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RHOMBUS FILTRATIONS AND RAUZY ALGEBRAS

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Abstract. Peach introduced rhombal algebras associated to quivers given by tilings of the plane by rhombi. We develop general techniques to analyze rhombal algebras, including a filtration by what we call rhombus modules.

We introduce a way to relate the infinite-dimensional rhombal algebra corresponding to a complete tiling of the plane to finite-dimensional algebras corresponding to finite portions of the tiling. Throughout, we apply our general techniques to the special case of the Rauzy tiling, which is built in stages reflecting an underlying self-similarity. Exploiting this self-similar structure allows us to uncover interesting features of the associated finite-dimensional algebras, including some of the tree classes in the stable Auslander-Reiten quiver.

1. Introduction

Peach introduced rhombal algebras in his thesis [10] by imposing certain relations on quivers corresponding to tilings of the plane by rhombi. He shows that quotients of these rhombal algebras model parts of weight 2 blocks of symmetric groups. Ringel [12] and Turner [13] have further analysed these rhombal algebras, and Chuang and Turner [5] generalised them to higher dimensions. Among the general methods to analyse infinite-dimensional rhombal algebras we develop, the most important tool is a filtration by rhombus modules. In particular, to each rhombus in the quiver we associate a module, which we call a rhombus module. We then determine how the projective indecomposable modules and others are filtered. In addition to more precise information on the module structure of the modules analysed, these rhombus filtrations provide insight into the structure of the module category.

We apply our methods to the infinite-dimensional rhombal algebra associated to a special rhombal tiling known as the Rauzy tiling. The Rauzy tiling is constructed in stages by a substitution rule that reflects the underlying self-similarity of the tiling. The dynamics and combinatorics of the Rauzy tiling have been extensively studied (see, for example, [1],[4] or [6]), and we make use of its special properties for the solution of many of the representation theoretic problems we encounter.

More precisely, we develop techniques relating the infinite-dimensional Rauzy algebra to finite-dimensional algebras related to the finite portions of the complete tiling that occur in its construction. Comparing different possible truncation methods for making such relations

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leads to a natural choice of truncating the infinite-dimensional algebra by an idempotent associated to the finite portions, resulting in what we call the finite-dimensional Rauzy algebras. Applying our technique of rhombus filtrations to analyse Rauzy algebras leads to an understanding of many of the modules that arise in this setting. Special features of the Rauzy tiling and our general techniques allow us to reveal many interesting properties of the finite-dimensional Rauzy algebras. In particular, we are able to identify modules with periodic orbits under application of the Heller operator. These periodic orbits occur along lines in the quiver that are analogues of the 'induced lines' introduced in [8]. Modules with periodic orbits are crucial for understanding the graph structure of the stable Auslander-Reiten quivers of the finite-dimensional Rauzy algebras. Through our results we are able to identify the tree classes and hence the graph structure of many components. Some of the features which we found lead to a general conjecture, including the occurrence of trees of class A_∞^∞.

2. The Rauzy tiling

Rauzy [11] introduced a fractal domain arising in a natural way from a substitution. Both the fractal and the substitution now bear his name. To better understand the Rauzy fractal and its dynamics, alternative geometric interpretations have been introduced. As shown in [1], [6, Chapt. 8], there is an increasing sequence of patches $\mathcal{P}_i$ of tiles whose union $\bigcup \mathcal{P}_i$ forms a complete tiling of the plane. Appropriate renormalizations of the patches $\mathcal{P}_i$ converge to the Rauzy fractal, but we shall only be interested in the tiling of the plane, which we refer to as the *Rauzy tiling*. We shall describe a recursive method for obtaining the increasing sequence of patches $\mathcal{P}_i$, the edges and vertices of which yield the quivers used to construct the Rauzy algebra. This recursive construction reflects the original substitution and produces a tiling with a hierarchical, self-similar structure.

The tiles in $\mathcal{P}_i$ are projections of faces of unit cubes in $\mathbb{R}^3$ with vertices in $\mathbb{Z}^3$. Specifically, we define the linear map $\mathrm{p} : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathrm{p}(\mathbf{e}_1) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right),\ \mathrm{p}(\mathbf{e}_2) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right),\ \text{and } \mathrm{p}(\mathbf{e}_3) = (0,1) = -(\mathrm{p}(\mathbf{e}_1) + \mathrm{p}(\mathbf{e}_2)),$$

where the $\mathbf{e}_i$ are the standard basis elements of $\mathbb{R}^3$. Then p projects the lattice $\mathbb{Z}^3$ onto the planar lattice

$$\mathcal{L} = \{m\,\mathrm{p}(\mathbf{e}_2) + n\,\mathrm{p}(\mathbf{e}_3) \ :\ m, n \in \mathbb{Z}\},$$

and all tiles will be rhombi with vertices in $\mathcal{L}$.

In fact, each rhombus of the tiling is a translation of one of the three rhombi R_1, R_2 or R_3 of the initial patch $\mathcal{P}_0$ pictured below, where the common point of all three is the origin. We shall describe a rhombus in $\mathcal{P}_i$ as a translation of one of the original rhombi of the

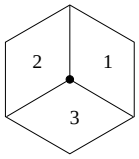


FIGURE 1. $\mathcal{P}_0$

form $\mathrm{p}(z) + R_k$ for some $z \in \mathbb{Z}^3$. As p is not injective, this representation is not unique. However, to each rhombus we will recursively assign a specific representation of this form, where we identify R_k with $\mathrm{p}(\mathbf{0}) + R_k$. The recursive process in going from $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$ is determined by the function $\mathcal{R}$ from the set of rhombi in $\mathcal{P}_i$ to the collection of rhombi in $\mathcal{P}_{i+1}$. A key element in the definition of $\mathcal{R}$ is the matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ derived from the Rauzy substitution. We denote the columns of $\mathbf{M}$ by $\mathbf{c}_i$, considered as elements of $\mathbb{Z}^3$. Then one obtains the rhombi in $\mathcal{P}_1$ by taking the union of the following sets of rhombi:

$$\mathcal{R}(R_1) = \{R_3,\, \mathrm{p}(\mathbf{c}_2) + R_1,\, \mathrm{p}(\mathbf{c}_3) + R_2\},\ \mathcal{R}(R_2) = \{R_1\},\ \text{and } \mathcal{R}(R_3) = \{R_2\}$$

In general, once the rhombi in $\mathcal{P}_i$ have been determined, one defines on each rhombus $\mathrm{p}(z) + R_k \in \mathcal{P}_i$

$$\mathcal{R}(\mathrm{p}(z) + R_k) := \mathrm{p}(\mathbf{M}z) + \mathcal{R}(R_k),$$

where the translation by $\mathrm{p}(\mathbf{M}z)$ applies to each rhombus in $\mathcal{R}(R_k)$, and one represents $\mathrm{p}(\mathbf{M}z) + \mathrm{p}(\mathbf{c}_{k+1}) + R_k$ as $\mathrm{p}(\mathbf{M}z + \mathbf{c}_{k+1}) + R_k$. One then obtains the rhombi in $\mathcal{P}_{i+1}$ by taking the union of all the rhombi in $\mathcal{R}$ applied to the rhombi in $\mathcal{P}_i$.

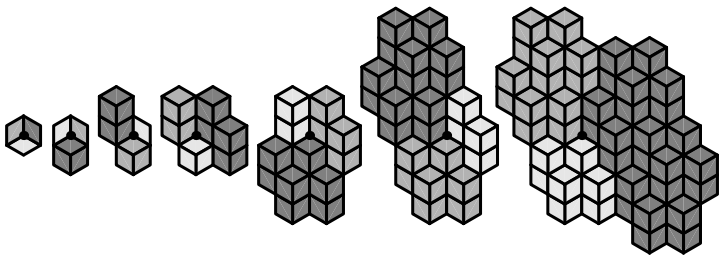


FIGURE 2. $\mathcal{P}_0$ through $\mathcal{P}_6$, with $\mathcal{R}^i(R_k)$ in $\mathcal{P}_i$ in darkness descending as k increases.

3. Rhombal algebras associated to the Rauzy tiling

3.1. **Overview of path algebras of quivers.** We survey the properties of the representation theory of quivers and paths algebras of quivers vital to our investigations. For a more detailed reference, we refer the reader to [2] and [3].

A quiver Q is an oriented graph. A path in the quiver is a sequence of arrows such that each arrow begins at the vertex at which the preceding arrow ended. Furthermore, to each vertex z we associate a trivial path e_z. For a field k, the path algebra kQ of the quiver is the algebra generated by the set of all paths of Q, where the multiplication of two paths is given by concatenation of the two paths if they concatenate and by zero otherwise. Note that for each vertex z of Q, e_z is an idempotent in kQ.
Given a set of relations R on the paths of Q, we can define a two sided ideal I generated by R, leading to the quotient algebra kQ/I. Then by a fundamental theorem of Gabriel [9], every basic algebra over an algebraically closed field is isomorphic to the path algebra of a quiver with relations.
The radical of kQ/I is spanned by the image of all paths of length ≥ 1. The socle of kQ/I is spanned by all paths b such that $ab = 0$ for all arrows a.
Furthermore, for every vertex z of Q, there is a simple kQ/I-module S_z of dimension one associated to z such that $S_z e_z = S_z$ and $S_z e_x = 0$ for all $x \neq z$, and for any arrow of Q the corresponding element in kQ/I acts as zero. Then $e_z(kQ/I)$ is the projective indecomposable corresponding to S_x. For each vertex x, the dimension of $e_z(kQ/I)e_x$ is equal to the composition multiplicity of S_x as a composition factor of $e_z(kQ/I)$.

3.2. **Infinite-dimensional Rauzy algebra.** The graph formed by the edges and vertices of the Rauzy tiling is a member of the family of graphs investigated by Peach [10]. We form a quiver Q from the Rauzy tiling by replacing each edge with a double arrow in opposite directions. In representing quivers, letters towards the end of the alphabet are reserved for vertices, while letters towards the beginning of the alphabet are used for arrows. When focusing on a particular vertex z, arrows in figures will be drawn with tail given by the vertex that is closest to z, and arrows and vertices around z will be labelled with indices increasing in the counter-clockwise direction, as indicated in figure 3. The arrow pointing in the opposite direction from the arrow b is denoted $\bar{b}$.

For the path algebra kQ associated to the field k, following Peach [10] we define a set of relations R on kQ as follows:
1. (Two rhombus relation) Any path of length two in kQ that borders more than one rhombus is zero.
2. (Mirror relation) Any two paths of length two connecting opposite vertices of the same rhombus are equal.
3. (Star relation) For any vertex z and labelling as indicated in figure 3, the following relations hold, where we replace a path by zero if there is no corresponding edge in the

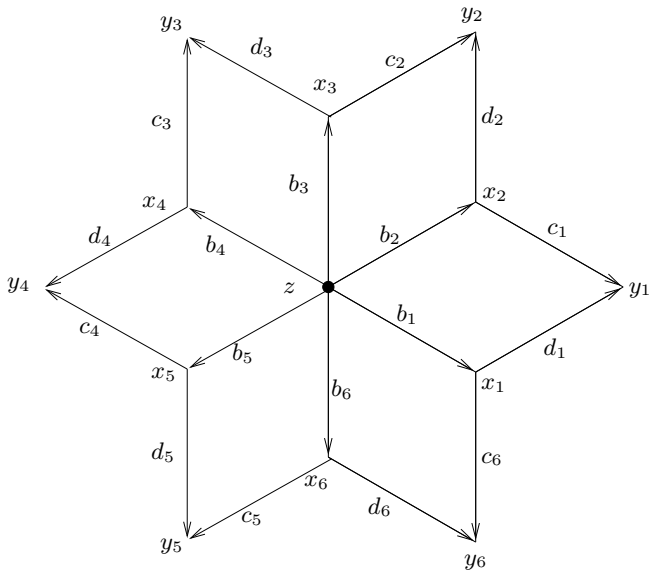


FIGURE 3. Labelling of the vertices and paths neighbouring z

quiver.

$$\begin{aligned} b_2\bar{b_2} - b_5\bar{b_5} &= \overset{|}{\varepsilon}_z(b_4\bar{b_4} - b_1\bar{b_1}) \\ b_6\bar{b_6} - b_3\bar{b_3} &= \overset{\diagdown}{\varepsilon}_z(b_2\bar{b_2} - b_5\bar{b_5}) \\ b_4\bar{b_4} - b_1\bar{b_1} &= \overset{\diagup}{\varepsilon}_z(b_6\bar{b_6} - b_3\bar{b_3}) \end{aligned}$$

where $\overset{|}{\varepsilon}_z, \overset{\diagdown}{\varepsilon}_z, \overset{\diagup}{\varepsilon}_z \in \{-1, 1\}$ and $\overset{|}{\varepsilon}_z\overset{\diagdown}{\varepsilon}_z\overset{\diagup}{\varepsilon}_z = 1$. A choice of signs at a single vertex z determines the signs at all vertices of Q. For more detail we refer the reader to [10].

Definition 1. The infinite-dimensional *Rauzy algebra* A is defined to be kQ/I, where I is the ideal generated by the relations R.

Let z be a vertex and $P_z = e_zA$ the corresponding projective module of A. Peach [10, Ch. 2] has shown many facts about paths in A and P_z that we list here for convenience.

Any path which does not border a single rhombus is zero in A. Furthermore, all paths of length ≥ 5 are zero in A, and all paths of length 4 which do not start and end at the same vertex are zero in A. The socle of P_z is simple and isomorphic to S_z. In fact, A is a symmetric algebra. The image of any path p of length four around a rhombus starting and ending at z spans the socle of P_z. Any two paths of length 3 around a single rhombus are linearly dependent. In particular, if x is a vertex such that there is an arrow b from z to x, then the space e_zAe_x has dimension 2, and it is spanned by b together with any path of length three around a rhombus which has corners z, x.

3.3. **Finite-dimensional Rauzy algebras.** Each patch $\mathcal{P}_i$ in the formation of the Rauzy tiling as described in section 2 gives rise to a quiver Q_i with vertex set V_i. Since $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ for each i and since the Rauzy tiling itself is given by $\cup\mathcal{P}_i$, the underlying graph of each Q_i embeds naturally in the infinite graph given by the Rauzy tiling. Two natural possibilities then arise for relating the finite-dimensional path algebras associated to the Q_i with the Rauzy algebra A. We can either truncate the vertices but allow all paths that start and end in a given V_i, or we can truncate the relations together with the vertices.

The first option corresponds to cutting A by idempotents given by the sum of the primitive idempotents of the vertices in V_i. With this approach, Q_i yields the algebra

$$A_i = e_i A e_i, \text{where } e_i = \textstyle\sum_{v \in V_i} e_v \text{ and } e_v \text{ is the primitive idempotent at } v.$$

Thus, there will be paths which are non-zero in A_i that go through a vertex not contained in V_i. The dotted lines in figure 4 indicate possible edges for non-zero paths of A_0 starting and ending at vertices in V_0 which are not paths of Q_0.

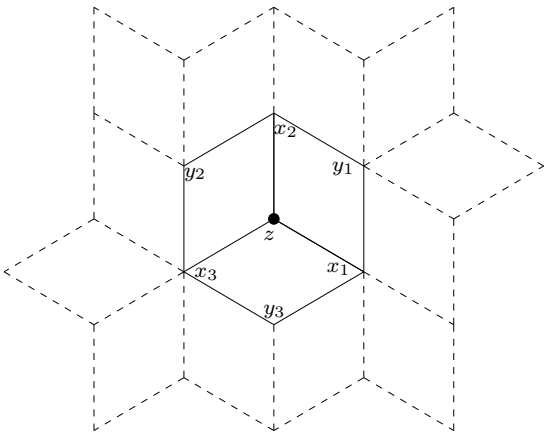


FIGURE 4. The vertices x_i, y_i and z are the vertices of Q_0

The second option corresponds to a truncation of the underlying graph on which we then impose the relations R, yielding the algebras $B_i = kQ_i/(I \cap kQ_i)$ with the relations R.
To decide which of these approaches to choose in order to define the finite-dimensional path algebras, we examine the indecomposable projective modules.

3.4. **Which type of truncation?** Let z be a vertex and $P_z = e_z A$ the corresponding indecomposable projective module of A, where e_z is the primitive idempotent at z. The star relations imply that the crucial portion of P_z is the space $e_z P_z e_z$. More precisely, the following subspace of $e_z P_z e_z$ and P_z proves to be most important.

Definition 2. Let X_z be the subspace of P_z spanned by paths of length two starting and ending at z.

We calculate the dimension of X_z as a subspace of A and as a subspace of the finite-dimensional algebras $A_i = e_iAe_i$. In fact, we show that in the latter case, X_z is also a subspace of a projective module at z if the primitive idempotent at z is a summand of the idempotent e_i. We wil also show that imposing the star relations on B_i produces a finite-dimensional algebra which is not symmetric, which we will therefore not consider in later sections.

The key difference in the relations for the two types of truncations lies in the star relations.

(I) Cutting A by the idempotent e_i. Recall that e_i is the sum of the primitive idempotents in Q_i and that $A_i = e_iAe_i$. In this case the star relations are as given in section 3.2. Let z be a vertex such that $e_ze_i = e_z$. Then e_z is a primitive idempotent of the algebra e_iAe_i, and we have the indecomposable projective modules $P_z = e_zA$ of A, and $e_zAe_i = e_z(e_iAe_i)$ of e_iAe_i. We now compare X_z with the subspace of e_zAe_i given by paths of length two. Since $e_ie_z = e_ze_i = e_z$, we have $e_zAe_ze_i = e_zAe_z$, and as a vector space e_zAe_z decomposes into $e_zAe_z = \langle e_z\rangle \oplus X_z \oplus \langle X_z^2\rangle$. It follows from [10, 2.4.12] that the last summand is 1-dimensional. The decomposition still holds when we multiply by e_i on both sides. Therefore, in e_zAe_i the space spanned by paths of length two beginning and ending at z is equal to X_z, and X_z has the following dimensions:

$$\dim(X_z) = \begin{cases} 1 & z \text{ is a 3-vertex} \\ 2 & z \text{ is a 4-vertex} \\ 3 & z \text{ is a 5-vertex} \\ 4 & z \text{ is a 6-vertex} \end{cases}$$

To examine the second type of truncation, the following definition is essential.

Definition 3. Let z be a vertex of Q_i. We say that z is a (k, n)-*vertex* if z is an n-vertex in the untruncated quiver Q and a k-vertex in Q_i.

For example, z in figure 7 is a $(3, 4)$-vertex, where the dotted arrows are outside Q_i.
We label the patch in the quiver around z as indicated in figure 5, with the same condition on arrows as before. Then if for example z is a three vertex, the patch in the quiver around z will be labelled as in figure 6.

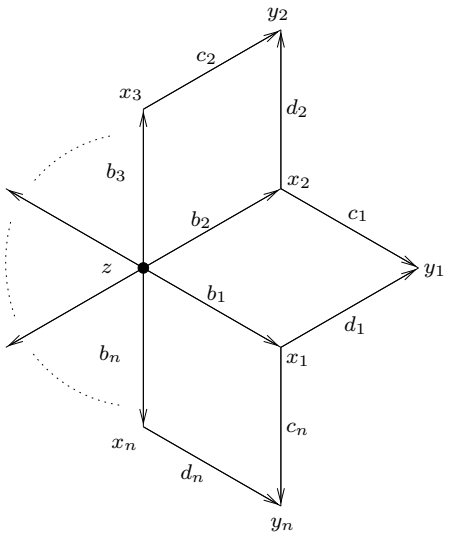


FIGURE 5.

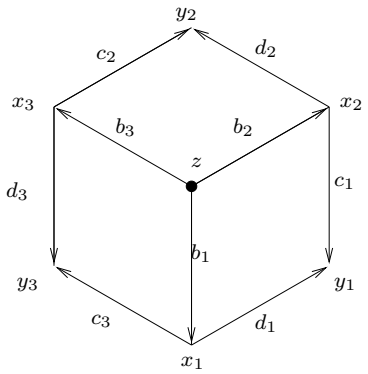


FIGURE 6.

(II) Imposing the relations on Q_i. Recall that $B_i = kQ_i/(I \cap kQ_i)$; that is, we impose the relations R on the truncated quiver. If z is a vertex of Q_i, we calculate the dimension of X_z as a subspace of B_i.

(a) Suppose z is a $(2,5)$-vertex in Q_i. Then X_z is spanned in A by two paths, and one pair of opposite paths is completely missing. Also, the opposite of each existing path is missing. If we now impose the star relation on the truncated quiver Q_i, all of the two paths in X_z are actually equal to zero.

(b) Suppose z is a $(6,3)$-vertex in Q_i. Then as a subspace of A, X_z is spanned by three paths. If we impose the star relations on Q_i, then any two of these paths are linearly dependent, and thus X_z as a subspace of B_i is 1-dimensional.

(c) Suppose z is a $(4,3)$-vertex in Q_i.

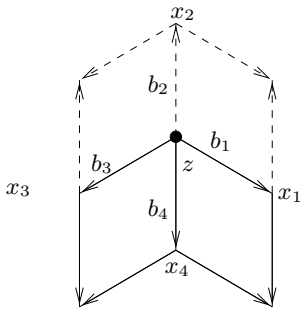


FIGURE 7. (4,3)-vertex

With the notations of figure 7 the star relations in A for such a vertex are

$$\begin{aligned} -b_3\bar{b}_3 &= \overset{|}{\varepsilon}_z(-b_1\bar{b}_1) \\ b_4\bar{b}_4 - b_2\bar{b}_2 &= \overset{\diagdown}{\varepsilon}_z(-b_3\bar{b}_3) \\ -b_1\bar{b}_1 &= \overset{\diagup}{\varepsilon}_z(b_4\bar{b}_4 - b_2\bar{b}_2) \end{aligned}$$

However, in the truncated quiver b_2 does not exist. So the imposed star relations give that any two of the three paths are linearly dependent and X_z as a subspace of B_i is again 1-dimensional.

As a special case, we consider the following.

Imposing star relations on Q_1.

Let z be the $(5,2)$-vertex of Q_1. Then with the notation established in section 3.2, we get a labelling as in figure 8.

Now consider the space $P'_z := e_z(B_1)$. We have seen that for the vertex z, the paths $b_3\bar{b}_3 = 0 = b_4\bar{b}_4$, where $b_3, \bar{b}_3, b_4$ and $\bar{b}_4$ are paths inside B_1. By the mirror relation, $b_3d_3 = b_4c_3$. Thus, for a basis of P_z the only path of length two we need is b_3d_3. Now consider paths of length three. There are only two such paths to consider: $b_3d_3\bar{d}_3$ and $b_3d_3\bar{c}_3$ (all others are zero). But $d_3\bar{d}_3$ occurs in a star relation starting at x_3, and from II(c) above we know that any two paths of length two are dependent. So $d_3\bar{d}_3 = e\bar{e}$, where $e \neq d_3$ is some path starting at x_3. But then $b_3d_3\bar{d}_3 = b_3e\bar{e} = 0$.
By the mirror relation $b_3d_3\bar{c}_3 = b_4c_3\bar{c}_3$. We have an imposed star relation at x_4 of the form

$$(0 - f\bar{f}) = \overset{|}{\varepsilon}_{x_4}(c_3\bar{c}_3 - 0).$$

Hence, by the two rhombus relation, $b_4c_3\bar{c}_3 = \pm b_4f\bar{f} = 0$.

This has proved that $(b_3d_3)J = 0$ (with $J = \mathrm{rad}(e_1Ae_1)$, and thus b_3d_3 lies in the (right) socle. So the projective P'_z has Loewy length at most 3 (most likely, it is 3); and, moreover,

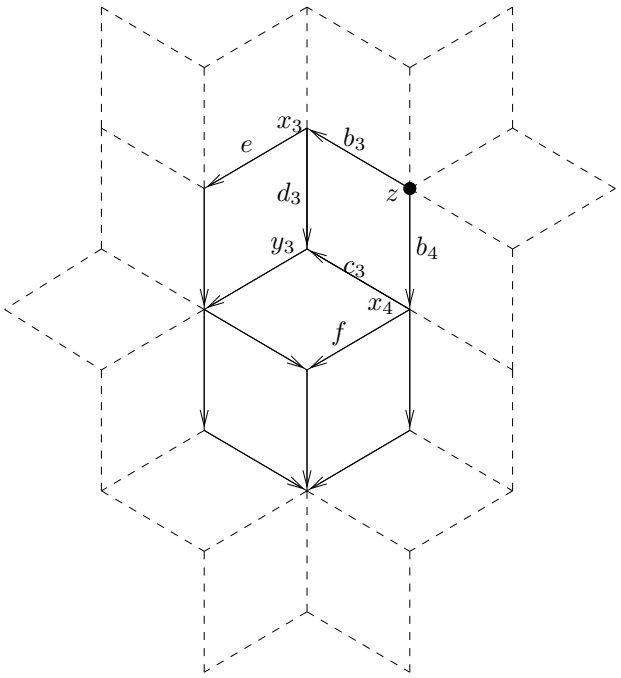


FIGURE 8. The solid arrows are in B_1 and the dotted lines represent arrows that are in A but not in B_1. The arrows are defined with respect to z, and we have only indicated at most one direction for each arrow.

the socle is not isomorphic to the top. Therefore, this is not a projective module for a symmetric algebra! (Although, the algebra is quite possibly self-injective).

Conclusion : Imposing the star relations on the truncated quiver does not produce symmetric algebras, and hereafter the only finite-dimensional algebras we consider are the A_i.

4. Rhombus modules and rhombus filtrations

In this section we develop a general technique that applies to all infinite-dimensional rhombal algebras. We describe how to associate modules, which we call rhombus modules, to any rhombus of the quiver associated to the algebra. These modules may be thought of as building blocks of the rhombal algebras. We show that the projective modules of the algebra A have filtrations by such rhombus modules.

4.1. **Rhombus modules.** Consider a rhombus in the quiver labelled as in figure 9.

Lemma 1. *For any rhombus in the quiver, there is a unique module R with top isomorphic to S_y and socle isomorphic to S_z and $rad(R)/soc(R) \cong S_{x_1} \oplus S_{x_2}$. Explicitly, $R = b_1 dA = b_2 cA$.*

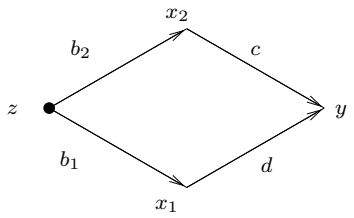


FIGURE 9.

Definition 4. We say that a module R such as in Lemma 1 is a *rhombus module* and denote it by $R = R^y_z$.

Proof (1) Existence: The space e_zAe_y is the 1-dimensional space spanned by b_1d. The module $R := (b_1d)A$ is contained in e_zA and therefore has socle S_z. The top of R is isomorphic to S_y as it is generated by an element x with $x = xe_y$. Furthermore, R has basis

$$b_1d, b_1d\bar{d}, b_1d\bar{c}, b_1d\bar{d}\,\bar{b}_1$$

That the elements listed are all non-zero follows from [10, 2.4.13].
(2) Uniqueness: Any such module must be isomorphic to a submodule of the injective module with socle S_z; that is, a submodule of e_zA. As it is generated by an element of e_zAe_y and this is 1-dimensional and spanned by b_1d, it follows that $R \cong b_1dA$. □

Remark There are four such rhombus modules for each rhombus in the quiver. (This may be considered as an analogue of the modelling of different $\mathcal{O}$-forms for a liftable module with an irreducible character.)

4.2. **Rhombus filtrations.** We shall show that several modules that arise naturally in our setting, including the modules generated by arrows and indecomposable modules, have rhombus filtrations as defined below.

Definition 5. We say that a module M has a *rhombus filtration* if there is a sequence of submodules $0 = M_0 \subset M_1 \subset \ldots M_k = M$, where M_i/M_{i-1} is a rhombus module for each i.

However, as the following lemma shows, 'rhombus filtration multiplicities' are not well-defined if one allows all rhombus modules as quotients.

Throughout this section we refer to figure 5 for notation.

Lemma 2. (Arrow Lemma) *Suppose b is an arrow in the quiver. Then the module bA has two rhombus filtrations, each with two quotients.*

Proof Use the standard labelling of vertices and arrows as defined in figure 5 and set $b = b_1$.
(1) There is an exact sequence

$$0 \to bc_nA \to bA \to \bar{b}_2bA \to 0.$$

Clearly bA contains the rhombus module bc_nA, and furthermore left multiplication by $\bar{b}_2$ induces a surjection $\pi : bA \to \bar{b}_2bA$. We also have $\bar{b}_2bc_n = 0$, so $bc_nA \subseteq \ker \pi$. Then equality holds by dimensions, since $bA/(bc_n)A$ has basis the cosets of $b, bd_1, b\bar{b}, bd_1\bar{c}_1$.
(2) Similarly there is an exact sequence

$$0 \to bd_1A \to bA \to \bar{b}_nbA \to 0.$$

□

Next we consider submodules of projectives which are generated by two arrows.

Lemma 3. (Two-arrow lemma) *With the notation of figure 5, assume z is an n-vertex and $n \geq 4$. Then the module $b_1A + b_2A$ has a rhombus filtration, with three rhombus quotients. Explicitly, we have an exact sequence*

$$0 \to b_1d_1A \ \to \ b_1A + b_2A \to \bar{b}_nb_1A \oplus \bar{b}_3b_2A \to 0$$

Proof Since $b_1d_1 = b_2c_1$, we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker p_1 & \longrightarrow & \ker p & \longrightarrow & \ker \phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & b_1d_1A \oplus b_2c_1A & \longrightarrow & b_1A \oplus b_2A & \longrightarrow & \bar{b}_nb_1A \oplus \bar{b}_3b_2A & \longrightarrow & 0 \\
 & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & b_1d_1A & \xrightarrow{j} & b_1A + b_2A & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Namely, take as the middle row the direct sum of the two arrow sequences. Then take for p and p_1 the addition maps, and the map j is inclusion. Then the left lower square commutes, and hence it induces the map ϕ making the right lower square commute as well. Clearly, p and p_1 are surjective, and then ϕ is also surjective. Thus, the top row is exact by the Snake Lemma.

By definition, we have $\ker p = \{(x, -x) : x \in b_1A \cap b_2A\}$, which is isomorphic to $b_1A \cap b_2A$. Similarly, $\ker p_1 = (b_1d_1, -b_2c_1)A \cong b_1d_1A$, which is a rhombus module.

If $n \geq 4$, we now show that $b_1A \cap b_2A = b_1d_1A$. To see this, note that $b_1d_1 = b_2c_1$, and b_1d_1A is thus contained in the intersection.
Suppose to the contrary that the intersection were not contained in b_1d_1A. Then there would be a simple module contained in both b_1A/b_1d_1A and b_2A/b_2c_1A. From the Two-arrow Lemma, we know that $b_1A/b_1d_1A \cong \bar{b}_n b_1 A$ and that it has simple socle isomorphic to S_{x_n}. Similarly, $b_2A/b_2c_1A \cong \bar{b}_3 b_2 A$, which has simple socle isomorphic to S_{x_3}. For $n \geq 4$ the vertices x_n and x_3 are distinct, and then S_{x_n} is not isomorphic to S_{x_3}, a contradiction that demonstrates the desired equality.
Hence in this case, $\ker \phi = 0$ and ϕ is therefore an isomorphism. □

Remark. The condition that $n \geq 4$ is necessary. If z is a 3-vertex, then the intersection of b_1A and b_2A properly contains the rhombus b_1d_1A and ϕ has a kernel of dimension two.

We now prove a 'dual' version of the previous lemma. As the arrow $\bar{b}_1$ starts at the vertex x_1 and the arrow $\bar{b}_2$ starts at the vertex x_2, $(\bar{b}_1, \bar{b}_2)A$ is a submodule of $e_{x_1}A \oplus e_{x_2}A$.

Lemma 4. (Dual two-arrow lemma) *Suppose z is an n-vertex with $n \geq 4$. Then $(\bar{b}_1, \bar{b}_2)A$ has a rhombus filtration with three rhombus quotients. Explicitly, there is an exact sequence*

$$0 \to \bar{b}_1 b_n A \oplus \bar{b}_2 b_3 A \to (\bar{b}_1, \bar{b}_2)A \to \bar{d}_1 \bar{b}_1 A \to 0$$

Proof We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \pi_1 & \longrightarrow & (\bar{b}_1, \bar{b}_2)A & \xrightarrow{\pi_1} & (\bar{d}_1\bar{b}_1, \bar{c}_1\bar{b}_2)A & \longrightarrow & 0 \\
 & & {\scriptstyle\phi}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j_1}\downarrow & & \\
0 & \longrightarrow & \bar{b}_1 b_n A \oplus \bar{b}_2 b_3 A & \longrightarrow & \bar{b}_1 A \oplus \bar{b}_2 A & \xrightarrow{\pi} & \bar{d}_1\bar{b}_1 A \oplus \bar{c}_1\bar{b}_2 A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Cok}\phi & \longrightarrow & \operatorname{Cok} j & \longrightarrow & \operatorname{Cok} j_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Namely, start with the middle row which is a direct sum of the two arrow sequences. Then take for j and j_1 the inclusion and for π_1 the restriction of π. Then the top right square commutes and induces the map ϕ, which then also is 1-1. Now the Snake Lemma gives

the lower row. Since $\bar{d}_1\bar{b}_1A = \bar{c}_1\bar{b}_2A$, it follows that $\operatorname{Coker} j \cong \bar{d}_1\bar{b}_1A$ which is a rhombus module. If $n \geq 4$, then we check that $\operatorname{Cok} j \cong \operatorname{Cok} j_1$ (for example, by dimensions), and then ϕ is an isomorphism. □

As before, when $n = 3$ this does not hold.

Lemma 5. (Three-arrow lemma) *Suppose $n \geq 5$. Then the module $M = b_1A + b_2A + b_3A$ has a rhombus filtration with four quotients. Explicitly, we have an exact sequence*

$$0 \to b_1A + b_2A \to M \to \bar{b}_4b_3A \to 0.$$

Proof The module M is contained in e_zA. Define $\pi : M \to \bar{b}_4A$ to be multiplication by $\bar{b}_4$, a homomorphism from M onto $\bar{b}_4b_3A$. From [10] (see the summary in section 3.2), we know that $b_1A + b_2A$ is contained in the kernel of π. It remains to show that equality holds. Restrict π to b_3A, the Arrow Lemma shows that $\ker \pi \cap b_3A = b_3c_2A$. But $b_3c_2 = b_2d_2 \in b_2A$, and so the intersection of $\ker \pi$ with b_3A is contained in $b_1A + b_2A$. Now it follows that $\ker \pi \subseteq b_1A + b_2A$. The claim follows. □

Below is a dual version. The proof, which is similar, is left to the reader.

Lemma 6. (Dual three-arrow lemma) *Let $n \geq 5$. The module $M := (\bar{b}_1, \bar{b}_2, \bar{b}_3)A$ has a rhombus filtration with four quotients, and we have an exact sequence*

$$0 \to \bar{b}_3b_4A \to M \to (\bar{b}_1, \bar{b}_2)A \to 0.$$

4.3. **Indecomposable projective modules.** First we observe that the composition factors of projective modules have an easy description in terms of possible rhombus filtrations. For this we use the following standard notation from the representation theory of quivers: for an A module M, $\underline{\dim}\ M = (m_x)_x$, where x ranges over the set of vertices of Q and $m_x = \dim Me_x$ for the primitive idempotent e_x at x.

Recall that $[e_zA : S_x] = \dim e_zAe_x$. For this to be nonzero, there must be a rhombus which has corners z and x. As noted in 3.2, if there is an arrow from z to x, the dimension is 2. If z and x are opposite corners, then the dimension is 1; specifically, any two paths of length 2 are the same by the mirror relation, and there are no non-zero paths of length > 2. Furthermore, the dimension of e_zAe_z is n if z is an n-vertex. Together these facts imply the following lemma.

Lemma 7. (Multiplicity Lemma) *The composition factors of e_zA are given by*

$$\underline{\dim}\, e_zA = \sum_i \underline{\dim}\, (R^z_{a_i}).$$

That is, $[e_zA : S_z] = n$ *where* z *is an* n*-vertex,* $[e_zA : S_{x_i}] = 2$ *and* $[e_zA : S_{y_i}] = 1$*, for each* i*.*

The same applies to the modules e_zA_i in A_i for z sufficiently far from the boundary of Q_i.

We now describe rhombus filtrations of indecomposable projectives with the aid of the the *Heller operator* Ω. Recall that for a module M with projective cover $p : P_M \to M$, $\Omega(M) = \ker p$. As the number of rhombi at a given vertex depends on the type of vertex, we must deal with the cases individually.

Lemma 8. (3-vertex projectives) *Suppose* z *is a 3-vertex. Then we have*

$$0 \to b_1A \to e_zA \to \bar{c}_2\bar{b}_3A \to 0$$

and

$$0 \to \bar{d}_1\bar{b}_1A \to e_zA \to b_3A \to 0.$$

In particular, $\Omega(\bar{c}_2\bar{b}_3)A \cong b_1A$ *and* $\Omega(b_3A) \cong \bar{d}_1\bar{b}_1A$*.*

Proof Left multiplication by $\bar{c}_2\bar{b}_3$ gives an epimorphism from e_zA onto $\bar{c}_2\bar{b}_3A$, and since $\bar{c}_2\bar{b}_3b_1 = 0$ we know that b_1A is contained in the kernel of this map. The description of the multiplicities of e_zA from the Multiplicity Lemma, together with the Arrow Lemma show that $\underline{\dim}\, e_zA = \underline{\dim}\, b_1A + \underline{\dim}\, \bar{c}_2\bar{b}_3A$, and hence the first sequence is exact. Similar arguments yield the second sequence. □

Lemma 9. (4-vertex projectives) *Suppose* z *is a 4-vertex. Then there is a short exact sequence*

$$0 \to b_1A \to e_zA \to \bar{b}_3A \to 0$$

In particular, $\Omega(b_1A) \cong \bar{b}_3A$*.*

Proof The arrow $\bar{b}_3$ ends at vertex z, and we have a surjection $e_zA \to \bar{b}_3A$ given by left multiplication with $\bar{b}_3$. Since $\bar{b}_3b_1 = 0$, we see that b_1A is contained in the kernel of this map. Again, the Multiplicity Lemma and the Arrow Lemma imply that b_1A must be equal to the kernel. □

Lemma 10. (5-vertex projectives) *We have a short exact sequence of* A*-modules*

$$0 \to b_1A + b_2A \to e_zA \to \bar{b}_4A \to 0$$

Hence e_zA *has a rhombus filtration. Moreover,* $\Omega(\bar{b}_4A) \cong b_1A + b_2A$*.*

Proof We have a surjective homomorphism $e_zA \to \bar{b}_4A$ given by left multiplication. Then $b_1A + b_2A$ is contained in the kernel. It follows from the Two-arrow Lemma that $b_1A + b_2A$ has a rhombus filtration with three quotients. Furthermore, from the Arrow Lemma we know that $\bar{b}_4A$ has a rhombus filtration with two quotients. In each case we know the composition factors, and it follows that the sequence is exact. In particular, e_zA has a rhombus filtration. □

Lemma 11. (6-vertex projectives) *Assume z is a 6-vertex. There is a short exact sequence*

$$0 \to b_1A + b_2A \to e_zA \to (\bar{b}_4, \bar{b}_5)A \to 0.$$

The kernel and the cokernel both have a rhombus filtration with three quotients.

Proof There is an obvious surjection from e_zA onto $W := (\bar{b}_4, \bar{b}_5)A$, and $b_1A + b_2A$ is contained in the kernel. From the Dual two-arrow Lemma it follows that W has a rhombus filtration with three quotients, and we also know that $b_1A + b_2A$ has a rhombus filtration with three quotients (see the Two-arrow Lemma). The Multiplicity Lemma now shows that the sequence is exact. □

Remark By the same arguments as in the proofs of Lemmas 9 and Lemmas 10, if z is a 6-vertex, then there also is a short exact sequence

$$0 \to b_1A + b_2A + b_3A \to e_zA \to \bar{b}_5A \to 0$$

such that the kernel has a rhombus filtration with four quotients, the cokernel a rhombus filtration with one quotient and $\Omega(\bar{b}_5A) \cong b_1A + b_2A + b_3A$.

5. Truncations

We now examine the finite-dimensional Rauzy algebras $A_i = e_iAe_i$ in the light of rhombus filtrations. Recall that if z is a vertex in V_i, then $e_ze_i = e_z$, and furthermore we have that e_zAe_i is the indecomposable projective (and injective) module of A_i corresponding to z. In particular, it has simple top and socle.

The functor $(-)e_i$ from the module category of right A-modules to the module category of right A_i-modules is exact. As a result, for any vertex z of Q_i, the projective e_zA_i has a filtration by truncations of rhombus modules, and these are easy to write down. Furthermore, if R is a rhombus module and $Re_i \neq 0$, then Re_i is indecomposable (this follows from the shape of Q_i).

5.1. **Truncation of arrow modules.** The truncation of an arrow module bA when the arrow b starts or ends at some vertex z in Q_i has interesting properties. If b starts at vertex z, then bAe_i is a submodule of e_zAe_i, and hence bAe_i has a simple socle. Similarly, if b ends at z, then bAe_i has a simple top. In either case, bAe_i is indecomposable.
The module bA has a filtration different from the rhombus filtration which gives more precise information about bAe_i.
Let $b = b_1$ with $b : z \to x_1$. We need to distinguish between two cases determined by the angles at z of the two rhombi which have b in common (see figure 10).

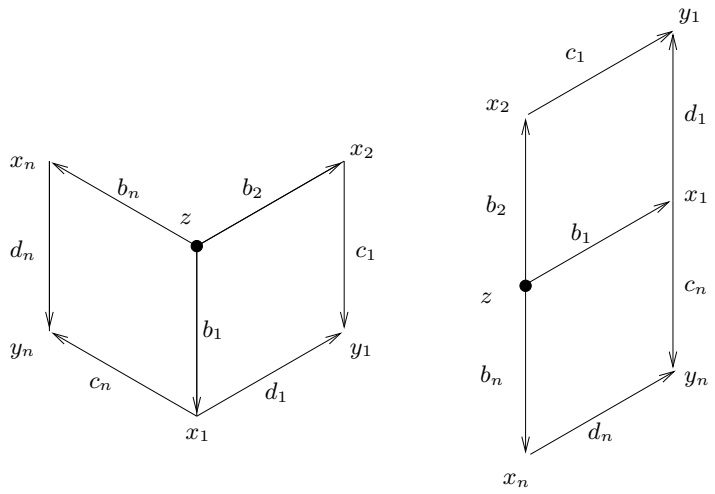


FIGURE 10. An illustration of adjacent acute and obtuse angles

(i) either both angles are obtuse, or both angles are acute;
(ii) one of the angles is obtuse, and the other is acute.

(i) Consider the first case. Then we have an exact sequence

$$0 \to \bar{b}bd_1A + \bar{b}bc_nA \to \bar{b}A \to b\bar{b}A \to 0. \tag{1i}$$

The kernel is 3-dimensional with a simple socle and top of length two, and its composition factors are S_{y_1}, S_{y_n} and S_{x_1}. There is also an exact sequence

$$0 \to \bar{b}bA \to \bar{b}A \to (\bar{b}_2b\bar{b}, \bar{b}_nb\bar{b})A \to 0. \tag{2i}$$

Here, the cokernel is 3-dimensional with a simple top and socle of length two. Its composition factors are S_{x_2}, S_{x_n} and S_z.

Dually, we have exact sequences

$$0 \to b\bar{b}b_nA + b\bar{b}b_2A \to bA \to \bar{b}bA \to 0 \tag{3i}$$

with kernel of length three and composition factors S_{x_n}, S_{x_2} and S_z; and

$$0 \to b\bar{b}A \to bA \to (\bar{d}_1\bar{b}b, \bar{c}_n\bar{b}b)A \to 0 \tag{4i}$$

with cokernel of length three, and composition factors S_{y_1}, S_{y_n} and S_{x_1}.

(ii) Consider the second case, which can only occur if z is a 5-vertex or a 4-vertex.
(a) Assume b starts at a 5-vertex z, as in the right of figure 10 with $n = 5$ and $b = b_1$. Then the star relation at the vertex z gives

$$b\bar{b} - b_4\bar{b}_4 = \pm b_3\bar{b}_3.$$

Since $\bar{b}_4 b = 0$ and $\bar{b}_3 b = 0$, right multiplication by b shows that $b\bar{b}b = 0$.
Similarly we have $\bar{b}b_4 = 0$ and $\bar{b}b_3 = 0$, and hence $\bar{b}b\bar{b} = 0$.
(b) Assume b starts at a 4-vertex z and a rhombus adjacent to b has an obtuse angle at z. As follows from the general list of possible configurations of rhombi as detailed in Table 1 of section 7.2, the only type of 4-vertex which occurs in the Rauzy algebra is the completition to a 4-vertex as in the left of figure 10.
Assume $b = b_2$. Then the star relation at z gives

$$b\bar{b} = \pm b_3\bar{b}_3,$$

and $\bar{b}_3 b = 0$ implies $b\bar{b}b = 0$.
Now we have seen that in case (ii) $b\bar{b}b = 0$, and similarly $\bar{b}b\bar{b} = 0$. Therefore we have the exact sequences

$$0 \to \bar{b}bA \to \bar{b}A \to b\bar{b}A \to 0, \tag{1ii}$$

$$0 \to b\bar{b}A \to bA \to \bar{b}bA \to 0. \tag{2ii}$$

Lemma 12. (Arrow truncation) *Let z be a 4-vertex.*
(a) Suppose $e_{x_1}e_i = 0 = e_{y_1}e_i = e_{y_4}e_i$ but $e_z e_i \neq 0$. Then we have $bAe_i = b\bar{b}Ae_i = b\bar{b}A_i = \bar{b}Ae_i$.
(b) Suppose $e_z e_i = 0 = e_{x_4}e_i = e_{x_2}e_i$ but $e_{x_1}e_i \neq 0$. Then we have $bAe_i = \bar{b}bAe_i = \bar{b}bA_i = \bar{b}Ae_i$.

Note that b and $\bar{b}$ do not belong to A_i, but in part (a) we have $b\bar{b} \in A_i$ and in part (b) we have $\bar{b}b \in A_i$.

Proof (i) Assume first that z is as in case (i).
For part (a), apply the exact functor $(-)e_i$ to the exact sequence (1i). The kernel becomes zero by the hypotheses of part (a), and therefore

$$\bar{b}Ae_i = b\bar{b}Ae_i = b\bar{b}A_i.$$

Furthermore, we take the exact sequence (4i), which has bA as the middle term, and apply the functor $(-)e_i$. By the hypotheses of part (a), this time the cokernel becomes zero. So we have $bAe_i = b\bar{b}Ae_i = b\bar{b}A_i$.
Part (b) is proved similarly by using the exact sequences (2i) and (3i).
(ii) Now assume z is as in case (ii).
We apply the exact functor $(-)e_i$ to the exact sequences (1ii) and (2ii). With the hypotheses of part (a), we have $\bar{b}bAe_i = 0$, and the claim follows. To prove (b), we note that $b\bar{b}Ae_i = 0$ in this case. □

Suppose $0 \neq \beta = \bar{b}b \in A_i$, with b and $\bar{b}$ arrows in A but not in A_i, then β may or may not belong to $\operatorname{rad}^2 A_i$. This leads to the following definition.

Definition 6. We call $\beta \in A_i$ a *loop* whenever $\beta = \bar{b}b$, $\beta \notin \operatorname{rad}^2 A_i$, and b and $\bar{b}$ are arrows in A but not in A_i.

5.2. **Identifying loops in A_i.** To properly understand the structure of A_i, we must determine which vertices of Q_i admit loops.

Definition 7. We call a (k, n)-vertex z *isolated* if $k = 2$. We call an isolated vertex *acute* (resp. *obtuse*) if the corresponding angle at z is acute (resp. obtuse).

Note that isolated vertices always lie on the boundary of the corresponding $\mathcal{P}_i$. We use the notation of section 2, and rhombi are considered to be of type $1, 2$ or 3 as indicated in figure 1.
A $\mathcal{C}$-patch denotes a translation of $\mathcal{P}_0$ occuring as a patch in some $\mathcal{P}_i$. And a $\mathcal{C}'$-patch denotes a translation of the patch corresponding to $\mathcal{R}(R_1)$.

Lemma 13. *If a rhombus R in $\mathcal{P}_i$ occurs as the image under either one or two iterations of $\mathcal{R}$ of a rhombus in some $\mathcal{C}$-patch or $\mathcal{C}'$-patch, then there is no acute isolated vertex in R.*

Proof The function $\mathcal{R}$ induces a function on patches as well as on individual tiles. Since the image under either one or two iterations of $\mathcal{R}$ on either a $\mathcal{C}$-patch or a $\mathcal{C}'$-patch has no isolated vertex and since the occurence of a patch in $\mathcal{P}_i$ can only transform isolated vertices into unisolated vertices, one obtains the lemma. □

Lemma 14. *For any $i \geq 0$, no acute isolated vertex occurs in $\mathcal{P}_i$.*

Proof The proposition is clear for $i = 0$.
For $i \geq 1$, a rhombus in $\mathcal{P}_i$ of type 3 can only occur as the image of a rhombus of type 1. Hence, any type 3 rhombus in $\mathcal{P}_i$ must occur as part of a $\mathcal{C}'$-patch, which can have no acute isolated vertex.
For $i \geq 1$, a rhombus R of type 2 can only occur as an image of a rhombus of type 1 or 3. If R is the image of type 1 rhombus, R must occur as part of a $\mathcal{C}'$-patch. If R occurs as the image of a rhombus R' of type 3 in $\mathcal{P}_{i-1}$, then R' must be part of a $\mathcal{C}$-patch or a $\mathcal{C}'$-patch, and so by lemma 13 R can have no isolated acute vertex.
For $i \geq 1$, a rhombus R of type 1 can only occur as an image of a rhombus of type 1 or 2. If R is the image of a type 1 rhombus, R must occur as part of a $\mathcal{C}'$-patch. If R occurs as the image of R' of type 2, by considering the previous case, one sees that lemma 13 applies to R as well. □

Lemma 15. *For any (k, n)-vertex z, $n - k \leq 3$.*

Proof As every vertex is the vertex in a rhombus, there are always at least two edges and at most six edges at z; thus, $2 \leq n, k \leq 6$. Therefore we always have $n - k \leq 4$, and we only have to check that $n - k = 4$ does not occur. As $n \geq k$, the only possibility is a $(2, 6)$-vertex. But a $(2, 6)$-vertex corresponds to an acute isolated vertex in $\mathcal{P}_i$, which cannot occur by lemma 14. □

Proposition 16. *The (k, n)-vertex z yields a loop in A_i if and only if $k < n - 2$.*

Proof As there are n different paths of length two from and to z in Q and there are two independent relations, the dimension of the space X_z is $n - 2$.
Then $\mathrm{rad}^2(A_i) \cap X_z$ is spanned by k elements and hence has dimension $\leq k$. If $k < n - 2$, then this space is strictly contained in X_z and there are loops. If $n - 2 - k = 1$ then there is just one.
Suppose $k \geq n - 2$. One checks easily, case by case, that always $\mathrm{rad}^2(A_i) \cap X_z = X_z$, and hence there is no loop in this case. □

By lemma 15, $(3, 6)$-vertices and $(2, 5)$-vertices are the only such vertices. In these cases there is just one loop at such a vertex.
If P is indecomposable projective and injective, the subquotient $\mathrm{rad}P/\mathrm{soc}P$ is often called the 'heart' of P.

Proposition 17. *If there is a loop at the vertex z, then the projective P_z has decomposable heart, and one of the summands is simple and isomorphic to S_z.*

Proof Suppose z is a $(3,6)$-vertex. Let x_1, x_2, x_3 be the neighbouring vertices of z in Q_i and let $y_1, y_2, b_1, \ldots$ be as before (see figure 3). Assume that b_1, b_2, b_3 are neighbouring arrows which are nonzero in A_i. Then $X_z \cap (\sum_{i=1}^{3} b_i A)$ is spanned by $b_i\bar{b}_i$ for $i = 1, 2, 3$, and they are linearly independent. But X_z has dimension four, so we need some element in X_z to generate the radical of $e_z A_i$.
We can take as the additional generator the path $\gamma = b_5\bar{b}_5$. Then $\gamma b_1 = \gamma b_2 = \gamma b_3 = 0$.
Let $M := \sum_{i=1}^{3} b_i A$, then $\mathrm{rad}P_z = M + \gamma A_i$ and $M \cap \gamma A_i = \gamma^2 A_i = \mathrm{soc}P_z$. That is, $\mathrm{rad}P_z/\mathrm{soc}P_z$ is the direct sum of a 1-dimensional module with $M/\mathrm{soc}P_z$, and the 1-dimensional summand is isomorphic to S_z.
Similarly, if z is a $(2,5)$-vertex, then the loop gives rise to a 1-dimensional direct summand of $rad(P_z)/soc(P_z)$. □

6. The stable Auslander-Reiten quiver of A_i

The *Auslander-Reiten quiver* $\Gamma(\Lambda)$ for a finite-dimensional symmetric k-algebra Λ is the quiver with vertices the isomorphism classes of indecomposable A-modules and arrows $[M] \to [N]$ corresponding to irreducible maps $f : M \to N$, where $[M]$ is the class of the module M. Thus, by providing generators and some relations, $\Gamma(\Lambda)$ may be thought of as yielding part of a presentation of the module category of Λ. The irreducible maps are occur in the setting of almost split sequences. The almost split sequence ending in N is a non-split short exact sequence of A-modules

$$0 \to \Omega^2(N) \xrightarrow{g} X \xrightarrow{h} N \to 0$$

such that any map $\Omega^2(N) \to Y$ which is not a split monomorphism factors through g. Equivalently, any map $Y \to N$ which is not a split epimorphism factors through h.
There is an irreducible map $M \to N$ if and only if M is isomorphic to a direct summand of X, and if so then the irreducible map occurs as a component of h. Dually, there is an irreducible map $\Omega^2(N) \to M$ if and only if M is a direct summand of X, and if so then the irreducible map occurs as a component of g; see, for example, [2] or [3].

Any indecomposable projective module P of Λ with simple top S has a unique almost split sequence where P occurs, namely

$$(*) \qquad 0 \to \Omega(S) \to P \oplus \mathrm{rad}P/\mathrm{soc}P \to \Omega^{-1}(S) \to 0$$

The *stable Auslander-Reiten quiver* $\Gamma_s(\Lambda)$ of Λ is obtained from $\Gamma(\Lambda)$ by removing the projective modules and adjacent arrows. Any connected component of $\Gamma_s(\Lambda)$ is, as a graph, isomorphic to $\mathbb{Z}T/\Pi$, where Π is some admissible group of automorphisms. The graph $\bar{T}$

associated to T is uniquely determined by the component, and is called its *tree class*, hence one wants to identify $\bar{T}$. The main method to do this, for symmetric (or self-injective) algebras, uses subadditive functions, and for these one needs to establish the existence of suitable Ω-periodic modules, which we herafter refer to as *periodic modules*. For more detail, see [3] and [7].

Lemma 18. ([7]) *Suppose Θ is a component of the stable Auslander-Reiten quiver of a symmetric algebra. If there is some periodic module W such that $\underline{\mathrm{Hom}}_\Lambda(W, X) \neq 0$ for some module X in Θ, then the tree class of Θ is Dynkin, Euclidean, or one of the infinite trees $A_\infty, A_\infty^\infty$, D_∞, or possibly B_∞ or C_∞.*

With this hypothesis, one takes as (sub)additive function the map

$$d_W(-) := \dim(\underline{\mathrm{Hom}}(\hat{W}, -),$$

where $\hat{W}$ is the direct sum of all distinct Ω-translates of W.

Types B_∞ or C_∞ cannot occur for algebraically closed fields. For group algebras or other symmetric algebras arising naturally where tree classes are known, one usually has the tree class of A_∞, and one might expect this to be the case more generally. Now we focus our attention again on the algebras A_i to determine whether suitable (sub)additive functions exist and to see if the tree classes of simple modules can be identified.

Proposition 19. *Suppose x is a vertex of the quiver of A_i, and assume that there is a periodic module W such that $\underline{\mathrm{Hom}}(W, S_x) \neq 0$. Let $H_x := \mathrm{rad}(P_x)/\mathrm{soc}(P_x)$, and let $\bar{T}$ be the tree class of the stable component containing S_x. If $\bar{T}$ is an infinite tree, we have the following dichotomy.*
(a) If H_x is indecomposable, then $\bar{T} \cong A_\infty$.
(b) If H_x is decomposable, then the tree class is not A_∞.

Proof (a) We work over $\bar{k}$, the algebraic closure of k. In this case, B_∞ and C_∞ are excluded. If $\bar{T} = A_\infty$ over $\bar{k}$, then it follows from 'Galois descent' (see [3]) that $\bar{T} = A_\infty$ over k as well.

The components of S_x and of H_x have the same tree class. This holds since H_x is in the component of $\Omega(S_x)$, as the standard sequence shows, and Ω induces a stable self-equivalence of A_i, hence it induces a graph isomorphism of the stable Auslander-Reiten quiver.
So we consider the tree class of the component of H_x and therefore of $\Omega(S_x)$. Recall that the vertices of $\bar{T}$ correspond to the Ω^2-orbits on the component. Since H_x is indecomposable,

the middle term of the almost split sequence starting in $\Omega(S_x)$ has indecomposable non-projective part. This means that the corresponding vertex of $\bar{T}$ must be an end vertex. Suppose $\bar{T} \neq A_\infty$, then the only other possibility is D_∞. Since the vertex of $\bar{T}$ corresponding to $\Omega(S_x)$ is an end vertex, it follows that the vertex of $\bar{T}$ corresponding to H_x is the branch vertex. Therefore, by the definition of $\mathbb{Z}D_\infty$, there is an almost split sequence of the form

$$(**) \qquad 0 \to \Omega^2(U) \to H_x \xrightarrow{g} U \to 0$$

where $U \ncong \Omega^{-1}(S_x)$.
We have $\mathrm{soc}(H_x) \cong \oplus S_{x_i}$, where x_i ranges over all the neighbours of x in Q_i. Now, $\mathrm{soc}(\Omega^2(U))$ is contained in $\mathrm{soc}(H_x)$. On the other hand, we have $\mathrm{Ext}^1(U, S_x) \neq 0$, and a non-zero element in this space exists, namely the inverse image of U in P_x. But

$$\mathrm{Ext}^1(U, S_x) \cong \underline{\mathrm{Hom}}(\Omega(U), S_x) \cong \mathrm{Hom}(\Omega(U), S_x)$$

and hence S_x occurs in the top of $\Omega(U)$, which is isomorphic to the socle of $\Omega^2(U)$, a contradiction.

(b) Suppose now that H_x is a direct sum. This happens only when the quiver of A_i has a loop at vertex x. In this case we know that H_x has a direct summand isomorphic to S_x. Then by considering the standard sequence (*) we see that there is an irreducible map $\Omega(S_x) \to S_x$. This shows that the graph automorphism of the stable Auslander-Reiten-quiver fixes the component of S_x, but it induces a non-trivial automorphism of the tree $\bar{T}$ as the Ω^2-orbits of S_x and of $\Omega(S_x)$ are distinct. But A_∞ does not have a non-trivial automorphism. □

By Auslanders's theorem, an indecomposable agebra of infinite type does not have a component where the tree class is Dynkin.

Conjecture *(a) For any simple module S_x there is a periodic module W with $\underline{\mathrm{Hom}}(W, S_x) \neq 0$.*
(c) Euclidean components do not occur in the stable Auslander-Reiten-quiver of A_i.
(c) If A_i has a loop at the vertex x, then the tree class of the component of S_x is A_∞^∞.

To prove this, we would need suitable periodic modules. We will next show that we have periodic modules, and this will allow us to identify tree classes of some components of simple modules.

7. Search for periodic modules

In this section we examine when arrow modules are periodic.

7.1. **Ω-Translates of arrow modules.** For an arrow $b : v \to z$, the arrow module depends on z. Below is a list of the different possibilities.

Arrow translates:

(1) If b is an arrow which ends at a 5-vertex z (see figure 5 with $n = 5$ and $b = \bar{b}_1$), then we have seen that $\Omega(bA) = b_3A + b_4A$.
(2) If b ends at a 4-vertex (see figure 10 with $b = \bar{b_1}$), then $\Omega(bA) = b_4A$. (This then determines the Ω-translates for arrow modules along any path in the quiver that goes only between 4-vertices).
(3) If the arrows b_3 and b_4 in (1) bound the same rhombus and both end at a 3-vertex, then $\Omega(b_3A + b_4A) = (c_3, -d_3)A$.
(4) If two adjacent arrows $\bar{b}_1$ and $\bar{b}_2$ end at a 5-vertex (see figure 5 with $n = 5$ with an obtuse angle between b_4 and b_5 or b_3 and b_4), then $\Omega\left((\bar{b}_1, \bar{b}_2)A\right) = b_4A$.
(5) If two adjacent arrows $\bar{b}_1$ and $\bar{b}_2$ end at a 6-vertex (see figure 3), then $\Omega\left((\bar{b}_1, \bar{b}_2)A\right) = b_4A + b_5A$.

7.2. **Horizontal lines.** In order to identify periodic modules, we need to introduce some terminology that describes the geometry of the Ω-translates of arrow modules.

We call an edge in $\mathcal{P}_i$ *horizontal* if it is not vertical; that is, if it is parallel to $\mathrm{p}(\mathbf{e}_1)$ or $\mathrm{p}(\mathbf{e}_2)$. We say that two vertices are *on the same level* if there is a sequence of horizontal edges joining them. Then we refer to a collection of edges between two vertices v and z on the same level as a *horizontal line* if it contains all horizontal edges joining vertices that are on the same level as and between v and z. Figures 11 and 12 provide some examples of horizontal lines in bold. Finally, the *width* of Q_i at the level of a vertex z is the minimum number of horizontal edges joining boundary points on the same level as z.

This new terminology allows us to describe the results of the previous section in greater detail as follows.

Proposition 20. *If b is a horizontal arrow in Q_i which starts at the left or right boundary of Q_i and if k is the width of Q_i at the level of b, then the first k Ω-translates of bA are one of the arrow translates as described in section 7.1 and lie along a horizontal line.*

We now study the Ω-translate of a module as the horizontal line reaches the boundary of Q_i. The important question is whether the Ω-translates after step k are still small.

Lemma 21. *Let b be an arrow in Q_i starting at a 4-vertex and ending at the 4-vertex z. Suppose that in A we have $\Omega(bA) = hA$ for some path h of A such that $he_i = 0$. Then $\Omega(bA_i) \cong h\bar{h}A_i$ and $\Omega^2(bA_i) \cong \bar{b}A_i$.*

Proof We then have an exact sequence of A-modules

$$0 \to hA \to e_zA \to bA \to 0.$$

The module $bAe_i = bA_i$ is still an arrow module in the algebra A_i. The functor $(-)e_i$ is exact, and therefore e_zAe_i, the projective indecomposable attached to z in A_i, is a projective cover of bA_i. Hence in A_i the Ω-translate of bA_i is equal to hAe_i. It follows from the Arrow truncation Lemma that

$$hAe_i = h\bar{h}Ae_i = \bar{h}Ae_i.$$

Furthermore, we have an exact sequence in A

$$0 \to \bar{b}A \to e_zA \to \bar{h}A \to 0.$$

Applying the exact functor $(-)e_i$ gives the projective cover for $\bar{h}Ae_i$, demonstrating that $\Omega(\bar{h}Ae_i) = \bar{b}A_i$. □

Hence, we have the following.

Theorem 22. *Suppose the horizontal line L in Q_i joins two 4-vertices on the boundary. Let b be an arrow on L starting or ending at a 4-vertex. Then bA_i is periodic of period $2k + 2$, where k is the width of the quiver at this level. Furthermore, the Ω-translates of bA_i lie along L: two modules are as in lemma 21 (corresponding to the ends of L), while the other translates are as described in section 7.1.*

Thus, we have an entirely combinatorial criterion for periodicity that allows us to identify periodic modules in each A_i for sufficiently large i.

Theorem 23. *For each $i \geq 6$, A_i contains a periodic module.*

Proof We demonstrate the existence of periodic modules in each algebra A_i for $i \geq 6$ directly. First, we show how to find horizontal lines corresponding to two such modules in A_i $6 \leq i \leq 9$, then we describe how the construction of A_{i+1} from A_i guarantees the existence of two similar horizontal lines for all $i \geq 9$. Finally, by examining the limited number of configurations of tiles in the Rauzy tiling, we show that these horizontal lines will always yield periodic modules.

Consider the two indicated horizontal lines in $\mathcal{P}_6$ that end in what can be seen to be 4-vertices by examining the surrounding tiles in $\mathcal{P}_9$ as illustrated in figure 11. These horizontal lines correspond to periodic modules. Translations of these horizontal lines can be found

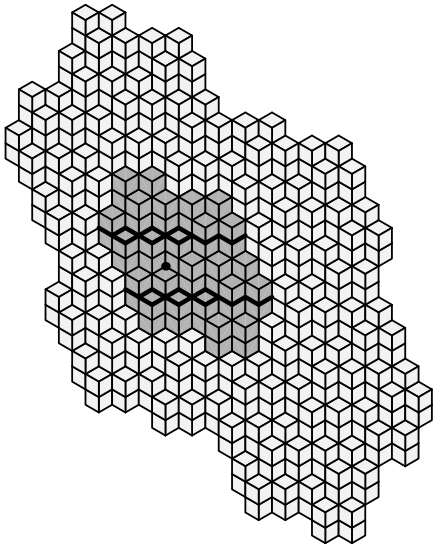


FIGURE 11. $\mathcal{P}_6$ within $\mathcal{P}_9$ and two horizontal lines corresponding to periodic modules.

in $\mathcal{P}_7$ and $\mathcal{P}_8$ joining 4-vertices on the boundary. But in $\mathcal{P}_9$, there are such horizontal lines in *each* $\mathcal{R}^9(R_k)$ joining 4-vertices of the boundary of the corresponding patch $\mathcal{R}^9(R_k)$, as indicated in figure 12.

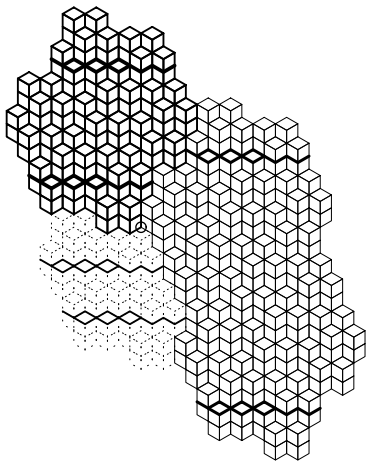


FIGURE 12. $\mathcal{P}_9$ with relevant horizontal lines indicated in each $\mathcal{R}^9(R_i)$

Since $\mathcal{R}^{10}(R_k)$ for $k = 2, 3$ are simply $\mathcal{R}^9(R_k)$ for $k = 1, 2$, these patches will also contain such horizontal lines joining 4-vertices on the boundary. As $\mathcal{R}^{10}(R_1)$ is formed as a combination of translations of the $\mathcal{R}^9(R_k)$, it too will have such horizontal lines. By induction, this will be the case for each subsequent $\mathcal{P}_i$. Then for each such i, $\mathcal{P}_i$ will have at least two horizontal lines going from one boundary vertex to another. (Precisely which patches $\mathcal{R}^i(R_k)$ contribute these horizontal lines depends on the class of $i \mod 3$).

However, in order to conclude that these horizontal lines correspond to periodic modules, we must know that each of these horizontal lines begin and end at a 4-vertex. For this we will need the results of [4], which tells us how many configurations of rhombi of a certain size the Rauzy tiling contains. Recall the lattice $\mathcal{L}$, on which the vertices of the rhombi lie, has basis $\{\mathbf{b}_1 = \mathrm{p}(\mathbf{e}_2)\ , \mathbf{b}_2 = \mathrm{p}(\mathbf{e}_3)\}$. When we consider the rhombi of the three types to have bases as indicated in figure 14, each vertex of $\mathcal{L}$ is the base for exactly one rhombus [4].

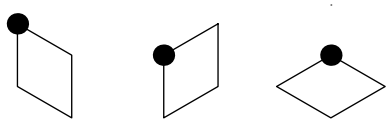


FIGURE 13. Each R_k with its base indicated

Then there are $mn + m + n$ different configurations of rhombi based in (m, n)-sections of $\mathcal{L}$ composed of vertices of the form

$$\{(k+i)\,\mathbf{b}_1 + (\ell + j)\,\mathbf{b}_2 : i = 1, \ldots, m\,; j = 1, \ldots, n\}$$

for some k and ℓ [4].

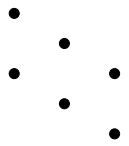


FIGURE 14. Example of a $(3, 2)$-section of $\mathcal{L}$

Below we include a table of these configurations for select (m, n), where the entry a_{ij} of the matrix representing a configuration corresponds to the type of rhombus (1, 2 or 3) based at $(k+i)\,\mathbf{b}_1 + (\ell + (n-j))\,\mathbf{b}_2$.

(m,n)	$P(m,n)$	List of possible configurations of rhombi
$(1,1)$	3	(1), (2), (3)
$(1,2)$	5	$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}$
$(2,1)$	5	$(1,1)$, $(1,2)$, $(3,1)$, $(2,1)$, $(2,3)$
$(2,2)$	8	$\begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}2&1\\2&3\end{pmatrix}, \begin{pmatrix}1&2\\1&1\end{pmatrix}, \begin{pmatrix}1&2\\3&1\end{pmatrix}, \begin{pmatrix}1&1\\3&1\end{pmatrix}, \begin{pmatrix}2&3\\1&2\end{pmatrix}, \begin{pmatrix}3&1\\2&3\end{pmatrix}, \begin{pmatrix}3&1\\2&1\end{pmatrix}$
$(3,2)$	11	$\begin{pmatrix}1&2&1\\1&2&3\end{pmatrix}, \begin{pmatrix}1&1&2\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}2&1&1\\2&3&1\end{pmatrix}, \begin{pmatrix}2&1&2\\2&3&1\end{pmatrix}, \begin{pmatrix}2&3&1\\1&2&1\end{pmatrix}, \begin{pmatrix}2&3&1\\1&2&3\end{pmatrix}, \begin{pmatrix}3&1&2\\2&1&1\end{pmatrix}, \begin{pmatrix}3&1&2\\2&1&2\end{pmatrix}, \begin{pmatrix}3&1&2\\2&3&1\end{pmatrix}$

Table 1

Each of the horizontal lines under consideration ends on the right with a $\begin{pmatrix}3\\2\end{pmatrix}$ configuration above, and so consideration of the possible $(2,2)$-configurations and geometry forces this to be completed as $\begin{pmatrix}3&1\\2&1\end{pmatrix}$, corresponding to the desired 4-vertex. On the left, the horizontal lines end cutting through a $\begin{pmatrix}1&2\\1&1\end{pmatrix}$ configuration, and so consideration of the possible $(3,2)$-configurations shows that it must be completed on the left as $\begin{pmatrix}3&1&2\\2&1&1\end{pmatrix}$. This means that the configuration on the left end of the horizontal line containing the type 1 rhombus based at the left endpoint has the form $\begin{pmatrix}2&1\\x&y\end{pmatrix}$, which can only be completed as $\begin{pmatrix}2&1\\2&3\end{pmatrix}$, yielding the desired 4-vertex. □

We call a horizontal line *periodic* if it corresponds to a periodic orbit. In what follows, L denotes a periodic line in Q_i.

Definition 8. We say that an A_i-module W is *along* L if it has arrow translates as described in 7.1 or if it is of the form $W = (b\bar{b})A_i$, where b starts at L but does not belong to A_i and $b\bar{b}$ is in A_i.

By theorem 22, the modules along L form one Ω-orbit.

Corollary 24. *Suppose M is an indecomposable non-projective A_i-module and suppose for some W along the line L we have $\underline{\mathrm{Hom}}(W, \Omega^t(M)) \neq 0$ for some $t \geq 0$. Then the tree class of the component of M belongs to the list as in lemma 18.*

We can completely answer for which simple modules this applies.

Theorem 25. *Suppose S is a simple module. Then there is a periodic W along a periodic line L such that $\underline{\mathrm{Hom}}(W, \Omega^t(S)) \neq 0$ for some t if and only if $S = S_x$ and x is a vertex on L.*

Proof (1) Suppose $S = S_x$, where x is a 4-vertex or 5-vertex on a periodic line L. Then $\Omega(S)$ contains $W = bA_i$ for b an arrow starting at x, hence W is along L. An inclusion map does not factor through a projective module, and hence

$$0 \neq \underline{\mathrm{Hom}}(W, \Omega(S)).$$

(2) Suppose $S = S_x$ where x is a 6-vertex on a periodic line L. Then $\Omega(S)$ contains $W = hA_i + kA_i$, where h, k are arrows starting at x and W is along L. Then similarly $\underline{\mathrm{Hom}}(W, \Omega(S)) \neq 0$.
(3) Suppose $S = S_x$ where x is a 3-vertex which occurs along a periodic line L. Then S is a top composition factor of $W = hA_i + kA_i$, where W is along L and where either h or k ends at x. The epimorphism from W onto S does not factor through a projective module. Hence, as before, $\underline{\mathrm{Hom}}(W, S) \neq 0$.

For the converse, suppose that S is a simple module and that there is some W along L such that $\underline{\mathrm{Hom}}(W, \Omega^t(S)) \neq 0$ for some t. This space is isomorphic to

$$\underline{\mathrm{Hom}}(\Omega^{-t}(W), S) \cong \mathrm{Hom}(\Omega^{-t}(W), S)$$

where the first isomorphism is by dimension shift, and the second holds because S is a simple module of a symmetric algebra and $\Omega^{-t}(W)$ does not have a projective summand. Now, $\Omega^{-t}(W)$ also is a module along L. Since there is a non-zero homomorphism from this module onto S it follows that S is a top composition factor. But all modules along L have top composition factors S_x with x on L. Hence $S = S_x$ as stated. □

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