

Transition mechanisms within the boundary-layer flow over slender vs. broad rotating cones

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We describe progress in classifying the convective instability characteristics of the boundary-layer flow over the family of rotating cones. Existing experimental and theoretical studies are discussed which lead to the clear hypothesis of a centrifugal instability mode that dominates over slender cones and manifests as Görtler-type spiral vortices. Although a formulation consistent with the classic rotating-disk problem has been successful in predicting the stability characteristics over broad cones, it is unable to identify such a centrifugal mode as the half-angle is reduced. An alternative formulation is introduced and an asymptotic analysis summarized which identifies such a centrifugal mode.

I. Introduction

This paper describes recent advances in the ongoing study of boundary-layer transition over rotating cones. In particular, we are concerned with the distinct stability characteristics of slender and broad cones and have undertaken an extended theoretical project with the ultimate aim of understanding the transition mechanisms within the boundary-layer flow over cones of all half-angles, ψ . We describe the experimental motivation for the investigation, summarize relevant previous work and present the latest findings in our attempt to identify a hypothesized convective mode that is expected to dominate over slender cones.

Our interest in rotating cones is motivated by the flow around nose cones in aeroengine and spinning projectile applications where laminar-turbulent transition within the boundary layer can lead to significant increases in drag. For aeroengine applications this has negative implications for the fuel efficiency through increased noise and energy dissipation, and for projectile applications this has negative implications for control and accurate targeting. The ultimate understanding of the transition of such flows will enable the development of strategies to maintain the laminar flow, leading to modifications in design and significant cost savings.

It is clear that the linear-stability analyses presented here for cones rotating within incompressible and otherwise still fluids are of limited relevance in terms of the motivating applications. However, this work should be considered as a first step towards fully classifying the instability mechanics at play, with a view to further extensions at a later date. Indeed, studies of the effects of enforced axial flow over broad cones have already been published¹⁻⁴ and a study into compressibility is underway. Such incremental advances in knowledge is consistent with the historical development of the rotating-disk problem in the literature with regards its application to swept-wing flows.

II. Experimental motivation

Experimental observations of the transitional region over rotating cones first highlighted the distinction between the transition characteristics of slender and broad cones. For example, the experimental studies of Kobayashi & co-workers^{5,6} of cones with slender half-angles rotating in still fluid show the existence of pairs of counter-rotating Görtler-type vortices. These arise from a dynamic instability induced by the centrifugal

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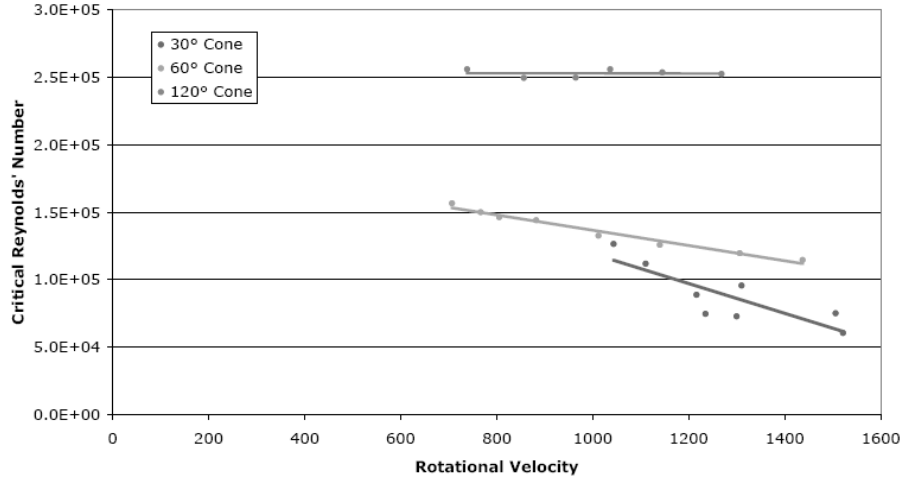


Figure 1. Experimental data due to Nickels for the onset of turbulence on rotating cones, uppermost plot is $\psi = 60^\circ$. (Cone angle = 2ψ)

force of the flow field. However, as the half-angle is increased beyond $\psi = 30^\circ$, their visualizations clearly show that these vortices change from pairs of counter-rotating vortices to co-rotating crossflow vortices, as observed on rotating disks^{7–12} and spheres.^{13,14} Indeed, their measurements of the orientation angle of the vortices and other characteristics approach those observed for a rotating disk as the half-angle tends to 90° . It is well known from the studies of Gregory *et al.*¹¹ and Hall,¹⁵ for example, that the stationary spiral vortices observed on the rotating disk are in fact co-rotating vortices attributed to an underlying crossflow instability that arises from an unstable inflexion point in the crossflow component of the flow field. The observed centrifugal instability for cones with smaller half-angles stems from an inherently different process to that governing the crossflow instability for cones of larger half-angles. Therefore, there appears to be a distinct variation in the underlying physical mechanism governing the instability for more slender cones.

Further evidence is obtained from the more recent experimental measurements for the onset of turbulence by Nickels (personal communication, 2007). Nickels' motivation was to recreate Lingwood's experiments¹⁶ into the existence (or otherwise) of streamwise absolute instability within the boundary-layer flow over rotating cones. Although the exact involvement of absolute instability is now known to be less clear than originally proposed by Lingwood,^{17–23} the fact remains that the theoretical onset of local absolute instability is tantalizingly close to numerous consistent measurements of the onset of turbulence over rotating disks.^{7,8} For this reason further study into local absolute instability and its relationship to the onset of turbulence over rotating cones is worthwhile. Garrett & co-workers^{1,4,24} have demonstrated the existence of local absolute instability in the rotating-cone boundary layer at all half-angles (reproducing Lingwood's results in the limiting case of $\psi = 90^\circ$). They find the theoretical prediction for the onset of absolute instability to occur at a Reynolds number $R_X \approx 2.5 \times 10^5$, independent of half-angle. This Reynolds number is based on distance from the apex and local surface speeds. Figure 1 shows Nickels' measurements for a variety of cones at different rotation rates and we see that the onset of turbulence for the broad cone ($\psi = 60^\circ$) is in good agreement with the predicted onset of absolute instability and is independent of rotation rate. However, the onset of turbulence for slender cones ($\psi = 30^\circ$ and 15°) is well in advance of the predicted onset of absolute instability and dependent on the rotation rate. Furthermore, Nickels notes different behaviour in the turbulent intensity through transition in the case of the most slender cone. These observations again demonstrate that transition over slender cones is inconsistent with the mechanism that occurs over broad cones and disks.

It is interesting to note that for very slender cones ($\psi \leq 15^\circ$), two physical cases of vortices can exist: spiral and circular vortices. These are distinguished by non-zero and zero waveangles respectively, and have been observed by Kobayashi and co-workers for cones rotating in both otherwise still fluid and enforced axial flow.^{5,6} The theoretical study of these two cases is slightly different and we consider only spiral vortices here. Our work in §IV is therefore applicable to slender cones, but with half-angles $\psi > 15^\circ$. Circular waves

for $\psi < 15^\circ$ have been studied by Hussain³ and will be subject of future publications.

III. Previous theoretical work

Garrett^{1,24} uses a parallel-flow approximation to formulate the rotating-cone problem in still fluid and enforced axial flow. Including viscous and streamline-curvature effects the resulting eigenvalue problem is solved using a numerical approach. In addition to absolute instability (discussed in §II), Garrett considers convective modes of instability of type I and II (also known as crossflow and streamline curvature, respectively) and computes full neutral curves and critical Reynolds numbers for half-angles $\psi = 10^\circ$ – 90° . The results agree with theoretical analyses in the literature in the limiting case of the rotating disk.^{23,25–27}

Although use of parallel-flow assumption is well established in the literature, it leads to a mathematical formulation that is inconsistent at $O(R_L^{-1})^a$, the order at which viscous and streamline-curvature effects occur. This has implications for the accuracy of quantitative predictions. For this reason a rigorous analysis using asymptotic methods has been conducted by the present authors²⁸ where expressions for neutral type I and II branches are derived in the high Reynolds-number limit.

As shown in Figures 2 and 3, comparisons of orientation angle, ϵ , and effective wavenumber, k_δ , of the vortices arising from the asymptotic and numerical analyses are in excellent agreement for large Reynolds number. The figures show that increasing ψ leads to a rise in both type I and II waveangles, which are attributed to the increased rotational shear effect: the spiral vortices are swept more in an azimuthal direction, due to the steep angle of the cone surface, resulting in a wider orientation angle with respect to the cone meridian. Conversely, for smaller ψ , the spiral vortices undergo a stronger forcing in the streamwise direction and hence wrap around the surface in a helical nature, propagating at a lower deviation angle from the streamwise direction. Importantly, it is observed that an increase in ψ has the effect of stabilizing both the type I and type II modes by increasing the predicted value of the critical Reynolds number at the onset of instability. This effectively expands the region of stable flow and gives rise to fewer wavenumbers in the unstable area to the right of the neutrally-stable modes.

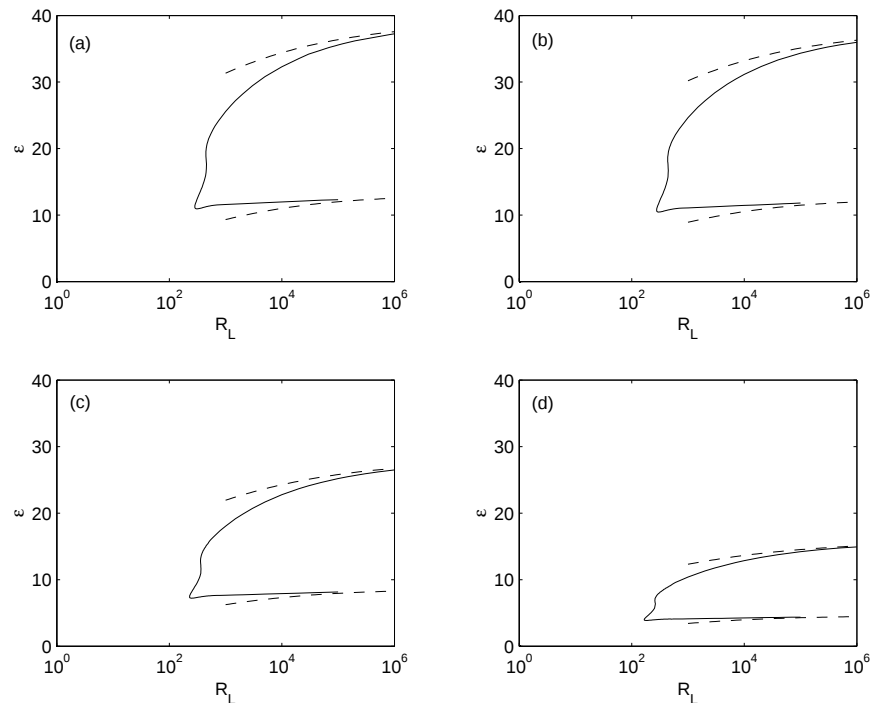


Figure 2. Neutral-stability curves for stationary modes in terms of predicted waveangle (solid line numerical, dashed line asymptotic) (a) $\psi = 80^\circ$; (b) $\psi = 70^\circ$; (c) $\psi = 40^\circ$; (d) $\psi = 20^\circ$. Figure taken from Garrett *et al.*²⁸

The computed critical Reynolds numbers compare well with experimental observations of the appearance

^aNote that $R_L = \sqrt{R_x^2} \sin \psi$ is the Reynolds number based on boundary-layer thickness and local surface speed.

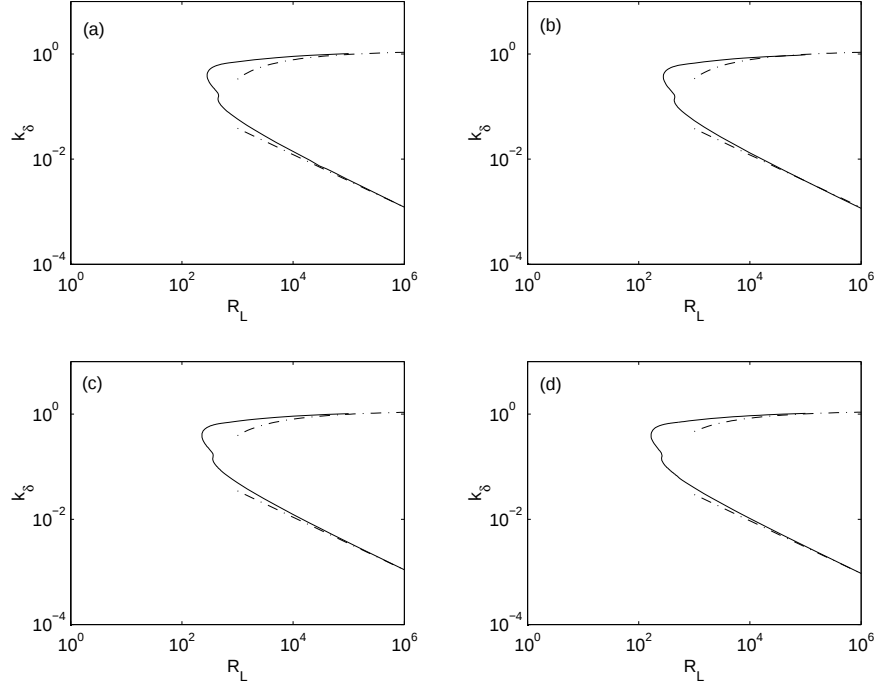


Figure 3. Neutral-stability curves for stationary modes in terms of predicted wavenumber (solid line numerical, dashed line asymptotic) (a) $\psi = 80^\circ$; (b) $\psi = 70^\circ$; (c) $\psi = 40^\circ$; (d) $\psi = 20^\circ$. Figure taken from Garrett *et al.*²⁸

of spiral vortices for broad cones with $\psi \geq 50^\circ$.^{6,29,30} However, for more slender half-angles an increasing discrepancy is found. This suggests an apparent change in the physical nature of the instability, not attributable to the type I and II modes that were explicitly sought. Such observations, when taken into account with Kobayashi's physical descriptions of the vortices, suggest the possibility of a viscous-mode dominated structure at work, pertaining to the onset of the centrifugal Görtler-type instability. This mode was hypothesized by Garrett *et al.*²⁸ and further elaborated by Garrett.³¹ In particular, Garrett extended the existing numerical results to consider the amplification rates of the stationary and traveling type I and II modes through the convectively-unstable region. He found that the amplification rates of both modes reduce with decreased half-angle, as shown in Figure 4. This behaviour adds weight to the hypothesis of a centrifugal mode that dominates at low half-angles but becomes less amplified as the half-angle increases. As the half-angle is increased we expect the existence of a possible critical half-angle for the variation from a predominantly centrifugal instability (manifested in the appearance of counter-rotating vortices) to the crossflow instability (manifested in the appearance of co-rotating vortices). Experiments suggest that this critical half-angle is around 40° but a theoretical prediction of this is an intriguing possibility which motivates the remainder of this paper and further work in progress.

IV. Analysis of the centrifugal mode

In this section we present the latest progress in identifying the hypothesized centrifugal mode. An asymptotic study is currently underway, as summarized in §IV.B, and a numerical analysis is intended to commence shortly.

IV.A. Formulation

The formulation and methods are significantly different from those used in our previous studies.^{4,24,28,31} With regards the formulation, we had previously fixed the coordinate axes with respect to the axis of rotation of the cone and with origin at the apex. However we now have axes aligned with the spiral vortices and with origin O' placed at the position with local surface radius $r_0^* = x_0^* \sin \psi$, where x^* is the Cartesian axis along the axis of rotation (note that a $*$ represents a dimensional quantity). As shown in Figure 5, we introduce

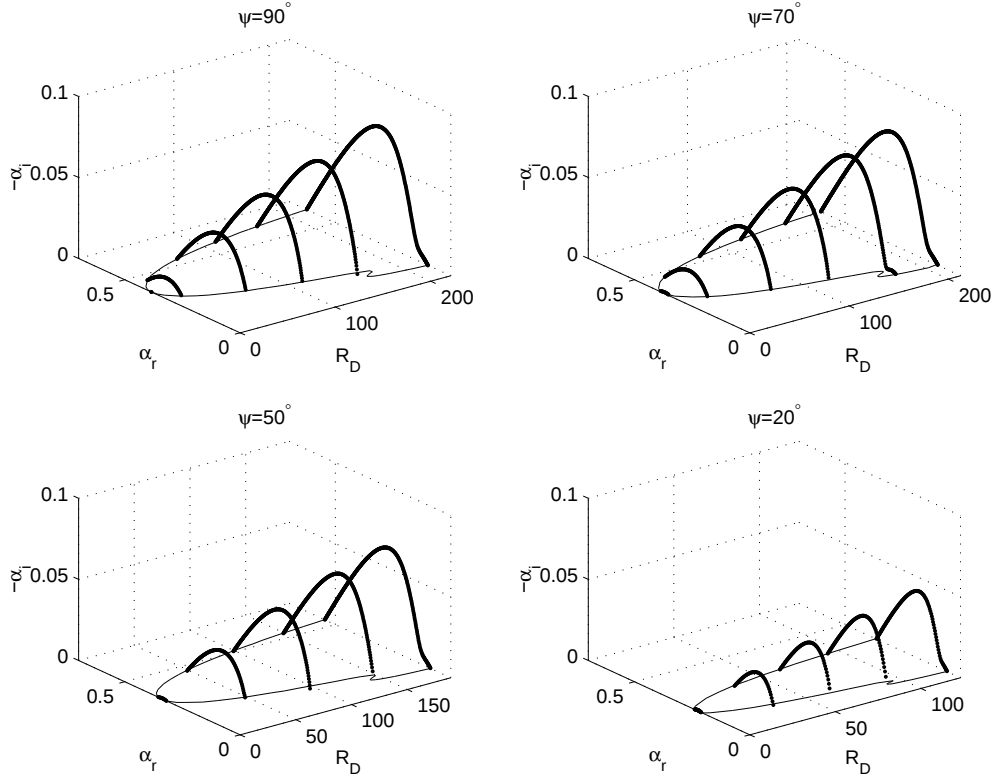


Figure 4. Linear convective growth rates for stationary-mode disturbances of type I through the convectively unstable region at various ψ . $R_D = R_L - R_{L,C}$. Figure taken from Garrett.³¹

$\hat{x}^*$ - and y^* -axes that coincide with the direction of propagation of the spiral vortices and the tangent curve to the spiral vortices, respectively; the z^* -axis is normal to the cone surface. The coordinate system $(\hat{x}^*, y^*, z^*)$ rotates with the cone surface at constant angular frequency Ω^* . Importantly, the logarithmic spirals are directed such that the y^* -axis has a positive projection with the direction of rotation of the cone. This requires that the $\hat{x}^*$ -axis has positive projection onto the axis of rotation and the y^* -axis to have negative projection, as seen in Figure 5. The spiral vortices are shed at a waveangle ϕ , and coordinates (r^*, θ, z^*) used in the previous studies are transformed to

$$r^* = r_0^* \exp\left(\frac{1}{r_0^*}(\hat{x}^* \cos \phi - y^* \sin \phi)\right), \quad (1)$$

$$\theta = \frac{1}{r_0^* \sin \psi}(\hat{x}^* \sin \phi + y^* \cos \phi), \quad (2)$$

$$z^* = z^*, \quad (3)$$

from which the governing dimensional equations can be derived with appropriate scale factors. This formulation is similar to Kobayashi's³² but differs by a sign of the Coriolis terms in the momentum equations.

We non-dimensionalize lengths on a characteristic distance along the cone l^* , so that $\hat{x}^* = l^* \hat{x}$ and $y^* = l^* y$. Furthermore, we scale both logarithmic coordinates $\hat{x}$ and y , as well as the normal coordinate z^* , on the boundary-layer thickness, leading to the scaled coordinate system $(\tilde{x}, \tilde{y}, \eta) = R^{1/2}(\hat{x}, y, z)$ where R is the Reynolds number given by

$$R = \frac{\Omega^* l^{*2} \sin \psi}{\nu^*}.$$

This enables the vortex structure in both logarithmic directions to be analyzed on the same level as the smallest length scale in the surface-normal direction. Usually a Görtler-mode analysis requires only lengths normal to the surface and spanwise to the vortices to be scaled on boundary layer thickness (see Hall³³ and Denier *et al.*³⁴). However, here the counter-rotating vortices are characterized by both logarithmic coordinates which require this scaling.

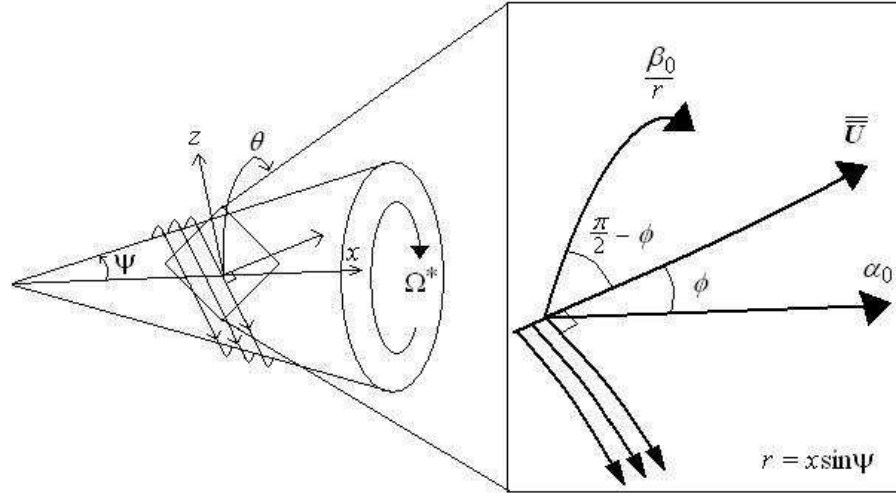


Figure 5. Diagram of spiral vortex instability of a rotating cone (left) and the detailed physical interpretation (right) showing streamwise, azimuthal and effective velocity directions. Note the cone is rotating anticlockwise when viewed from the nose tip.

Next we use the fact that the spiral waves are periodic in the effective velocity direction to introduce periodicity into the perturbation quantities of vortex $\tilde{x}$ -wavenumber a and $\tilde{y}$ -wavenumber b . Scaling our perturbing quantities on the boundary-layer thickness, we introduce a perturbed flow of the form

$$\tilde{\mathbf{u}}^* = \Omega^* l^* \sin \psi [\{x\tilde{U}(\eta), x\tilde{V}(\eta), R^{-\frac{1}{2}}W\} + R^{-\frac{1}{2}}\{\tilde{u}(\eta), \tilde{v}(\eta), \tilde{w}(\eta)\}\exp(ia\tilde{x} + ib\tilde{y})],$$

The pressure perturbation term scales as

$$p^* = (\rho^* \Omega^{*2} l^{*2} \sin^2 \psi) R^{-1} \tilde{p}(\eta) \exp(ia\tilde{x} + ib\tilde{y}).$$

The shifted steady-flow components in the $\tilde{x}$ - and $\tilde{y}$ -directions are $\bar{U}$ and $\bar{V}$, and are expressed as

$$\begin{aligned} \bar{U} &= x(U(\eta) \cos \phi + V(\eta) \sin \phi) = x\tilde{U}(\eta), \\ \bar{V} &= x(U(\eta) \sin \phi + V(\eta) \cos \phi) = x\tilde{V}(\eta), \end{aligned}$$

where $U(\eta)$ and $V(\eta)$ are scaled velocities in the streamwise and azimuthal directions in our previous formulation²⁸ and are the familiar von Kàrmàn profiles. We therefore see that although the formulation is different to that used previously, the steady flows on which the new analysis is conducted are consistent. Note that $\tilde{U}$ and $\tilde{V}$ can be interpreted as effective velocities in the ϕ -direction. Figure 6 gives the steady-flow profiles for $\phi = 0^\circ, 5^\circ$ and 10° . There is a minor increase in the velocity in the $\tilde{y}$ -direction. However, for the $\tilde{x}$ -direction, we observe a greater change with the velocity decreasing and becoming negative as the waveangle increases. It is clear that the profiles in the case of $\phi = 0^\circ$ correspond to those used in the previous analyses and are not considered further here.

In order to derive the governing perturbation equations for spiral vortices with $\phi \neq 0$ it is necessary to make a number of mathematical approximations to the scale factors. These are based on the assumption of large Reynolds number and small waveangle which are justified from the experimental observations of Kobayashi & Izumi⁶ where $\phi \approx 0^\circ - 2.7^\circ$. We wish to investigate the short-wavelength asymptotic structure of the centrifugal instability and hence identify the spiral vortex wavenumber in the $\tilde{x}$ -direction as $a = \epsilon^{-1}$, where ϵ is a small parameter. Here, $b = O(1)$ is the wavenumber in the $\tilde{y}$ -direction. Full details of the mathematical manipulations are given by Hussain,³ and we arrive at the governing stability equations also stated in his thesis.

IV.B. Asymptotic analysis

In our previous analyses of the type I and II modes^{3,28} we introduced a small parameter given by inverse powers of the Reynolds number as the basis of the asymptotic structure. The governing perturbation

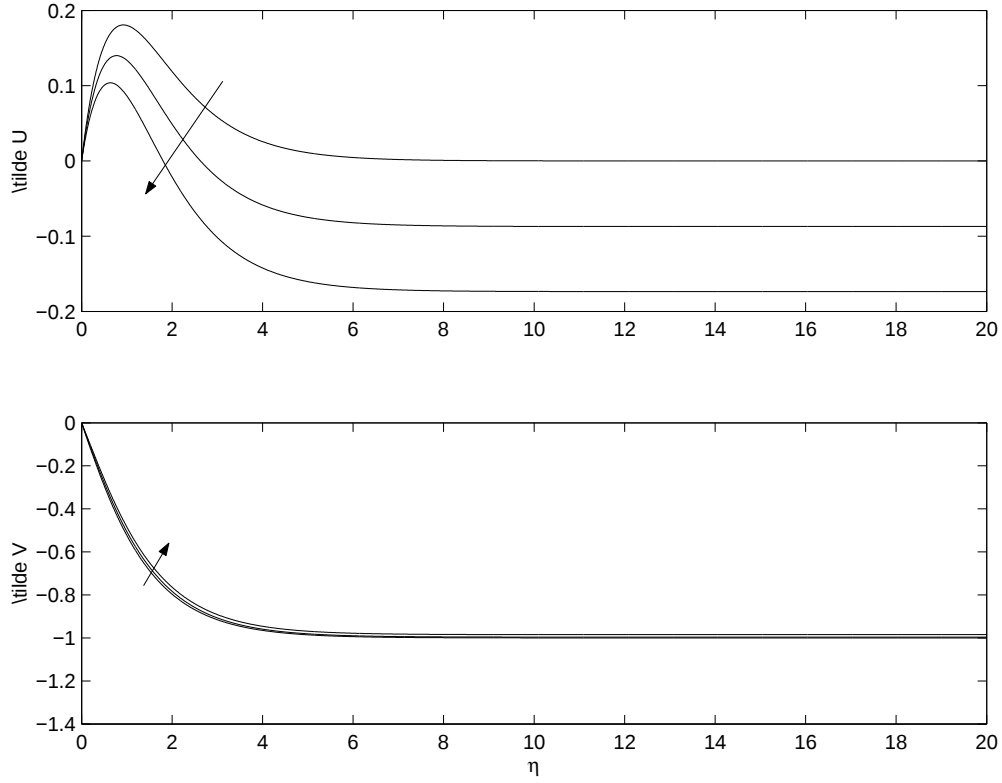


Figure 6. Effective velocity profiles $\tilde{U}$ and $\tilde{V}$ for waveangles $\phi = 0^\circ, 5^\circ$ and 10° (in direction of arrow).

equations were then solved to form leading- and first-order estimates of the wavenumbers for neutrally-stable modes. This resulted in explicit expressions (applicable in the large Reynolds-number limit) that enable simple comparisons with the upper and lower branches of the numerically-computed neutral curves, as can be seen in Figure 3, for example. However, in this analysis it is inverse powers of wavenumber which form the basis of our asymptotic structure. The governing equations are then solved to determine leading- and next-order estimates of a scaled Taylor number for neutrally-stable modes. Hussain³ shows that the Taylor number (as defined in equation (4)) is linearly related to the rotational Reynolds number used by Kobayashi & Izumi.⁶ This asymptotic approach is therefore quite distinct to that used previously and follows Hall³³ for the Taylor problem of flow between concentric rotating cylinders. Indeed, for slender rotating cones, the half-angle is sufficiently small that the formulation resembles that for flow moving axially over a rotating cylinder.

We wish to investigate the short-wavelength asymptotic structure of the centrifugal instability and hence identify the spiral vortex wavenumber in the $\tilde{x}$ -direction as $a = \epsilon^{-1}$, where ϵ is a small parameter which forms the basis of our asymptotic analysis. The wavenumber in the $\tilde{y}$ -direction is $b = O(1)$. The Taylor number characterizes the importance of centrifugal forces relative to viscous forces in this problem, and is given as a function of half-angle for each ϕ under consideration:

$$T = \frac{2 \cot \psi \cos \phi}{\sin^5 \psi}. \quad (4)$$

IV.B.1. Leading-order solution

The perturbation quantities are expanded and we consider a WKB solution for small values of ϵ . The dominant terms in the governing equations balance if we scale $T \sim \epsilon^{-4}$ and $W/V \sim O(\epsilon^{-2})$, resulting in

$$\begin{aligned}\tilde{u} &= E(u_0(\eta) + \epsilon u_1(\eta) + \epsilon^2 u_2(\eta) + \dots), \\ \tilde{v} &= \epsilon^2 E(v_0(\eta) + \epsilon v_1(\eta) + \epsilon^2 v_2(\eta) + \dots), \\ \tilde{w} &= E(w_0(\eta) + \epsilon w_1(\eta) + \epsilon^2 w_2(\eta) + \dots), \\ T &= \epsilon^{-4}(\lambda_0 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \dots),\end{aligned}$$

where $\lambda = \lambda_0 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \dots$, $E = \exp \frac{i}{\epsilon} \int^\varphi K(\tau) d\tau$ and $\varphi = \frac{\sin \psi}{\bar{h}_1} \eta$.

After substitution of these expansions into the governing equations and some simplification owing to the assumption of small waveangle, we arrive at an eigenrelation at leading order which can be solved to give the scaled leading-order eigenvalue estimate

$$\bar{\lambda}_0 = \lambda_0 \bar{h}_1^4 (1 + \check{x} \cos \phi - \bar{y} \sin \phi) = \frac{1}{\left[(\tilde{V} \cos \phi + 1) \frac{\partial \tilde{V}}{\partial \eta} \right]_{\min}},$$

where $\bar{h}_1$ is a scale factor defined as $\bar{h}_1 = 1 + \check{x} \cos \phi - \bar{y} \sin \phi + \eta \cos \psi \sin^2 \phi$.

The minimum value of the denominator occurs at the wall ($\eta = 0$) in the range of small waveangles that have been experimentally observed for slender cones. In order to evaluate the denominator, we note that $\tilde{V}(0) = 0$ and so we are required to evaluate only $\frac{\partial \tilde{V}}{\partial \eta}(0)$ for varying ϕ . Although not shown here, this quantity is found to decrease as ϕ is increased and our leading-order eigenvalue estimate increases in magnitude as the waveangle is increases. Numerical values for $\bar{\lambda}_0$ are given in Table 1.

IV.B.2. First-order solution

Following Hall's method, we expand the Taylor number in the form

$$T = \epsilon^{-4}(\lambda_0 + \lambda_1 \epsilon^{\frac{2}{3}} + \dots)$$

and re-scale the normal variable on an appropriate thickness $\xi = \frac{\varphi}{3^{\frac{1}{3}} \epsilon^{\frac{2}{3}}}$. The normal perturbation velocity is then expanded as

$$\tilde{w} = w_0(\xi) + \epsilon^{\frac{2}{3}} w_1(\xi) + \dots,$$

with $\tilde{u} = O(1)$ and $\tilde{v} = O(\epsilon^2)$ as in the leading-order analysis. After substituting these expressions into the governing equations and equating terms of $O(\epsilon^{\frac{2}{3}})$ we obtain an eigenvalue relation at first order. This can be solved to give a first-order estimate of our scaled Taylor-number eigenvalue as

$$\bar{\lambda}_1 = \frac{2.3381 \times 3^{\frac{1}{3}}}{|\tilde{V}'(0)|} \left[\frac{\tilde{V}''(0)}{\tilde{V}'(0)} + \tilde{V}'(0) \cos \phi \right]^2.$$

The mathematics is very involved and full details are given by Hussain.³ In fact we obtain an infinite sequence of eigenvalues $\{\lambda_{1n}\}$, corresponding to the zeros of an Airy function on the negative real axis. Numerical values for the most dangerous $\bar{\lambda}_1$ are given in Table 1 for various ψ and ϕ .

IV.B.3. Asymptotic estimate of the Taylor number

Combining the leading- and next-order solutions, the most dangerous instability mode has a scaled Taylor-number expansion given by

$$\begin{aligned}\bar{T} &= T \bar{h}_1^4 (1 + \check{x} \cos \phi - \bar{y} \sin \phi) \\ &= \epsilon^{-4} \left(\frac{1}{|\tilde{V}'(0)|} + \frac{2.3381 \times 3^{\frac{1}{3}} \epsilon^{\frac{2}{3}}}{|\tilde{V}'(0)|} \left[\frac{\tilde{V}''(0)}{\tilde{V}'(0)} + \tilde{V}'(0) \cos \phi \right]^2 + \dots \right).\end{aligned}$$

For large n , the eigenvalues may be approximated by

$$2\left(\frac{\bar{\lambda}_{1n}}{3|\tilde{V}'(0)|^3}\left[\frac{\tilde{V}''(0)}{\tilde{V}'(0)} + \tilde{V}'(0)\cos\phi\right]^6\right)^{\frac{3}{2}} \sim \pi\left(n - \frac{1}{4}\right),$$

and so the scaled Taylor-number estimate has the form

$$\begin{aligned}\bar{T}_n &= T\bar{h}_1^4(1 + \tilde{x}\cos\phi - \tilde{y}\sin\phi), \\ &= \epsilon^{-4}\left[\frac{1}{|\tilde{V}'(0)|} + 3\left(\frac{\epsilon|\tilde{V}'(0)|^3}{\left[\frac{\tilde{V}''(0)}{\tilde{V}'(0)} + \tilde{V}'(0)\cos\phi\right]^6}\frac{\pi}{2}\left(n - \frac{1}{4}\right)\right)^{\frac{2}{3}} + \dots\right], \\ &= \epsilon^{-4}\left(\bar{\lambda}_0 + \bar{\lambda}_{1n}\epsilon^{\frac{2}{3}} + \dots\right).\end{aligned}\tag{5}$$

Numerical estimates of the leading- and first-order eigenvalues corresponding to the scaled Taylor number are shown in Table 1 for parameter values in the range of those observed by Kobayashi & Izumi.⁶

ψ	ϕ	$\bar{\lambda}_0$	$\bar{\lambda}_1$
15°	0°	1.6236	2.0769
30°	1°	1.6477	1.8567
	2°	1.6731	1.6389
	2.7°	1.6915	1.4887
	4°	1.7277	1.2162
	5°	1.7572	1.0146
	6°	1.7883	0.8221
	8°	1.8556	0.4747
45°	8.5°	1.8735	0.3979
	10°	1.9305	0.1991

Table 1. Leading- and first-order eigenvalue estimates of the scaled Taylor number, $\bar{\lambda}_0, \bar{\lambda}_1$, for a range of slender rotating cones in still fluid.

Plots of the scaled Taylor number against wavenumber, $\epsilon^{-1} = a$, for $\phi = 0^\circ - 10^\circ$ in increments of 2.5° are shown in Figure 7. The unstable region is above the curves and the stable region below. We see that there is slight variation with waveangle which reduces with increased wavenumber. The definition of the Taylor number is such that increased $\bar{T}$ corresponds to reduced half-angle (for any fixed waveangle), the results may therefore suggest that the most unstable case (in the sense of a broader range of unstable wavenumbers) for the centrifugal-instability mode is for small half-angles; this, however, assumes a particular behaviour for the upper neutrally-stable branch in that it is at worst parallel to the computed lower branch.

IV.C. Numerical analysis

The asymptotic analysis has identified the centrifugal mode, which is the first result in this study. However, in order to judge its importance at each half-angle and waveangle, it is necessary to investigate the onset and subsequent amplification of the mode. These properties are unable to be determined by the asymptotic approach and work is underway on a numerical solution of the governing stability equations which will result in neutral-stability curves parameterized by Taylor number. Such an analysis will lead to predictions of amplification rates and other measurable quantities for experimental comparison.

V. Conclusion

In this paper we have highlighted the experimental motivation for the hypothesis of a centrifugal-instability mode within the rotating-cone boundary-layer flow. Furthermore, the results of previous theoretical studies have been summarized which add weight to the idea that an alternative mode exists. In particular, previous studies have been formulated with a view to studying the type I and II modes and note a discrepancy between the predicted onset of these modes and experimental measurements of the appearance

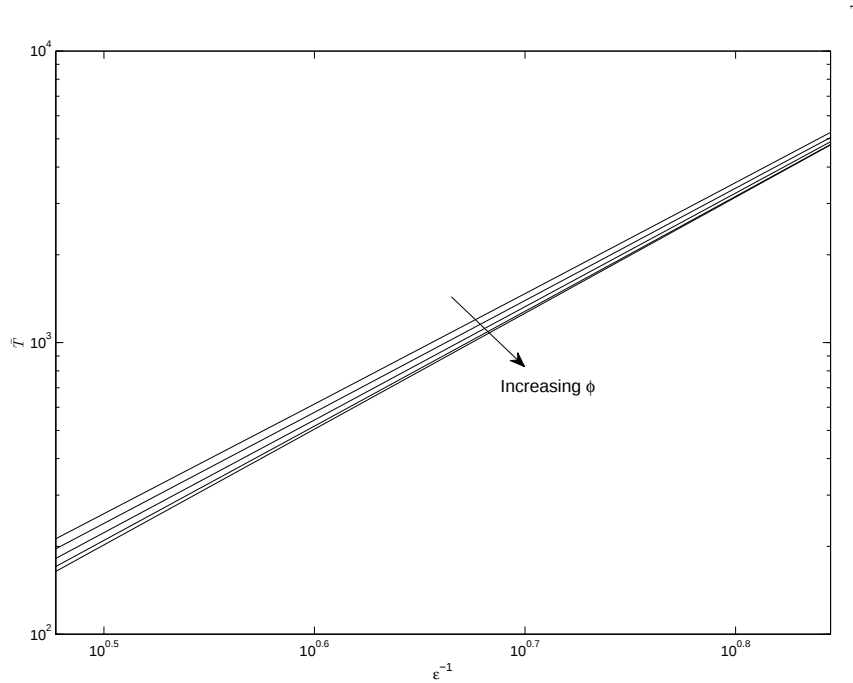


Figure 7. Asymptotic scaled Taylor number $\bar{T}$ as a function of non-dimensional vortex wavenumber ϵ^{-1} for $\phi = 0^\circ - 10^\circ$ in increments of 2.5° . The unstable region is above the neutral curves.

of spiral vortices with reduced half-angle.^{1,24,28} In addition, the recent discovery that the amplification rates of type I and II modes reduce with decreased half-angle is consistent with the notion of an alternative, competing mode with amplification rate that dominates for small half-angles.³¹

An alternative formulation that focuses on centrifugal effects has been developed and an asymptotic analysis conducted. Although the analysis requires the use of various scalings which make interpretation in terms of measurable quantities difficult, a centrifugal mode has been identified and an indication of the range of unstable wavenumbers against half-angle has been presented. It is suggested that the range of unstable wavenumbers increases with reduced half-angle (which is, in a sense, destabilizing), however the asymptotic approach is unable to calculate critical Reynolds numbers and amplification rates. A numerical analysis of the governing equations under this formulation is underway. The hypothesized properties of the centrifugal mode lead us to expect either a reduction in critical Reynolds number or an increase in amplification rates with reduced half-angle, or both.

We conclude that three instability modes exist within the boundary-layer flow over rotating cones: cross-flow (type I), streamline curvature (type II) and centrifugal. Although it is tempting to label this new mode as *type III*, we note that this is already used in the literature to denote the convectively-stable mode involved in the local absolute instability over broad cones and disks.^{1,4,23,24} Ultimately we wish to compare the onset and subsequent amplification of these three convectively-unstable modes for each cone in order to predict the half-angle at which the dominant instability mode changes; current experimental observations suggest that this is around $\psi = 40^\circ$.

Acknowledgments

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