
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4 + h^4)^{-1}$	$h \approx 9.004$	≈ 6.367	≈ 81.06
$\frac{3\sqrt{3}h^2}{2\pi}(x^3 + h^3)^{-1}$	$h \approx 8.266$	≈ 8.266	N/A
$\frac{2h}{\pi}(x^2 + h^2)^{-1}$	$h = 6.314$	N/A	N/A

TABLE .3. The case when the integration between 0 and y is equal to a constant, $C = \frac{1}{10}$. We have the values of the parameter, mean distance and square mean distance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4 + h^4)^{-1}$, $(x^6 + h^6)^{-1}$, $(x^3 + h^3)^{-1}$ and $(x^2 + h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

Distributions	$y = 1$ and $C = \frac{9}{10}$	Variance	Fastest
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = (\sqrt{2}erf^{-1}(\frac{9}{10}))^{-1}$	$\frac{\pi-2}{2\pi(erf^{-1}(\frac{9}{10}))^2} \approx 0.1343$	11th
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}e^{-ax^{1.5}}$	$a = (\Gamma^{-1}(\frac{2}{3}, \frac{1}{10}(\Gamma(\frac{2}{3}))))$	$\frac{6\Gamma(\frac{5}{3})-2\Gamma(\frac{4}{3})}{9(\Gamma^{-1}(\frac{2}{3}, \frac{1}{10}(\Gamma(\frac{2}{3}))))^{\frac{4}{3}}(\Gamma(\frac{5}{3}))^2} \approx 0.1505$	12th
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \ln(10)$	$\frac{1}{(\ln(10))^2} \approx 0.1886$	10th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = (W(\frac{-1}{10e}) + 1)$	$\frac{84}{(W(\frac{-1}{10e})+1)^4} \approx 0.3669$	7th
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a \approx 5.322$	≈ 0.7288	6th
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$(\sqrt{10}-1)^{-1} < h$ $= ((\frac{1}{10})^{\frac{1}{1-\mu}} - 1)^{-1}$	$\frac{\mu-1}{(\mu-3)(\mu-2)^2((\frac{1}{10})^{\frac{1}{1-\mu}} - 1)^2}$	Between 5th and 13th
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$h = (\sqrt{10}-1)^{-1}$	N/A	4th
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$h = \frac{1}{9}$	N/A	1st
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h \approx 0.9185$	≈ 0.1406	9th
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$h \approx 0.7174$	≈ 0.2573	8th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	$h \approx 0.5039$	N/A	3rd
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	$h = 0.1584$	N/A	2nd

TABLE .4. The case when the integration between 0 and y is equal to a constant, $C = \frac{9}{10}$. We have the values of the parameter and variance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

Distributions	$y = 1$ and $C = \frac{1}{10}$	Variance	Fastest
$\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ or Half-Normal	$\sigma = (\sqrt{2}erf^{-1}(\frac{1}{10}))^{-1}$	$\frac{\pi-2}{2\pi(erf^{-1}(\frac{1}{10}))^2} \approx 23.01$	11th
$\frac{a^{\frac{2}{3}}}{\Gamma(\frac{2}{3})}e^{-ax^{1.5}}$	$a = (\Gamma^{-1}(\frac{2}{3}, \frac{9}{10}(\Gamma(\frac{2}{3}))))$	$\frac{6\Gamma(\frac{5}{3})-2\Gamma(\frac{4}{3})}{9(\Gamma^{-1}(\frac{2}{3}, \frac{9}{10}(\Gamma(\frac{2}{3}))))^{\frac{4}{3}}(\Gamma(\frac{5}{3}))^2} \approx 36.44$	9th
$\lambda e^{-\lambda x}$ or Exponential	$\lambda = \ln(\frac{10}{9})$	$\frac{1}{(\ln(\frac{10}{9}))^2} \approx 90.08$	8th
$\frac{a^2}{2}e^{-a\sqrt{x}}$	$a = (W(\frac{-9}{10e}) + 1)$	$\frac{84}{(W(\frac{-9}{10e}) + 1)^4} \approx 1050$	6th
$\frac{a^3}{6}e^{-a\sqrt[3]{x}}$	$a \approx 1.102$	≈ 9246	3rd
$\frac{\mu-1}{h^{1-\mu}}(x+h)^{-\mu}$ or Power-Law $\mu > 3$	$\left(\sqrt{\frac{10}{9}} - 1\right)^{-1} < h$ $= \left(\left(\frac{9}{10}\right)^{\frac{1}{1-\mu}} - 1\right)^{-1}$	$\frac{\mu-1}{(\mu-3)(\mu-2)^2 \left(\left(\frac{9}{10}\right)^{\frac{1}{1-\mu}} - 1\right)^2}$	Between 5th and 13th
$2h^2(x+h)^{-3}$ or Power-Law $\mu = 3$	$h = \left(\sqrt{\frac{10}{9}} - 1\right)^{-1}$	N/A	4th
$h(x+h)^{-2}$ or Power-Law $\mu = 2$	$h = 9$	N/A	1st
$\frac{3h^5}{\pi}(x^6+h^6)^{-1}$	$h \approx 9.550$	≈ 15.20	12th
$\frac{2^{\frac{3}{2}}h^3}{\pi}(x^4+h^4)^{-1}$	$h \approx 9.004$	≈ 40.53	10th
$\frac{3\sqrt{3}h^2}{2\pi}(x^3+h^3)^{-1}$	$h \approx 8.266$	N/A	7th
$\frac{2h}{\pi}(x^2+h^2)^{-1}$	$h = 6.314$	N/A	2nd

TABLE .5. The case when the integration between 0 and y is equal to a constant, $C = \frac{1}{10}$. We have the values of the parameter and variance for distributions based on $e^{-ax^{1.5}}$, $e^{-a\sqrt{x}}$, $e^{-a\sqrt[3]{x}}$, $(x^4+h^4)^{-1}$, $(x^6+h^6)^{-1}$, $(x^3+h^3)^{-1}$ and $(x^2+h^2)^{-1}$ and the half-normal, exponential and standard power-law distributions.

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