University of Leicester

DEPARTMENT OF ECONOMICS

On the utility representation of asymmetric single-peaked preferences*

Francisco Martínez-Mora, University of Leicester, UK

M. Socorro Puy, Universidad de Málaga, ES

On the utility representation of asymmetric single-peaked preferences[*]

Francisco Martínez-Mora[†]

University of Leicester and FEDEA

M. Socorro Puy[‡]

Universidad de Málaga

January 27, 2011

Abstract

We introduce two natural types of asymmetric single-peaked preferences, which we name *biased-above* and *biased-below*, depending on whether the asymmetry (or preference-bias) favors alternatives above or below the peak. We define a rich family of utility functions, *the generalized distance-metric utility functions,* that can represent preferences biased-above or biased-below, besides accommodating any degree of asymmetry. We also identify restrictions on differentiable utility representations that guarantee the underlying preferences to be biased-above or below, and allow to compare degrees of asymmetry. Finally, we consider a specific application –agents preferences over government size– to illustrate the role of factors such as risk aversion and tax distortions in shaping asymmetric preferences.

Key-words: Single-peaked preferences, asymmetric preferences, quadratic preferences, risk aversion, prudence.
JEL classification numbers: D72, H31, H5.

1   Introduction

Most models of political economy assume that preferences of policymakers and voters over policies are single-peaked and symmetric around the peak, i.e., there is a most preferred policy (the *peak*) and a symmetric excess or shortfall with respect to this peak results in an equal loss. In fact, the assumption that agent possess quadratic or euclidean preferences over policies is the most standard one in models of political competition.[1] However, as noted by some authors (see e.g., Milyo; 2000) these types of symmetric single-peaked preferences cannot be deduced from well-behaved public sector preferences.

In most cases, the symmetric single-peaked preferences are only justified by their analytical tractability. Some economist analyzing policymakers' preferences over target values such as inflation or public debt suggest that the symmetry of preferences around the target requires more serious thinking (Blinder, 1997; Heller, 1975). In words of Blinder: "*Academic macroeconomists tend to use quadratic loss functions for reason of mathematical convenience, without thinking much about their substantive implications. The assumption is not innocuous, [...] practical central bankers and academics would benefit from more serious thinking about the functional form of the loss function*". Thus, authors such as Blinder (1997) or Ruge-Murcia (2003) propose central banks endowed with asymmetric single-peaked preferences.

According to Surico (2007) there is empirical evidence showing that preferences of Fed have been asymmetric with respect to target values: there has been more responses to output contraction than to output expansion. Surico departs from the conventional quadratic set up in that policy makers are allowed, but not required, to treat differently positive and negative deviations of inflation and output from the target. This author shows that the hypothesis of asymmetric single-peaked preferences over inflation (and output) is not only empirically relevant but also theoretically important to explain the design of monetary policy. As claimed by Surico: " [...] *potential evidence of asymmetries in the central bank objective may be interpreted as*

---

[1] Among the models that consider symmetric preferences: Enelow and Hinich (1982), Ansolabehere and Snyder (2000), Groseclose (2001), Aragonés and Palfrey (2002) who analyze the effect of voters preferences on candidates' personal characteristics; Palfrey (1983), Osborne and Slivinsky (1996) who study the the strategic entry of candidates; Calvert (1985), Bernhardt, Duggan and Squintani (2007) who account for candidates' uncertainty about the location of the median voter.

evidence of asymmetries in the representative agent's utility."

Likewise, within the literature on fiscal response to foreign-aid, Heller (1975) and Feeny (2006) highlight the relevance of policymakers' asymmetric single-peaked preferences over deviations with respect to target spending or tax revenues.[2]

Within the family of single-peaked preferences, we introduce two natural types of asymmetric preferences, which we name *biased-above* and *biased-below*, depending on whether the asymmetry (or preference-bias) is above or below the peak. Our main objective is to study the extent to which we can propose utility functions representing asymmetric single-peaked preferences, without giving up the analytical tractability.

We also propose general conditions on the utility representation of preferences that guarantee that the underlying preferences are biased-above or biased-below. Moreover, we identify conditions on the utility representation to compare (if possible) degrees of asymmetry.

For those economic settings in which the analysis of asymmetric single-peaked preferences is not innocuous, we propose concrete utility specifications that can accommodate any degree of asymmetry. In particular, we propose a generalization of the distance-metric utility functions which inherits their analytical tractability besides representing any asymmetric single-peaked preferences.

Finally, we consider a specific application to analyze which factors may drive single-peaked preferences to be biased-above or biased-below. In a standard model of public good provision, we find that, among others, risk aversion with CARA or DARA specifications of consumers' risk over private consumption, as well as tax distortions, induce preferences over public good to be biased-below.

The remaining of the paper is organized as follows. The next section presents the environment and definitions. Section 3 studies conditions on the utility representation yielding alternative asymmetric single-peaked preferences and compares preferences in terms of degrees of asymmetry. Section 4 proposes the generalized distance-metric utility representation. Section 5 analyzes the concrete application. Section 6 concludes.

---

[2]Prospect theory also accounts for asymmetric preferences, though in a different sense. Following empirical and experimental evidence, this literature assumes that agents are more sensitive to losses than to gains with respect to a reference point (Kahneman and Tversky, 1991; Benartzi and Thaer 1995). In contrast to our analysis, the reference point is not a maximizer.

2 The environment and definitions

An agent has preferences defined over alternatives in the interval $[0, \bar{e}]$. The space of alternatives we consider is exogenous, that is, it has a specific and meaningful metric; thus, it is not possible to alter the spatial location of the alternatives.[3] The preference R of the agent on the set of alternatives is a complete preorder. The set of complete preorders on the set of alternatives is $\mathcal{R}$.

The strict and indifference preference relations induced by R are denoted by P and I respectively. Given $R \in \mathcal{R}$, the peak of R, when it exists, is an alternative e strictly preferred to any other in $[0, \bar{e}]$. Let e^p denote the peak of R.

Definition 1 *A preference $R \in \mathcal{R}$ satisfies single-peakedness (SP) if there exists a peak of R, and for all $d_0, d_1 > 0$ such that $d_0 < d_1$, with $e^p - d_1, e^p + d_1 \in [0, \bar{e}]$, we have $e^p - d_0 \ P \ e^p - d_1$ and $e^p + d_0 \ P \ e^p + d_1$.*

The SP property of preferences requires that, at each side of the peak, alternatives located closer to the peak are preferred to those located further away from it. The set of complete preorders satisfying SP on the set of alternatives is denoted by $\mathcal{R}^{SP}$. We refer to a generic element of $\mathcal{R}^{SP}$ as a SP preference.

We assume that $\bar{e}$ is sufficiently large as to guarantee that there exists an alternative $\tilde{e} \leq \bar{e}$ such that $0 \ I \ \tilde{e}$, i.e., such that the agent is indifferent between 0 and $\tilde{e}$. This implies that $e^p \in (0, \tilde{e})$. Thus, every alternative below the peak can be associated to another alternative above the peak according to the indifference preference relation. In what follows, we interpret $d \geq 0$ as a deviation off the peak where $d \in [0, e^p]$. We next define a function that assigns to every deviation below the peak, a deviation above the peak according to the indifference preference relation.[4]

Definition 2 *The preference-bias function $\delta : [0, e^p] \rightarrow [0, \tilde{e} - e^p]$ associated to $R \in \mathcal{R}^{SP}$ assigns to every deviation d, the corresponding deviation $\delta(d)$ for which $e^p - d \ I \ e^p + \delta(d)$.*

---

[3]In the specific application we consider later on in the paper this is guaranteed, as alternatives are levels of spending in a publicly provided good. See Eguia (2010) for an analysis that endogeneizes the spatial representation of the set of alternatives.

[4]The assumption that $0 \ I \ \tilde{e}$, $\tilde{e} \in (e^p, \bar{e}]$ is made only to simplify the exposition. Alternatively, if there exists an alternative $\hat{e} \in [0, e^p)$ such that $\bar{e} \ I \ \hat{e}$ then the preference-bias function is defined on $\delta : [0, e^p - \hat{e}] \rightarrow [0, \bar{e} - e^p]$ and the analysis remains unchanged.

Observe that every preference-bias function associated to a SP preference is a strictly increasing function.

We propose three types of SP preferences in terms of its associated preference-bias function; these types are symmetric, biased-below and biased-above.

Definition 3 *We say that the preference $R \in \mathcal{R}^{SP}$ is symmetric when $\delta(d) = d$ for all $d \in [0, e^p]$. We say that the preference $R \in \mathcal{R}^{SP}$ is biased-above when $\delta(d) > d$ for all $d \in (0, e^p]$. We say that the preference $R \in \mathcal{R}^{SP}$ is biased-below when $\delta(d) < d$ for all $d \in (0, e^p]$.*

Symmetric SP preferences are those inducing indifference between alternatives located symmetrically around the peak. When a SP preference is biased-above, the comparison between two alternatives symmetrically located at each side of the peak is such that the alternative located above the peak is higher in the preference ordering than the alternative located below the peak. Of course, it is the opposite when a SP preference is biased-below. The proposed types of SP preferences do not fully classify the set $\mathcal{R}^{SP}$. Our aim, however, is to capture two natural and meaningful types of preference bias in the set of SP preferences.[5]

Because the preference-bias function is independent of the location of the peak, one can compare degrees of asymmetry between pairs of preferences even when their respective peaks do not coincide. The following definition establishes the binary relation *more biased than* on $\mathcal{R}^{SP}$.

Definition 4 *Let $R_1, R_2 \in \mathcal{R}^{SP}$ with δ_1, δ_2 and e_1^p, e_2^p denoting their respective preference-bias functions and peaks. We say that R_2 is more biased-above than R_1 when $\delta_1(d) \leq \delta_2(d)$ for all $d \in [0, \min\{e_1^p, e_2^p\}]$ (or equivalently, R_1 is more biased-below than R_2).*

Thus, according to this definition, degrees of asymmetry are comparable across pairs of preference relations when their associated preference-bias functions satisfy that one is above the other, i.e., $\delta_1(d) \leq \delta_2(d)$ for all d in the domain. Observe, therefore, that the proposed binary relations, more

---

[5]In particular, these definitions do not cover SP preferences which are biased-above in some ranges of the domain and biased-below in others. Therefore, they only cover a subset of $\mathcal{R}$. It would of course be possible to generalize them to account for any SP preference in $\mathcal{R}$.

biased-above and more biased-below, are partial preorders (transitive but not complete) on the sets of SP preferences which are biased-above or biased-below.[6]

3 Conditions on the utility representation

Every preference $R \in \mathcal{R}^{SP}$ can be represented by a strictly quasi-concave utility function $V : [0, \bar{e}] \to \mathbb{R}$ which has a maximizer. The maximizer (or peak of R) satisfies:

$$e^p = \underset{e \in [0, \bar{e}]}{\arg\max} \; V(e) \tag{1}$$

In what follows, we consider differentiable utility representations so that the peak of R is an interior solution which satisfies $V'(e^p) = 0$.

3.1 Conditions for asymmetric SP preferences

In terms of the utility representation, when the SP preference is biased-above we have that $V(e^p - d) < V(e^p + d)$ for all $d \in (0, e^p]$, and when the SP preference is biased-below, in turn, $V(e^p - d) > V(e^p + d)$ holds for all $d \in (0, e^p]$. Figure 1 depicts two examples of utility functions representing SP preferences that are biased-below and biased-above respectively.

Let V' denote the first derivative of V. For SP preferences, $V'(e) > 0$ holds for alternatives below the peak, whereas $V'(e) < 0$ is true for alternatives above the peak.

A characterization of symmetric single-peaked preferences in terms of the properties of the utility representation V is straightforward to establish. Preferences are symmetric if and only if $V'(e^p - d) = -V'(e^p + d)$ for all $d \in [0, e^p]$.[7] That is to say, if and only if marginal utility at every pair of symmetric deviations above and below the peak coincide. Similarly, certain properties of the slope of V guarantee that preferences are biased-above or biased-below.

Proposition 1 *Let V be the utility representation of $R \in \mathcal{R}^{SP}$.*
Each of the following properties on V guarantee that R is biased-above:

---

[6] Again, the definition could be generalized to compare degrees of asymmetry in specific intervals of the domain of d.

[7] This result is straightforward to derive by differentiating $V(e^p - d) - V(e^p + d) = 0$ with respect to d. Observe that this condition does not restrict the shape of V in $e \in (\tilde{e}, \bar{e}]$.

(1a) $V'(e^p - d) > -V'(e^p + d)$ for all $d \in (0, e^p]$,
(2a) V' strictly convex for all $e \in [0, \bar{e}]$,
where (2a) $\Rightarrow$(1a).
Each of the following properties on V guarantee that R is biased-below:
(1b) $V'(e^p - d) < -V'(e^p + d)$ for all $d \in (0, e^p]$,
(2b) V' strictly concave for all $e \in [0, \bar{e}]$,
where (2b) $\Rightarrow$(1b).

Proof. The following claims prove our statement.[8]

Claim 1: $(1a) \Rightarrow \delta(d) > d$ for all $d \in (0, e^p]$.
Proof of Claim 1: Utility loss derived from reducing e^p to $e^p - d$ is measured by $\int_{e^p - d}^{e^p} V'(e)de$, whereas utility loss from increasing e^p to $e^p + d$ is measured by $-\int_{e^p}^{e^p + d} V'(e)de$. By (1a), $\int_{e^p - d}^{e^p} V'(e)de > -\int_{e^p}^{e^p + d} V'(e)de$ for all $d \in (0, e^p]$. Solving for the integral and simplifying $V(e^p - d) < V(e^p + d)$ for all $d \in (0, e^p]$, i.e., preferences are biased-above.

Claim 2: $(2a) \Rightarrow \delta(d) > d$ for all $d \in (0, e^p]$.
Proof of Claim 2: If V' is strictly convex in $e \in [0, \bar{e}]$, by Jensens' inequality, the expected value of V' over the interval[9] $[0, 2e^p]$ is above the value of V' in the mean of $[e^p - d, e^p + d]$ for all $d \in (0, e^p]$. Thus, $\int_{e^p - d}^{e^p + d} \frac{V'(e)}{2d}de > V'(e^p)$. Solving for the integral, $\frac{1}{2d}[V(e^p + d) - V(e^p - d)] > V'(e^p)$. By SP, $V'(e^p) = 0$, and substituting in the inequality, $V(e^p - d) < V(e^p + d)$ for all $d \in (0, e^p]$, i.e., preferences are biased-above.

Claim 3: $(2a) \Rightarrow (1a)$
Proof of Claim 3: By strict convexity of V' we have $V'(e^p) < \frac{V'(e^p - d) + V'(e^p + d)}{2}$ for all $d \in (0, e^p]$. By SP, $V'(e^p) = 0$, and substituting in the inequality yields $V'(e^p - d) > -V'(e^p + d)$ for all $d \in (0, e^p]$.

∎

By conditions (1a) and (1b), the comparison of the slopes of the utility representation of SP preferences at every symmetric deviation with respect to the peak reveals the direction of preference-bias. Conditions (2a) and (2b) also reveal the direction of preference-bias by checking whether marginal utility is a strictly concave or strictly convex function.

---

[8]The analogous statements on biased-below preferences can be proved following a similar reasoning (that we omit in the interest of brevity).

[9]If $2e^p > \bar{e}$, the proof accommodates by considering instead the interval $[0, \bar{e}]$.

3.2 Conditions for *more biased* SP preferences

Figure 2 depicts two utility representations of preferences $R_1, R_2 \in \mathcal{R}^{SP}$ where R_2 is more biased-below than R_1. We put together both utility representations around their respective peaks in order to compare their associated preference-biased function.

Next, we show that the comparison between the slopes of different utility representations indicates the strength of preference-bias across different preferences. For each pair of preferences $R_1, R_2 \in \mathcal{R}^{SP}$, we denote by e_1^p, e_2^p their respective peaks.

Proposition 2 *Let V_1, V_2 be two utility representations of $R_1, R_2 \in \mathcal{R}^{SP}$, respectively. If $V_2'(e_2^p + \gamma) \leq V_1'(e_1^p + \gamma)$ for all $\gamma \in [-\min\{e_1^p, e_2^p\}, \min\{e_1^p, e_2^p\}]$, then R_2 is more biased-below than R_1 (or equivalently, R_1 is more biased-above than R_2).*

Proof. Condition $V_2'(e_2^p - d) \leq V_1'(e_1^p - d)$ for all $d \leq \min\{e_1^p, e_2^p\}$ implies that the utility loss derived from reducing e_2^p to $e_2^p - d$, which is measured by $\int_{e_2^p - d}^{e_2^p} V_2'(e)de = B$, in comparison to the utility loss derived from reducing e_1^p to $e_1^p - d$, which is measured by $\int_{e_1^p - d}^{e_1^p} V_1'(e,)de = A$, is such that $B \leq A$. By definition of the preference-bias function, there is δ_1 for which $A = \int_{e_1^p}^{e_1^p + \delta_1} V_1'(e)de$, at the same time, and given that $0 > V_1'(e_1^p + d) \geq V_2'(e_2^p + d)$ for all $d \leq \min\{e_1^p, e_2^p\}$, we have

$$\int_{e_2^p}^{e_2^p + \delta_1} V_2'(e)de \geq A. \tag{2}$$

By definition of the preference-bias function there is δ_2 for which

$$\int_{e_2^p}^{e_2^p + \delta_2} V_2'(e)de = B. \tag{3}$$

Because $B \leq A$, conditions (2) and (3) imply $\int_{e_2^p}^{e_2^p + \delta_2} V_2'(e)de \leq \int_{e_2^p}^{e_2^p + \delta_1} V_2'(e)de$ from where we derive that $\delta_2(d) \leq \delta_1(d)$ for all $d \in [0, \min\{e_1^p, e_2^p\}]$. ∎

In other terms, the proposed sufficient condition for R_2 to be more biased-below than R_1 implies that $V_2'(e_2^p - d) \leq V_1'(e_1^p - d)$ and $V_2'(e_2^p + d) \leq V_1'(e_1^p + d)$,

8

$d > 0$; i.e., below the peak, equal-distance deviations generate more disutility with V_1 than with V_2, and above the peak, equal-distance deviation generate more disutility with V_2 than with V_1.

The degree of preference-bias can also be compared using the degree of concavity or convexity of the marginal utility function. For this, one of the marginal utility specifications must be obtained as an increasing transformation of the other.

Proposition 3 *Let V_1, V_2 be strictly concave utility representations of $R_1, R_2 \in \mathcal{R}^{SP}$. If V_1', V_2' are strictly concave and $-\frac{V_2'''(e)}{V_2''(e)} \leq -\frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)}$ for all $e < min\{2e_1^p, 2e_2^p\}$ where $\Lambda = e_2^p - e_1^p$, then R_2 is more biased-below than R_1.*[10]

Proof. By strict concavity of V_1 and V_2, the functions V_1' and V_2' are strictly decreasing functions and they can be related by a strictly increasing transformation g such that $V_2'(e) = g(V_1'(e - \Lambda))$ where $\Lambda = e_2^p - e_1^p$. This implies that when $e = e_2^p$, $g(0) = 0$.[11] Differentiating the expression,

$$V_2''(e) = g'(V_1'(e - \Lambda))V_1''(e - \Lambda)$$
$$V_2'''(e) = g''(V_1'(e - \Lambda))[V_1''(e - \Lambda)]^2 + g'(V_1'(e - \Lambda))V_1'''(e - \Lambda).$$

From where

$$-\frac{V_2'''(e)}{V_2''(e)} = -\frac{V_1'''(e - \Lambda)}{V_1''(e - \Lambda)} - \frac{g''(V_1'(e - \Lambda))V_1''(e - \Lambda)}{g'(V_1'(e - \Lambda))}.$$

By strict concavity of V_1' and V_2', we have $V_1''' < 0$ and $V_2''' < 0$. Then, $-\frac{V_2'''(e)}{V_2''(e)} \leq -\frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)}$ implies $g''(V_1'(e - \Lambda)) \leq 0$ (g concave).
By definition of δ_2, it follows that $V_2(e_2^p - d) = V_2(e_2^p + \delta_2(d))$, or equivalently, $\int_{e_2^p - d}^{e_2^p + \delta_2(d)} V_2'(e)de = 0$. Substituting function g,

$$0 = \int_{e_2^p - d}^{e_2^p + \delta_2(d)} V_2'(e)de = \int_{e_2^p - d}^{e_2^p + \delta_2(d)} g(V_1'(e - \Lambda))de.$$

---

[10]Similarly, if V_1', V_2' are strictly convex and $-\frac{V_2'''(e)}{V_2''(e)} \geq -\frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)}$ for all $e < min\{2e_1^p, 2e_2^p\}$, then R_2 is more biased-above than R_1.

[11]If $e_2^p = e_1^p$, then $\Lambda = e_2^p - e_1^p = 0$, and it is possible to compare degrees of asymmetry just by comparing $-\frac{V_1'''(e)}{V_1''(e)}$ to $-\frac{V_2'''(e)}{V_2''(e)}$ for every $e < 2e_1^p$.

By concavity of g, $\int_{e_2^p-d}^{e_2^p+\delta_2(d)} g(V_1'(e-\Lambda))de \leq g(\int_{e_2^p-d}^{e_2^p+\delta_2(d)} V_1'(e-\Lambda)de)$, where $\int_{e_2^p-d}^{e_2^p+\delta_2(d)} V_1'(e-\Lambda)de = V_1(e_1^p+\delta_2(d)) - V_1(e_1^p-d)$. Since g is strictly increasing and $g(0) = 0$, then $V_1(e_1^p + \delta_2(d)) - V_1(e_1^p - d) \geq 0$. Since $V_1' < 0$ for all $e > e_1^p$, and by definition of $\delta_1(d)$, $V_1(e_1^p - d) = V_1(e_1^p + \delta_1(d))$, we deduce that $\delta_1(d) \geq \delta_2(d)$ for all $d \leq \min\{e_1^p, e_2^p\}$. ∎

This proposition reveals an analogy between the conditions for degrees of preference-bias and the theories developed by Arrow (1971), Pratt (1964) and Kimball (1990). According to Arrow-Pratt's theory of risk aversion, concavity of a utility function over consumption indicates the presence of risk aversion, while according to Kimball's theory of precautionary savings, concavity of the marginal utility function entails precautionary saving behavior. In each case, the degree of concavity of the utility function or the degree of concavity of the marginal utility function measures risk aversion or precautionary savings respectively.[12] These behavioral traits become thus comparable across pairs of concave functions such that one is a concave transformation of the other. In our context, as long as V is strictly concave, the curvature of the marginal utility function determines the degree of preference-bias. We can therefore apply the coefficient of prudence proposed by Kimball (1990) to measure the level of preference-bias: preferences are more biased-above for the *more convex* marginal utility representation, and more biased-below for the *more concave* marginal utility representation.

4   Asymmetric SP utility representation

In this section, we show that a generalization of any distance-metric utility function allows for the utility-representation of any asymmetric SP preference relation.

A *distance-metric utility function* is defined by $V(e) = -f(e - e^p)$ where f is a continuous and strictly increasing distance-function between the peak e^p and the alternative e. Particular examples of f are the quadratic function, in which $f(e - e^p) = [e - e^p]^2$, or the distance function induced by a norm, in which $f(e - e^p) = \|e - e^p\|$, or any function $f(e - e^p) = |e - e^p|^\lambda$ where $\lambda > 0$.

---

[12] Arrow-Pratt's coefficient of risk aversion is defined by $-\frac{u''}{u'}$, whereas Kimball's coefficient of prudence is defined by $-\frac{u'''}{u''}$ where u measures utility over private consumption.

Given a preference $R \in \mathcal{R}^{SP}$, the preference-bias function δ associated to R assigns to each deviation below the peak, a deviation above the peak for which the agent is indifferent. Let $\delta^{-1} : [0, \tilde{e} - e^p] \rightarrow [0, e^p]$ be the inverse of the preference-bias function;[13] δ^{-1} is bounded above by $\delta^{-1}(\tilde{e} - e^p) = e^p$, which corresponds to the indifference relation $0 \, I \, \tilde{e}$. We extend the domain of δ^{-1} to every $d \in (\tilde{e} - e^p, \bar{e} - e^p]$. The *generalized distance-metric utility function* is defined by

$$V(e) = \begin{cases} -f(e - e^p) & \text{when } e \in [0, e^p] \\ -f(\delta^{-1}(e - e^p)) & \text{when } e \in (e^p, \bar{e}] \end{cases}.$$

Below the peak, the proposed utility function coincides with the distance-metric utility function. In order to capture the degree of preference-bias, the level of utility derived from any alternative above the peak $e \in [e^p, \tilde{e}]$ is equal to the corresponding distance-metric utility value at its indifferent alternative below the peak. Finally, for alternatives $e \in (\tilde{e}, \bar{e}]$, the utility derived is below $V(\tilde{e})$ and the function is strictly decreasing. If the SP preference relation is symmetric, i.e., $\delta^{-1}(d) = d$, the generalized distance-metric utility function collapses to the distance-metric utility function.

Theorem 1: *Every preference $R \in \mathcal{R}^{SP}$ can be represented by the generalized distance-metric utility function. Furthermore, this utility specification can be used to compare pairs of preferences such that one of them is more biased-above or more biased-below than the other.*

Proof. First, we show that every preference relation $R \in R^{SP}$ is represented by the generalized distance-metric utility function. The preference ordering across alternatives located at the same side of the peak is captured by the distance-metric utility function below the peak, and above the peak, by a function that is strictly decreasing in distance (given that δ^{-1} is a strictly increasing function in all its domain). The preference ordering of pairs of alternatives located at opposite sides of the peak can be deduced by identifying those pairs of alternatives yielding equal utility. Thus, $V(e^p + \delta(d)) = -f\left(\delta^{-1}(e^p + \delta(d) - e^p)\right)$ and simplifying $V(e^p + \delta(d)) = -f(d)$. Since $V(e^p - d) = -f(d)$, we deduce that $e^p - d \, I \, e^p + \delta(d)$ for all $d \in (0, e^p]$.

Second, we show that the generalized distance-metric utility function can be

---

[13]Because the preference satisfies the SP condition, the preference-bias function δ is biyective and it has an inverse.

used to compare degrees of asymmetries across pairs of preferences. Suppose that $\delta_1(d) \leq \delta_2(d)$ for all $d \in (0, \min\{e_1^p, e_2^p\})$. Then, because δ is always a strictly increasing function, $\delta_1(d) \leq \delta_2(d)$ implies that $\delta_1^{-1}(d) \geq \delta_2^{-1}(d)$. Plugging this inequality into the generalized distance-metric utility function we obtain that $V_2'(e^p + d) \leq V_1'(e^p + d)$ for all $d \in (0, \min\{e_1^p, e_2^p\})$. In addition, $V_2'(e^p - d) = V_1'(e^p - d)$ for all $d \in (0, \min\{e_1^p, e_2^p\})$.[14] We deduce, according to Proposition 2, that R_2 is more biased-above than R_1. ∎

The generalized distance-metric utility function can accommodate every SP preference, in particular those which are biased-above, or biased-below according to Definition 3. For instance, if we take a linear preference bias-function $\delta(d) = kd$ with $k > 0$, the corresponding generalized distance-metric utility function is such that

$$V(e) = \begin{cases} -f(e - e^p) & \text{when } e \in [0, e^p] \\ -f\left(\frac{e - e^p}{k}\right) & \text{when } e \in (e^p, \overline{e}] \end{cases}.$$

Observe that this function extends, in a natural way, the domain of δ^{-1} to alternatives in the interval $e \in (\tilde{e}, \overline{e}]$. According to this utility specification, $k > 1$ represents a particular class of SP preferences that are biased-above, and $k < 1$ represents another particular class of SP preferences that are biased-below.

Our proposal accommodates every continuous and strictly increasing distance-metric function. For instance, a rich family of utility specifications are given by:

$$V(e) = \begin{cases} -|e - e^p|^\lambda & \text{when } e \in [0, e^p] \\ -\left|\delta^{-1}(e - e^p)\right|^\lambda & \text{when } e \in (e^p, \overline{e}] \end{cases}$$

where different values of $\lambda > 0$ yield different utility functions.

5 Application: public sector preferences

In this section, we illustrate how the conditions stated in Section 3 can be used to determine whether specific induced utility functions over public consumption qualify into one of the proposed types of SP preferences: biased-below or biased-above.

---

[14]Observe that, off the peak, all the proposed distance-functions are differentiable.

Let $y > 0$ be the income of an agent and consider that there are two goods: private consumption x and public expenditure e. Public expenditure is financed through taxation: let $\tau(e, y) > 0$ be the tax-bill of an agent with income y when the amount of public expenditure is e, and its first and second derivatives with respect to e satisfy $\tau' > 0$ and $\tau'' \geq 0$. The utility representation of preferences is defined by $U(x, e) = u(x) + e$ where $u' > 0$ and $u'' < 0$. The induced indirect utility function is

$$V(e) \equiv u\left(y - \tau(e)\right) + e. \tag{4}$$

Thus, V is strictly concave and has a maximizer, i.e., this specification represents SP preferences.[15]

Solving for the third derivative of V with respect to e, we get:

$$V''' = -u''' \left[\tau'\right]^3 + 3u''\tau'\tau'' - u'\tau'''. \tag{5}$$

The generic tax-bill function τ may represent different finance schemes, with different degrees of public sector effectiveness in transforming tax revenues into public spending. We say that the *government is efficient* when the scheme τ is linear in e (i.e., $\tau'' = 0$).[16] When τ is strictly convex in e (i.e., $\tau'' > 0$) we say that the *government is inefficient*.[17]

Risk neutrality is characterized by $u'' = 0$, whereas risk aversion by $u'' < 0$. According to Kimball (1990), precautionary saving behavior (or prudence) is characterized by $u''' > 0$. In particular, CARA, DARA or CRRA specifications of risk imply prudence.

According to condition (2b) of Proposition 1 (requiring concavity of the marginal utility function), Remark 1 identifies conditions that unequivocally generate SP preferences over the size of the public sector that are biased-below.

---

[15]The primitive function u is assumed to be a C^3 function so that V is C^3 as well.

[16]Particular examples of efficient tax-bill functions are the lump-sum tax or the proportional income tax. Progressive or regressive tax schemes can also be efficient. For instance, the tax bill function $\tau = e\rho(y)$ with ρ strictly increasing in income is linear in e and progressive whenever $\rho'(y) > \frac{\rho(y)}{y}$, or regressive if $\rho'(y) < \frac{\rho(y)}{y}$. The education finance scheme studied by Bénabou (2002) implies a non-proportional tax scheme of that form.

[17]Convexity of τ in e may be due to congestion effects in the government ability to transform tax revenue into public expenditure, tax distortions, or corruption of public officials. In the latter case, the convexity derives from the deviation of tax revenues away from public expenditure (see, e.g., the model on career concerns proposed by Persson and Tabellini, 2000, chapter 4).

Remark 1 *Consider the utility representation of Equation (4). Each of the following conditions generate preferences biased-below:*
i) risk neutrality and an inefficient government with $\tau''' > 0$,
ii) risk aversion (according to CARA, DARA or CRRA condition) and an efficient government
iii) risk aversion (according to CARA, DARA or CRRA condition) and an inefficient government with $\tau''' > 0$.

Risk neutrality, as well as risk aversion with CARA, DARA or CRRA specifications of risk, are the assumptions more generally invoked in economic applications.[18] Therefore, Remark 1 covers most standard utility specifications as well as widely applicable tax bill functions.

It is intuitive that consumers exhibiting CARA, DARA or CRRA specifications of risk, which imply prudence ($u''' > 0$), contribute to SP preferences being biased-below. According to Eeckhoudt and Schlesinger (2006), prudence implies that an agent prefers to accept an extra risk when private consumption is higher (that is, when public expenditure is lower), rather than when private consumption is lower (and public expenditure is higher). Thus, marginal utility falls slower at low level of public expenditure and faster at higher levels of public expenditure. Likewise, an inefficient government with $\tau''' > 0$, contributes to SP preferences being biased-below given that the cost of raising public funds (or corruption) increases faster at higher levels of public expenditure.

Of particular interest is the case ii) of Remark 1 in which $V''' = -u''' [\tau']^3$. According to Proposition 3, we can directly compare the asymmetry of preferences by means of the coefficient $-\frac{V'''}{V''}$ which, in this case, equals $\frac{u'''}{u''}\tau'$ (as $\tau'' = 0$). Because the term $-\frac{u'''}{u''}$ defines the coefficient of absolute prudence, the higher the coefficient of prudence of the agent over private consumption, the more biased-below are her preferences over the size of the public sector. Furthermore, as τ' measures the tax price of an additional unit of spending, the greater such tax price for the agent, the more biased-below are her preferences over the size of the public sector.

---

[18]On the one hand, recent empirical evidence presented by Chiappori and Paiella (2008) does not give support to the IARA condition. On the other hand, DARA is widely considered to be reasonable (see Arrow 1971) and, as postulated by Pratt (1964): DARA is implied by such behavior of investing in risky securities as one becomes richer.

6   Conclusion

Symmetric single-peaked preferences are usually represented by distance-metric utility functions. Our analysis shows that such utility specifications can be easily transformed to accommodate preferences biased-above or biased-below, besides representing any degree of asymmetry. Our utility specification, that we call *generalized distance-metric utility function*, can be used to capture the preferences of policy-makers, investors, politicians, or the media, among others, when bias-preferences is not an innocuous modeling assumption. This is for instance the case in the models proposed by Blinder (1997), Ruge-Murcia (2003), Surico (2007), or Feeny (2006) among others.

We also describe sufficient conditions on standard utility representations that reveal the direction (if any) of the asymmetry of preferences. These conditions can indicate what factors of the primitive decision problem induce preference-bias. We prove that the analog of Kimball's coefficient of prudence can be used to compare (if possible) degrees of asymmetry across different utility specification. Finally, to illustrate these points we analyze a standard model of public good provision. The analysis of this model reveals that risk aversion and government inefficiencies induce single-peaked preferences of agents to be biased-below and suggests a role for the progressivity of the tax system on shaping single-peaked preferences.

References

[1] Ansolabehere S. and J.M. Snyder (2000), Valence Politics and Equilibrium in Spatial Election Models, *Public Choice* 103: 327–336.

[2] Aragonés E. and T. R. Palfrey (2002), Mixed Equilibrium in a Downsian Model with a Favored Candidate, *Journal of Economic Theory* 103: 131-161.

[3] Arrow K. (1971), Essays in the Theory of Risk-Bearing. Chicago, IL: Markham Publishing Company.

[4] Benabou R. (2002), Tax and Education Policy in a Heterogeneous-Agent Economy: What Levels of Redistribution Maximize Growth and Efficiency?, *Econometrica* 70 (2): 481-517.

[5] Benartzi S. and R.H. Thaler (1995), Myopic Loss Aversion and the Equity Premium Puzzle, *Quarterly Journal of Economics* 110: 73-92.

[6] Bernhardt D., J. Duggan and F. Squintani (2009), The Case of Responsible Parties, *American Political Science Review* 103: 570–587.

[7] Blinder A. (1997), Distinguished lecture on economics and government: what central bankers could learn from academics and vice versa, *Journal of Economic Perspective* 11: 3–19.

[8] Calvert R. (1985), Robustness of the Multidimensional Voting Model: Candidate Motivations, Uncertainty, and Convergence, *American Journal of Political Science* 29: 69-95.

[9] Eeckhout L. and H. Schlesinger (2006), Putting Risk in Its Proper Place, *American Economic Review* 96: 280-289.

[10] Eguia J. X. (2010), Foundations of Spatial Preferences, *Journal of Mathematical Economics, forthcoming.*

[11] Enelow J.M. and M.J. Hinich (1982), Nonspatial Candidate Characteristics and Electoral Competition, *Journal of Politics* 44: 115-130.

[12] Feeny S. (2006), Policy Preferences in Fiscal Response Studies, *Journal of International Development* 18: 1167-1175.

[13] Groseclose T. (2001), A Model of Candidate Location When One Candidate Has a Valence Advantage, *American Journal of Political Science* 45: 862-886.

[14] Heller P.S. (1975) A model of Public Fiscal Behavior in Developing Countries: Aid, Investment, and Taxation, *American Economic Review* 65: 429-445.

[15] Kahneman D. and A. Tversky (1991), Loss Aversion and Riskless Choice: a Reference Dependent Model, *Quarterly Journal of Economics* 106: 1039-1061.

[16] Kimball M.S. (1990), Precautionary Saving in the Small and in the Large, *Econometrica*, 58: 53-73.

[17] Milyo J. (2000), A problem with Euclidean preferences in saptial models of politics, *Economics Letters* 66, 179-182.

[18] Osborne M.J., Slivinski, A. (1996), A Model of Political Competition with Citizen-Candidates, *Quarterly Journal of Economics* 111: 65-96.

[19] Palfrey T.R. (1984), Spatial Equilibrium with Entry, *Review of Economic Studies* 51: 139-156.

[20] Persson T. and G. Tabellini (2000), *Political Economics. Explaining Economic Policy.* The MIT Press: Cambridge.

[21] Pratt J.W. (1964), Risk Aversion in the Small and in the Large, *Econometrica* 32: 122-136.

[22] Pratt J.W. and R.J. Zeckhauser (1987) Proper Risk Aversion, *Econometrica* 55: 143-154.

[23] Ruge-Murcia F.J. (2003), Inflation Targeting under Asymmetric Preferences, *Journal of Money, Credit and Banking* 35: 763-785.

[24] Surico P. (2007), The Fed's Monetary Policy Rule and U.S. Inflation: The Case of Asymmetric Preferences, *Journal of Economic Dynamics and Control* 31: 305-324.

[25] Wittman D. (1983), Candidate Motivation: A Synthesis of Alternative Theories, *American Political Science Review* 77: 142-157.