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Abstract

Prospect theory is the main behavioral alternative to expected utility. Tversky and Kahnemann (1992) motivate the utility function for gains and losses under prospect theory by using the axiom of preference homogeneity. However, they do not provide the formal proof. We provide the relevant proof. Furthermore, we show that the utility function under preference homogeneity obeys an additional and important restriction that is not noted by Tversky and Kahnemann (1992). This simplifies the use of prospect theory by reducing the number of free parameters by one.

Keywords: Prospect Theory, Preference homogeneity, Functional equations.

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1. Introduction

It is well known that the axioms of expected utility are violated in a range of experiments and surveys. This has been well known for a long time; for instance, Luce and Raiffa (1957).[1] For a more recent and definitive treatment, see Kahnemann and Tversky (2000).

The main behavioral alternative to expected utility is prospect theory. The earliest version was given by Kahneman and Tversky (1979). A later version that takes account of cumulative transformations of probability and, hence, the insights developed in rank dependent expected utility[2] was provided by Tversky and Kahneman (1992).

Prospect theory has proven extremely influential in explaining a range of phenomena that could not be otherwise explained within an expected utility framework. These include the disposition effect, asymmetric price elasticities, elasticities of labour supply that are inconsistent with standard models of labour supply and the excess sensitivity of consumption to income; see, for example, Camerer (2000). Further applications include the explanation of tax evasion (Dhami and al-Nowaihi (2006)), insurance (al-Nowaihi and Dhami (2006)), failure of the Becker proposition (Dhami and al-Nowaihi (2006)) and several applications to finance (Thaler (2005)) among others.

A critical aspect in successfully applying prospect theory, particularly in quantitative applications, is the form of the utility function for gains and losses. Tversky and Kahneman (1992) state, without proof, that if preference homogeneity[3] holds, then the value function of prospect theory has the power function form[4]

$$\begin{aligned} v(x) &= x^{\alpha}, \text{ for } x \geq 0, \\ v(x) &= -\lambda(-x)^{\beta}, \text{ for } x < 0, \\ &\text{where } 0 < \alpha \leq 1,\ 0 < \beta \leq 1,\ \lambda > 1. \end{aligned} \tag{1.1}$$

The contribution of our paper is twofold. First we give a simple proof which shows that preference homogeneity is a sufficient condition for the preferences given in (1.1). Second, we show that, necessarily, $\alpha = \beta$. Tversky and Kahneman (1992) report that $\alpha = \beta$ is in agreement with the experimental evidence. This facilitates the use of prospect theory in quantitative and qualitative exercises by reducing the number of parameters.

The plan of this note is as follows. Section 2 gives the basic definitions that we need for our main theorem, which is derived in Section 3.

[1] Luce and Raiffa (1957) wrote explicitly, page 35, that "A second difficulty in attempting to ascertain a utility function is the fact that reported preferences almost never satisfy the axioms..."

[2] This mainly had to do with a transformation of cumulative rather than objective probabilities; see Quiggin (1993) for the details.

[3] Preference homogeniety is formally defined below. It essentially implies that when all prizes in a lottery are scaled up by a factor, say k, then the certaintly equivalent of the lottery is also scaled up by a factor k.

[4] Under expected utility theory, preference homogeniety gives rise to CRRA preferences.

2. Preliminary definitions

Let $v : \mathbf{R} \to \mathbf{R}$ be a value function. Then v satisfies:

1. $v(0) = 0$ (reference dependence),
2. v is strictly increasing (monotonicity),
3. v is concave for $x \geq 0$ (declining sensitivity for gains),
4. v is convex for $x < 0$ (declining sensitivity for losses),
5. $|v(-x)| > -v(x)$ for $x > 0$ (loss aversion).

Definition 1 *Let (x, p) be the lottery that pays $x \in \mathbf{R}$ with probability $p \in [0, 1]$. Then the value function, v, satisfies 'preference homogeneity' if for all such lotteries, if c is the certainty equivalent of (x, p) then, for all $k \in R_+$, kc is the certainty equivalent of (kx, p).*

Definition 2 *: By a probability weighting function we mean a strictly increasing function $w : [0, 1] \to [0, 1], w(0) = 0, w(1) = 1$.*

The domain of a probability weighting function is the unit interval of cumulative probabilities. Several probability weighting functions have been proposed in the literature.[5]

3. Derivation of the power form for the value function

We derive the main set of our results in this section.

Theorem 1 *:Suppose the value function, v, satisfies preference homogeneity. Then, necessarily, v takes the form:*

$$\begin{aligned} v(x) &= x^{\alpha}, \text{ for } x \geq 0, \\ v(x) &= -\lambda(-x)^{\alpha}, \text{ for } x < 0, \\ \text{where } 0 &< \alpha \leq 1, \ \lambda > 1. \end{aligned} \tag{3.1}$$

Proof of Theorem 1: Let w be the probability weighting function. Then w is a strictly increasing function from $[0, 1]$ onto $[0, 1]$. Let $0 \leq c \leq 1$. By reference dependence and monotonicity, $0 = v(0) \leq v(c) \leq v(1)$ and $v(1) > 0$. Hence, $0 \leq \frac{v(c)}{v(1)} \leq 1$. Let $p = w^{-1}\left(\frac{v(c)}{v(1)}\right)$. Hence, $w(p) = \frac{v(c)}{v(1)}$ and, hence,

$$w(p)\, v(1) = v(c). \tag{3.2}$$

[5] For a critical discussion of the features involved with traditional probability functions and a proposal for a new set of probability weighting functions see al-Nowaihi and Dhami (2006).

Hence, c is the certainty equivalent of the lottery $(1, p)$. Preference homogeneity then implies,

$$w(p)\,v(k) = v(ck)\,,\text{ for all } k \geq 0. \tag{3.3}$$

Substitute $w(p) = \frac{v(c)}{v(1)}$ from (3.2) into (3.3) to get

$$v(ck) = \frac{v(c)\,v(k)}{v(1)},\text{ for all } c \in [0,1] \text{ and all } k \geq 0. \tag{3.4}$$

Define $u : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$u(x) = \frac{v(x)}{v(1)},\ x \geq 0. \tag{3.5}$$

In particular, for $x = 1$, (3.5) gives

$$u(1) = 1. \tag{3.6}$$

From (3.4) and (3.5) we get $u(ck) = \frac{v(ck)}{v(1)} = \frac{v(c)v(k)}{v(1)v(1)} = u(c)\,u(k)$. Hence,

$$u(ck) = u(c)\,u(k)\,,\text{ for all } c \in [0,1] \text{ and all } k \geq 0. \tag{3.7}$$

Equation (3.7) holds for any numbers c, k such that $c \in [0,1]$ and all $k \geq 0$. In what follows, c does not necessarily have the interpretation of a certainty equivalent.

Let $x > 0$. If $0 < x \leq 1$, let $c = x$ and $k = \frac{1}{x}$. If $x > 1$, let $c = \frac{1}{x}$ and $k = x$. In either case, (3.6) and (3.7) give $1 = u(1) = u\left(x\frac{1}{x}\right) = u\left(\frac{1}{x}\right)u(x)$. Hence,

$$u\left(\frac{1}{x}\right) = \frac{1}{u(x)},\text{ for all } x > 0. \tag{3.8}$$

Let $x \geq 0$ and $y \geq 0$. If $x \leq 1$, take $c = x$ and $k = y$. If $y \leq 1$, take $c = y$ and $k = x$. In either case, (3.7) gives $u(xy) = u(x)\,u(y)$. Suppose now $x > 1$ and $y > 1$. Then (3.7) and (3.8) give $u(xy) = \frac{1}{u\left(\frac{1}{xy}\right)} = \frac{1}{u\left(\frac{1}{x}\right)}\frac{1}{u\left(\frac{1}{y}\right)} = u(x)\,u(y)$. Hence,

$$u(xy) = u(x)\,u(y) \text{ for all } x \geq 0 \text{ and all } y \geq 0 \tag{3.9}$$

Since v is strictly increasing, so u is also strictly increasing (from (3.5), since $v(1) > 0$). Hence, (3.9) has the unique solution[6]:

$$u(x) = x^{\alpha},\text{ for some } \alpha > 0. \tag{3.10}$$

Putting $a = v(1)$, (3.5) and (3.10) give:

$$v(x) = ax^{\alpha},\ a > 0,\ \alpha > 0,\ x \geq 0 \tag{3.11}$$

[6] See, for example, Eichhorn (1978, Theorem 1.9.13).

Similarly, by now taking $u(x) = \frac{v(-x)}{v(-1)}$, $x \geq 0$, we get

$$v(x) = -\lambda(-x)^{\beta}, \ b > 0, \ \lambda > 0, \ x \leq 0 \tag{3.12}$$

Without loss of generality, we can take $a = 1$, so that

$$\begin{aligned} v(x) &= x^{\alpha}, \text{ for } x \geq 0, \\ v(x) &= -\lambda(-x)^{\beta}, \text{ for } x < 0, \\ \text{where } \alpha &> 0, \ \beta > 0, \ \lambda > 0. \end{aligned} \tag{3.13}$$

Loss aversion then implies

$$\lambda x^{\beta} > x^{\alpha} \text{ for all } x > 0. \tag{3.14}$$

For $x = 1$, (3.14) gives

$$\lambda > 1 \tag{3.15}$$

Also from (3.14)

$$\ln \lambda > (\alpha - \beta) \ln x \text{ for all } x > 0. \tag{3.16}$$

We will now prove that $\alpha = \beta$. Suppose $\alpha \neq \beta$. Then either $\alpha > \beta$ or $\beta > \alpha$. If $\alpha > \beta$, then we can make $(\alpha - \beta) \ln x$ as large as we like by choosing x to be sufficiently large. But this cannot be because, by (3.16), $(\alpha - \beta) \ln x$ is bounded above by $\ln \lambda$. If $\beta > \alpha$, then we can make $(\alpha - \beta) \ln x$ as large as we like by choosing $x > 0$ sufficiently close to 0. But this cannot be true either. Hence $\alpha = \beta$. Finally, by declining sensitivity we must have $\alpha \leq 1$. It follows that v must take the form (3.1).

4. Conclusions

We provide a formal proof which shows that preference homogeneity is a sufficient condition for the utility function proposed in Tversky and Kahneman (1992). We also show that preference homogeneity gives rise to a more parsimonious utility function than that proposed by Tversky and Kahneman (1992). By reducing the number of free parameters, this is expected to simplify the application of prospect theory.

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