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Abstract

Rational expectations is typically taken to mean that, conditional on the information set and the relevant economic theory, the expectation formed by an economic agent should be equal to its mathematical expectation. This is correct only when actual inflation is "linear" in the aggregate inflationary expectation or if it is non-linear then forecasters are "small" and use "causal reasoning". We show that if actual inflation is non-linear in expected inflation and (1) there are "large" forecasters, or, (2) small/ large forecasters who use "evidential reasoning", then the optimal forecast does not equal the mathematical expectation of the variable being forecast. We also show that when actual inflation is non-linear in aggregate inflation there might be no solution if one identifies rational expectations with equating the expectations to the mathematical average, while there is a solution using the "correct" forecasting rule under rational expectations. Furthermore, results suggest that published forecasts of inflation may be systematically different from the statistical averages of actual inflation and output, on average, need not equal the natural rate. The paper has fundamental implications for macroeconomic forecasting and policy, testing the assumptions and implications of market efficiency and for rational expectations in general.

Keywords: Non-linearities, large forecasters, evidential reasoning, rational expectations, endogenous forecasts, classical and behavioral game theory.

JEL Classification: C53 (Forecasting and Other Model Applications), D84 (Expectations; Speculations), E47 (Forecasting and Simulation), E63 (Comparative or Joint Analysis of Fiscal and Monetary Policy).

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1. Introduction

The assumption of rational expectations has been identified, almost exclusively, with the requirement that the expectation formed by an economic agent of a random variable should be equal to the mathematical expectation of that variable as given by the relevant model and conditional on the agent's information set. This result is derived, as in Muth (1961), by assuming that a forecaster aims to minimize the sum of the squares of the forecasting errors, an assumption that we retain in our model.

For the optimal forecast to be equal to the mathematical expectation, either of the following two assumptions, A1 or A2, is crucial, but these are usually left implicit.

A1. **Forecasters are "small" and they use "causal reasoning"**
Forecasters are assumed to be *small* in the sense that their own forecast has no practical significance for the aggregate value of the variable being forecast. If small forecasters fully understand this causal link, we then say that they use *causal reasoning* because they, correctly, perceive that their own forecast will not have a causal affect on the variable being forecast. We define an *exogenous forecast* to be the forecast produced by a (small) forecaster who perceives that her own individual forecast has no practical influence on the value of the random variable being forecast.

A2. **Forecasters are "large" and they use "linear" inflation processes**
The forecaster can be *large*, for instance, a central bank, who should perceive that its forecast will significantly affect the value of variable being forecast[1]. However, the process governing actual inflation is often taken to be a linear function of aggregate inflationary expectations, either by assumption or on account of the structure of the model, for instance in a linear-quadratic framework. The original exposition of Muth (1961) also relied on a linear process for inflation[2].

The purpose of this paper is to highlight the importance of these implicit assumptions in macroeconomic forecasting, to relax them and to examine the resulting implications. We show that, in general, if either of these assumptions is relaxed on its own, or in conjunction with the other, the optimal forecast is no longer the mathematical expectation of the variable. In particular, when the underlying inflation process is non-linear in aggregate inflationary expectations then the optimal forecast is not the mathematical expectation of

[1]For instance, existing homeowners with mortgages as well as entrants into the housing market might alter their plans in light of these forecasts. Credit, and credit rates, extended by commercial banks, and the financial and investment decisions of firms can also by potentially affected by central bank forecasts.

[2]Muth (1961: pp. 317) makes the following assumptions. (1)The random disturbances are normally distributed. (2)Certainty equivalence exists for the variables to be predicted. (3)The equations of the system, including the expectations formulas, are linear.

the variable under either of the following two cases. (1) Large forecasters. (2) Forecasters (large or small) using non-causal reasoning. In particular, we examine the implications of "evidential reasoning", which is defined below. We address each of these issues in turn.

1.1. "Non-Linearities" in Inflation

We relax the linear-quadratic framework, widely used in the literature and which generates an inflation process that is linear in aggregate inflationary expectations. We do so in the following two ways. First we examine the implications of a general, non-linear inflationary process. Second, we construct an example of a non-linear inflation process by considering the possibility of a "liquidity trap" in a standard aggregate demand - aggregate supply model. In its classical form, the liquidity trap is a situation where an economy is caught up in a recession and the nominal interest rates have been driven down to zero. This eliminates the possibility of using monetary policy as a potent instrument.

Interest in the liquidity trap has revived in recent years due, in no small measure, to the experience of Japan since 1990. There is also concern that Germany and France may be heading in a similar direction. Because monetary policy is ineffective in a liquidity trap, reliance must be placed on other policies, such as fiscal policy, exchange rate policy, unconventional monetary policies or more fundamental structural policies; see for example Blinder (2000). Furthermore, in light of the theoretical reformulation by Krugman (1998), the liquidity trap has received new interest in its own right, see for instance, Dhami and al-Nowaihi (2004), Eggerston, G. and Woodford, M. (2003), Svenson (2003), McCallum (2000), and Benhabib, J. Schmitt-Grohe, S. and Uribe, M. (1999). Our approach is neither contrived, nor specific to the model we use; the same results can be achieved by several possible relaxations of the linear-quadratic framework. Rather, our point is more fundamental.

1.2. "Large forecasters", "endogenous forecasts" and "evidential reasoning"

We shall call a forecast produced by a forecaster who perceives that her individual forecast has a significant affect on the value of the random variable being forecast as an *endogenous forecast*. A large forecaster will produce endogenous forecasts.

Evidential reasoning is a well known method of reasoning and inference in the psychological and sociological literature. Unlike under causal reasoning, a forecaster using evidential reasoning takes her own forecast as evidence of how the other forecasters will forecast. *Therefore, a small forecaster who uses evidential reasoning will behave like a large forecaster and produce endogenous forecasts.*

Evidential reasoning was used by Lewis (1979) to explain cooperation in the one shot prisoner's dilemma game. If a player uses causal reasoning then she should defect, because defect is a strictly dominant strategy. However, she would realize that mutual defection is inferior to mutual cooperation. If she uses evidential reasoning, she might take her preference for mutual cooperation as evidence that her rival also has a preference for mutual cooperation, in which case, both players cooperate.

Evidential reasoning was also used by Quattrone and Tversky (1984, 1988) to explain the voting paradox. Given that any one voter is most unlikely to be pivotal, why do so many people actually vote? The explanation runs something like this. If I do not bother to vote for my preferred party, then probably most like-minded people will not, and my preferred party will loose. On the other hand, if I decide to vote then, probably, other like-minded people will also vote and my preferred party has a better chance of winning. So I vote. Quattrone and Tversky (1984) show that experimental evidence is supportive of this view. Hence, these findings show that, in some circumstances at least, people behave as though their actions are causal even when they know them to be merely diagnostic or evidential.

Evidential reasoning can also be used to explain other paradoxes, like giving to charity and paying for public goods. Another application of evidential reasoning is the following. According to the Calvinist doctrine of "predestination", those who are to be saved have been chosen by God at the beginning of time, and nothing that one can do will lead to salvation unless one has been chosen. Although, one cannot increase the chances of salvation by good works, one can produce the evidence of having been chosen by engaging in acts of piety, devotion to duty, hard work and self denial. According to Max Weber, this is exactly how millions of people responded to the Calvinist doctrine and why capitalism developed more quickly in Protestant rather than Catholic countries, an explanation that is widely accepted in sociology; see for example, Nozick (1993).

1.3. Results and schematic outline

We illustrate our arguments at two levels. First, we use a general, abstract, non-linear model of an inflationary process. Second, we use a simple aggregate supply - aggregate demand model with a liquidity trap as an illustrative example of a non-linear inflationary process. Individuals produce the optimal inflation forecast, given the model, in the sense of minimizing the mean of the squares of the forecasting errors. We show that, in the presence of non-linearity in the inflation process, and under either of the following two cases, the optimal forecast of inflation is not equal to the mathematical expectation of inflation.

B1. **Large forecasters**.

B2. **Forecasters using evidential reasoning**.

Hence observed departures of the expectations of agents from the mathematical expectation do not necessarily constitute violation of rationality or market efficiency.

In our illustrative example, we show that a unique solution always exists under "endogenous forecasting", which, in the presence of non-linearities, arises under either of the two cases, B1 or B2, above. However, under some reasonable parameter values, our model fails to have a solution under "exogenous forecasting", which results from using, for instance, assumption A1 above. For other, also reasonable, parameter values the solution under exogenous forecasting exists but differs from that under endogenous forecasting. Furthermore, our results suggest that published forecasts of inflation may be systematically different from the statistical averages of actual inflation and output, on average, need not equal the natural rate.

The rest of the paper is organized as follows. Exogenous and endogenous forecasts, in conjunction with causal and evidential reasoning, are discussed in a general, abstract, model in Section 2. A specific model of aggregate demand - aggregate supply with a liquidity trap is outlined in Section 3 and its results are derived in Section 4 while Section 5 gives some auxiliary results. Section 6 discusses game theoretic foundations in terms of static and dynamic games with small and large forecasters. It also comments on issues relating to common knowledge, equilibrium selection, coordination and the game theoretic consequences of using evidential reasoning. Section 7 summarizes and concludes. All proofs are contained in the appendix.

2. A General Framework

Consider a forecaster wishing to forecast the level of inflation, π. Assume that our forecaster chooses the standard criterion, to minimize the sum of squares of the forecasting errors. Thus, our forecaster adopts the following loss function:

$$\mathcal{L}(\pi^e) = \frac{1}{2}E\left[(\pi^e - \pi)^2\right] \tag{2.1}$$

where π, the actual rate of inflation, is a random variable and depends on the "state of the world". π^e is the forecaster's expectation, or prediction of π, and is constant across "states of the world". E is the expectation operator that takes expectations across states of the world. We will allow π to depend on π^e, which is the case in standard macroeconomic models[3]. The consequences for forecasting, of this dependence of π on π^e, will be the main focus of this paper.

[3]This is consistent with a wide range of macroeconomic models in which the sequence of moves is as follows. Economic agents form expectations of inflation followed by the realization of some random shocks. The actual inflation rate then materializes, for each possible realization of the random shocks, and, possibly, as a function of actual monetary/ fiscal policies chosen by the appropriate authorities.

Of course, other loss functions are possible and have been used, for example

$$\mathcal{L}(\pi^e) = \frac{1}{2}E\left[|\pi^e - \pi|^{\alpha}\right]; \quad 0 < \alpha < \infty \tag{2.2}$$

However, we shall concentrate on the special case, $\alpha = 2$, when (2.2) reduces to (2.1). The latter has several attractive properties, which is probably the reason it has become standard in the literature and in the practice of forecasting in econometrics, macroeconomics and finance.

For simplicity, assume a discrete probability distribution for π. Then (2.1) can be written as

$$\mathcal{L}(\pi^e) = \frac{1}{2}\sum\nolimits_{i=1}^{\infty} p_i (\pi^e - \pi_i)^2, \tag{2.3}$$

where p_i is the probability of state i of the world and π_i is the level of inflation in that state. It is only appropriate to use (2.3) if p_i is independent of π_i. However, π_i may, and typically does, depend on π^e. Differentiating (2.3) with respect to π^e

$$\frac{d\mathcal{L}(\pi^e)}{d\pi^e} = \sum\nolimits_{i=1}^{\infty} p_i (\pi^e - \pi_i)\left(1 - \frac{d\pi_i}{d\pi^e}\right) \tag{2.4}$$

If π^e is unrestricted, then minimizing $\mathcal{L}(\pi^e)$ is equivalent to setting $\frac{d\mathcal{L}(\pi^e)}{d\pi^e} = 0$, which, from (2.4) gives

$$\sum\nolimits_{i=1}^{\infty} p_i (\pi^e - \pi_i)\left(1 - \frac{d\pi_i}{d\pi^e}\right) = 0 \tag{2.5}$$

If $\frac{d\pi_i}{d\pi^e}$ is a constant across states of the world, then (2.5) gives

$$\sum\nolimits_{i=1}^{\infty} p_i (\pi^e - \pi) = 0 \tag{2.6}$$

Since, $\sum\nolimits_{i=1}^{\infty} p_i (\pi^e - \pi_i) = \pi^e \sum\nolimits_{i=1}^{\infty} p_i - \sum\nolimits_{i=1}^{\infty} p_i \pi_i = \pi^e - E[\pi]$, (2.6) gives the following well known result that forms the basis of much of macroeconomic forecasting,

$$\pi^e = E[\pi] \tag{2.7}$$

However, if $\frac{d\pi_i}{d\pi^e}$ is not constant, across states of the world, then the standard implication of rational expectations, namely (2.7), is not correct; the correct forecasting rule in this case is given by solving (2.5). These issues are taken up below.

2.1. Macroeconomic forecasting when the inflation process is linear

The underlying model of inflation could be *linear* in π^e, for instance,

$$\pi_i = \alpha_i + \beta\pi^e, \tag{2.8}$$

where α_i, β are some constants such that $\beta \neq 1$ and β is the same in all states of the world.

2.1.1. Causal reasoning

To explain causal reasoning, consider a *large* forecaster, for example, a central bank or a major institution, whose forecast would significantly affect expected aggregate inflation, π^e. It is reasonable to assume that this forecaster will perceive the effect of its own forecast on π^e and, hence, π. We then say that this forecaster is using causal reasoning because she, correctly, perceives that her own forecast will have a causal affect on π. For the latter reason, we say that this forecaster uses *endogenous forecasting.* Hence, using (2.8),

$$\frac{d\pi_i}{d\pi^e} = \beta, \tag{2.9}$$

which when substituted in (2.5) gives the standard forecasting rule under rational expectations, (2.7).

Now consider a *small* forecaster, one whose own forecast of π is only an insignificant component of expected aggregate inflation, π^e. If this forecaster uses causal reasoning, she should conclude that her own forecast has no practical bearing on π. If all other forecasters are small, she should also conclude that none of the other forecasters has a significant influence on π. In this case we say that she uses *exogenous forecasts* and, hence, she sets

$$\frac{d\pi_i}{d\pi^e} = 0 \tag{2.10}$$

in (2.5) which leads to (2.7).

2.1.2. Evidential reasoning

Suppose now that forecasters (either large or small) use evidential reasoning. Forecasters using evidential reasoning will take their own forecasts as evidence that others are using similar forecasts. Hence, each forecaster behaves as if her own individual forecast has a causal effect on the aggregate forecast, π^e. In this case we say forecasters, large or small, use *endogenous forecasts*, and hence $\frac{d\pi_i}{d\pi^e} = \beta$,which when substituted in (2.5) gives (2.7).

The results in this subsection are summarized in the following Proposition

Proposition 1 *: When the process of actual inflation is linear, as described in (2.8), the endogenous and exogenous forecasts are identical and are given by* $\pi^e = E[\pi]$.

Proposition 1 shows that there is no loss in generality with respect to large forecasters and/ or evidential reasoning in using the forecasting rule $\pi^e = E[\pi]$ provided the underlying inflation process is linear.

2.2. Macroeconomic forecasting when the inflation process is non-linear

Suppose that the process of actual inflation is non-linear and is given by

$$\pi = h\left(\pi^e; \boldsymbol{\epsilon}\right)$$

where $h\left(\pi^e; \boldsymbol{\epsilon}\right)$ is a non-linear differentiable function conditional on some vector of random shocks, $\boldsymbol{\epsilon}$. Hence, while the aggregate inflationary expectation, π^e, is invariant to the actual realization of $\boldsymbol{\epsilon}$, actual inflation, π, is conditional on the actual realization of $\boldsymbol{\epsilon}$. The analogue of (2.5) in this case is given by

$$\sum\nolimits_{i=1}^{\infty} p_i \left(\pi^e - \pi_i\right)\left(1 - h'\left(\pi^e; \epsilon_i\right)\right) = 0 \tag{2.11}$$

where ϵ_i is the realization of $\boldsymbol{\epsilon}$ in state of the world, i. If a forecaster is small and uses causal reasoning to produce an exogenous forecast, then $h' = 0$ and so (2.11) reduces to the standard rational expectation condition (2.7).

However, if the forecaster is large, or uses evidential reasoning (despite being small) then in each case the forecaster believes that her individual forecast will have an affect on the aggregate forecast and so $h'\left(\pi^e; \epsilon_i\right) \neq 0$. Such forecasters are said to be using "endogenous" forecasts. In this case, (2.11) is the correct forecasting rule. It does not reduce to (2.6) and, therefore, (2.7) is not the correct forecasting rule in general.

The results in this subsection are summarized in the following Proposition.

Proposition 2 *: In a model with two states of the world, when the process of actual inflation is non-linear in aggregate inflationary expectation, π^e, the endogenous and exogenous forecasts are different i.e. endogenous forecasts will imply $\pi^e \neq E\left[\pi\right]$.*

The proof does not extend to a world with more than two states of nature. For example, suppose that we have three states of the world in which $\boldsymbol{\epsilon}$ takes respective values ϵ_1, ϵ_2 and ϵ_3, all with equal probability. Then (2.11) gives

$$\frac{1}{3}\left(\bar{\pi} - \pi_1\right)\left(1 - h'\left(\pi^e; \epsilon_1\right)\right) + \frac{1}{3}\left(\bar{\pi} - \pi_2\right)\left(1 - h'\left(\pi^e; \epsilon_2\right)\right) + \frac{1}{3}\left(\bar{\pi} - \pi_3\right)\left(1 - h'\left(\pi^e; \epsilon_3\right)\right) = 0 \tag{2.12}$$

Even if $h'\left(\pi^e; \epsilon_1\right) \neq h'\left(\pi^e; \epsilon_2\right)$, then (2.12) can still hold if $h'\left(\pi^e; \epsilon_3\right)$ has a suitable value, but this is very unlikely to occur. This can be stated formally as follows. In a suitable measure space of models, the set of counterexamples to Proposition 2 (with more than two states) is of measure zero.

Insofar as one believes in the presence of large forecasters in the economy, such as central banks, and/ or in the psychological evidence on evidential reasoning, then, relaxing the restrictiveness of linear inflation processes clearly identifies the inappropriateness of the traditionally used rule of rational expectations, $\pi^e = E[\pi]$. Indeed, in this case, $\pi^e = E[\pi]$ could coincide with irrational expectations, because it could be inconsistent with optimization behavior, while the correct forecasting rule is given by solving (2.11). This has fundamental implications for macroeconomic forecasting and for tests of rational expectations and market efficiency, which have equated departures from $\pi^e = E[\pi]$ with violations of rationality.

Although, Muth (1961) relied on linear inflation processes for all his results, he showed acute awareness of the possibility that the inflation process could be non-linear as the following general definition of rational expectations in the same paper reveals. "The hypothesis can be rephrased a little more precisely as follows: that expectations of firms...tend to be distributed, for the same information set, about the prediction of the theory (or the "objective" probability distributions of outcomes)." Muth (1961: pp.316). However, even this definition is clearly inadequate as a description of (2.5) or (2.11) because Muth was interested in a single representative forecaster, while more interesting game theoretic issues arise in our model; see Section 6 below.

One might pose the following question. Given that macroeconomic departures from equilibrium are usually only of the order of a few percent, why can we not use a linear-quadratic approximation, in which case the expectation rule $\pi^e = E[\pi]$ might be a tolerable approximation? We have the following two answers.

1. If the model equations are not analytic (i.e. do not have a Taylor expansion), then the linear-quadratic approximation is not valid. This is the case with our illustrative example in Section 3 below.

2. Even if the equations were analytic, linearizing the behavioral equations (as is often done, explicitly or implicitly) is not correct. What should be linearized are the final form equations. Linearizing the behavioral equations can radically alter the dynamics of the model.

3. An Example: Model

3.1. Background to the model

As an illustrative example, we use an aggregate supply - aggregate demand macroeconomic model with a liquidity trap. The liquidity trap introduces a non-linearity, which causes exogenous and endogenous forecasts to be different, for some parameter values (see Proposition 2). In our example, as we shall see below, non-linearity takes the specific form

$$\pi_i = \alpha_i + \beta_i \pi^e$$

where β_i is now the value of β (compare with (2.8)) in state i of the world.

An interesting feature of the model is that, for some reasonable parameter values, a solution under exogenous forecasts does not exist. On the other hand, a unique solution under endogenous forecasts always exists. So this example forces the issue: it is vital to consider endogenous forecasts.

3.2. Structure of the economy

The four main equations of the model are the following: an aggregate demand equation, an aggregate supply equation, a social welfare function and a forecaster's loss function.

Aggregate demand and supply are given by, respectively

$$AD : y = f - (i - \pi^e) + \epsilon \tag{3.1}$$

$$AS : y = \pi - \pi^e \tag{3.2}$$

where y is the deviation of aggregate output from the natural level of output. f is a fiscal policy variable. For example, $f > 0$ could be a fiscal deficit and $f < 0$, a fiscal surplus. The deficit could either be debt financed or money financed. Alternatively, f could also be a proxy for a temporary balanced budget reallocation of taxes and subsidies that has a net expansionary ($f > 0$) or contractionary ($f < 0$) effect; see for instance Dixit and Lambertini (2000). $i \geq 0$ is the nominal interest rate, π is the rate of inflation, and π^e is expected inflation. Hence, $i - \pi^e$ is the real rate of interest. The instruments of policy are i and f. ϵ is a demand shock[4] that takes two values, $1, -1$, with equal probability, hence

$$E[\epsilon] = 0, \, Var[\epsilon] = 1. \tag{3.3}$$

The aggregate demand equation reflects the fact that demand is increasing in the fiscal impulse, f, and decreasing in the real interest rate; it is also affected by demand shocks. The aggregate supply equation shows that deviations of output around the natural rate are caused by unexpected movements in the rate of inflation; a formulation that is widespread in the policy literature on rational expectations.

A number of models in the literature on monetary and fiscal policy interaction can provide microfoundations for (3.1) and (3.2); see for instance Dixit and Lambertini (2000). Variations of the model in (3.1) and (3.2) are also used by Ceccheti (2000), Lambertini and Rovelli (2003) and Uhlig (2002). Our intent, therefore, is to use a fairly standard model in the macroeconomic policy literature to formally illustrate our arguments. The

[4] We could also introduce a supply shock; this merely complicates the algebra without changing the conclusions. The modern literature on the liquidity trap stresses demand over supply shocks as major contributory factors.

microfoundations rely on some form of wage-price rigidity. On the other hand, in our model, the nominal interest rate, i, is completely flexible in the region $i > 0$. Hence, for a sufficiently negative shock, the equilibrium real interest rate may be negative[5].

Note the absence of parameters in (3.1),(3.2). This is because our conclusions do not qualitatively depend on the values of such parameters. So we have suppressed them to improve readability.

3.3. Preferences of the policymaker

We assume that a unitary authority, the Treasury, controls both fiscal and monetary policy instruments, f, i, and seeks to maximize the social welfare function,

$$W = -\frac{1}{2}(y - y_F)^2 - \frac{1}{2}\pi^2 - f^2;\;\; y_F \geq 0 \tag{3.4}$$

where y_F is the difference between desired output and the natural rate, hence, we allow desired output to be different from the natural rate. The first term captures the cost to society from output being different from its desired level, y_F. The second term reflects losses that are quadratic in the inflation rate, relative to the desired rate of zero inflation. The use of the fiscal instrument, f, is more costly than the use of the monetary instrument, i[6]. We model this as imposing a strictly positive cost of fiscal policy, equal to f^2, but no cost of using the monetary policy[7].

We assume that the Treasury aims to maximize society's welfare function (3.4) (we ignore the important question of whether this is true or, indeed, if such a social welfare function can be constructed, however, see Rotemberg and Woodford (1999)).

Our model contains only one parameter, y_F. We could complicate the model by including further parameters, a general probability distribution for the shocks, international trade, structural dynamics etc. However, none of these affect our story.

3.4. Objective of private forecasters

We assume that each forecaster aims to minimize the loss function

$$\mathcal{L}(\pi^e) = \frac{1}{2}E\left[(\pi^e - \pi(\pi^e))^2\right] \tag{3.5}$$

[5] We conjecture that the combination of rigid wages-prices and a flexible nominal interest rate has the effect that the real interest rate, $i - \pi^e$, overshoots so as to equilibrate the economy.

[6] Fiscal policy is typically more cumbersome to alter, on account of the cost of changing it (balanced budget requirements, lobby groups etc.) while monetary policy is more flexible and less costly to alter. Indeed the monetary policy committee in the UK as well as the Fed in the USA meet on a regular basis to make decisions on the interest rate and one observes a much greater frequency of interest rate changes relative to tax/debt changes.

[7] Strictly speaking, for our qualitative results to hold, we only require that fiscal policy be relatively more expensive than the (possibly strictly positive) cost of using monetary policy. Normalising the cost of using monetary policy to zero, however, ensures greater tractability and transparency of the results.

where E is the expectation operator that is taken over the two states of the world arising from $\epsilon = 1$ and $\epsilon = -1$.

We will concentrate on the discretionary solution. This is because our main focus is on the effect of different forecasting rules rather than on the problems associated with the time-inconsistency of the optimal solution and the large number of proposals to deal with it; see for instance, Clarida, Gali and Gertler (1999).

3.5. Sequence of Moves

The sequence of moves is as follows. First, the private sector forms its expectations, π^e, of inflation, π, synonymous with the signing of nominal wage contracts in anticipation of future inflation. The demand shock, ϵ, is then realized ($\epsilon = \pm 1$ with equal probability). Finally, the Treasury chooses the optimal values of the policy variables, f, i, given π^e and ϵ. These moves are shown in Figure 3.1.

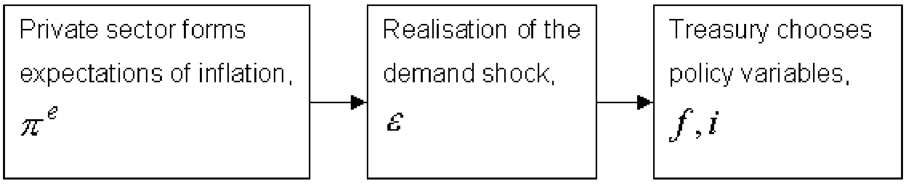

Figure 3.1: **Sequence Of Moves**

3.6. Reduced Form Equations

Equating aggregate demand and supply from (3.1) and (3.2), we get our reduced form equations for output and inflation.

$$y = f - i + \pi^e + \epsilon \tag{3.6}$$

$$\pi = f - i + 2\pi^e + \epsilon \tag{3.7}$$

Substituting the reduced form equations (3.6) and (3.7) into the social welfare function (3.4) we get

$$W = -\frac{1}{2}\left(f - i + \pi^e + \epsilon - y_F\right)^2 - \frac{1}{2}\left(f - i + 2\pi^e + \epsilon\right)^2 - f^2 \tag{3.8}$$

3.7. Method of Solution

We solve the model backwards. First, we determine f, i as functions of π^e and ϵ by maximizing social welfare (3.8). This corresponds to the Treasury's actions. We then work out π^e, using (3.7) and the "correct" forecasting rule, which is found by minimizing (3.5); we also contrast this with the standard use of rational expectations, $\pi^e = E[\pi]$. This corresponds to the calculations of the private sector.

The partial derivatives of W with respect to f, i are given by:

$$\frac{\partial W}{\partial f} = y_F - 2\epsilon - 4f - 3\pi^e + 2i \tag{3.9}$$

$$\frac{\partial W}{\partial i} = 2f - 2i + 3\pi^e + 2\epsilon - y_F \tag{3.10}$$

Since f is unrestricted, maximization of W with respect to f yields $\frac{\partial W}{\partial f} = 0$ and, hence, from (3.9):

$$f = \frac{1}{4}y_F - \frac{3}{4}\pi^e + \frac{1}{2}i - \frac{1}{2}\epsilon \tag{3.11}$$

However, the nominal interest rate is constrained to be non-negative. Hence, the relevant first order condition is

$$\frac{\partial W}{\partial i} \leq 0, \quad i\frac{\partial W}{\partial i} = 0, \quad , i \geq 0 \tag{3.12}$$

Using (3.12) in conjunction with (3.10) it follows that

$$i = f + \frac{3}{2}\pi^e - \frac{1}{2}y_F + \epsilon \text{ (not liquidity trapped)} \tag{3.13}$$

$$i = 0 \qquad\qquad \text{(liquidity trapped)} \tag{3.14}$$

Since (3.11) always holds (because f is unrestricted), we can use it to eliminate f from (3.10) and (3.13). This gives

$$\frac{\partial W}{\partial i} = \frac{3}{2}\pi^e - \frac{1}{2}y_F - i + \epsilon \tag{3.15}$$

and, hence, if the economy is not liquidity trapped,

$$i = \frac{3}{2}\pi^e - \frac{1}{2}y_F + 1 \text{ (not liquidity trapped)} \tag{3.16}$$

$$f = 0 \text{ (not liquidity trapped)} \tag{3.17}$$

while if it is liquidity trapped,

$$i = 0 \text{ (liquidity trapped)} \tag{3.18}$$

$$f = \frac{1}{4}y_F - \frac{3}{4}\pi^e + \frac{1}{2} \text{ (liquidity trapped)} \tag{3.19}$$

From (3.16), (3.17) we see that when the economy is not liquidity trapped, then the Treasury stabilizes the economy by exclusively using, the less expensive, monetary policy. However, from (3.18), (3.19), when the economy is liquidity trapped, then monetary policy is impotent and the Treasury is forced to use the costly fiscal policy.

The following notation will be convenient. $f_+, i_+, y_+, \pi_+, W_+, \mathcal{L}_+$ denote the values of $f, i, y, \pi, W, \mathcal{L}$ when the good state of the world, $\epsilon = 1$, occurs. Similarly, $f_-, i_-, y_-, \pi_-, W_-, \mathcal{L}_-$ denote the values of $f, i, y, \pi, W, \mathcal{L}$ when the bad state of the world, $\epsilon = -1$, occurs.

We shall use an overbar to denote the expected value (or the average/ mean value) of a variable. So $\bar{\pi} = \frac{1}{2}\pi_+ + \frac{1}{2}\pi_-$, $\bar{y} = \frac{1}{2}y_+ + \frac{1}{2}y_-$ and so on.

On the other hand, we shall use π^e exclusively to denote the optimal forecast of π. So, for exogenous forecasts, we get, from Propositions 1, 2,

$$\pi^e = \bar{\pi} = \frac{1}{2}\pi_+ + \frac{1}{2}\pi_- \text{ (exogenous forecasts)} \tag{3.20}$$

However, for endogenous forecasts, π^e is obtained by solving (2.5) which, for our example, gives

$$\frac{1}{2}\left(\pi^e - \pi_+\right)\left(1 - \frac{d\pi_+}{d\pi^e}\right) + \frac{1}{2}\left(\pi^e - \pi_-\right)\left(1 - \frac{d\pi_-}{d\pi^e}\right) = 0 \text{ (endogenous forecasts)} \tag{3.21}$$

We know from Proposition 2 of the general model that in the case of endogenous forecasts, $\pi^e \neq \bar{\pi}$.

4. Main Results

The solution is now examined under the two cases of exogenous and endogenous forecasts.

4.1. Solution under exogenous forecasts

Under exogenous forecasting, each agent takes π as given exogenously, independent of the method she uses for forecasting. Suppose that a generic variable, x, takes respective values x_- and x_+ when respectively $\epsilon = -1$ and $\epsilon = 1$. Recall that we have chosen to denote the average (across states of the world) value of variable, x, by $\bar{x}$, such that $\bar{x} = \frac{1}{2}x_- + \frac{1}{2}x_+$. Note that the average value $\bar{x}$ need not be equal to the forecasted value of x which is given by x^e. This distinction turns out to be important in interpreting the differences between results arising from exogenous and endogenous forecasts respectively.

As shown in Proposition 1, an agent using exogenous forecasts relies on the forecasting rule $\pi^e = E[\pi]$. The most important result in this section is that there might be no optimal solution if agents use exogenous forecasting; this is shown in Proposition 3.

Proposition 3 *: If $y_F < \frac{1}{2}$ then there is no (optimal) solution under exogenous forecasting.*

Contingent on parameter values, the economy may or may not be liquidity trapped under exogenous forecasts. For completeness, these cases are considered in the auxiliary results in Section 5.1.

4.2. Solution under endogenous forecasts

Here we consider the case where forecasters perceive the dependence of π on their own method of forecasting. This could arise from a variety of cases, such as the following. (1) Some forecasters are large, such as central banks and some financial institutions. If there are small forecasters they simply follow the large forecasters. (2) Forecasters (large or small) use evidential reasoning rather than causal reasoning.

Because the underlying process of inflation is non-linear in our example, it follows from Proposition 2 that (2.7) is not the correct forecasting rule. If π^e is not restricted, then the correct forecasting rule can be found by solving (2.5). The results are as follows.

Proposition 4 *: If $0 \leq y_F < 1$, then the economy is liquidity trapped for $\epsilon = -1$ but not for $\epsilon = 1$. The solution in this case is given by*

$\boldsymbol{\epsilon = -1}$	$\boldsymbol{\epsilon = 1}$	$\boldsymbol{\bar{\epsilon} = 0}$
$i_- = 0$	$i_+ = \frac{1}{5}(8 + 2y_F) > 0$	$\bar{i} = \frac{1}{5}(4 + y_F) > 0$
$f_- = \frac{1}{5}(1 - y_F) > 0$	$f_+ = 0$	$\bar{f} = \frac{1}{10}(1 - y_F) > 0$
$y_- = \frac{2}{5}(y_F - 1) < 0$	$y_+ = \frac{1}{5}(y_F - 1) < 0$	$\bar{y} = \frac{3}{10}(y_F - 1) < 0$
$\pi_- = y_F \geq 0$	$\pi_+ = \frac{1}{5}(1 + 4y_F) > 0$	$\bar{\pi} = \frac{1}{10}(1 + 9y_F) > 0$
$W_- = -\frac{1}{25}(3 + 4y_F + 18y_F^2)$	$W_+ = -\frac{1}{25}(1 + 8y_F + 16y_F^2)$	$\bar{W} = -\frac{1}{25}(2 + 6y_F + 17y_F^2)$
$i_- - \pi^e = -\frac{1}{5}(2 + 3y_F) < 0$	$i_+ - \pi^e = \frac{1}{5}(6 - y_F) > 0$	$i^e - \pi^e = \frac{2}{5}(1 - y_F) > 0$

Furthermore, the aggregate expectation of inflation is given by $\pi^e = \frac{1}{5}(2 + 3y_F)$ and $\bar{\pi} < \pi^e$.

The result $\bar{\pi} < \pi^e$ is interesting. The economic intuition behind this result is as follows. Under discretion, the Treasury accommodates higher inflation expectations with higher actual inflation. In turn, higher actual inflation results in a higher forecast of inflation.

This positive feedback is internalized by endogenous forecasts. However, it is absent in exogenous forecasts because the forecasters take inflation as an exogenous random variable, unaffected by inflation expectations.

If this result is generally true, then published forecasts of inflation may be systematically different from the statistical averages of actual inflation.

Another interesting result is that output is below the natural rate in each state of the world, and on average. Thus, discretionary policy can be deflationary when there is the possibility of a liquidity trap. This runs counter to the standard result in natural rate models with exogenous forecasts that $\bar{y} = 0$. More generally, in a non-linear rational expectations model with endogenous forecasting, average output need not be at the natural rate. This has clear implications for estimating the natural rate of output based on aggregate output data. In particular, the natural rate should not be estimated using purely statistical methods but, rather, should be estimated using an appropriately specified macroeconomic model.

The case $y_F \geq 1$ is less interesting because the economy is not liquidity trapped at all in this case and so the non-linearity in the inflation process disappears. This result is presented in the auxiliary results in Section 5.2.

Proposition 4 formalizes the main result of this section in terms of a simple but widely used model in macroeconomics that allows for the possibility of a liquidity trap. *It suggests that empirical rejections of $\pi^e = \bar{\pi}$ do not necessarily constitute a failing of rational expectations.* Indeed in the presence of non-linear inflation processes, and either of the following two assumptions, large forecasters or evidential reasoning, an empirical finding of $\pi^e = \bar{\pi}$ may constitute a violation of rational expectations.

5. Auxiliary Results

5.1. Auxiliary results under exogenous forecasts

The two propositions below complete the solution to the model in the case of exogenous forecasts.

Proposition 5 *: If $\frac{1}{2} \leq y_F < 1$, then the economy is liquidity trapped for $\epsilon = -1$ but not for $\epsilon = 1$. The complete solution is given by*

$\epsilon = -1$	$\epsilon = 1$	$\bar{\epsilon} = 0$
$i_- = 0$	$i_+ = 4y_F - 2$	$\bar{i} = 2y_F - 1$
$f_- = 2(1 - y_F)$	$f_+ = 0$	$\bar{f} = 1 - y_F$
$y_- = y_F - 1$	$y_+ = 1 - y_F$	$\bar{y} = 0$
$\pi_- = 4y_F - 3$	$\pi_+ = 2y_F - 1$	$\bar{\pi} = 3y_F - 2 = \pi^e$
$W_- = 20y_F - 12y_F^2 - 9$	$W_+ = 4y_F - 4y_F^2 - 1$	$\bar{W} = 12y_F - 8y_F^2 - 5$
$i_- - \bar{\pi} = 2 - 3y_F$	$i_+ - \bar{\pi} = y_F$	$i^e - \bar{\pi} = 1 - y_F$

Furthermore, when $\epsilon = -1$, $\frac{\partial W_-}{\partial i_-} = 4(y_F - 1)$.

Proposition 6 *: If* $y_F \geq 1$*, then the economy is not liquidity trapped in either state. The solution in this case is given by*

$\epsilon = -1$	$\epsilon = 1$	$\bar{\epsilon} = 0$
$i_- = y_F - 1 > 0$	$i_+ = 1 + y_F$	$\bar{i} = y_F$
$f_- = 0$	$f_+ = 0$	$\bar{f} = 0$
$y_- = 0$	$y_+ = 0$	$\bar{y} = 0$
$\pi_- = y_F$	$\pi_+ = y_F$	$\bar{\pi} = y_F = \pi^e$
$W_- = -y_F^2$	$W_+ = -y_F^2$	$\bar{W} = -y_F^2$
$i_- - \bar{\pi} = -1$	$i_+ - \bar{\pi} = 1$	$i^e - \bar{\pi} = 0$

5.2. Auxiliary results under endogenous forecasts

Proposition 7 *: If* $y_F \geq 1$*, then the economy is not liquidity trapped in either state. The results are as in Proposition 6. i.e. the same as under exogenous forecasts.*

In the cases considered in Propositions 5 and 6, the non-linearity in the inflation process disappears and so, as claimed in the general model (see Proposition 1), $\bar{\pi} = \pi^e$. Furthermore, $\bar{y} = 0$ unlike Proposition (4) where it is negative.

6. Game Theoretic Foundations: Common knowledge, equilibrium selection and evidential reasoning

A number of different game theoretic foundations can support Proposition 3-7. These are discussed in sections 6.1 to 6.3. Section 6.4 discusses problems of common knowledge, equilibrium selection and coordination in the context of the model of this paper. All this is carried out in a completely classical game theoretic framework. Section 6.4 discusses the consequences of evidential reasoning for these issues, thus introducing an element of behavioral game theory.

6.1. Static games

In the simplest game theoretic formulation we have two players. One of the players is the Treasury who controls both fiscal and monetary instruments f, i and whose payoff is social welfare, W, given by (3.4). The second player is the private sector, treated as a single player, who chooses her expectation of inflation, π^e, so as to minimize her forecasting loss function (2.1).

The timing of events is given by Figure 3.1. First, the private sector chooses π^e. Next, nature chooses ϵ. Finally, the Treasury chooses f, i. However, if π^e is unknown to the Treasury when it chooses f, i then, technically, this is a simultaneous move game. Analogously, when the private sector chooses π^e, it does not know whether the payoff of the Treasury is W_+ or W_- (corresponding to $\epsilon = 1$ or $\epsilon = -1$, respectively).

Hence, this is a static game of incomplete information and the relevant solution concept is "Bayesian Nash equilibrium". However, there is only one type for the private sector, hence, Bayes law automatically holds. Thus any Nash equilibrium of this game is also a Bayesian Nash equilibrium,

Since the above game is static, it follows that, in a Nash equilibrium, the private sector takes f_+, f_-, i_+, i_-, and hence π_+, π_-, as given. Thus, all forecasts are "exogenous". There are two Nash equilibria for $y_F > \frac{1}{2}$ but one of them strictly Pareto dominates the other. It is the "Pareto dominant equilibrium" that is reported in Proposition 5 (for $\frac{1}{2} \leq y_F < 1$) and Proposition 6 (for $y_F \geq 1$). For $y_F < \frac{1}{2}$, there is no pure strategy Nash equilibrium for this game; a fact reported in Proposition 3.

Now let us suppose that the private sector, infact, consists of many forecasters. Suppose that the Treasury observes none of these forecasts, nor the aggregate forecast when it chooses f, i. Then the analysis and the results are still exactly as above.

6.2. Dynamic games with small forecasters

Suppose the private sector consists of a large number of forecasters, each of them small. Suppose they all move simultaneously, forming their forecasts independently and before the realization of the demand shock, ϵ. Suppose that the Treasury chooses f, i having observed the realization of the demand shock, ϵ, and having observed the forecasts of all forecasters.

This is clearly a dynamic game. It is a game of incomplete information because when each member of the private sector chooses her forecasts, she does not know whether the payoff of the Treasury is W_+ or W_- (corresponding to $\epsilon = 1$ or $\epsilon = -1$, respectively). Hence the relevant solution concept is the "perfect Bayesian equilibrium" or one of its refinements. Each player moves just once. Each member of the private sector is of one type. The Treasury, however, is of two types, W_+, W_- but moves after the private sector.

From these considerations, it follows that Bayes law is automatically satisfied. Thus, any subgame perfect equilibrium of this game is also a perfect Bayesian equilibrium.

Since each forecaster is small she perceives, correctly, that her own forecast is only an insignificant component of the aggregate forecast, π^e. Hence she, correctly, perceives that her own individual forecast has no practical effect on π^e and, hence, π. Therefore, each forecaster takes π as given, exactly as in the static model of Section 6.1, despite the dynamic nature of the present model. Thus, forecasters produce exogenous forecasts and we have exactly the same outcomes as in Section 6.1, and Propositions 3, 5, 6, again hold.

Now suppose the Treasury observes aggregate π^e but not the individual forecasts of the private sector. The relevant solution concept is now the "perfect public equilibrium". However, the outcomes are exactly as above, and are described in Propositions 3, 5, 6.

6.3. Dynamic games with large forecasters

Reconsider the dynamic game of Section 6.2 but with the private sector now treated as a single player. The private sector is now a large forecaster who, correctly, perceives the dependence of π on her forecast π^e. Hence, she produces endogenous forecasts.

The private sector is of a single type and moves just once, at the start of the game. The Treasury, which is of two types, W_+, W_-, moves subsequent to the private sector and after having observed π^e. Hence, this is a dynamic game of perfect, though incomplete information. Hence it always has a subgame perfect equilibrium (which is automatically a perfect Bayesian equilibrium, see Section 6.2). This equilibrium can be found by backward induction and is, infact, unique. It is reported in Proposition 4 (for $0 \leq y_F < 1$) and Proposition 7 (for $y_F \geq 1$).

Suppose the private sector consists of many players but just one (large) forecaster, e.g., the Central Bank. First, the forecaster forms her forecast of inflation, π^e, and makes it public. Second, the other players make their decisions having observed π^e but before the Treasury chooses f, i. Finally, the Treasury chooses f, i having observed π^e. Just as above, there is a unique subgame perfect equilibrium whose outcome is described by Proposition 4 (for $0 \leq y_F < 1$) and Proposition 7 (for $y_F \geq 1$).

However, if there is more than one large forecaster, then the game theoretic interactions between the players have to be remodelled with results, not necessarily, as above. But this lies beyond the scope of this paper.

6.4. Common knowledge, equilibrium selection and coordination

Common knowledge of the game is an implicit assumption above, as it is in classical game theory. This means that each player knows the game being played, knows that all other players know the game being played, knows that all players know that all other players know that all players know the game being played and so ad infinitum. The common knowledge assumption is a strong one to make. Furthermore, if players do not have common knowledge of the game being played at the outset, it is difficult to see how they can arrive at it.

Quite frequently a game has more than one equilibrium. Therefore, equilibrium selection criteria are needed. This is one of the motivations behind the 'refinements of the Nash' programme. These selection criteria have to be part of the common knowledge of the game.

If we have just one large forecaster, whose forecast is made public before the other players take their decisions, then our model has a unique subgame perfect equilibrium, described by Propositions 4, 7 (see section 6.3 above).

On the other hand we get multiple equilibria in each of the following two cases: (1) all forecasters are small, (2) forecasters are large or small but forecasts are not observed by the Treasury when it sets f, i. However, in the first case, payoff dominance selects a unique subgame perfect equilibrium (section 6.2 above) and in the second case, payoff dominance also selects a unique Nash equilibrium (section 6.1 above).

Of course, many games have multiple equilibria that are not Pareto ranked. Moreover, even when "payoff dominance" is applicable, it can be in conflict with other selection criteria, for example, "risk dominance". A major theme in game theory has been the search for an acceptable equilibrium selection criterion. A notable candidate is the equilibrium selection procedure of Harsanyi and Selten (1988) which will always select a unique equilibrium. However, Norde, Potters, Reijnierse and Vermeulen (1996) show that the only solution concept that satisfies consistency, nonemptiness and one-person rationality is that of Nash. So if one thinks that these properties are desirable then one has to be pessimistic about the success of the equilibrium selection programme.

Over and above the problem of equilibrium selection, there is the problem of coordination. Even if there were a preferred equilibrium, how can the players coordinate their actions to achieve the desired outcome?

6.5. Consequence of evidential reasoning

Suppose that each player knows the game, and uses "evidential reasoning". Each player will take her knowledge of the game as evidence that the other players also know the game. Since she now knows that other players also know the game, she will take this latter conclusion as evidence that each of the other players knows that all players know that all players know the game and so on ad infinitum. In short, private knowledge of the game together with evidential reasoning implies common knowledge of the game.

One can speculate on the emergence of evidential reasoning (which is irrational) and its survival alongside causal reasoning (which is rational). In a large number of applications, private knowledge of the game is inadequate for action. The players also need common knowledge of the game. As an example, consider the following version of the well known "coordinated attack" problem; see for instance, Halpern (1986). (W)ellington and (B)lucher wish to attack their common enemy (N)apoleon. If W or B attack on their own, N will win. But if W or B attack together, they will win. W sends a message to B saying he will attack, but only if he receives confirmation from B that B will also attack. B replies that he will attack, but only if he receives confirmation that his message has reached W, and so on. Under causal reasoning, neither W or B would attack. However, under evidential reasoning, W and B will both attack because each uses his own reasoning as evidence that the other is similarly minded (and, maybe, a finite number of messages is sufficient to enforce this psychological mode of reasoning)[8].

Thus evidential reasoning, while not a logically correct method of reasoning may, nevertheless, have practical utility.

Evidential reasoning can help small players behave strategically. Assume that all forecasters are small. If they use causal reasoning then they will produce exogenous forecasts. Consequently, an equilibrium may not exist (see Proposition 3). However, if they use evidential reasoning then they will behave as a single large forecaster and produce endogenous forecasts for which a unique equilibrium exists in our model (see Propositions 4, 7).

Evidential reasoning can also change a static game into a dynamic game. Reconsider the static games of section 6.1. Although π^e is determined before the Treasury sets f, i the Treasury sets its policies without knowledge of π^e. However, the Treasury can calculate π^e, and know what it is, without observing it. The calculation of π^e by the Treasury can itself be foreseen by each forecaster. This can be seen as evidence by each forecaster that the Treasury will behave as if it observes π^e.

Finally, evidential reasoning can lead to the sustainability of the non-Nash outcomes. For example, the cooperative outcome in the prisoner's dilemma game, discussed in section 1.2.

To summarize, evidential reasoning can help establish common knowledge of the game, in coordinating players, in enabling small players to behave strategically, can turn static games into dynamic games and can sustain non-Nash outcomes.

7. Conclusions

Rational expectations is almost always taken to mean that, given the information set and the relevant economic theory, the expectation formed by an economic agent, of a random

[8] Eventually, Wellington and Blucher did attack Napoleon with decisive consequences in the Battle of Waterloo.

variable, should be equal to its mathematical expectation. This is only true if the process governing inflation is linear in the aggregate inflationary expectations.

For many problems, especially where the underlying structure is not linear-quadratic, however, actual inflation might be non-linear in expectations of inflation. In this case, if either forecasters are large, or use evidential reasoning (despite being large or small) then the optimal forecast does not equal the mathematical expectation of the variable being forecast. Indeed, under these circumstances, an observation of the forecast being equal to its mathematical expectation could constitute a violation of rational expectations; the correct general forecasting equation that is consistent with rational expectations is then given by (2.11).

We show in our illustrative example that there might be no solution to the model if rational expectations are interpreted as equating expectations with the average, while there is a solution using the correct forecasting rule. This clearly has serious implications for the existence of solutions if rational expectations is interpreted as $\bar{\pi} = \pi^e$. Furthermore, results suggest that published forecasts of inflation may be systematically different from the statistical averages of actual inflation and output, on average, need not equal the natural rate.

In order to demonstrate our results, we use a simple, standard, model of aggregate demand - aggregate supply with a liquidity trap, which gives rise to non-linearities in the inflation process. We also illustrate our arguments by using a general, abstract, framework where inflation is a non-linear function of inflation expectations. Several possible departures from the linear-quadratic framework, which typifies a large literature on macroeconomic policy, could have been used to derive our results. However, our chosen exposition is much simpler, tractable and topical. Variants of the model that we use have been used recently in several papers that examine the interaction between monetary and fiscal policy, hence, our example is not contrived. The paper has deep, fundamental, implications for forecasting using rational expectations in several areas, but, in particular macroeconomics and finance.

8. Appendix

Proof of Proposition 1:

Suppose that the process of actual inflation is described in (2.8). If forecasts are endogenous then $\frac{d\pi_i}{d\pi^e} = \beta$, which when substituted into (2.5), gives (2.7), and so $\pi^e = E[\pi]$. When forecasts are exogenous (2.10) holds, which when substituted into (2.5), gives (2.7), and so $\pi^e = E[\pi]$, once again. ■

Proof of Proposition 2:

Suppose that there are only two states of the world in which $\boldsymbol{\epsilon}$ takes values ϵ_1 and ϵ_2. Assume, contrary to the claim in the Proposition, that $E[\pi] = \pi^e$. Equation (2.11) then gives

$$p_1 \left(\bar{\pi} - \pi_1\right) \left(1 - h' \left(\pi^e; \epsilon_1\right)\right) + p_2 \left(\bar{\pi} - \pi_2\right) \left(1 - h' \left(\pi^e; \epsilon_2\right)\right) = 0 \tag{8.1}$$

Substituting $E[\pi] = p_1\pi_1 + p_2\pi_2$ in (8.1) and simplifying gives

$$h' \left(\pi^e; \epsilon_1\right) = h' \left(\pi^e; \epsilon_2\right),$$

which implies that the actual process of inflation is linear, as in (2.8). However, this contradicts the assumption of a non-linear inflation process and, hence, establishes Proposition 2.■

As preparation for the proofs of the main propositions under exogenous and endogenous forecasts we first give the intermediate results in Lemmas 1 through 4.

Lemma 1 : *If the economy is liquidity trapped in the good state, $\epsilon = 1$, then it is also liquidity trapped in the bad state, $\epsilon = -1$.*

Proof: Put $\epsilon = 1$, then $\epsilon = -1$, in (3.15), to get

$$\frac{\partial W_+}{\partial i_+} = \frac{3}{2}\pi^e - \frac{1}{2}y_F - i_+ + 1 \tag{8.2}$$

$$\frac{\partial W_-}{\partial i_-} = \frac{3}{2}\pi^e - \frac{1}{2}y_F - i_- - 1 \tag{8.3}$$

Suppose that the economy is liquidity trapped in the good state, $\epsilon = 1$. We start by showing that $\frac{\partial W_+}{\partial i_+} < 0$, for all $i_+ \geq 0$. Suppose not. Then $\frac{\partial W_+}{\partial i_+} \geq 0$ for some $i_+ = i_0 \geq 0$. From (8.2) we get $\frac{3}{2}\pi^e - \frac{1}{2}y_F - i_0 + 1 \geq 0$, $i_0 \geq 0$, hence, $\frac{3}{2}\pi^e - \frac{1}{2}y_F + 1 \geq 0$. Consider $i_+ = \frac{3}{2}\pi^e - \frac{1}{2}y_F + 1$. Then $i_+ \geq 0$ and , from (8.2) $\frac{\partial W_+}{\partial i_+} = \frac{3}{2}\pi^e - \frac{1}{2}y_F - \left(\frac{3}{2}\pi^e - \frac{1}{2}y_F + 1\right) + 1 = 0$. This means that the economy is not liquidity trapped in the good state, $\epsilon = 1$, contrary to the assumption of the lemma. Hence, $\frac{\partial W_+}{\partial i_+} < 0$ for all $i_+ \geq 0$, as claimed. In particular, for $i_+ = 0$, (8.2) now gives $\frac{3}{2}\pi^e - \frac{1}{2}y_F + 1 < 0$. Hence, also, $\frac{3}{2}\pi^e - \frac{1}{2}y_F - 1 < 0$. Thus, $\frac{3}{2}\pi^e - \frac{1}{2}y_F - i_- - 1 < 0$, for all $i_- \geq 0$, which establishes that the economy is also liquidity trapped in the bad state, $\epsilon = -1$.■

From Lemma 1 it follows that if the economy is not liquidity trapped in the bad state, $\epsilon = -1$, it is not liquidity trapped in the good state either, Hence, we have 3, and only 3, cases to consider.

1. The economy is liquidity trapped in the good state.

2. The economy is not liquidity trapped in the bad state.

3. The economy is liquidity trapped in the bad state but not in the good state.

These three cases are covered in Lemmas 2 - 4 below.

Lemma 2 *: If the economy is liquidity trapped in the good state,* $\epsilon = 1$, *then*
(a) $\frac{d\pi}{d\pi^e}$ *is constant across states of the world and* $\pi = \frac{5}{4}\pi^e + \frac{1}{4}y_F + \frac{1}{2}\epsilon$.
(b) Exogenous and endogenous forecasts are the same and given by $\pi^e = \bar{\pi} = -y_F$.
(c) $y_F > \frac{1}{2}$.
(d) The full set of results is given by

$\boldsymbol{\epsilon = -1}$	$\boldsymbol{\epsilon = 1}$	$\bar{\boldsymbol{\epsilon}}\mathbf{= 0}$
$i_- = 0$	$i_+ = 0$	$\bar{i} = 0$
$f_- = y_F + \frac{1}{2}$	$f_+ = y_F - \frac{1}{2}$	$\bar{f} = y_F$
$y_- = -\frac{1}{2}$	$y_+ = \frac{1}{2}$	$\bar{y} = 0$
$\pi_- = -y_F - \frac{1}{2}$	$\pi_+ = -y_F + \frac{1}{2}$	$\bar{\pi} = -y_F = \pi^e$
$W_- = -2\left(y_F + \frac{1}{2}\right)^2$	$W_+ = -2\left(y_F - \frac{1}{2}\right)^2$	$\bar{W} = -2y_F^2 - \frac{1}{2}$
$i_- - \pi^e = y_F$	$i_+ - \pi^e = y_F$	$i^e - \pi^e = y_F$

Furthermore, the magnitude of the forecasters loss is given by $\mathcal{L} = \frac{1}{8}$.

Proof: From Lemma 1, the economy is liquidity trapped in both states of the world. Hence, (3.18), (3.19) apply in both states of the world. Substituting from (3.18), (3.19), for i, f into (3.7), then simplifying, gives

$$\pi = \frac{5}{4}\pi^e + \frac{1}{4}y_F + \frac{1}{2}\epsilon, \ \forall\epsilon \tag{8.4}$$

Part (a) follows from (8.4). Part (b) then follows from Proposition 1 and taking expectations of both sides of (8.4), then solving for π^e. Part (c) follows from the requirement that $\frac{\partial W_+}{\partial i_+} < 0$ at $i_+ = 0$. Finally, part (d) follows from part (b), (3.18), (3.19) and (3.6) - (3.8). Substituting π_-, π_+, π^e in (2.1), one obtains the magnitude of the forecaster's loss. ■

Lemma 3 *: If the economy is not liquidity trapped in the bad state,* $\epsilon = -1$, *then*
(a) $\frac{d\pi}{d\pi^e}$ *is constant across states of the world and* $\pi = \frac{1}{2}\pi^e + \frac{1}{2}y_F$.
(b) Exogenous and endogenous forecasts are the same and given by $\pi^e = \bar{\pi} = y_F$.
(c) $y_F \geq 1$.
(d) The full set of results is given by

$\boldsymbol{\epsilon = -1}$	$\boldsymbol{\epsilon = 1}$	$\bar{\boldsymbol{\epsilon}}\mathbf{= 0}$
$i_- = y_F - 1$	$i_+ = y_F + 1$	$\bar{i} = y_F$
$f_- = 0$	$f_+ = 0$	$\bar{f} = 0$
$y_- = 0$	$y_+ = 0$	$\bar{y} = 0$
$\pi_- = y_F$	$\pi_+ = y_F$	$\bar{\pi} = y_F = \pi^e$
$W_- = -y_F^2$	$W_+ = -y_F^2$	$\bar{W} = -y_F^2$
$i_- - \pi^e = -1$	$i_+ - \pi^e = 1$	$i^e - \pi^e = 0$

Furthermore, the magnitude of the forecasters loss is given by $\mathcal{L} = 0$.

Proof: From Lemma 1, the economy is not liquidity trapped in either state of the world. Hence, (3.16), (3.17) apply in both states of the world. Substituting from (3.16), (3.17), for i, f into (3.7), then simplify to get

$$\pi = \frac{1}{2}\pi^e + \frac{1}{2}y_F, \ \forall\epsilon \tag{8.5}$$

Part (a) follows from (8.5). Part (b) follows from Proposition 1 and taking expectations of both sides of (8.5), then solving for π^e. Part (c) follows from the requirement that $i_- \geq 0$. Finally, part (d) follows from (3.16), (3.17) and (3.6) - (3.8). Substituting π_-, π_+, π^e in (2.1), one obtains the magnitude of the forecaster's loss.■

Lemma 4 : *If the economy is liquidity trapped in the bad state,* $\epsilon = -1$*, but not in the good state,* $\epsilon = 1$*, then,*
(a) $\frac{d\pi}{d\pi^e}$ *is not constant across states of the world and*

$$\pi_+ = \frac{1}{2}\pi^e + \frac{1}{2}y_F \quad (\epsilon = 1)$$

$$\pi_- = \frac{5}{4}\pi^e + \frac{1}{4}y_F - \frac{1}{2} \quad (\epsilon = -1)$$

(b) Exogenous and endogenous forecasts are different.
(c) For exogenous forecasts
(i) $\pi^e = \bar{\pi} = 3y_F - 2$
(ii) $\frac{1}{2} \leq y_F < 1$
(iii) The full set of results under exogenous forecasts are given by

$\boldsymbol{\epsilon = -1}$	$\boldsymbol{\epsilon = 1}$	$\boldsymbol{\bar{\epsilon} = 0}$
$i_- = 0$	$i_+ = 4y_F - 2$	$\bar{i} = 2y_F - 1$
$f_- = 2(1 - y_F)$	$f_+ = 0$	$\bar{f} = 1 - y_F$
$y_- = y_F - 1$	$y_+ = 1 - y_F$	$\bar{y} = 0$
$\pi_- = 4y_F - 3$	$\pi_+ = 2y_F - 1$	$\bar{\pi} = 3y_F - 2 = \pi^e$
$W_- = 20y_F - 12y_F^2 - 9$	$W_+ = 4y_F - 4y_F^2 - 1$	$\bar{W} = 12y_F - 8y_F^2 - 5$
$i_- - \pi^e = 2 - 3y_F$	$i_+ - \pi^e = y_F$	$i^e - \pi^e = 1 - y_F$

Furthermore, the magnitude of the forecasters loss is given by $\mathcal{L} = \frac{1}{2}(1 - y_F)^2$.
(d) For endogenous forecasts
(i) $\pi^e = \frac{3}{5}y_F + \frac{2}{5}$
(ii) $\bar{\pi} = \frac{9}{10}y_F + \frac{1}{10}$
(iii) $y_F < 1$
(iv) The full set of results are given by

$\boldsymbol{\epsilon = -1}$	$\boldsymbol{\epsilon = 1}$	$\boldsymbol{\bar{\epsilon} = 0}$
$i_- = 0$	$i_+ = \frac{1}{5}(8 + 2y_F)$	$\bar{i} = \frac{1}{5}(4 + y_F)$
$f_- = \frac{1}{5}(1 - y_F)$	$f_+ = 0$	$\bar{f} = \frac{1}{10}(1 - y_F)$
$y_- = \frac{2}{5}(y_F - 1)$	$y_+ = \frac{1}{5}(y_F - 1)$	$\bar{y} = \frac{3}{10}(y_F - 1)$
$\pi_- = y_F$	$\pi_+ = \frac{1}{5}(1 + 4y_F)$	$\bar{\pi} = \frac{1}{10}(1 + 9y_F)$
$W_- = -\frac{1}{25}(3 + 4y_F + 18y_F^2)$	$W_+ = -\frac{1}{25}(1 + 8y_F + 16y_F^2)$	$\bar{W} = -\frac{1}{25}(2 + 6y_F + 17y_F^2)$
$i_- - \pi^e = -\frac{1}{5}(2 + 3y_F)$	$i_+ - \pi^e = \frac{1}{5}(6 - y_F)$	$i^e - \pi^e = \frac{2}{5}(1 - y_F)$

Furthermore, the magnitude of the forecasters loss is given by $\mathcal{L} = \frac{1}{20}(1 - y_F)^2 \leq \frac{1}{20}$.

Proof: Since the economy is not liquidity trapped in the good state, $\epsilon = 1$, we substitute from (3.16), (3.17) into (3.7), then simplify to get

$$\pi_+ = \frac{1}{2}\pi^e + \frac{1}{2}y_F \quad (\epsilon = 1) \tag{8.6}$$

Since the economy is liquidity trapped in the bad state, $\epsilon = -1$, we substitute from (3.18), (3.19)into (3.7), then simplify to get

$$\pi_- = \frac{5}{4}\pi^e + \frac{1}{4}y_F - \frac{1}{2} \quad (\epsilon = -1) \tag{8.7}$$

Part (a) follows from (8.6), (8.7). Part (b) then follows from Proposition 1 and is verified by direct calculation below.

Part c(i) follows from substituting for π_+, π_- from (8.6), (8.7) into (3.20), then solving for $\pi^e (= \bar{\pi})$. The two requirements $\left(\frac{\partial W_-}{\partial i_-}\right)_{i_-=0} < 0$ and $i_+ \geq 0$ lead to c(ii). Part c(iii) follows from c(i), (3.16) - (3.19) and (3.6) - (3.8). Substituting π_-, π_+, π^e in (2.1), one obtains the magnitude of the forecaster's loss, $\frac{1}{2}(1 - y_F)^2$.

Part d(i) follows from (8.6), (8.7) and (3.21). Part d(ii) follows from d(i), (8.6), (8.7) and $\bar{\pi} = \frac{1}{2}\pi_+ + \frac{1}{2}\pi_-$. Part d(iii) follows from (3.15) and the requirement that $\left(\frac{\partial W_-}{\partial i_-}\right)_{i_-=0} < 0$. Finally d(iv) follows from d(i), (3.16) - (3.19) and (3.6) - (3.8). Substituting π_-, π_+, π^e in (2.1), one obtains the magnitude of the forecaster's loss, $\frac{1}{20}(1 - y_F)^2$.■

Proof of Proposition 3

We have 3, and only 3, cases to consider.

1. The economy is liquidity trapped in the good state.

2. The economy is not liquidity trapped in the bad state.

3. The economy is liquidity trapped in the bad state but not in the good state.

These three cases are covered by Lemmas 3 to 4, respectively. From Lemma 2(e), $y_F > \frac{1}{2}$. From Lemma 3(c), $y_F \geq 1$. From Lemma 4c(ii), $\frac{1}{2} \leq y_F < 1$. Hence, for $y_F < \frac{1}{2}$, none of the three cases apply and Proposition 3 follows. $\blacksquare$

Proof of Proposition 4

If $0 \leq y_F \leq \frac{1}{2}$, then only Lemma 4(d) applies and Proposition 4 follows. If $\frac{1}{2} < y_F < 1$, then both Lemmas 2 and 4(d) apply. The forecaster's loss under Lemma 4(d) is $\frac{1}{20}(1 - y_F)^2 \leq \frac{1}{20}$ which is clearly less than the forecaster's loss under Lemma 2, which is $\mathcal{L} = \frac{1}{8}$. Hence, the forecaster chooses $\pi^e = \frac{3}{5}y_F + \frac{2}{5}$ from Lemma 4d (i). Since the Treasury chooses f, i having observed π^e, it chooses f, i as given in Lemma 4d (iv). $\blacksquare$

Proof of Proposition 5

Since $\frac{1}{2} \leq y_F < 1$, Lemma 3 clearly does not apply, i.e., the economy must be liquidity trapped in, at least, one state of the world. For $y_F = \frac{1}{2}$, only Lemma 4(c) applies. For $\frac{1}{2} < y_F < 1$, both Lemmas 2 and 4(c) apply. A simple calculation shows that the payoff to a forecaster, and also to each type of Treasury (W_+, W_-), is higher under the solution of Lemma 4(c) than it is for the solution to Lemma 2. If we adopt payoff dominance as our selection criterion, then we select the solution of 4(c).

We need a method of coordination between the players. Here evidential reasoning can help in the following manner. The Treasury selects the payoff dominant solution. It takes this as evidence that each member of the private sector will likewise select the payoff dominant solution. Furthermore, each member of the private sector takes her own calculation as evidence that all other players are carrying out similar calculations.

Actually, there are infinitely many solutions, depending on the fraction of the population believing whether the economy will be liquidity trapped in both states of the world or just the bad state. However, "payoff dominance" will select the solution where the economy is liquidity trapped in the bad state only.$\blacksquare$

Proof of Proposition 6

Since $y_F \geq 1$, Lemmas 2 and 3 apply but not Lemma 4. A simple calculation shows that it is the solution to Lemma 3 that satisfies payoff dominance. Selecting that solution establishes Proposition 6. $\blacksquare$

Proof of Proposition 7

Since $y_F \geq 1$, Lemmas 2 and 3 apply but not Lemma 4. A forecaster's loss under Lemma 3, $\mathcal{L} = 0$, is clearly less than under Lemma 2, which is $\mathcal{L} = \frac{1}{8}$. Hence, the forecaster chooses $\pi^e = y_F$ from Lemma 3(b). Since the Treasury chooses f, i having observed π^e, it chooses f, i as given in Lemma 3d. $\blacksquare$

9. References

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