Error Estimates for Interpolation of Rough and Smooth Functions using Radial Basis Functions

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Abstract

In this thesis we are concerned with the approximation of functions by radial basis function interpolants. There is a plethora of results about the asymptotic behaviour of the error between appropriately smooth functions and their interpolants, as the interpolation points fill out a bounded domain in Euclidean space. In all of these cases, the analysis takes place in a natural function space dictated by the choice of radial basis function—the native space.

This work establishes L_p-error estimates, for $1 \leq p \leq \infty$, when the function being interpolated fails to have the required smoothness to lie in the corresponding native space; therefore, providing error estimates for a class of rougher functions than previously known. Such estimates have application in the numerical analysis of solving partial differential equations using radial basis function collocation methods. At first our discussion focusses on the popular polyharmonic splines. A more general class of radial basis functions is admitted into exposition later on, this class being characterised by the algebraic decay of the Fourier transform of the radial basis function. The new estimates presented here offer some improvement on recent contributions from other authors by having wider applicability and a more satisfactory form. The method of proof employed is not restricted to interpolation alone. Rather, the technique provides error estimates for the approximation of rough functions for a variety of related approximation schemes as well.

For the previously mentioned class of radial basis functions, this work also gives error estimates when the function being interpolated has some additional smoothness. We find that the usual L_p-error estimate, for $1 \leq p \leq \infty$, where the approximand belongs to the corresponding native space, can be doubled. Furthermore, error estimates are established for functions with smoothness intermediate to that of the native space and the subspace of the native space where double the error is observed.

In memory of Will Light.

Preface

This thesis concerns approximation theory, a subject first introduced to the author by means of an undergraduate course taught by Will Light in 1998. In particular, this work is devoted to the study of the error induced by the recovery of data from a scattered data set, using radial basis functions. This again is a topic first introduced to the author by Will, this time in his role as my supervisor for my final year undergraduate project. I had, by this time, been snared by Will's unique, captivating and inspiring teaching style, so Will became a very natural choice as my PhD supervisor.

I acknowledge with deep gratitude Will's supervision. It will be a lasting memory that, despite what seemed like an eon of reporting to Will that I had got nowhere since he and I last spoke (but not through lack of endeavour), Will was never disappointed and always maintained (a sometimes questionable) faith in my ability. This thesis is dedicated to the memory of Will, who died on December 8, 2002.

I also acknowledge with deep gratitude, the role that Jeremy Levesley played as my supervisor after Will's death. Jeremy showed great interest in my research and, at a tricky time, provided me with renewed enthusiasm and new direction for my work.

Of all my postgraduate colleagues, fellow approximation theory PhD students David

Hunt and Michelle Vail must be accredited special praise. Absolutely invaluable were the many informal research related discussions that we shared. The friendship and support of Richard Bruce and Steve Lakin is also deserving of special mention, as too is that of former house mates, Sib Foster, Toby Reeve and Chris Davies, and more recently, Matthew Shephard, Chris Young and Clare Coen. I am especially thankful to David and Steve for their careful proof reading of my thesis. My appreciation is extended to Fran Narcowich for helpful personal communications whilst at MATA03 in Cancún, and to Michael Johnson for similarly helpful email communications. Obligatory thanks must go to EPSRC for providing the means for my study.

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Contents

Chapter 1

Introduction

Interpolation using the translates of a radial basis function is a popular method for the reconstruction of a multivariate function from a scattered data set. The scheme, which we describe shortly, has its beginnings in the empirical discovery of the multiquadric approach by Hardy [35]. One of the next major events on the radial basis function timeline was the publication of the seminal papers of Duchon [20, 21, 22] which gave a theoretical foundation for interpolation using the so-called polyharmonic splines. Now, having been studied for many years, there is a great abundance of results concerning radial basis functions, a sample of which are easily accessible through the review articles of Buhmann [11, 12], Dyn [24] and Powell [61], as well as Buhmann's recent monograph [13].

Radial basis functions have found themselves useful in many interesting applications. Communicated below are a selection of these. For example, mapping of images such as underwater sonar scans, aerial photographs or X-rays into another image for the purpose of comparison, lends itself well to radial basis function interpolation. Interpolation is important in this application as the user may wish to retain, under the mapping, certain special features which do not change over time. Zala and Barrodale [83] report that in two-dimensions the aforementioned polyharmonic splines are well suited to such problems. Interpolation is also important when reconstructing a temperature or barometric pressure profile on the globe using data obtained from monitoring stations in scattered locations. Hardy [36] reports that multiquadric interpolation performs well here. Another application for interpolation arises in the training of artificial neural networks. Here, using radial basis functions for the problem of learning to perform a particular task from a set of examples, is particularly alluring. One reason for this is that radial basis function interpolation is dimensionally independent and such neural networks usually generate high-dimensional interpolation problems (see Girosi [30]).

Finite element methods are historically the most popular methods for solving partial differential equations numerically. Here, the underlying principle is that one breaks up the computational domain, using a mesh, into small manageable pieces and approximates the solution of the partial differential equation on each of these cells. This approximation usually comprises of low order polynomials. Finite element methods have undeniably enjoyed much success, however, the generation of appropriate meshes in three or more space dimensions is a nontrivial problem. Further, in situations where the computational domain has a complicated structure, or the size and shape of the domain changes with time, the generation of a good quality computational mesh is also a challenging task. To overcome the difficulties just cited, there has been much interest in methods for solving such equations which avoid the generation of a computational mesh—*meshless methods*. One method is radial basis function collocation, which was pioneered by Kansa [45]. The method has enjoyed much applied success, for example in Dubal, Oliveira and Matzner [19] the method is used to find initial data for the configuration of two black holes.

Examples of the current nature of radial basis function research include: generating fast algorithms for approximation such as iterative methods for implementation (e.g. Beatson, Goodsell and Powell [3]) or fast evaluation techniques (e.g. Beatson and Light [4]), and providing more sophisticated error estimates than the ones arising naturally from the basic theory. The latter of these is precisely what this thesis concerns. The theory we glean and tools we obtain from this chapter are aimed at easing the discussion of such error estimates later. In this introductory chapter the intention is to set up and discuss some of the fundamental theory surrounding radial basis functions. In particular, we discuss the error in interpolation, and introduce certain important seminorms and spaces of functions which arise naturally in the study of interpolation using radial basis functions.

1.1 The interpolation problem

Let us describe the interpolation problem that is central to this work. We are supplied with two pieces of data. Firstly, we are given a finite set of pairwise distinct nodes, $\mathcal{A}$, which are possibly wildly scattered in $\mathbb{R}^d$, with no restriction on the ambient space dimension $d \geq 1$. Secondly, to each $a \in \mathcal{A}$, we are supplied a corresponding real-value. In other words we are given a *data function* $f : \mathcal{A} \to \mathbb{R}$. We seek to construct a function that agrees with this given data exactly—that is, we wish to find a function $Sf : \mathbb{R}^d \to \mathbb{R}$ that satisfies the *interpolation equations*,

$$(Sf)(a) = f(a), \qquad \text{for all } a \in \mathcal{A}. \tag{1.1.1}$$

The name given to this process is *interpolation* and Sf is called an *interpolant* to f on $\mathcal{A}$. The members of $\mathcal{A}$ are called *interpolation points.* We shall assume that the search for our interpolant is confined to a fixed space of real-valued functions on $\mathbb{R}^d$ that we shall call X.

It would not be unreasonable to expect that if the data function f were replaced by αf, $\alpha \in \mathbb{R}$, then the interpolant to the new data should be given by the equation $S(\alpha f) = \alpha Sf$. Similarly, one would hope that given another data function $g : \mathcal{A} \to \mathbb{R}$, we would have $S(f + g) = Sf + Sg$. These sensible desires restrict our choice for X to be a linear space of functions. A drawback of having a linear space is that interpolation may not be uniquely specified. For, if there is a nonzero function $T \in X$ that vanishes at all the interpolation points, then $Sf + \alpha T \in X$ is an interpolant to the data for all $\alpha \in \mathbb{R}$. In order to provide numerical stability to the system (1.1.1) it is important that Sf is uniquely specified; therefore, we demand that $\mathcal{A}$ is unisolvent with respect to X.

Definition 1.1.1. *If X is a space of functions on $\mathbb{R}^d$, then a subset $\mathcal{A}$ of $\mathbb{R}^d$ is called unisolvent with respect to X if zero is the only function in X that vanishes on $\mathcal{A}$.*

The theory we are describing seems to be progressing well; however, there is a surprising difficulty when considering a truly multivariate situation. Indeed, it is disappointing to learn that the following theorem, often attributed to Mairhuber, is true.

Theorem 1.1.2 (Mairhuber [55])**.** *Let X be an n-dimensional space of continuous real valued functions on $\mathbb{R}^d$. Assume that every set of points $\mathcal{A} \subset \mathbb{R}^d$ with $|\mathcal{A}| = n$ is unisolvent with respect to X. Then either $n = 1$ or $d = 1$.*

Fortunately, there is a readily available remedy to circumvent Mairhuber's theorem—we choose X to depend on the nodes $\mathcal{A}$. Arguably the simplest way to achieve this is to use the translates of a single multivariate *basis function*, $\psi : \mathbb{R}^d \to \mathbb{R}$, to define the space X as the linear span

$$X = \operatorname{span}\Big\{\psi(\cdot\ - a) :\ a \in \mathcal{A}\Big\}.$$

One could generalise this setting by using $\psi(x, y)$ instead of $\psi(x - y)$, with ψ now defined on $\mathbb{R}^d \times \mathbb{R}^d$, but we do not encounter such kernels in this work. The particular choice of ψ can be highly dependent on the application the user has in mind. For example, she may require the interpolant to have a certain degree of smoothness or a certain behaviour at infinity.

A function ψ is called *radial* if the value of the function depends only on the Euclidean distance of the argument. More precisely, if there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that $\psi = \phi \circ |\cdot|$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. Naturally, we refer to those basis functions which are radial as *radial basis functions*. It is convenient to be a little sloppy and refer to the underlying univariate function ϕ that generates ψ as the radial basis function as well. In most of the common applications the function ψ is a radial basis function. Our attention now turns to the solvability of the interpolation equations (1.1.1).

1.2 Solving the interpolation equations

As introduced in the previous section, we have the following interpolation problem. Let ψ be a real-valued function defined on $\mathbb{R}^d$ and let $\mathcal{A} \subset \mathbb{R}^d$ be pairwise distinct. Given the real-values $f(a)$, $a \in \mathcal{A}$, we wish to construct

$$(Sf)(x) = \sum_{a \in \mathcal{A}} \lambda_a \psi(x - a), \qquad \text{for } x \in \mathbb{R}^d, \tag{1.2.1}$$

where the coefficients λ_a are determined by the interpolation equations $(Sf)(a) = f(a)$, $a \in \mathcal{A}$. This is equivalent to solving the linear system

$$\sum_{b \in \mathcal{A}} \lambda_b \psi(a-b) = f(a), \qquad \text{for all } a \in \mathcal{A}.$$

Rewriting this system in self-evident matrix notation we have

$$\Psi\lambda = f, \tag{1.2.2}$$

where λ and f are vectors in $\mathbb{R}^{|\mathcal{A}|}$, and Ψ is the $|\mathcal{A}| \times |\mathcal{A}|$ matrix with entries $\Psi_{ab} = \psi(a-b)$, for $a, b \in \mathcal{A}$. Here, we are using $|\mathcal{A}|$ to denote the cardinality of $\mathcal{A}$. Of course, the interpolant Sf will be uniquely specified if the *interpolation matrix*, Ψ, is nonsingular. We mostly restrict our discussions to a particular class of nonsingular matrix.

1.2.1 Strictly positive definite functions

One way to render the interpolation matrix nonsingular is to introduce the concept of strictly positive definite functions.

Definition 1.2.1. *Let $\psi : \mathbb{R}^d \to \mathbb{R}$. If, for any finite set of pairwise distinct points $\mathcal{A} \subset \mathbb{R}^d$ and any vector $\lambda \in \mathbb{R}^{|\mathcal{A}|} \setminus 0$, we have*

$$\lambda^T \Psi \lambda > 0,$$

then we say ψ is strictly positive definite.

The linear system (1.2.2) is uniquely solvable provided the function ψ is strictly positive definite. This is because Ψ is a positive definite matrix in this situation and so, in particular, is nonsingular. The Gaussian, defined by $\psi(x) = \mathrm{e}^{-\alpha|x|^2}$, $\alpha > 0$, is an example of a strictly positive definite function (see Powell [61, Page 118]). This is our first example of a radial basis function since $\psi(x) = \phi(|x|)$ for $\phi(r) = \mathrm{e}^{-\alpha r^2}$, $r \geq 0$. Another example comes from those functions that can be written in the special form

$$\phi(r) = \int_0^\infty \mathrm{e}^{-\alpha r^2} w(\alpha)\,\mathrm{d}\alpha, \qquad r \geq 0, \tag{1.2.3}$$

for some nonnegative function w. Then, the proof that $\psi(x) = \phi(|x|)$ is strictly positive definite follows immediately from the established strict positive definiteness of $\mathrm{e}^{-\alpha|x|^2}$ (see Powell [61]). Substituting $w(\alpha) = (\pi\alpha)^{-1/2}\mathrm{e}^{-\alpha c^2}$, for $c > 0$, into (1.2.3) leads to the inverse multiquadric basis function $\phi(r) = (c^2 + r^2)^{-1/2}$. This is the second example of a radial basis function that we can add to our list. An alternative proof that the Gaussian and inverse multiquadric are strictly positive definite comes from what turns out to be a characterisation of strictly positive definite functions, first given by Schoenberg [69].

Definition 1.2.2. *A function ϕ is said to be completely monotone if $\phi \in C^\infty(0, \infty)$ and $(-1)^j \phi^{(j)}$ is nonnegative for all $j \in \mathbb{Z}_+$.*

Theorem 1.2.3 (Schoenberg [69])**.** *Let $\phi \in C[0, \infty)$ be completely monotone. Then the function defined by $\psi = \phi \circ |\cdot|^2$ is strictly positive definite.*

Now we see, for example, that the inverse multiquadric is strictly positive definite as the function $\phi = (c^2 + \cdot)^{-1/2}$ gladly satisfies the hypotheses of Theorem 1.2.3, for all $c > 0$.

1.2.2 Conditionally strictly positive definite functions

Curiously, some of the first radial basis functions to be used successfully were not strictly positive definite functions at all. These include the thin-plate spline and the multiquadric which we shall encounter shortly. These particular functions generate interpolation matrices which are positive definite only on a specific proper subspace.

Definition 1.2.4. *Let $\psi : \mathbb{R}^d \to \mathbb{R}$. If, for any finite set of pairwise distinct points $\mathcal{A} \subset \mathbb{R}^d$ and any vector $\lambda \in \mathbb{R}^{|\mathcal{A}|} \setminus 0$, satisfying*

$$\sum_{a \in \mathcal{A}} \lambda_a p(a) = 0, \qquad \text{for all } p \in \Pi_{m-1}(\mathbb{R}^d),$$

we have

$$\lambda^T \Psi \lambda > 0,$$

then we say ψ is conditionally strictly positive definite of order m.

The notation $\Pi_{m-1}(\mathbb{R}^d)$ in Definition 1.2.4 is used to denote the space of real-valued polynomials on $\mathbb{R}^d$ of degree at most $m-1$. To ensure there is no confusion when $m = 0$ we adopt the convention $\Pi_{-1}(\mathbb{R}^d) = 0$. Then, we see that conditionally strictly positive definite functions of order 0 are a guise for strictly positive definite functions.

It is important to observe that we can interpolate uniquely with a conditionally strictly positive definite function ψ of order m. When supplied with such a basis function we augment the sum of translates (1.2.1) with a polynomial p of degree $m-1$. Our new interpolant will now have the form

$$(Sf)(x) = \sum_{a \in \mathcal{A}} \lambda_a \psi(x-a) + p(x), \qquad x \in \mathbb{R}^d, \tag{1.2.4}$$

and the linear interpolation system becomes

$$\sum_{b \in \mathcal{A}} \lambda_b \psi(a-b) + p(a) = f(a), \qquad a \in \mathcal{A}. \tag{1.2.5a}$$

To recover a square system we must add precisely

$$\ell = \dim\left(\Pi_{m-1}(\mathbb{R}^d)\right) = \binom{d+m-1}{m-1}$$

more equations. A most convenient way to do this is to demand that

$$\sum_{a\in\mathcal{A}} \lambda_a q(a) = 0, \qquad \text{for all } q \in \Pi_{m-1}(\mathbb{R}^d). \tag{1.2.5b}$$

We can now rewrite the system (1.2.5) in self-evident matrix notation as

$$\begin{bmatrix} \Psi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{1.2.6}$$

where λ and f are vectors in $\mathbb{R}^{|\mathcal{A}|}$, $\mu \in \mathbb{R}^\ell$, Ψ is as in (1.2.2) and P is an $|\mathcal{A}|\times\ell$ matrix. We now verify that we can interpolate uniquely with a conditionally strictly positive definite function.

Theorem 1.2.5. *Let ψ be a conditionally strictly positive definite of order m and let $\mathcal{A} \subset \mathbb{R}^d$ be unisolvent with respect to $\Pi_{m-1}(\mathbb{R}^d)$. Then the system* (1.2.6) *is uniquely solvable.*

Proof. Let $(\lambda,\mu) \in \mathbb{R}^M \times \mathbb{R}^\ell$ be a member of $\ker \left[\begin{smallmatrix} \Psi & P \\ P^T & 0 \end{smallmatrix}\right]$. Then, by multiplying out the system we have $\Psi\lambda + P\mu = 0$ and $P^T\lambda = 0$. Premultiplying the first equation by λ^T and using the second of these equations we find that $\lambda^T\Psi\lambda = 0$. Since ψ is conditionally strictly positive definite of order m we conclude that $\lambda = 0$. Thus $P\mu = 0$. This last equation can be written in full as

$$\sum_{|\beta|<m} \mu_\beta a^\beta = 0, \qquad a \in \mathcal{A},$$

for some numbers μ_β. Here, $\beta \in \mathbb{Z}_+^d$ is a *multi-integer* and $|\beta|$ denotes the order of β (this concept is defined in Section 1.5). Since $\mathcal{A}$ is unisolvent with respect to $\Pi_{m-1}(\mathbb{R}^d)$ it follows that $\mu_\beta = 0$ for $|\beta| < m$. Thus $\mu = 0$. Therefore, the kernel contains only the zero vector. This in turn implies that $\left[\begin{smallmatrix} \Psi & P \\ P^T & 0 \end{smallmatrix}\right]$ is surjective; hence, the system (1.2.6) is uniquely solvable. □

The unisolvency condition we impose on the interpolation points in Theorem 1.2.5 is, in practice, not problematic. This is because the popular conditionally strictly positive definite functions, a selection of which we see shortly, are typically of low order. For example, with $d = m = 2$ we must ensure that $\mathcal{A}$ contains three non-collinear points. For $m = 1$ the condition is trivial for any $d \geq 1$. For $m = 0$, as we expect, there is no requirement made of the points.

The following result, due to Micchelli [57], generalises Theorem 1.2.3 to include conditionally strictly positive definite functions. Interestingly, the converse is also true and was supplied later by Guo, Hu and Sun [33]; therefore, Theorem 1.2.6 genuinely constitutes a characterisation of conditionally strictly positive definite functions.

Theorem 1.2.6 (Micchelli [57])**.** *Let $\phi \in C[0,\infty)$ and suppose $(-1)^m \phi^{(m)}$ is completely monotone but not constant. Then the function defined by $\psi = \phi \circ |\cdot|^2$ is conditionally strictly positive definite of order m.*

It follows that the radial basis function $\phi(r) = -(c^2 + r^2)^{1/2}$, $c \geq 0$, is conditionally strictly positive definite of order 1. When $c = 0$ we call this radial basis function the bare norm, and when $c > 0$ we call it the multiquadric. The thin-plate spline $\phi(r) = r^2 \log r$ and the cubic radial basis function $\phi(r) = r^3$, which are members of the polyharmonic

Radial basis function	$\phi(r)$, $r \geq 0$	m
Cubic	r^3	2
Thin-plate spline	$r^2 \log r$	2
Bare norm	$-r$	1
Multiquadric	$-(c^2 + r^2)^{1/2}$, $c > 0$	1
Gaussian	$\mathrm{e}^{-\alpha r^2}$, $\alpha > 0$	0
Inverse multiquadric	$(c^2 + r^2)^{-1/2}$, $c > 0$	0

Table 1.1: Popular radial basis functions and their respective orders.

spline family, both turn out to be conditionally strictly positive definite of order 2. It is worth noting that interpolating using the bare norm with $d = 1$ produces the natural linear spline. Similarly, interpolation with the cubic radial basis function with $d = 1$ leads to the natural cubic spline. So, interpolating with these basis functions, for $d > 1$, can be thought of as a multivariate analogy of the well-loved univariate natural splines.

Micchelli also established that for the radial basis function $\phi(r) = (c^2 + r^2)^{1/2}$, $c \geq 0$, the related interpolation matrix Ψ alone is nonsingular for all choices of finite pairwise distinct interpolation points.

Theorem 1.2.7 (Micchelli [57]). *Let $\phi \in C[0, \infty)$ be such that ϕ' is completely monotone but not constant and $\phi(0) \geq 0$. If $\psi = \phi \circ |\cdot|^2$ then Ψ is nonsingular for every set of finite pairwise distinct points.*

Collected in Table 1.1 is a summary of the radial basis functions we have encountered so far, and a list of their respective orders. We now have a nice sized collection of basis functions with which to interpolate scattered data; however, the detour we have taken to introduce conditionally strictly positive definite functions looks a little contrived. Indeed, it is unclear why adding a polynomial term to the sum of translates (1.2.1) is a sensible idea. Partly to address this issue, our discussion now moves to explain how conditionally strictly positive definite functions can arise as part of a beautiful variational approach to interpolation.

1.3 A variational approach to interpolation

Among the first classes of radial basis functions to be actively researched were the polyharmonic splines, and one of the earliest references to them comes from the aeronautical industry in the form of a paper by Harder and Desmarais [34]. The problem there involved finding interpolants which minimised the bending energy of thin-plates subject to some interpolation conditions—that is, an interpolant that minimises the seminorm

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}\right|^2 \mathrm{d}x\mathrm{d}y.$$

Duchon, building on earlier work by Atteia [2], extended this idea to higher dimensions. His papers [20, 21, 22] were seminal in the development of a variational approach to interpolation. The exposition we see in this section is based, fundamentally, on the work of Golomb and Weinberger [32] and, more recently, on the contribution by Light and Wayne [49].

We assume we have a linear space X of continuous functions in which to carry out our variational arguments. The space X is assumed to have a real-valued semi-inner product $(\,\cdot\,,\,\cdot\,)_X$ and associated seminorm $|\cdot|_X = \sqrt{(\,\cdot\,,\,\cdot\,)_X}$ with ℓ-dimensional kernel K. In every one of our applications, K will turn out to be a space of polynomials. The spaces that Duchon considers are, in fact, spaces of distributions which he is able to embed in $C(\mathbb{R}^d)$. We use $\mathscr{S}'$ to denote the space of all distributions, and for those unfamiliar with this notion, Section 1.5 is provided and contains the essentials. These particular spaces of distributions are called Beppo Levi spaces in honour, it seems, of the person who was first to study them (see Deny and Lions [17]). For $m \in \mathbb{Z}_+$ with $m > d/2$, the m-th integer order Beppo Levi space is denoted by $BL^m(\mathbb{R}^d)$ and defined as

$$BL^m(\mathbb{R}^d) = \Big\{ f \in \mathscr{S}' : \; D^\alpha f \in L_2(\mathbb{R}^d), \; \alpha \in \mathbb{Z}_+^d, \; |\alpha| = m \Big\},$$

with semi-inner product

$$(f, g)_{m,\mathbb{R}^d} = \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} (D^\alpha f)(x)(\overline{D^\alpha g})(x)\,\mathrm{d}x, \qquad \text{for } f, g \in BL^m(\mathbb{R}^d), \tag{1.3.1}$$

and associated seminorm $|\cdot|_{m,\mathbb{R}^d}$. The constants c_α are chosen so that the seminorm is rotationally invariant:

$$\sum_{|\alpha|=m} c_\alpha x^{2\alpha} = |x|^{2m}, \qquad \text{for all } x \in \mathbb{R}^d.$$

Precisely, we choose $c_\alpha = \binom{m}{\alpha!}$. The kernel of this seminorm is $\Pi_{m-1}(\mathbb{R}^d)$. We have assumed that $m > d/2$ here, because this has the effect that $BL^m(\mathbb{R}^d)$ is embedded in the continuous functions (Duchon [21]).

Returning to our general setup, we always have in mind a set of pairwise distinct points $\mathcal{A} \subset \mathbb{R}^d$, containing a unisolvent subset with respect to K, on which we wish to interpolate. Now, we choose to alter the semi-inner product $(\,\cdot\,,\,\cdot\,)_X$ in a special way to form an authentic inner product on X. This is achieved by selecting, from $\mathcal{A}$, a subset $\mathcal{A}'$

of size ℓ which is unisolvent with respect to K. We then define

$$\langle f, g \rangle_X = (f, g)_X + \sum_{a \in \mathcal{A}'} f(a)\overline{g}(a), \qquad \text{for } f, g \in X.$$

We assume X is complete with respect to the induced norm $\|\cdot\|_X = \sqrt{\langle\cdot\,,\,\cdot\rangle_X}$. Our final assumption concerning X is that, given $x \in \mathbb{R}^d$, there exists a $C > 0$ such that

$$|f(x)| \leq C\|f\|_X, \qquad \text{for all } f \in X.$$

It follows from a classical result, called the Riesz representation theorem (Adams [1, Page 5] for instance), that a Hilbert space is a reproducing kernel Hilbert space if and only if the point evaluation functional δ_x is a bounded linear functional.

Definition 1.3.1. *Let X be a Hilbert space of real-valued functions on $\mathbb{R}^d$, with inner product $\langle\cdot\,,\,\cdot\rangle_X$. Then X is called a reproducing kernel Hilbert space if, given $x \in \mathbb{R}^d$, there is a unique function $R_x \in X$ such that*

$$f(x) = \langle f, R_x \rangle_X, \qquad \text{for all } f \in X.$$

The function R_x is called the reproducing kernel for x in X.

1.3.1 Minimal norm interpolants

The so-called minimal norm interpolant to $f : \mathcal{A} \to \mathbb{R}$ on $\mathcal{A}$ from X is the function $Sf \in X$ satisfying

$$(Sf)(a) = f(a), \qquad \text{for all } a \in \mathcal{A}; \tag{1.3.2a}$$

$$\|Sf\|_X \leq \|g\|_X, \qquad \text{for all } g \in X \text{ such that } g(a) = f(a) \text{ for all } a \in \mathcal{A}.$$

The interpolation constraints reduce the second condition to

$$|Sf|_X \leq |g|_X, \qquad \text{for all } g \in X \text{ such that } g(a) = f(a) \text{ for all } a \in \mathcal{A}. \tag{1.3.2b}$$

One can demonstrate that the set of all interpolants to f on $\mathcal{A}$ from X is both closed and convex. So, we are assured that the minimal norm interpolant exists and is unique (see Cheney [14, Page 22]). Furthermore, we have at our disposal a method for calculating Sf.

Theorem 1.3.2. *Let $f \in X$. Let $\mathcal{A} \subset \mathbb{R}^d$ be a finite set of pairwise distinct points containing a unisolvent subset with respect to K. For each $x \in \mathbb{R}^d$, let R_x be the reproducing kernel for x in X. Let $Sf \in X$ be the minimal norm interpolant to f on $\mathcal{A}$. Then $Sf = \sum_{a \in \mathcal{A}} \lambda_a R_a$, where the numbers λ_a are determined by the linear system*

$$\sum_{b \in \mathcal{A}} \lambda_b \langle R_b, R_a \rangle_X = \langle f, R_a \rangle_X, \qquad \textit{for all } a \in \mathcal{A}.$$

Proof. Firstly, let us define the subset $V = \bigcap_{a \in \mathcal{A}} \{ v \in X : \ v(a) = 0 \}$. We have chosen Sf to minimise $\|Sf\|_X$ such that $Sf(a) = f(a)$, $a \in \mathcal{A}$. This is equivalent to minimising $\|Sf\|_X$ subject to the condition $f - Sf \in V$. Writing $v = f - Sf$, we see that we are minimising $\|f - v\|_X$ subject to the condition that $v \in V$. This is a standard problem of best approximation (see Cheney and Light [15, Page 210]), and the solution, v, is characterised by the conditions $v \in V$ and $f - v \perp V$. Hence, $f - Sf \in V$ and $Sf \perp V$. The set V can be rewritten using the reproducing kernels for $a \in \mathcal{A}$ in X as

$$V = \bigcap_{a \in \mathcal{A}} \{ v \in X : \ \langle v, R_a \rangle_X = 0 \} = \bigcap_{a \in \mathcal{A}} R_a^{\perp} = \left(\operatorname{span} \left\{ R_a : \ a \in \mathcal{A} \right\} \right)^{\perp}.$$

Hence, $Sf \in \operatorname{span} \{ R_a : \ a \in \mathcal{A} \}$ and the coefficients in the span are determined by the interpolation equations. $\square$

A very useful orthogonality property of the minimal norm interpolant emerges in the proof of Theorem 1.3.2. In particular, we have $Sf \perp (f - Sf)$ for all $f \in X$. Therefore,

$$\begin{aligned}\|f - Sf\|_X^2 = \langle f - Sf, f - Sf\rangle_X &= \langle f, f\rangle_X - \langle Sf, f\rangle_X \\ &= \langle f, f\rangle_X - \langle Sf, Sf\rangle_X = \|f\|_X^2 - \|Sf\|_X^2,\end{aligned}$$

for all $f \in X$. The interpolation equations reduce this display to

$$|f - Sf|_X^2 = |f|_X^2 - |Sf|_X^2, \qquad \text{for all } f \in X. \tag{1.3.3}$$

We make good use this *Pythagorean* property later.

1.3.2 Surface splines

As an example to help facilitate our discussion, we return to the case $X = BL^m(\mathbb{R}^d)$, which happily satisfies all of our assumptions. To compute the minimal norm interpolant we desire an expression for the reproducing kernel for x in $BL^m(\mathbb{R}^d)$. Such an expression can be readily extracted from, amongst other places, Light and Wayne [49]. We have, for $x, y \in \mathbb{R}^d$,

$$\begin{aligned}R_x(y) = \phi(|y-x|) - \sum_{a\in\mathcal{A}'} p_a(x)\phi(|y-a|) - \sum_{a\in\mathcal{A}'} p_a(y)\phi(|x-a|) \\ + \sum_{a\in\mathcal{A}'}\sum_{b\in\mathcal{A}'} p_a(x)p_b(y)\phi(|a-b|) + \sum_{a\in\mathcal{A}'} p_a(x)p_a(y).\end{aligned} \tag{1.3.4}$$

Here, the functions p_a, $a \in \mathcal{A}'$, form a Lagrange basis for $\Pi_{m-1}(\mathbb{R}^d)$ based on the points $\mathcal{A}'$. This means that if $b \in \mathcal{A}'$ then $p_a(b)$ is 1 if $b = a$ and 0 if $b \neq a$. The function ϕ depends on the parity of d and is defined, up to a known constant, by

$$\phi(r) \doteq \begin{cases} r^{2m-d}\log r, & \text{if } d \text{ is even,} \\ r^{2m-d}, & \text{otherwise,}\end{cases} \qquad r \geq 0. \tag{1.3.5}$$

We now see, from the form of (1.3.4), that for each $a \in \mathcal{A}$, R_a is a linear combination of $\phi(|\cdot - a|)$, $\phi(|\cdot - b|)$ and p_b, $b \in \mathcal{A}'$. It follows from Theorem 1.3.2 that we can write the minimal norm interpolant in the form

$$Sf = \sum_{a \in \mathcal{A}} \lambda_a \phi(|\cdot - a|) + \sum_{a \in \mathcal{A}'} \mu_a p_a.$$

It transpires that the coefficients appearing in the above display solve the system of equations

$$\begin{aligned} \sum_{b \in \mathcal{A}} \lambda_b \phi(|a - b|) + \sum_{b \in \mathcal{A}'} \mu_b p_b(a) &= f(a), \qquad && a \in \mathcal{A}, \\ \sum_{b \in \mathcal{A}} \lambda_b p_a(b) &= 0, && a \in \mathcal{A}'. \end{aligned}$$

Now, we observe that the unique solution of the variational problem (1.3.2) is one which is precisely of the form (1.2.4). Furthermore, the coefficients are selected in the same manner as solving (1.2.5). It is perhaps no surprise therefore to learn that the radial basis function (1.3.5) is conditionally strictly positive definite (of order m). The related radial basis function interpolants are known as polyharmonic splines. The name polyharmonic spline arises because the function ϕ in (1.3.5) is the fundamental solution of the m-th iterated Laplacian:

$$\Delta^m \phi(|x|) \doteq \delta_x.$$

1.3.3 Fundamental estimates for the error in interpolation

Let us return once again to our general setup. It is of central importance to understand the behaviour of the error between a function $f : \Omega \to \mathbb{R}$ and its interpolant, as the set

$\mathcal{A} \subset \Omega$ becomes "dense" in Ω. The measure of density we employ is the *fill-distance*

$$h = \sup_{x \in \overline{\Omega}} \min_{a \in \mathcal{A}} |x - a|.$$

This can be thought of as the radius of the largest open ball, with centre inside Ω, that does not intersect $\mathcal{A}$. It is also of interest to learn that the balls $\{x \in \mathbb{R}^d : |x - a| \leq h\}$ cover Ω. One might hope that for some suitable norm $\|\cdot\|$ there is a positive constant γ, independent of f and h, such that

$$\|f - Sf\| = \mathcal{O}(h^\gamma), \qquad \text{as } h \to 0.$$

The reproducing kernel provides us with a useful tool to establish such error estimates whenever f lies in X. We let $V = \big\{g \in X : \ g(a) = 0, \ a \in \mathcal{A}'\big\}$ and define the *power function*, $P : \mathbb{R}^d \to \mathbb{R}$, via

$$P(x) = \sup \Big\{|v(x)| : \ v \in V, \ |v|_X = 1\Big\}, \qquad \text{for } x \in \mathbb{R}^d.$$

This immediately provides us with the error estimate

$$|f(x) - (Sf)(x)| \leq P(x)|f - Sf|_X, \qquad \text{for all } x \in \mathbb{R}^d, \ f \in X.$$

Our hypotheses on $\big(X, \|\cdot\|_X\big)$ imply that $\big(V, \|\cdot\|_X\big)$ is a reproducing kernel Hilbert space in its own right. We shall use r_x to denote the reproducing kernel for x in V. Using r_x we can rewrite the power function as

$$P(x) = \sup \Big\{|\langle v, r_x\rangle_X| : \ v \in V, \ |v|_X = 1\Big\}, \qquad \text{for } x \in \mathbb{R}^d.$$

Let $v \in V$ with $|v|_X = 1$. By using the Cauchy–Schwarz inequality we obtain

$$|\langle v, r_x\rangle_X| \leq \|v\|_X \|r_x\|_X = |v|_X \|r_x\|_X = \|r_x\|_X,$$

which instructs us that $P(x) \leq \|r_x\|_X$, for all $x \in \mathbb{R}^d$. On the other hand, since $r_x \in V$,

$$P(x) \geq \frac{|\langle r_x, r_x \rangle_X|}{|r_x|_X} = \frac{\|r_x\|_X^2}{\|r_x\|_X} = \|r_x\|_X.$$

Hence, $P(x) = \|r_x\|_X = \sqrt{r_x(x)}$, $x \in \mathbb{R}^d$, and we have the error estimate

$$|f(x) - (Sf)(x)| \leq \sqrt{r_x(x)}|f - Sf|_X, \qquad \text{for all } x \in \mathbb{R}^d,\ f \in X. \tag{1.3.6}$$

One can replace $|f - Sf|_X$ by $|f|_X$ in this error estimate by using equality (1.3.3). Therefore, if one can show that, locally at least, we have the asymptotic behaviour $\sqrt{r_x(x)} = \mathcal{O}(h^\gamma)$ for some $\gamma > 0$, we would obtain the sought after error estimate.

For the polyharmonic splines of Section 1.3.2, it transpires that r_x is given by

$$r_x(y) = t_x(y) - \sum_{a \in \mathcal{A}'} t_x(a) p_a(y), \qquad x, y \in \mathbb{R}^d,$$

where

$$t_x(y) = \phi(|y - x|) - \sum_{a \in \mathcal{A}'} p_a(x) \phi(|y - a|), \qquad x, y \in \mathbb{R}^d,$$

with ϕ as in (1.3.5). The function t_x belongs to X and is such that $f(x) = \langle t_x, f \rangle_{m,\mathbb{R}^d}$ for all $f \in V$. To obtain r_x one takes the image of t_x under a suitable projection. A local asymptotic bound for

$$\sqrt{r_x(x)} = \left(-2 \sum_{a \in \mathcal{A}'} p_a(x) \phi(|x - a|) + \sum_{a \in \mathcal{A}'} \sum_{b \in \mathcal{A}'} p_a(x) p_b(x) \phi(|a - b|) \right)^{1/2}$$

is available as $\mathcal{A}$ fills out an open, bounded, connected domain $\Omega \subset \mathbb{R}^d$ satisfying the cone property (Definition 2.1.1), and so an $L_\infty(\Omega)$ estimate is obtained. Then, by splitting Ω into balls in a special way and estimating the error separately on each of the balls one can obtain the following L_p-error estimate for $f \in X$ and $2 \leq p \leq \infty$.

Theorem 1.3.3 (Duchon [22]). *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz-continuous boundary. Suppose $m > d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, Π_{m-1}-unisolvent subset of Ω with fill-distance h. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h, such that, for $2 \leq p \leq \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \leq C h^{m-|\alpha|-d/2+d/p} |f|_{m,\Omega}, \qquad |\alpha| \leq m - d/2 + d/p, \tag{1.3.7}$$

for all $f \in BL^m(\Omega)$, as $h \to 0$.

Here, the notation $BL^m(\Omega)$ is the manifestly obvious local variant of the space $BL^m(\mathbb{R}^d)$ with seminorm defined by

$$|f|_{m,\Omega} = \left(\sum_{|\alpha|=m} c_\alpha \int_\Omega |(D^\alpha f)(x)|^2 \, \mathrm{d}x \right)^{1/2}, \qquad \text{for } f \in BL^m(\Omega).$$

1.3.4 Native spaces

We have seen previously that the polyharmonic splines arise as the solution of a variational problem set in a certain natural space of functions. This space was generated by a seminorm. The work of Light and Wayne [47] considers the seminorm

$$\left(\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(\widehat{D^\alpha f})(x)|^2 w(x) \, \mathrm{d}x \right)^{1/2},$$

and, much like the integer order Beppo Levi spaces, this seminorm is used to construct a natural, or native space of functions to carry out a variational argument. The notation $\widehat{\ }$ is used to denote the (distributional) Fourier transform (this is defined in Section 1.5). The *weight function* $w : \mathbb{R}^d \to \mathbb{R}$ satisfies a list of reasonable axioms that we describe, in detail, in Chapter 3. If the weight function $w = 1$ is selected we would return to the polyharmonic spline setting via the Plancherel theorem (see Section 1.4). These axioms ensure, amongst other things, that the ensuing space is continuously embedded in the continuous functions, so that point evaluation makes sense. Just like in the polyharmonic spline case, the minimal norm interpolant has precisely the form of (1.2.4) that we desire. Here, the basis function ψ satisfies the distributional equation

$$\widehat{\psi}|\cdot|^{2m} = \frac{1}{w}.$$

Further, ψ turns out to be a continuous conditionally strictly positive definite function of some appropriate order.

The work of Madych and Nelson [51, 52, 53] contains an alternative approach. They choose instead to start with a conditionally strictly positive definite function and use it to construct a space of functions in which to carry out a variational argument. Other papers that embrace this approach include those of Dyn [23, 24] and several papers by Schaback which are accessible through the survey paper [64]. As the author prefers the line of attack described in Section 1.3, we omit the details of this construction and instead comment that these two approaches—given a space that dictates the basis function or given a basis function that dictate the space—are equivalent (see Buhmann [13, Page 121]). In either case, the space is given the apt name *native space*.

1.4 Error estimates for rough and smooth functions

Look at our fundamental error estimate (1.3.6); the analysis to arrive at this bound is highly reliant on f belonging to X—the native space. Indeed, the bound is meaningless if f lies outside of X, as $|f - Sf|_X$ is not necessarily defined. To illustrate this point further, consider again our example of the polyharmonic splines and Duchon's error estimate (1.3.7). It is a natural question to ask: what happens if the function f does not possess sufficient smoothness to lie in $BL^m(\Omega)$? It may well be that f lies in $BL^k(\Omega)$, where $2m > 2k > d$. The condition $2k > d$ ensures that $f(a)$ exists for each $a \in \mathcal{A}$, and so $S_m^h f$ certainly exists. It is simple to conjecture that the new estimate for the error in interpolation of this rougher function might be, for $2 \le p \le \infty$,

$$\|f - S_m^h f\|_{L_p(\Omega)} \le Ch^{k-d/2+d/p}|f|_{k,\Omega}, \qquad \text{as } h \to 0. \tag{1.4.1}$$

Chapter 2 of this work endeavours to establish the conjectured bound (1.4.1). Theorem 2.2.8 is the definitive result we obtain. Chapter 3 again tackles the problem of approximation orders for the error in interpolation of rough functions, but this time for the weighted Beppo Levi spaces that we saw briefly in Section 1.3.4. Here, Theorem 3.2.13 is the definitive result we obtain. The method of proof we employ is not restricted to interpolation alone. The technique will succeed in providing error estimates for the approximation of rough functions for a variety of related approximation schemes as well (see Page 48). The error estimates presented in Chapter 2 and Chapter 3 offer some improvement on recent contributions from other authors who have also considered interpolation of rough functions. In particular, the results here have wider applicability than those of Yoon [80, 81, 82], and the form of the estimates is more satisfactory than those of Narcowich and Ward [59, 60]. These assertions form part of a discussion starting on Page 49.

Let us look once again at the fundamental error estimate (1.3.6). As we have already commented, the factor $|f - Sf|_X$ can be thrown away and replaced with $|f|_X$—a term which contains no further approximating power whatsoever. If we instead hesitate, and choose not to dispense with $|f - Sf|_X$, it may well be possible to extract some extra approximating power from this term. In Chapter 4, we see that if we impose certain supplementary smoothness and boundary conditions on f, then the fundamental error estimate for the polyharmonic splines can be effectively doubled. Again, this treatment is applied to the weighted Beppo Levi space setting too. The result is essentially delivered by an application of what has become known as the Aubin–Nitsche trick (see Ciarlet [16, Page 136]). This idea has been approached before with regards to radial basis functions (see Schaback [66], Wendland [74] and Yoon [82]). However, the work in this chapter goes further and establishes error estimates for the interpolation of functions with smoothness lying (in some sense) between that of the usual native space and the subspace which enjoys the error doubling property. The thesis is concluded in Chapter 5 with some commentary regarding conclusions and further work.

1.5 Distribution theory, the Fourier transform and notation

To close this chapter we introduce some notation and standard results that will be employed throughout this work. All of the content here can be found in Rudin [63], in which further details and proofs of the results stated here can be found. We use $\mathbb{Z}_+^d$ to denote the multi-integers with nonnegative entries. For $\alpha = (\alpha_1, \dots, \alpha_d)^T \in \mathbb{Z}_+^d$, we define the order of α to be $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and the factorial by $\alpha! = \alpha_1!, \dots, \alpha_d!$. If $\alpha, \beta \in \mathbb{Z}_+^d$, and we write $\alpha \leq \beta$, then this is to be understood to be componentwise inequality. We define the differential operator D^α as

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}.$$

If $|\alpha| = 0$ then $D^\alpha = I$, the identity operator. If $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}_+^d$, the monomial x^α is defined by $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

The Schwartz space of rapidly decreasing test functions consists of those members of $C^\infty(\mathbb{R}^d)$ for which

$$\sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N |(D^\alpha f)(x)| < \infty, \qquad \text{for all } N \in \mathbb{Z}_+. \tag{1.5.1}$$

These functions form a vector space, denoted by $\mathscr{S}$, whose topology is given by the countable collection of norms (1.5.1). We denote by $\mathscr{S}'$ the space of (tempered) distributions—that is, those linear functionals on $\mathscr{S}$ which are continuous with respect to this topology. If Λ is a distribution and $\phi \in \mathscr{S}$, then we use the notation $[\Lambda, \phi]$ to denote the action of the distribution on the test function. If Λ is a distribution and $f \in C^\infty(\mathbb{R}^d)$ then $f\Lambda$ is the distribution whose action is defined by

$$[f\Lambda, \phi] = [\Lambda, f\phi], \qquad \text{for all } \phi \in \mathscr{S}.$$

Also, for $\alpha \in \mathbb{Z}_+^d$, the distribution $D^\alpha\Lambda$ has the action

$$[D^\alpha\Lambda, \phi] = [\Lambda, (-1)^{|\alpha|} D^\alpha\phi], \qquad \text{for all } \phi \in \mathscr{S}.$$

When we write $\widehat{f}$ we mean the Fourier transform of f. The context will clarify whether the Fourier transform is the natural one on $L_1(\mathbb{R}^d)$,

$$\widehat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y) \mathrm{e}^{-\mathrm{i}xy}\, \mathrm{d}y, \qquad \text{for all } x \in \mathbb{R}^d,$$

or one of its several extensions to $L_2(\mathbb{R}^d)$ or $\mathscr{S}'$. The Fourier transform is a continuous, linear, one-to-one mapping of $\mathscr{S}$ onto $\mathscr{S}$ whose inverse is also continuous, and we have $\widehat{\widehat{f}} = f(-\,\cdot\,)$. A routine calculation reveals the useful identity $\widehat{D^\alpha f} = (\mathrm{i}\,\cdot\,)^\alpha \widehat{f}$, for all $\alpha \in \mathbb{Z}_+^d$. We extend the Fourier transform to distributions by defining, for $\Lambda \in \mathscr{S}'$,

$$[\widehat{\Lambda}, \phi] = [\Lambda, \widehat{\phi}], \qquad \text{for all } \phi \in \mathscr{S}.$$

The distributional Fourier transform is a continuous, linear, one-to-one mapping of $\mathscr{S}'$ onto $\mathscr{S}'$ whose inverse is also continuous.

We shall, at some stage, call upon the Parseval formula,

$$\int_{\mathbb{R}^d} \widehat{f}(x)\overline{\widehat{g}}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\overline{g}(x)\,\mathrm{d}x, \qquad \text{for all } f, g \in L_2(\mathbb{R}^d).$$

The special case $f = g$ yields the Plancherel theorem, which states that the Fourier transform is an L_2-isometry:

$$\|\widehat{f}\|_{L_2(\mathbb{R}^d)} = \|f\|_{L_2(\mathbb{R}^d)}, \qquad \text{for all } f \in L_2(\mathbb{R}^d).$$

We occasionally invoke the Riemann–Lebesgue lemma which states that if $f \in L_1(\mathbb{R}^d)$, then $\widehat{f} \in C(\mathbb{R}^d)$ and vanishes at infinity.

The convolution of two functions f and g on $\mathbb{R}^d$ is given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y, \qquad \text{for all } x \in \mathbb{R}^d,$$

whenever the integral exists. For ϕ and ψ in $\mathscr{S}$ the convolution $\phi * \psi$ is well-defined and is itself an element of $\mathscr{S}$. The convolution enjoys the properties $\widehat{\phi * \psi} = \widehat{\phi}\widehat{\psi}$ and $\widehat{\phi\psi} = \widehat{\phi} * \widehat{\psi}$ as well as $D^\alpha(\phi * \psi) = (D^\alpha \phi) * \psi = \phi * (D^\alpha \psi)$, for all $\alpha \in \mathbb{Z}_+^d$.

The support of a function $\phi : \mathbb{R}^d \to \mathbb{R}$ is defined to be the closure of the set $\{x \in \mathbb{R}^d : \ \phi(x) \neq 0\}$, and is denoted by $\operatorname{supp}(\phi)$. The volume of a bounded set Ω is the quantity $\int_\Omega \mathrm{d}x$ and will be denoted $\mathrm{vol}(\Omega)$. The closed ball $B(a,r)$, $r>0$, is the set $\{x \in \mathbb{R}^d : \ |x-a| \le r\}$. The space $L_1^{\mathrm{loc}}(\mathbb{R}^d)$ consists of all those measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $f|_K \in L_1(K)$ for all compact subsets $K \subset \mathbb{R}^d$.

Finally, we end with a minor remark on notation. We admit there has been flagrant overworking of the notation $|\cdot|$ in this chapter. It has been used to denote the Euclidean norm on $\mathbb{R}^d$, the order of a multi-integer and the cardinality of a set. We have chosen not to introduce new notation for each as we hope the intended meaning is clear from the context in all of these cases. The notation $|\cdot|$ is also used to denote various seminorms throughout this thesis. However, to avoid further labouring of the notation, we always include a subscript which is intended to identify the seminorm.

Chapter 2

Interpolation of rough functions using polyharmonic splines

In this chapter we orchestrate an escape from the native space of the polyharmonic spline. In the case of the integer order Beppo Levi spaces, there is a considerable freedom of choice for the norm in which the error between f and Sf is measured. The most widely quoted result concerns the norm $\|\cdot\|_{L_\infty(\Omega)}$, but for variety we prefer to deal with the L_p-norm, for $1 \le p \le \infty$. To do this it is helpful to assume $\Omega \subset \mathbb{R}^d$ is a bounded domain, whose boundary is sufficiently smooth.

The form of the error estimate we seek is motivated by our discussion in Section 1.4. Hence we will endeavour to establish the following,

$$\|f - S_m^h f\|_{L_p(\Omega)} \le C h^{k-d/2+d/p} |f|_{k,\Omega}, \qquad \text{for } 2 \le p \le \infty, \text{ as } h \to 0. \tag{2.0.1}$$

In fact we choose to be a little bolder and establish estimates for the derivatives of the error as well. Here, $C > 0$ is a number independent of both f and h. Let us also remind ourselves that $\mathcal{A}$ is a finite subset of Ω with fill-distance h, and $S_m^h f \in BL^m(\mathbb{R}^d)$ is the m-th order polyharmonic spline interpolant (minimal norm interpolant) to $f \in BL^k(\Omega)$ on $\mathcal{A}$, where m and k are integers satisfying $m > k > d/2$. Furthermore, in light of a recent result from Johnson [43], which we shortly describe, we will only consider interpolation points that possess a certain property. We define the separation distance of $\mathcal{A}$ as the quantity

$$q = \min_{\substack{a,b\in\mathcal{A} \\ a\neq b}} \frac{|a-b|}{2},$$

which is half the smallest distance between any two points in $\mathcal{A}$. Alternatively, this gives the maximum radius $r > 0$ such that all the balls $\{x \in \mathbb{R}^d : |x-a| < r\}$, $a \in \mathcal{A}$, are disjoint. We will consider those point sets where the ratio h/q is bounded above by some fixed number for all $h > 0$.

Let us recall the familiar definition of a Sobolev space. Let $W_2^k(\Omega)$ denote the k-th integer order Sobolev space, which consists of functions all of whose distributional partial derivatives up to and including order k are in $L_2(\Omega)$. It is a Banach space under the norm

$$\|f\|_{k,\Omega} = \left(\sum_{i=0}^{k} |f|_{i,\Omega}^2\right)^{1/2}, \qquad \text{where } f \in W_2^k(\Omega).$$

We have already, in Section 1.3, tacitly alluded to the Sobolev embedding theorem which states that when Ω is reasonably regular (for example, when Ω possesses a Lipschitz continuous boundary) and $k > d/2$, then the space $W_2^k(\Omega)$ can be embedded in $C(\Omega)$ (see [1, Page 97]). One can also define $W_p^k(\Omega)$, for $1 \le p < \infty$, in an analogous way and $W_\infty^k(\Omega)$ by the usual convention. Now Johnson's result is as follows.

Theorem 2.0.1 (Johnson [43]). *Let Ω be the unit ball in $\mathbb{R}^d$ and assume $m > k > d/2$. For every $h_0 > 0$, there exists an $f \in W_2^k(\mathbb{R}^d)$ and a sequence of sets $\{\mathcal{A}_n\}_{n\in\mathbb{Z}_+}$ with the* *following properties:*

1. *each set $\mathcal{A}_n$ consists of finitely many points contained in Ω;*

2. *the fill-distance of each set $\mathcal{A}_n$ is at most h_0;*

3. *if $S_m^n f$ is the polyharmonic spline interpolant to f from $BL^m(\mathbb{R}^d)$ associated with $\mathcal{A}_n$, for each $n \in \mathbb{Z}_+$, then $\|S_m^n f\|_{L_1(\Omega)} \to \infty$ as $n \to \infty$.*

If the L_1-norm of the polyharmonic spline interpolation operator is unbounded, there is of course no possibility of getting an error estimate of the kind stated in (2.0.1). However, Johnson's proof uses point sets which have a special feature. Let the separation distance and fill-distance of each $\mathcal{A}_n$ be q_n and h_n, respectively. In Johnson's proof, the construction of $\mathcal{A}_n$ is such that $q_n/h_n \to 0$ as $n \to \infty$. We make this remark, because Johnson's result in one-dimension refers to interpolation by natural splines, and in this setting the connection between the separation distance and the unboundedness of S_m^n has been known for some time. What is also known in the one-dimensional case is that if the separation distance is tied to the fill-distance, then a result of the type we are seeking is true (see Schumaker [70, Page 210]).

2.1 Sobolev extension theory

In this section we intend to collect together a number of useful results, chiefly about the sorts of extensions which can be carried out on Sobolev spaces. We begin with a well-known result which can be found in many of the standard texts. Of course, the precise nature of the set Ω in the following theorem may vary from book to book, and we have not striven here for the utmost generality, because that is not really part of our agenda.

Definition 2.1.1. *The domain $\Omega \subset \mathbb{R}^d$ has the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone $C_x \subset \Omega$ and congruent to C. A bounded domain Ω has the uniform cone property if there exists numbers $\delta > 0$, $L > 0$, an open cover $U_1, \ldots, U_N$ of $\partial\Omega$ and a corresponding collection of finite cones $C_1, \ldots, C_N$, each congruent to a fixed cone C, such that:*

1. *every U_j has diameter less than L;*
2. *for any $x \in \Omega$ whose distance from $\partial\Omega$ is less than δ, we have $x \in \bigcup_{j=1}^{N} U_j$;*
3. *$\bigcup_{x \in \Omega \cap U_j}(x + C_j) \subset \Omega$, for $1 \leq j \leq N$.*

A bounded domain Ω has a Lipschitz continuous boundary if each point $x \in \partial\Omega$ has a neighbourhood U_x such that $\partial\Omega \cap U_x$ is the graph of a Lipschitz continuous function. Every domain with a Lipschitz continuous boundary has the uniform cone property (see Adams [1, Page 67]).

Theorem 2.1.2 (Sobolev extension theorem [1, Page 91])**.** *Let Ω be an open, bounded subset of $\mathbb{R}^d$ satisfying the uniform cone property. For every $f \in W_2^k(\Omega)$ there is an $f^\Omega \in W_2^k(\mathbb{R}^d)$ satisfying $f^\Omega|_\Omega = f$. Moreover, there is a positive constant $K = K(\Omega)$ such that for all $f \in W_2^m(\Omega)$,*

$$\|f^\Omega\|_{k,\mathbb{R}^d} \leq K\|f\|_{m,\Omega}.$$

We remark that the extension f^Ω can be chosen to be supported on any compact subset of $\mathbb{R}^d$ containing Ω. To see this, we construct f^Ω in accordance with Theorem 2.1.2, then select $\eta \in C_0^m(\mathbb{R}^d)$ such that $\eta(x) = 1$ for $x \in \Omega$. Now, if we consider the compactly supported function $f_0^\Omega = \eta f^\Omega \in W_2^k(\mathbb{R}^d)$, we have $f_0^\Omega|_\Omega = f$. An elementary application of the Leibniz formula gives

$$\|f_0^\Omega\|_{k,\mathbb{R}^d} \le C\|f\|_{k,\Omega}, \qquad \text{where } C = C(\Omega, \eta).$$

One of the nice features of the above extension is that the behaviour of the constant $K(\Omega)$ can be understood for simple choices of Ω. The reason for this is, of course, the choice of Ω and the way the seminorms defining the Sobolev norms behave under dilations and translations of Ω.

Lemma 2.1.3. *Let Ω be a measurable subset of $\mathbb{R}^d$. Define the mapping $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ by $\sigma(x) = a + h(x - t)$, where $h > 0$, and a, t, $x \in \mathbb{R}^d$. Then for all $f \in W_p^k(\sigma(\Omega))$,*

$$|f \circ \sigma|_{k,\Omega} = h^{k-d/p}|f|_{k,\sigma(\Omega)}.$$

Proof. We have, for $|\alpha| = k$,

$$D^\alpha(f \circ \sigma) = h^k(D^\alpha f)(\sigma(\,\cdot\,)).$$

Thus,

$$\begin{aligned}
|f \circ \sigma|_{k,\Omega}^p &= \sum_{|\alpha|=k} c_\alpha \int_\Omega |(D^\alpha(f \circ \sigma))(x)|^p \,\mathrm{d}x \\
&= h^{pk} \sum_{|\alpha|=k} c_\alpha \int_\Omega |(D^\alpha f)(\sigma(x))|^p \,\mathrm{d}x.
\end{aligned}$$

Now, using the change of variables $y = \sigma(x)$,

$$|f \circ \sigma|_{k,\Omega}^p = h^{pk-d} \sum_{|\alpha|=k} c_\alpha \int_{\sigma(\Omega)} |(D^\alpha f)(y)|^p \,\mathrm{d}y = h^{pk-d}|f|_{k,\sigma(\Omega)}^p. \qquad \square$$

Unfortunately, the Sobolev extension refers to the Sobolev norm. We want to work with a norm which is more convenient for our purposes. This norm is in fact equivalent to the Sobolev norm, as we shall now see. Throughout this chapter we make much use of the space $\Pi_{k-1}(\mathbb{R}^d)$, so for brevity we fix ℓ as the dimension of this space, we also henceforth assume that m and k are fixed integers satisfying the relation $m \geq k > d/2$.

Lemma 2.1.4. *Let Ω be an open subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz continuous boundary. Let $b_1, \ldots, b_\ell \in \Omega$ be unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^d)$. Define a norm on $W_2^k(\Omega)$ via*

$$\|f\|_\Omega = \left(|f|_{k,\Omega}^2 + \sum_{i=1}^{\ell} |f(b_i)|^2 \right)^{1/2}, \qquad \text{for } f \in W_2^k(\Omega).$$

There are positive constants K_1 and K_2 such that for all $f \in W_2^k(\Omega)$,

$$K_1 \|f\|_{k,\Omega} \leq \|f\|_\Omega \leq K_2 \|f\|_{k,\Omega}.$$

Proof. The conditions imposed on k and Ω ensure that $W_2^k(\Omega)$ is continuously embedded in $C(\Omega)$ (see Adams [1, Page 97]). So, given $x \in \Omega$, there is a constant C such that $|f(x)| \leq C\|f\|_{k,\Omega}$ for all $f \in W_2^k(\Omega)$. Thus, there are constants $C_1, \ldots, C_\ell$ such that

$$\|f\|_\Omega^2 \leq |f|_{k,\Omega}^2 + \sum_{i=1}^{\ell} C_i \|f\|_{k,\Omega}^2 \leq \left(1 + \sum_{i=1}^{\ell} C_i\right) \|f\|_{k,\Omega}^2. \tag{2.1.1}$$

On the other hand, suppose there is no positive number K with $\|f\|_{k,\Omega} \leq K\|f\|_\Omega$ for all $f \in W_2^k(\Omega)$. Then there is a sequence $\{f_j\}$ in $W_2^k(\Omega)$ with

$$\|f_j\|_{k,\Omega} = 1 \qquad \text{and} \qquad \|f_j\|_\Omega \leq \frac{1}{j}, \qquad \text{for } j = 1, 2, \ldots.$$

The Rellich selection theorem [6, Page 32] states that $W_2^k(\Omega)$ is compactly embedded in $W_2^{k-1}(\Omega)$. Therefore, as $\{f_j\}$ is bounded in $W_2^k(\Omega)$, this sequence must contain a convergent subsequence in $W_2^{k-1}(\Omega)$. With no loss of generality we shall assume $\{f_j\}$

itself converges in $W_2^{k-1}(\Omega)$. Thus $\{f_j\}$ is a Cauchy sequence in $W_2^{k-1}(\Omega)$. Next, as $\|f_j\|_\Omega \to 0$, it follows that $|f_j|_{k,\Omega} \to 0$. Moreover,

$$\begin{aligned}\|f_i - f_j\|_{k,\Omega}^2 &= \|f_i - f_j\|_{k-1,\Omega}^2 + |f_i - f_j|_{k,\Omega}^2 \\ &\leq \|f_i - f_j\|_{k-1,\Omega}^2 + 2|f_i|_{k,\Omega}^2 + 2|f_j|_{k,\Omega}^2.\end{aligned}$$

Since $\{f_j\}$ is a Cauchy sequence in $W_2^{k-1}(\Omega)$, and $|f_j|_{k,\Omega} \to 0$, it follows that $\{f_j\}$ is a Cauchy sequence in $W_2^k(\Omega)$. Since $W_2^k(\Omega)$ is complete with respect to $\|\cdot\|_{k,\Omega}$, this sequence converges to a limit $f \in W_2^k(\Omega)$. By (2.1.1),

$$\|f - f_j\|_\Omega^2 \leq \left(1 + \sum_{i=1}^{\ell} C_i\right) \|f - f_j\|_{k,\Omega}^2,$$

and hence $\|f - f_j\|_\Omega \to 0$ as $j \to \infty$. Since $\|f_j\|_\Omega \to 0$, it follows that $f = 0$. Because $\|f_j\|_{k,\Omega} = 1$, $j = 1, 2, \ldots$, it follows that $\|f\|_{k,\Omega} = 1$. This contradiction establishes the result. □

We are almost ready to state the key result which we will employ in our later proofs concerning error estimates. Before we do this, let us make a simple observation. Look at the unisolvent points $b_1, \ldots, b_\ell$ in the statement of the previous lemma. Since $W_2^k(\Omega)$ can be embedded in $C(\Omega)$, it makes sense to talk about the interpolation projection $P : W_2^k(\Omega) \to \Pi_{k-1}(\mathbb{R}^d)$ based on these points. Furthermore, under certain nice conditions (for example Ω being a bounded domain), P is the orthogonal projection of $W_2^k(\Omega)$ onto $\Pi_{k-1}(\mathbb{R}^d)$.

Lemma 2.1.5. *Let B be any ball of radius h and centre $a \in \mathbb{R}^d$, and let $f \in W_2^k(B)$. Whenever $b_1, \ldots, b_\ell \in \mathbb{R}^d$ are unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^d)$, let $P_b : C(\mathbb{R}^d) \to$*

$\Pi_{k-1}(\mathbb{R}^d)$ *be the Lagrange interpolation operator on* $b_1, \ldots, b_\ell$. *Then there exists* $c = (c_1, \ldots, c_\ell) \in B^\ell$ *and* $g \in W_2^k(\mathbb{R}^d)$ *such that:*

1. $g(x) = (f - P_c f)(x)$ *for all* $x \in B$;

2. $g(x) = 0$ *for all* $|x - a| > 2h$;

3. *there exists a* $C > 0$, *independent of* f *and* B, *such that* $|g|_{k,\mathbb{R}^d} \leq C|f|_{k,B}$.

Furthermore, $c_1, \ldots, c_\ell$ *can be arranged so that* $c_1 = a$.

Proof. Let B_1 be the unit ball in $\mathbb{R}^d$ and let $B_2 = 2B_1$. Let $b_1, \ldots, b_\ell \in B_1$ be unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^d)$. Define $\sigma(x) = h^{-1}(x - a)$ for all $x \in \mathbb{R}^d$. Set $c_i = \sigma^{-1}(b_i)$ for $i = 1, \ldots, \ell$ so that $c_1, \ldots, c_\ell \in B$ are unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^d)$. Take $f \in W_2^k(B)$. Then $(f - P_c f) \circ \sigma^{-1} \in W_2^k(B_1)$. Set $F = (f - P_c f) \circ \sigma^{-1}$. Let F^{B_1} be constructed as an extension to F on B_1. By Theorem 2.1.2 and the remarks following it, we can assume F^{B_1} is supported on B_2. Define $g = F^{B_1} \circ \sigma \in W_2^k(\mathbb{R}^d)$. Let $x \in B$. Since $\sigma(B) = B_1$ there is a $y \in B_1$ such that $x = \sigma^{-1}(y)$. Then,

$$g(x) = (F^{B_1} \circ \sigma)(x) = F^{B_1}(y) = ((f - P_c f) \circ \sigma^{-1})(y) = (f - P_c f)(x).$$

Also, for $x \in \mathbb{R}^d$ with $|x - a| > 2h$, we have $|\sigma(x)| > 2$. Since F^{B_1} is supported on B_2, $g(x) = 0$ for $|x - a| > 2h$. Hence, g satisfies properties 1 and 2. By Theorem 2.1.2 there is a K_1, independent of f and B, such that

$$\|F^{B_1}\|_{k,B_2} = \|F^{B_1}\|_{k,\mathbb{R}^d} \leq K_1 \|F\|_{k,B_1}.$$

We have seen in Lemma 2.1.4 that if we endow $W_2^k(B_1)$ and $W_2^k(B_2)$ with the norms

$$\|v\|_{B_i} = \left(|v|_{k,B_i}^2 + \sum_{i=1}^{\ell} |v(b_i)|^2 \right)^{1/2}, \qquad \text{for } i = 1, 2,$$

then $\|\cdot\|_{B_i}$ and $\|\cdot\|_{m,B_i}$ are equivalent for $i = 1, 2$. Thus, there are constants K_2 and K_3, independent of f and B, such that

$$\|F^{B_1}\|_{B_2} \leq K_2\|F^{B_1}\|_{k,B_2} \leq K_1K_2\|F\|_{k,B_1} \leq K_1K_2K_3\|F\|_{B_1}.$$

Set $C = K_1K_2K_3$. Since $F^{B_1}(b_i) = F(b_i) = (f - P_cf)(\sigma^{-1}(b_i)) = (f - P_cf)(c_i) = 0$, for $i = 1, \ldots, \ell$, it follows that $|F^{B_1}|_{m,B_2} \leq C|F|_{m,B_1}$. Thus, $|g\circ\sigma^{-1}|_{k,\mathbb{R}^d} \leq C|(f - P_cf)\circ\sigma^{-1}|_{k,B_1}$. Now, Lemma 2.1.3 can be employed twice to give

$$|g|_{k,\mathbb{R}^d} = h^{d/2-k}|g \circ \sigma^{-1}|_{k,\mathbb{R}^d} \leq Ch^{d/2-k}|(f - P_cf) \circ \sigma^{-1}|_{k,B_1} = C|f - P_cf|_{k,B}.$$

Finally, we observe that $|f - P_cf|_{k,B} = |f|_{k,B}$ to complete the first part of the proof. The remaining part follows by selecting $b_1 = 0$ and choosing $b_2, \ldots, b_\ell$ accordingly in the above construction. □

We end this section by stating a useful result concerning seminorm extension theorems. The result is quoted from Duchon [22] and we omit the proof.

Lemma 2.1.6 (Duchon [22])**.** *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz continuous boundary. Let $f \in W_2^k(\Omega)$. Then there exists a unique element $f^\Omega \in BL^k(\mathbb{R}^d)$ such that $f^\Omega|_\Omega = f$, and amongst all elements of $BL^k(\mathbb{R}^d)$ satisfying this condition, $|f^\Omega|_{k,\mathbb{R}^d}$ is minimal. Furthermore, there exists a constant $K = K(\Omega)$ such that, for all $f \in W_2^k(\Omega)$,*

$$|f^\Omega|_{k,\mathbb{R}^d} \leq K|f|_{k,\Omega}.$$

It is of interest to note that if one chooses the domain in the previous theorem to be any ball in $\mathbb{R}^d$, then the *embedding constant*, K, can be selected independent of that ball.

This follows by combining Lemma 2.1.6 with the change of variables result (Lemma 2.1.3) in an appropriate way.

2.2 L_p-error estimates

We arrive now at our main section, in which we derive the required error estimates. We begin with a function f in $BL^k(\mathbb{R}^d)$. We want to estimate $\|f - S_m f\|$ for some suitable norm $\|\cdot\|$, where S_m is the minimal norm interpolation operator from $BL^m(\mathbb{R}^d)$, and $m > k$. We suppose that we already have an error bound using the norm $\|\cdot\|$ for all functions $g \in BL^m(\mathbb{R}^d)$. Our proof now proceeds as follows. Firstly, we adjust f in a somewhat delicate manner, obtaining a function F, still in $BL^k(\mathbb{R}^d)$, and with seminorm in $BL^k(\mathbb{R}^d)$ not too far away from that of f. We then smooth F by convolving it with an approximate identity function $\phi \in C_0^\infty(\mathbb{R}^d)$. The key feature of the adjustment of f to F is that $(\phi * F)(a) = f(a)$ for every point a in our set of interpolation points. Furthermore, it follows that $\phi * F \in BL^m(\mathbb{R}^d)$. We then use the usual error estimate in $BL^m(\mathbb{R}^d)$. A standard procedure (Lemma 2.2.3) then takes us back to an error estimate in $BL^k(\mathbb{R}^d)$. Before we see this let us gather two useful classical results.

Theorem 2.2.1 (Fubini's theorem [62, Page 164])**.** *Let f be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m$ and suppose at least one of the integrals*

$$I_1 = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)| \, \mathrm{d}x \right) \mathrm{d}y \qquad \text{or} \qquad I_2 = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)| \, \mathrm{d}y \right) \mathrm{d}x$$

exists and is finite. Then $I_1 = I_2$.

Theorem 2.2.2 (Lebesgue's dominated convergence theorem [62, Page 26])**.** *Let $\{f_j\}$ be a sequence in $L_1(\mathbb{R}^d)$ such that:*

1. *$f_n \to f$ almost everywhere;*

2. *there exists a nonnegative $g \in L_1(\mathbb{R}^d)$ such that $|f_n| \le g$ almost everywhere for $n = 1, 2, \ldots$.*

Then $f \in L_1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x, \qquad \text{as } n \to \infty.$$

Lemma 2.2.3. *Let $k \le m$ and let $\phi \in C_0^\infty(\mathbb{R}^d)$. For each $h > 0$, let $\phi_h(x) = h^{-d}\phi(x/h)$ for $x \in \mathbb{R}^d$. Then there exists a constant $C > 0$, independent of h, such that for all $f \in BL^k(\mathbb{R}^d)$,*

$$|\phi_h * f|_{m,\mathbb{R}^d} \le Ch^{k-m}|f|_{k,\mathbb{R}^d}.$$

Furthermore, we have $|\phi_h * f|_{m,\mathbb{R}^d} = o(h^{k-m})$, as $h \to 0$.

Proof. The chain rule for differentiation gives $(D^\gamma\phi_h)(x) = h^{-(d+|\gamma|)}(D^\gamma\phi)(x/h)$ for all $x \in \mathbb{R}^d$, and $\gamma \in \mathbb{Z}_+^d$. Thus, for $\beta \in \mathbb{Z}_+^d$ with $|\beta| = k$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |(D^\gamma\phi_h * D^\beta f)(x)|^2\,\mathrm{d}x &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(D^\gamma\phi_h)(x-y)(D^\beta f)(y)\,\mathrm{d}y\right|^2\mathrm{d}x \\
&= h^{-2(d+|\gamma|)}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(D^\gamma\phi)\Big(\frac{x-y}{h}\Big)(D^\beta f)(y)\,\mathrm{d}y\right|^2\mathrm{d}x \\
&= h^{-2|\gamma|}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(D^\gamma\phi)(t)(D^\beta f)(x-ht)\,\mathrm{d}t\right|^2\mathrm{d}x \\
&= h^{-2|\gamma|}\int_{\mathbb{R}^d}\left|\int_{K}(D^\gamma\phi)(t)(D^\beta f)(x-ht)\,\mathrm{d}t\right|^2\mathrm{d}x, \qquad (2.2.1)
\end{aligned}$$

where $K = \operatorname{supp}(\phi)$. An application of the Cauchy–Schwarz inequality gives

$$\int_{\mathbb{R}^d}|(D^\gamma\phi_h * D^\beta f)(x)|^2\,\mathrm{d}x \le h^{-2|\gamma|}\int_{\mathbb{R}^d}\left(\int_K |(D^\gamma\phi)(t)|^2\,\mathrm{d}t\right)\left(\int_K |(D^\beta f)(x-ht)|^2\,\mathrm{d}t\right)\mathrm{d}x,$$

and so,

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \,\mathrm{d}x \le h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \,\mathrm{d}t \int_{\mathbb{R}^d} \int_K |(D^\beta f)(x - ht)|^2 \,\mathrm{d}t\mathrm{d}x. \tag{2.2.2}$$

The Parseval formula together with the relation $(D^\alpha(\phi_h * f))^\frown = (\mathrm{i}\,\cdot)^\alpha(\phi_h * f)^\frown$ provide us with the equality

$$\begin{aligned} \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x &= \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(\mathrm{i}x)^\alpha (\phi_h * f)^\frown(x)|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \sum_{|\alpha|=m} c_\alpha x^{2\alpha} |(\phi_h * f)^\frown(x)|^2 \,\mathrm{d}x. \end{aligned} \tag{2.2.3}$$

Now, when (2.2.3) is used in conjunction with the relation

$$\sum_{|\alpha|=m} c_\alpha x^{2\alpha} = |x|^{2m} = |x|^{2(k+m-k)} = \sum_{|\beta|=k} c_\beta x^{2\beta} \sum_{|\gamma|=m-k} c_\gamma x^{2\gamma},$$

we obtain

$$\begin{aligned} \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^d} \sum_{|\beta|=k} c_\beta x^{2\beta} \sum_{|\gamma|=m-k} c_\gamma x^{2\gamma} |(\phi_h * f)^\frown(x)|^2 \,\mathrm{d}x \\ &= \sum_{|\beta|=k} c_\beta \int_{\mathbb{R}^d} \sum_{|\gamma|=m-k} c_\gamma x^{2\gamma} |(\mathrm{i}x)^\beta (\phi_h * f)^\frown(x)|^2 \,\mathrm{d}x \\ &= \sum_{|\beta|=k} c_\beta \int_{\mathbb{R}^d} \sum_{|\gamma|=m-k} c_\gamma x^{2\gamma} |(D^\beta(\phi_h * f))^\frown(x)|^2 \,\mathrm{d}x \\ &= \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(\mathrm{i}x)^\gamma (D^\beta(\phi_h * f))^\frown(x)|^2 \,\mathrm{d}x \\ &= \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma(D^\beta(\phi_h * f)))^\frown(x)|^2 \,\mathrm{d}x \\ &= \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma(D^\beta(\phi_h * f)))(x)|^2 \,\mathrm{d}x. \end{aligned}$$

Since the operation of differentiation commutes with convolution, we have that

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x = \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \,\mathrm{d}x. \tag{2.2.4}$$

Combining equation (2.2.2) with equation (2.2.4) we deduce that

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x$$
$$\leq \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \,\mathrm{d}t \int_{\mathbb{R}^d} \int_K |(D^\beta f)(x-ht)|^2 \,\mathrm{d}t\mathrm{d}x$$
$$= h^{2(k-m)} |\phi|^2_{m-k,\mathbb{R}^d} \sum_{|\beta|=k} c_\beta \int_{\mathbb{R}^d} \int_K |(D^\beta f)(x-ht)|^2 \,\mathrm{d}t\mathrm{d}x.$$

Fubini's theorem permits us to change the order of integration in the previous inequality; thus,

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x$$
$$\leq h^{2(k-m)} |\phi|^2_{m-k,\mathbb{R}^d} \sum_{|\beta|=k} c_\beta \int_K \int_{\mathbb{R}^d} |(D^\beta f)(x-ht)|^2 \,\mathrm{d}x\mathrm{d}t.$$

Finally, a change of variables in the inner integral above yields

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x \leq h^{2(k-m)} |\phi|^2_{m-k,\mathbb{R}^d} \sum_{|\beta|=k} c_\beta \int_K \int_{\mathbb{R}^d} |(D^\beta f)(z)|^2 \,\mathrm{d}z\mathrm{d}t.$$

Setting $C = |\phi|_{m-k,\mathbb{R}^d} \sqrt{\mathrm{vol}(K)}$ we conclude that $|\phi_h * f|_{m,\mathbb{R}^d} \leq C h^{k-m} |f|_{k,\mathbb{R}^d}$ as required.

To deal with the remaining statement of the lemma, we observe that for $\gamma \neq 0$ we have

$$\int_K (D^\gamma \phi)(t) \,\mathrm{d}t = \int_{\mathbb{R}^d} (D^\gamma \phi)(t) \,\mathrm{d}t = (\widehat{D^\gamma \phi})(0) = ((\mathrm{i}\,\cdot)^\gamma \widehat{\phi})(0) = 0.$$

Then it follows from (2.2.1) that for $|\beta| = k$,

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \,\mathrm{d}x$$
$$= h^{-2|\gamma|} \int_{\mathbb{R}^d} \left| \int_K (D^\gamma \phi)(t)((D^\beta f)(x-ht) - (D^\beta f)(x)) \,\mathrm{d}t \right|^2 \mathrm{d}x.$$

Now, if we apply the Cauchy–Schwarz inequality in a different manner than we did before, we obtain

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \,\mathrm{d}x$$

$$\begin{aligned}
&\leq h^{-2|\gamma|} \int_{\mathbb{R}^d} \left(\int_K 1^2 \,\mathrm{d}t\right)\left(\int_K |(D^\gamma \phi)(t)((D^\beta f)(x-ht) - (D^\beta f)(x))|^2 \,\mathrm{d}t\right) \mathrm{d}x \\
&= \mathrm{vol}(K) h^{-2|\gamma|} \int_{\mathbb{R}^d} \int_K |(D^\gamma \phi)(t)((D^\beta f)(x-ht) - (D^\beta f)(x))|^2 \,\mathrm{d}t\mathrm{d}x \\
&= \mathrm{vol}(K) h^{-2|\gamma|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)((D^\beta f)(x-ht) - (D^\beta f)(x))|^2 \,\mathrm{d}t\mathrm{d}x.
\end{aligned}$$

An application of Fubini's theorem gives us

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \,\mathrm{d}x \leq \mathrm{vol}(K) h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \|(D^\beta f)(\,\cdot - ht) - D^\beta f\|^2_{L_2(\mathbb{R}^d)} \,\mathrm{d}t.$$

Hence, by (2.2.4) we have

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \,\mathrm{d}x$$

$$\leq \mathrm{vol}(K) h^{2(k-m)} \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \|(D^\beta f)(\,\cdot - ht) - D^\beta f\|^2_{L_2(\mathbb{R}^d)} \,\mathrm{d}t.$$

Since $D^\beta f \in L_2(\mathbb{R}^d)$ for each $\beta \in \mathbb{Z}_+^d$ with $|\beta| = k$, it follows from Folland [26, Page 238] that for all $t \in \mathbb{R}^d$,

$$\|(D^\beta f)(\,\cdot - ht) - D^\beta f\|_{L_2(\mathbb{R}^d)} \to 0, \qquad \text{as } h \to 0.$$

Furthermore, setting

$$g(t) = 4\|D^\beta f\|^2_{L_2(\mathbb{R}^d)} |(D^\gamma \phi)(t)|^2, \qquad \text{for all } t \in \mathbb{R}^d,$$

we see that $g \in L_1(\mathbb{R}^d)$ and

$$|(D^\gamma \phi)(t)|^2 \|(D^\beta f)(\,\cdot - ht) - D^\beta f\|^2_{L_2(\mathbb{R}^d)} \leq g(t),$$

for each $h > 0$. Applying Lebesgue's dominated convergence theorem (Theorem 2.2.2), we obtain

$$\int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \|(D^\beta f)(\cdot - ht) - D^\beta f\|^2_{L_2(\mathbb{R}^d)} \, \mathrm{d}t \to 0, \qquad \text{as } h \to 0.$$

Hence, for $k < m$, $|\phi_h * f|_{k,\mathbb{R}^d} = o(h^{k-m})$ as $h \to 0$. $\square$

Lemma 2.2.4. *Suppose $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported on the unit ball and satisfies*

$$\int_{\mathbb{R}^d} \phi(x) \, \mathrm{d}x = 1 \qquad \textit{and} \qquad \int_{\mathbb{R}^d} \phi(x) x^\alpha \, \mathrm{d}x = 0, \qquad \textit{for all } 0 < |\alpha| \le k-1.$$

For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, let $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$. Let B be any ball of radius h and centre $a \in \mathbb{R}^d$. For a fixed $p \in \Pi_{k-1}(\mathbb{R}^d)$ let f be a mapping from $\mathbb{R}^d$ to $\mathbb{R}$ such that $f(x) = p(x)$ for all $x \in B$. Then $(\phi_\varepsilon * f)(a) = p(a)$ for all $\varepsilon \le h$.

Proof. Let B_1 denote the unit ball in $\mathbb{R}^d$. We begin by employing a change of variables to deduce that

$$\begin{aligned}
(\phi_\varepsilon * f)(a) &= \int_{\mathbb{R}^d} \phi_\varepsilon(a-y) f(y) \, \mathrm{d}y \\
&= \varepsilon^{-d} \int_{\mathbb{R}^d} \phi\Big(\frac{a-y}{\varepsilon}\Big) f(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} \phi(x) f(a - x\varepsilon) \, \mathrm{d}x \\
&= \int_{B_1} \phi(x) f(a - x\varepsilon) \, \mathrm{d}x.
\end{aligned}$$

Then, for $x \in B_1$, $|(a - x\varepsilon) - a| \le \varepsilon \le h$. Thus, $f(a - x\varepsilon) = p(a - x\varepsilon)$ for all $x \in B_1$; moreover, there are numbers b_α such that $p(a - x\varepsilon) = p(a) + \sum_{0<|\alpha|\le k-1} b_\alpha x^\alpha$. Hence,

$$(\phi_\varepsilon * f)(a) = \int_{B_1} \phi(x) p(a - x\varepsilon) \, \mathrm{d}x = \int_{\mathbb{R}^d} \phi(x) \Big(p(a) + \sum_{0<\alpha\le m-1} b_\alpha x^\alpha \Big) \, \mathrm{d}x = p(a). \quad \square$$

Definition 2.2.5. *Let Ω be an open, bounded subset of $\mathbb{R}^d$. Let $\mathcal{A}$ be a finite set of pairwise distinct points in Ω with fill-distance $h > 0$ and separation distance $q > 0$. The quantity h/q will be called the mesh-ratio of $\mathcal{A}$. The mesh-ratio measures to what extent points in $\mathcal{A}$ uniformly cover Ω.*

The mesh-ratio is always bounded below by 1 for any reasonable domain Ω, for example Ω open, connected and satisfying the cone property. To see this, let $b \in \mathcal{A}$ and choose $x \in \Omega$ with $|b - x| \geq q$. Then, for $a \in \mathcal{A}$ with $a \neq b$ we have

$$|x - a| \geq |a - b| - |b - x| \geq 2q - q = q.$$

Thus, $h \geq q$. Equality only holds for uniform data in $\mathbb{R}$, so the mesh-ratio is strictly bigger than 1 in all but this special situation.

Theorem 2.2.6. *Let $\mathcal{A}$ be a finite subset of $\mathbb{R}^d$ of separation $q > 0$ and let $d < 2k \leq 2m$. Then for all $f \in BL^k(\mathbb{R}^d)$ there exists an $F \in BL^m(\mathbb{R}^d)$ such that:*

1. *$F(a) = f(a)$ for all $a \in \mathcal{A}$;*
2. *there exists a $C > 0$, independent of f and q, such that $|F|_{k,\mathbb{R}^d} \leq C|f|_{k,\mathbb{R}^d}$ and $|F|_{m,\mathbb{R}^d} \leq Cq^{k-m}|f|_{k,\mathbb{R}^d}$.*

Proof. Take $f \in BL^k(\mathbb{R}^d)$. For each $a \in \mathcal{A}$, let $B_a \subset \mathbb{R}^d$ denote the ball of radius $\delta = q/4$ centred at a. For each B_a, let g_a be constructed in accordance with Lemma 2.1.5. That is, for each $a \in \mathcal{A}$, take $c' = (c_2, \dots, c_\ell) \in B_a^{\ell-1}$ and $g_a \in W_2^k(\mathbb{R}^d)$ such that:

1. $a, c_2, \dots, c_\ell$ are unisolvent with respect to Π_{k-1};
2. $g_a(x) = (f - P_{(a,c')}f)(x)$ for all $x \in B_a$;

3. $P_{(a,c')}f \in \Pi_{k-1}$ and $(P_{(a,c')}f)(a) = f(a)$;

4. $g_a(x) = 0$ for all $|x - a| > 2\delta$;

5. there exists a $C_1 > 0$, independent of f and B_a, such that $|g_a|_{m,\mathbb{R}^d} \le C_1 |f|_{m,B_a}$.

Note that if $a \neq b$, then $\operatorname{supp}(g_a)$ does not intersect $\operatorname{supp}(g_b)$, because if $x \in \operatorname{supp}(g_a)$ then

$$|x - b| > |b - a| - |x - a| \ge 2q - 2\delta = 6\delta.$$

Using the observation above regarding the support of g_a it follows that

$$\begin{aligned}
\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{k,\mathbb{R}^d} &= \sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} \left|\sum_{a\in\mathcal{A}} (D^\alpha g_a)(x)\right|^2 \, \mathrm{d}x \\
&= \sum_{|\alpha|=k} c_\alpha \sum_{b\in\mathcal{A}} \int_{\operatorname{supp}(g_b)} \left|\sum_{a\in\mathcal{A}} (D^\alpha g_a)(x)\right|^2 \, \mathrm{d}x \\
&= \sum_{|\alpha|=k} c_\alpha \sum_{b\in\mathcal{A}} \int_{\operatorname{supp}(g_b)} |(D^\alpha g_b)(x)|^2 \, \mathrm{d}x \\
&= \sum_{a\in\mathcal{A}} |g_a|^2_{k,\mathbb{R}^d}.
\end{aligned}$$

Applying condition 5 to the above equality we have

$$\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{k,\mathbb{R}^d} \le C_1^2 \sum_{a\in\mathcal{A}} |f|^2_{k,B_a} \le C_1^2 |f|^2_{k,\mathbb{R}^d}.$$

Now set $H = f - \sum_{a\in\mathcal{A}} g_a$. It then follows from condition 2 that $H(x) = (P_{(a,c')}f)(x)$ for all $x \in B_a$, and from condition 3 that $H(a) = f(a)$ for all $a \in \mathcal{A}$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be supported on the unit ball and enjoy the properties

$$\int_{\mathbb{R}^d} \phi(x) \, \mathrm{d}x = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \phi(x) x^\alpha \, \mathrm{d}x = 0, \qquad \text{for all } 0 < |\alpha| \le k-1.$$

Now set $F = \phi_\delta * H$. Using Lemma 2.2.3, there is a constant $C_2 > 0$, independent of q and f, such that

$$\begin{aligned}
|F|^2_{m,\mathbb{R}^d} &\le C_2 \delta^{2(k-m)} \Big| f - \sum_{a\in\mathcal{A}} g_a \Big|^2_{k,\mathbb{R}^d} \\
&\le 2C_2 \delta^{2(k-m)} \left(|f|^2_{k,\mathbb{R}^d} + \Big| \sum_{a\in\mathcal{A}} g_a \Big|^2_{k,\mathbb{R}^d} \right) \\
&\le 2C_2 (1 + C_1^2) \delta^{2(k-m)} |f|^2_{k,\mathbb{R}^d}.
\end{aligned}$$

Similarly, there is a constant $C_3 > 0$, independent of q and f, such that

$$\begin{aligned}
|F|^2_{k,\mathbb{R}^d} &\le C_3 \Big| f - \sum_{a\in\mathcal{A}} g_a \Big|^2_{k,\mathbb{R}^d} \\
&\le 2C_3 (1 + C_1^2) |f|^2_{k,\mathbb{R}^d}.
\end{aligned}$$

Thus $|F|_{m,\mathbb{R}^d} \le Cq^{k-m}|f|_{k,\mathbb{R}^d}$ and $|F|_{k,\mathbb{R}^d} \le C|f|_{k,\mathbb{R}^d}$ for some appropriate constant $C > 0$. Since $F = \phi_\delta * H$ and $H|_{B_a} \in \Pi_{k-1}$ for each $a \in \mathcal{A}$, it follows from Lemma 2.2.4 that $F(a) = H(a) = f(a)$ for all $a \in \mathcal{A}$. □

We are almost ready to state and prove our conjectured error estimate for polyharmonic splines. As outlined at the beginning of this section, in proving this result we make use of an existing error estimate for polyharmonic splines. In particular, we shall employ the L_p-error estimate of Duchon for $2 \le p \le \infty$ (Theorem 1.3.3). We wish to treat the case $1 \le p < 2$ as well, which is not significantly different. An example of how to deal with this case can be gleaned from Light and Wayne [49].

Under the assumptions of Theorem 1.3.3, Duchon's result, including the case $1 \le p < 2$ and stated without derivatives of the error, is

$$\|f - S_m f\|_{L_p(\Omega)} \le Ch^{\beta(m)} |f|_{m,\Omega}, \qquad \text{as } h \to 0,$$

for all $f \in BL^m(\Omega)$. Where,

$$\beta(m) = \min\{m, m - d/2 + d/p\}.$$

The reader is wise to wonder if this approximation order is optimal. Duchon's result only says that the optimal L_p-approximation order is at least $\beta(m)$, for $1 \le p \le \infty$. Duchon's error estimate is a *direct theorem*, in the sense that an error estimate is concluded from information about the approximand. To discover if the L_p-approximation order given by Duchon is optimal, one has to conjecture and prove an *inverse theorem*. The following inverse theorem was established by Schaback and Wendland in [68]:

Theorem 2.2.7 (Schaback & Wendland [68])**.** *Let Ω be an open, bounded subset of $\mathbb{R}^d$ satisfying the cone property and let $m > d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_{m-1}(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^m(\mathbb{R}^d)$. Suppose that for some $f \in C(\Omega)$ there exists a $\mu > 0$ and $C = C(f) > 0$ such that*

$$\|f - S_m^h f\|_{L_\infty(\Omega)} \le C(f)h^\mu,$$

for all $\mathcal{A}_h \subset \Omega$ with sufficiently small h. If $\mu > m$, then $f \in BL^m(\Omega)$.

This theorem shows, in the uniform norm setting, that the condition $\mu > m$ is necessary for showing $f \in BL^m(\Omega)$. Hence, there is a gap of $d/2$ between the necessary and sufficient L_∞-approximation orders. Johnson [42] goes some way to closing this gap by showing that, for sufficiently smooth $\partial\Omega$ and f, the optimal L_p-approximation order for polyharmonic spline interpolation is at least $\beta(m) + 1/2$, for $1 \le p \le \infty$. Johnson also shows, this time in [40], that the optimal L_p-approximation order is at most $m + 1/p$, for $1 \le p \le \infty$.

These two results coincide when $p = 2$ to imply that the optimal L_2-approximation order for polyharmonic spline interpolation is precisely $m + 1/2$. Most recently, Johnson [44] has further improved the lower bound on the L_p-approximation order, for $1 \le p \le 2$, to $m + 1/p$. So the question of optimality is settled for $1 \le p \le 2$. Furthermore, Johnson conjectures on the basis of experimental evidence that the L_p-approximation order for $2 < p \le \infty$ is $m + 1/p$ as well. We make these remarks so that it is clear to the reader that, because we intend to employ Duchon's error estimate and because the issue of optimal L_p-approximation orders is currently unsettled, the L_p-approximation orders we give in the theorem below are not necessarily optimal for $2 < p \le \infty$, and not optimal in the case $1 \le p \le 2$.

Theorem 2.2.8. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz continuous boundary. Suppose also that k and m are such that $d/2 < k \le m$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_{m-1}(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h and mesh-ratio ρ. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{k-|\alpha|-d/2+d/p} \rho^{m-k} |f|_{k,\Omega}, \qquad |\alpha| \le k - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{k-|\alpha|} \rho^{m-k} |f|_{k,\Omega}, \qquad |\alpha| \le k - d/2,$$

for all $f \in BL^k(\Omega)$, as $h \to 0$.

Proof. Take $f \in BL^k(\Omega)$. By [21], $f \in W_2^k(\Omega)$. We define f^Ω in accordance with

Lemma 2.1.6. For most of this proof we wish to work with f^Ω and not f, so for convenience we shall write f instead of f^Ω. Construct F in accordance with Theorem 2.2.6 and set $G = f - F$. Then $F(a) = f(a)$ and $G(a) = 0$ for all $a \in \mathcal{A}_h$. Furthermore, there is a constant $C_1 > 0$, independent of f and h, such that

$$|F|_{m,\mathbb{R}^d} \le C_1 \Big(\frac{h}{\rho}\Big)^{k-m} |f|_{k,\mathbb{R}^d}, \tag{2.2.5a}$$

$$|G|_{k,\mathbb{R}^d} \le |f|_{k,\mathbb{R}^d} + |F|_{k,\mathbb{R}^d} \le (1 + C_1)|f|_{k,\mathbb{R}^d}. \tag{2.2.5b}$$

Thus $S^h_m f = S^h_m F$ and $S^h_k G = 0$, where we have adopted the obvious notation for S^h_k; hence,

$$\begin{aligned} \|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} &= \|D^\alpha(f - S^h_m F)\|_{L_p(\Omega)} \\ &= \|D^\alpha(F + G - S^h_m F)\|_{L_p(\Omega)} \\ &\le \|D^\alpha(F - S^h_m F)\|_{L_p(\Omega)} + \|D^\alpha(G - S^h_k G)\|_{L_p(\Omega)}. \end{aligned}$$

Now, employing Duchon's error estimate for polyharmonic splines (1.3.7), there are positive constants $C_2 > 0$ and $C_3 > 0$, independent of h and f, such that

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \le C_2 h^{\beta(m)}|F|_{m,\Omega} + C_3 h^{\beta(k)}|G|_{k,\Omega}, \qquad \text{as } h \to 0,$$

where we have defined

$$\beta(n) = \begin{cases} n - |\alpha| - d/2 + d/p, & |\alpha| \le k - d/2 + d/p,\ 2 \le p \le \infty, \\ n - |\alpha|, & |\alpha| \le k - d/2,\ 1 \le p < 2, \end{cases} \qquad \text{for } n = 1, 2, \ldots.$$

Finally, using the bounds in (2.2.5) we have

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \le C_4 h^{\beta(k)}(\rho^{m-k} + 1)|f|_{k,\mathbb{R}^d} \le 2C_4 h^{\beta(k)}\rho^{m-k}|f|_{k,\mathbb{R}^d}, \qquad \text{as } h \to 0,$$

for some appropriate $C_4 > 0$. To complete the proof we remind ourselves that we have substituted f^Ω with f, and so an application of Lemma 2.1.6 shows that we can find $C_5 > 0$ such that

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le 2C_4 h^{\beta(k)} \rho^{m-k} |f^\Omega|_{k,\mathbb{R}^d} \le 2C_4 C_5 h^{\beta(k)} \rho^{m-k} |f|_{k,\Omega}, \qquad \text{as } h \to 0. \ \square$$

Corollary 2.2.9. *With the notation and assumptions of Theorem 2.2.8, suppose there is a quantity $r > 0$ such that the mesh-ratio of each $\mathcal{A}_h$ is bounded above by r for all $h > 0$. Then there exists a constant $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{k-|\alpha|-d/2+d/p} |f|_{k,\Omega}, \qquad |\alpha| \le k - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{k-|\alpha|} |f|_{k,\Omega}, \qquad |\alpha| \le k - d/2,$$

for all $f \in BL^k(\Omega)$, as $h \to 0$.

Interestingly, the proof technique employed to prove Theorem 2.2.8 is applicable to more than just polyharmonic splines. Indeed, the technique will succeed for a range of approximation schemes. To see this, suppose that for each $m > d/2$ we have an operator

$$U_m : C(\Omega) \to BL^m(\Omega),$$

which is uniquely defined by the values of f on $\mathcal{A}$, and that vanishes if f is zero on $\mathcal{A}$. Suppose further that whenever $\mathcal{A}$ has fill-distance h and $m > d/2$, there is a $C_1 > 0$ and $\gamma(m) > 0$, independent of h, and a (semi)norm $\|\cdot\|$, such that for all $f \in BL^m(\Omega)$,

$$\|f - U_m f\| \le C_1 h^{\gamma(m)} |f|_{m,\Omega}, \qquad \text{as } h \to 0.$$

Then by proceeding in precisely the same manner as in the proof of Theorem 2.2.8 we have, for all $f \in BL^k(\Omega)$ with $m \geq k > d/2$,

$$\|f - U_m f\| \leq C_2 h^{\min(\gamma(m)+k-m, \gamma(k))} \rho^{m-k} |f|_{k,\Omega}, \qquad \text{as } h \to 0,$$

for an appropriate constant $C_2 > 0$. Therefore, by assuming that $\gamma(m) \geq m - k$ we obtain a useable error estimate for our approximation scheme—an error estimate for a class of rougher functions than previously known.

Subsequent to completing this work, the author became aware of independent work by Yoon [80, 81, 82]. In these papers, error bounds for the case we consider here are also offered. Because of Yoon's technique of proof, which is considerably different to our own, he obtains error bounds for functions f with the additional restriction that f lies in $W^k_\infty(\Omega)$, so the results here have wider applicability. However, Yoon does consider *shifted* polyharmonic splines, which employ the translates of the usual polyharmonic splines shifted by a positive parameter:

$$\phi(r) \doteq \begin{cases} (r^2 + c^2)^{m-d/2} \log(r^2 + c^2), & \text{if } d \text{ is even}, \\ (r^2 + c^2)^{m-d/2}, & \text{otherwise}, \end{cases} \qquad r \geq 0.$$

To obtain the result one is instructed to scale the parameter c with the fill-distance of the interpolation points. This so-called *stationary analysis* introduces artificial homogeneity into these basis functions except for a possible log-term, and undoubtedly simplifies the ensuing analysis. We have only considered the unshifted polyharmonic splines as an exemplar of what can be achieved. In Chapter 3 we will see our technique applied to a wider class of basis functions.

We conclude this section with a brief commentary on the approach of Yoon. It is hardly surprising that Yoon's technique also utilises a smoothing via convolution with a smooth kernel function corresponding closely to our function ϕ used in the proof of Theorem 2.2.8. However, Yoon's approach is simply to smooth at this stage, obtaining the equivalent of our function F in the proof of Theorem 2.2.8. Because there is no preprocessing of f to H, Yoon's function F does not enjoy the nice property $F(a) = f(a)$ for all $a \in \mathcal{A}$. It is this property which makes the following step, where we treat $G = f - F$, a fairly simple process. Correspondingly, Yoon has considerably more difficulty treating his function G. Our method also yields the same bound as that in Yoon's work, but for a wider class of functions. Indeed we would suggest that $BL^k(\Omega)$ is the natural class of functions for which one would wish an error estimate of the type given in Theorem 2.2.8.

Recently, Narcowich and Ward [59, 60] have contributed to the problem of providing error estimates for interpolation by radial basis functions when the function is not in the appropriate native space. Their approach, like ours, involves smoothing out the function with the hope that one can control the error in interpolating to the smooth function. However, the technique employed to smooth is to chop off the Fourier transform of the function outside a compact set—which leads to bandlimited functions. The authors show that if f is a 'non-smooth' function then there is a bandlimited function f_σ, where σ is the radius of the ball of support of $\widehat{f_\sigma}$, satisfying the requirement that f_σ interpolates f on the required set, and f_σ is also close to f, where the closeness depends on the separation distance of the interpolation points. Now if Sf is the radial basis function interpolant to f then because f_σ interpolates f, they can write $Sf = Sf_\sigma$. This is the same crucial technique we exploited to obtain our result, and arguably, the proof is more accessible and natural than the methods employed by Yoon. Narcowich and Ward obtain the same

L_p-error estimate for rough functions as we do; however, they must work in $C^k(\overline{\Omega})$, rather than $BL^k(\Omega)$. Consequently, they do not obtain the explicit dependence of the constant in the error estimate on $|f|_{k,\Omega}$ as we do. Therefore, the result given in Theorem 2.2.8 can be considered more satisfactory than the corresponding result given by Narcowich and Ward.

2.2.1 ℓ_p-error estimates

In practice one would not calculate the continuous L_p-norm that we investigated in the previous section; instead, one actually considers a discrete version—the ℓ_p-norm. Let $\mathcal{B}$ be a finite subset of Ω. For a continuous function $f : \Omega \to \mathbb{R}$, we define

$$\|f\|_{\ell_p(\mathcal{B})} = \left(\frac{1}{|\mathcal{B}|} \sum_{b\in\mathcal{B}} |f(b)|^p \right)^{1/p}, \qquad \text{for } 1 \le p \le \infty,$$

with the norm $\|\cdot\|_{\ell_\infty(\mathcal{B})}$ defined by the usual convention. The purpose of this section is to derive a theorem analogous to Theorem 2.2.8 for this discrete norm. To do so, we will need an ℓ_p-estimate for the error in interpolation when the target function belongs to the appropriate native space. Then, as in the proof of Theorem 2.2.8, it will be Theorem 2.2.6 that will provide the estimate when f lies outside the native space.

Theorem 2.2.10 (Duchon [22]). *Let Ω be an open, bounded subset of $\mathbb{R}^d$ satisfying the cone property. Then there are constants M_1, M_2, M_3 and h_0 such that for each $0 < h < h_0$ there corresponds a finite set $T_h \subset \Omega$ such that:*

1. *$B(t,h) \subset \Omega$ for all $t \in T_h$;*
2. *$\Omega \subset \bigcup_{t\in T_h} B(t, M_1 h)$;*
3. *$\sum_{t\in T_h} \chi_{B(t,M_1 h)} \le M_2$;*

4. $|T_h| \leq M_3 h^{-d}$.

Lemma 2.2.11 (Duchon [22]). *Let $m > d/2$, $p \geq 2$ and $|\alpha| \leq m - d/2 + d/p$. There exists an $R > 0$, and for all $M \geq 1$, there exists a $C > 0$ satisfying the following: for each $h > 0$, $t \in \mathbb{R}^d$ the ball $B(t, Rh)$ contains ℓ closed balls $B_1, \ldots, B_\ell$ each of radius h such that,*

$$\|D^\alpha f\|_{L_p(B(t,MRh))} \leq C h^{m-|\alpha|-d/2+d/p} |f|_{m,B(t,MRh)},$$

for all $f \in BL^m(B(t, MRh))$ that vanish at at least one point in each of the balls B_i.

Theorem 2.2.12. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz continuous boundary. Suppose also that k and m are such that $d/2 < k \leq m$ and let $|\alpha| \leq k - d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_{m-1}(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h and mesh-ratio ρ. Let $\mathcal{B}$ be another finite subset of Ω, with mesh-ratio σ and separation distance $q_{\mathcal{B}} \leq h$. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h and $\mathcal{B}$, such that,*

$$\|D^\alpha(f - S_m^h f)\|_{\ell_p(\mathcal{B})} \leq \begin{cases} C h^{k-|\alpha|-d/2+d/p} \rho^{m-k} \sigma^{d/p} |f|_{k,\Omega}, & 2 \leq p \leq \infty, \\ C h^{k-|\alpha|} \rho^{m-k} \sigma^{d/p} |f|_{k,\Omega}, & 1 \leq p < 2, \end{cases}$$

for all $f \in BL^k(\Omega)$, as $h \to 0$.

Proof. Firstly, we observe that the result is true for $p = \infty$ by Theorem 2.2.8. Next, we deal with the case $2 \leq p < \infty$. Suppose $f \in BL^m(\mathbb{R}^d)$. Let us construct R and C_1 in accordance with Lemma 2.2.11 and implement Lemma 2.2.10 with Rh in place of h. Then, there are constants M_1, M_2, M_3, h_0 such that for each $0 < h < h_0/R$ there corresponds a finite set $T_h \subset \Omega$ such that:

1. $B(t, Rh) \subset \Omega$ for all $t \in T_h$;

2. $\Omega \subset \bigcup_{t \in T_h} B(t, M_1 Rh)$;

3. $\sum_{t \in T_h} \chi_{B(t, M_1 Rh)} \leq M_2$;

4. $|T_h| \leq M_3 (Rh)^{-d}$.

As $\Omega \subset \bigcup_{t \in T_h} B(t, M_1 Rh)$ it follows that

$$\sum_{b \in \mathcal{B}} |(D^\alpha (f - S_m^h f))(b)|^p \leq \sum_{t \in T_h} \sum_{b \in \mathcal{B} \cap B(t, M_1 Rh)} |(D^\alpha (f - S_m^h f))(b)|^p.$$

Fix $0 < h < h_0/R$ and $t \in T_h$. Then $B(t, Rh) \subset \Omega$ contains ℓ balls of radius h. Since $\mathcal{A}_h$ has fill-distance h, each of these balls contains at least one member of $\mathcal{A}_h$. Hence, by employing the L_∞-bound offered in Lemma 2.2.11, we have

$$\begin{aligned} \sum_{b \in \mathcal{B}} &|(D^\alpha (f - S_m^h f))(b)|^p \\ &\leq C_1^p h^{(m - |\alpha| - d/2)p} \sum_{t \in T_h} \sum_{b \in \mathcal{B} \cap B(t, M_1 Rh)} |f - S_m^h f|^p_{m, B(t, M_1 Rh)} \\ &\leq C_1^p h^{(m - |\alpha| - d/2)p} \sum_{t \in T_h} |\mathcal{B} \cap B(t, M_1 Rh)| |f - S_m^h f|^p_{m, B(t, M_1 Rh)}, \end{aligned} \tag{2.2.6}$$

where C_1 is independent of both h and f. Let $b \in \mathcal{B} \cap B(t, M_1 Rh)$. Since $q_{\mathcal{B}} \leq h$ and $M_1, R \geq 1$ we have $B(b, q_{\mathcal{B}}) \subset B(t, 2M_1 Rh)$. Therefore, by comparing volumes, we deduce that

$$|\mathcal{B} \cap B(t, M_1 Rh)| \operatorname{vol}(B(b, q_{\mathcal{B}})) \leq \operatorname{vol}(B(t, 2M_1 Rh)).$$

Hence there is a constant $C_2 > 0$ independent of h, $q_{\mathcal{B}}$, t and b such that

$$|\mathcal{B} \cap B(t, M_1 Rh)| \leq C_2 \Big(\frac{h}{q_{\mathcal{B}}} \Big)^d.$$

Inserting this bound into (2.2.6) we obtain

$$\sum_{b\in\mathcal{B}} |(D^\alpha(f - S_m^h f))(b)|^p \le C_1^p C_2 h^{(m-|\alpha|-d/2+d/p)p} q_{\mathcal{B}}{}^{-d} \sum_{t\in T_h} |f - S_m^h f|^p_{m,B(t,M_1 Rh)}. \quad (2.2.7)$$

Using the fact that if $v \in \mathbb{R}^n$ then $\|v\|_p \le \|v\|_2$ for $2 \le p \le \infty$, we have

$$\begin{aligned}
\left(\sum_{t\in T_h} |f - S_m^h f|^p_{m,B(t,M_1 Rh)}\right)^{2/p} &\le \sum_{t\in T_h} |f - S_m^h f|^2_{m,B(t,M_1 Rh)} \\
&\le \sum_{t\in T_h} \sum_{|\alpha|=m} c_\alpha \int_{B(t,M_1 Rh)} |(D^\alpha(f - S_m^h f))(x)|^2 \,\mathrm{d}x \\
&\le \sum_{t\in T_h} \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \chi_{B(t,M_1 Rh)}(x) |(D^\alpha(f - S_m^h f))(x)|^2 \,\mathrm{d}x \\
&= \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \left(\sum_{t\in T_h} \chi_{B(t,M_1 Rh)}(x)\right) |(D^\alpha(f - S_m^h f))(x)|^2 \,\mathrm{d}x \\
&\le M_2 \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(f - S_m^h f))(x)|^2 \,\mathrm{d}x \\
&\le M_2 \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha f)(x)|^2 \,\mathrm{d}x,
\end{aligned}$$

where the last inequality is by virtue of the Pythagorean property (1.3.3). Substituting this bound into (2.2.7), we have, for all $f \in BL^m(\mathbb{R}^d)$, $m > d/2$ and all $|\alpha| \le m - d/2$,

$$\sum_{b\in\mathcal{B}} |(D^\alpha(f - S_m^h f))(b)|^p \le C_3 h^{(m-|\alpha|-d/2+d/p)p} q_{\mathcal{B}}^{-d} |f|^p_{m,\mathbb{R}^d}, \quad (2.2.8)$$

where we have set $C_3 = C_1^p C_2 M_2^{p/2}$. Suppose now that $f \in BL^k(\mathbb{R}^d)$ with $d/2 < k \le m$ and $|\alpha| \le k - d/2$. Let us construct F in accordance with Theorem 2.2.6 and set $G = f - F$. Then $F(a) = f(a)$ and $G(a) = 0$ for all $a \in \mathcal{A}_h$. There is a constant $C_4 > 0$, independent of f and h, such that

$$|F|_{m,\mathbb{R}^d} \le C_4 \left(\frac{h}{\rho}\right)^{k-m} |f|_{k,\mathbb{R}^d}, \quad (2.2.9a)$$

$$|G|_{k,\mathbb{R}^d} \le |f|_{k,\mathbb{R}^d} + |F|_{k,\mathbb{R}^d} \le (1 + C_4)|f|_{k,\mathbb{R}^d}. \quad (2.2.9b)$$

By Minkowski's inequality, which is valid for $p \geq 1$,

$$\begin{aligned}
&\left(\frac{1}{|\mathcal{B}|}\sum_{b\in\mathcal{B}}|(D^\alpha(f - S_m^h f))(b)|^p\right)^{1/p} \\
&\qquad = \left(\frac{1}{|\mathcal{B}|}\sum_{b\in\mathcal{B}}|(D^\alpha(F - S_m^h F))(b) + (D^\alpha(G - S_k^h G))(b)|^p\right)^{1/p} \\
&\qquad \leq \left(\frac{1}{|\mathcal{B}|}\sum_{b\in\mathcal{B}}|(D^\alpha(F - S_m^h F))(b)|^p\right)^{1/p} + \left(\frac{1}{|\mathcal{B}|}\sum_{b\in\mathcal{B}}|(D^\alpha(G - S_k^h G))(b)|^p\right)^{1/p}.
\end{aligned}$$

Hence, by (2.2.8) there is a constant C_5, independent of h and f, such that

$$\|D^\alpha(f - S_m^h f)\|_{\ell_p(\mathcal{B})} \leq C_5|\mathcal{B}|^{-1/p} q_{\mathcal{B}}^{-d/p}\Big(h^{m-|\alpha|-d/2+d/p}|F|_{m,\mathbb{R}^d} + h^{k-|\alpha|-d/2+d/p}|G|_{k,\mathbb{R}^d}\Big).$$

Therefore, it follows by applying the bounds given in (2.2.9) that for all $f \in BL^k(\mathbb{R}^d)$,

$$\begin{aligned}
\|D^\alpha(f - S_m^h f)\|_{\ell_p(\mathcal{B})} &\leq C_6|\mathcal{B}|^{-1/p} h^{k-|\alpha|-d/2+d/p}(\rho^{m-k} + 1) q_{\mathcal{B}}^{-d/p}|f|_{k,\mathbb{R}^d} \\
&\leq 2C_6|\mathcal{B}|^{-1/p} h^{k-|\alpha|-d/2+d/p}\rho^{m-k} q_{\mathcal{B}}^{-d/p}|f|_{k,\mathbb{R}^d},
\end{aligned}$$

for an appropriate constant $C_6 > 0$. Let $h_{\mathcal{B}}$ be the fill-distance of $\mathcal{B}$ in Ω. Then, the balls $B(b, h_{\mathcal{B}})$, $b \in \mathcal{B}$, cover Ω. So, by comparing volumes once again we find that

$$\operatorname{vol}(\Omega) \leq |\mathcal{B}| \operatorname{vol}(B(b, h_{\mathcal{B}})), \qquad \text{for } b \in \mathcal{B}.$$

Hence there is a constant $C_7 > 0$, independent of $\mathcal{B}$, such that $C_7 \leq |\mathcal{B}| h_{\mathcal{B}}^d$. The proof is now complete for $2 \leq p < \infty$ because for $f \in BL^k(\Omega)$ we would work instead with the extension f^Ω described in Lemma 2.1.6.

For $1 \leq p < 2$ we need a slightly different technique. Let $f \in BL^m(\mathbb{R}^d)$ again. We follow the proof as far as (2.2.7) and let r be such that $p/2 + 1/r = 1$. Now, using Hölder's inequality,

$$\begin{aligned}\sum_{t\in T_h}|f-S_m^hf|^p_{m,B(t,M_1Rh)} &\le \left(\sum_{t\in T_h}1^r\right)^{1/r}\left(\sum_{t\in T_h}\left(|f-S_m^hf|^p_{m,B(t,M_1Rh)}\right)^{2/p}\right)^{p/2}\\ &\le \left(\sum_{t\in T_h}1^r\right)^{1/r}\left(\sum_{t\in T_h}|f-S_m^hf|^2_{m,B(t,M_1Rh)}\right)^{p/2}\\ &= |T_h|^{1/r}\left(\sum_{t\in T_h}|f-S_m^hf|^2_{m,B(t,M_1Rh)}\right)^{p/2}.\end{aligned}$$

We have already established in this proof that

$$\sum_{t\in T_h}|f-S_m^hf|^2_{m,B(t,M_1Rh)} \le M_2|f|^2_{m,\mathbb{R}^d}.$$

As for $|T_h|$, we know from Lemma 2.2.10 that $|T_h|\le M_3h^{-d}$. Set $C_8=M_2^{p/2}M_3^{1/r}$, then

$$\sum_{t\in T_h}|f-S_m^hf|^p_{m,B(t,M_1Rh)} \le C_8h^{-d/r}|f-S_m^hf|^p_{m,\mathbb{R}^d} = C_8h^{(-d/p+d/2)p}|f|^p_{m,\mathbb{R}^d}.$$

Hence, by (2.2.7), which is valid for all $p\ge 1$,

$$\sum_{b\in\mathcal{B}}|(D^\alpha(f-S_m^hf))(b)|^p \le C_9h^{(m-|\alpha|)p}q_{\mathcal{B}}^{-d}|f|^p_{m,\mathbb{R}^d},$$

with $C_9=C_1^pC_2C_8$. The proof now continues in precisely the same manner as for the case $2\le p<\infty$. □

Corollary 2.2.13. *With the notation and assumptions of Theorem 2.2.12, suppose there is a quantity $r>0$ such that the mesh-ratio of each $\mathcal{A}_h$ is bounded above by r for all $h>0$. Then there exists a constant $C>0$, independent of h and $\mathcal{B}$, such that,*

$$\|D^\alpha(f-S_m^hf)\|_{\ell_p(\mathcal{B})}\le\begin{cases}Ch^{k-|\alpha|-d/2+d/p}\sigma^{d/p}|f|_{k,\Omega}, & 2\le p\le\infty,\\ Ch^{k-|\alpha|}\sigma^{d/p}|f|_{k,\Omega}, & 1\le p<2,\end{cases}$$

for all $f\in BL^k(\Omega)$, as $h\to 0$.

2.2.2 The polyharmonic spline interpolation operator

Look at the statement of Theorem 2.2.8. We can choose to view S_m^h as an operator from $BL^k(\Omega)$ to $BL^m(\Omega)$, where $m \geq k > d/2$. The boundedness of this operator is well known in the case $k = m$. To see this, we first define f^Ω in accordance with Lemma 2.1.6. Then, using the seminorm minimisation property (1.3.2b), we have

$$|S_m^h f|_{m,\Omega} = |S_m^h(f^\Omega)|_{m,\Omega} \leq |S_m^h(f^\Omega)|_{m,\mathbb{R}^d} \leq |f^\Omega|_{m,\mathbb{R}^d} \leq C|f|_{m,\Omega}.$$

Using the technique developed in Section 2.2, we can show that S_m^h is a bounded operator for $m > k$ too.

Theorem 2.2.14. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz continuous boundary. Suppose that k and m are such that $d/2 < k \leq m$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_{m-1}(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h and mesh-ratio ρ. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^m(\mathbb{R}^d)$. Then there exists constants $C > 0$ and $h_0 > 0$, independent of h, such that,*

$$|S_m^h f|_{k,\Omega} \leq C\rho^{m-k}|f|_{k,\Omega},$$

for all $f \in BL^k(\Omega)$, and $h < h_0$.

Proof. Let $f \in BL^k(\Omega)$. Let f^Ω be constructed in accordance with Lemma 2.1.6. For convenience we shall write f instead of f^Ω. Let us now construct F in accordance with Theorem 2.2.6. Let $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| = k$ and let $x \in \mathbb{R}^d$. Since F interpolates f on $\mathcal{A}_h$, we have

$$|(D^\alpha(S_m^h f))(x)|^2 = |(D^\alpha(S_m^h F))(x)|^2$$
$$= |(D^\alpha(F - S_m^h F))(x) - (D^\alpha F)(x)|^2$$
$$\leq 2|(D^\alpha(F - S_m^h F))(x)|^2 + 2|(D^\alpha F)(x)|^2.$$

Now, by premultiplying the previous inequality by c_α, summing over all $|\alpha| = k$ and integrating over Ω, we obtain

$$|S_m^h f|_{k,\Omega}^2 \leq 2 \sum_{|\alpha|=k} c_\alpha \|D^\alpha(F - S_m^h F)\|_{L_2(\Omega)}^2 + 2|F|_{k,\Omega}^2.$$

By Theorem 1.3.3, there is a constant $C_1 > 0$, independent of h and f, such that

$$\|D^\alpha(F - S_m^h F)\|_{L_2(\Omega)}^2 \leq C_1^2 h^{2(m-|\alpha|)} |F|_{m,\Omega}^2, \qquad \text{as } h \to 0.$$

Hence,

$$|S_m^h f|_{m,\Omega}^2 \leq 2C_1^2 d^k h^{2(m-k)} |F|_{m,\Omega}^2 + 2|F|_{k,\Omega}^2, \qquad \text{as } h \to 0. \tag{2.2.10}$$

By construction there are constants $C_2 > 0$ and $C_3 > 0$, independent of h and f, such that $|F|_{k,\mathbb{R}^d} \leq C_2 |f|_{k,\mathbb{R}^d}$ and $|F|_{m,\mathbb{R}^d} \leq C_3 (h/\rho)^{k-m} |f|_{k,\mathbb{R}^d}$. Inserting these bounds into (2.2.10) yields

$$|S_k^h f|_{m,\Omega} \leq C_4(\rho^{m-k} + 1)|f|_{k,\mathbb{R}^d} \leq 2C_4 \rho^{m-k} |f|_{k,\mathbb{R}^d}, \qquad \text{as } h \to 0,$$

for an appropriate constant C_4, independent of h and f. The proof is complete once we recall that we have been using f to denote f^Ω. □

Corollary 2.2.15. *With the notation and assumptions of Theorem 2.2.14, suppose there is a quantity $r > 0$ such that the mesh-ratio of each $\mathcal{A}_h$ is bounded above by r for all $h > 0$.*

Then there exists constants $C > 0$ and $h_0 > 0$, independent of h, such that,

$$|S^h_m f|_{k,\Omega} \le C|f|_{k,\Omega},$$

for all $f \in BL^k(\Omega)$, and $h < h_0$.

It is of interest to note that Theorem 2.2.14 provides a neat alternative proof of Theorem 2.2.8. To see this, let us adopt the notation and hypotheses of Theorem 2.2.8 and let us define, as we have done before, the number $\beta(n)$ by

$$\beta(n) = \begin{cases} n - |\alpha| - d/2 + d/p, & |\alpha| \le k - d/2 + d/p,\ 2 \le p \le \infty, \\ n - |\alpha|, & |\alpha| \le k - d/2,\ 1 \le p < 2, \end{cases} \qquad \text{for } n = 1, 2, \ldots.$$

Now, for $f \in BL^k(\Omega)$ we observe that

$$\begin{aligned} \|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} &\le \|D^\alpha(f - S^h_k f)\|_{L_p(\Omega)} + \|D^\alpha(S^h_k f - S^h_m f)\|_{L_p(\Omega)} \\ &= \|D^\alpha(f - S^h_k f)\|_{L_p(\Omega)} + \|D^\alpha(S^h_k(S^h_m f) - S^h_m f)\|_{L_p(\Omega)}, \end{aligned}$$

where the last equality is valid because $S^h_m f$ interpolates f on $\mathcal{A}$. Now, using the fundamental error estimate (Theorem 1.3.3), there exists an appropriate generic constant $C > 0$, independent of h and f, such that

$$\begin{aligned} \|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} &\le Ch^{\beta(k)}|f|_{k,\Omega} + Ch^{\beta(k)}|S^h_m f|_{k,\Omega} \\ &\le Ch^{\beta(k)}|f|_{k,\Omega} + Ch^{\beta(k)}\rho^{m-k}|f|_{k,\Omega} \\ &\le Ch^{\beta(k)}(1 + \rho^{m-k})|f|_{k,\Omega} \le Ch^{\beta(k)}\rho^{m-k}|f|_{k,\Omega}, \qquad \text{as } h \to 0. \end{aligned}$$

Chapter 3

Interpolation of rough functions using the scattered shifts of a basis function

In this chapter we extend the work of the previous chapter to more general classes of radial basis functions. To this end, we now introduce those native spaces and related spaces that we shall be concerned with.

The work of Light and Wayne in [50] can be viewed as a successful attempt to generalise the setup used by Duchon [22]. The authors consider generalised Beppo Levi spaces which arise from the introduction into the semi-inner product (1.3.1) of a *weight function* $w : \mathbb{R}^d \to \mathbb{R}$ that is positive almost everywhere. Specifically, for $m \in \mathbb{Z}_+$, these spaces are

$$\mathcal{Z}^m(\mathbb{R}^d) = \left\{ f \in \mathscr{S}' : \ \widehat{D^\alpha f} \in L_1^{\mathrm{loc}}(\mathbb{R}^d), \ \int_{\mathbb{R}^d} |(\widehat{D^\alpha f}\,)(x)|^2 w(x)\,\mathrm{d}x < \infty, \ |\alpha| = m \right\},$$

and carry the semi-inner product

$$(f,g)_{m,w,\mathbb{R}^d} = \left(\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} (\widehat{D^\alpha f}\,)(x)(\overline{\widehat{D^\alpha g}}\,)(x) w(x)\,\mathrm{d}x \right)^{1/2}, \qquad \text{for } f,g \in \mathscr{Z}^m(\mathbb{R}^d).$$

Recall that $\mathscr{S}'$ is the space of distributions we introduced in Section 1.5. It should be clear from the definition that the Fourier transform that appears in this semi-inner product is taken in the distributional sense. The constants c_α are selected as in Section 1.3.

So that we have an interpolation problem that we can handle, we need to fit the space $\mathscr{Z}^m(\mathbb{R}^d)$ into the variational framework described in Section 1.3. Recall that this variational approach demands that the space $\mathscr{Z}^m(\mathbb{R}^d)$ can be continuously embedded in the continuous functions. In order for this to occur, the weight function is initially chosen to satisfy the axioms:

(W0) $w \in C(\mathbb{R}^d \setminus 0)$;

(W1) $w(x) > 0$ if $x \neq 0$;

(W2) $1/w \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$;

(W3) there is a positive $\mu \in \mathbb{R}$ such that $(w(x))^{-1} = \mathcal{O}(|x|^{-2\mu})$ as $|x| \to \infty$.

Consequently, the space $\mathscr{Z}^m(\mathbb{R}^d)$ is complete with respect to $|\cdot|_{m,w,\mathbb{R}^d}$, and whenever we have $m + \mu > d/2$ then $\mathscr{Z}^m(\mathbb{R}^d)$ is continuously embedded in the continuous functions (see Light and Wayne [50]).

The kernel of the semi-inner product certainly contains $\Pi_{m-1}(\mathbb{R}^d)$ and it transpires that the kernel is precisely $\Pi_m(\mathbb{R}^d)$. To see this clearly, it is helpful to rewrite the seminorm in so-called *direct form*—that is, without the Fourier transform of f appearing explicitly. We demand that w satisfies the additional axioms:

(W4) $w(y) = w(-y)$ for all $y \in \mathbb{R}^d$;

(W5) $w(0) = 0$ and $\widehat{w}\,(x) \leq 0$ for almost all $x \in \mathbb{R}^d$;

(W6) $\widehat{w}$ is a measurable function and for any neighbourhood N of the origin, we have $\widehat{w} \in L_1(\mathbb{R}^d \setminus N)$;

(W7) $|\widehat{w}\,(y)| = \mathcal{O}(|y|^\lambda)$ as $y \to 0$, where $\lambda + d + 2 > 0$.

Once armed with axioms (W1) and (W4)–(W7) it follows from the work of Levesley and Light [46] that

$$|f|^2_{m,w,\mathbb{R}^d} = -\frac{1}{2}\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{w}\,(x-y)|(D^\alpha f)(x) - (D^\alpha f)(y)|^2 \,\mathrm{d}x\mathrm{d}y, \qquad f \in \mathcal{Z}^m(\mathbb{R}^d). \tag{3.0.1}$$

We see immediately that the kernel of the seminorm, and that of the associated semi-inner product, is $\Pi_m(\mathbb{R}^d)$. It is also shown in [46] that (W6) and (W7) together imply that $w \in \mathscr{S}' \cap C(\mathbb{R}^d)$. This means that axiom (W0) becomes redundant if these two axioms are assumed.

Theorem 3.0.1. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W0)–(W3) and let $m + \mu > d/2$. Let $\mathcal{A} \subset \mathbb{R}^d$ be unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Then $\mathcal{Z}^m(\mathbb{R}^d)$ is complete with respect to the norm*

$$\|f\|_{\mathbb{R}^d} = \left(|f|^2_{m,w,\mathbb{R}^d} + \sum_{a\in\mathcal{A}} |f(a)|^2\right)^{1/2}.$$

Further, there is a constant $C > 0$ such that $|f(x)| \leq C\|f\|_{\mathbb{R}^d}$ for all $f \in \mathcal{Z}^m(\mathbb{R}^d)$.

Proof. Let $\{f_j\}$ be a Cauchy sequence in $\mathcal{Z}^m(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{\mathbb{R}^d}$. Then the sequence is Cauchy with respect to $|\cdot|_{m,w,\mathbb{R}^d}$, so there is an element $f \in \mathcal{Z}^m(\mathbb{R}^d)$ with $|f_j - f|_{m,w,\mathbb{R}^d} \to 0$ as $j \to \infty$. For each $a \in \mathcal{A}$ we also have $\{f_j(a)\}$ as a Cauchy sequence in $\mathbb{R}$, so there is a number $f_a \in \mathbb{R}$ with $|f_j(a) - f_a| \to 0$ as $j \to \infty$. Since the set $\mathcal{A}$ is unisolvent with respect to $\Pi_m(\mathbb{R}^d)$, there is a unique polynomial $p \in \Pi_m(\mathbb{R}^d)$ such that $p(a) = f(a) - f_a$, for each $a \in \mathcal{A}$ (see Cheney and Light [15, Page 4]). Thus,

$$\begin{aligned} \|f_j - (f-p)\|^2_{\mathbb{R}^d} &= |f_j - (f-p)|^2_{m,w,\mathbb{R}^d} + \sum_{a\in\mathcal{A}} |f_j(a) - (f(a) - p(a))|^2 \\ &= |f_j - (f-p)|^2_{m,w,\mathbb{R}^d} + \sum_{a\in\mathcal{A}} |f_j(a) - f_a|^2 \\ &= |f_j - f|^2_{m,w,\mathbb{R}^d} + \sum_{a\in\mathcal{A}} |f_j(a) - f_a|^2. \end{aligned}$$

Finally, a constant $C > 0$ such that $|f(x)| \le C\|f\|_{\mathbb{R}^d}$, for all $f \in \mathcal{Z}^m(\mathbb{R}^d)$, is provided in [50]. □

We hope that our setup admits minimal norm interpolants of the form

$$S_m f = \sum_{a\in\mathcal{A}} \lambda_a \psi(\cdot - a) + p,$$

with the extra degrees of freedom taken up, in the usual manner, by the *natural boundary conditions*. We are not disappointed. Light and Wayne are able to show this by constructing the reproducing kernel for x in $\mathcal{Z}^m(\mathbb{R}^d)$ explicitly. Here, p is a polynomial of appropriate degree and ψ is a distribution which satisfies the equation

$$\widehat{\psi}|\cdot|^{2m} = \frac{1}{w}.$$

In addition, ψ is a continuous function and is conditionally strictly positive definite of some appropriate order. Notice that the basis function is not necessarily radial since w itself is not necessarily radial. However, the archetypal case the author has in mind is

$w(x) = |x|^{2\mu}$, for $x \in \mathbb{R}^d$ and $0 < \mu < d/2$, which is radial. Here, the basis function is

$$\psi(x) \doteq \begin{cases} |x|^{2m+2\mu-d} \log|x|, & \text{if } 2m+2\mu-d \text{ is even,} \\ |x|^{2m+2\mu-d}, & \text{otherwise,} \end{cases} \qquad x \in \mathbb{R}^d.$$

Let us now recapitulate the problem we posed in Chapter 2. We intend to question what happens if the function being approximated is conspicuously rough. If $f \in \mathcal{Z}^k(\mathbb{R}^d)$ with $k < m$ and we measure the error $f - S_m f$ in some appropriate norm, then we predict the approximation order achieved if we had instead approximated f with the minimal norm interpolant to f on $\mathcal{A}$ from $\mathcal{Z}^k(\mathbb{R}^d)$.

3.1 Local native spaces

To establish the conjecture we have just outlined, we will mimic the technique of Chapter 2. This in turn requires us to establish some new tools, whose analogous variants in the Sobolev setting were available for free—for example, the notion of local native spaces and the existence of various extension and embedding theorems.

Let $\widehat{w} : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that is nonpositive almost everywhere. We shall define, for $m \in \mathbb{Z}_+$ and for any domain $\Omega \subset \mathbb{R}^d$, the following local space,

$$X^m(\Omega) = \Big\{ f|_\Omega \colon \; f \in C_0^m(\mathbb{R}^d), \; |f|_{m,w,\Omega} < \infty \Big\},$$

where

$$|f|_{m,w,\Omega} = \left(-\frac{1}{2} \sum_{|\alpha|=m} c_\alpha \int_\Omega \int_\Omega \widehat{w}\,(x-y)|(D^\alpha f)(x) - (D^\alpha f)(y)|^2 \, \mathrm{d}x\mathrm{d}y \right)^{1/2}, \qquad f \in X^m(\Omega).$$

It is convenient for us to work, from now on, with a more common norm on $W_2^m(\Omega)$ than we have previously been working. This is given by

$$\|f\|_{m,\Omega} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|^2_{L_2(\Omega)} \right)^{1/2}, \qquad \text{for } f \in W_2^m(\Omega).$$

Clearly, this norm is equivalent to the norm we first imposed on $W_2^m(\Omega)$ on Page 28. Now, we impose a norm on $X^m(\Omega)$ via

$$\|f\|_{m,w,\Omega} = \left(\|f\|^2_{m,\Omega} + |f|^2_{m,w,\Omega} \right)^{1/2}, \qquad \text{for } f \in X^m(\Omega).$$

The notation $\mathcal{X}^m(\Omega)$ is used to denote the completion of $X^m(\Omega)$ with respect to $\|\cdot\|_{m,w,\Omega}$, while $\mathcal{Y}^m(\Omega)$ denotes the completion of $X^m(\Omega)$ with respect to $|\cdot|_{m,w,\Omega}$. It is these spaces that we call the *local native spaces.*

Theorem 3.1.1. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W7). Then $\mathcal{X}^m(\mathbb{R}^d) \subset \mathcal{Y}^m(\mathbb{R}^d) \subset \mathcal{Z}^m(\mathbb{R}^d)$.*

Proof. The inclusion $\mathcal{X}^m(\mathbb{R}^d) \subset \mathcal{Y}^m(\mathbb{R}^d)$ is clear as a consequence of the definition of the norm $\|\cdot\|_{m,w,\mathbb{R}^d}$. To see the other inclusion we first let $f \in X^m(\mathbb{R}^d)$ and $|\alpha| = m$. Then $D^\alpha f \in C_0(\mathbb{R}^d)$. It follows that $\widehat{D^\alpha f}$ is certainly in $L_1^{\text{loc}}(\mathbb{R}^d)$. Furthermore, it is manifestly true that for all $f \in X^m(\mathbb{R}^d)$ we have $|f|_{m,w,\mathbb{R}^d} < \infty$; hence, $X^m(\mathbb{R}^d) \subset \mathcal{Z}^m(\mathbb{R}^d)$. We have, in an earlier comment, acknowledged that $\mathcal{Z}^m(\mathbb{R}^d)$ is complete with respect to $|\cdot|_{m,w,\mathbb{R}^d}$. Hence, $\mathcal{Y}^m(\mathbb{R}^d) \subset \mathcal{Z}^m(\mathbb{R}^d)$. $\square$

For our later work on extension theorems, it is necessary at this point to take on board four additional axioms and introduce an important type of domain:

Definition 3.1.2. *Let Ω_1 and Ω_2 be domains in $\mathbb{R}^d$, and Φ a bijection from Ω_1 to Ω_2. We say that Φ is m-smooth if, writing $\Phi(x) = (\phi_1(x_1, \ldots, x_d), \ldots, \phi_d(x_1, \ldots, x_d))$ and*

$\Phi^{-1}(x) = \Psi(x) = (\psi_1(x_1, \ldots, x_d), \ldots, \psi_d(x_1, \ldots, x_d))$, *then the functions* $\phi_1, \ldots, \phi_d$ *belong to* $C^m(\overline{\Omega}_1)$ *and* $\psi_1, \ldots, \psi_d$ *belong to* $C^m(\overline{\Omega}_2)$. *Let* Φ *be a bijection from* $\mathbb{R}^d$ *to* $\mathbb{R}^d$. *We say* Φ *is locally* m*-smooth if* Φ *is* m*-smooth on every bounded domain in* $\mathbb{R}^d$.

(W8) for every locally $(m+1)$-smooth map ϕ on $\mathbb{R}^d$, and every bounded subset Ω of $\mathbb{R}^d$, there is a $C_1 > 0$ such that $\widehat{w}\,(\phi(x) - \phi(y)) \leq C_1 \widehat{w}\,(x - y)$, for all $x, y \in \Omega$;

(W9) there exists a constant $C_2 > 0$ such that if $x = (x', x_d) \in \mathbb{R}^d$ and $y = (x', y_d) \in \mathbb{R}^d$ with $|x_d| \geq |y_d|$, then $\widehat{w}\,(x) \leq C_2 \widehat{w}\,(y)$;

(W10) $\int_A \widehat{w} < 0$ whenever A has positive measure;

(W11) $\widehat{w}\,(y) = \widehat{w}\,(-y)$ for all $y \in \mathbb{R}^d$.

Definition 3.1.3. *Let* $B = \{(y_1, y_2, \ldots, y_d) \in \mathbb{R}^d : \ |y_j| < 1, \ 1 \leq j \leq d\}$, *and set* $B_+ = \{y \in B : \ y = (y', y_d)$ *and* $y_d > 0\}$ *and* $B_0 = \{y \in B : \ y = (y', y_n)$ *and* $y_n = 0\}$. *An open, bounded, connected and convex set* Ω *in* $\mathbb{R}^d$ *with boundary* $\partial\Omega$ *will be called a V-domain if the following all hold:*

(A1) there exist open sets $G_1, \ldots, G_N \subset \mathbb{R}^d$ *such that* $\partial\Omega \subset \bigcup_{j=1}^N G_j$;

(A2) there exist locally (m+1)-smooth maps $\phi_j : \mathbb{R}^d \to \mathbb{R}^d$ *such that* $\phi_j(B) = G_j$, $\phi_j(B_+) = G_j \cap \Omega$ *and* $\phi_j(B_0) = G_j \cap \partial\Omega$, $j = 1, \ldots, N$;

(A3) let Ω_δ *be the set of all points in* Ω *whose distance from* $\partial\Omega$ *is less than* δ. *Then for some* $\delta > 0$,

$$\Omega_\delta \subset \bigcup_{j=1}^N \phi_j\left(\left\{(y_1, y_2, \ldots, y_d) \in \mathbb{R}^d : |y_j| < \frac{1}{m+1}, \ 1 \leq j \leq d\right\}\right).$$

The definition of a V-domain is taken from a paper by Light and Vail [48] in which extension theorems for our local native spaces are considered. Indeed, the name lends itself to one of the authors of that paper—Michelle Vail. It is not immediately obvious how to exemplify the conditions we have placed on the domain. However, useful information that we feel we should convey to the reader is that a V-domain is an open, bounded, convex subset of $\mathbb{R}^d$ with a "pleasantly smooth boundary". In particular, a V-domain has the cone property (see Wloka [76, Page 35]) and every open ball is itself a V-domain. The conditions are a fairly stringent requirement on $\partial\Omega$, and for further details the reader is directed to the thesis of Vail [72, Page 63].

Theorem 3.1.4 (Light & Vail [48]). *Let $\Omega \subset \mathbb{R}^d$ be a V-domain. Let $\widehat{w} : \mathbb{R}^d \to \mathbb{R}$ satisfy (W6)–(W11). There exists a continuous linear operator $L : \mathcal{X}^m(\Omega) \to \mathcal{X}^m(\mathbb{R}^d)$ such that, for all $f \in \mathcal{X}^m(\Omega)$:*

1. *$Lf = f$ on Ω;*

2. *$\operatorname{supp}(Lf)$ is compact and independent of f;*

3. *$\|Lf\|_{m,w,\mathbb{R}^d} \leq K\|f\|_{m,w,\Omega}$, for some positive constant $K = K(\Omega)$ independent of f.*

A very useful feature of the construction of the extension operator in Theorem 3.1.4 is that Lf can be chosen to be supported on any compact subset of $\mathbb{R}^d$ containing Ω. For the precise details of the construction of the extension operator, the reader should consult [48]. Also at our disposal is a seminorm version:

Theorem 3.1.5 (Light & Vail [48]). *Let $\Omega \subset \mathbb{R}^d$ be a V-domain. Let $\widehat{w} : \mathbb{R}^d \to \mathbb{R}$ satisfy (W6)–(W11). Given $f \in \mathcal{Y}^m(\Omega)$ there exists a function $f^\Omega \in \mathcal{Y}^m(\mathbb{R}^d)$ such that:*

1. $f^\Omega = f$ *on* Ω;

2. $|f^\Omega|_{m,w,\mathbb{R}^d} \le C|f|_{m,w,\Omega}$, *for some positive constant* $C = C(\Omega)$ *independent of* f.

3.1.1 A compact embedding theorem

The purpose of this section of work is to replace appropriately the Rellich selection theorem that we employed without hesitancy in the proof of Lemma 2.1.4. That theorem states that $W_2^m(\Omega)$ is compactly embedded in $W_2^{m-1}(\Omega)$. The analogous result we desire is that $\mathcal{X}^m(\Omega)$ is compactly embedded in $W_2^m(\Omega)$.

Theorem 3.1.6. *Let* $\widehat{w} : \mathbb{R}^d \to \mathbb{R}$ *be a measurable function that is nonpositive almost everywhere and let* $\Omega \subset \mathbb{R}^d$ *be a domain. Then for all* $f \in \mathcal{X}^m(\Omega)$,

$$\|f\|_{m,\Omega} \le \|f\|_{m,w,\Omega}.$$

Proof. Let $f \in \mathcal{X}^m(\Omega)$. Then $\|f\|^2_{m,\Omega} \le \|f\|^2_{m,\Omega} + |f|^2_{m,w,\Omega}$. □

More succinctly, Theorem 3.1.6 states that $\mathcal{X}^m(\Omega)$ is continuously embedded in $W_2^m(\Omega)$. As we have already mentioned, it will be useful in the sequel to know that this embedding is also compact. We remind ourselves that this means that every bounded sequence in $\mathcal{X}^m(\Omega)$ has a convergent subsequence in $W_2^m(\Omega)$.

Lemma 3.1.7. *There exists a constant* $C > 0$ *such that for all* $f \in W_2^m(\mathbb{R}^d)$,

$$\|f\|_{m,\mathbb{R}^d} \le \|(1 + |\cdot|)^m \widehat{f}\|_{L_2(\mathbb{R}^d)} \le C\|f\|_{m,\mathbb{R}^d}.$$

Proof. Let $x \in \mathbb{R}^d$. Then $(1 + |x|)^{2m} = \sum_{j=0}^{2m} \binom{2m}{j} |x|^j$. Using this expansion it follows that,

$$\sum_{|\alpha|\le m} x^{2\alpha} = \sum_{j=0}^{m}\sum_{|\alpha|=j} x^{2\alpha} \le \sum_{j=0}^{m}\sum_{|\alpha|=j} c_\alpha x^{2\alpha}$$
$$= \sum_{j=0}^{m}|x|^{2j} \le \sum_{j=0}^{2m}|x|^{j} \le \sum_{j=0}^{2m}\binom{2m}{j}|x|^{j} = (1+|x|)^{2m}.$$

Furthermore,

$$\begin{aligned}
(1+|x|)^{2m+1} &= \sum_{j=0}^{2m+1}\binom{2m+1}{j}|x|^j \\
&\le \max_{0\le j\le 2m+1}\binom{2m+1}{j}\sum_{j=0}^{2m+1}|x|^j \\
&= \max_{0\le j\le 2m+1}\binom{2m+1}{j}\left(\sum_{j=0}^{m}|x|^{2j}+\sum_{j=0}^{m}|x|^{2j+1}\right) \\
&= \max_{0\le j\le 2m+1}\binom{2m+1}{j}\sum_{j=0}^{m}|x|^{2j}(1+|x|) \\
&= \max_{0\le j\le 2m+1}\binom{2m+1}{j}\sum_{j=0}^{m}\sum_{|\alpha|=j} c_\alpha x^{2\alpha}(1+|x|) \\
&\le \max_{\substack{0\le j\le 2m+1\\ |\alpha|\le m}} c_\alpha\binom{2m+1}{j}\sum_{j=0}^{m}\sum_{|\alpha|=j} x^{2\alpha}(1+|x|) \\
&= \max_{\substack{0\le j\le 2m+1\\ |\alpha|\le m}} c_\alpha\binom{2m+1}{j}\sum_{|\alpha|\le m} x^{2\alpha}(1+|x|).
\end{aligned}$$

Hence, there is a $C>0$ such that for all $x\in\mathbb{R}^d$,

$$\sum_{|\alpha|\le m} x^{2\alpha} \le (1+|x|)^{2m} \le C\sum_{|\alpha|\le m} x^{2\alpha}. \tag{3.1.1}$$

Let $f\in W_2^m(\mathbb{R}^d)$. The Parseval formula together with the relation $\widehat{D^\alpha f} = (\mathrm{i}\,\cdot)^\alpha \widehat{f}$ provides us with the equality

$$\|f\|^2_{m,\mathbb{R}^d} = \sum_{|\alpha|\le m}\int_{\mathbb{R}^d}|(D^\alpha f)(x)|^2\,\mathrm{d}x$$

$$= \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} |(\mathrm{i}x)^\alpha \widehat{f}(x)|^2 \,\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \Big(\sum_{|\alpha|\le m} x^{2\alpha} \Big) |\widehat{f}(x)|^2 \,\mathrm{d}x.$$

Hence, by (3.1.1),

$$\|f\|^2_{m,\mathbb{R}^d} \le \int_{\mathbb{R}^d} (1+|x|)^{2m} |\widehat{f}(x)|^2 \,\mathrm{d}x \le C\|f\|^2_{m,\mathbb{R}^d}. \qquad \square$$

Lemma 3.1.8. *Let $f_1, f_2, \ldots$ form a sequence in $W_2^m(\mathbb{R}^d)$ all of whose supports are contained in a fixed compact set and let $\alpha \in \mathbb{Z}_+^d$. Then there is a $C > 0$ such that for all $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}^d$,*

$$|(D^\alpha \widehat{f_n})(x)| \le C\|f_n\|_{m,\mathbb{R}^d}.$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\phi = 1$ on $\bigcup_{n\in\mathbb{Z}_+} \operatorname{supp}(f_n)$. Fix $n \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^d$. Then

$$D^\alpha \widehat{f_n} = D^\alpha(\widehat{\phi f_n}) = D^\alpha(\widehat{\phi} * \widehat{f_n}) = (D^\alpha \widehat{\phi}) * \widehat{f_n}.$$

Using the Cauchy–Schwarz inequality it follows that, for $x \in \mathbb{R}^d$,

$$\begin{aligned}
|(D^\alpha \widehat{f_n})(x)|^2 &= \left| \int_{\mathbb{R}^d} (D^\alpha \widehat{\phi})(x-y) \widehat{f_n}(y)\ \mathrm{d}y \right|^2 \\
&= \left| \int_{\mathbb{R}^d} (1+|y|)^{-m} (D^\alpha \widehat{\phi})(x-y)(1+|y|)^m \widehat{f_n}(y)\ \mathrm{d}y \right|^2 \\
&\le \int_{\mathbb{R}^d} (1+|y|)^{-2m} |(D^\alpha \widehat{\phi})(x-y)|^2 \,\mathrm{d}y \int_{\mathbb{R}^d} (1+|y|)^{2m} |\widehat{f_n}(y)|^2\ \mathrm{d}y \\
&\le C_1 \|f_n\|^2_{m,\mathbb{R}^d} \int_{\mathbb{R}^d} (1+|y|)^{-2m} |(D^\alpha \widehat{\phi})(x-y)|^2 \,\mathrm{d}y,
\end{aligned}$$

for an appropriate number $C_1 > 0$. For $y \in \mathbb{R}^d$, $(1+|y|)^{-2m} \le 1$. Therefore, by using a change of variables,

$$
\begin{aligned}
|(D^\alpha \widehat{f_n})(x)|^2 &\le C_1 \|f_n\|^2_{m,\mathbb{R}^d} \int_{\mathbb{R}^d} |(D^\alpha \widehat{\phi})(x-y)|^2 \,\mathrm{d}y \\
&= C_1 \|f_n\|^2_{m,\mathbb{R}^d} \|D^\alpha \widehat{\phi}\|_{L_2(\mathbb{R}^d)}. \qquad \square
\end{aligned}
$$

The following theorem is the mean value theorem in $\mathbb{R}^d$, a proof of which can be found in almost all elementary texts on real analysis.

Theorem 3.1.9 (Mean value theorem [56, Page 121])**.** *Let f be a continuous real-valued and differentiable function defined on $\mathbb{R}^d$. Then there is a $0 < t < 1$ such that*

$$
f(y) - f(x) = \sum_{j=1}^{d} \Big(\frac{\partial f}{\partial x_j}\Big)(x + t(y-x))(y_j - x_j).
$$

We also take this opportunity to state another classical result from real analysis—the Arzelá–Ascoli theorem. Our chosen reference for this theorem is again McShane and Botts [56].

Theorem 3.1.10 (Arzelá–Ascoli theorem [56, Page 92])**.** *Let K be a compact subset of $\mathbb{R}^d$ and let $\mathcal{F}$ be a collection of real-valued functions defined on K. If $\mathcal{F}$ is uniformly bounded and equicontinuous on K, then every sequence of functions in $\mathcal{F}$ contains a uniformly convergent subsequence.*

Remember we say that a collection $\mathcal{F}$ of real-valued functions defined on Ω is *equicontinuous* if to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for every $f \in \mathcal{F}$ and all pairs of points $x, y \in \Omega$ with $|x - y| < \delta$. It is the Arzelá–Ascoli theorem that offers us the opportunity to find the convergent subsequence that will establish the sought after compact embedding. In the next lemma we talk about a collection being *uniformly Lipschitz.* This means that each member of the collection is Lipschitz continuous, and that the same Lipschitz constant can be used for each member. A collection which is uniformly Lipschitz is automatically equicontinuous.

Lemma 3.1.11. *Let K be a compact subset of $\mathbb{R}^d$. Let $\{f_n\}$ be a collection of real-valued functions in $C^1(\mathbb{R}^d)$. Suppose that for all $|\alpha| = 1$, the collection $\{D^\alpha f_n\}$ is uniformly bounded on compact subsets of $\mathbb{R}^d$. Then $\{f_n\}$ is uniformly Lipschitz on K; hence, $\{f_n\}$ is equicontinuous on K.*

Proof. Let $x, y \in K$ and $n \in \mathbb{Z}_+$. By the mean value theorem in $\mathbb{R}^d$ (Theorem 3.1.9) there is a $0 < t < 1$ such that

$$|f_n(y) - f_n(x)| \leq \sum_{j=1}^{d} \Big| \Big(\frac{\partial f_n}{\partial x_j}\Big)(x + t(y-x)) \Big| |y_j - x_j|.$$

The line segment $\{x + t(y-x) : \ 0 \leq t \leq 1\}$ is a compact subset of $\mathbb{R}^d$, so by assumption there is an appropriate $C > 0$ such that

$$|f_n(y) - f_n(x)| \leq C \sum_{j=1}^{d} |y_j - x_j| \leq C\sqrt{d}|y - x|.$$

Hence, each f_n satisfies a Lipschitz condition with constant $C\sqrt{d}$ on K. Now, let $\varepsilon > 0$ and choose $\delta = \varepsilon / C\sqrt{d}$. Then, $|f_n(y) - f_n(x)| < \varepsilon$ whenever $|x - y| < \delta$. □

Lemma 3.1.12. *Let $f_1, f_2, \ldots$ form a sequence in $W_2^m(\mathbb{R}^d)$ all of whose supports are contained in some fixed compact set. Let us assume there is a $B > 0$ such that $\|f_n\|_{m,\mathbb{R}^d} \leq B$ for all $n \in \mathbb{Z}_+$. Let K be a compact subset of $\mathbb{R}^d$. Then $\{\widehat{f_n}\}$ contains a subsequence uniformly convergent on K.*

Proof. Let K be a compact subset of $\mathbb{R}^d$ and let $\alpha \in \mathbb{Z}_+^d$. Using the uniform boundedness of $\|f_n\|_{m,\mathbb{R}^d}$ and Lemma 3.1.8 we may assert the existence of a $C > 0$ such that for all

$n \in \mathbb{Z}_+$ and $x \in \mathbb{R}^d$,

$$|(D^\alpha \widehat{f_n})(x)| \leq C.$$

Hence, $\{D^\alpha \widehat{f_n}\}$ is uniformly bounded on $\mathbb{R}^d$. Now, Lemma 3.1.11 states that $\{\widehat{f_n}\}$ is equicontinuous on K. Finally, an application of the Arzelá–Ascoli theorem (Theorem 3.1.10) delivers the sought after uniformly convergent subsequence. □

Theorem 3.1.13. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W7). Let $f_1, f_2, \ldots$ form a sequence in $\mathcal{X}^m(\mathbb{R}^d)$ all of whose supports are contained in some fixed compact set. Let us assume there is a $B > 0$ such that $\|f_n\|_{m,w,\mathbb{R}^d} \leq B$ for all $n \in \mathbb{Z}_+$. Then $\{f_n\}$ contains a convergent subsequence in $W_2^m(\mathbb{R}^d)$.*

Proof. By Lemma 3.1.12 we know that $\{\widehat{f_n}\}$ contains a subsequence uniformly convergent on the compact set $B(0, r)$. Here, $r > 0$ is a parameter to be chosen shortly. For brevity of our exposition, and without loss of generality, we shall call this subsequence $\{\widehat{f_n}\}$ as well. Now, using Lemma 3.1.7,

$$\|f_s - f_t\|^2_{m,\mathbb{R}^d} \leq \int_{\mathbb{R}^d} (1 + |x|)^{2m} |\widehat{f_s}(x) - \widehat{f_t}(x)|^2 \, \mathrm{d}x.$$

Let $\varepsilon > 0$. Since $\{\widehat{f_n}\}$ converges uniformly on $B(0, r)$, there is an $N > 0$ such that for all $s, t > N$,

$$\int_{B(0,r)} (1 + |x|)^{2m} |\widehat{f_s}(x) - \widehat{f_t}(x)|^2 \, \mathrm{d}x < \frac{\varepsilon^2}{2}. \tag{3.1.2}$$

Furthermore, by using (3.1.1), there is a constant $C_1 > 0$ such that

$$\int_{|x|>r} (1 + |x|)^{2m} |\widehat{f_s}(x) - \widehat{f_t}(x)|^2 \, \mathrm{d}x$$
$$\leq C_1 \sum_{|\alpha| \leq m} \int_{|x|>r} |(D^\alpha (f_s - f_t))\widehat{\ }(x)|^2 \, \mathrm{d}x$$

$$\leq C_1\Big(\int_{|x|>r}(1+|x|)^{2(m-1)}|\widehat{f_s}\,(x)-\widehat{f_t}\,(x)|^2\ \mathrm{d}x+\sum_{|\alpha|=m}\int_{|x|>r}|(D^\alpha(f_s-f_t))^\wedge(x)|^2\ \mathrm{d}x\Big).$$

$$(3.1.3)$$

Let us consider each part of the last inequality separately. Firstly, there is a $C_2>0$ such that,

$$\begin{aligned}\int_{|x|>r}(1+|x|)^{2(m-1)}|\widehat{f_s}\,(x)-\widehat{f_t}\,(x)|^2\ \mathrm{d}x &\leq r^{-2}\int_{|x|>r}(1+|x|)^{2m}|\widehat{f_s}\,(x)-\widehat{f_t}\,(x)|^2\ \mathrm{d}x\\ &\leq C_2r^{-2}\|f_s-f_t\|^2_{m,\mathbb{R}^d}. \qquad (3.1.4)\end{aligned}$$

Secondly, using axiom (W3),

$$\begin{aligned}\sum_{|\alpha|=m}\int_{|x|>r}|(D^\alpha(f_s-f_t))^\wedge(x)|^2\,\mathrm{d}x &= \sum_{|\alpha|=m}\int_{|x|>r}|(D^\alpha(f_s-f_t))^\wedge(x)|^2w(x)(w(x))^{-1}\,\mathrm{d}x\\ &\leq C_3r^{-2\mu}\sum_{|\alpha|=m}\int_{|x|>r}|(D^\alpha(f_s-f_t))^\wedge(x)|^2w(x)\,\mathrm{d}x\\ &\leq C_3r^{-2\mu}\sum_{|\alpha|=m}\int_{\mathbb{R}^d}|(D^\alpha(f_s-f_t))^\wedge(x)|^2w(x)\,\mathrm{d}x,\end{aligned}$$

for an appropriate $C_3>0$. By assumption we are in a position whereby we can move freely to the direct form of the seminorm (3.0.1). Hence

$$\sum_{|\alpha|=m}\int_{|x|>r}|(D^\alpha(f_s-f_t))^\wedge(x)|^2\,\mathrm{d}x \leq C_3r^{-2\mu}|f_s-f_t|^2_{m,w,\mathbb{R}^d}. \qquad (3.1.5)$$

Now, by combining (3.1.3), (3.1.4) and (3.1.5), there is a $C_4>0$ such that

$$\int_{|x|>r}(1+|x|)^{2m}|\widehat{f_s}\,(x)-\widehat{f_t}\,(x)|^2\,\mathrm{d}x \leq C_4(r^{-2}+r^{-2\mu})\|f_s-f_t\|^2_{m,w,\mathbb{R}^d} \leq 4B^2C_4(r^{-2}+r^{-2\mu}).$$

Set $C_5=4B^2C_4$ and choose $r>\max\{(C_5/\varepsilon^2)^{1/2},(C_5/\varepsilon^2)^{1/2}\}$. It follows that

$$\int_{|x|>r}(1+|x|)^{2m}|\widehat{f_s}\,(x)-\widehat{f_t}\,(x)|^2\ \mathrm{d}x<\frac{\varepsilon^2}{2}. \qquad (3.1.6)$$

Combining (3.1.2) with (3.1.6) we see that, for $s, t > N$,

$$\|f_s - f_t\|_{m,\mathbb{R}^d} < \varepsilon.$$

Hence, $\{f_n\}$ is a Cauchy sequence in $W_2^m(\mathbb{R}^d)$. Since $W_2^m(\mathbb{R}^d)$ is complete the result follows. $\square$

We are finally in a position to state and prove the following compact embedding theorem.

Theorem 3.1.14. *Let $\Omega \subset \mathbb{R}^d$ be a V-domain. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W11). Let $\{f_n\}$ be a sequence in $\mathcal{X}^m(\Omega)$. Suppose there is a $B > 0$ such that $\|f_n\|_{m,w,\Omega} \leq B$ for all $n \in \mathbb{Z}_+$. Then $\{f_n\}$ contains a convergent subsequence in $W_2^m(\Omega)$.*

Proof. Let $\{f_n\}$ be a uniformly bounded sequence in $\mathcal{X}^m(\Omega)$. For each $n \in \mathbb{Z}_+$ let us construct an f_n^Ω in accordance with Theorem 3.1.4. That is, $f_n^\Omega \in \mathcal{X}^m(\mathbb{R}^d)$ such that:

1. $f_n^\Omega = f_n$ on Ω;
2. $\operatorname{supp}(f_n^\Omega)$ is compact and independent of f_n;
3. $\|f_n^\Omega\|_{m,w,\mathbb{R}^d} \leq C\|f_n\|_{m,w,\Omega}$, for some constant $C > 0$ independent of f_n.

Now, $f_1^\Omega, f_2^\Omega, \ldots$ forms a uniformly bounded sequence in $\mathcal{X}^m(\mathbb{R}^d)$ all of whose supports are contained in some fixed compact set. Hence, $\{f_n^\Omega\}$ contains a convergent subsequence in $W_2^m(\mathbb{R}^d)$ by Theorem 3.1.13. We label this subsequence $\{f_n^\Omega\}$ too and let $f \in W_2^m(\mathbb{R}^d)$ be the limit of this sequence. Now, because $\|f_n - f\|_{m,\Omega} = \|f_n^\Omega - f\|_{m,\Omega} \leq \|f_n^\Omega - f\|_{m,\mathbb{R}^d}$, the proof is complete. $\square$

3.1.2 Extension theorems for native spaces

In this section we collect various useful results concerning extension theorems cast on $\mathcal{X}^m(\Omega)$. Fortunately, we already have at our disposal the very useful extension operator of Light and Vail (Theorem 3.1.4). That theorem refers to a norm not too dissimilar to the intrinsic fractional order Sobolev norm. We prefer to work with an alternative equivalent norm. Note that for brevity we shall fix ℓ as the dimension of $\Pi_m(\mathbb{R}^d)$ for the remainder of this chapter, unless otherwise stated.

Lemma 3.1.15. *Let $\Omega \subset \mathbb{R}^d$ be a V-domain. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W11) and let $m + \mu > d/2$. Let $b_1, \ldots, b_\ell \in \Omega$ be unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Define a norm on $\mathcal{X}^m(\Omega)$ via*

$$\|f\|_\Omega = \left(|f|^2_{m,w,\Omega} + \sum_{i=1}^{\ell} |f(b_i)|^2 \right)^{1/2}, \qquad \text{for } f \in \mathcal{X}^m(\Omega).$$

Then there are positive constants K_1 and K_2 such that for all $f \in \mathcal{X}^m(\Omega)$,

$$K_1 \|f\|_{m,w,\Omega} \leq \|f\|_\Omega \leq K_2 \|f\|_{m,w,\Omega}.$$

Proof. The conditions imposed on m and Ω ensure that $\mathcal{X}^m(\Omega)$ is continuously embedded in $C(\Omega)$. So, given $x \in \Omega$, there is a constant C such that $|f(x)| \leq C\|f\|_{m,w,\Omega}$ for all $f \in \mathcal{X}^m(\Omega)$. Thus, there are constants $C_1, \ldots, C_\ell$ such that

$$\|f\|^2_\Omega \leq |f|^2_{m,w,\Omega} + \sum_{i=1}^{\ell} C_i \|f\|^2_{m,w,\Omega} \leq \left(1 + \sum_{i=1}^{\ell} C_i\right) \|f\|^2_{m,w,\Omega}. \tag{3.1.7}$$

On the other hand, suppose there is no positive number K with $\|f\|_{m,w,\Omega} \leq K\|f\|_\Omega$ for all $f \in \mathcal{X}^m(\Omega)$. Then there is a sequence $\{f_j\}$ in $\mathcal{X}^m(\Omega)$ with

$$\|f_j\|_{m,w,\Omega} = 1 \qquad \text{and} \qquad \|f_j\|_\Omega \leq \frac{1}{j}, \qquad \text{for } j = 1, 2, \ldots.$$

Theorem 3.1.14 states that $\mathcal{X}^m(\Omega)$ is compactly embedded in $W_2^m(\Omega)$. Therefore, as $\{f_j\}$ is bounded in $\mathcal{X}^m(\Omega)$, this sequence must contain a convergent subsequence in $W_2^m(\Omega)$. With no loss of generality we shall assume $\{f_j\}$ itself converges in $W_2^m(\Omega)$. Thus $\{f_j\}$ is a Cauchy sequence in $W_2^m(\Omega)$. Next, as $\|f_j\|_\Omega \to 0$ it follows that $|f_j|_{m,w,\Omega} \to 0$. Moreover,

$$
\begin{aligned}
\|f_i - f_j\|_{m,w,\Omega}^2 &= \|f_i - f_j\|_{m,\Omega}^2 + |f_i - f_j|_{m,w,\Omega}^2 \\
&\leq \|f_i - f_j\|_{m,\Omega}^2 + 2|f_i|_{m,w,\Omega}^2 + 2|f_j|_{m,w,\Omega}^2.
\end{aligned}
$$

Since $\{f_j\}$ is a Cauchy sequence in $W_2^m(\Omega)$, and $|f_j|_{m,w,\Omega} \to 0$, it follows that $\{f_j\}$ is a Cauchy sequence in $\mathcal{X}^m(\Omega)$. Since $\mathcal{X}^m(\Omega)$ is complete with respect to $\|\cdot\|_{m,w,\Omega}$, this sequence converges to a limit $f \in \mathcal{X}^m(\Omega)$. By (3.1.7),

$$
\|f - f_j\|_\Omega^2 \leq \left(1 + \sum_{i=1}^{\ell} C_i\right) \|f - f_j\|_{m,w,\Omega}^2,
$$

and hence $\|f - f_j\|_\Omega \to 0$ as $j \to \infty$. Since $\|f_j\|_\Omega \to 0$, it follows that $f = 0$. Because $\|f_j\|_{m,w,\Omega} = 1$, $j = 1, 2, \ldots$, it follows that $\|f\|_{m,w,\Omega} = 1$. This contradiction establishes the result. $\square$

In the above proof we stated, with impunity, that $\mathcal{X}^m(\Omega)$ is continuously embedded in the continuous functions for $m + \mu > d/2$. We feel that we should say something more on this matter. To prove this claim, one can first show that $\mathcal{X}^m(\mathbb{R}^d)$ is continuously embedded in the fractional order Sobolev space $W_2^{m+\mu}(\mathbb{R}^d)$. Then, since $W_2^{m+\mu}(\mathbb{R}^d)$ is continuously embedded in $C(\mathbb{R}^d)$ for $m + \mu > d/2$ (see Adams [1, Page 217]) we obtain the required embedding for $\mathcal{X}^m(\mathbb{R}^d)$. The local result then follows by appealing to the extension operator from Theorem 3.1.4. In more detail, the intrinsic norm on $W_2^{m+\mu}(\mathbb{R}^d)$

can be written in the form

$$\left(\|f\|_{m,\mathbb{R}^d}^2 + \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(D^\alpha f)(x) - (D^\alpha f)(y)|^2}{|x-y|^{2\mu+d}} \, \mathrm{d}x \mathrm{d}y \right)^{1/2}.$$

Now observe that for $f \in \mathcal{X}^m(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(D^\alpha f)(x) - (D^\alpha f)(y)|^2}{|x-y|^{2\mu+d}} \, \mathrm{d}x \mathrm{d}y \\
&\qquad \doteq \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(\widehat{D^\alpha f}\,)(x)|^2 |x|^{2\mu} \, \mathrm{d}x \\
&\qquad = \sum_{|\alpha|=m} c_\alpha \int_{B(0,1)} |(\widehat{D^\alpha f}\,)(x)|^2 |x|^{2\mu} \, \mathrm{d}x + \sum_{|\alpha|=m} c_\alpha \int_{|x|>1} |(\widehat{D^\alpha f}\,)(x)|^2 |x|^{2\mu} \, \mathrm{d}x \\
&\qquad \leq \sum_{|\alpha|=m} c_\alpha \int_{B(0,1)} |(\widehat{D^\alpha f}\,)(x)|^2 \, \mathrm{d}x + C \sum_{|\alpha|=m} c_\alpha \int_{|x|>1} |(\widehat{D^\alpha f}\,)(x)|^2 w(x) \, \mathrm{d}x \\
&\qquad \leq \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha f)(x)|^2 \, \mathrm{d}x + C|f|_{m,w,\mathbb{R}^d}^2,
\end{aligned}$$

for an appropriate generic constant $C > 0$. Hence,

$$\|f\|_{m,\mathbb{R}^d}^2 + \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(D^\alpha f)(x) - (D^\alpha f)(y)|^2}{|x-y|^{2\mu+d}} \, \mathrm{d}x \mathrm{d}y \leq C\|f\|_{m,w,\mathbb{R}^d}^2.$$

Returning to our exposition, we shall shortly wish to understand the behaviour of the constant $K(\Omega)$ in the previous theorem for simple choices of Ω. To realise this, we shall require that our weight function satisfies one further and final axiom:

(W12) there exists C_1, $C_2 > 0$ such that $C_1 h^\lambda \widehat{w}\,(x) \leq \widehat{w}\,(hx) \leq C_2 h^\lambda \widehat{w}\,(x)$, for all $h > 0$, $x \in \mathbb{R}^d$.

Note that when (W7) holds in addition to the above axiom, the number λ in (W12) is understood to be precisely the same λ that appears in (W7).

Lemma 3.1.16. *Let $\widehat{w} : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that is nonpositive almost everywhere and satisfies (W12). Let Ω be a measurable subset of $\mathbb{R}^d$. Define the mapping $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ by $\sigma(x) = a + h(x - t)$, where $h > 0$, and a, t, $x \in \mathbb{R}^d$. Then there exists constants K_1, $K_2 > 0$, independent of Ω, such that for all $f \in \mathcal{Y}^m(\sigma(\Omega))$,*

$$K_1 h^{m-\lambda/2-d} |f|_{m,w,\sigma(\Omega)} \le |f \circ \sigma|_{m,w,\Omega} \le K_2 h^{m-\lambda/2-d} |f|_{m,w,\sigma(\Omega)}.$$

Proof. We have, for $|\alpha| = m$,

$$(D^\alpha(f \circ \sigma))(x) = h^m (D^\alpha f)(\sigma(x)).$$

Thus,

$$\begin{aligned} |f \circ \sigma|^2_{m,w,\Omega} &= -\frac{1}{2} \sum_{|\alpha|=m} c_\alpha \int_\Omega \int_\Omega \widehat{w}\,(x-y) |(D^\alpha(f \circ \sigma))(x) - (D^\alpha(f \circ \sigma))(y)|^2 \, \mathrm{d}x\mathrm{d}y \\ &= -\frac{1}{2} h^{2m} \sum_{|\alpha|=m} c_\alpha \int_\Omega \int_\Omega \widehat{w}\,(x-y) |(D^\alpha f)(\sigma(x)) - (D^\alpha f)(\sigma(y))|^2 \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Now, from the change of variables $u = \sigma(x)$, $v = \sigma(y)$ we obtain

$$|f \circ \sigma|^2_{m,w,\Omega} = \\ -\frac{1}{2} h^{2(m-d)} \sum_{|\alpha|=m} c_\alpha \int_{\sigma(\Omega)} \int_{\sigma(\Omega)} \widehat{w}\,(\sigma^{-1}(u) - \sigma^{-1}(v)) |(D^\alpha f)(u) - (D^\alpha f)(v)|^2 \, \mathrm{d}u\mathrm{d}v.$$

A simple calculation reveals that $\widehat{w}\,(\sigma^{-1}(u) - \sigma^{-1}(v)) = \widehat{w}\,((u-v)/h)$ for all $u, v \in \mathbb{R}^d$. By (W12), there exists C_1, $C_2 > 0$ such that

$$C_1 h^{-\lambda} \widehat{w}\,(u-v) \le \widehat{w}\,((u-v)/h) \le C_2 h^{-\lambda} \widehat{w}\,(u-v), \qquad \text{for all } u, v \in \mathbb{R}^d.$$

Hence,

$$C_1 h^{2(m-d)-\lambda} |f|^2_{m,w,\sigma(\Omega)} \le |f \circ \sigma|^2_{m,w,\Omega} \le C_2 h^{2(m-d)-\lambda} |f|^2_{m,w,\sigma(\Omega)} \qquad \square.$$

Lemma 3.1.17. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12). Let B be any ball of radius h and centre $a \in \mathbb{R}^d$, and let $f \in \mathcal{X}^m(B)$. Whenever $b_1, \ldots, b_\ell \in \mathbb{R}^d$ are unisolvent with respect to $\Pi_m(\mathbb{R}^d)$, let $P_b : C(\mathbb{R}^d) \to \Pi_m(\mathbb{R}^d)$ be the Lagrange interpolation operator on $b_1, \ldots, b_\ell$. Then there exists $c = (c_1, \ldots, c_\ell) \in B^\ell$ and $g \in \mathcal{X}^m(\mathbb{R}^d)$ such that:*

1. *$g(x) = (f - P_c f)(x)$ for all $x \in B$;*
2. *$g(x) = 0$ for all $|x - a| > 2h$;*
3. *there exists a $C > 0$, independent of f and B, such that $|g|_{m,w,\mathbb{R}^d} \le C|f|_{m,w,B}$.*

Furthermore, $c_1, \ldots, c_\ell$ can be arranged so that $c_1 = a$.

Proof. The proof uses the same framework as the proof of Lemma 2.1.5. Let B_1 be the unit ball in $\mathbb{R}^d$ and let $B_2 = 2B_1$. Let $b_1, \ldots, b_\ell \in B_1$ be unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Define $\sigma(x) = h^{-1}(x - a)$ for all $x \in \mathbb{R}^d$. Set $c_i = \sigma^{-1}(b_i)$ for $i = 1, \ldots, \ell$ so that $c_1, \ldots, c_\ell \in B$ are unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Take $f \in \mathcal{X}^m(B)$. Then $(f - P_c f) \circ \sigma^{-1} \in \mathcal{X}^m(B_1)$. Set $F = (f - P_c f) \circ \sigma^{-1}$. Let F^{B_1} be constructed as an extension to F on B_1. By Theorem 3.1.4 and the remark following it, we can assume F^{B_1} is supported on B_2. Define $g = F^{B_1} \circ \sigma \in \mathcal{X}^m(\mathbb{R}^d)$. Let $x \in B$. Since $\sigma(B) = B_1$ there is a $y \in B_1$ such that $x = \sigma^{-1}(y)$. Then,

$$g(x) = (F^{B_1} \circ \sigma)(x) = F^{B_1}(y) = ((f - P_c f) \circ \sigma^{-1})(y) = (f - P_c f)(x).$$

Also, for $x \in \mathbb{R}^d$ with $|x - a| > 2h$, we have $|\sigma(x)| > 2$. Since F^{B_1} is supported on B_2, $g(x) = 0$ for $|x - a| > 2h$. Hence, g satisfies properties 1 and 2. By Theorem 3.1.4 there is a K_1, independent of f and B, such that

$$\|F^{B_1}\|_{m,w,B_2} \le \|F^{B_1}\|_{m,w,\mathbb{R}^d} \le K_1 \|F\|_{m,w,B_1}.$$

We have seen in Lemma 3.1.15 that if we endow $\mathcal{X}^m(B_1)$ and $\mathcal{X}^m(B_2)$ with the norms

$$\|v\|_{B_i} = \left(|v|^2_{m,w,B_i} + \sum_{i=1}^{\ell} |v(b_i)|^2 \right)^{1/2}, \qquad \text{for } i = 1, 2,$$

then $\|\cdot\|_{B_i}$ and $\|\cdot\|_{m,B_i}$ are equivalent for $i = 1, 2$. Thus, there are constants K_2 and K_3, independent of f and B, such that

$$\|F^{B_1}\|_{B_2} \leq K_2 \|F^{B_1}\|_{m,w,B_2} \leq K_1 K_2 \|F\|_{m,w,B_1} \leq K_1 K_2 K_3 \|F\|_{B_1}.$$

Set $C = K_1 K_2 K_3$. Since $F^{B_1}(b_i) = F(b_i) = (f - P_c f)(\sigma^{-1}(b_i)) = (f - P_c f)(c_i) = 0$, for $i = 1, \ldots, \ell$, it follows that $|F^{B_1}|_{m,w,B_2} \leq C |F|_{m,w,B_1}$. Thus, $|g \circ \sigma^{-1}|_{m,w,\mathbb{R}^d} \leq C |(f - P_c f) \circ \sigma^{-1}|_{m,w,B_1}$. Now, Lemma 3.1.16 can be employed twice to provide us with constants C_2 and $C_3 > 0$, independent of f and B, such that

$$\begin{aligned} |g|_{m,w,\mathbb{R}^d} &\leq C_2 h^{d+\lambda/2-m} |g \circ \sigma^{-1}|_{m,w,\mathbb{R}^d} \\ &\leq C_1 C_2 h^{d+\lambda/2-m} |(f - P_c f) \circ \sigma^{-1}|_{m,w,B_1} \leq C_1 C_2 C_3 |f - P_c f|_{m,w,B}. \end{aligned}$$

Finally, we observe that $|f - P_c f|_{m,w,B} = |f|_{m,w,B}$ to complete the first part of the proof. The remaining part follows by selecting $b_1 = 0$ and choosing $b_2, \ldots, b_\ell$ accordingly in the above construction. $\square$

3.2 Error estimates

Let us remind ourselves that the goal of this chapter is to provide error estimates for functions that lie outside the native space of a particular class of interpolants. Specifically, if $f \in \mathcal{Z}^k(\mathbb{R}^d)$, we want to estimate $\|f - S_m f\|_{L_p(\Omega)}$, where S_m is the minimal norm interpolation operator from $\mathcal{Z}^m(\mathbb{R}^d)$ based on $\mathcal{A}$, and $m > k$. The technique that we intend to use hinges on the existence of a fundamental error bound for all functions in $\mathcal{Z}^m(\mathbb{R}^d)$.

3.2.1 Fundamental estimates

The purpose of this section is to establish the fundamental error estimate on which our technique for providing estimates for rough functions is so reliant. Our proof mirrors the framework used by Duchon to establish his fundamental error estimates for the polyharmonic splines.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{R}^d$ be a V-domain. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W11) and let $m + \mu > d/2$. Let us introduce the following pair of well-defined norms on $\mathcal{X}^m(\Omega)/\Pi_m(\Omega)$,*

$$\|f + \Pi_m(\Omega)\|_1 = \inf_{p \in \Pi_m(\Omega)} \|f - p\|_{m,w,\Omega},$$

and

$$\|f + \Pi_m(\Omega)\|_2 = |f|_{m,w,\Omega}.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. Let $f \in \mathcal{X}^m(\Omega)$. Then, for all $p \in \Pi_m(\Omega)$,

$$\|f - p\|^2_{m,w,\Omega} = \|f - p\|^2_{m,\Omega} + |f - p|^2_{m,w,\Omega} \geq |f - p|^2_{m,w,\Omega} = |f|^2_{m,w,\Omega}.$$

Thus, $\|f + \Pi_m(\Omega)\|_2 \leq \|f + \Pi_m(\Omega)\|_1$. On the other hand, by Lemma 3.1.15, for all $g \in \mathcal{X}^m(\Omega)$ there is a $C > 0$, independent of g, such that

$$\|g\|^2_{m,w,\Omega} \leq C\bigg(|g|^2_{m,w,\Omega} + \sum_{i=1}^{\ell} |g(b_i)|^2\bigg),$$

where $b_1, \ldots, b_\ell \in \Omega$ are any fixed set of points which are unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Let $P : C(\mathbb{R}^d) \to \Pi_m(\mathbb{R}^d)$ be the Lagrange interpolation operator based on $b_1, \ldots, b_\ell$. Then,

$$\|f - Pf\|^2_{m,w,\Omega} \le C\bigg(|f - Pf|^2_{m,w,\Omega} + \sum_{i=1}^{\ell} |(f - Pf)(b_i)|^2\bigg) = C|f - Pf|^2_{m,w,\Omega} = C|f|^2_{m,w,\Omega}.$$

Hence, $\|f + \Pi_m(\Omega)\|_1 \le C\|f + \Pi_m(\Omega)\|_2$. $\square$

The norm $\|\cdot\|_1$ appearing in Lemma 3.2.1 is often called the Hilbert quotient norm and makes $\mathcal{X}^m(\Omega)/\Pi_m(\Omega)$ into a Banach space. This will be important when we apply the uniform boundedness theorem (Jameson [39, Page 183] for instance) in the proof of Lemma 3.2.3.

Lemma 3.2.2 (Jameson [39, Pages 72 & 180])**.** *Let S be a normed vector space and T a complete normed space. Let D be a dense linear subspace of S and let Λ_0 be a continuous mapping of D into T. Then there is a unique continuous extension Λ from S to T that extends Λ_0. Further, Λ is linear and $\|\Lambda\| = \|\Lambda_0\|$.*

The above result will also hold when one has a seminormed rather than a normed space; however, in this situation the extension Λ will no longer be uniquely defined.

Lemma 3.2.3. *Let $\Omega \subset \mathbb{R}^d$ be a V-domain. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W11) and let $m + \mu > d/2$. Let $B \subset \Omega^\ell$ be a compact subset of $(\mathbb{R}^d)^\ell$ with the property that if $b = (b_1, \ldots, b_\ell) \in B$ then $b_1, \ldots, b_\ell$ are unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Let $P_b : C(\mathbb{R}^d) \to \Pi_m(\mathbb{R}^d)$ be the Lagrange interpolation operator based on $b_1, \ldots, b_\ell$. Finally, let $p \ge 2$ and $|\alpha| \le m - d/2 + d/p$. Then there exists a $C > 0$ such that for all $f \in \mathcal{Y}^m(\Omega)$*

and $b \in B$,

$$\|D^\alpha(f - P_b f)\|_{L_p(\Omega)} \le C|f|_{m,w,\Omega}.$$

Proof. Let $p_1, \ldots, p_\ell$ denote the Lagrange basis for $\Pi_m(\mathbb{R}^d)$, and for each $b \in B$ let $A(b)$ denote the $\ell \times \ell$ matrix with entries $A(b)_{ij} = p_j(b_i)$. Let $A(b)^{-1}_{ij}$ denote the entries of $A(b)^{-1}$, then

$$P_b f = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} A(b)^{-1}_{ij} f(b_j) p_i, \qquad \text{for all } f \in C(\mathbb{R}^d).$$

Let $f \in \mathcal{X}^m(\Omega)$, then

$$\|f - P_b f\|_{m,w,\Omega} \le \|f\|_{m,w,\Omega} + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} |A(b)^{-1}_{ij}||f(b_j)|\|p_i\|_{m,w,\Omega}.$$

The matrix $A(b)$ depends continuously on b, and the operation of matrix inversion is itself continuous. Therefore, since B is compact, $\{|A(b)^{-1}_{ij}| :\ b \in B\}$ is bounded in $\mathbb{R}$. Similarly, $\{|f(b_j)| :\ b \in B\}$ is bounded in $\mathbb{R}$; thus, for each $f \in \mathcal{X}^m(\Omega)$ there is a $C_1 > 0$ such that $\|f - P_b f\|_{m,w,\Omega} \le C_1$ for all $b \in B$. For each $b \in B$, the mapping

$$T_b :\ \mathcal{X}^m(\Omega)/\Pi_m(\Omega) \to \mathcal{X}^m(\Omega),$$

defined by

$$T_b :\ f + \Pi_m(\Omega) \mapsto f - P_b f,$$

is well-defined, linear and continuous. Now, the uniform boundedness theorem states that there is a $C_2 > 0$ such that

$$\|f - P_b f\|_{m,w,\Omega} \le C_2 \|f + \Pi_m(\Omega)\|_1,$$

for all $f \in \mathcal{X}^m(\Omega)$ and all $b \in B$. An application of Lemma 3.2.1 subsequently provides us with a $C_3 > 0$ such that for all $f \in \mathcal{X}^m(\Omega)$ and all $b \in B$,

$$\|f - P_b f\|_{m,w,\Omega} \le C_3 |f|_{m,w,\Omega}. \tag{3.2.1}$$

If $p \geq 2$ then the condition $|\alpha| - d/p \leq m - d/2$ is precisely the condition needed in the Sobolev embedding theorem (Adams [1, Page 97]) to ensure $W_2^m(\Omega)$ is continuously embedded in $W_p^{|\alpha|}(\Omega)$. Now, it follows by combining (3.2.1) and Theorem 3.1.6 that there is a constant $C_4 > 0$ satisfying

$$\begin{aligned} \|D^\alpha(f - P_b f)\|_{L_p(\Omega)} &\leq C_4 \|f - P_b f\|_{m,\Omega} \\ &\leq C_4 \|f - P_b f\|_{m,w,\Omega} \\ &\leq C_3 C_4 |f|_{m,w,\Omega}, \end{aligned} \tag{3.2.2}$$

for all $f \in \mathcal{X}^m(\Omega)$ and all $b \in B$. Clearly (3.2.2) must also hold for all f in the precomplete space $X^m(\Omega)$. Finally, we are in a position to invoke Lemma 3.2.2 and its subsequent remark to complete the proof. More precisely, in the notation of Lemma 3.2.2, we let $D = X^m(\Omega)$, $S = \mathcal{Y}^m(\Omega)$, $T = W_p^{|\alpha|}(\Omega)$ and $\Lambda_0 : f \mapsto f - P_b f$. □

Lemma 3.2.4 (Light & Wayne [49])**.** *Let $v_1, \ldots, v_\ell \in \mathbb{R}^d$ be unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Then there exists a $\delta > 0$ such that if $(b_1, \ldots, b_\ell) \in B(v_1, \delta) \times \cdots \times B(v_\ell, \delta)$, then $b_1, \ldots, b_\ell$ form a set of unisolvent points with respect to $\Pi_m(\mathbb{R}^d)$.*

The next lemma involves an inequality which one could aptly label "Duchon's inequality". This is because it constitutes, in our setting, a generalised version of the important inequality that Duchon establishes in [22]. Informally, the inequality says that a function with many zeros in a ball cannot get too large. Another generalisation of this important inequality can be found in Narcowich, Ward and Wendland [60] who consider a fractional order Sobolev space setting and more general domains.

Lemma 3.2.5. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12) and let $m + \mu > d/2$. Let $p \geq 2$*

and $|\alpha| \leq m - d/2 + d/p$. There exists an $R > 0$, and for all $M \geq 1$, there exists a $C > 0$ satisfying the following: for each $h > 0$ and $t \in \mathbb{R}^d$, the ball $B(t, Rh)$ contains ℓ closed balls $B_1, \ldots, B_\ell$ each of radius h such that,

$$\|D^\alpha f\|_{L_p(B(t,MRh))} \leq Ch^{m-|\alpha|-\lambda/2-d+d/p}|f|_{m,w,B(t,MRh)},$$

for all $f \in \mathcal{Y}^m(B(t, MRh))$ that vanish at at least one point in each of the balls B_i.

Proof. Take $v_1, \ldots, v_\ell \in \mathbb{R}^d$ to form a set of unisolvent points with respect to $\Pi_m(\mathbb{R}^d)$. Then there is a $\delta > 0$ such that if $(b_1, \ldots, b_\ell) \in B(v_1, \delta) \times \cdots \times B(v_\ell, \delta)$, then $b_1, \ldots, b_\ell$ form a set of unisolvent points with respect to $\Pi_m(\mathbb{R}^d)$ (Lemma 3.2.4). Thus, scaling the points $v_1, \ldots, v_\ell$ by a factor of $1/\delta$ gives us points $u_1, \ldots, u_\ell$ such that, if $(b_1, \ldots, b_\ell) \in B(u_1, 1) \times \cdots \times B(u_\ell, 1)$, then $b_1, \ldots, b_\ell$ form a set of unisolvent points with respect to $\Pi_m(\mathbb{R}^d)$. Choose $R > 0$ such that

$$\bigcup_{i=1}^{\ell} B(u_i, 1) \subset B(0, R).$$

Let $M \geq 1$ and let $f \in \mathcal{Y}^m(B(0, MR))$. By applying Lemma 3.2.3 to the ball centred at 0 of radius MR, there is a $C_1 > 0$, independent of f, such that

$$\|D^\alpha(f - P_b f)\|_{L_p(B(0,MR))} \leq C_1|f|_{m,w,B(0,MR)},$$

for all $b \in B(u_1, 1) \times \cdots \times B(u_\ell, 1)$. Now, let $f \in \mathcal{Y}^m(B(0, MR))$ vanish at at least one point in each $B(u_i, 1)$. There exists a $b_0 \in B(u_1, 1) \times \cdots \times B(u_\ell, 1)$ such that $P_{b_0} f = 0$. Thus

$$\|D^\alpha f\|_{L_p(B(0,MR))} \leq C_1|f|_{m,w,B(0,MR)}.$$

Let $h > 0$ and let $t \in \mathbb{R}^d$. Set $\sigma(x) = (x - t)/h$ for all $x \in \mathbb{R}^d$ and set $B_i = \sigma^{-1}(B(u_i, 1)) = B(hu_i + t, h)$ for $i = 1, \ldots, \ell$, so that each B_i is a closed ball of radius h. Let $x \in B_i$, then $\sigma(x) \in B(u_i, 1) \subset B(0, R)$. Thus, $x \in \sigma^{-1}(B(0, R)) = B(t, Rh)$ so that $B_i \subset B(t, Rh)$. Finally, let $f \in \mathcal{Y}^m(B(t, MRh))$ vanish at at least one point in each B_i. Then $f \circ \sigma^{-1} \in \mathcal{Y}^m(B(0, MR))$ and vanishes at at least one point in each $B(u_i, 1)$. Hence, by systematically applying the change of variable result of Lemma 2.1.3 followed by Lemma 3.1.16, it follows that

$$\begin{aligned} h^{|\alpha|-d/p} \|D^\alpha f\|_{L_p(B(t,MRh))} &= \|D^\alpha (f \circ \sigma^{-1})\|_{L_p(B(0,MR))} \\ &\leq C_1 |f \circ \sigma^{-1}|_{m,w,B(0,MR)} \\ &\leq C_1 C_2 h^{m-\lambda/2-d} |f|_{m,w,B(t,MRh)}, \end{aligned}$$

for some constant $C_2 > 0$, independent of f. $\square$

It is worth commenting that one only obtains a useful estimate in the previous lemma when $-\lambda/2 - d/2 \geq 0$ (in other words when $\lambda + d \leq 0$).

Theorem 3.2.6 (Light and Wayne [50])**.** *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W0) and (W1). Then the set $\{f \in \mathcal{Z}^m(\mathbb{R}^d) : \widehat{f} \in C_0^\infty(\mathbb{R}^d)\}$ is dense in $\mathcal{Z}^m(\mathbb{R}^d)$.*

Lemma 3.2.7. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1) and (W4)–(W7). Let Ω be a bounded domain in $\mathbb{R}^d$ and let $f \in \mathcal{Z}^m(\mathbb{R}^d)$. Then $f|_\Omega \in \mathcal{Y}^m(\Omega)$.*

Proof. Let $f \in \mathcal{Z}^m(\mathbb{R}^d)$. By virtue of Theorem 3.2.6, for each $n \in \mathbb{Z}_+$ there is a $g_n \in \mathscr{S} \cap \mathcal{Z}^m(\mathbb{R}^d)$ with $|f - g_n|_{m,w,\mathbb{R}^d} < 1/n$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\phi = 1$ on Ω and set $f_n = \phi g_n \in C_0^\infty(\mathbb{R}^d)$. Now, for each $n \in \mathbb{Z}_+$, $|f_n|_{m,w,\Omega} = |g_n|_{m,w,\Omega} \leq |g_n|_{m,w,\mathbb{R}^d} < \infty$. Thus, $\{f_n\}$ is a sequence in $X^m(\Omega)$. Let $\varepsilon > 0$ and choose $s, t > 2/\varepsilon$, then we have

$$\begin{aligned} |f_s - f_t|_{m,w,\Omega} = |g_s - g_t|_{m,w,\Omega} &\leq |g_s - g_t|_{m,w,\mathbb{R}^d} \\ &\leq |f - g_s|_{m,w,\mathbb{R}^d} + |f - g_t|_{m,w,\mathbb{R}^d} < 1/s + 1/t < \varepsilon. \end{aligned}$$

Hence, $\{f_n\}$ is a Cauchy sequence in $X^m(\Omega)$. Furthermore, $f|_\Omega$ is a limit of this sequence; hence, $f|_\Omega \in \mathcal{Y}^m(\Omega)$. $\square$

It is now time to demonstrate the purpose of all our preparatory steps.

Theorem 3.2.8. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12) and let $m + \mu > d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_m(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $\mathcal{Z}^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{m-|\alpha|-\lambda/2-d+d/p} |f - S_m^h f|_{m,w,\mathbb{R}^d}, \qquad |\alpha| \le m - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{m-|\alpha|-\lambda/2-d/2} |f - S_m^h f|_{m,w,\mathbb{R}^d}, \qquad |\alpha| \le m - d/2,$$

for all $f \in \mathcal{Y}^m(\mathbb{R}^d)$, as $h \to 0$.

Proof. We choose to deal with the case $2 \le p < \infty$ first, so we fix $|\alpha| \le m - d/2 + d/p$. Let us begin by invoking Lemma 3.2.5. So, there exists an $R > 0$, and for all $M \ge 1$, there exists a $C_1 > 0$ satisfying the following: for each $h > 0$, $t \in \mathbb{R}^d$ the ball $B(t, Rh)$ contains ℓ closed balls $B_1, \ldots, B_\ell$ each of radius h such that

$$\|D^\alpha f\|_{L_p(B(t,MRh))} \le C_1 h^{m-|\alpha|-\lambda/2-d+d/p} |f|_{m,w,B(t,MRh)}, \tag{3.2.3}$$

for all $f \in \mathcal{Y}^m(B(t, MRh))$ that vanish at at least one point in each of the balls B_i. Now, let us invoke Lemma 2.2.10 with Rh in place of h. Then there are constants M_1, M_2, M_3 and h_0 such that for each $0 < h < h_0/R$ there corresponds a set $T_h \subset \Omega$ such that:

1. $B(t, Rh) \subset \Omega$ for all $t \in T_h$;

2. $\Omega \subset \bigcup_{t \in T_h} B(t, M_1 Rh)$;

3. $\sum_{t \in T_h} \chi_{B(t, M_1 Rh)} \leq M_2$;

4. $|T_h| \leq M_3 h^{-d}$.

Fix $0 < h < h_0/R$ and $t \in T_h$. Then $B(t, Rh) \subset \Omega$ contains ℓ balls of radius h. Since $\sup_{x \in \overline{\Omega}} \min_{a \in \mathcal{A}_h} |x - a| \leq h$, each of these balls contains at least one member of $\mathcal{A}_h$. Let $f \in \mathcal{Y}^m(\mathbb{R}^d)$. Then the function $f - S_m^h f \in \mathcal{Z}^m(\mathbb{R}^d)$ vanishes on $\mathcal{A}_h$, and by Lemma 3.2.7 we are assured $(f - S_m^h f)|_{B(t, M_1 Rh)} \in \mathcal{Y}^m(B(t, M_1 Rh))$. Hence,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(B(t, M_1 Rh))}^p \leq C_1^p h^{(m - |\alpha| - \lambda/2 - d + d/p)p} |f - S_m^h f|_{m, w, B(t, M_1 Rh)}^p. \tag{3.2.4}$$

As a consequence of property 2 and (3.2.4) we obtain

$$\begin{aligned} \|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)}^p &\leq \sum_{t \in T_h} \|D^\alpha(f - S_m^h f)\|_{L_p(B(t, M_1 Rh))}^p \\ &\leq C_1^p h^{(m - |\alpha| - \lambda/2 - d + d/p)p} \sum_{t \in T_h} |f - S_m^h f|_{m, w, B(t, M_1 Rh)}^p. \end{aligned}$$

Set $v(x, y) = -\widehat{w}\,(x - y)/2$. Using the fact that if $b \in \mathbb{R}^n$ then $\|b\|_p \leq \|b\|_2$, for $2 \leq p \leq \infty$, we have

$$\begin{aligned} &\left(\sum_{t \in T_h} |f - S_m^h f|_{m, w, B(t, M_1 Rh)}^p \right)^{2/p} \\ &\leq \sum_{t \in T_h} \sum_{|\alpha| = m} c_\alpha \\ &\qquad \int_{B(t, M_1 Rh)} \int_{B(t, M_1 Rh)} v(x, y) |(D^\alpha(f - S_m^h f))(x) - (D^\alpha(f - S_m^h f))(y)|^2 \,\mathrm{d}x \mathrm{d}y \\ &\leq \sum_{t \in T_h} \sum_{|\alpha| = m} c_\alpha \int_{\mathbb{R}^d} \int_{B(t, M_1 Rh)} v(x, y) |(D^\alpha(f - S_m^h f))(x) - (D^\alpha(f - S_m^h f))(y)|^2 \,\mathrm{d}x \mathrm{d}y \end{aligned}$$

$$= \sum_{t\in T_h} \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B(t,M_1Rh)}(x)$$

$$v(x,y)|(D^\alpha(f - S_m^h f))(x) - (D^\alpha(f - S_m^h f))(y)|^2 \,\mathrm{d}x\mathrm{d}y.$$

$$= \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{t\in T_h} \chi_{B(t,M_1Rh)}(x) \right)$$

$$v(x,y)|(D^\alpha(f - S_m^h f))(x) - (D^\alpha(f - S_m^h f))(y)|^2 \,\mathrm{d}x\mathrm{d}y.$$

Continuing, it follows from property 3 that,

$$\sum_{t\in T_h} |f - S_m^h f|^p_{m,w,B(t,M_1Rh)} \leq M_2^{p/2} |f - S_m^h f|^p_{m,w,\mathbb{R}^d}.$$

Hence, upon setting $C_2 = C_1 M_2^{1/2}$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \leq C_2 h^{m-|\alpha|-\lambda/2-d+d/p} |f - S_m^h f|_{m,w,\mathbb{R}^d}.$$

We fix $|\alpha| \leq m - d/2$ for the remainder of the proof. For the case $p = \infty$ we simply observe that

$$\|D^\alpha(f - S_m^h f)\|_{L_\infty(\Omega)} \leq \sup_{t\in T_h} \|D^\alpha(f - S_m^h f)\|_{L_\infty(B(t,M_1Rh)}.$$

Now, we can use (3.2.3), which is valid for $p = \infty$, to obtain

$$\begin{aligned}
\|D^\alpha(f - S_m^h f)\|_{L_\infty(\Omega)} &\leq C_1 h^{m-|\alpha|-\lambda/2-d} \sup_{t\in T_h} |f - S_m^h f|_{m,w,B(t,M_1Rh)} \\
&\leq C_1 h^{m-|\alpha|-\lambda/2-d} \left(\sum_{t\in T_h} |f - S_m^h f|^2_{m,w,B(t,M_1Rh)} \right)^{1/2} \\
&\leq C_2 h^{m-|\alpha|-\lambda/2-d} |f - S_m^h f|_{m,w,\mathbb{R}^d}.
\end{aligned}$$

The remaining part of the theorem concerns the case $1 \leq p < 2$. Here, we have

$$\|D^\alpha(f - S_m^h f)\|^p_{L_p(\Omega)} \leq \sum_{t\in T_h} \int_{B(t,M_1Rh)} |(D^\alpha(f - S_m^h f))(x)|^p \,\mathrm{d}x.$$

Employing the L_∞-bound in (3.2.3) once again, we find that

$$\begin{aligned}\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)}^p &\le C_1^p h^{(m-|\alpha|-\lambda/2-d)p} \sum_{t\in T_h} \mathrm{vol}(B(t, M_1Rh))|f - S_m^h f|_{m,w,B(t,M_1Rh)}^p \\ &\le C_1^p C_3 h^{(m-|\alpha|-\lambda/2-d+d/p)p} \sum_{t\in T_h} |f - S_m^h f|_{m,w,B(t,M_1Rh)}^p, \qquad (3.2.5)\end{aligned}$$

for an appropriate constant $C_3 > 0$, independent of h and f. Let r be the number satisfying $p/2 + 1/r = 1$. Now, using Hölder's inequality,

$$\begin{aligned}\sum_{t\in T_h} |f - S_m^h f|_{m,w,B(t,M_1Rh)}^p &\le \left(\sum_{t\in T_h} 1^r\right)^{1/r} \left(\sum_{t\in T_h} \left(|f - S_m^h f|_{m,w,B(t,M_1Rh)}^p\right)^{2/p}\right)^{p/2} \\ &\le \left(\sum_{t\in T_h} 1^r\right)^{1/r} \left(\sum_{t\in T_h} |f - S_m^h f|_{m,w,B(t,M_1Rh)}^2\right)^{p/2} \\ &= |T_h|^{1/r} \left(\sum_{t\in T_h} |f - S_m^h f|_{m,w,B(t,M_1Rh)}^2\right)^{p/2}.\end{aligned}$$

We have already established in this proof that

$$\sum_{t\in T_h} |f - S_m^h f|_{m,B(t,M_1Rh)}^2 \le M_2 |f - S_m^h f|_{m,w,\mathbb{R}^d}^2.$$

As for $|T_h|$, we know from property 4 that $|T_h| \le M_3 h^{-d}$. Set $C_4 = M_2^{p/2} M_3^{1/r}$, then

$$\sum_{t\in T_h} |f - S_m^h f|_{m,w,B(t,M_1Rh)}^p \le C_4 h^{-d/r} |f - S_m^h f|_{m,w,\mathbb{R}^d}^p = C_4 h^{(-d/p+d/2)p} |f - S_m^h f|_{m,w,\mathbb{R}^d}^p.$$

Hence, by (3.2.5),

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)}^p \le C_5 h^{(m-|\alpha|-\lambda/2-d/2)p} |f - S_m^h f|_{m,w,\mathbb{R}^d}^p,$$

with $C_5 = C_1^p C_3 C_4$. □

Our fundamental error estimate for $f \in \mathcal{Y}^m(\Omega)$ now follows from the previous theorem by first extending f in accordance with Lemma 3.1.5 and then using the characterising Pythagorean property (1.3.3). Let us summarise this as a corollary.

Corollary 3.2.9. *Assume the hypotheses of Theorem 3.2.8. Suppose, in addition, that Ω is a V-domain. Then there exists a constant $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{m-|\alpha|-\lambda/2-d+d/p} |f|_{m,w,\Omega}, \qquad |\alpha| \le m - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{m-|\alpha|-\lambda/2-d/2} |f|_{m,w,\Omega}, \qquad |\alpha| \le m - d/2,$$

for all $f \in \mathcal{Y}^m(\Omega)$, as $h \to 0$.

The L_p-approximation orders given in Corollary 3.2.9 are not known to be optimal. For one reason this is because the complete version of Theorem 2.2.7 that appears in [68] holds for the class of radial basis functions with algebraically decaying Fourier transforms. Therefore, the theorem reveals the same $d/2$-gap in the necessary and sufficient L_∞-approximation order that appears in the polyharmonic spline setting.

3.2.2 Estimates for rough data

As before, our rough target function $f \in \mathcal{Z}^k(\mathbb{R}^d)$ is adjusted to obtain another function in $\mathcal{Z}^k(\mathbb{R}^d)$, with seminorm in $\mathcal{Z}^k(\mathbb{R}^d)$ close to f. If $k < m$, an interpolant $F \in \mathcal{Z}^m(\mathbb{R}^d)$ to f on $\mathcal{A}$ is then produced by convolving our adjusted function with an appropriate approximate identity. We then apply the fundamental error estimate in $\mathcal{Z}^m(\mathbb{R}^d)$ to $\|F - S_m F\|_{L_p(\Omega)}$ (Corollary 3.2.9) before a standard procedure (Lemma 3.2.10) returns us to an error estimate in $\mathcal{Z}^k(\mathbb{R}^d)$.

Although the following results bare a sizeable resemblance to those we have already seen in Section 2.2, we feel we should nevertheless entertain their proofs where appropriate. This is because the seminorms involved are very different.

Lemma 3.2.10. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W0) and (W1). Let $k \le m$ and $\phi \in C_0^\infty(\mathbb{R}^d)$. For each $h > 0$ let $\phi_h(x) = h^{-d}\phi(x/h)$ for $x \in \mathbb{R}^d$. Then there exists a constant $C > 0$, independent of h, such that for all $f \in \mathcal{Z}^k(\mathbb{R}^d)$,*

$$|\phi_h * f|_{m,w,\mathbb{R}^d} \le Ch^{k-m}|f|_{k,w,\mathbb{R}^d}.$$

Furthermore, we have $|\phi_h * f|_{m,w,\mathbb{R}^d} = o(h^{k-m})$, as $h \to 0$.

Proof. Let $f \in \mathcal{Z}^k(\mathbb{R}^d)$ and $\beta \in \mathbb{Z}_+^d$ with $|\beta| = k$. Then, $(\widehat{D^\beta f}\,)\sqrt{w} \in L_2(\mathbb{R}^d)$. Let us define $(Bf)(x) = f(-x)$ for all $x \in \mathbb{R}^d$. Then $(B((\widehat{D^\beta f}\,)\sqrt{w}))\hat{} \in L_2(\mathbb{R}^d)$. The chain rule for differentiation gives $(D^\gamma\phi_h)(x) = h^{-(d+|\gamma|)}(D^\gamma\phi)(x/h)$ for all $x \in \mathbb{R}^d$, $\gamma \in \mathbb{Z}_+^d$. Thus, for any $\gamma \in \mathbb{Z}_+^d$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(D^\gamma\phi_h * (B((\widehat{D^\beta f}\,)\sqrt{w}))\hat{}\,)(x)|^2 \,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(D^\gamma\phi_h)(x-y)(B((\widehat{D^\beta f}\,)\sqrt{v}))\hat{}\,(y)\,\mathrm{d}y\right|^2 \mathrm{d}x \\
&\qquad = h^{-2(d+|\gamma|)}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(D^\gamma\phi)\Big(\frac{x-y}{h}\Big)(B((\widehat{D^\beta f}\,)\sqrt{w}))\hat{}\,(y)\,\mathrm{d}y\right|^2 \mathrm{d}x \\
&\qquad = h^{-2|\gamma|}\int_{\mathbb{R}^d}\left|\int_K (D^\gamma\phi)(t)(B((\widehat{D^\beta f}\,)\sqrt{w}))\hat{}\,(x-ht)\,\mathrm{d}t\right|^2 \mathrm{d}x,
\end{aligned}$$

where $K = \operatorname{supp}(\phi)$. An application of the Cauchy–Schwarz inequality provides us with the inequality

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(D^\gamma\phi_h * (B((\widehat{D^\beta f}\,)\sqrt{w}))\hat{}\,)(x)|^2 \,\mathrm{d}x \\
&\qquad \le h^{-2|\gamma|}\int_K |(D^\gamma\phi)(t)|^2\,\mathrm{d}t \int_{\mathbb{R}^d}\int_K |(B((\widehat{D^\beta f}\,)\sqrt{w}))\hat{}\,(x-ht)|^2\,\mathrm{d}t\mathrm{d}x.
\end{aligned}$$

Employing the Parseval formula and Fubini's theorem in the previous inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(D^\gamma \phi_h * (B((\widehat{D^\beta f}\,)\sqrt{w}))^\frown)(x)|^2 \,\mathrm{d}x \\
&\qquad \le h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \,\mathrm{d}t \int_K \int_{\mathbb{R}^d} |(B((\widehat{D^\beta f}\,)\sqrt{w}))(x-ht)|^2 \,\mathrm{d}x\mathrm{d}t. \\
&\qquad \le h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \,\mathrm{d}t \int_K \int_{\mathbb{R}^d} |(\widehat{D^\beta f}\,)(y)|^2 w(y) \,\mathrm{d}y \ \mathrm{d}t \\
&\qquad = \mathrm{vol}(K) h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \,\mathrm{d}t \int_{\mathbb{R}^d} |(\widehat{D^\beta f}\,)(y)|^2 w(y) \,\mathrm{d}y, \qquad (3.2.6)
\end{aligned}$$

where we have used the change of variables $y = ht - x$ in the inner integral above. Now, using routine calculations in conjunction with the relation

$$\sum_{|\alpha|=m} c_\alpha x^{2\alpha} = |x|^{2m} = |x|^{2(k+m-k)} = \sum_{|\beta|=k} c_\beta x^{2\beta} \sum_{|\gamma|=m-k} c_\gamma x^{2\gamma},$$

we obtain

$$\begin{aligned}
&\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))^\frown(x)|^2 w(x) \,\mathrm{d}x \\
&\qquad = \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} x^{2\alpha} |(\phi_h * f)^\frown(x)|^2 w(x) \,\mathrm{d}x \\
&\qquad = \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} x^{2\beta} x^{2\gamma} |(\phi_h * f)^\frown(x)|^2 w(x) \,\mathrm{d}x \\
&\qquad = \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma(D^\beta(\phi_h * f)))^\frown(x)|^2 w(x) \,\mathrm{d}x \\
&\qquad = \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)^\frown(x)|^2 w(x) \,\mathrm{d}x.
\end{aligned}$$

Therefore, using the identity

$$(D^\gamma \phi_h * D^\beta f)^\frown \sqrt{w} = (\widehat{D^\gamma \phi_h}\,)(\widehat{D^\beta f}\,)\sqrt{w} = (D^\gamma \phi_h * (B((\widehat{D^\beta f}\,)\sqrt{w}))^\frown)^\frown,$$

and the Parseval formula, we have

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))\widehat{\ }(x)|^2 w(x)\,\mathrm{d}x$$
$$= \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma \phi_h * (B((\widehat{D^\beta f}\,)\sqrt{w}))\widehat{\ })(x)|^2\,\mathrm{d}x. \tag{3.2.7}$$

Combining (3.2.6) with (3.2.7) we deduce that

$$\sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))\widehat{\ }(x)|^2 w(x)\,\mathrm{d}x$$
$$\le \sum_{|\beta|=k} c_\beta \sum_{|\gamma|=m-k} c_\gamma h^{-2|\gamma|} \operatorname{vol}(K) \int_{\mathbb{R}^d} |(D^\gamma\phi)(t)|^2\,\mathrm{d}t \int_{\mathbb{R}^d} |(\widehat{D^\beta f}\,)(y)|^2 w(y)\,\mathrm{d}y$$
$$= \operatorname{vol}(K) h^{2(k-m)} \sum_{|\gamma|=m-k} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma\phi)(t)|^2\,\mathrm{d}t \sum_{|\beta|=k} c_\beta \int_{\mathbb{R}^d} |(\widehat{D^\beta f}\,)(y)|^2 w(y)\,\mathrm{d}y.$$

This deals with the first part of the statement of the lemma, and we have already seen how to deal with the remaining part of the lemma in the proof of Lemma 2.2.3. □

For our own convenience we restate Lemma 2.2.4 below in terms of k instead of $k-1$.

Lemma 3.2.11. *Suppose $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported on the unit ball and satisfies*

$$\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1 \qquad \textit{and} \qquad \int_{\mathbb{R}^d} \phi(x) x^\alpha\,\mathrm{d}x = 0, \qquad \textit{for all } 0 < |\alpha| \le k.$$

For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, let $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$. Let B be any ball of radius h and centre $a \in \mathbb{R}^d$. For a fixed $p \in \Pi_k(\mathbb{R}^d)$ let f be a mapping from $\mathbb{R}^d$ to $\mathbb{R}$ such that $f(x) = p(x)$ for all $x \in B$. Then $(\phi_\varepsilon * f)(a) = p(a)$ for all $\varepsilon \le h$.

Theorem 3.2.12. *Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12) with respect to k. Let $k+\mu > d/2$ and $m \ge k$. Let $\mathcal{A}$ be a finite subset of $\mathbb{R}^d$ of separation $q > 0$. Then for all $f \in \mathcal{X}^k(\mathbb{R}^d)$ there exists an $F \in \mathcal{X}^m(\mathbb{R}^d)$ such that:*

1. *$F(a) = f(a)$ for all $a \in \mathcal{A}$;*

2. *there exists a $C > 0$, independent of f and q, such that $|F|_{k,w,\mathbb{R}^d} \le C|f|_{k,w,\mathbb{R}^d}$ and $|F|_{m,w,\mathbb{R}^d} \le Cq^{k-m}|f|_{k,w,\mathbb{R}^d}$.*

Proof. Take $f \in \mathcal{X}^k(\mathbb{R}^d)$. For each $a \in \mathcal{A}$ let $B_a \subset \mathbb{R}^d$ denote the ball of radius $\delta = q/4$ centred at a. Let $\ell = \dim(\Pi_k(\mathbb{R}^d))$. For each B_a let g_a be constructed in accordance with Lemma 3.1.17. That is, for each $a \in \mathcal{A}$ take $c' = (c_2, \ldots, c_\ell) \in B_a^{\ell-1}$ and $g_a \in \mathcal{X}^k(\mathbb{R}^d)$ such that:

1. $a, c_2, \ldots, c_\ell$ are unisolvent with respect to $\Pi_k(\mathbb{R}^d)$;

2. $g_a(x) = (f - P_{(a,c')}f)(x)$ for all $x \in B_a$;

3. $P_{(a,c')}f \in \Pi_k(\mathbb{R}^d)$ and $(P_{(a,c')}f)(a) = f(a)$;

4. $g_a(x) = 0$ for all $|x - a| > 2\delta$;

5. there exists a $C_1 > 0$, independent of f and B_a, such that $|g_a|_{k,w,\mathbb{R}^d} \le C_1|f|_{k,w,B_a}$.

Note that if $a \neq b$, then $\operatorname{supp}(g_a)$ does not intersect $\operatorname{supp}(g_b)$, because

$$|x - b| > |b - a| - |x - a| \ge 2q - 2\delta = 6\delta,$$

for all $x \in \operatorname{supp}(g_a)$. Let $U = \bigcup_{b\in\mathcal{A}} \operatorname{supp}(g_b)$ and set $v(x, y) = -\widehat{w}\,(x - y)/2$. Then, by splitting up the double integral, we obtain

$$\begin{aligned}
\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{k,w,\mathbb{R}^d} &= \sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(x, y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x) - (D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y \\
&= \sum_{|\alpha|=k} c_\alpha \bigg(\int_{\mathbb{R}^d\setminus U}\int_{\mathbb{R}^d\setminus U} v(x, y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x) - (D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$+\int_{\mathbb{R}^d\setminus U}\int_U v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y$$

$$+\int_U\int_{\mathbb{R}^d\setminus U} v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y$$

$$+\int_U\int_U v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y\Bigg).$$

We now consider separately each of these four double integrals. Firstly, the integral over $\mathbb{R}^d\setminus U\times\mathbb{R}^d\setminus U$ is zero because $\sum_{a\in\mathcal{A}} g_a$ is supported on U. Next, using the observation above regarding the support of g_a it follows that

$$\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d\setminus U}\int_U v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y$$

$$=\sum_{b\in\mathcal{A}}\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d\setminus U}\int_{\mathrm{supp}\,(g_b)} v(x,y)\left|\sum_{a\in\mathcal{A}}(D^\alpha g_a)(x)\right|^2 \mathrm{d}x\mathrm{d}y$$

$$=\sum_{b\in\mathcal{A}}\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d\setminus U}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)|^2 \mathrm{d}x\mathrm{d}y$$

$$=\sum_{b\in\mathcal{A}}\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d\setminus U}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_b)(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$\leq \sum_{b\in\mathcal{A}}|g_b|^2_{k,w,\mathbb{R}^d}.$$

Similarly,

$$\sum_{|\alpha|=k} c_\alpha \int_U\int_{\mathbb{R}^d\setminus U} v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2 \mathrm{d}x\mathrm{d}y \leq \sum_{b\in\mathcal{A}}|g_b|^2_{k,w,\mathbb{R}^d}.$$

Before calculating the final integral let us examine the following expression. Let $b\in\mathcal{A}$ and $\alpha\in\mathbb{Z}^d_+$ with $|\alpha|=k$ be fixed, then

$$\sum_{\substack{c\in\mathcal{A}\\ c\neq b}}\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_c)(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$=\sum_{\substack{c\in\mathcal{A}\\ c\neq b}}\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_b)(y)+(D^\alpha g_c)(x)-(D^\alpha g_c)(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$\leq 2\sum_{\substack{c\in\mathcal{A}\\c\neq b}}\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_b)(y)|^2\,\mathrm{d}x\mathrm{d}y$$

$$+\,2\sum_{\substack{c\in\mathcal{A}\\c\neq b}}\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_c)(x)-(D^\alpha g_c)(y)|^2\,\mathrm{d}x\mathrm{d}y$$

$$\leq 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_b)(y)|^2\,\mathrm{d}x\mathrm{d}y$$

$$+\,2\sum_{\substack{c\in\mathcal{A}\\c\neq b}}\int_{\mathbb{R}^d}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_c)(x)-(D^\alpha g_c)(y)|^2\,\mathrm{d}x\mathrm{d}y.$$

Now, by using our observation regarding the support of g_a once again, it follows that

$$\sum_{|\alpha|=k} c_\alpha\int_U\int_U v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2\mathrm{d}x\mathrm{d}y$$

$$=\sum_{b\in\mathcal{A}}\sum_{c\in\mathcal{A}}\sum_{|\alpha|=k} c_\alpha\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)\left|\sum_{a\in\mathcal{A}}((D^\alpha g_a)(x)-(D^\alpha g_a)(y))\right|^2\mathrm{d}x\mathrm{d}y$$

$$=\sum_{b\in\mathcal{A}}\sum_{c\in\mathcal{A}}\sum_{|\alpha|=k} c_\alpha\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_c)(y)|^2\,\mathrm{d}x\mathrm{d}y$$

$$=\sum_{b\in\mathcal{A}}\sum_{|\alpha|=k} c_\alpha\int_{\mathrm{supp}\,(g_b)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_b)(y)|^2\,\mathrm{d}x\mathrm{d}y$$

$$+\sum_{b\in\mathcal{A}}\sum_{\substack{c\in\mathcal{A}\\c\neq b}}\sum_{|\alpha|=k} c_\alpha\int_{\mathrm{supp}\,(g_c)}\int_{\mathrm{supp}\,(g_b)} v(x,y)|(D^\alpha g_b)(x)-(D^\alpha g_c)(y)|^2\,\mathrm{d}x\mathrm{d}y$$

$$\leq\sum_{b\in\mathcal{A}}|g_b|^2_{k,w,\mathbb{R}^d}+2\sum_{b\in\mathcal{A}}|g_b|^2_{k,w,\mathbb{R}^d}+2\sum_{c\in\mathcal{A}}|g_c|^2_{k,w,\mathbb{R}^d}$$

$$\leq 5\sum_{b\in\mathcal{A}}|g_b|^2_{k,w,\mathbb{R}^d}.$$

Hence,

$$\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{k,w,\mathbb{R}^d}\leq 7\sum_{a\in\mathcal{A}}|g_a|^2_{k,w,\mathbb{R}^d}.$$

Applying Condition 5 to the above equality we have

$$\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{k,w,\mathbb{R}^d}\leq 7C_1^2\sum_{a\in\mathcal{A}}|f|^2_{k,w,B_a}\leq 7C_1^2|f|^2_{k,w,\mathbb{R}^d}.$$

Now, set $H = f - \sum_{a \in \mathcal{A}} g_a$. It then follows from Condition 1 that $H(x) = (P_{(a,c')}f)(x)$ for all $x \in B_a$, and from Condition 3 that $H(a) = f(a)$ for all $a \in \mathcal{A}$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be supported on the unit ball and enjoy the properties

$$\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \phi(x) x^\alpha \,\mathrm{d}x = 0, \qquad \text{for all } 0 < |\alpha| \le k.$$

Let $F = \phi_\delta * H$. By Lemma 3.2.10, there is a constant $C_2 > 0$, independent of q and f, such that

$$\begin{aligned} |F|^2_{m,w,\mathbb{R}^d} \le C_2 \delta^{2(k-m)} \Big| f - \sum_{a \in \mathcal{A}} g_a \Big|^2_{k,w,\mathbb{R}^d} &\le 2C_2 \delta^{2(k-m)} \left(|f|^2_{k,w,\mathbb{R}^d} + \Big| \sum_{a \in \mathcal{A}} g_a \Big|^2_{k,w,\mathbb{R}^d} \right) \\ &\le 2C_2(1 + 7C_1^2)\delta^{2(k-m)} |f|^2_{k,w,\mathbb{R}^d}. \end{aligned}$$

Similarly, there is a constant $C_3 > 0$, independent of q and f, such that

$$|F|^2_{k,w,\mathbb{R}^d} \le C_3 \Big| f - \sum_{a \in \mathcal{A}} g_a \Big|^2_{k,w,\mathbb{R}^d} \le 2C_3(1 + 7C_1^2)|f|^2_{k,w,\mathbb{R}^d}.$$

Thus, $|F|_{m,w,\mathbb{R}^d} \le C q^{k-m} |f|_{k,w,\mathbb{R}^d}$ and $|F|_{k,w,\mathbb{R}^d} \le C|f|_{k,w,\mathbb{R}^d}$ for some appropriate constant $C > 0$. Finally, because $F = \phi_\delta * H$ and $H|_{B_a} \in \Pi_k(\mathbb{R}^d)$ for each $a \in \mathcal{A}$, it follows from Lemma 3.2.11 that $F(a) = H(a) = f(a)$ for all $a \in \mathcal{A}$. $\square$

Theorem 3.2.13. *Let $\Omega \subset \mathbb{R}^d$ be a V-domain and let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12) with respect to m. Let $k + \mu > d/2$ and $m \ge k$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_m(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h and mesh-ratio ρ. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $\mathcal{Z}^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{k-|\alpha|-\lambda/2-d+d/p} \rho^{m-k} |f|_{k,w,\Omega}, \qquad |\alpha| \le k - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{k-|\alpha|-\lambda/2-d/2} \rho^{m-k} |f|_{k,w,\Omega}, \qquad |\alpha| \le k - d/2,$$

for all $f \in \mathcal{Y}^k(\Omega)$, *as* $h \to 0$.

Proof. Take $f \in \mathcal{X}^k(\mathbb{R}^d)$. Construct F in accordance with Theorem 3.2.12 and set $G = f - F$. Then $F(a) = f(a)$ and $G(a) = 0$ for all $a \in \mathcal{A}_h$. Furthermore, there is a constant $C_1 > 0$, independent of f and h, such that

$$|F|_{m,w,\mathbb{R}^d} \le C_1 \left(\frac{h}{\rho}\right)^{k-m} |f|_{k,w,\mathbb{R}^d}, \tag{3.2.8a}$$

$$|G|_{k,w,\mathbb{R}^d} \le |f|_{k,w,\mathbb{R}^d} + |F|_{k,w,\mathbb{R}^d} \le (1 + C_1)|f|_{k,w,\mathbb{R}^d}. \tag{3.2.8b}$$

Thus $S_m^h f = S_m^h F$ and $S_k^h G = 0$, where we have adopted the obvious notation for S_k^h; hence,

$$\begin{aligned}\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} &= \|D^\alpha((F + G) - S_m^h F)\|_{L_p(\Omega)} \\ &\le \|D^\alpha(F - S_m^h F)\|_{L_p(\Omega)} + \|D^\alpha(G - S_k^h G)\|_{L_p(\Omega)}.\end{aligned}$$

Now, combining the error estimate in Corollary 3.2.9 with the bounds in (3.2.8) we obtain the result for all $f \in \mathcal{X}^k(\mathbb{R}^d)$. In particular, we have established the result for all $f \in X^k(\mathbb{R}^d)$. As $X^k(\mathbb{R}^d)$ is a dense linear subspace of $\mathcal{Y}^k(\mathbb{R}^d)$ the result extends to hold for all $f \in \mathcal{Y}^k(\mathbb{R}^d)$ using a standard normed space argument (Theorem 3.2.2). To complete the proof we merely let $f \in \mathcal{Y}^k(\Omega)$ and define f^Ω in accordance with Theorem 3.1.5. $\square$

Corollary 3.2.14. *With the notation and assumptions of Theorem 3.2.13, suppose there is a quantity* $r > 0$ *such that the mesh-ratio of each* $\mathcal{A}_h$ *is bounded above by* r *for all* $h > 0$.

Then there exists a constant $C > 0$, independent of h, such that, for $2 \le p \le \infty$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le Ch^{k-|\alpha|-\lambda/2-d+d/p}|f|_{k,w,\Omega}, \qquad |\alpha| \le k - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le Ch^{k-|\alpha|-\lambda/2-d/2}|f|_{k,w,\Omega}, \qquad |\alpha| \le k - d/2,$$

for all $f \in \mathcal{Y}^k(\Omega)$, as $h \to 0$.

3.2.3 Auxiliary results

After we established the L_p-estimate for rough functions in Section 2.2, we went on to generate ℓ_p-estimates and also prove the boundedness of the associated interpolation operator. For the ℓ_p-estimates, the same can be done here in this more general setting. We choose to omit the proof, which follows the same pattern as Theorem 2.2.12, and merely state the result below. However, following the proof pattern of Theorem 2.2.14, in an attempt to show the boundedness of the interpolation operator in this setting, will fail. This is because we do not possess an estimate for the seminorm $|\cdot|_{k,w,\Omega}$ like we did, analogously, in the polyharmonic spline setup.

Theorem 3.2.15. *Let $\Omega \subset \mathbb{R}^d$ be a V-domain and let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12) with respect to m. Let $k + \mu > d/2$ with $m \ge k$ and let $|\alpha| \le k - d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_m(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h and mesh-ratio ρ. Let $\mathcal{B}$ be another finite subset of Ω, with mesh-ratio σ and separation distance $q_{\mathcal{B}} \le h$. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from*

$\mathcal{Z}^m(\mathbb{R}^d)$. *Then there exists a constant $C > 0$, independent of h and $\mathcal{B}$, such that,*

$$\|D^\alpha(f - S_m^h f)\|_{\ell_p(\mathcal{B})} \leq \begin{cases} Ch^{k-|\alpha|-\lambda/2-d+d/p}\rho^{m-k}\sigma^{d/p}|f|_{k,w,\Omega}, & 2 \leq p \leq \infty, \\ Ch^{k-|\alpha|-\lambda/2-d/2}\rho^{m-k}\sigma^{d/p}|f|_{k,w,\Omega}, & 1 \leq p < 2, \end{cases}$$

for all $f \in \mathcal{Y}^k(\Omega)$, as $h \to 0$.

Corollary 3.2.16. *With the notation and assumptions of Theorem 3.2.15, suppose there is a quantity $r > 0$ such that the mesh-ratio of each $\mathcal{A}_h$ is bounded above by r for all $h > 0$. Then there exists a constant $C > 0$, independent of h and $\mathcal{B}$, such that,*

$$\|D^\alpha(f - S_m^h f)\|_{\ell_p(\mathcal{B})} \leq \begin{cases} Ch^{k-|\alpha|-\lambda/2-d+d/p}\sigma^{d/p}|f|_{k,w,\Omega}, & 2 \leq p \leq \infty, \\ Ch^{k-|\alpha|-\lambda/2-d/2}\sigma^{d/p}|f|_{k,w,\Omega}, & 1 \leq p < 2, \end{cases}$$

for all $f \in \mathcal{Y}^k(\Omega)$, as $h \to 0$.

Finally, we close this chapter by noting that the comment given on Page 48, when rewritten in the weighted Beppo Levi setting, remains valid. That is, the method of proof we employ to establish Theorem 3.2.13 is not limited to interpolation, but can be used to deliver an error estimate for rough functions for a class of approximation schemes as well.

Chapter 4

Interpolation of smooth functions using the scattered shifts of a basis function

Whenever the interpolation points are taken to be an infinite grid of fill-distance h and Ω is taken as the whole of $\mathbb{R}^d$, it can be shown that the m-th order polyharmonic spline interpolant enjoys, in the uniform norm, the convergence order $\mathcal{O}(h^{2m})$, provided f is sufficiently smooth (see Buhmann [11]). Notice that this is around twice the established approximation order that one has in the bounded domain setting. It is owing to edge effects that one loses around half the accuracy near the boundary. This corroborated experimental evidence first reported by Powell and Beatson [61].

The purpose of this chapter is twofold, firstly we would like to show that if certain additional smoothness requirements and boundary conditions are met, the fundamental error estimate can be at least doubled. This is achieved by an elegant trick, reminiscent of the Aubin–Nitsche trick which arises in the theory of partial differential equations (see Ciarlet [16, Page 136]). We consider both the polyharmonic spline and generalised polyharmonic spline settings that have dominated this work so far. Secondly, we add new estimates for classes of functions with smoothness lying, *in some sense*, between the native space and the space of functions that enjoy this super-convergence property.

4.1 Improved error estimates

Although applying the Aubin–Nitsche trick to radial basis functions is not a particularly new idea (see Schaback [66], Wendland [74] and Yoon [82]), we still repeat it here. However, we do so not unnecessarily, for we also close a gap missed in these papers by considering derivatives of the error as well. It is this step that allows us to achieve our second goal of the chapter.

We begin by working in the setting described in Section 3.1. For a domain $\Omega \subset \mathbb{R}^d$ and $\beta \in \mathbb{Z}_+^d$, we define the space $\mathcal{Y}^m(\beta, \Omega)$ by

$$\mathcal{Y}^m(\beta,\Omega) = \left\{ f \in \mathcal{Y}^m(\mathbb{R}^d) : \; g \in L_2(\mathbb{R}^d) \text{ and } \operatorname{supp}(\widehat{g}\,) \subset \Omega \text{ where } g = \frac{|\cdot|^{2m} \widehat{f}\, w}{(\,\cdot\,)^\beta} \right\},$$

and define a seminorm on $\mathcal{Y}^m(\beta,\Omega)$ via

$$|f|_{w^2,\beta} = \left(\sum_{|\alpha|=2m} c_\alpha \int_{\mathbb{R}^d} \frac{|(\widehat{D^\alpha f}\,)(x)|^2 w^2(x)}{x^{2\beta}} \, \mathrm{d}x \right)^{1/2}, \qquad \text{for } f \in \mathcal{Y}^m(\beta,\Omega).$$

A helpful routine calculation reveals that

$$|f|_{w^2,\beta} = \left\| \frac{|\cdot|^{2m} \widehat{f}\, w}{(\,\cdot\,)^\beta} \right\|_{L_2(\mathbb{R}^d)}.$$

Theorem 4.1.1 (Lebesgue's monotone convergence theorem [62, Page 21]). *Let $\{f_j\}$ be a sequence of measurable functions on $\mathbb{R}^d$, and suppose that:*

1. $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$ *for every* $x \in \mathbb{R}^d$*;*

2. $f_n(x) \to f(x)$ *as* $n \to \infty$*, for every* $x \in \mathbb{R}^d$*.*

Then f is measurable, and

$$\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x, \qquad \text{as } n \to \infty.$$

Theorem 4.1.2. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W12) and let $m + \mu > d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_m(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S_m^h f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $\mathcal{Z}^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{2m-|\alpha|-|\beta|-\lambda-3d/2+d/p} |f|_{w^2,\beta}, \qquad |\alpha| \le m - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le C h^{2m-|\alpha|-|\beta|-\lambda-d} |f|_{w^2,\beta}, \qquad |\alpha| \le m - d/2,$$

for all $|\beta| \le m$ and all $f \in \mathcal{Y}^m(\beta, \Omega)$, as $h \to 0$.

Proof. Let $f \in \mathcal{Y}^m(\beta, \Omega)$ and set $E = f - S_m^h f$. By invoking Theorem 3.2.8 we may assert the existence of positive constants C_1 and h_0, independent of f and h, such that, for a fixed $h < h_0$, we have, for $2 \le p \le \infty$,

$$\|D^\alpha E\|_{L_p(\Omega)} \le C_1 h^{m-|\alpha|-\lambda/2-d+d/p} |E|_{m,w,\mathbb{R}^d}, \qquad |\alpha| \le m - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha E\|_{L_p(\Omega)} \le C_1 h^{m-|\alpha|-\lambda/2-d/2} |E|_{m,w,\mathbb{R}^d}, \qquad |\alpha| \le m - d/2,$$

Since $f \in \mathcal{Y}^m(\mathbb{R}^d)$ the interpolant $S_m^h f$ has the characterising Pythagorean property (1.3.3). In other words, for all $|\alpha| = m$,

$$\int_{\mathbb{R}^d} (\widehat{D^\alpha E}\,)(x)(\overline{(D^\alpha(S_m^h f))\widehat{\ }})(x) w(x)\, \mathrm{d}x = 0.$$

Consequently,

$$\begin{aligned} |E|^2_{m,w,\mathbb{R}^d} &= \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} (\widehat{D^\alpha E}\,)(x)(\overline{\widehat{D^\alpha f}}\,)(x) w(x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{f}}\,(x) \widehat{E}\,(x) w(x)\, \mathrm{d}x. \end{aligned}$$

Fix $|\beta| \le m$. If we could establish the inequality

$$\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{f}}\,(x) \widehat{E}\,(x) w(x)\, \mathrm{d}x \le C_2 h^{m-|\beta|-\lambda/2-d/2} |f|_{w^2,\beta} |E|_{m,w,\mathbb{R}^d}, \tag{4.1.1}$$

for an appropriate constant $C_2 > 0$, as $h \to 0$, then the proof would be complete. The remainder of the proof endeavours to do this. Take $F \in \mathscr{S}$ with $\widehat{F} \in C_0^\infty(\mathbb{R}^d)$ and $\operatorname{supp}(\widehat{F}\,) \subset K$. Let $\phi \in C(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{F}}\,(x) \widehat{E}\,(x) \phi(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \left(\frac{|\cdot|^m \overline{\widehat{F}}\,\phi}{\sqrt{w}} \right)(x) (|\cdot|^m \widehat{E}\,\sqrt{w})(x)\, \mathrm{d}x. \tag{4.1.2}$$

Because $E \in \mathcal{Z}^m(\mathbb{R}^d)$ we know that $|\cdot|^m \widehat{E}\,\sqrt{w} \in L_2(\mathbb{R}^d)$. Therefore, the integral (4.1.2) will be finite if we can show that $|\cdot|^m \overline{\widehat{F}}\,\phi/\sqrt{w} \in L_2(\mathbb{R}^d)$. To see this, we recall that

$1/w \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. It then follows that

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|x|^{2m} |\overline{\widehat{F}}(x)\phi(x)|^2}{w(x)}\,\mathrm{d}x &= \int_K \frac{|x|^{2m} |\overline{\widehat{F}}(x)\phi(x)|^2}{w(x)}\,\mathrm{d}x \\ &\le C_1 \int_K \frac{1}{w(x)}\,\mathrm{d}x \\ &< \infty,\end{aligned}$$

for an appropriate constant $C_1 > 0$. Now, let $0 \le w_j \in C^\infty(\mathbb{R}^d)$ form a monotone increasing sequence of functions converging uniformly to w on compact subsets of $\mathbb{R}^d$. Set $G_j = |\cdot|^{2m} \overline{\widehat{F}}\, w_j/(\mathrm{i}\,\cdot)^\beta$. Since (4.1.2) is finite for all $\phi \in C(\mathbb{R}^d)$, there is no problem with the finiteness of the following integral,

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{F}}(x) \widehat{E}(x) w_j(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} G_j(x)(\mathrm{i}x)^\beta \widehat{E}(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} G_j(x)(\widehat{D^\beta E})(x)\,\mathrm{d}x.\end{aligned}$$

For $\Lambda \in \mathscr{S}'$ we denote the action of Λ on $\phi \in \mathscr{S}$ by $[\Lambda, \phi]$. The function $\widehat{D^\beta E}$ is locally integrable except in a neighbourhood of the origin. To see this we observe that

$$\widehat{D^\beta E} = (\mathrm{i}\,\cdot)^\beta \widehat{E} = \frac{(\mathrm{i}\,\cdot)^\beta}{(\mathrm{i}\,\cdot)^\alpha} (\mathrm{i}\,\cdot)^\alpha \widehat{E} = \frac{(\mathrm{i}\,\cdot)^\beta}{(\mathrm{i}\,\cdot)^\alpha} \widehat{D^\alpha E},$$

and acknowledge that $\widehat{D^\alpha E} \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. Furthermore, the singularity is not too bad since the function $(\,\cdot\,)^{\alpha-\beta} \widehat{D^\beta E}$ is integrable near the origin. A feature of how such functions are viewed as distributions is as follows; if the integral of the distribution against a particular test function exists, then this is precisely the action of that distribution on that test function (see Donoghue [18, Page 114] for details of the analytic continuation of distributions). Hence,

$$\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{F}}(x) \widehat{E}(x) w_j(x)\,\mathrm{d}x = [\widehat{D^\beta E}, G_j] = [D^\beta E, \widehat{G_j}].$$

The distribution $D^\beta E$ is in $L_1^{\text{loc}}(\mathbb{R}^d)$; therefore,

$$\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{F}}(x) \widehat{E}(x) w_j(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{G_j}(x)(D^\beta E)(x)\,\mathrm{d}x.$$

Hence, by Lebesgue's monotone convergence theorem (Theorem 4.1.1),

$$\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{F}}(x) \widehat{E}(x) w(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{G}(x)(D^\beta E)(x)\,\mathrm{d}x,$$

where we have set $G = |\cdot|^{2m} \overline{\widehat{F}}\, w/(\mathrm{i}\,\cdot)^\beta$. Now, by virtue of the density result (Theorem 3.2.6), we can replace F by f in the previous equality and obtain

$$\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{f}}(x) \widehat{E}(x) w(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{g}(x)(D^\beta E)(x)\,\mathrm{d}x,$$

with $g = |\cdot|^{2m} \overline{\widehat{f}}\, w/(\mathrm{i}\,\cdot)^\beta$. By assumption $\operatorname{supp}(\widehat{g}) \subset \Omega$. Thus, by using the Cauchy–Schwarz inequality and Parseval formula, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |x|^{2m} \overline{\widehat{f}}(x) \widehat{E}(x) w(x)\,\mathrm{d}x &\le \left(\int_\Omega |\widehat{g}(x)|^2\,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |(D^\beta E)(x)|^2\,\mathrm{d}x\right)^{1/2} \\
&= \left\| \frac{|\cdot|^{2m} \overline{\widehat{f}}\, w}{(\mathrm{i}\,\cdot)^\beta} \right\|_{L_2(\mathbb{R}^d)} \|D^\beta E\|_{L_2(\Omega)} \\
&= \left\| \frac{|\cdot|^{2m} \widehat{f}\, w}{(\,\cdot\,)^\beta} \right\|_{L_2(\mathbb{R}^d)} \|D^\beta E\|_{L_2(\Omega)} \\
&= |f|_{w^2,\beta} \|D^\beta E\|_{L_2(\Omega)}.
\end{aligned}$$

Finally, applying Theorem 3.2.8 to the term $\|D^\beta E\|_{L_2(\Omega)}$ in the previous inequality establishes (4.1.1). □

Theorem 4.1.2 yields two interesting corollaries. The first of these is the error doubling result that we have already alluded to. This follows by setting $\beta = \alpha$ in Theorem 4.1.2. The second result, although not strictly speaking a corollary, provides at most three halves of the approximation order predicated by the fundamental error estimate.

Corollary 4.1.3. *Under the same assumptions as Theorem 4.1.2, there exists a $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le Ch^{2(m-|\alpha|-\lambda/2-d/2)-d/2+d/p}|f|_{w^2,\alpha}, \qquad |\alpha| \le m - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le Ch^{2(m-|\alpha|-\lambda/2-d/2)}|f|_{w^2,\alpha}, \qquad |\alpha| \le m - d/2,$$

for all $f \in \mathcal{Y}^m(\alpha, \Omega)$, as $h \to 0$.

Corollary 4.1.4. *Assume the hypotheses of Theorem 4.1.2. Let $n \ge 2$ and suppose $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}_+^d$ with $|\alpha_2|, \ldots, |\alpha_n| \le m$. Then there exists a $C > 0$, independent of h, such that, for $2 \le p \le \infty$,*

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le Ch^{\sigma(h)-d/2+d/p}|f|_{m,w,\mathbb{R}^d}^{1/2^{n-1}} \prod_{j=2}^{n} |f|_{w^2,\alpha_j}^{1/2^{j-1}}, \qquad |\alpha| \le m - d/2 + d/p,$$

and, for $1 \le p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \le Ch^{\sigma(h)}|f|_{m,w,\mathbb{R}^d}^{1/2^{n-1}} \prod_{j=2}^{n} |f|_{w^2,\alpha_j}^{1/2^{j-1}}, \qquad |\alpha| \le m - d/2,$$

for all $f \in \bigcap_{j=2}^n \mathcal{Y}^m(\alpha_j, \Omega)$, as $h \to 0$. Where $\sigma(h) = \sum_{j=1}^n \frac{m-|\alpha_j|-\lambda/2-d/2}{2^{j-1}}$.

Proof. We give the proof for $2 \le p \le \infty$ and comment that the case $1 \le p < 2$ is established similarly. Follow the proof of Theorem 4.1.2 until we arrive at the inequality

$$|f - S_m^h f|_{m,w,\mathbb{R}^d}^2 \le |f|_{w^2,\beta}\|D^\beta(f - S_m^h f)\|_{L_2(\mathbb{R}^d)}, \qquad \text{for all } |\beta| \le m. \tag{4.1.3}$$

Let $|\alpha_1| \le m - d/2 + d/p$, $|\alpha_2|, \ldots, |\alpha_n| \le m$ and $f \in \bigcap_{j=2}^n \mathcal{Y}^m(\alpha_j, \Omega)$. It follows by combining (4.1.3) with Theorem 3.2.8 that there exits a $C_1 > 0$ and $h_1 > 0$, independent of h and f, such that, for $h < h_1$,

$$\|D^{\alpha_1}(f - S_m^h f)\|^2_{L_p(\Omega)} \le C_1 h^{2(m-|\alpha_1|-\lambda/2-d+d/p)} |f|_{w^2,\alpha_2} \|D^{\alpha_2}(f - S_m^h f)\|_{L_2(\Omega)},$$

Similarly, there exists appropriate numbers $h_2 > 0$ and $C_2 > 0$ such that, for all $h < h_2$,

$$\|D^{\alpha_2}(f - S_m^h f)\|^2_{L_2(\Omega)} \le C_2 h^{2(m-|\alpha_2|-\lambda/2-d/2)} |f|_{w^2,\alpha_3} \|D^{\alpha_3}(f - S_m^h f)\|_{L_2(\Omega)}.$$

If we continue to split in this manner until we can no longer split anymore, we end up with a total of $n-1$ inequalities, the last of which being, for $h < h_{n-1}$,

$$\|D^{\alpha_{n-1}}(f - S_m^h f)\|^2_{L_2(\Omega)} \le C_{n-1} h^{2(m-|\alpha_{n-1}|-\lambda/2-d/2)} |f|_{w^2,\alpha_n} \|D^{\alpha_n}(f - S_m^h f)\|_{L_2(\Omega)}.$$

Using Theorem 3.2.8 for a final time, there exists a $C_n > 0$ and $h_n > 0$, independent of f and h, such that, for all $h < h_n$,

$$\begin{aligned}\|D^{\alpha_n}(f - S_m^h f)\|_{L_2(\Omega)} &\le C_n h^{m-|\alpha_n|-\lambda/2-d/2} |f - S_m^h f|_{w,k,\mathbb{R}^d} \\ &\le C_n h^{m-|\alpha_n|-\lambda/2-d/2} |f|_{w,k,\mathbb{R}^d}.\end{aligned}$$

Set $C = \prod_{j=1}^n C_j^{1/2^j}$. We now have a total of n inequalities and combining them establishes the result. $\square$

We do not want to neglect the polyharmonic splines in this chapter, so set out below is the analogous restatement of Theorem 4.1.2 and its associated corollaries in that setting. Of course, the definition of $\mathcal{Y}^m(\beta, \Omega)$ must be changed accordingly to

$$BL^m(\beta, \Omega) = \left\{ f \in BL^m(\mathbb{R}^d) : \ g \in L_2(\mathbb{R}^d) \text{ and } \operatorname{supp}(\widehat{g}) \subset \Omega \text{ where } g = \frac{|\cdot|^{2m}\widehat{f}}{(\cdot)^\beta} \right\},$$

with seminorm

$$|f|_\beta = \left(\sum_{|\alpha|=2m} c_\alpha \int_{\mathbb{R}^d} \frac{|(\widehat{D^\alpha f})(x)|^2}{x^{2\beta}} \, \mathrm{d}x \right)^{1/2}, \qquad \text{for } f \in BL^m(\beta, \Omega).$$

Theorem 4.1.5. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and let $m > d/2$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, $\Pi_{m-1}(\mathbb{R}^d)$-unisolvent subset of Ω with fill-distance h. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S^h_m f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h, such that, for $2 \leq p \leq \infty$,*

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \leq C h^{2m-|\alpha|-|\beta|-d/2+d/p} |f|_\beta, \qquad |\alpha| \leq m - d/2 + d/p,$$

and, for $1 \leq p < 2$,

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \leq C h^{2m-|\alpha|-|\beta|} |f|_\beta, \qquad |\alpha| \leq m - d/2,$$

for all $f \in BL^m(\beta, \Omega)$, as $h \to 0$.

Corollary 4.1.6. *Under the same assumptions as Theorem 4.1.5, there exists a $C > 0$, independent of h, such that, for $2 \leq p \leq \infty$,*

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \leq C h^{2(m-|\alpha|)-d/2+d/p} |f|_\alpha, \qquad |\alpha| \leq m - d/2 + d/p,$$

and, for $1 \leq p < 2$,

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \leq C h^{2(m-|\alpha|)} |f|_\alpha, \qquad |\alpha| \leq m - d/2,$$

for all $|\alpha| \leq m$ and all $f \in BL^m(\alpha, \Omega)$, as $h \to 0$.

Corollary 4.1.7. *Assume the hypotheses of Theorem 4.1.5. Let $n \geq 2$ and suppose $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}^d_+$ with $|\alpha_2|, \ldots, |\alpha_n| \leq m$. Then there exists a $C > 0$, independent of h, such that, for $2 \leq p \leq \infty$,*

$$\|D^\alpha(f - S^h_m f)\|_{L_p(\Omega)} \leq C h^{\sigma(h)-d/2+d/p} |f|^{1/2^{n-1}}_{m,\mathbb{R}^d} \prod_{j=2}^{n} |f|^{1/2^{j-1}}_{\alpha_j}, \qquad |\alpha| \leq m - d/2 + d/p,$$

and, for $1 \leq p < 2$,

$$\|D^\alpha(f - S_m^h f)\|_{L_p(\Omega)} \leq Ch^{\sigma(h)} |f|_{m,\mathbb{R}^d}^{1/2^{n-1}} \prod_{j=2}^{n} |f|_{\alpha_j}^{1/2^{j-1}}, \qquad |\alpha| \leq m - d/2,$$

for all $f \in \bigcap_{j=2}^n BL^m(\alpha_j, \Omega)$, *as* $h \to 0$. *Where* $\sigma(h) = \sum_{j=1}^n \frac{m-|\alpha_j|}{2^{j-1}}$.

Chapter 5

Conclusions and further work

The bulk of this thesis has focussed on obtaining estimates for the error in interpolation using radial basis functions when the approximand lies outside the interpolant's native space. Chapters 2 and 3 have been very successful in addressing this problem for the polyharmonic splines and its cousins.

Some important radial basis functions fail to fit into our theory. In particular, the Gaussian and (inverse) multiquadric, which we discussed in the introductory chapter, are not included. These basis functions belong to the class of radial basis functions whose Fourier transform exhibits exponential decay. Two reasons why our theory, as it stands, can not be directly applied to basis functions, like the Gaussian and (inverse) multiquadric, are as follows. Firstly, extension theorems are an integral part of our technique. It is shown in Levesley and Light [46] that the seminorm for the Gaussian,

$$|f|_X = \left(\int_{\mathbb{R}^d} |\widehat{f}(x)| \mathrm{e}^{|x|^2} \, \mathrm{d}x \right)^{1/2},$$

for example, can be rewritten in direct form as

$$|f|_X = \left(\sum_{j=0}^{\infty} \frac{1}{j!} |f|^2_{j,\mathbb{R}^d} \right)^{1/2}.$$

This calculation enables us to define the associated local native space in the obvious way. Extension theorems for this local native space will need to be established. Secondly, the technique of smoothing a rough function by convolution with an approximate identity, like that used in Lemma 2.2.3 and Lemma 3.2.10, will need to be replaced appropriately in this different setting. This is because it is unable to render a function, say $f \in BL^k(\mathbb{R}^d)$, smooth enough so that $|\phi_h * f|_X < \infty$.

For the class of radial basis functions with exponentially decaying Fourier transforms, only the spectral approximation orders that occur when the approximand belongs to the appropriate native space are known. These date back to the work of Madych and Nelson [54]. However, for a radial basis function belonging to this class, there is acknowledged a genuine scarcity of functions in its associated native space. One could almost say that such spaces consists of little more than the bandlimited functions alone. To go some way to completing the theory, it would be very desirable to orchestrate an escape from the native space of this class of smooth radial basis functions as well as for the polyharmonic splines and the generalisation of them contained in Chapter 3. It is disappointing that for the Gaussian and inverse multiquadric there are currently no results of this flavour, and for the multiquadric there are only the non-stationary results of Yoon [80, 81, 82].

Another reason why we would like to have established error estimates for rough functions using smooth basis functions emerges when solving partial differential equations (PDEs) via the so-called RBF collocation method (see Kansa [45] and Fasshauer [25]).

Here, one considers a linear PDE of the form

$$Lu = f, \qquad \text{in } \Omega \subset \mathbb{R}^d,$$

$$Bu = g, \qquad \text{on } \partial\Omega,$$

where Ω is a bounded domain and $\partial\Omega$ is at least Lipschitz-continuous. The functions f and g are supplied. The operators L and B are linear differential and boundary operators, respectively. The approach is to select two disjoint sets of nodes $\mathcal{A}_1 \subset \Omega$, $\mathcal{A}_2 \subset \partial\Omega$ and solve the following interpolatory system

$$\lambda_a U = f(a), \qquad a \in \mathcal{A}_1,$$

$$\mu_b U = g(b), \qquad b \in \mathcal{A}_2.$$

Here U is our approximation to the solution of the PDE and is chosen to be of the specific form

$$U = \sum_{a\in\mathcal{A}_1} c_a(\lambda_a\psi)(\cdot - a) + \sum_{b\in\mathcal{A}_2} d_b(\mu_b\psi)(\cdot - b).$$

The linear functionals λ_a and μ_b are selected to describe, in a discrete sense, the operators L and B respectively. Thus to find U we must solve a square system of linear equations—the collocation matrix. Choosing the radial basis function ψ to be strictly positive definite and smooth is one way of rendering the collocation matrix nonsingular (see Wu [77]). As mentioned in the introduction to Chapter 1, the method has enjoyed applied success. Further, the method has been given theoretical foundation by Franke and Schaback [27, 28]. The use of smooth basis functions is particularly alluring because experimental evidence suggests that, in this situation, appropriate convergence orders for the method can be obtained without implicitly assuming a highly regular solution of the PDE (see Golberg, Chen and Karur [31]). If one instead employs finite smoothness basis functions, such as the polyharmonic splines, then, by virtue of the fixed smoothness, one potentially restricts convergence order.

In practice the solutions of linear PDEs can be genuinely rough. So, for one to perform the numerical analysis of the RBF collocation method, estimates of the type sought after for rough functions are essential. With such estimates one has the prospect of fitting them into the PDE context and establishing new convergence estimates for the method. The existing convergence analysis for the method, belonging to Franke and Schaback, applies to problems with very regular solutions.

Chapter 4 addressed the problem of what can be said if the function being approximated has additional smoothness properties, and this was successful for the polyharmonic splines and family. That the Gaussian and multiquadrics do not enter our discussion here is not so important. This is because there is little room for improvement in the fundamental approximation orders that these basis functions provide, being as they are of the form $\mathcal{O}(\mathrm{e}^{-1/h})$.

List of notation and symbols

$\mathbb{Z}_+$	set of all nonnegative integers
$\mathbb{Z}_+^d$	set of d-tuples of nonnegative integers (multi-integers)
D^α	partial differentiation operator
$\mathbb{R}$	the real number system
$\mathbb{R}_+$	set of all nonnegative real numbers
$\mathbb{R}^d$	d-dimensional Euclidean space
$\lvert\cdot\rvert$	the Euclidean norm on $\mathbb{R}^d$, the order of a multi-integer or the cardinality of a set
$\Pi_m(\Omega)$	the space of real-valued polynomials on Ω of degree at most m
$C(\Omega)$	the set of continuous functions on Ω
$C^m(\Omega)$	the set of functions on Ω with continuous derivatives up to the m-th order
$C^\infty(\Omega)$	the set of infinitely differentiable continuous functions on Ω
$C_0(\mathbb{R}^d)$	space of all compactly supported, continuous functions on $\mathbb{R}^d$
$C_0^m(\mathbb{R}^d)$	the set $C_0(\mathbb{R}^d) \cap C^m(\mathbb{R}^d)$
$C_0^\infty(\mathbb{R}^d)$	the set $C_0(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$

Table A: List of standard notation and their meanings.

$L_p(\Omega)$	set of all Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ for which $\int_\Omega \lvert f\rvert^p < \infty$
$\lVert\cdot\rVert_{L_p(\Omega)}$	norm on $L_p(\Omega)$
$L_1^{\mathrm{loc}}(\mathbb{R}^d)$	the space of locally integrable functions
$\mathscr{S}$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$
$\mathscr{S}'$	the space of continuous linear functionals on $\mathscr{S}$
$\widehat{\ }$	the (distributional) Fourier transform
$*$	convolution operator
$W_p^m(\Omega)$	Sobolev space
$\lVert\cdot\rVert_{m,\Omega}$	norm on $W_2^m(\Omega)$

Table A: Continued.

$BL^m(\mathbb{R}^d)$	13	$\lvert\cdot\rvert_{m,w,\Omega}$	64
$BL^m(\Omega)$	20	$\lVert\cdot\rVert_{m,w,\Omega}$	65
$\lvert\cdot\rvert_{m,\Omega}$	20	$\mathcal{X}^m(\Omega)$	65
$\widehat{\ }$	24	$\mathcal{Y}^m(\Omega)$	65
$*$	25	$\mathcal{Y}^m(\beta,\Omega)$	104
$\lVert\cdot\rVert_\Omega$	32, 76	$\lvert\cdot\rvert_{w^2,\beta}$	104
$\mathcal{Z}^m(\mathbb{R}^d)$	60	$BL^m(\beta,\Omega)$	110
$X^m(\Omega)$	64	$\lvert\cdot\rvert_\beta$	110

Table B: List of special symbols and the page numbers on which they first appear.

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