Universal coefficient theorem in triangulated categories

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Abstract. We consider a homology theory $h : \mathsf{T} \to \mathsf{A}$ on a triangulated category T with values in a graded abelian category A. If the functor h reflects isomorphisms, is full and is such that for any object x in A there is an object X in T with an isomorphism between $h(X)$ and x, we prove that A is a hereditary abelian category, all idempotents in T split and the kernel of h is a square zero ideal which as a bifunctor on T is isomorphic to $\mathsf{Ext}^1_{\mathscr{A}}(h(-)[1], h(-))$.

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We assume that the reader is familiar with triangulated categories (see [7], [4]). Let us just recall that the triangulated categories were introduced independently by Puppe [6] and by Verdier [7]. Following to Puppe we do not assume that the octahedral axiom holds.

If T is a triangulated category, the shifting of an object $X \in \mathsf{T}$ is denoted by $X[1]$. Assume an abelian category A is given, which is equipped with an auto-equivalence $x \mapsto x[1]$. Objects of A are denoted by the small letters x, y, z, etc, while objects of T are denoted by the capital letters X, Y, Z, etc. A *homology theory on* T *with values in* A is a functor $h : \mathsf{T} \to \mathsf{A}$ such that h commutes with shifting (up to an equivalence) and for any distinguished triangle $X \to Y \to Z \to X[1]$ in T the induced sequence $h(X) \to h(Y) \to h(Z)$ is exact. It follows that then one has the following long exact sequence

$$\cdots \to h(Z)[-1] \to h(X) \to h(Y) \to h(Z) \to h(X)[1] \to \cdots$$

In what follows $\mathsf{Ext}^1_{\mathscr{A}}(x, y)$ denotes the equivalence classes of extensions of x by y in the category A and we assume that these classes form a set.

In this paper we prove the following result:

THEOREM 1. *Let $h : \mathsf{T} \to \mathsf{A}$ be a homology theory. Assume the following conditions hold*

i) h reflects isomorphisms,

ii) h is full.

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Then the ideal

$$\mathbb{I} = \{f \in \mathsf{Hom}_{\mathscr{T}}(X,Y) \mid h(f) = 0\}$$

is a square zero ideal. Suppose additionally the following condition holds

> *iii) for any short exact sequence* $0 \to x \to y \to z \to 0$ *in* A *with* $x \cong h(X)$ *and* $z \cong h(Z)$ *there is an object* $Y \in \mathsf{T}$ *and an isomorphism* $h(Y) \cong y$ *in* A *.*

Then $\mathbb{I}$ *is isomorphic as a bifunctor on* T *to*

$$(X,Y) \mapsto \mathsf{Ext}^1_{\mathscr{A}}(h(X)[1], h(Y)).$$

In particular for any $X, Y \in \mathsf{T}$ *one has the following short exact sequence*

$$0 \to \mathsf{Ext}^1_{\mathscr{A}}(h(X)[1], h(Y)) \to \mathsf{T}\ (X,Y) \to \mathsf{Hom}_{\mathscr{A}}(h(X), h(Y)) \to 0.$$

Moreover, if we replace condition (iii) by the stronger condition

> *iv) for any object* $x \in \mathsf{A}$ *there is an object* $X \in \mathsf{T}$ *and an isomorphism* $h(X) \cong x$ *in* A *,*

then A *is a hereditary abelian category and all idempotents in* T *split.*

Thus this is a sort of "universal coefficient theorem" in triangulated categories.

Our result is a one step generalization of a well-known result which claims that if h is an equivalence of categories then A is *semi-simple* meaning that $\mathsf{Ext}^1_{\mathscr{A}} = 0$ (see for example [4, p. 250]). As was pointed out by J. Daniel Christensen our theorem generalizes Theorem 1.2 and Theorem 1.3 of [3] on phantom maps. Indeed let S be the homotopy category of spectra or, more generally, a triangulated category satisfying axioms 2.1 of [3] and let A be the category of additive functors from finite objects of S to the category of abelian groups. The category A has a shifting, which is given by $(F[1])(X) = F(X[1])$, $F \in \mathsf{A}$. Moreover let $h : \mathsf{S} \to \mathsf{A}$ be a functor given by $h(X) = \pi_0(X \wedge (-))$. Then h is a homology theory for which the assertions i)-iii) hold and $\mathbb{I}(X,Y)$ consists of phantom maps from X to Y. Hence by the first part of theorem we obtain the familiar properties of phantom maps.

Before we give a proof of the Theorem, let us explain notations involved on it. The functor h reflects isomorphisms, this means that $f \in \mathsf{Hom}_{\mathscr{T}}(X,Y)$ is an isomorphism provided $h(f)$ is an isomorphism in A .

This holds if and only if $X=0$ as soon as $h(X)=0$. Moreover h is full, this means that the homomorphism $\mathsf{T}\,(X,Y)\to \mathsf{Hom}_{\mathscr{A}}(h(X),h(Y))$ given by $f\mapsto h(f)$ is surjective for all $X,Y\in\mathsf{T}$. Furthermore an abelian category A is *hereditary* provided for any two-fold extension

$$0\longrightarrow u\xrightarrow{\hat\alpha} v\xrightarrow{\hat\beta} w\xrightarrow{\hat\gamma} x\longrightarrow 0 \tag{1}$$

there exists a commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & v \arrow[r] \arrow[d, "\mathsf{Id}"'] & z \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, "\mathsf{Id}"'] & 0 \\
0 \arrow[r] & u \arrow[r] & v \arrow[r] & w \arrow[r] & x \arrow[r] & 0
\end{tikzcd}$$

This exactly means that $\mathsf{Ext}^2_{\mathscr{A}}=0$, where Ext is understood a lá Yoneda. Let us also recall that an ideal $\mathbb{I}$ in an additive category $\mathbb{A}$ is a sub-bifunctor of the bifunctor $\mathsf{Hom}_{\mathbb{A}}(-,-):\mathbb{A}^{op}\times\mathbb{A}\to\mathsf{Ab}$. It follows that $\mathbb{I}$ is an additive bifunctor. One can form the quotient category $\mathbb{A}/\mathbb{I}$ in an obvious way, which is an additive category. One says that $\mathbb{I}^2=0$ provided $gf=0$ as soon as $f\in\mathbb{I}(A,B)$ and $g\in\mathbb{I}(B,C)$. In this case the bifunctor $\mathbb{I}:\mathbb{A}^{op}\times\mathbb{A}\to\mathsf{Ab}$ factors through the quotient category $\mathbb{A}/\mathbb{I}$ in a unique way.

Proof. It is done in several steps.
First step. The equality $\mathbb{I}^2=0$. To make notations easier we denote $h(X),h(Y)$ simply by x,y, etc. Moreover, for a morphism $\alpha:X\to Y$, we let $\hat\alpha:x\to y$ be the morphism $h(\alpha)$. Suppose $\alpha:X\to Y$ and $\beta:Y\to Z$ are morphisms such that $\hat\alpha=0$ and $\hat\beta=0$. We have to prove that $\gamma:=\beta\alpha$ is the zero morphism. By the morphisms axiom there is a diagram of distinguished triangles

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\mathsf{Id}"] & Y \arrow[r] \arrow[d, "\beta"] & U \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "\mathsf{Id}"] \\
X \arrow[r, "\gamma"] & Z \arrow[r, "\omega"] & V \arrow[r, "\nu"] & X[1]
\end{tikzcd}$$

Apply h to get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & y \arrow[r] \arrow[d, "0"] & u \arrow[r] \arrow[d] & x[1] \arrow[r] \arrow[d, "\mathsf{Id}"] & 0 \\
0 \arrow[r] & z \arrow[r, "\hat\omega"] & v \arrow[r, "\hat\nu"] & x[1] \arrow[r] & 0
\end{tikzcd}$$

It follows that there is a morphism $\hat\mu:x[1]\to v$ in A such that $\hat\nu\hat\mu=\mathsf{Id}_{x[1]}$. Thus $(\hat\omega,\hat\mu):z\oplus x[1]\to v$ is an isomorphism. Since h is full, we can find $\mu : X[1] \to V$ which realizes $\hat{\mu}$, meaning that $h(\mu) = \hat{\mu}$. The morphism $(\omega, \mu) : Z \oplus X[1] \to V$ is an isomorphism, because h reflects isomorphisms. In particular ω is a monomorphism and therefore $\gamma = 0$ and first step is done.

For objects $X, Y \in \mathsf{T}$ we put

$$\mathbb{I}(X,Y) := \{\alpha \in \mathsf{Hom}_{\mathscr{A}}(X,Y) \mid h(\alpha) = 0\}.$$

We have just proved that $\mathbb{I}^2 = 0$. In particular $\mathbb{I}$ as a bifunctor factors through the category $\mathsf{T}/\mathbb{I}$. The next step shows that it indeed factors through the category A and a quite explicit description of this bifunctor is given.

Second step. Bifunctorial isomorphism $\mathbb{I}(X,Y) \cong \mathsf{Ext}^1_{\mathscr{A}}(h(X)[1], h(Y))$. We put as usual $x = h(X)$, $y = h(Y)$, etc. Let $\alpha : X \to Y$ be an element of $\mathbb{I}(X,Y)$. Consider a distinguished triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]. \tag{2}$$

By applying h one obtains the following short exact sequence

$$0 \to y \xrightarrow{\hat{\beta}} z \xrightarrow{\hat{\gamma}} x[1] \to 0 \tag{3}$$

whose class in $\mathsf{Ext}^1_{\mathscr{A}}(x[1], y)$ is independent on the choice of the triangle in (2) and it is denoted by $\Xi(\alpha)$. In this way one obtains the binatural transformation $\Xi : \mathbb{I} \to \mathsf{Ext}^1_{\mathscr{A}}((-)[1], (-))$. We claim that Ξ is an isomorphism. Indeed, if $\Xi(\alpha) = 0$, then there exists a section $\hat{\mu} : x[1] \to z$ of $\hat{\gamma}$ in (3). Then $(\hat{\beta}, \hat{\mu}) : y \oplus x[1] \to z$ is an isomorphism. Since h is full, we can find $\mu : X[1] \to Z$ which realizes $\hat{\mu}$. The morphism $(\beta, \mu) : Y \oplus X[1] \to Z$ is an isomorphism, because h reflects isomorphisms. In particular β is a monomorphism and therefore $\alpha = 0$. Hence Ξ is a monomorphism. Let us take any element in $\mathsf{Ext}^1_{\mathscr{A}}(x[1], y)$, which is represented by a short exact sequence, say the sequence (3). Take any realization $\beta : Y \to Z$ of $\hat{\beta}$. By Lemma 1 below we obtain the following distinguished triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$$

containing β. It follows that $\Xi(\alpha)$ represents our original element in $\mathsf{Ext}^1_{\mathscr{A}}(x[1], y)$. Hence Ξ is an isomorphism.

Third step. A *is hereditary.* Let (1) be a two-fold extension in A. We put $y = \mathsf{Im}(\hat{\alpha})$. Thus the exact sequence (1) splits in the following two short exact sequences

$$0 \to u \xrightarrow{\hat{\alpha}} v \xrightarrow{\hat{\mu}} y \to 0$$

and

$$0 \to y \xrightarrow{\hat{\nu}} w \xrightarrow{\hat{\gamma}} x \to 0$$

with $\hat{\beta} = \hat{\nu}\hat{\mu}$. Using Assumption iii) and without loss of generality we can assume that u, v, w, x as well as $\hat{\alpha}$ and $\hat{\gamma}$ have realizations. By Lemma 1 below we obtain the following distinguished triangles

$$U \xrightarrow{\alpha} V \xrightarrow{\mu} Y \xrightarrow{\xi} U[1]$$

and

$$Y \xrightarrow{\nu} W \xrightarrow{\gamma} X \xrightarrow{\chi} Y[1].$$

Since $\hat{\mu}$ is an epimorphism and $\hat{\nu}$ is a monomorphism it follows that $h(\xi) = 0$ and $h(\chi) = 0$. Thus $\xi \circ \chi[-1] = 0$ thanks to the fact that $\mathbb{I}^2 = 0$. Therefore there exists $\lambda : X[-1] \to V$ such that $\mu \circ \lambda = \chi[-1]$, in other words one has the following commutative diagram

$$\begin{tikzcd}
 & & X[-1] \arrow[dl, "\lambda"'] \arrow[d, "{\chi[-1]}"] & \\
U \arrow[r, "\alpha"] & V \arrow[r, "\mu"] & Y \arrow[r, "\xi"] & U[1]
\end{tikzcd}$$

We claim that one can always find λ with property $h(\lambda) = 0$. Indeed, for a given λ with $\mu \circ \lambda = \chi[-1]$ one obtains the following diagram after applying h:

$$\begin{tikzcd}
 & & & x[-1] \arrow[dl, "\hat{\lambda}"'] \arrow[d, "0"] & \\
0 \arrow[r] & u \arrow[r, "\hat{\alpha}"] & v \arrow[r, "\hat{\mu}"] & y \arrow[r] & 0
\end{tikzcd}$$

Thus $\hat{\lambda} = \hat{\alpha} \circ \hat{\phi}$, for some $\phi : X[-1] \to U$. Now it is clear that $\lambda' = \lambda - \alpha \circ \phi$ has the expected properties $h(\lambda') = 0$ and $\mu \circ \lambda' = \chi[-1]$, and the claim is proved.

One can use the morphisms axiom to conclude that there exists a commutative diagram

$$\begin{tikzcd}
X[-1] \arrow[r, "\lambda"] \arrow[d, "\mathrm{Id}"] & V \arrow[r] \arrow[d, "\mu"] & Z \arrow[r] \arrow[d, dashed] & X \arrow[d, "\mathrm{Id}"] \\
X[-1] \arrow[r, "{\chi[-1]}"] & Y \arrow[r, "\nu"] & W \arrow[r, "\gamma"] & X
\end{tikzcd}$$

Since $h(\lambda) = 0$, by applying h one obtains the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & v \arrow[r] \arrow[d, "\hat{\mu}"'] & z \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & y \arrow[r, "\hat{\nu}"] & w \arrow[r, "\hat{\gamma}"] & x \arrow[r] & 0
\end{tikzcd}$$

which shows that one has a commutative diagram with exact rows

$$\begin{tikzcd} & 0 \arrow[r] & v \arrow[r] \arrow[d, "\mathsf{Id}"'] & z \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, "\mathsf{Id}"'] & 0 \\ 0 \arrow[r] & u \arrow[r, "\hat{\alpha}"] & v \arrow[r, "\hat{\beta}"] & w \arrow[r, "\hat{\gamma}"] & x \arrow[r] & 0 \end{tikzcd}$$

Thus A is hereditary.

Forth step. Idempotents split in T. Let $\mathsf{Idem}(\mathsf{T})$ be the idempotent completion of T (see [5] or [1]). We have to show that the canonical functor $\mathsf{T} \to \mathsf{Idem}(\mathsf{T})$ is an equivalence of categories. One can summarize the previous steps saying that the category T is a linear extension of A by the bifunctor $(X,Y) \mapsto \mathsf{Ext}^1_{\mathscr{A}}(h(X)[1], h(Y))$ in the sense of Baues and Wirsching [2]. Now one can use Proposition 3.2 of [5] to conclude that $\mathsf{T} \to \mathsf{Idem}(\mathsf{T})$ is indeed an equivalence of categories.

An alternative proof can be done using the result of [1] and Corollary 2 below which uses only the first three steps. Indeed, by [1], the category $\mathsf{T}' = \mathsf{Idem}(\mathsf{T})$ carries a natural triangulated structure. Since A is an abelian category, all idempotents in A split and it follows from the universal property of the idempotent completion that the functor h has a unique extension $\mathsf{T}' \to \mathsf{A}$, which is denoted by h'. We claim that the functor h' reflects isomorphisms. Indeed, if X' is an object in T such that $h'(X') = 0$, then there exits an object Y' such that $Z = X' \oplus Y'$ lies in T. Let $e : Z \to Z$ be given by $e(x,y) = (0,y)$. Then $h(Z) = h'(Y')$ and therefore $h(e)$ is an isomorphism. By our assumption on h it follows that e is an isomorphism and hence $X' = 0$. It is clear that h' is full and realizes all objects of A. Hence the conditions of Corollary 2 below hold and therefore $\mathsf{T} \to \mathsf{Idem}(\mathsf{T})$ is an equivalence of categories. $\square$

LEMMA 1. *Let $h : \mathsf{T} \to \mathsf{A}$ be a homology theory. Assume h reflects isomorphisms and is full. Suppose there is given a morphism $\alpha : U \to V$, an object W in T and a short exact sequence*

$$0 \to u \xrightarrow{\hat{\alpha}} v \xrightarrow{\hat{\beta}} w \to 0$$

in A, where as usual $u = h(U), v = h(V), w = h(W)$ and $\hat{\alpha} = h(\alpha)$. Then there exists a distinguished triangle

$$U \xrightarrow{\alpha} V \xrightarrow{\beta} W \xrightarrow{\gamma} U[1]$$

such that $h(\beta) = \hat{\beta}$. The dual statement is also true: Suppose there is given a morphism $\beta : V \to W$, an object U in T and a short exact sequence

$$0 \to u \xrightarrow{\hat{\alpha}} v \xrightarrow{\hat{\beta}} w \to 0$$

in A , *where* $\hat{\beta} = h(\beta)$. *Then there exists a distinguished triangle*

$$U \overset{\alpha}{\to} V \overset{\beta}{\to} W \overset{\gamma}{\to} U[1]$$

such that $h(\alpha) = \hat{\alpha}$.

Proof. Take any distinguished triangle containing α,

$$U \overset{\alpha}{\to} V \overset{\eta}{\to} Z \overset{\epsilon}{\to} U[1].$$

Apply h to get a short exact sequence

$$0 \to u \overset{\hat{\alpha}}{\to} v \overset{\hat{\eta}}{\to} z \to 0.$$

Then we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & u \arrow[r, "\hat{\alpha}"] \arrow[d, "\mathsf{Id}"] & v \arrow[r, "\hat{\eta}"] \arrow[d, "\mathsf{Id}"] & z \arrow[r] \arrow[d, "\hat{\delta}"] & 0 \\
0 \arrow[r] & u \arrow[r, "\hat{\alpha}"] & v \arrow[r, "\hat{\beta}"] & w \arrow[r] & 0
\end{tikzcd}$$

with $\hat{\delta}$ an isomorphism. By assumption one can realize $\hat{\delta}$ to obtain an isomorphism $\delta : Z \to W$, $h(\delta) = \hat{\delta}$. Then we have an isomorphism of triangles

$$\begin{tikzcd}
U \arrow[r, "\alpha"] \arrow[d, "\mathsf{Id}"] & V \arrow[r, "\eta"] \arrow[d, "\mathsf{Id}"] & Z \arrow[r, "\epsilon"] \arrow[d, "\delta"] & U[1] \arrow[d, "\mathsf{Id}"] \\
U \arrow[r, "\alpha"] & V \arrow[r, "\beta"] & W \arrow[r, "\gamma"] & U[1]
\end{tikzcd}$$

where $\beta = \delta\eta$ and $\gamma = \epsilon \circ \delta^{-1}$. It follows that the triangle

$$U \overset{\alpha}{\to} V \overset{\beta}{\to} W \overset{\gamma}{\to} U[1]$$

is also a distinguished triangle. Thus the first statement is proved. The dual argument gives the second result. □

COROLLARY 2. *Let* $j : \mathsf{T} \to \mathsf{T}'$ *be a triangulated functor between triangulated categories. Assume* $h' : \mathsf{T}' \to \mathsf{A}$ *is a homological functor satisfying the conditions i), ii) and iv) of Theorem 1. If the homology functor* $h = h' \circ j : \mathsf{T} \to \mathsf{A}$ *also satisfies the same conditions then* j *is an equivalence of categories.*

Proof. First observe that the functor j is full and faithful because for any pair of objects $X, Y \in \mathsf{T}$ both abelian groups $\mathsf{T}(X, Y)$ and

$\mathsf{T}'(jX, jY)$ are part of the equivalent extensions of $\mathsf{Hom}_{\mathscr{A}}(h(X), h(Y))$ by $\mathsf{Ext}^1_{\mathscr{A}}(h(X)[1], h(Y))$. If now X' is an object in T' then there is an object X in T and an isomorphism $\hat{\alpha} : h(X) \to h'(X')$ in A. But $h(X) = h'(j(X))$ and h' is full so $\hat{\alpha} = h'(\alpha)$ for a morphism $\alpha : jX \to X'$, which is an isomorphism because h' reflects isomorphisms. □

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