DEGREES OF UNSOLVABILITY.

by

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To my parents for all their help over the years.

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Degrees of unsolvability complementary between recursively enumerable degrees.

Given a set S of mutually incomparable degrees and a pair of degrees $\underline{a}$ and $\underline{b}$ we say that S is complementary between $\underline{a}$ and $\underline{b}$ whenever $\underline{a}$ is the greatest lower bound of the members of S and $\underline{b}$ is the least upper bound. A degree $\underline{a}$ is minimal if $\underline{0}$ is the least upper bound of the degrees strictly less than $\underline{a}$. We obtain an indication of the variety of decision problems to be found amongst degrees of a particular type by looking at the pairs of degrees between which sets of degrees of that type are complementary. If S complementary between $\underline{0}$ and $\underline{a}$ we say that S is complementary below $\underline{a}$ and we prove below that there is a pair of minimal degrees complementary below $\underline{0}'$.

Spector [8] showed that minimal degrees exist and Sacks [6] constructed one below $\underline{0}'$, the largest recursively enumerable degree. Shoenfield [7] proved that given any degree strictly between $\underline{0}$ and $\underline{0}'$ we may find a minimal degree below $\underline{0}'$ which is incomparable with it. Lachlan [3] proved that no pair of recursively enumerable (r.e.) degrees is complementary below $\underline{0}'$ even though there is a pair of r.e. degrees complementary below some r.e. degree (see Yates [10] and Lachlan [3]). We construct below a pair of minimal degrees with join $\underline{0}'$. Shoenfield's theorem is an immediate corollary of this. Since the theorem yields a pair complementary below $\underline{0}'$ we have that no dramatic generalisation of Lachlan's theorem is possible. Related results proved elsewhere are: (1) there is a pair of degrees complementary below any given r.e. degree other than $\underline{0}$, (2) there is a r.e.

degree other than $\underline{0}$ below which no set of minimal degrees is complementary (although Yates [11] has shown there to be countably many minimal predecessors for each non-zero r.e. degree), (3) there are three r.e. degrees complementary below $\underline{0}'$.

We take $\{\Phi_e | e \geqslant 0\}$ to be a standard enumeration of the partial recursive functionals. $\{\Phi_{e,s} | e,s \geqslant 0\}$ is a double sequence of finite approximations to these functionals satisfying the following: (i) $\{\Phi_{e,s}\}$ is a recursive set, (ii) $\Phi_{e,s} \subseteq \Phi_{e,s+1}$ for each e and each $s \geqslant 0$, (iii) $\Phi_e = \bigcup_{s \geqslant 0} \Phi_{e,s}$ for each $e \geqslant 0$, (iv) for each s $\Phi_{e,s}$ is empty for all but a finite set of numbers. The last condition is included in order to avoid an infinite search occurring at a stage of the construction. $\{R_e\}$ will be a standard list of the recursively enumerable sets with double sequence $\{R_{e,s}\}$ of approximations with properties similar to (i) - (iv) above for $\{\Phi_{e,s}\}$. And $\{F_e\}$ is an enumeration of the partial recursive functions, each F_e having its recursive tower $\{F_{e,s} | s \geqslant 0\}$ of finite approximations.

σ is said to be a <u>string of length</u> n+1 if it is an initial segment (or <u>beginning</u>) C[n] of a characteristic function C defined on exactly n+1 numbers. If σ is a string of length n+1 and $m \leqslant n$ we write $\sigma[m]$ for the beginning of σ of length m+1. If we write $lh(\sigma)$ for the length of σ and $y(\sigma_1, \sigma_2)$ for the least number y for which $\sigma_1(y) \neq \sigma_2(y)$, there is a natural ordering $\leqslant$ of the strings defined by:

$$\sigma_1 \leqslant \sigma_2 \longleftrightarrow$$

$\sigma_1 = \sigma_2$ or $\mathrm{lh}(\sigma_1) < \mathrm{lh}(\sigma_2)$ or $\mathrm{lh}(\sigma_1) =$

$\mathrm{lh}(\sigma_2)$ and $\sigma_1(y(\sigma_1,\sigma_2)) < \sigma_2(y(\sigma_1,\sigma_2))$.

Define an ordering $\leqslant$ on the ordered pairs of strings by:

$$(\sigma_1,\sigma_2) \leqslant (\pi_1,\pi_2) \longleftrightarrow$$

$\sigma_1[y(\sigma_1,\sigma_2) - 1] < \pi_1[y(\pi_1,\pi_2) - 1]$ or

$\sigma_1[y(\sigma_1,\sigma_2) - 1] = \pi_1[y(\pi_1,\pi_2) - 1]$ and $\sigma_1 < \pi_1$ or $\sigma_1 = \pi_1$ and $\sigma_2 \leqslant \pi_2$.

This will enable us to talk of the least pair of strings with a given property.

ϕ is the string defined nowhere and 0 and 1 are the strings with domain $\{0\}$ and respective ranges $\{0\}$ and $\{1\}$.

$\sigma * \tau$ is the string defined by:

$$\sigma * \tau(x) = \begin{cases} \sigma(x) \text{ if } x < \mathrm{lh}\sigma, \\ \tau(x - \mathrm{lh}\sigma) \text{ if } \mathrm{lh}\sigma \leqslant x < \mathrm{lh}\sigma + \mathrm{lh}\tau, \\ \text{undefined otherwise.} \end{cases}$$

If σ and τ are beginnings of some characteristic function C then we say that σ and τ are <u>compatible</u>, and write $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$ according as $\mathrm{lh}\sigma \leqslant \mathrm{lh}\tau$ or $\mathrm{lh}\tau \leqslant \mathrm{lh}\sigma$. Otherwise σ and τ are <u>incompatible</u>.

A <u>tree</u> T is a mapping from the strings into the strings of C, such that if $T(\tau * i)$ is defined where i is 0 or 1 then so are $T(\tau * 1 - i)$ and $T(\tau)$, and such that the partial ordering induced on the range of T by the ordering $\subseteq$ on the domain of T coincides with the ordering $\subseteq$ on the range of T. The terms 'recursive tree' and 'partial recursive tree' will be used a natural informal way.

If $T(\tau * 0)$, $(\tau * 1)$ $(=T(\tau * 0), T(\tau * 1))$ are defined then they comprise the <u>syzygy on T based on $T(\tau)$</u>. Otherwise if $T(\tau)$ is defined then $T(\tau)$ is an <u>end string for T</u>. A string σ is <u>compatible with</u> a tree T if σ lies on T (i.e., is in the range of T) or is an extension of an end string for T. T' is <u>compatible with T</u> if every string on T' is compatible with T.

We say that two strings σ_1, σ_2 <u>split τ for e through x</u> if $\sigma_1, \sigma_2 \supset \tau$ and $\Phi_e(\sigma_1, x)$ (σ_2, x) and $\Phi_e(\sigma_1, x)$, (σ_2, x) are defined and unequal. σ_1, σ_2 <u>split τ for e through x at stage s</u> if $\sigma_1, \sigma_2 \supset \tau$ and $\Phi_{e,s}(\sigma_1, x)$, (σ_2, x) are defined and unequal Then σ_1, σ_2 split τ for e through x if and only if σ_1, σ_2 split τ for e through x at some stage $s \geqslant 0$ since $\Phi_e = \bigcup_{s \geqslant 0} \Phi_{e,s}$, and that if σ_1, σ_2 split τ for e through x at stage s then σ_1, σ_2 split τ for e through x at every stage $s' > s$ because $\Phi_{e,s'} \supseteq \Phi_{e,s}$.

Before proving the main theorem we give a short proof of a weaker result.

THEOREM 1. There is a pair of degrees complementary below $\underline{0}'$.

PROOF: Let D be a set of degree $\underline{0}'$ such that D is recursive in every infinite subset of D (i.e., D is intro-reducible in the sense of [2]). We construct at stages $n \geqslant 0$ beginnings α_n, β_n of characteristic functions A and B respectively and take the required pair to be the degrees of A and B. For each n we will have $lh(\alpha_n) = lh(\beta_n)$. Strings α and β with $\alpha \supset \alpha_n$ and $\beta \supset \beta_n$ are said to be <u>admissable at stage n+1</u> if for no $x \geqslant lh(\alpha_n)$ do we have $\alpha(x)$ and $\beta(x)$ defined and each equal to 0.

<u>Stage 4e of the construction</u>.

Define

$$x_0 = \text{the least number in } D,$$

$$x_{n+1} = \text{the least element of } D \text{ greater than } lh(\alpha_{4n+3}).$$

Let $\alpha \supseteq \alpha_{4e-1}$ and $\beta \supseteq \beta_{4e-1}$ comprise the least pair of strings admissable at stage 4e with $lh\alpha = lh\beta = x_e$.

Define

$$\alpha_{4e}, \beta_{4e} = \alpha * 0, \beta * 0 \quad \text{respectively}.$$

<u>Stage 4_{e+1}</u>.

Look for the least triple (β,x,s) (under some recursive ordering) for which $\beta \supset \beta_{4e}$ and $\Phi_{e,s}(\beta,x)$ is defined and such that if $\Phi_e(\beta,x) = 1$ then $\beta(x) \neq 0$.

If no such (β,x,s) exists set

$$\alpha_{4e+1}, \beta_{4e+1} = \alpha_{4e} * 1, \beta_{4e} * 1 \quad \text{respectively.}$$

Otherwise let α, β' be the least pair with $\alpha \supset \alpha_{4e}$, $\beta' \supset \beta_{4e}$, α,β admissable at stage $4e+1$ with $\beta' \supseteq \beta$, $lh(\alpha) = lh(\beta')$ and such that $\alpha(x)$ is defined and is not equal to $\Phi_e(\beta,x)$.

Define

$$\alpha_{4e+1},\ \beta_{4e+1} = \alpha\ ,\ \beta' \qquad \text{respectively.}$$

Stage $4e+2$.

The same as stage $4e+1$ but with α and β interchanged and $4e+2$, $4e+1$ written for $4e+1$, $4e$ respectively.

Stage $4e+3$.

Let (m,n) be the e^{th} pair of numbers (in some recursive ordering).

We look for the least quadruple (β^1, β^2, x, s) for which β^1, β^2 split $4e+2$ for n through x at stage s.

If (β^1, β^2, x, s) does not exist set

$$\alpha_{4e+3},\ \beta_{4e+3} = \alpha_{4e+2} * 1,\ \beta_{4e+2} * 1 \text{ respectively, and}$$

otherwise look for the least pair (α, s) with $\alpha \supset \alpha_{4e+2}$ such that α, β^1 and α, β^2 are admissable pairs and $\Phi_{m,s}(\alpha, x)$ is defined.

If α exists let β^i be the least of the strings β^1, β^2 such that

$$\Phi_m(\alpha\ ,\ x) \neq \Phi_n(\beta^i, x)$$

and take α^*, β^* to be the least admissable pair of strings of equal length with $\alpha^* \supseteq \alpha$ and $\beta^* \supseteq \beta^i$.

Define

$$\alpha_{4e+3},\ \beta_{4e+3} = \alpha^*,\ \beta^* \qquad \text{respectively.}$$

Otherwise take α_{4e+3}, β_{4e+3} to be the least pair of admissable strings of equal length with $\alpha_{4e+3} \supset \alpha_{4e+2}$ and $\beta_{4e+3} \supset \beta_{4e+2}$ and with

$lh(\alpha_{4e+3}) \geqslant lh(\beta^1) + lh(\beta^2)$.

LEMMA 1. A and B are recursive in $\underline{0}'$.

PROOF: We examine the questions asked during the construction. The result will follow from the fact that they are uniformly recursive in $\underline{0}'$ and in what we have defined at previous stages of the construction so that we could define α_n, β_n by a recursion schema using $\underline{0}'$ recursive functions.

(1) - (4) below correspond to the stages 4e to 4e+3 of the construction.

(1) We require the number x_e, which depends only on $D \in \underline{0}'$ and on the strings α_{4e-1} and β_{4e-1} already defined (the admissable pairs form a recursive set).

(2) The set of triples (β, x, s) that we are interested in is a r.e. set qualified by a predicate recursive in α_{4e} and β_{4e},

(3) Similarly for the triples (α, x, s).

(4) The quadruples (β^1, β^2, x, s) and the pairs (α, s) each form the intersection of an α_n, β_n recursive set and a fixed r.e. set.

It follows that if we write $\underline{a} = \deg A$ and $\underline{b} = \deg B$ then $\underline{a} \cup \underline{b} \leqslant \underline{0}'$.

LEMMA 2. $\underline{0}' \leqslant \underline{a} \cup \underline{b}$.

PROOF: If we inspect the construction we find that the only stages at which we fail to choose an admissable pair α, β

as extensions of α_n, β_n respectively are the stages $4e \geqslant 0$ when α_{4e}, β_{4e} are chosen to be admissable apart from the fact that

$$\alpha_{4e}(x_e) = \beta_{4e}(x_e) = 0.$$

This means that $A \cap B$ is a subset of D and is infinite since infinitely many numbers x_e are chosen. Since D is intro-reducible we have $D \leqslant_T A \cap B$ where $\deg A \cup B \leqslant \underline{a} \cup \underline{b}$.

It follows from lemmas 1 and 2 that $\underline{0}' = \underline{a} \cup \underline{b}$.

LEMMA 3. $\underline{a}$ and $\underline{b}$ are incomparable.

PROOF: Assume that

$$A = \Phi_e(B)$$

for some number e.

If a triple (β, x, s) exists satisfying stage $4e+1$ of the construction then we have that $\Phi_e(\beta_{4e+1}, x)$ is defined and is not equal to α_{4e+1}, which would mean that $\Phi_e(B;x) \neq A(x)$.

So for every pair (β, x) such that $\beta \supset \beta_{4e}$ and $\Phi_e(\beta, x)$ is defined we have that $\Phi_e(\beta, x) = 1$ which implies that A is empty, contradicting the fact that $A \cap B$ is an infinite subset of D.

LEMMA 4. If $\Phi_m(A)$, $\Phi_n(B)$ are total and $\Phi_m(A) = \Phi_n(B)$ then $\Phi_m(A)$ is recursive.

PROOF: Let (m,n) be the e^{th} pair of numbers. Then at stage

4e+3 we look for a pair β^1, β^2 which split β_{4e+2} for n through some number x at a stage $s \geqslant 0$. If β^1, β^2 do not exist then $\Phi_n(B)$ will be recursive. In order to compute $\Phi_n(B, x)$ for a given number x we need only generate recursively the functionals $\Phi_{n,s}$ and also the extensions σ of β_{4e+2}, and if for some such σ and some $s \geqslant 0$ we have

$$\Phi_{n,s}(\sigma, x) = \delta$$

then we have that

$$\Phi_n(B, x) = \delta.$$

Otherwise there is a beginning β of B, which we can choose to properly extend β_{4e+2}, for which

$$\Phi_n(\beta, x) = \delta' \neq \delta,$$

so that for some $s^* > s$ we have

$$\Phi_{n,s^*}(\beta, x) = \delta' \neq \delta = \Phi_{n,s^*}(\sigma, x)$$

(since $\Phi_n = \bigcup_{s \geqslant 0} \Phi_{n,s}$ and $\Phi_{n,s} \subseteq \Phi_{n,s+1}$ for each s) and so β, σ split β_{4e+2} through x for n at stage s^*.

Say (β^1, β^2, x, s) exists.

If (α, s) does not exist then since β^1, α_{4e+3} and β^2, α_{4e+3} are admissable pairs at stage 4e+3 and $\mathrm{lh}\,\alpha_{4e+3} \geqslant \max\{\mathrm{lh}\,\beta^i \mid i=1 \text{ or } 2\}$ there can be no extension α' of α_{4e+3} for which $\Phi_m(\alpha', x)$ is defined, so that $\Phi_m(A, x)$ is not defined.

If α exists then by choice of α_{4e+3} and β_{4e+3} we have that

$$\Phi_n(\beta_{4e+3}, x), \quad \Phi_m(\alpha_{4e+3}, x)$$

are defined and unequal so that

$$\Phi_n(B) \neq \Phi_m(A).$$

It follows from the lemma that $\underline{a} \cap \underline{b}$ exists and is equal to $\underline{0}$.

We can adapt the proof so as to replace $\underline{0}, \underline{0}'$ by $\underline{c}$, $\underline{c}'$ for any given $\underline{c} \geqslant \underline{0}$. This has the corollary that every degree is a non-trivial meet of a pair of degrees. Lachlan [3] has shown that if $\underline{c}$ is r.e. and strictly below $\underline{0}'$ then we cannot in general choose the pair of degrees to be r.e. But we can ask:

(1) Is every degree below $\underline{0}'$ a non-trivial meet of two degrees below $\underline{0}'$? , or

(2) Is there some general class of r.e. degrees with non-trivial r.e.meets (e.g., Robert Robinson's low degrees [5])?

Sacks [6] examines lattice embeddings for the degrees as a whole and Lachlan [4] and Thomason [9] obtain results about lattice embeddings in the r.e. degrees, but little is known about embeddings which preserve greatest and least elements in the degrees below $\underline{0}'$ or in the r.e. degrees between two comparable r.e. degrees.

THEOREM 2. There exists a pair of minimal degrees with least upper bound $\underline{0}'$.

PROOF: Let f be a recursive function which enumerates without repetitions a r.e. set D of degree $\underline{0}'$. At stages $s \geqslant 0$ we construct strings α^0_s and α^1_s and take the pair of degrees to be the degrees of A^0 and A^1 where

$$A^i(x) = \lim_s \alpha^i_s(x)$$

for each $i \leqslant 1$ and each x. The strings α^0_s and α^1_s will be chosen to lie on certain finite trees $T^i_{e,s}$ *with* $i \leqslant 1$ where at any given stage $s \geqslant 0$ there will only be a finite number of these trees different from ϕ.

If $\sigma \subseteq \alpha^p_s$ for some $p \leqslant 1$ then σ is said to have <u>rank e of the p^{th} kind at stage s+1</u> where e is the least number for which

$$\sigma \subseteq T^p_{e,s}(\delta)$$

for some $\delta \leqslant 1$. We order the pairs (e,p) lexicographically upwards.

The method by which we make A^0, A^1 to be of minimal degree is a constructivisation of that of Spector's in [8] but different from that of [11] in that not every syzygy defined on a tree $T^p_{e,s}$ at a stage $2s + p - 1 \geqslant 0$ will be a splitting pair for e, and also in that we will not expect the limit trees

$$T^p_e = \lim_s T^p_{e,s}$$

to be partial recursive, although if A^p lies on an infinite splitting portion of T^p_e then we will be able to select a partial recursive splitting subtree of T^p_e on which A^p also lies.

If $T^p_{e,s}(\tau)$, say, is defined and has been chosen as a member of syzygy which splits for e then if there is no syzygy for $T^p_{e,s}$ based on $T^p_{e,s}(\tau)$ which splits for e at stage s we say that $T^p_{e,s}(\tau)$ is a <u>boundary string for $T^p_{e,s}$ at stage s</u>.

The method by which we make D recursive in the join of the degrees of A^0 and A^1 is to ensure that if there is a stage s such that $T^0_{e+1,s}(\phi)$ and $T^1_{e+1,s}(\phi)$ are beginnings of A^0 and A^1 respectively then

$$D_s(e) = D(e)$$

where $D_s = \{f(k) \mid k \leqslant s\}$.

<u>Stage 0 of the construction.</u>

Define

$$T^p_{-1,0} = I \qquad \text{(the identity tree)}$$

for each p = 0 or 1.

$$T^p_{e,0} = \phi \qquad \text{otherwise.}$$

Define

$$\alpha_0^p = \phi \qquad \text{for each } p = 0 \text{ or } 1.$$

<u>Stage 2s + p + 1.</u>

Define

$$T^p_{-1,s+1} = I.$$

Assume that $T^p_{i,s+1}$ has been defined for each $i < e$ and that $T^p_{e,s+1}(\tau)$ has been defined where τ is a string other than ϕ and that

$$T^p_{e,s+1}(\tau) = T^p_{e,s}(\tau).$$

We may now base a syzygy on $T^p_{e,s}(\tau)$ at stage s+1 through one of the following cases:

Case I.

Let $T^p_{e,s}(\tau)$ have rank k of the p^{th} kind at stage s+1.

Assume that $T^p_{e,s}(\tau * 0)$, $(\tau * 1)$ are defined and are compatible with each tree $T^p_{i,s+1}$ with $i < e$.

Also assume that one of the following hold:

(1) $T^p_{e,s}(\tau * 0)$, $(\tau * 1)$ split for e at stage s+1, or

(2) there is no pair of strings $\sigma_1, \sigma_2 \supset T^p_{e,s}(\tau)$ which split for e at stage s+1 and which satisfy the following conditions:

(i) σ_1, σ_2 are compatible with every tree $T^p_{i,s+1}$ with $i < e$ and neither of σ_1, σ_2 properly extend a boundary string $T^p_{i,s+1}(\pi)$ with $i < e$ and

$$T^p_{e,s}(\tau) \subset T^p_{i,s+1}(\pi),$$

(ii) if σ_1 or σ_2 extends some <u>prohibited string</u> π (a term to be defined later) where

$$T^p_{e,s}(\tau) \subset \pi$$

then we may <u>free</u> π by <u>stretching</u> a string of rank k^* of the

$(1-p)^{th}$ kind where

$$(k, p) < (k^*, 1-p),$$

(iii) by defining

$$\sigma_1, \sigma_2 = T^p_{e,s+1}(\tau * 0), (\tau * 1)$$

respectively we do not make some string π of rank k^* of the q^{th} kind at stage s+1 <u>liable to require attention at a stage greater than 2s + p + 1</u> (again a term to be defined later) through a number $e' > k^*$ where

$$(k,p) \geqslant (k^*, q) \text{ and } q \leqslant 1 \text{, or}$$

(3) $T^p_{e,s}(\tau) \not\subseteq \alpha^p_s$.

We define

$$T^p_{e,s+1}(\tau * 0), (\tau * 1) = T^p_{e,s}(\tau * 0), (\tau * 1) \quad \text{respectively.}$$

Case II.

Assume that case I does not hold and that none of (1) - (3) of case I holds.

So there does exist a pair σ_1, σ_2 as described in (2).

We define

$$T^p_{e,s+1}(\tau * 0), (\tau * 1) = \sigma_1, \sigma_2$$

respectively, and we require a string of rank k^* of the $(1-p)^{th}$ kind at stage s+1 to free all the prohibited strings π such that

$$T^p_{e,s}(\tau) \subset \pi \subseteq \sigma_1$$

or $\quad T^p_{e,s}(\tau) \subset \pi \subseteq \sigma_2$,

where we choose k^* to be the largest possible such number.

Case III.

If cases I and II do not hold but

$$T^p_{e,s}(\tau) \subseteq \alpha^p_s$$

define

$$T^p_{e,s+1}\ (\tau * 0),\ (\tau * 1) = \sigma_1' \ ,\ \sigma_2'$$

respectively where σ_1' , σ_2' is the least pair of incompatible strings which extend $T^p_{e,s}(\tau)$ and which are compatible with every tree $T^p_{i,s+1}$ with $i < e$. This concludes case III.

We say that <u>e^* is liable to require attention through $x-1$ for q at stage $2s+p+1$</u> if

$$D_s(x-1) = 1$$

and e^* is the largest number for which there is a string $\sigma \supset T^q_{e^*, s+p \dot{-} q}(0)$ which is incompatible with each $T^q_{x,w}$ (0), $w \leqslant s$,

such that

$$T^{1-q}_{x,w}(0) \subseteq \alpha^{1-q}_{s+p \dot{-} (1-q)} \quad ,$$

and which is compatible with each tree $T^q_{i,s+p\dot{-}q}$ such that $i < e^*$.

At stage 2s+p+1 we make a string π of rank k^* of the q^{th} kind <u>liable to require attention at a stage greater than 2s+p+1</u>

if at the end of stage 2s+p+1 we have that k^*, k^{**} are liable to require attention through some $x-1$ for q, $1-q$ respectively at stage $2s+p+2$, and

$$(k^*, q) > (k^{**}, 1-q).$$

Assume now that the extensions σ_1, σ_2 of $T^p_{e,s+1}(\tau)$ as described in I(2) do exist except that (iii) fails to hold. Then σ_1 or σ_2 extends a string $T^p_{x,t}(0)$ where $t \leqslant s$ and $x-1$ is greater than the rank of $T^p_{e,s}(\tau)$. If e^* is liable to require attention through $x-1$ for p at stage $2s+p+1$ we require $T^p_{e^*,s}(0)$ to be stretched at stage at stage $2s+p+1$ unless this has already been done at some earlier stage for the potential syzygy σ_1, σ_2.

The new number enumerated in D at stage s+1

Let

$$f(s+1) = x-1.$$

If $T^p_{e^*,s}(0)$ is liable to require attention through $x-1$ at stage $2s+p+1$ for some $e^* \geqslant 0$ then $T^p_{e^*,s}(0)$ requires attention at stage $2s+p+1$ through $x-1$. We will try to ensure at every subsequent stage $w > s$ that we either have

$$T^p_{x,t}(0) \not\subseteq \alpha^p_\omega \text{ or } T^{1-p}_{x,t}(0) \not\subseteq \alpha^{1-p}_w$$

for each $t \leqslant s$, and so as to achieve this certain strings $T^p_{x,t}(0)$, $T^{1-p}_{x,t}(0)$ with $t \leqslant s$ may become strings prohibited through x.

At stage $2s+p$ we may have required some string to free a prohibited string π where we defined extensions of some string through case II at stage $2s+p$ one of which extended π. Assume that π was prohibited at stage $2s+p$ by virtue of being a string $T^{1-p}_{y,t}(0)$ for some y, t where $t \leqslant t'$ and $f(t'+1) = y-1$.

Then we choose $T^{p}_{e^*,s}(0)$ in a similar way to that above to be a string for which there is a proper extension σ compatible with all the trees $T^{p}_{i,s}$ with $i < e^*$ and incompatible with each string $T^{p}_{y,t}(0)$ such that $t \leqslant t'$ and

$$T^{1-p}_{y,t}(0) \subseteq \alpha^{1-p}_{s+p-1}.$$

And $T^{p}_{e^*,s}(0)$ is the string which is required to free π at stage $2s+p+1$ if (and only if) $T^{p}_{e^*,s+1}(\phi)$ is defined and $T^{p}_{e^*,s}(0)$, (1) and σ are compatible with each tree $T^{p}_{i,s+1}$ with $i < e$. Also we have that each string $T^{p}_{y,t}(0)$ with $t \leqslant t'$ and

$$T^{1-p}_{y,t}(0) \subseteq \alpha^{1-p}_{s+p-1}$$

is prohibited through y at each stage $t^* > 2s+p$ such that we have not required $T^{p}_{y,t}(0)$ to be freed at a stage t^{**} such that

$$t^* > t^{**} > 2s+p.$$

<u>We define $T^{p}_{e,s+1}(0)$, (1) at stage 2s+p+1 if $T^{p}_{e,s+1}(\phi)$ is defined and is not equal to α^{p}_{s+1}</u>

Case (a).

Assume that $T^{p}_{e,s}(0)$, (1) are defined and are compatible with every tree $T^{p}_{i,s+1}$ with $i < e$.

If e requires attention because $f(s+1) = x-1$ or if $T^p_{e,s}(0)$ is required to free a string π where π is prohibited by virtue of being a string $T^{1-p}_{x,t}(0)$ for some $t \leqslant s$, or if $T_{e,s}(0)$ is required to be stretched because we would have defined strings $T^p_{i,s+1}(\tau * 0)$, $(\tau * 1)$ through case II apart from the fact that $T^p_{i,s+1}(\tau * 0)$ or $(\tau * 1)$ extends a string $T^p_{x,t}(0)$ for some $t \leqslant s$ then let $\sigma \supset T^p_{e,s}(0)$ be the least string incompatible with each $T^p_{y,t}(0)$ for

$$t \leqslant \mu\, t'(f(t'+1) = y-1 \text{ or } t' = s)$$

and $$T^{1-p}_{y,t}(0) \subseteq \alpha^{1-p}_{s+p} \quad ,$$

where $z \leqslant y \leqslant x$ where z is chosen to be the least number for which there exists such a string and such that σ is compatible with each tree $T^p_{i,s+1}$ with $i < e$.

Define

$$T^p_{e,s+1}(0),\ (1) = \sigma,\quad T^p_{e,s}\ (1)$$

respectively and in the former case every string $T^q_{x,t}(0)$ with $t \leqslant s$ and

$$T^{1-q}_{x,t}(0) \subseteq \alpha^{1-q}_{sg(p+q)+s}$$

becomes prohibited through x.

We now inductively make changes in the definitions of some of the strings $T^p_{i,s+1}(\tau)$, $i < e$. Assume that the necessary changes have been made on all trees $T^p_{j,s+1}$, $j < i$. Let $T^p_{i,s+1}(\tau)$ be the least string such that either $T^p_{i,s+1}(\tau)$ is not compatible with some tree $T^p_{j,s+1}$ with $j < i$, or

$$T^p_{e,s}(0) \subseteq T^p_{i,s+1}(\tau) \subset \sigma$$

and $T^p_{i,s+1}(\tau)$ is a boundary string for i at stage $s+1$. If no such string exists we make no changes. Otherwise we re-define

$$T^p_{i,s+1}(\tau) = \sigma,$$

and $T^p_{i,s+1}(\tau * \pi)$ is undefined for each $\pi \supset \phi$. We say that <u>$T^p_{e,s}(0)$ is stretched to σ</u> $(= T^p_{e,s+1}(0))$.

Otherwise we define

$$T^p_{e,s+1}(0),\ (1) = T^p_{e,s}(0),\ (1) \quad \text{respectively.}$$

Case (b).

If $T^p_{e,s}(0)$, (1) are not defined and compatible with every tree $T^p_{i,s+1}$, $i<e$, let σ_1, σ_2 be the least pair of incompatible extensions of $T^p_{e,s+1}(\phi)$ compatible with every tree $T^p_{i,s+1}$ with $i < e$ where if one of these strings extends no string $T^p_{e,t}(0)$ with $t \leqslant s$ we take it to be σ_2.

Define

$$T^p_{e,s+1}(0),\ (1) = \sigma_1, \sigma_2 \quad \text{respectively.}$$

In either of cases (a) or (b) if

$$T^p_{e,s+1}(0),\ (1) \neq T^p_{e,s}(0),\ (1),$$

define

$$T^p_{e,s+1}(0) = \alpha^p_{s+1} = T^p_{e+1,s+1}(\phi),$$

otherwise merely defining

$$T^p_{e,s+1}(0) = T^p_{e+1,s+1}(\phi).$$

LEMMA 5. For each number $e \geqslant 0$ and each $p \leqslant 1$ $T^p_e(0) = \lim_s T^p_{e,s}(0)$ is defined.

PROOF: First of all we show that there is a stage after which $T^p_{e,s}(0)$, (1) do not change other than by being stretched. As inductive hypothesis we take:

(i) for all $s >$ some t $T^q_{i,s}(0)$, (1) change value only through being stretched if $(i,q) < (e,p)$,

(ii) $D_{t^*}[e] = D[e]$

where $t = 2t^* + q^* + 1$,

(iii) for each $i <$ some $e' < e$, if $s > t$ then $T^p_{e,s}(0)$, (1) do not change value because of the definition of a new syzygy for $T^p_{i,w}$ at a stage $2w + p - 1 > 2s + p - 1$.

We inductively verify the validity of the hypothesis for every $e' < e$ and from this obtain the first part of the step in the main induction.

We may assume that at no stage $s > t^*$ is $T^p_{e-1,s}(0)$ stretched. To see this we look at the three ways in which

$T^{p}_{e-1,s}(0)$ might be stretched:

1. $T^{p}_{e-1,s}(0)$ may be required to free some prohibited string π through the definition of strings on a tree through case II.

But in order that this should happen the string for which new extensions are defined must have rank k where

$$(k, 1-p) < (e-1, p).$$

And this means that some string $T^{1-p}_{e^*,s}(0)$ where

$$(e^*, 1-p) < (e, p)$$

changes at a stage $s > t^*$ and not through being stretched which contradicts (i) of the inductive hypothesis.

2. $e-1$ may require attention at some stage greater than t for p.

We show that this can happen at most a finite number of times. At stage t e-1 can only be liable to require attention through a finite number of numbers x-1 since $T^{p}_{x,t'}(0)$ is only defined for a finite number of numbers x with $t' \leqslant t$, and $e-1$ can only require attention at most once through each of these numbers. Also it is easy to see that if $T^{p}_{x,t'}(0)$ is defined for no $t' \leqslant t$ then e-1 cannot require attention through x-1 at a stage $2s+p+1 > t$. Since $e-1$ is not liable to require attention through x-1 at stage t and since

$$D_{t^*}[e] = D[e],$$

we must define extensions at some stage $> t$ of some string $T^q_{i,s}(\tau)$ of rank e' which renders $e-1$ liable to require attention. This is because if $e-1$ becomes liable to require attention through $x-1$ through some e' requiring attention at a stage $t' > t$ through a number $x'-1$ then we have $x < x'$, since if a string of rank e' is stretched to be incompatible with each string onto which the x'^{th} tree of the r^{th} kind maps 0 at stages $2u+r+1 < t' = 2s'+r+1$ where

$$\alpha^{1-r}_{s'+r} \supseteq T^{1-r}_{x',u+1}(0)$$

then it will be stretched to be incompatible with all such strings of greater rank. And $x > x'$ since otherwise by the construction there can have been no string of rank $\geqslant e'$ of the r^{th} kind incompatible with each $T^r_{x,u}(0)$ defined before stage t' with

$$\alpha^{1-r}_{s'+r} \supseteq T^{1-r}_{x,u}(0)$$

and compatible with all the i^{th} trees at stage t' with $i < e'$.

So at some stage $t' > t$ we base a syzygy on a string $T^q_{i,s+1}(\tau)$ of rank e' of the q^{th} kind at stage $s+1$ which renders $e-1$ liable to require attention at some stage greater than t. There are two possibilities:

(a) $(e', q) \geqslant (e-1, p)$.

But this cannot happen since

$$D_{t^*}[e] = D[e]$$

and because $T^q_{i,s+1}$ $(\tau * 0)$, $(\tau * 1)$ are defined through case II and must satisfy condition (2) (iii) of the construction at stage t'.

(b) As for case 1. above we cannot have

$$(e', q) < (e-1, p)$$

because of (i) of the inductive hypothesis.

3. We may require $T^p_{e-1,s}(0)$ to be stretched at a stage $2s + p + 1 > t$.

This means that there are potential extensions σ_1, σ_2 of a string $T^p_{e',s}$ (τ) which satisfy all the requirements of case II at stage $2s + p + 1$ except for (iii) where $T^p_{e',s}(\tau)$ is of rank $\leqslant e-1$ of the p^{th} kind at stage $s+1$. Then the assumption implies that if e^* is liable to require attention through $x-1$ for $1-p$ at stage $2s+p+1$ where $T^p_{e-1,s}(0)$ is required to be stretched because one of the potential extensions σ_1 or σ_2 extends a string $T^p_{x,w}$ (0) with $w \leqslant s$ then $x > e$ and

$$(e^*, 1-p) < (e-1, p)$$

and $e-1$ is liable to require attention through $x-1$ for p at stage $2s+p+1$.

$$(e^*, 1-p) < (e-1, p)$$

since no alterations are made to trees of the $(1-p)^{th}$ kind at stage $2s+p+1$ and so if by taking

$$\sigma_1, \sigma_2 = T^p_{e',s+1} \quad (\tau * 0), (\tau * 1)$$

respectively we would have made a string π of rank k^* of the q^{th} kind liable to require attention at some stage greater than $2s+p+1$ then we have

$$(e-1,p) > (k^*,q) \geqslant (e^*, 1-p).$$

We show that there can only be finitely many such numbers x, or more specifically, if e^*, e-1 are liable to require attention for 1-p, p respectively through x-1 at stage $2s+p+1 > t$ where

$$(e^*, 1-p) < (e-1, p)$$

then e^*, e-1 are liable to require attention for 1-p, p respectively through x-1 at stage t. This is because if the former holds then e-1 is liable to require attention through x-1 at stage $2s+p+1$ and from part 2. we know that in this case e-1 must have been liable to require attention through x-1 at stage t.

Lastly we notice that $T^p_{e-1,s}(0)$ can only be stretched by being required to be stretched at a finite number of stages through a given number x-1, for if $T^p_{e-1,s}(0)$ is stretched through being required to be stretched at a stage $2s+p+1 > t$ through x-1 then e-1 is not liable to require attention for p through x-1 at stage $2s+p+3$ since $x > e$, and in fact is not liable to require attention for p through x-1 at a stage $> 2s+p+3$ by a similar argument to that in which we limited the relevant numbers x-1 to a finite set.

So $T^p_{e-1,s}(0)$ is stretched at no stage $2s+p+1 > t$ and hence by the inductive hypothesis $T^p_e(\phi)$ exists where

$$T^p_e(\phi) = \lim_s T^p_{e,s}(\phi) = \lim_s T^p_{e-1,s}(0)$$

and $\qquad T^p_{e,t}(\phi) = T^p_e(\phi).$

We may assume that for all $i < e$ either there is a string τ^i for which

$$T^p_{i,s}(\tau^i) = T^p_e(\phi)$$

for all $s > t^*$ or else $T^p_{e,s}(\phi)$ lies on $T^p_{i,s}$ for no $s > t^*$.

If $T^p_{e,s}(0), (1)$ are to change at a stage $2s+p+1 > t$ other than through being stretched we must at stage $2s+p+1$ have

$$T^p_{i,s+1}(\tau^i * 0), (\tau^i * 1) \neq T^p_{i,s}(\tau^i * 0), (\tau^i * 1)$$

respectively for some $i < e$.

We take as the hypothesis for a sub-induction:

There is a stage $2t(i) + p + 1 > t$ such that for each $j < i$ either for each $s > t(i)$ $T^p_{j,s}(\tau^j * 0), (\tau^j * 1)$ split $T^p_{j,s}(\tau^j)$ for j at stage $s+1$ or $T^p_{j,s}(\tau^j * 0), (\tau^j * 1)$ split for j at no stage $s+1 > t(i)$;

and also for each $j < i$, each $\pi \supset \phi$, if for some $s > t(i)$ and every $\pi' * q$ with $q \leqslant 1$ and $\pi' * q \subseteq \pi$ we have that $T^p_{j,s}(\tau^j * \pi' * q)$ $(\tau^j * \pi' * 1-q)$ split $T^p_{j,s}(\tau^j)$ for j at stage $s+1$ and are not boundary strings for a tree $T^p_{k,s}$ with $k < j$ then $T^p_{j,w}(\tau^j * \pi)$ changes at no stage $2w+p+1 > 2s+p+1$ except as a result of being stretched.

There are two possibilities for the number i :

(a) at no stage $2s+p+1 > 2t(i)+p+1$ do we define strings $T^p_{i,s+1}(\tau^i * 0), (\tau^i * 1)$ which split $T^p_{i,s}(\tau^i)$ for i at stage $s+1$. In this case the next stage of the induction follows immediately.

(b) at a stage $2s+p+1 > 2t(i)+p+1$ the strings $T^p_{i,s+1}(\tau^i * 0), (\tau^i * 1)$ are defined and split for i at stage $s+1$.

If σ_1, σ_2 are respective extensions of $T^p_{i,s+1}(\tau^i * 0), (\tau^i * 1)$ then σ_1, σ_2 split for i at each stage $w+1 \geqslant s+1$. This means that if there is a stage $w+1 > s+1$ such that $T^p_{i,w+1}(\tau^i * 0), (\tau^i * 1)$ do not split for i at stage $w+1$ then at some stage $2u+p+1 > 2s+p+1$ we must have

$$(T^p_{i,u+1}(\tau^i * 0), (\tau^i * 1)) \neq (T^p_{i,u}(\tau^i * 0), (\tau^i * 1))$$

other than as a result of a member of the latter syzygy being stretched at a stage $2u+p+1$. That is we must define strings $T^p_{j,u+1}(\pi * 0), (\pi * 1)$ through case II at stage $2u+p+1$ where $j < i$ and $T^p_{j,u}(\pi)$ is a boundary string for some tree $T^p_{k,u}$ with $k \leqslant j$ and where

$$T^p_{j,u}(\pi) \subset T^p_{i,u}(\tau^i * q)$$

for some $q \leqslant 1$ (If $T^p_{j,u}(\pi)$ is not such a boundary string then

~~we would define strings $T^p_{j,u}(\pi)$ is a boundary string for some tree $T^p_{k,u}$ with $k \leqslant j$ and where~~

$$\cancel{T^p_{j,u}(\pi) \subset T^p_{i,u}(\tau^i * q)}$$

~~for some $q \leqslant 1$ (If $T^p_{j,u}(\pi)$ is not such a boundary string then~~ we would define strings $T^p_{j,u+1}$ $(\pi' * 0)$, $(\pi' * 1)$ through case II where $\pi' \subset \pi$ and by the construction this would preclude such a definition for $T^p_{j,u+1}$ $(\pi * 0)$, $(\pi * 1)$ at stage $2u + p + 1$).

Since $u > t^*$ we have

$$T^p_{i,u}(\tau^i) = T^p_i(\tau^i)$$

and so

$$T^p_{i,u}(\tau^i) \subseteq T^p_{j,u}(\pi),$$

and since $u > t(i)$ we cannot have $\pi = \tau^i$ by the inductive hypothesis which means that

$$T^p_{i,u}(\tau^i) \subset T^p_{j,u}(\pi) \subset T^p_{i,u}(\tau^i * q)$$

for some $q \leqslant 1$.

Choose $v \geqslant s$ to be the least number for which we have that $T^p_{j,v+1}(\pi)$ is a boundary string for a tree $T^p_{k,v+1}$ with $k \leqslant j$ and for which we have that

$$T^p_{i,v+1}(\tau^i) \subset T^p_{j,v+1}(\pi) \subset T^p_{i,v+1}(\tau^i * q).$$

Let

$$T^p_{j,v+1}(\pi) = T^p_{k,v+1}(\pi^*).$$

There are now three possible ways in which the first part of the next step of the sub-induction can fail with $t(i+1) = s$:

(i) either

$$T^p_{k,v}(\pi^*) \subseteq T^p_{i,v}(\tau^i)$$

and $T^p_{k,v}(\pi^*)$ alters through stretching at stage $2v+p+1$, or

$$T^p_{i,v}(\tau^i * q) \subseteq T^p_{k,v}(\pi^*)$$

and $T^p_{i,v}\ (\tau^i * q)$ alters through stretching at stage $2v+p+1$,

(ii) $T^p_{k,v+1}\ (\pi^*)$ is defined at stage $2v+p+1$ through case II of the construction,

(iii) $T^p_{k,v}\ (\pi^* * 0)$, $(\pi^* * 1)$ split for k at stage v but $T^{\ p}_{k,v+1}\ (\pi^* * 0)$, $(\pi^* * 1)$ do not split for k at stage v+1.

If the first part of (i) occurs then

$$T^p_{k,v+1}\ (\pi^*) = T^p_{i,v+1}\ (\tau^i)$$

if the latter is to be defined.

For the second part we notice that if

$$T^p_{i,v+1}\ (\tau^i * q) \supset T^p_{k,v+1}\ (\pi^*)$$

then by the nature of the stretching operation $T^p_{k,v+1}\ (\pi^*)$ cannot be a boundary string for $T^p_{k,v+1}$.

If (ii) holds then there is a $\pi' \subset \pi^*$ such that

$T^p_{k,v}(\pi') \subset T^p_{k,v+1}(\pi^*) \subseteq T^p_{i,v+1}(\tau^i * q)$ and such that

$T^p_{k,v}(\pi')$ is a boundary string for a tree $T^p_{k',v+1}$ with

$$k' \leqslant k \leqslant j.$$

Arguing as above we must also have

$$T^p_{i,v}(\tau^i) \subset T_{k,v}(\pi')$$

which contradicts the choice of v.

Finally (iii) cannot occur since by the second part of the hypothesis of the sub-induction it would mean that there is a $\pi' * q'$ where $q' \leqslant 1$ such that

$$\tau^k \subset \pi' * q' \subset \pi^* * r$$

for some $r \leqslant 1$ and such that $T^p_{k,v}(\pi' * q')$, $(\pi' * 1 - q')$ do not split for k at stage v. And this would imply by definition of case II of the construction that we have a string

$$T^p_{k,v}(\tau^k * \sigma) \subseteq T^p_{k,v}(\pi^*)$$

with $\sigma \supset \phi$ which is a boundary string for some tree $T^p_{k',v}$ with

$$k' < k \leqslant j.$$

Since

$$T^p_{k,v}(\tau^k * \sigma) \supset T^p_{k,v}(\tau^k) = T^p_{i,v}(\tau^i)$$

and

$$T^p_{k,v}(\pi^*) \subset T^p_{i,v}(\tau^i * q)$$

this contradicts the definition of v again.

The second half of the $(i+1)^{th}$ step of the sub-induction proceeds exactly as does the proof of the first half when case (b) applies. The only difficulty is that we must deal with the relevant splitting pairs $T^p_{i,s+1}(\pi * 0), (\pi * 1)$ on $T^p_{i,s+1}$ above $T^p_{i,s+1}(\tau^i)$ by induction on the length of π where the base of the induction is given by the first part of the sub-induction.

It follows that $t(e)$ exists.

Let $i < e$ be the greatest number for which $T^p_{i,t(e)+1}(\tau^i * 0), (\tau^i * 1)$ are defined and split for i at stage $t(e) + 1$. Then from the proof of the sub-induction, for each $w > t(e)$ we have

$$T^p_{i,w}(\tau^i * 0), (\tau^i * 1) \subseteq T^p_{i,w+1}(\tau^i * 0), (\tau^i * 1)$$

respectively and if $i < j < e$ and $T^p_{j,w+1}(\tau^j * 0), (\tau^j * 1)$ are defined then

$$T^p_{j,w+1}(\tau^j * 0), (\tau^j * 1) = T^p_{i,w+1}(\tau^i * 0), (\tau^i * 1)$$

respectively.

So at each stage $2w + p + 1 > 2t(e) + p + 1$ we have

$$T^p_{e,w+1}(0), (1) \supseteq T^p_{i,t(e)+1}(0), (1)$$

respectively where we only fail to have equality when

$T^p_{e,w+1}(0)$, (1) have been stretched for some reason.

As in the proof of the first part of the sub-induction we never have a boundary string π for a tree $T^p_{j,w+1}$ with $j < e$ where

$$T^p_{e,w+1}(\phi) \subset \pi \subset T^p_{e,w+1}(0)$$

or $$T^p_{e,w+1}(\phi) \subset \pi \subset T^p_{e,w+1}(1)$$

and hence

$$T^p_{e,w+1}(0),\ (1) \supseteq T^p_{e,w}(0),\ (1) \text{ respectively for each } w > t(e)$$

and $T^p_{e,w}(0)$, (1) only change value at a stage $2w+p+1$ through being stretched.

It follows easily from the lemma that $\lim_s T^p_{e,s}$ exists for all e and for each $p \leqslant 1$.

From the proof of lemma 5 we have that $\lim_s T^p_{e,s}(0)$ exists for each e, p. If there is a stage t such that

$$T^p_{e,s}(\tau) \subseteq \alpha^p_s$$

for no $s > t$ then by the construction if

$$(T^p_{e,s+1}(\tau * 0),(\tau * 1)) \neq (T^p_{e,s}(\tau * 0),(\tau * 1))$$

for some $s > t$ other than through a member of the syzygy being stretched we have that $T^p_{e,w}(\tau * 0),(\tau * 1)$ are defined for no $w > s$. And since we only stretch strings $T^p_{i,s}(0)$ such that

$$T^p_{i,s}(0) \subseteq \alpha^p_s$$

at stage $2s + p + 1$, we cannot stretch $T^p_{e,s}(\tau * 0)$, $(\tau * 1)$ at a stage $s > t$.

If

$$T^p_{e,s}(\tau) \subseteq \alpha^p_s$$

for each $s >$ a stage t we notice that if τ has length K then neither of $T^p_{e,s}(\tau * 0)$ or $(\tau * 1)$ have rank greater than $e + K + 1$ at any stage $s \geqslant 0$. Hence $\lim_s T^p_{e,s}(\tau * 0),(\tau * 1)$ exist since $\lim_s T^p_{e+K+1,s}(0)$ exists.

LEMMA 6. D is recursive in the recursive join of A^0 and A^1.

PROOF: Since $\lim_s T^p_{e,s}(0)$ exists for each $e \geqslant 0$ and each $p = 0$ or 1 we have that if

$$A^0, A^1 = \lim_s \alpha_s^0, \lim_s \alpha_s^p$$

respectively then A^0, A^1 are well defined sets of degree less than or equal to $\underline{0}'$.

We show that whenever $s(e)$ is a number for which $T^0_{e+1,s(e)}(0)$ and $T^1_{e+1,s(e)}(0)$ are respective beginnings of A^0 and A^1 it happens that

$$D_{s(e)}(e) = D(e).$$

The lemma follows from the fact that the whole construction proceeds uniformly recursively and from the fact that there always exists such a number $s(e)$.

Assume that there are numbers s and e such that

$$e \in D$$

but for which

$$D_s(e) = 1$$

and $T^p_{e+1,s}(0)$ is a beginning of A^p for each number $p \leqslant 1$.

Let

$$s^* = \mu s \, (e \in D_{s+1})$$

so that $s \leqslant s^*$ and either some number $e(0)$ requires attention through e at step $2s^* + 1$ or some number $e(1)$ requires attention through e at step $2s^* + 2$.

We need only verify that some number $e^* \geqslant 0$ is liable to require attention through e for 0 or 1 at stage $2s^* + 1$ or stage $2s^* + 2$ respectively, which is easy since at worst we can take

$$e(p) = 0$$

for each $p = 0$ or 1.

To prove this for each $p \leqslant 1$ take as inductive hypothesis:

$T^p_{o,w}(0)$ is defined and if

$$T^p_{o,w}(0) = \pi$$

then for some string σ we have that $\pi * \sigma$ is incompatible with each $T^p_{e+1,u}(0)$ with $u \leqslant w$.

The base of the induction is given by $w = 1$ since $T^p_{o,1}(0),(1)$ are defined for each $p \leqslant 1$ but $T^p_{y,u}(0)$ is defined for no numbers y, u, p where

$$y > 0, \quad 0 \leqslant p \leqslant 1 \quad \text{and} \quad 0 \leqslant u \leqslant 1.$$

Assuming that the induction fails let the hypothesis hold for $w = W$ but not for $w = W + 1$, and let

$$T^p_{o,W}(0) = \Pi$$

and let $\Pi * \Sigma$ be incompatible with each $T^p_{e+1,u}(0)$ with $u \leqslant W$. So

$$\Pi * \Sigma \subseteq T^p_{e+1,W+1}(0)$$

or

$$T^p_{o,W+1}(0) \neq \Pi .$$

If

$$\Pi * \Sigma \subset T^p_{e+1,W+1}(0)$$

then they hypothesis holds for $w = W + 1$ with

$$\pi * \sigma = T^p_{e+1,W+1}(1).$$

We cannot have

$$\Pi * \Sigma = T^p_{e+1,W+1}(0)$$

unless the hypothesis hold for $w = W+1$ with more than one string σ (say Σ and Σ^*) since by the construction of $T^p_{e+1,W+1}(0),(1)$ we would not have a $u < W+1$ for which

$$T^p_{e+1,u}(0) \subseteq T^p_{e+1,W+1}(1)$$

unless

$$T^p_{e+1,u}(0) \subseteq T^p_{e+1,W+1}(0).$$

So if

$$\Pi * \Sigma = T^p_{e+1,W+1}(0)$$

the hypothesis would follow for $w = W+1$ with

$$\sigma = \Sigma^*.$$

If

$$T^p_{o,W+1}(0) \not\supseteq \Pi$$

then since

$$T^p_{-1,W} = T^p_{-1,W+1} = I$$

for each $p \leqslant 1$ it must happen that $T^p_{o,W}(0)$ is stretched to $T^p_{o,W+1}(0)$ at stage $2W + p + 1$. If

$$T^p_{o,W+1}(0) \subset \Pi * \Sigma$$

then the inductive step follows using

$$\pi * \sigma = \Pi * \Sigma$$

again. If

$$T^p_{o,W+1}(0) \supseteq \Pi * \Sigma$$

then we may take for $w = W+1$

$$\pi = T^p_{o,W+1}(0) \;, \sigma = \pi * q$$

for some $q \leqslant 1$ such that α^p_{W+1} is incompatible with $\pi * q$.

By the construction if $T^p_{o,W+1}(0)$ is incompatible with $\Pi * \Sigma$ then since $\Pi * \Sigma$ satisfies the hypothesis for $w = W$ we must have that $T^p_{o,W+1}(0) * q$ satisfies the hypothesis for $w = W + 1$ for some $q \leqslant 1$.

So $e(p)$ requires attention at step $2s^* + p + 1$ for some $p \leqslant 1$ which means that

$$T^q_{e+1,s}(0) \not\subseteq \alpha^q_{s^*+1}$$

for some $q \leqslant 1$.

Let $t^* > s^*$ be the least number such that

$$T^q_{e+1,s}(0) \subseteq \alpha^q_{w+1}$$

for each $q \leqslant 1$ and each $w \geqslant t^*$.

Inspection of the construction gives us that at each stage greater than $2s^* + p + 1$ for each $u < s^* + 1$ if

$$T^q_{e+1,u}(0) \subseteq \alpha^q_w$$

for some $q \leqslant 1$ then $T^{q'}_{e+1,u}(0)$ is a string prohibited through $e + 1$ for some $q' \leqslant 1$, and so at each stage $2w + p + 1 > 2t^* + p + 1$ there is a string σ prohibited through $e + 1$ such that

$$\sigma \subseteq \alpha^o_{w+1} \quad \text{or} \quad \sigma \subseteq \alpha^1_{w+1}.$$

By the construction if there is a string σ prohibited through $e + 1$ for q at the end of stage $2t^* + q + 1$ where

$$\sigma \subseteq \alpha^q_{t^*+1}$$

but

$$\sigma \not\subseteq \alpha^q_{t^*}$$

then this cannot occur through a string $T^q_{e*,t*}(\tau)$ being stretched where

$$T^q_{e*,t*}(\tau) \subseteq \alpha^q_{t*}$$

and

$$\sigma \subseteq T^q_{e*,\ t*+1}(\tau) \subseteq \alpha^q_{t*+1} .$$

This is because as in the proof of the above ~~of the above~~ induction we can show that there is an extension of $T^q_{e*,\ t*}(\tau)$ compatible with each tree $T^q_{i,t*+1}$ with $i < e^*$ but incompatible with each string $T^q_{e+1,u}(0)$ such that

$$T^q_{e+1,u}(0) \not\subseteq T^q_{e*,t*}(\tau)$$

and $u < s^* + 1$. By the choice of t^* there is no string $T^q_{e+1,v}(0)$ with $v < s^* + 1$ and

$$T^q_{e+1,v}(0) \subseteq T^q_{e*,\ t*}$$

and so by the definition of the stretching operation

$$\sigma \not\subseteq T^q_{e*,t*+1} .$$

This means that we require a string to free σ at stage $2t^* + q + 2$. And each string $T^{1-q}_{e+1,u}(0)$ with $u \leqslant s^*$ and

$$T^q_{e+1,u}(0) \subseteq \alpha^q_{t*+1}$$

becomes prohibited for $1 - q$ at stage $2t^* + q + 2$.

We construct a function $E(2w + r + 1)$ where $r \leqslant 1$ which we take to be undefined for

$$2w + r + 1 \leqslant 2t^* + q + 1 ,$$

and take as inductive hypothesis :

At stage $2w + r + 1 > 2t^* + q + 1$ we define strings $T^r_{e',w+1}(\tau * 0),(\tau * 1)$ through case II of the construction resulting in a requirement for a string to free a string prohibited through $e + 1$ at stage $2w + r + 2$ where $T^r_{e',w+1}$ has rank $E(2w + r + 1)$ and

$$(E(2w + r + 1),r) < (E(2w + r),1 - r)$$

if $E(2w + r)$ is defined.

We examine stage $2W + R + 1$ assuming the result for each stage $2w + r + 1$ with

$$2W + R + 1 > 2w + r + 1 > 2t^* + q + 1 .$$

At stage $2W + R + 1$ a string of rank k is required to free a string σ prohibited through $e + 1$ and all strings $T^R_{e+1,u}(0)$ with $u \leqslant s^*$ and

$$T^{1-R}_{e+1,u}(0) \subseteq \alpha^{1-R}_{W+R}$$

are prohibited for R at stage $2W + R + 1$ by virtue of the fact that extensions of some string of rank k' were defined at stage $2W + R$ through case II where

$$(k',1-R) < (k,R) .$$

We can only fail to free σ if we define strings $T^R_{e',W+1}(\tau * 0),(\tau * 1)$ through case II for some $e' \geqslant 0$ one of which extends a string σ' prohibited through $e + 1$. But in this case we require a string to free σ' at stage $2W + R + 2$, and since such a string cannot have rank greater than k', and by the conditions laid down for case II of the construction we must have that

$$(k',1-R) > (\text{rank } T^R_{e',W+1}(\tau),R) .$$

If $E(2W + R)$ is defined so that

$$k' = E(2W + R)$$

we obtain the result by defining

$$E(2W + R + 1) = \text{rank } T^{R}_{e',W+1}(\tau) \ .$$

But from this we see that we have obtained an infinite descending sequence of numbers and so there is no such t^* and the lemma follows.

LEMMA 7. A^0 and A^1 are of minimal degree.

PROOF: We show for each $p \leqslant 1$ and each $e \geqslant 0$ that if $\Phi_e(A^p)$ is total then either $\Phi_e(A^p)$ is recursive or A^p is recursive in $\Phi_e(A^p)$. It will follow that the degrees of A^0 and A^1 are minimal by lemma 6 and from the fact that $\underline{0}'$ is neither recursive nor minimal.

We say that trees <u>T and T' are mutually compatible</u> if $T(\phi)$ and $T'(\phi)$ are compatible and (considering a tree as an array of strings) we have that

$$\{\sigma \mid \sigma \in T \text{ and } \sigma \supseteq T'(\phi)\}$$

is compatible with

$$\{\sigma \mid \sigma \in T' \text{ and } \sigma \supseteq T(\phi)\}$$

and vice-versa. We write $T \simeq T'$.

We describe a uniformly recursive set of trees

$$\{\Psi^{p}_{e,s} \mid e,s \geqslant 0,\ 1 \geqslant p \geqslant 0\}$$

whose members have the following properties:

(1) $\sigma \in \Psi^{p}_{e,s} - \Psi^{p}_{e,s+1} \rightarrow \sigma$ is an end string for $\Psi^{p}_{e,s}$ and there is a string σ' such that $\sigma \subseteq \sigma'$ and

$$\sigma' \in \Psi^{p}_{e,s+1} \ ,$$

(2) $\Psi^p_{e+1,s} \subseteq \Psi^p_{e,s}$

for each e, s, p,

(3) $\Psi^p_{e,s} \simeq T^p_{e,s}$

for each e, s, p and no string σ on $\Psi^p_{e,s}$ is a boundary string for a tree $T^p_{i,s}$ with $i \leqslant e$ unless σ is an end string for $\Psi^p_{e,s}$,

(4) either $\Psi^p_{e,s}$ is a splitting tree for e at stages $s \geqslant 0$ or there are only finitely many pairs of strings σ_1 , σ_2 such that for some $s \geqslant 0$

$$\sigma_1 , \sigma_2 \in \Psi^p_{e,s}$$

and σ_1 , σ_2 split for e at stage s ,

(5) for each e, p we have that

$$\Psi^p_e = \lim_s \Psi^p_{e,s}$$

exists and contains infinitely many beginnings of A^p .

Assume that $\Psi^p_{e,s}$ has been defined for each $e < e^* + 1$ and each $s \geqslant 0$ for some given $p \leqslant 1$ (We take $\Psi^p_{-1,s} = I$ for each $s \geqslant 0$ and each $p \leqslant 1$).

If for every

$$\pi \in \Psi^p_{e^*} \cap \{A^p[n] \mid n \geqslant 0\}$$

there is a pair

$$T^p_{e^*+1}(\tau * 0), (\tau * 1) \in \Psi^p_{e^*}$$

which split π for $e^* + 1$ define $s(e^* + 1)$ to be the least number for which there is a string

$$T^p_{e^*+1,s(e^*+1)}(\tau) = T^p_{e^*+1}(\tau) \in \Psi^p_{e^*} \cap \Psi^p_{e^*,s(e^*+1)}$$

and take $\pi(e^* + 1)$ to be the least such string $T^p_{e^*+1,s(e^*+1)}(\tau)$ which is a beginning of A^p. There must be such a string as long as we can prove (5) for $\Psi^p_{e^*}$ and since by the construction every beginning of A^p is compatible with $T^p_{e^*+1}$ and since by assumption there is a string

$$T^p_{e^*+1}(\tau) \in \Psi^p_{e^*} .$$

Then $\Psi^p_{e^*+1,s}$ is defined to be empty if $s < s(e^* + 1)$ and otherwise is the set of strings

$\{T^p_{e^*+1,s}(\tau) \in \Psi^p_{e^*,s} \mid$ for each $T^p_{e^*+1,s}(\tau' * q)$ with $q \leqslant 1$ and $\pi \subset T^p_{e^*+1,s}(\tau' * q) \subseteq T^p_{e^*+1,s}(\tau)$ we have that $T^p_{e^*+1,s}(\tau' * q)$, $(\tau' * 1-q)$ split for $e^* + 1\}$ arranged in a tree-like array.

Otherwise choose a

$$\pi \in \Psi^p_{e^*} \cap \{A^p[n] \mid n \geqslant 0\}$$

such that no pair

$$T^p_{e^*+1}(\tau * 0),\ (\tau * 1) \in \Psi^p_{e^*}$$

split π for $e^* + 1$.

Define $s(e^* + 1)$ to be the least number for which there is a

$$T^p_{e^*+1,s(e^*+1)}(\tau) = T^p_{e^*+1}(\tau) \in \Psi^p_{e^*} \cap \Psi^p_{e^*,s(e^*+1)}$$

with

$$T^p_{e^*+1}(\tau) \supset \pi$$

if such a number exists and take $\pi(e^* + 1)$ to be the least such string

$$T^p_{e^*+1}(\tau) \subset A^p .$$

And if $s(e^* + 1)$ is still not determined take it to be $s(e^*)$ and take $\pi(e^* + 1) = \pi$.

In both of the latter cases $\Psi^p_{e^*+1,s}$ is nowhere defined for $s < s(e^* + 1)$ and is

$$\{\Psi^p_{e^*,s}(\tau) \supseteq \pi(e^* + 1)\}$$

otherwise with the tree ordering induced by $\Psi^p_{e^*,s}$.

We now verify the facts (1) - (5) for

$$\{\Psi^p_{e^*+1,s} \mid s \geqslant 0\}$$

using these facts for each set

$$\{\Psi^p_{e,s} \mid s \geqslant 0\}$$

with $e \leqslant e^*$ and also using any relevant details arising from the inductive definitions.

From the uniform recursiveness of the approximating trees and from (1) it will follow that each Ψ^p_e is 'almost' partial recursive so that by a modified Spector-type argument the lemma will follow from (4) and (5).

We distinguish three cases in the definition of $\{\Psi^p_{e^*+1,s}\}_{s\geqslant 0}$ and treat each in turn.

<u>Case 1</u>. Say

$$\Psi^p_{e^*+1,s}(\tau) = T^p_{e^*+1,s}(\tau') \not\in \Psi^p_{e^*+1,s+1} .$$

From the definition of $\Psi^p_{e,s}$ for $e \leqslant e^* + 1$ we see that if $T^p_{e,s}(\sigma)$ is a boundary string for $T^p_{e,s}$ and

$$T^p_{e,s}(\sigma) \subseteq \Sigma$$

for some string $\Sigma \in \Psi^p_{e,s}$ then

$$T^p_{e,s}(\sigma) \in \Psi^p_{e,s}$$

or

$$T^p_{e,s}(\sigma) \subset \pi(e) \ .$$

In the former case $T^p_{e,s}(\sigma)$ is an end string for $\Psi^p_{e,s}$ by (3) and so

$$T^p_{e,s}(\sigma) \not\subset \Psi^p_{e^*+1,s}(\tau)$$

and in the latter case $T^p_{e,s}(\sigma)$ is a boundary string for $T^p_{e,s+1}$ by the choice of $\pi(e)$. This means that $T^p_{e^*+1,s+1}(\tau')$ is defined and

$$T^p_{e^*+1,s+1}(\tau') \supset T^p_{e^*+1,s}(\tau')$$

since $T^p_{e^*+1,s}(\tau')$ can only change through being stretched. And since only boundary strings are stretched we have that $T^p_{e^*+1,s}(\tau')$ is a boundary string for some tree $T^p_{e,s}$ with $e \leqslant e^* + 1$ and so by (3) and the definition of $\Psi^p_{e^*+1,s}$ we have that $T^p_{e^*+1,s}(\tau')$ is an end string for $\Psi^p_{e^*+1,s}$. Since $T^p_{e^*+1,s}(\tau')$ is a member of a splitting syzygy for $e^* + 1$ at stage s, $T^p_{e^*+1,s+1}(\tau')$ is a member of syzygy splitting for $e^* + 1$ at stage $s + 1$.

Finally

$$T^p_{e^*+1,s+1}(\tau') \in \Psi^p_{e^*,s+1}$$

since otherwise let $e < e^* + 1$ be the least number for which

$$T^p_{e^*+1,s+1}(\tau') \not\subseteq \Psi^p_{e,s+1} .$$

Say there is a string Π which is a boundary string for $T^p_{e,s+1}$ where

$$\Pi \subset T^p_{e^*+1,s+1}(\tau') .$$

Then by definition of the stretching operation we must have

$$\Pi \subset T^p_{e^*+1,s}(\tau') = \Psi^p_{e^*+1,s}(\tau)$$

which contradicts (3) by definition of $\Psi^p_{e^*+1,s}$.

$\Psi^p_{e,s+1}$ will be defined through case (1) since otherwise every end string for $\Psi^p_{e,s+1}$ is an end string for $\Psi^p_{e-1,s+1}$.

So $T^p_{e^*+1,s+1}(\tau')$ lies on $T^p_{e,s+1}$ and there is an end string Π for a tree $\Psi^p_{e',s+1}$ with $e' < e$ such that

$$\Pi \subset T^p_{e^*+1,s+1}(\tau')$$

which contradicts the way in which we chose e.

This proves (1) for $e^* + 1$.

We obtain $\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e^*,s}$ directly from the construction.

To see that

$$\Psi^p_{e^*+1,s} \simeq T^p_{e^*+1,s}$$

for each s we first note that every string on $\Psi^p_{e^*+1,s}$ also lies on $T^p_{e^*+1,s}$ and so $\Psi^p_{e^*+1,s}$ is compatible with $T^p_{e^*+1,s}$ and $T^p_{e^*+1,s}(\phi)$ and $\Psi^p_{e^*+1,s}(\phi)$ are compatible by the construction.

Assume that

$$\{\sigma \mid \sigma \in T^{p}_{e^*+1,s} \text{ and } \sigma \supseteq \Psi^{p}_{e^*+1,s}(\phi)\}$$

is not compatible with $\Psi^{p}_{e^*+1,s}$.

Then for some $T^{p}_{e^*+1,s}$ with

$$T^{p}_{e^*+1,s}(\tau) \supset \Psi^{p}_{e^*+1,s}(\phi)$$

we have that $T^{p}_{e^*+1,s}(\tau)$ neither lies on $\Psi^{p}_{e^*+1,s}$ nor extends an end string for $\Psi^{p}_{e^*+1,s}$.

So for some

$$\Psi^{p}_{e^*+1}(\pi) = T^{p}_{e^*+1,s}(\tau')$$

we have that

$$\Psi^{p}_{e^*+1,s}(\phi) \subset T^{p}_{e^*+1,s}(\tau) \subset \Psi^{p}_{e^*+1,s}(\pi)$$

which by the definition of $\Psi^{p}_{e^*+1,s}$ implies that

$$T^{p}_{e^*+1,s}(\tau) \notin \Psi^{p}_{e^*,s}.$$

Let e be the least number such that

$$T^{p}_{e^*+1,s}(\tau) \notin \Psi^{p}_{e,s}.$$

Since

$$\Psi^{p}_{e^*+1,s} \subseteq \Psi^{p}_{e,s}$$

so that

$$T^{p}_{e^*+1,s}(\tau) \supseteq \Psi^{p}_{e,s}(\phi)$$

we must have $\Psi^{p}_{e,s}$ defined by means of case 1 and so by the definition of $\Psi^{p}_{e,s}$ and the fact that

$$T^{p}_{e^*+1,s}(\tau) \subset \Psi^{p}_{e^*+1,s}(\pi)$$

we have that $T^{p}_{e^*+1,s}(\tau)$ lies on $T^{p}_{e,s}$. Say

$$T^{p}_{e^*+1,s}(\tau) = T^{p}_{e,s}(\pi')$$

where

$$\Psi^p_{e,s}(\phi) \subset T^p_{e,s}(\pi') \subset \Psi^p_{e,s}(\pi'')$$

some π'' . Then by the definition of $\Psi^p_{e,s}$

$$T^p_{e,s}(\pi') \in \Psi^p_{e,s}$$

since

$$T^p_{e^*+1,s}(\tau) \in \Psi^p_{e',s}$$

for each $e' < e$, which is a contradiction.

Now let

$$\Psi^p_{e^*+1,s}(\tau) \subset \Psi^p_{e^*+1,s}(\tau') ,$$

some τ', be a boundary string for a tree $T^p_{e,s}$ with $e \leqslant e^* + 1$, and choose e to be the least such number. Since

$$\Psi^p_{e',s} \supseteq \Psi^p_{e^*+1,s}$$

for each $e' \leqslant e^* + 1$, $\Psi^p_{e^*+1,s}(\tau)$ is an end string for no tree $\Psi^p_{e',s}$ with $e' \leqslant e^* + 1$. By the definition of a case 1 construction $\Psi^p_{e,s}$ cannot be defined as a splitting tree for e. But neither of the other cases can hold since $\Psi^p_{e^*+1,s}(\tau)$ being a boundary string for $T^p_{e,s}$ would contradict the choice of $s(e)$ and $\pi(e)$.

By the definition of $\Psi^p_{e^*+1,s}$ we have that $\Psi^p_{e^*+1,s}$ is a splitting tree for $e^* + 1$ at each stage $s \geqslant 0$.

From the proof of (1) we see that if $\Psi^p_{e^*+1,s}(\tau)$ is defined and is not an end string for $\Psi^p_{e^*+1,s}$ then

$$\Psi^p_{e^*+1,s}(\tau) = \Psi^p_{e^*+1,w}(\tau)$$

for each $w \geqslant s$, and if $\Psi^p_{e^*+1,s}(\tau)$ is an end string for $\Psi^p_{e^*+1,w}$ then for some σ we have that for each $w \geqslant s$

$$\Psi^p_{e^*+1,w}(\tau) = T^p_{e^*+1,w}(\sigma)$$

where $T^p_{e^*+1,w}(\sigma)$ is defined and changes only by virtue of being stretched. Since $\lim_s T^p_{e^*+1,s}(\sigma)$ exists so does $\lim_s \Psi^p_{e^*+1,s}(\tau)$.

By definition

$$\pi(e^* + 1) = \Psi^p_{e^*+1}(\phi)$$

is a beginning of A^p. Let $\Psi^p_{e^*+1}(\tau)$ be some beginning of A^p where

$$\Psi^p_{e^*+1}(\tau) = T^p_{e^*+1}(\sigma) .$$

Since case 1 applies there is a pair

$$T^p_{e^*+1}(\sigma * \rho * 0),\ (\sigma * \rho * 1) \in \Psi^p_{e^*}$$

which split $T^p_{e^*+1}(\sigma)$ for $e^* + 1$. By the second part of (3) we deduce that $T^p_{e^*+1}(\sigma * 0), (\sigma * 1)$ split $T^p_{e^*+1}(\sigma)$ for $e^* + 1$, and since

$$T^p_{e^*+1}(\sigma) \subset A^p$$

$T^p_{e^*+1}(\sigma * q)$ is a beginning of A^p for some $q \leqslant 1$. So as in the proof of the first part of (3) and by (5) for each tree T^p_e with $e < e^* + 1$ we have that

$$T^p_{e^*+1}(\sigma * q) \in \Psi^p_e$$

for each $e < e^* + 1$. This means that $T^p_{e^*+1}(\sigma)$ is a boundary string for no tree T^p_e with $e \leqslant e^* + 1$.

We show that $T^p_{e^*+1}(\sigma * 1 - q)$ lies on each tree Ψ^p_e with $e < e^* + 1$.

Assume that e is the least number for which

$$T^p_{e^*+1}(\sigma * 1 - q) \notin \Psi^p_e ,$$

so that Ψ^p_e is defined by case 1 and

$$T^p_{e^*+1}(\sigma) = T^p_e(\rho)$$

for some ρ.

Since $T^p_{e^*+1}(\sigma * 0), (\sigma * 1)$ split for $e^* + 1$ and since $T^p_{e^*+1}(\sigma)$ is not a boundary string for T^p_e but $T^p_e(\rho)$ is a member of a pair which splits for e by definition of Ψ^p_e we have that

$$T^p_{e^*+1}(\sigma * 1 - q) \in T^p_e .$$

Otherwise we would have that for some string π $T^p_e(\rho * \pi)$ is a boundary string for T^p_e

and

$$T^p_{e^*+1}(\sigma) \subset T^p_e(\rho * \pi) \subset T^p_{e^*+1}(\sigma * 1 - q)$$

which would contradict condition (i) of case II of the main construction. From this we get

$$T^p_{e^*+1}(\sigma * 1 - q) \in \Psi^p_e ,$$

a contradiction. So the definition of $\Psi^\rho_{e^*+1}$ implies that

$$T^p_{e^*+1}(\sigma * 0),\ (\sigma * 1) \in \Psi^p_{e^*+1}$$

and so there are beginnings of A^p of arbitrarily long length on $\Psi^p_{e^*+1}$.

Cases 2 and 3.

The only real difference between these cases lies in the definition of $\pi(e^* + 1)$, which will appear in the proof of (5).

If

$$\sigma \in \Psi^p_{e^*+1,s} - \Psi^p_{e^*+1,s+1}$$

then by the definition of $\Psi^p_{e^*+1,s+1}$ we have that

$$\sigma \in \Psi^p_{e^*,s} - \Psi^p_{e^*,s+1}$$

and so by the inductive hypothesis σ is an end string for $\Psi^p_{e^*,s}$ and for some ρ we have that

$$\Psi^p_{e^*,s+1}(\rho) \supset \sigma .$$

By the definition of $\Psi^p_{e^*+1,s+1}$

$$\Psi^p_{e^*,s+1}(\rho) \in \Psi^p_{e^*+1,s+1}$$

since

$$\Psi^p_{e^*,s+1}(\rho) \supset \pi(s^* + 1) .$$

By definition we have

$$\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e^*,s} .$$

By the choice of $\pi(e^* + 1)$ there is no pair

$$T^p_{e^*+1,s+1}(\tau * 0),\ (\tau * 1) \in \Psi^p_{e^*,s+1}$$

above $\pi(e^* + 1)$ which is defined through case II. So for each string τ and each number s such that $T^p_{e^*+1,s+1}(\tau * 0)$, $(\tau * 1)$ are defined and compatible with $\pi(e^* + 1)$ and are beginnings of strings on $\Psi^p_{e^*,s+1}$ there is a string π and a number $e < e^* + 1$ for which

$T^p_{e,s+1}(\pi), (\pi * 0), (\pi * 1)$ are defined and equal to $T^p_{e^*+1,s+1}(\tau), (\tau * 0), (\tau * 1)$ respectively. So the tree T consisting of those strings σ such that

$$\sigma \in T^p_{e^*+1,s}$$

and σ is compatible with $\pi(e^* + 1)$ and σ is a beginning of a string on $\Psi^p_{e^*+1,s}$ is mutually compatible with $T^p_{e^*,s}$. Also

$$T^p_{e^*,s} \simeq \Psi^p_{e^*,s}$$

by the inductive hypothesis and

$$\Psi^p_{e^*,s} \simeq \Psi^p_{e^*+1,s}$$

by definition of $\Psi^p_{e^*+1,s}$. Hence

$$\Psi^p_{e^*+1,s} \simeq T$$

which implies that

$$T^p_{e^*+1,s} \simeq \Psi^p_{e^*+1,s}$$

Since

$$T^p_{e^*+1,s} \neq \phi$$

implies that

$$T^p_{e^*+1,s}(\phi) = \pi(e^* + 1) \in \Psi^p_{e^*+1,s}$$

the first part of (3) follows for $e^* + 1$.

Since

$$\Psi^p_{e^*+1,s} \subseteq \Psi^p_{e^*,s}$$

and there are no boundary strings for trees $T^p_{e,s}$ with $e \leqslant e^*$ on $\Psi^p_{e^*,s}$ other than end strings, and since there are no boundary strings for $T^p_{e^*+1,s}$ on

$\Psi^p_{e^*,s}$ since case 2 or 3 applies, the second part of (3) follows.

We show that the second part of (4) holds for $\Psi^p_{e^*+1}$ and treat cases 2 and 3 separately.

Assume that $\Psi^p_{e^*+1,s}$ is defined through case 2 at each stage $s \geqslant 0$ but that there are infinitely many pairs

$$\sigma_1, \sigma_2 \in \Psi^p_{e^*+1}$$

which split for $e^* + 1$.

We know that $\Psi^p_{e^*+1}(\phi)$ is a beginning of A^p and lies on $T^p_{e^*+1}$ and that no string on $\Psi^p_{e^*+1}$ which is not an end string for $\Psi^p_{e^*+1}$ can be a boundary string for a tree T^p_e with $e \leqslant e^* + 1$. Also we know that there is no pair

$$T^p_{e^*+1}(\tau * 0), (\tau * 1) \in \Psi^p_{e^*+1}$$

which split for $e^* + 1$.

So there are infinitely many pairs σ_1, σ_2 such that at some stage $s \geqslant 0$ we have :

(a) $\sigma_1, \sigma_2 \in \Psi^p_{e^*+1,s}$,

(b) σ_1, σ_2 split $\Psi^p_{e^*+1,s}(\phi)$ for $e^* + 1$ at stage s where

$$\Psi^p_{e^*+1,s}(\phi) \subseteq \alpha^p_s ,$$

(c) $T^p_{e^*+1,s}(\tau)$ is defined and

$$T^p_{e^*+1,s}(\tau) = \Psi^p_{e^*+1,s}(\phi) = T^p_{e^*+1}(\tau) ,$$

(d) if $\pi \subseteq \sigma_1$ or σ_2 and π is a boundary string for a tree $T^p_{e,s}$ for some $e \leqslant e^* + 1$ then

$$\pi \subseteq T^p_{e^*+1,s} \quad .$$

Since we have (3) for each $e \leqslant e^* + 1$ (a) gives that σ_1 and σ_2 are compatible with each tree $T^p_{e,s}$ with $e \leqslant e^* + 1$.

Looking at case II of the main construction we see that either:

[1] there are infinitely many beginnings of A^p which are beginnings of strings π prohibited at a stage $s \geqslant 0$ where we are unable to free π at stage $s + 1$ other than by stretching a string of rank k^* of the $(1-p)^{th}$ kind where

$$(k^*, 1-p) < (\text{rank } T^p_{e^*+1,s} \text{ , } p)$$

(since by lemma 6 no beginning of A^p is prohibited at infinitely many stages), or

[2] at stage $2s + p + 1$ we have

$$T^p_{e^*+1,s}(\tau) = T^p_{e^*+1}(\tau)$$

and there are strings σ_1 and σ_2 om $\Psi^p_{e^*+1,s}$ which we would define to be $T^p_{e^*+1,s+1}(\tau * 0), (\tau * 1)$ respectively if it were not for the fact that condition (iii) for case II does not hold for σ_1, σ_2, where we can choose (σ_1, σ_2) and s to be as large as we like.

To see that [1] does not apply we notice that for each x there can only be finitely many prohibited strings $T^p_{x,t}(0)$ and that since

$$T^p_e = \lim_s T^p_{e,s}$$

exists for each e there are only finitely many strings

$$T^{1-p}_{x,t}(0) \subseteq T^{1-p}_{e,s}(0)$$

at some stage $s \geqslant 0$ with

$$(e-1, 1-p) < (\text{rank}\, T^{p}_{e^*+1,s}(\tau), p) .$$

So eventually we must be able to choose our splitting pair σ_1, σ_2 such that if

$$T^{p}_{x,t}(0) \subseteq \sigma_1, \text{ or } T^{p}_{x,t}(0) \subseteq \sigma_2$$

where $T^{p}_{x,t}(0)$ is prohibited then $T^{p}_{x,t}(0)$ can be freed by stretching a string $T^{1-p}_{e',s}(0)$ where

$$(e-1, 1-p) < (e', 1-p) .$$

Again the fact that there are only finitely many strings

$$T^{1-p}_{x,t}(0) \subseteq T^{1-p}_{e,s}(0)$$

at some stage $s \geqslant 0$ with

$$(e-1, 1-p) < (\text{rank}\, T^{p}_{e^*+1,s}(\tau), p)$$

implies that we can only make strings of rank e with

$$(e, 1-p) < (\text{rank}\, T^{p}_{e^*+1,s}(\tau), p)$$

liable to require attention through a finite set of numbers. Let $X - 1$ be the largest such number. If we take t^* to be a stage such that

$$T^{q}_{x,s}(0) = T^{q}_{x}(0)$$

for each $q \leqslant 1$ each $s > t^*$ then [2] cannot occur at a stage $2s + p + 1 > 2t^* + p + 1$ since in this case a string of rank less than X of the p^{th} kind would be required to be stretched at stage $2s + p + 1$.

If the second part of (4) does not hold for $\Psi^p_{e^*+1}$, then $\Psi^p_{e^*+1}$ is not defined through case 3. If there is no string $T^p_{e^*+1}(\tau)$ such that

$$T^p_{e^*+1}(\tau) \in \Psi^p_{e^*}$$

then since A lies on $\Psi^p_{e^*}$ and by the construction either A lies on $T^p_{e^*+1}$ or some beginning of A is an end string for $T^p_{e^*+1}$ we have that for some $t^* > 0$, some τ, each $s > t^*$, $T^p_{e^*+1,s}(\tau)$ is defined and

$$T^p_{e^*+1,s}(\tau) = T^p_{e^*+1,s-1}(\tau)$$

and there is no syzygy for $T^p_{e^*+1,s}$ based on $T^p_{e^*+1,s}(\tau)$ which contradicts case III of the construction of $T^p_{e^*+1,s}$

Since (5) holds for $e = e^*$ (5) holds for $e = e^* + 1$.

The end of the proof is a straight-forward modification of the arguments of [8].

Assume that Ψ^p_{e+1} is defined through case 2 or case 3. Choose a $\pi \supseteq \Psi^p_{e+1}(\phi)$ above which no pair of strings on Ψ^p_{e+1} split for e .

Define

$s(x) = \mu s[\Phi_{e,s}(\sigma,x)$ is defined with $\sigma \in \Psi^p_{e+1,s}$ and $\sigma \supset \pi]$ and $\sigma_x = \mu\sigma[\Phi_{e,s(x)}(\sigma,x)$ is defined with $\sigma \in \Psi^p_{e+1,s(x)}, \sigma \supset \pi]$ and

$$f(x) = \Phi_{e,s(x)}(\sigma_x, x) .$$

f is partial recursive and since A^p is on Ψ^p_{e+1} if $\Phi_e(A)$ is total then f is recursive.

Say $f \neq \Phi_e(A)$. Then for some beginning $A[n]$ of A and some $x \geq 0$ we have $A[n] \in \Psi^p_{e+1}$ and $\Phi_e(A[n],x)$ is defined and

$$\Phi_{e,s(x)}(\sigma_x, x) \neq \Phi_e(A[n],x) .$$

So by (1) and (5) there is a $\sigma \supseteq \sigma_x$ such that $\sigma \in \Psi^p_e$ and $\sigma, A[n]$ split π for e, a contradiction.

Assume that Ψ^p_{e+1} is defined through case 1. We show how to compute arbitrarily large beginnings of A whenever $\Phi_e(A)$ is total by asking questions uniformly recursive in $\Phi_e(A)$. Assume that $A[n]$ is given where

$$A[n] = \Psi^p_{e+1,s}(\tau)$$

for some $s \geq 0$, some τ.

Wait until $\Psi^p_{e+1,t}(\tau * 0), (\tau * 1)$ are defined for some $t \geq s$, so that

$$\Psi^p_{e+1,t}(\tau) \supseteq A[n]$$

by (1) and is a beginning of A by (5), which implies that

$$\Psi^p_{e+1,t}(\tau * q) \subset A$$

for some $q \leq 1$. By the construction $\Psi^p_{e+1,t}(\tau * 0), (\tau * 1)$ split for e through some $x \geq 0$ at stage t and so $\Psi^p_{e+1,t}(\tau * q)$ is a beginning of A where

$$\Psi^p_{e+1,t}(\tau * q) \supset A[n]$$

and

$$\Phi_{e,t}(\Psi^p_{e+1,t} x) = \Phi_e(A,x) .$$

Hence

$$A \leqslant_T \Phi_e(A) .$$

COROLLARY (Shoenfield). There is a minimal degree below $\underline{0}'$ incomparable with any given degree strictly between $\underline{0}$ and $\underline{0}'$.

Another problem concerning joins is that of characterising the joins of degrees of sets satisfying particular separation properties. Also does theorem 2 remain true when we include the degrees of partial functions? Case [1] has shown that the degrees constructed in the proof of theorem 2 will not be minimal partial degrees.

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Minimal degrees and the jump operator

The jump $\underline{a}'$ of a degree $\underline{a}$ is defined to be the largest degree recursively enumerable in $\underline{a}$ in the upper semi-lattice of degrees of unsolvability. Friedberg [1] showed that the equation $\underline{a} = \underline{x}'$ is solvable if and only if $\underline{a} \geqslant \underline{0}'$. Sacks [5] showed that we can find a solution of $\underline{a} = \underline{x}'$ which is $\leqslant \underline{0}'$ (and in fact is r.e.) if and only if $\underline{a} \geqslant \underline{0}'$ and is r.e. in $\underline{0}'$. Spector [7] constructed a minimal degree and Sacks [5] constructed one $\leqslant \underline{0}'$. So far the only result concerning the relationship between minimal degrees and the jump operator is one due to Yates [9] who showed that there is a minimal predecessor for each non-recursive r.e. degree, and hence that there is a minimal degree with jump $\underline{0}'$. In §1 we obtain an analogue of Friedberg's theorem by constructing a minimal degree solution for $\underline{a} = \underline{x}'$ whenever $\underline{a} \geqslant \underline{0}'$. We incorporate Friedberg's original number theoretic device with a complicated sequence of approximations to the nest of trees necessary for the construction of a minimal degree. In §2 we show that any hope for a result analagous to that of Sacks on the jumps of r.e. degrees cannot be fulfilled since $\underline{0}''$ is not the jump of any minimal degree below $\underline{0}'$.

We use a characterisation of the degrees below $\underline{0}'$ with jump $\underline{0}''$ similar to that found for r.e. degrees with jump $\underline{0}''$ by Robert Robinson [4]. Finally in §3 we give a proof that every degree $\underline{a} \leqslant \underline{0}'$ with $\underline{a}' = \underline{0}''$ has a minimal predecessor. Yates [9] has already shown that every non-zero r.e. degree has a minimal predecessor, but that [ ] there is a non-zero degree $\leqslant \underline{0}'$ with no minimal predecessor. ~~Yates [9] has already shown that every non-zero r.e. degree has a minimal predecessor, but that [ ] there is a non-zero degree $\leqslant \underline{0}'$ with no minimal predecessor.~~

§1 <u>Every complete degree is the completion of a minimal degree.</u>

As usual we will construct a sequence of partial recursive trees. It will be necessary to resort to stage by stage approximations to these trees, and we present an alternative method to that of Yates in [9].

If $A \in \underline{a}$ we may take $\underline{a}'$ to be the degree of the domain of $J(A)$ where J is the partial recursive functional defined by

$$J(A;x) = \mu y\, T_1^{A}((x)_1,(x)_2,y)$$

where we write $J(A;x)$ for $J(A)(x)$. It is easily seen that $\mathrm{dom}\, J(A)$ is r.e. in A and that every set r.e. in A (and so every set r.e. in any $S \in \underline{a}$) is one-one reducible to $\mathrm{dom}\, J(A)$. We define

$J_s(A;x) = \mu y \ (y \leqslant s$ and $T_1^A((x)_1, (x)_2, y))$.

Then dom J_s is finite and we have

$$J(A; x) = \lim_s J_s(A; x).$$

<u>THEOREM 1</u> Let $\underline{c} > \underline{0}'$. Then there exists a minimal degree $\underline{b}$ such that $\underline{b}' = \underline{c}$.

<u>PROOF:</u> Using Friedberg's completeness theorem let $\underline{a}$ be a degree such that $\underline{a}' = \underline{c}$. Let $\{C_s\}$ be an enumeration recursive in $\underline{a}$ of a set C of degree $\underline{c}$. With regard to strings and trees we follow the notation of [8]. We write $T(\tau)$, (τ') for $T(\tau)$, $T(\tau')$. At the end of stage $s \geqslant 0$ there will be constructed a finite number of finite trees $T\sigma, s$, σ a string, and a string β_s will be chosen to lie on certain of these trees. Let $T\sigma = \lim_s T\sigma, s$ and $B = \lim_s \beta_s$. Then if $\sigma = C[m]$ some m, either B will lie on T_σ or there will be an end string (i.e., a well defined string $T_\sigma(\tau)$ for which $T(\tau * 0)$, $(\tau * 1)$ are not defined) on T_σ which is a beginning of B. Let $lh(\sigma)$ = the length of σ. We will try to ensure that if $lh(\sigma) = e+1$, then $T\sigma, s$ is a splitting tree for e, but this will not necessarily happen. $T\sigma, s\ (\tau)$ is said to be a <u>boundary string</u> for σ at stage s if $T_{\sigma,s}(\tau)$ was defined as a member of a splitting pair for e, but $T_{\sigma,s}(\tau * 0)$, $(\tau * 1)$ were not. A string σ is said to be <u>compatible with</u> a tree T if either σ lies on T or σ is an

extension of some end string for T. The strings, S, will be ordered lexicographically upwards. Let $y(\sigma_1, \sigma_2) = \mu y\ (\sigma_1(y) \neq \sigma_2(y))$.

A partial ordering $\leq$ is defined on the pairs of strings as on page 3 of the previous chapter.

At stage s let σ be the longest string such that $\tau \supset T_{\sigma,s}(n)$, where n is the null string. Then we say that τ is of rank σ at stage s and we write $R(\tau, s) = \sigma$. It will follow from the construction that R is a recursive function. We now give the construction.

Stage 0

Define $T_{n,o}(n) = \beta_0$. $T_{\sigma,o}(\tau)$ is undefined otherwise.

Stage s+1

Definition of Tn, s+1

For each τ such that $T_{n,s}(\tau)$ is defined, set

$$T_{n,s+1}(\tau) = T_{n,s}(\tau).$$

If $T_{n,s}(\tau)$ is defined but $T_{n,s}(\tau * 0)$, $(\tau * 1)$ are not defined, let (σ_1, σ_2) be the least pair of incompatible strings $\supset T_{n,s}(\tau)$ such that if $R(T_{n,s}(\tau), s) = \sigma$, then for each p_m^k such that $lh(T_{n,s}(\tau)) < p_m^k \leq lh(\sigma_i)$, $i = 1,2$, $k > 0$, $m \leq lh(\sigma)$,

$$\sigma_i(P_m^k) = 0 \text{ iff } \sigma(m) = 0.$$

Define $T_{n,s+1}$ $(\tau * 0)$, $(\tau * 1) = \sigma_1, \sigma_2$ respectively.

<u>Definition of $T_{\sigma,s+1}$ where $\sigma \supset n$</u>

We assume that $T_{\pi,s+1}$ is defined for each $\pi \subset \sigma$. Assume that $\sigma = \sigma' * i$, $i = 0$ or 1, and that $T_{\sigma',s+1}(n)$ is not an end string for $T_{\sigma',s+1}$.

If σ has a valid follower π, set

$$T_{\sigma,s+1}(n) = \pi.$$

Otherwise set

$$T_{\sigma,s+1}(n) = T_{\sigma',s+1}(i).$$

Assume now that $T_{\sigma,s}$ has not been <u>cancelled</u>: otherwise stage s+1 is now completed for $T_{\sigma,s+1}$. If $T_{\sigma,s+1}(\tau)$ ~~some~~ is defined $= T_{\sigma,s}(\tau)$ some τ, we may define extensions through one of the following three cases:

I. $T_{\sigma,s}(\tau * 0)$, $(\tau * 1)$ are defined and compatible with each tree $T_{\pi,s+1}$, $\pi \subset \sigma$, and either:

(i) $\tau = n$,

or (ii) $T_{\sigma,s}$ $(\tau * 0)$, $(\tau * 1)$ were defined through case II,

or (iii) there do not exist strings σ_1, σ_2 such that

1) σ_1, σ_2 lie on $T_{n,s+1}$,

2) σ_1, σ_2 are compatible with every tree $T_{\pi,s+1}$, $\pi \subset \sigma$,

3) for every boundary string $\sigma' \subset \sigma_1$ or σ_2 on a tree $T_{\pi,s+1}$, $\pi \subset \sigma$, we have $\sigma' \subseteq T_{\sigma,s+1}(\tau)$,

4) σ_1, σ_2 split $T_{\sigma,s+1}$ (τ) for e at stage $s+1$,

5) no σ_i, i = 1,2, is prohibited for $T_{\sigma,s}(\tau)$ at stage s+1.

Define $T_{\sigma,s+1}$ $(\tau * 0)$, $(\tau * 0)$~~, $(\tau *)$ respectively.~~ $= T_{\sigma,s}(\tau * 0), (\tau * 1)$ respectively

In cases II, III we only consider strings $T_{\sigma,s+1}(\tau) \subseteq$ $T_{\sigma',s}$ (n) for some σ'.

II. Case I does not hold, $\tau \supset n$, and there do exist strings as defined in case I (iii).

Define

$T_{\sigma,s+1}$ $(\tau * 0)$, $(\tau * 1) = \sigma_1, \sigma_2$ respectively.

III. Cases I, II do not hold and either

(a) $\tau = n$,

or (b) there do exist strings as defined in (iii) except that 3) does not hold.

Let π_1, π_2 be the least pair of incompatible strings $\supset T_{\sigma,s}(\tau)$ compatible with every tree $T_{\pi,s+1}$, $\pi \subset \sigma$.

Define $T_{\sigma,s+1}(\tau * 0), (\tau * 1) = \pi_1, \pi_2$ respectively.

For each string $\sigma * 0$, $\sigma * 1$ <u>followers</u> may be appointed which will be concerned with ensuring that $\lim_s J_s(\beta_s; e)$ exists where $lh(\sigma) = e+1$.

<u>Prohibited strings</u> may be associated with a given follower. Once appointed a follower and its associated prohibitions will remain until cancelled. We say a follower τ of $\sigma * i$ is <u>valid at stage s+1</u> if $T_{\sigma,s+1}(0), (1) = T_{\sigma,s}(0), (1)$ respectively, and τ is compatible with each tree $T_{\pi,s+1}$, $\pi \subseteq \sigma$.

There are now, for each $T_{\sigma,s+1}(i)$, $i = 0,1$, two cases to consider:

<u>Case 1.</u> There exists a follower π of $\sigma * i$ which is not valid. Cancel π and all associated prohibitions.

<u>Case 2.</u> There exists no follower of $\sigma * i$, but there is a string $\pi \supset T_{\sigma,s+1}(i)$ such that π lies on $T_{n,s+1}$, π is compatible with every tree $T_{\tau,s+1}$, $\tau \subset \sigma$, and $J_{s+1}(\pi; e)$ is defined.

Appoint π the ~~follow~~ follower of $\sigma * i$. All strings $\supset T_{\sigma,s+1}(n)$ incompatible with π are prohibited at each stage w+1 for strings of rank $\supseteq \sigma$ at stage w, $w \geqslant s+1$, until π is cancelled. Cancel all trees $T_{\sigma', s+1}$, $\sigma' \supset \sigma$, and all followers and prohibititions associated with these trees.

Definition of β_{s+1}

Let σ be the largest string such that σ is a beginning of C_{s+1} and $T_{\sigma, s+1}(n)$ is defined.

Set $\quad \beta_{s+1} = T_{\sigma, s+1}(n)$.

LEMMA 1 For each string σ,

$$T_\sigma(n) = \lim_s T_{\sigma,s}(n) \text{ exists.}$$

PROOF: Let $\sigma = \sigma' * i$, i=0 or 1. It is sufficient to show that there is a stage s^* such that for each $s+1 > s^*$, $T_{\sigma', s+1}(i)$ is defined through case I and $T_{\sigma, s+1}$ is not cancelled. We assume that t is such that for each $\pi \subseteq \sigma'$, if there exists a $\tau(\pi)$ for which $T_\pi(\tau(\pi)) = T_{\sigma'}(n)$ then $T_{\pi, s+1}(\tau) = T_{\sigma'}(n)$ for each $s+1 > t$. If $T_{\sigma', s+1}(i)$ is defined other than through case I for $s+1 > t$ we must have either:

~~If $T_{\sigma', s+1}(i)$ is defined other than through case I for $s+1 > t$ we must have either:~~

(1) $T_{\sigma', s+1}$ is cancelled,

or (2) for some $\sigma_1 \subset \sigma'$ and $T_{\sigma_1, s}(\tau)$, where

$T_{\sigma', s}(n) \subseteq T_{\sigma_1, s+1}(\tau) \subset T_{\sigma', s}(i)$,

$T_{\sigma_1, s+1}$ $(\tau * 0)$, $(\tau * 1)$ are defined through case II.

We first note that no tree $T_{\sigma_1, s+1}$, $\sigma_1 \subseteq \sigma'$, is cancelled for $s+1 > t$, by the definition of t, and so (1) does not occur.

For inductive hypothesis we assume that for each $\pi \subset$ some $\sigma_1 \subseteq \sigma'$ there is a stage t^* such that either $T_{\pi, s+1}$ $(\tau(\pi) * 0)$, $(\tau(\pi) * 1)$ are defined for no $s+1 > t^*$ or $T_{\pi, s+1}$ $(\tau(\pi) * 0)$, $(\tau(\pi) * 1)$ $= T_{\pi, s}$ $(\tau(\pi) * 0)$, $(\tau(\pi) * 1)$ respectively for each $s > t^*$. We show that there is a stage s^* such that this holds with t^* replaced by s^* and π replaced by σ_1. The first stage of the induction is trivial since $T_{n,s}(\tau)$ defined $\rightarrow T_{n,v+1}(\tau) = T_{n,v}(\tau)$ for each $v \geqslant s$.

Say $T_{\sigma_1, s+1}$ $(\tau(\sigma_1) * 0)$, $(\tau(\sigma_1) * 1)$ become defined at stage $s+1 > t^*$ through case II. In order that these values be re-defined at a stage $u+1 > s+1$ we must have for some $\pi \subset \sigma_1$, some τ, some $v+1$ with $s+1 < v+1 \leqslant u+1$, that $T_{\pi,v}$ (τ) is a boundary string and $T_{\sigma_1}(\tau(\sigma_1)) \subseteq T_{\pi,v}(\tau) \subset T_{\sigma_1, s+1}$ $(\tau(\sigma_1) * 1)$, say, and $T_{\pi, u+1}$ $(\tau * 1)$, $(\tau * 0)$ are defined through case II at stage $u+1$. (If $T_{\pi,v}(\tau)$ is not a boundary string we would define extensions of some $T_{\pi,v}(\tau')$, $\tau' \subset \tau$, through case II at stage $u+1$).

We cannot have $\tau = \tau(\pi)$ by the inductive hypothesis. And so

we cannot have $T_{\pi,v}(\tau) \subset T_{\sigma_1,s+1}(\tau(\sigma_1) * 1)$ by requirement 3) of case II. So in this case
$T_{\sigma_1,s+1}(\tau(\sigma_1) * 0), (\tau(\sigma_1) * 1) = T_{\sigma_1}(\tau(\sigma_1) * 0), (\tau(\sigma_1) * 1)$
respectively.

Assume that $T_{\sigma_1,s+1}(\tau(\sigma_1) * 0), (\tau(\sigma_1) * 1)$ become defined at stage $s+1 > t^*$ through case III but never through case II at a stage $u+1 > s+1$. Then let $\pi \subset \sigma_1$ be the largest string for which $\tau(\pi)$ is defined and $T_{\pi}(\tau(\pi) * 0)$, $(\tau(\pi) * 1)$ exist. ~~We have $T_{\sigma_1,s+1}(\tau(\sigma_1) * 0), (\tau(\sigma_1) * 1)$ exist.~~ We have $T_{\sigma_1,s+1}(\tau(\sigma_1) * 0), (\tau(\sigma_1) * 1)$

$$= T_{\pi}(\tau(\pi) * 0), \quad (\tau(\pi) * 1) \text{ respectively}$$

$$= T_{\sigma_1}(\tau(\sigma_1) * 0), \quad (\tau(\sigma_1) * 1) \text{ respectively.}$$

(This follows from the minimality requirement for the extensions defined through case III).

If neither of our assumptions is correct the induction follows trivially.

In any case we have the stage s^* as required. From now on we take the s^* corresponding to $\sigma_1 = \sigma'$.
We now show that there is a stage $w > s^*$ such that $T_{\sigma,s+1}$ *is* cancelled for no $s+1 > w$. Assume that $T_{\sigma,s+1}$ is cancelled

at a stage $s+1 > s^*$. Then $T_{\sigma,u+1}$ can only be cancelled at a stage $u+1 > s+1$ if the follower of σ (say π) appointed at stage $s+1$ becomes invalid at some stage $v+1 > s+1$. By the definition of s^*, π cannot become incompatible with a tree $T_{\sigma_1, s+1}$, $\sigma_1 \subseteq \sigma$, through definition of extensions of a string on $T_{\sigma_1, s+1}$ of rank $\subset \sigma'$. And we prohibit all strings incompatible with π as candidates for extensions of strings of rank $\supseteq \sigma'$. So π never becomes invalid and $T_{\sigma, s'+1}$ is cancelled for no $s'+1 > s+1 > s^*$. This completes the proof.

We now note that for given σ, τ,

$$lh(R(T_{\sigma,s}(\tau), s)) \leqslant lh(\sigma) + lh(\tau),$$

so if $T_{\sigma,s+1}(\tau) \neq T_{\sigma,s}(\tau)$ for infinitely many s, we must have $T_{\sigma,s}(\tau) \subseteq T_{\sigma',s}(n)$ for some σ' and infinitely many s. But since $\lim_s T_{\sigma',s}(n)$ exists, this cannot happen, and so $\lim_s T_{\sigma,s}(\tau)$ exists for all σ, τ. It also follows from the lemma that $\lim_s \beta_s$ exists and so $B = \lim_s \beta_s$ is a set of degree $\leqslant \underline{a}' = \underline{c}$.

<u>LEMMA 2</u> $\underline{b}' \leqslant \underline{c}$.

<u>PROOF</u>: $\{\beta_s\}$ is an 'enumeration' of B uniformly recursive in $\underline{a}$. So we need only show that $\lim_s J_s(\beta_s, e)$ exists for each e to

prove that $\underline{b}' \leqslant \underline{a}' = \underline{c}$.

Assume that $T_{\sigma * i}(n)$ is a beginning of B where $lh(\sigma) = e+1$. Then there is a stage t such that $T_{\sigma * i, s}(n) = T_{\sigma * i}(n)$ and $\beta_s \supset T_{\sigma * i}(n)$ for all $s > t$. Say at stage $s > t$ we have $J_s(\beta_s; e)$ is defined. Then at stage s+1 we must have a valid follower π of $\sigma * i$ since $T_{\sigma * i, s+1}(n) = T_{\sigma * i}(n)$. It cannot become invalid, by a similar argument to that in lemma 1. And this means that for each $\sigma' \supseteq \sigma * i$, $T_{\sigma'}(n) \supseteq \pi$, and since $J_s(\pi; e)$ is defined we have $J_{s'}(\beta_{s'}; e)$ is defined for all $s' > s$, ~~and~~ $J_s(\beta_s; e)$

LEMMA 3 $\underline{c} \leqslant \underline{b}'$.

PROOF: We show that

$$e \in C \leftrightarrow (Ek^*)(k)_{k > k^*}(Pe^k \in B) \text{ and }$$

$$(k^*)(Ek)_{k > k^*}(Pe^k \in B).$$

Assume that $C[e] = \sigma$. Then there is a stage t and a string π such that $T_{n,s}(\pi) = T_{\sigma,s}(n) = T_\sigma(n)$ for all $s > t$. This means that all strings on T_σ will lie on T_n above $T_n(\pi)$, and since $R(T_{n,s}(\pi), s) = \sigma$ for $s > t$, for all strings $T_n(\pi * \tau)$ we have $T_n(\pi * \tau)(\rho e^k) = C(e)$ ~~whenever $T_n(\pi * \tau)(\rho e^k) = C(e)$~~ whenever $T_n(\pi * \tau)(\rho e^k)$ is defined, for large enough k. By our choice of β_s we eventually have $\beta_s \supset T_{\sigma,s}(n)$ for all large enough s, and for all $\pi \supset \sigma$, $T_\pi(n)$ lies on T_σ, and so the lemma follows.

<u>LEMMA 4</u> B is of minimal degree.

PROOF: B is not recursive since $\underline{b}' = \underline{c} > \underline{0}'$. We need only show that for each number e either $\Phi_e(B)$ is recursive or $B \leqslant_T \Phi_e(B)$.

We write $F_\tau(T)$ for the full subtree of T above τ.
We write $Sp_\pi(T_{\sigma,s})$ for the subtree of $T_{\sigma,s}$ above π given by: $T_{\sigma,s}(\tau * i) \in Sp_\pi(T_{\sigma,s})$ if and only if $T_{\sigma,s}(\tau * i) \supseteq \pi$, and either $\pi = T_{\sigma,s}(\tau * i)$ or $T_{\sigma,s}(\tau) \in Sp_\pi(T_{\sigma,s})$ and $T_{\sigma,s}(\tau * i)$ was defined through case II.

For each $e \geqslant 0$ we construct a tree $T^*_{e+1} = \lim_s T^*_{e+1,s}$ satisfying the following:

(1) for all s, $T^*_{e+1,s} \subseteq T^*_{e+1,s+1}$ (and so $T^*_{e+1,s}$ is partial recursive),

(2) either T^*_{e+1} is a splitting tree for e or there is a string $\beta \subset B$ on T_{e+1} such that no pair of strings on T^*_{e+1} above β split β for e.

(3) B lies on T^*_e, each e,

(4) if $T^*_{e+1,s}(\pi)$, $\pi \supset n$, is a boundary string for some $T_{\sigma,s}$, $lh(\sigma) \leqslant e+1$, then $T^*_{e+1,s}(\pi * 0)$, $\pi * 1)$ are not defined,

(5) If $T^*_{e+1,s}(\pi)$ has no other extensions defined then $T^*_{e+1,s}(\pi)$ is a boundary string for $T_{\sigma,s}$, some σ such that $lh(\sigma) = e+1$.

(6) if $T^*_{e+1,t}(n)$ is defined we do not define strings $T_{\sigma,s+1}(\tau * 0)$, $(\tau * 1)$ at step s+1 through cases II or III if $T_{\sigma,s+1}(\tau) \subset T^*_{e+1,s+1}(n)$ and $s+1 \geqslant t$.

We now give the construction of the nest of trees $\{T^*_{e,s}\}$. When defining $T^*_{e+1,s}$ we assume the properties (1) - (6) for trees $T^*_{i,s}$, $i < e + 1$. We verify these properties for $T^*_{e+1,s}$ after the construction.

First of all we define.

$$T^*_{o,s} = T_{n,s} \qquad \text{for all s.}$$

The properties (1) - (6) follow immediately for $T^*_{o,s}$.

Assume now that $\{T^*_{i,s}\}_{s \geqslant 0}$ have been defined for each $i < e + 1$.

Let σ be the beginning of C of length e+1. There are three cases to consider:

(i) there is a string π on T^*_e such that π is a beginning of B and for no $T_{\sigma,s}(\tau)$ do we have $T_{\sigma,s}(\tau) \supset \pi$,

(ii) B lies on T_σ and there is a string π on T_e^* such that π is a beginning of B and for no $T_{\sigma,s}(\tau)$ defined through case II do we have $T_{\sigma,s}(\tau) \supset \pi$ and $T_{\sigma,s}(\tau)$ lies on $T_{e,s}^*$,

(iii) otherwise.

Define

$$\pi(e+1) = \begin{cases} \text{the least } \pi \text{ satisfying (i) if (i) holds} \\ \text{the least } \pi \text{ satisfying (ii) if (ii) holds} \\ \text{the least } T_\sigma(\tau) \text{ such that } T_\sigma(\tau) \text{ lies on} \\ T_e^* \text{ and is a beginning of B and such that} \\ T_{\sigma,s}(\tau) \text{ is never cancelled if (iii) holds.} \end{cases}$$

$$s(e+1) = \mu s\ (T_{\sigma,s}(\tau) = T_\sigma(\tau) \text{ each } T_\sigma(\tau) \subseteq \pi(e+1))$$

$$T_{e+1,s} = \begin{cases} F_{\pi(e+1)}(T_{e,s}^*) \text{ if (i) or (ii) hold, for each} \\ \qquad\qquad\qquad\qquad s > s(e+1), \\ S_{\beta\pi(e+1)}(T_{\sigma,s}) \cap T_{e,s}^* \text{ otherwise, for each } s > s(e+1). \\ \text{undefined for } s \leqslant s(e+1). \end{cases}$$

We now verify facts (1) - (6) for T_{e+1}^*.

(1) If $T_{e+1,s}^* = F_{\pi(e+1)}(T_{e,s}^*)$ then the result follows from $T_{e,s}^* \subseteq T_{e,s+1}^*$.

Assume $T_{e+1,s}^*$ is defined through the second clause and $T_{e+1,s}^*(\tau)$ is defined $= T_{\sigma,s}(\omega)$, say. By the definition of $T_{e+1,s}^*$, $T_{\sigma,s}(\omega)$ is never cancelled. Since $T_{\sigma,s}(\omega)$ was defined through case II, we will have $T_{\sigma,s+1}(\omega) = T_{\sigma,s}(\omega)$ $\in T^*_{e,s+1} \supset T^*_{e,s}$ (and so $T^*_{e+1,s+1}(\tau) = T^*_{e+1,s}(\tau)$) unless there is some string $T_{\sigma',s}(\omega')$, $\sigma' \subset \sigma$, $T_{\sigma',s+1}(\omega') \subset T_{\sigma,s}(\omega)$, for which $T_{\sigma',s+1}(\omega' * 0)$, $(\omega' * 1)$ become defined $\neq T_{\sigma',s}(\omega' * 0)$, $(\omega' * 1)$ respectively.

Moreover $T_{\sigma',s+1}(\omega' * 0)$; $(\omega' * 1)$ must be defined through case II since otherwise the minimality requirement of case III implies that $T_{\sigma',s+1}(\omega')$ is incompatible with $T_{\sigma,s}(\omega)$, a contradiction. This means that $T_{\sigma',s}(\omega')$ is a boundary string for $T_{\sigma',s}$.

Because (4) holds for all $i < e+1$, $T_{\sigma',s}(\omega')$ does not lie on $T^*_{e,s}$.

Since (6) holds for all $i < e+1$ we do not have $T'_{\sigma,s}(\omega') \subset T^*_{e,s}(n)$

So no string $T_{\sigma',s}(\omega')$ exists.

(3) By the construction $T^*_{e+1}(n)$ is a beginning of B.

The result follows immediately if $T^*_{e+1} = F_{\pi(e+1)}(T^*_e)$ since B lies on T^*_e.

Assume T^*_{e+1} is defined through the second clause and $T^*_{e+1}(\tau)$ is an end string for T^*_{e+1} which is a beginning of B. Then we have $T_\sigma(\tau') = T^*_{e+1}(\tau)$ some τ'. Since T_σ satisfies case (iii) there is some $\omega \supseteq \tau'$ such that $T_\sigma(\omega * 0)$, $(\omega * 1)$ are defined through case II and $T_\sigma(\omega * i)$, say, lies

on T_e^*. Since (4) holds for T_e^*, we must have $\omega = \tau'$.
And since $T_\sigma(\tau' * 0)$, $(\tau' * 1)$ do not lie on T_{e+1}^*, we have $T_\sigma(\tau' * (1-i))$ does not lie on T_e^*. But $T_e^* \subseteq T_{\sigma'}$, some $\sigma' \subset \sigma$ and T_σ is compatible with each $T_{\sigma'}$, $\sigma' \subset \sigma$, so T_σ is compatible with T_e^*. This means that there is some end string $T_e^*(\sigma_1)$ such that $T_\sigma(\tau') \subset T_e^*(\sigma_1) \subset T_\sigma(\tau' * (1-i))$.
But (5) holds for T_e^* and this contradicts the definition of $T_\sigma(\tau' * (1-i))$ through case π.

(4) If (i) holds for T_σ, (4) follows for T_{e+1}^* from the fact that (4) holds for T_e^*.

If (ii) holds, we know that (4) holds for T_e^*.
Also we know that no $T_{\sigma,s+1}(\tau) \supset \pi(e+1)$ is defined through case II and lies on $T_{e,s+1}^*$. So no such string can be a boundary string for $T_{\sigma,s+1}$ and (4) follows for T_{e+1}^*.

If (iii) holds the result follows by the construction of T_σ and the definition of Sp.

(5) If $T_{e+1,s}^* = \cancel{F_{\pi(e+1)}(T_{e,s}^*)} = F_{\pi(e+1)}(T_{e,s}^*)$ the result follows immediately.
Otherwise, assume that $T_{e+1,s}^*(\tau)$ is an end string for

$T^*_{e+1,s}$ but not for $T^*_{e,s}$. By definition of S_β, we have $T^*_{e+1,s}(\tau) = T_{\sigma,s}(\pi)$ some π where $T_{\sigma,s}(\pi)$ was defined through case II. In order to contradict (5) we must have $T_{\sigma,s}$ $(\pi * 0)$, $(\pi * 1)$ defined through case II, but $T_{\sigma,s}$ $(\pi * i)$, say, $\notin$ $T^*_{e,s}$. Arguing as in the last part of (3) we see that there is an end string $T^*_{e,s}(\sigma_1)$ such that $T_{\sigma,s}(\pi) \subset T^*_{e,s}(\sigma_1) \subset T_{\sigma,s}(\pi * i)$.
Since (5) holds for $T^*_{e,s}$, this contradicts the definition of $T_{\sigma,s}$ $(\pi * i)$ through case II.

(6) This follows by the definition of $\pi(e+1)$, $s(e+1)$.

(2) If $T^*_{e+1} = Sp_{\pi(e+1)}(T_\sigma) \cap T^*_e$ then T^*_{e+1} is a splitting tree for e by the construction of T_σ and the definition of Sp.

Assume (i) holds and let $T_{\sigma,s}(\tau) \subseteq \pi(e+1)$, $s > s(e+1)$, be the end string on T_σ which is a beginning of B. Above we proved that there is a string σ' such that $R(T_{\sigma,s}(\tau), s) \leqslant \sigma'$ for all $s > s(e+1)$. This means that we only prohibit strings as extensions for $T_{\sigma,s}(\tau)$ through followers of strings $\sigma_1 \subseteq \sigma'$, and so such prohibitions are finite in number. Let β, $\pi(e+1) \subseteq \beta \subset B$, be a string such that no string above β is prohibited as an extension for $T_{\sigma,s}(\tau)$.

Say some pair of strings $\sigma_1, \sigma_2 \supset \beta$ on $T^*_{e,s+1}$ split for e at stage s+1. Then since $T^*_{e,s+1}$ is compatible with $T_{\sigma',s+1}$, each $\sigma' \subset \sigma$, we must define strings $T_{\sigma,s+1}$ $(\tau * 0)$, $(\tau * 1)$ through cases II or III, which is a contradiction.

If (ii) holds, we take $T_\sigma(\tau) \supset \pi(e+1)$ to be the least such beginning of B on T^*_e. Then $T_{\sigma,s}(\tau * 0)$, $(\tau * 1)$ never become defined through case II. Let β, $T_\sigma(\tau) \subseteq \beta \subset B$ be a string such that no string above β is prohibited as an extension for $T_\sigma(\tau)$.

Then say some pair of strings $\sigma_1, \sigma_2 \supset \beta$ on $T^*_{e,s+1}$ split for e at a stage $s+1 > s(e+1)$. We cannot define extensions of a string $T_{\sigma,s+1}(\tau') \subset T_\sigma(\tau)$ since $s+1 > s(e+1)$. So $T_{\sigma,s+1}$ $(\tau * 0)$, $(\tau * 1)$ are defined through case II since σ_1, σ_2 lie on $T^*_{e,s+1}$, $T^*_{e,s+1}$ is compatible with each $T_{\sigma',s+1}$ with $\sigma' \subset \sigma$, and since (4) holds for $T^*_{e,s+1}$. This completes the proof. In order to see that $\underline{b}$ is minimal, we need only apply lemmas 1,2 of [6].

We need to use Yates' theorem that there are minimal degrees with jump $\underline{0}'$ to complete our result since we have not taken any specific steps to ensure that B is not recursive. They would have been a fairly routine diagonalisation of the recursive functions, but cumbersome.

§2 No high degree is minimal

Let $A = \lim_s A_s$ be a set of degree $\leqslant \underline{0}'$. We define

$$C_A(n) = \mu s\,(A_s[n] = A[n])$$

The enumeration $\{A_s\}$ (in the weak sense) is said to be <u>high</u>. if C_A dominates every recursive function and A is said to be high if it has an enumeration (in the weak sense) which is high. For recursively enumerable sets this is the same as the definition of Robert Robinson [4]. First of all we need the following lemma which is analagous to theorem 3 of [4].

<u>LEMMA 5</u> If $\underline{a} \leqslant \underline{0}'$ and $\underline{a}' = 0''$, then there is a high set A recursive in $\underline{a}$.

<u>PROOF:</u> Let $\underline{a}$ satisfy the hypothesis of the lemma. From [4] we have a total function f of degree $\underline{a}$ which dominates every recursive function. Let $\{f_s\}$ be a sequence of finite functions such that for each n, $\lim_s f_s(n) = f(n)$. Define $p(m,e) = 2^e\, 3^m$

Let $\{e\}_s(y) = \mu z\ (z \leqslant s \text{ and } T_1(e, y, z))$.

Then $\lim_s \{e\}_s(y) = \{e\}(y)$ for each $e, y \geqslant 0$ where $\{e\}(y)$ is the e^{th} partial recursive function.

Define $s(m,e) = \mu s\ (\{e\}_s(y)$ is defined for all y with $p(m,e) < y \leqslant p(m+1,e))$

$$A_s(n) = \begin{cases} 0 & \text{if } n = p(m,e) \text{ some } m,e \text{ and } \\ & \{e\}_s(y) \text{ is defined for all } y,\ p(m,e) < y \leqslant p(m+1,e) \\ & \text{and } f_s(p(m,e)) \text{ and } s \text{ are both} > \max\{\{e\}(y) \mid p(m,e) < y \\ & \qquad \leqslant p(m+1,e)\} + s(m,e), \\ 1 & \text{otherwise}. \end{cases}$$

Let $A = \lim_s A_s$.

A is recursive in f since to find out whether $A(p(m,e)) = 0$ we look for the least stage s^* such that

$f(p(m,e)) \leqslant s^*$ or $s(m,e) = s^*$.

If the latter holds $A(p(m,e)) = 0$ if and only if $f(p(m,e)) > \max\{\{e\}(y) \mid p(m,e) < y \leqslant p(m+1,e)\} + s^*$.

If this occurs before $s(m,e)$ is defined then $A(p(m,e)) = 1$

Assume now that $\{e\}$ is a recursive function not dominated by C_A.

If $A(p(m,e)) = 0$, then $C_A(y) > \{e\}(y)$ for each y, $p(m,e) < y \leqslant p(m+1,e)$.

So $A(p(m,e)) = 1$ for infinitely many m.

But this means that for infinitely many m,

$f(p(m,e)) \leqslant \max\{\{e\}(y) \mid p(m,e) < y \leqslant p(m+1,e)\} + s(m,e)$.

Since $\{e\}$ is total, $s(m,e)$ is recursive for e fixed and so f fails to dominate a recursive function, which contradicts the choice of f.

<u>THEOREM 2</u> There is no minimal degree $\underline{a} \leqslant \underline{0}'$ such that $\underline{a}' = \underline{0}''$.

<u>PROOF:</u> Let $\underline{a} \leqslant \underline{0}'$ and $\underline{a}' = \underline{0}''$. Let A be a high set recursive in $\underline{a}$ and let $\{A_s\}_{s \geqslant 0}$ be a high enumeration for A.

With regard to functionals we follow the notation and practice of [3]. $\{\Phi_e\}_{e \geqslant 0}$ will be a standard enumeration of the partial recursive functionals, and $\{\Phi_{e,s}\}_{e,s \geqslant 0}$ will be a uniformly recursive sequence of approximations to these functionals. That is, $\lim_s \Phi_{e,s} = \Phi_e$ and $\Phi_{e,s} \subseteq \Phi_{e,s+1}$ each s. ~~$\{R_{e,s}\}_{e,s \geqslant 0}$ is a uniformly recursive set of approximations to a standard enumeration $\{R_e\}_{e \geqslant 0}$ will be a uniformly recursive sequence of approximations to these functionals. That is, $\lim_s \Phi_{e,s} = \Phi_e$ and $\Phi_{e,s} \subseteq \Phi_{e,s+1}$ each s.~~ $\{R_{e,s}\}_{e,s \geqslant 0}$ is a uniformly recursive set of approximations to a standard enumeration $\{R_e\}_{e \geqslant 0}$ of the recursively enumerable sets. For convenience we assume that for all but a finite set of numbers e, $R_{e,s}$ and $\Phi_{e,s}$ are empty at stage s. We construct stage by stage a non-recursive set recursive in A in which A is not recursive. As in [3] the device we use to demonstrate the recursiveness of this set in A is to define its characteristic function $= \Theta(A)$ where Θ is a partial recursive functional $= \lim_s \Theta_s$ where Θ_s is a finite set of axioms defined by stage s. It will be clear from the construction that Θ is consistent. Again following [3] we will use finite sets of numbers called <u>requirements</u> to preserve certain values $\Phi_{e,s}(\Theta_{\bar{s}}(A_s); y)$ with $y, s \geq 0$. We use the same symbol for a set or its characteristic function.

The conditions to be satisfied will be given the following orders

$$2e \quad : \qquad \overline{\Theta(A)} \neq R_e$$

(where $\bar{s}$ denotes the complement of s)

$$2e+1 \quad : \qquad A \neq \Phi_e(\Theta(A)).$$

We now give the construction.

<u>Step 0.</u> Define $\Theta(A_0[x];x) = 1$, all x.

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Step s+1. Followers may be appointed at step s+1.

We will say that a follower x of a number n is realised at step s+1 if either:

(i) $n = 2e$ and $x \in R_{e,s}$,

or (ii) if σ is the precondition of x, there are strings $\sigma_1, \sigma_2 \supset \sigma$ of equal length which split σ for e at stage s+1.

We define s(x) to be the least stage at which x is realised if such a stage exists. At stage s(x), if x follows a number 2e+1, σ_1, σ_2 are appointed to be partners of x and x is said to be the predecessor of (σ_1, σ_2). If σ_1, σ_2 split through w, say, (that is $\Phi_{e,s+1}(\sigma_1;w) \neq \Phi_{e,s+1}(\sigma_2;w)$) w is said to be the instigator of x.

Assume that σ_1 is such that $\Phi_{e,s(x)}(\sigma_1;w) \neq 0$. Then at stage s(x) we set up a requirement of order 2e+1 to make σ_1 a beginning of $\Theta(A)$ if $w \in A$ and σ_2 a beginning of $\Theta(A)$ otherwise.

If x follows a number 2e, then at stage s(x) we set up a requirement of order 2e to make $\Theta(A;x) = 0$. Followers x of n may become invalid at step s+1 if x is realised at step s+1 but $A_t[x] = A_{s+1}[x]$ for some $t < s(x)$, or if x has precondition σ and for some y such that $\sigma(y)$ is defined, $\sigma(y) \neq \Theta_s(A_{s+1}; y)$. A number n requires attention at step s+1 if a follower of order n becomes realised at step s+1 or if every follower of n invalid, and n is the least such number. If n requires attention at step s+1, cancel all followers and requirements of order $> n$. If n requires attention through the first part of the definition, let y be the least number greater than any number which has been in some requirement at a stage $< s+1$. Appoint y to follow n. If $n = 2e + 1$, let $x(y)$ be $y-1$ if there is no existing invalid follower of n, and otherwise let $x(y)$ be the least follower of n such that $\Theta_s(A_s)[x]$ has not been used as an intial segment of a precondition of a follower of n. The precondition of y (and of all requirements associated with y) is defined by

$$\sigma(z) = \begin{cases} \Theta_s(A_s;\ z) & \text{for } z \leqslant x(y) \\ 1 \quad \text{for} \quad x(y) < z < y. \end{cases}$$

A requirement of order $2e+1$ is associated with y to make $\Theta(A;z) = 1$ for all z, $x(y) < z < y$.

<u>Definition of new axioms for Θ at step s+1</u>

If z is in no requirement and $\Theta\ (A_{s+1};\ z)$ is as yet undefined, enumerate in Θ the axiom

$$\Theta(A_{s+1}[z];\ z) = 1$$

Assume that we have taken steps to satisfy all requirements of order $< n$.

If $n = 2e$, for each realised follower y for which $\Theta\ (A_{s+1};\ y)$ is undefined, set

$$\Theta(A_{s+1}[y];\ y) = 0.$$

If $n = 2e+1$, look for the least follower y (with precondition σ) of n such that for every z such that $\sigma(z)$ is defined, $\Theta(A_{s+1};\ z)$ is not defined or is defined $= \sigma(z)$, and if y is realised then $\Theta(A_{s+1};y)$ is not defined. For each of these z's such that $\Theta(A_{s+1};z)$ is not defined there will be a requirement to set $\Theta(A;\ z) = 1$, by definition of y.
For each such z enumerate in Θ the axiom

$$\Theta(A_{s+1}[z];\ z) = 1.$$

If y is realised, enumerate in Θ the axiom

$$\Theta(A_{s+1}[\max(\omega,y)];z) = \sigma_1(z) \text{ or } \sigma_2(z) \text{ according as}$$

$\omega \in A_{s+1}$ or $\omega \notin A_{s+1}$, for each $z \geq y$ such that $\sigma_1(z)$, $\sigma_2(z)$ are defined. For all z in a requirement of order n such that $\Theta(A_{s+1};\ z)$ is still not defined, set

$$\Theta(A_{s+1}[z];\ z) = 1.$$

This completes the construction.
It can be easily verified by following through the construction that Θ is a consistent partial recursive functional, and since $\Theta(A)$ is total, defines a set of degree $\leqslant \underline{a}$

<u>LEMMA 6</u> Each number n requires attention at only a finite number of stages.

PROOF: Assume the lemma for each number $< n$. If $n = 2e$ and n requires attention at an infinite number of stages, infinitely many followers y must be appointed to follow n, all but a finite number must be realised at a step $s(y)$ and for all but a finite number $A[y] = A_t[y]$ for some $t < s(y)$. Define a recursive function by

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is never a realised follower of } n, \\ s(x) \text{ otherwise.} \end{cases}$$

Then C_A does not dominate f which contradicts the fact that $\{A_s\}$ is a high enumeration for A. If $n = 2e+1$ and n requires attention at an infinite number of stages, there must be requirements of order n set up to make $\Theta(A;x) = 1$ for all but a finite number of x's ≥ 0. For suffiently large x, if a requirement is set up at stage $s(x)$ to make $\Theta(A;x) = 1$, it can only fail because $A_t[x]$ for some $t < s(x)$. This is because every action we take relative to the partners of a realised follower eventually fails. This means that for all sufficiently large x, $\Theta(A;x) = 1$, otherwise we could define a recursive function not dominated by C_A in the same way as above. By the construction infinitely many followers y of n will be defined with a precondition which is a beginning of $\Theta(A)$ and every one must become realised at a stage $s(y)$ but we must also have that $A[y] = A_t[y]$ for some $t < s(y)$ in order that more followers be appointed. And this means that we may define a recursive function f by

$$f(x) = \begin{cases} s(y) \text{ if } y \text{ is a follower whose precondition} \\ \text{is a beginning of } \Theta(A), \\ 0 \text{ otherwise.} \end{cases}$$

We see that C_A does not dominate f, which is a contradiction.

It follows from the lemma that $\overline{\Theta(A)} \neq R_e$ for all e and so $\Theta(A)$ is not recursive.

<u>LEMMA 7</u> $A \not\leq_T \Theta(A)$.

<u>PROOF</u>: $A = \Phi_e(\Theta(A))$ for no e since there must eventually be a follower y of $n = 2e+1$ which is valid at every sufficiently large stage, by lemma 1. If y is never realised, $\Phi_e(\Theta(A))$ is either not total or is recursive since the precondition of y is a beginning of $\Theta(A)$. If y is realised at some stage let ω be the instigator of y. Then we will have $A(\omega) \neq \Phi_e(\Theta(A); \omega)$ since we set up requirements to ensure this and $A_t[y] = A[y]$ for no $t < s(y)$.

This completes the proof of the theorem.

It seems possible to make $\Theta(A)$ high also, which means that <u>a</u> is very far from being minimal. It would be interesting to see some structural difference between the high degrees and the r.e. degrees with jump $\underline{0}''$, since they appear to have so much in common. For instance we conjecture that every high degree is the join of two degrees strictly less than it, and we show in §3 that every high degree has a minimal predecessor.

The question is still open as to what are the jumps of the minimal degrees below $\underline{0}'$, as it is for other types of degrees constructed by tree arguments. Jockush [2] showed that every degree with jump $\geqslant \underline{0}''$ is hyperhyper-immune, and that there is a hyperhyper immune degree $\underline{a}$ such that $\underline{a}'' = \underline{0}''$, but it is not certain whether there is a hyperhyper immune degree $\leqslant \underline{0}'$ with jump $< \underline{0}''$. We hope for strong answers and conjecture that every degree $\underline{c}$ r.e. in $\underline{0}'$ such that $\underline{0}' \leqslant \underline{c} < \underline{0}''$ is the jump of a minimal degree below $\underline{0}'$.

§3 Every high degree has a minimal predecessor

Combining techniques introduced in §1 and §2 we prove:

THEOREM 3 Let $\underline{a}$ be a degree $\leqslant \underline{0}'$ such that $\underline{a}' = \underline{0}''$. Then there exists a minimal degree $\underline{b}$ such that $\underline{b} \leqslant \underline{a}$.

PROOF As in theorem 2 we take A to be a high set of degree $\leqslant \underline{a}$. We include steps to make B not recursive and it will be seen how such steps could have been incorporated into the construction of theorem 1. We define certain trees $T_{e,s+1}$ at stage s+1 where a tree T_{2e+1} will be concerned with ensuring that if $\Phi_e(B)$ is the characteristic function of a set then either $\Phi_e(B)$ is recursive or B is recursive in $\Phi_e(B)$ and T_{2e+2} will be concerned with ensuring that if $\{e\}$ is the characteristic function of a set then $B \neq \{e\}$.

The boundary state, $R(T_{e+1,s}(\tau))$, of a string $T_{e+1,s}(\tau)$ is now defined to be the set

$$\{T_{e',s}(\pi) \mid e' \leqslant e \text{ and } T_{e',s}(\pi) \subseteq T_{e+1,s}(\tau) \text{ and } T_{e',s}(\pi) \text{ is a boundary string for } T_{e',s}\},$$

where $T_{e,s}(\tau)$ is now said to be a boundary string for $T_{e,s}$ if $T_{e,s}(\tau)$ was defined through case II originally but $T_{e,s}(\tau * 0)$, $(\tau * 1)$ were not.

We write $R(T_{e_1+1,s}(\tau_1)) \cong R(T_{e_2+1,s}(\tau_2))$ if

$$\{T_{e',s}(\pi) / T_{e',s}(\pi) \in R(T_{e_1+1,s}(\tau_1)) \text{ and } e' \leqslant \min(e_1, e_2)\} =$$

$$\{T_{e',s}(\pi) \mid \cancel{T_{e',s}(\pi) \mid} T_{e',s}(\pi) \in R(T_{e_2+1,s}(\tau_2)) \text{ and } e' \leqslant \min(e_1, e_2)\}.$$

We now give the construction.

Stage 0 Define $T_{0,s} = I$ (the identity tree) for all s.

$T_{e+1,0}$ is undefined for all $e \geqslant 0$. Define $\beta_0 = n$.

Stage s+1

Definition of $T_{e+1,s+1}$

We assume that $T_{e',s+1}$ is defined for each $e' < e+1$.

Assume that $T_{e,s+1}(n)$ is not an end string for $T_{e,s+1}$.

We write $\sigma \sim A_u$ if there is some $v < u$ such that $\sigma \subseteq \beta_v$ and $A_v[lh(\sigma)] = A_u[lh(\sigma)]$.

If for some $i \leqslant 1$ we have $T_{e,s+1}(i) \sim A_{s+1}$, define $T_{e+1,s+1}(n) = T_{e,s+1}(i)$.

Otherwise let $T_{e,s+1}(i)$ be a string which is not prohibited for $e+1$ at stage $s+1$ and set

$$T_{e+1,s+1}(n) = T_{e,s+1}(i).$$

(It will follow from the construction that such a string exists).

Assume now that $T_{e+1,s+1}(\tau)$ is defined $= T_{e+1,s}(\tau)$.

We may now define extensions of $T_{e+1,s+1}(\tau)$ through one of the following cases:

Case I $T_{e+1,s}(\tau * 0)$, $(\tau * 1)$ are defined and are compatible with every tree $T_{e',s+1}$, $e' < e+1$, and either: (i) $\tau = n$ or (ii) $T_{e+1,s}(\tau * 0)$, $(\tau * 1)$ are valid potential extensions of $T_{e+1,s}(\tau)$ with boundary state $R(T_{e+1,s+1}(\tau))$.,

or (iii) there do not exist valid potential extensions σ_1, σ_2 of $T_{e+1,s}(\tau)$ with boundary state $R(T_{e+1,s+1}(\tau))$.

Define $T_{e+1,s+1}(\tau*0), (\tau*1) = T_{e+1,s}(\tau*0), (\tau*1)$ respectively.

In cases II, III we only consider strings $T_{e+1,s+1}(\tau) \subseteq \beta_s$.

<u>Case II</u> Case I does not hold, $\tau \supset n$, and there do exist strings σ_1, σ_2 as defined in I(iii).

Define $T_{e+1,s+1}$ $(\tau * 0), (\tau * 1) = \sigma_1, \sigma_2$ respectively.

<u>Case III</u> Cases I, II do not hold and either

(a) $\tau = n$

or (b) there exist potential extensions of $T_{e+1,s}(\tau)$ with boundary state $R(T_{e+1,s+1}(\tau))$,

or (c) there exist strings σ_1, σ_2 satisfying the following:

(1) σ_1, σ_2 are compatible with every tree $T_{e',s+1}$, $e' < e+1$,

(2) if $R(T_{e+1,s+1}(\tau)) = \alpha$ then $\min(y(\beta_s,\sigma_1), y(\beta_s,\sigma_2)) >$ the largest number m for which $f_{e+1,\alpha}(m)$ is so far defined,

(3) if $e+1 = 2k+1$ some k, then σ_1, σ_2 split $T_{e+1,s+1}(\tau)$ for k at stage $s+1$,

(4) if $e+1 = 2k+2$ some k, then $T_{e+1,s+1}(\tau)$ is not a boundary string for $T_{e+1,s+1}$ (ie. for as much of $T_{e+1,s+1}$ as has been

constructed) and for some number i we have $\sigma_1(i)$, $\sigma_2(i)$, $\{k\}_{s+1}(i)$ are defined and $\sigma_1(i) = \sigma_2(i) \neq \{k\}_{s+1}(i)$.

Let π_1, π_2 be the least pair of incompatible strings $\supset T_{e+1,s}(\tau)$ compatible with every tree $T_{e',s+1}$, $e' < e+1$.

Define $T_{e+1,s+1}(\tau * 0)$, $(\tau * 1) = \pi_1$, π_2 respectively.

Definition of potential extensions for $T_{e+1,s+1}(\tau)$

Assume that $T_{e+1,s+1}(\tau)$ is defined and $R(T_{e+1,s+1}(\tau)) = \alpha$.

If $T_{e+1,s+1}(\tau)$ was originally defined through case III (a) or III (b), or $T_{e+1,s+1}(\tau)$ is a boundary string for some tree $T_{e',s+1}$ with $e' \leqslant e+1$, then we may define numbers $p(T_{e+1,s+1}(\tau),\alpha)$, $q(T_{e+1,s+1}(\tau), \alpha)$. We consider such strings $T_{e+1,s+1}(\tau)$ in their lexicographical ordering.

If $T_{e+1,s+1}(\tau)$ was originally defined through case III (a) or III (b) and $\tau = \tau' * i$, say, define

$$p(T_{e+1,s+1}(\tau),\alpha) = \max\{0, q(T_{e+1,s+1}(\tau'), R(T_{e+1,s+1}(\tau)))\}$$

If $T_{e+1,s+1}(\tau)$ is a boundary string for some $T_{e',s+1}$, $e' \leqslant e+1$, define

$$p(T_{e+1,s+1}(\tau),\alpha) = lh\,(T_{e+1,s+1}(\tau)).$$

If $f_{e+1,\alpha}(q(T_{e+1,s}(\tau),\alpha))$ is defined and $T_{e+1,s}(\tau)$ does not have potential extensions in boundary state α, or if $q(T_{e+1,s}(\tau), \alpha)$ is not defined, let m = the largest number m for which $f_{e+1,\alpha}(m)$ is so far defined, and set

$$q(T_{e+1,s+1}(\tau), \alpha) = m+1.$$

Otherwise set

$$q(T_{e+1,s+1}(\tau), \alpha) = q(T_{e+1,s}(\tau), \alpha).$$

For strings $T_{e+1,s+1}(\tau)$ satisfying the above we may now define strings Σ', II.

II $(T_{e+1,s+1}(\tau), \alpha)$ is defined to be the least string σ such that

(1) $lh(\sigma) \geqslant q(T_{e+1,s+1}(\tau),\alpha)$,

(2) $\sigma \supseteq T_{e+1,s+1}(\tau)$,

(3) σ is compatible with each tree $T_{e',s+1}$, $e' < e+1$.

$\Sigma'(T_{e+1,s+1}(\tau), \alpha)$ is defined to be the least string σ satisfying (1) - (3) above and also

(4) σ is compatible with any string π such that $\pi \sim A_{s+1}$ and $p(T_{e+1,s+1}(\tau), \alpha) < y(\sigma,\pi) \leqslant q(T_{e+1,s+1}(\tau), \alpha)$,

(5) for each $u < s+1$, if $T_{e+1,s+1}(\tau) = T_{e+1,u}(\tau)$, $R(T_{e+1,u}(\tau)) = \alpha$ ~~Ma~~ and $\Sigma'(T_{e+1,u}(\tau), \alpha)$ is defined, then $\Sigma'(T_{e+1,u}(\tau), \alpha) \subseteq \sigma$.

We say that each string π such that $y(\Sigma'(T_{e+1,s+1}(\tau),\alpha), \pi) > p(T_{e+1,s+1}(\tau), \alpha)$ and π is incompatible with $\Sigma'(T_{e+1,s+1}(\tau), \alpha)$ is prohibited ~~for~~

for each number $e' > e+1$ at stage $s+1$. Now assume that $T_{e+1,s+1}(\tau)$ satisfies the above hypothesis and that there are no potential extensions with boundary state α associated with $T_{e+1,s+1}(\tau)$, and that $T_{e+1,s+1}(\tau) \subseteq \beta_s$

Assume that there are strings σ_1, σ_2 satisfying III (c) (1), (3) and (4) above, and also satisfying

(2)' $\sigma_1, \sigma_2 \supset$ II $(T_{e+1,s+1}(\tau), \alpha)$,

(5) for every boundary string $\sigma' \subset \sigma_1$ or σ_2 on a tree $T_{e',s+1}$, $e' < e+1$, we have $\sigma' \subseteq T_{e+1,s+1}(\tau)$,

(6) σ_1, σ_2 are not prohibited for e+1 at stage s+1. σ_1, σ_2 now become <u>potential extensions</u> of $T_{e+1,s+1}(\tau)$ with boundary state α and remain so at each stage $t > s+1$ for which $T_{e+1,s+1}(\tau) = T_{e+1,t}(\tau)$.

We define

$f_{e+1,\alpha}(y) = s+1$ for each $y \leqslant q(T_{e+1,s+1}(\tau), \alpha)$ such that $f_{e+1,\alpha}(y)$ is not already defined.

σ_1, σ_2 are said to be <u>valid</u> at each stage $t > s+1$ for which $T_{e+1,t}(\tau) = T_{e+1,s+1}(\tau)$,

$(\pi)\ (T_{e+1,t}(\tau) \subset \pi \rightarrow (\pi \sim A_t \rightarrow \pi$ is compatible with σ_1 or $\sigma_2))$, σ_1, σ_2 are compatible with each tree $T_{e',t}$, $e' < e+1$, and neither of σ_1, σ_2 are prohibited for e+1 at stage t.

<u>Definition of β_{s+1}</u>

Define $\beta_{s+1} = T_{e,s+1}(n)$ where a is the largest number for which $T_{e,s+1}(n)$ is defined.

<u>LEMMA 8</u> $T_e = \lim_s T_{e,s}$ exists for each e.

<u>PROOF</u> By the construction $T_0 = I$.

Assume that $T_{e'}$ exists for each $e' < e+1$. Then $\lim_s$ $T_{e,s}(0),(1)$ exist and are defined. By the construction we have $T_{e+1}(n) = T_e(0)$ or $T_e(1)$.

Assume now that $T_{e+1}(\tau)$ exists and is defined. We prove that $\lim_s$ $T_{e+1,s}(\tau * 0)$, $(\tau * 1)$ exist. Let t be such that for each $e' \leqslant e+1$, if there exists a string $\pi(e')$ for which $T_{e'}(\pi(e')) = T_{e+1}(\tau)$ then $T_{e',s}(\pi(e')) = T_{e'}(e'))$ for each $s > t$. Also for each $e' < e+1$ if $T_{e'}(\pi(e') * 0)$, $(\pi(e') * 1)$ are defined then $T_{e',s}(\pi(e') * 0)$, $(\pi(e') * 1) = T_{e'}$ $(\pi(e') * 0)$, $(\pi(e') * 1)$ respectively for each $s > t$. If $R(T_{e+1,t+1}(\tau)) = \alpha$ then $R(T_{e+1,s}(\tau)) = \alpha$ for each $s > t$.

Now say there are potential extensions σ_1, σ_2 of $T_{e+1,s}(\tau)$ with boundary state α at a stage $s^* > t$. By the inductive hypothesis $\mathrm{Lim}_s R(T_{e',s}(\sigma))$ exists for each σ and each $e' < e+1$, and because of clause (4) in the definition of Σ, $\lim_s \Sigma'(T_{e',s}(\sigma), R(T_{e',s}(\sigma)\cancel{, R(T_{e',s}(\sigma))}$ [max $(\mathrm{lh}(\sigma_i) | i \leq 2)$] also exists. Therefore if σ_1 or σ_2 is prohibited at infinitely many stages through $T_{e',s}(\sigma)$, there is a stage $w > s^*$ such that σ_1 or σ_2 is prohibited for all $u > w$. Since by the definition of p, q, σ_1, σ_2 can only be prohibited through finitely many strings $T_{e',s}(\sigma)$, it follows that there is a stage t^* such that either σ_1, σ_2 are never prohibited for $u > t^*$, or such that σ_1, σ_2 are prohibited at every stage $u > t^*$. Finally we either have σ_1 or $\sigma_2 \sim A$ or not, and σ_1, σ_2 are compatible with every tree $T_{e'}$, $e' < e+1$, or not. So there is a stage $w^* > t^*$ such that σ_1, σ_2 are valid at no stage $u > w^*$ or such that σ_1, σ_2 are valid for all stages $u > w^*$. Let e' be the largest number $< e+1$ for which $T_{e'}(\pi(e') * 0)$, $(\pi(e') * 1)$ are defined. If the former holds then $T_{e+1}(\tau * 0)$, $(\tau * 1) = \sigma_1, \sigma_2$ respectively. If the latter holds, or if there do not exist such potential extensions,

$$T_{e+1}(\tau * 0),\ (\tau * 1) = T_{e'}(\pi(e') * 0),\ (\pi(e') * 1) \text{ respectively,}$$

or $T_{e+1}(\tau * 0)$, $(\tau * 1)$ exist and are not defined.

<u>LEMMA 9</u> B is recursive in A.

PROOF In order to compute $B[m]$, say, we merely follow through the construction until we have a stage s at which $lh(\beta_s) \geqslant m$ and $A_s[m] = A[m]$. Then $\beta_s[m] \sim A$ and $\beta_s[m] \subset B$. This is because of the way in which we define $T_{e+1,s+1}(n)$ is designed to achieve this, and because our objective cannot be defeated by what we do on trees $T_{e'}$, $e' < e+1$, by our definition of validity of potential extensions.

LEMMA 10 B is of minimal degree.

<u>PROOF</u> The proof follows that of lemma 4.

F_τ is defined as in lemma 4. $Sp_\pi(T_{e,s})$ is defined in the same way as we defined $Sp_\pi\ (T_{\sigma,s})$.

Again we construct trees T^*_{e+1} , $e \geq 0$, satisfying:

(1) as in lemma 4,

(2) as in lemma 4,

(3) as in lemma 4,

(4) if $T^*_{e+1}(\pi)$, $\pi \supset n$, is a boundary string for some $T_{e'}$, $e' \leq 2e+3$, then $T^*_{e+1}(\pi * 0)$, $(\pi * 1)$ are not defined,

(5) If $T^*_{e+1}\ (\pi)$ has no extensions defined then $T^*_{e+1}(\pi)$ is a boundary string for $T_{e'}$, some $e' \leq 2e+3$.

(6) if $T^*_{e+1,t}\ (n)$ is defined we do not define strings $T_{2e+3,s+1}\ (\tau * 0)$, $\cancel{(\sigma * 0)}$, $(\tau * 1)$ at step s+1 through cases II or III if $T_{2e+3,s+1}(\tau) \subset T^*_{e+1,s+1}(n)$ and $s+1 \geq t$.

Assume that $\{T^*_{i,s}\}_{s\geq 0}$ have been defined for each $i < e+1$.

There are three cases to consider:

(i) there is a string σ on T^*_e such that σ is a beginning of B and for no $T_{2e+3,s}(\tau)$ do we have $T_{2e+3,s}(\tau) \supset \sigma$,

(ii) B lies on T_{2e+3} and there is a string σ on T^*_e such that σ is a beginning of B and for no $T_{2e+3,s}(\tau) \supseteq \sigma$ do we define potential extensions for $T_{2e+3,s}(\tau)$ which lie on $T^*_{e,s}$,

(iii) otherwise.

Define

$\pi(e+1) =$

the least $\pi \supset \sigma$ such that no potential extensions which lie on $T^*_{e,s}$ are associated with strings $T_{2e+2,s}(\tau) \supseteq \pi$ at any stage $s \geqslant 0$ if (i) holds,

similarly if (ii) holds,

the least $\pi \subset B$ such that π lies on T^*_e and no potential extensions which lie on $T^*_{e,s}$ are associated with strings $T_{2e+2,s}(\tau) \supseteq \pi$ at any stage $s \geqslant 0$ and such that for each $T_{2e+3,s}(\tau) \supseteq \pi$ if potential extensions σ_1, σ_2 are defined for $T_{2e+3,s}(\tau)$ and lie on $T^*_{e,s}$

~~then $\sigma_1, \sigma_2 = T_{2e+3}(\sigma)$ and lie on $T^*_{e,s}$~~

~~then $\sigma_1, \sigma_2 = T_{2e+3,s}(\tau) \supseteq \pi$ if potential extensions σ_1, σ_2 are defined for $T_{2e+3,s}(\tau)$ and lie on $T^*_{e,s}$~~ then $\sigma_1, \sigma_2 = T_{2e+3}(\tau * 0), (\tau * 1)$

respectively, if (iii) holds.

$$s(e+1) = \mu s(u \geqslant s \to T_{2e+3,u}(\tau) \underset{}{\overset{= T_{2e+3}(\tau)}{}} \text{ for each } T_{2e+3}(\tau) \subseteq \pi(e+1))$$

$$T^*_{e+1,s} = \begin{cases} F_{\pi(e+1)}(T^*_{e,s}) \text{ if (i) or (ii) hold, for each} \\ \qquad s > s(e+1), \\ (S_{p\pi(e+1)}(T_{2e+3,s}) \cap T^*_{e,s}) \cup T^*_{e+1,s-1} \text{ otherwise,} \\ \qquad \text{for } s > s(e+1), \\ \text{undefined for } s \leqslant s(e+1). \end{cases}$$

The core of the argument is contained in the following:

<u>SUB-LEMMA</u> $\pi(e+1)$ is a well-defined string for each $e \geqslant 0$.

<u>PROOF</u> We first show that there is a string $\pi \subset B$ such that no potential extensions on $T^*_{e,s}$ are associated with a string $T_{2e+2,s}(\tau)$ for $T_{2e+2,s}(\tau) \supseteq \pi$ and $s \geqslant 0$.

Assume otherwise. For each $T_{2e+2}(\tau)$, $\lim_s R(T_{2e+2,s}(\tau))$ exists by lemma 8. Let $T_{2e+2}(\tau)$ be such that $T_{2e+2}(\tau) \subset B$, $T_{2e+2}(\tau) \in T^*_e$, $R(T_{2e+2}(\tau)) = \alpha$, say, and such that if $f_{2e+2,\alpha}$ is total then $R(T_{2e+2,s}(\tau * \sigma)) = \alpha \to q(T_{2e+2,s}(\tau * \sigma),\alpha) > X$ where $f_{2e+2,\alpha}(x) < \mu s(A_s[x] = A[x])$ for each $x > X$.

We show that there is some $T_{2e+2,s}(\tau') \supseteq T_{2e+2}(\tau)$ such that $T_{2e+2,s}(\tau') \subset B$, $R(T_{2e+2,s}(\tau')) = \alpha$, $\Pi(T_{2e+2,s}(\tau'), \alpha) \subset B$ and potential extensions σ_1, σ_2 of $T_{2e+2,s}(\tau')$ are appointed at stage s. By our assumption and since (4) holds for $e' < e + 1$ we must at some stage s appoint potential extensions of $T_{2e+2,s}(\tau)$ of boundary state α. Let τ' be such that $lh(T_{2e+2}(\tau')) > lh(\beta_s)$, $T_{2e+2}(\tau') \subset B$, $R(T_{2e+2}(\tau')) = \alpha$. Such a τ' exists because of the way in which we define p, by our assumption, and by (3) and (4) for $e' < e + 1$. We have $R(T_{2e+2}(\tau')) = \alpha$ since all potential extensions of strings $T_{2e+2}(\sigma) \subset T_{2e+2}(\tau')$ must eventually be always invalid by the assumption. It also follows from (4) and the assumption that the potential extensions σ_1, σ_2 are defined. We need only show that $\Pi(T_{2e+2}(\tau'), \alpha) \subset B$. To see this we note that for any string σ on T_e^* there is a 'minimal' path on T_e^* above σ, $M(T_e^*, \sigma)$, say. Then we will have

$\Pi(T_{2e+2}(\tau'), \alpha) \subseteq M(T_e^*, T_{2e+2}(\tau'))$ where

$\Pi(T_{2e+2}(\tau'), \alpha) = \lim_s \Pi(T_{2e+2,s}(\tau'), \alpha)$ (which exists since $T_{2e+2}(\tau')$ has potential extensions of boundary state α defined).

The only way in which we can have $\Pi(T_{2e+2}(\tau'),\alpha) \not\subset B$ is if we have some stage w such that $\beta_w \supset T_{2e+2}(\tau')$, β_w is incompatible with $M(T_e^*, T_{2e+2}(\tau'))$ and $\beta_w \sim A$. There are three possibilities:

(i) we define extensions σ_1, σ_2 through case case II on some tree at stage w where $\sigma_1, \sigma_2 \supset T_{2e+2}(\tau')$ and σ_1, σ_2 are incompatible with $\Pi(T_{2e+2,w}(\tau'), \alpha)$,

(ii) we define extensions σ_1, σ_2 through case II on some tree at stage w where $\sigma_1, \sigma_2 \supset T_{2e+2}(\tau')$, σ_1, σ_2 are compatible with $\Pi(T_{2e+2,w}(\tau'), \alpha)$ but $\Pi(T_{2e+2,w}(\tau'),\alpha) \not\subseteq \Pi(T_{2e+2}(\tau'), \alpha)$,

(iii) we define extensions σ_1, σ_2 through case II on some tree at stage w where $\sigma_1, \sigma_2 \supset T_{2e+2}(\tau')$, σ_1, σ_2 are compatible with $\Pi(T_{2e+2,w}(\tau'), \alpha)$ and $\bigcup_{\tau \subseteq \tau_1 \subseteq \tau'}$

$\Sigma'(T_{2e+2,w}(\tau_1)\alpha) \subset \Pi(T_{2e+2}(\tau'),\alpha)$.

Assume (i) holds. We are only interested in the case when σ_1, σ_2 are extensions for a string $T_{e,w}(\sigma) \subset \Pi(T_{2e+2,w}(\tau'),\alpha)$

We cannot have $e' < 2e + 2$ since $\Pi(T_{2e+2,w}(\tau'),\alpha)$ is compatible with each tree $T_{e',w}$, $e' < 2e + 2$. If $e' = 2e+2$ then our definition of extensions is only relevant if case (ii) or case (iii) holds. If we have $e' > 2e + 2$ we will have (by our use of prohibited strings)

$\Pi(T_{2e+2,w}(\tau'), \alpha) \neq \Sigma'(T_{2e+2,w}(\tau'), \alpha)$, and so case (i), (ii) or (iii) must occur at some stage $w' < w$. This means that we need only consider cases (ii) and (iii).

Assume (ii) holds. This can only happen if

$\Pi(T_{2e+2,w}(\tau'), \alpha) \supset T_{e',w}(\sigma)$, some σ, some $e' < 2e+2$, where $R(T_{e',w}(\sigma)) = \alpha' \sim \alpha$ and we eventually define $T_{e'}(\sigma * 0)$, $(\sigma * 1)$ through case II. By definition $\Pi(T_{e',w}(\sigma), \alpha')$ is compatible with $\Pi(T_{2e+2,w}(\tau'),\alpha)$. Also, by the construction we have $A_w[q(T_{e'}(\sigma), \alpha')] \neq A[q(T_{e'}(\sigma), \alpha')]$ (since C_A dominates $f_{e',\alpha'}$) and so we do not have $\Pi(T_{2e+2,w}(\tau'),\alpha) \sim A$.

Assume (iii) holds. We are only interested in extensions of strings $T_{e',w}(\sigma)$ where $e' \geqslant 2e+2$ and $T_{2e+2}(\tau') \subseteq T_{e',w}(\sigma) \subset \Pi(T_{2e+2}(\tau'), \alpha)$. To see the occurrence of (iii) does not lead to the assumed $\beta_w \sim A$, we merely follow through the construction and note that $q(T_{2e+2,s}(\tau),\alpha)$ is defined and each time we have

$q(T_{2e+2,w+1}(\tau_1), \alpha) > q(T_{2+2,w}(\tau_1), \alpha)$ or

$q(T_{2e+2,w+1}(\tau_1), \alpha)$ defined for the first time with

$\tau \subseteq \tau_1 \subseteq \tau'$, where $\tau_1 = \tau_2 * i$ say, we set

$f_{2e+2,\alpha}(q(T_{2e+2,w}(\tau_1),\alpha)) = w+1$ or $f_{2e+2,\alpha}(q(T_{2e+2,w}(\tau_2),\alpha) =$ $w+1$ respectively.

This ensures, since C_A dominates $f_{2e+2,\alpha}$, that

$\Sigma' (T_{2e+2}(\tau_1),\alpha) \subseteq \Pi(T_{2e+2}(\tau'),\alpha)$ for each $\tau_1 \subseteq \tau'$.

The rest of the proof of the sublemma follows by a similar argument. The hardest part, finding the <u>first</u> splitting pair which remains valid, follows the above proof. The extensions above the first pair will remain valid by our use of $f_{2e+3,}\alpha$.

It follows from the sublemma (incorporating the argument of (2) of lemma 4) that B is not recursive.
The proof of the facts (1) - (6) is very similar to that of lemma 4 and we leave the necessary adjustments to the reader.

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Initial segments of the degrees containing non-recursive recursively enumerable degrees

In recent work we have looked at the degrees below $\underline{0}'$ and tried to find some property of them not shared by the degrees below any other non-zero r.e. degree. We feel that we should be able to identify $\underline{0}'$ by some simple structure of the degrees surrounding it. This approach has had only limited success. It was shown by Yates [6] and by Lachlan [4] that there exist two non-zero r.e. degrees with g.l.b. $\underline{0}$, but by Lachlan [4] that there is no pair of incomparable r.e. degrees with g.l.b. $\underline{0}$ and l.u.b. $\underline{0}'$. The present author showed [2] that $\underline{0}'$ is the join of two minimal degrees but that not every non-zero r.e. degree is such a join. Another promising instrument is provided by a theorem of Yates (see [8]) that there is a non-zero r.e. degree which cannot be "cupped-up" to $\underline{0}'$ by any other r.e. degree $< \underline{0}'$.

We present here another approach, namely that of looking at the initial segments of the degrees containing $\underline{0}'$. We know, from Sacks' splitting theorem [5] and Friedberg's completeness theorem [3], that if an initial segment with a largest element contains $\underline{0}'$ then the largest element is the join of a pair of incomparable elements of the initial segment. We construct below a non-zero r.e. degree for which this does not hold. With respect to strings and trees we follow the notation of [7]. With respect to the partial recursive functionals Φ_i, $i \geq 0$, we follow the notation and practice of [4].

THEOREM. There is a non-recursive r.e. degree $\underline{a}$ and a degree $\underline{b}$, $\underline{a} < \underline{b} \leq \underline{0}'$, such that $\underline{a}$ is the l.u.b. of the degrees strictly less than $\underline{b}$.

PROOF: The construction will be carried out at stages 0,1,2,... . At stage s we will construct trees $T_{e,s,1}, T_{e,s,2}, \ldots, T_{e,s,m}$ for certain e's and define $T_{e,s} = T_{e,s,m}$. This will be done in such a way that $T_e = \mathrm{Lim}_s\, T_{e,s}$ exists for each e.

At stage s a string β_s will be chosen to lie on certain of the trees $T_{e,s}$. B will be the set whose characteristic function is $\mathrm{Lim}_s \beta_s$. If B lies on T_e then there will be a partial recursive sub-tree of T_e on which B also lies. When β_s is chosen there will be a string α_s of equal length associated with β_s (written $\alpha_s \sim \beta_s$). This will be done in such a way that: (1) if $\alpha_s \sim \beta_s$, $\alpha_t \sim \beta_t$ and $\beta_s \supset \beta_t$ then $\alpha_s \supset \alpha_t$, (2) if $\beta_s \sim \alpha_s$, $\beta_t \sim \alpha_t$, $t > s$, then $(\alpha_s(n) = 0 \rightarrow \alpha_t(n) = 0)$ for all n such that $\alpha_t(n)$ is defined. A_s consists of the numbers for which $\alpha_t(n) = 0$ for some $t \leq s$, and A will be the limit of A_s with respect to s. $T(\tau)$ is said to be an <u>end string</u> for T if $T(\tau)$ is defined and $T(\tau * 0)$, $(\tau * 1)$ $(= T(\tau * 0), T(\tau * 1))$ are not defined.

We say a string τ is <u>compatible with A_s</u> if for any α associated with an initial segment of τ, α is compatible with the characteristic function of A_s. τ is said to be <u>compatible with a tree T</u> if either τ lies on T or τ is an extension of some end string on T.

<u>Step 0.</u> Define $T_{-1,0} = I$ (the identity tree), $T_{e,o}$ being undefined for $e \geqslant 0$. Define $\beta_0 = \alpha_0 = \phi$ (the null-string).

<u>Step s+1.</u> As usual we say σ_1, σ_2 <u>split τ for e through x at step s + 1</u> if $\sigma_1, \sigma_2, \supset \tau$ and $\Phi_{e,s+1}(\sigma_i; x)$ are defined and unequal.

Firstly define $T_{-1,s+1,1} = T_{-1,s}$. Assume that $T_{-1,s+1,e+1}$, $T_{o,s+1,e}, \ldots\ldots\ldots, T_{i,s+1,e-i}, \ldots\ldots\ldots\ldots\ldots, T_{e-1,s+1,1}$ have been defined. Assume also that $T_{e,s+1,1}(\phi)$, (0), (1) have been defined and are equal to $T_{e,s}(\phi)$, (0), (1) respectively. Assume now that $T_{e,s+1,1}(\tau)$ has been defined but $T_{e,s+1}(\tau * 0)$, $(\tau * 1)$ have not and $T_{e,s+1,1}(\tau) = T_{e,s}(\tau)$. $T_{e,s+1,1}(\tau * 0)$, $(\tau * 1)$ may now become defined through one of the following cases.

<u>Case I.</u> Each of $T_{e,s}(\tau * 0)$, $(\tau * 1)$ is defined and is compatible with every tree $T_{i,s+1,e-i}$, $i < e$.

Define $T_{e,s+1,1}(\tau * 0), (\tau * 1) = T_{e,s}(\tau * 0), (\tau * 1)$ respectively.

<u>Case II.</u> Case I does not apply and $T_{e,s}(\tau)$ is not <u>terminal</u> or <u>dormant</u> (both terms to be defined later). Look for

$\sigma_1, \sigma_2 \supset T_{e,s}(\tau)$ such that σ_1, σ_2 are compatible with A_s, σ_1, σ_2 are compatible with each $T_{i,s+1,e-i}$, $i < e$, σ_1, σ_2 split $T_{e,s}(\tau)$ for e at stage s+1, and one of the following subcases holds:

(a) for each $i < e$ such that $T_{e,s}(\tau) \subset T_{i,s+1,e-i}(\pi * 0) \subset \sigma_1, \sigma_2$ for some string π, we either have that $T_{i,s+1,e-i}(\pi * 1)$ is incompatible with A_s or we have some j, $i < j < e$ and a string π' such that $T_{j,s+1,e-j}(\pi') \supseteq T_{i,s+1,e-i}(\pi * 0)$,

(b) σ_1, σ_2 do not satisfy (a).

We take action through case II according to the first of the two possibilities which obtains. We will have $T_{j,s+1,e+1-j} = T_{j,s+1,e-j}$ unless otherwise stated below.

(a) Define $T_{e,s+1,1}$ $(\tau * 0)$, $(\tau * 1) = \sigma_1, \sigma_2$ respectively. Also we inductively make changes in some of the values of $T_{i,s+1,e-i}$ above $T_{e,s}(\tau)$. Let j be the least number such that σ_1 properly extends an end string $T_{j,s+1,e-j}(\nu_1) \not\subset \sigma_2$. Define

$T_{j,s+1,e+1-j}(\nu_1) = \sigma_1$.

(We will say that $T_{j,s+1,e-j}(\nu_1)$ is __stretched__ to σ_1).

Assume that for some i, $j \leqslant i < e$, we have defined $T_{i,s+1,e+1-i}$ above $T_{e,s}(\tau)$ [illegible] (a)

(where a proper extension of $T_{e,s}(\tau)$ lies on $T_{i,s+1,e-i}$).

Let i' be the next such i. Let $S = \{T_{i',s+1,e-i'}(\pi) \mid (T_{i',s+1,e-i'}(\pi) \subseteq \sigma$, and $T_{i',s+1,e-i'}(\pi)$ does not lie on some $T_{h,s+1,e+1-h}$, $h \leqslant i$, such that σ_1 lies on $T_{h,s+1,e+1-h})$ and $(T_{i',s+1,e-i'}(\pi)$ is an end string $\subset \sigma_1$ but $\not\subset \sigma_2)\}$. If S is empty we make no alterations to the values of $T_{i's+1,e-i'}$ above $T_{e,s}(\tau)$. Otherwise let $T_{i's+1,e-i'}(\pi)$ be the string in S of least length. Define $T_{i',s+1,e+1-i'}(\pi) = \sigma_1$, $T_{i',s+1,e+1-i'}(\pi * \pi')$ undefined for all $\pi' \supset \phi$.

We deal similarly with σ_2.

(b) Let $j < e$ be the greater number such that there is a (least)string $T_{j,s+1,e-j}(\sigma) \supseteq T_{e,s}(\tau)$ such that, say $T_{j,s+1,e-j}(\sigma * 0)$ is defined and $\subset \sigma_1$, σ_2 and there is no proper extension of $T_{j,s+1,e-j}(\sigma)$ on any $T_{h,s+1,e-h}$, $j<h<e$, and $T_{j,s+1,e-j}(\sigma * 1)$ is compatible with A_s. Re-define $T_{e,s+1,1}(\tau) = T_{j,s+1,e-j}(\sigma * 1)$ and let $T_{e,s+1,1}(\tau)$ be <u>terminal</u>. Associated with the terminal string $T_{e,s+1,1}(\tau)$ will be the <u>predecessor</u> $T_{e,s}(\tau)$, <u>partners</u> σ_1, σ_2 and <u>instigator</u> x where σ_1, σ_2 split for e through x. $T_{e,s+1}(\tau)$ will remain terminal under all subsequent stretching until further notice. Also all strings on $T_{i,s+1,e-i}$, $i < j$, properly extending

$T_{e,s}(\tau)$ and compatible with σ, σ_2 become <u>dormant</u> until $T_{e,t}(\tau)$ changes at some step $t > s+1$ other than through stretching.

<u>Case III</u>. $T_{e,s+1,1}(\tau)$ is terminal, and there exists some $\sigma \supseteq T_{e,s+1,1}(\tau)$ such that σ is compatible with each $T_{i,s+1,e-i}$, $i<e$, σ is compatible with A_s and $\Phi_{e,s+1}(\sigma; x)$ is defined where x is the instigator of $T_{e,s+1,1}(\tau)$. We see if the partners and the predecessor of $T_{e,s+1,1}(\tau)$ are still compatible with every $T_{i,s+1,e-i}$, $i< e$. If not, we look for extensions of $T_{e,s+1,1}(\tau)$ through case II. Otherwise we re-define $T_{e,s+1,1}(\tau)$ = the predecessor of $T_{e,s+1,1}(\tau)$ and define $T_{e,s+1,1}(\tau * 0) = \sigma$ and stretch certain strings $T_{i,s+1,e-i}(\pi)$ with (the new) $T_{e,s+1,1}(\tau) \subset T_{i,s+1,e-i}(\pi) \subset \sigma$ to equal σ in the same way as we did for the σ_1 of case II(a).
Let σ_1, say, be a partner such that $\Phi_{e,s+1}(\sigma_1 \ ; x) \neq \Phi_{e,s+1}(\sigma \ ; x)$ where x is the instigator of $T_{e,s+1,1}(\tau)$. Define $T_{e,s+1,1}(\tau * 1) = \sigma_1$ and again stretch certain strings on proceeding trees to equal σ_1.
In either case $T_{e,s+1,1}(\tau)$ is no longer terminal and we no longer have any dormant strings associated with $T_{e,s+1,1}(\tau)$.

<u>Construction of $T_{e,s+1,1}(0)$, (1) when $T_{e,s+1,1}(\phi) \neq \beta_{s+1}$</u>

($T_{e,s+1,1}(0),(1)$ are not defined otherwise).

(i) $T_{e,s}(0),(1)$ satisfy case I above. Set $T_{e,s+1,1}(0),(1) = T_{e,s}(0),(1)$ respectively. If e does not require attention define $T_{e+1,s+1,1}(\phi) = T_{e+1,s}(\phi) = T_{e,s+1,1}(k)$ for some $k \leqslant 1$.

(ii) $T_{e,s}(0)$, (1) do not satisfy I and $e = 2f$, say.

Look for $\sigma_1, \sigma_2 \supset T_{e,s}(\phi)$ such that σ_1, σ_2 are compatible with A_s, σ_1, σ_2 are incompatible and compatible with each $T_{i,s+1,e-i}$, $i < e$, and such that σ_1, σ_2 satisfy II (a) above.

Action is taken under II(a) for σ_1, σ_2. We associate the initial segment of the characteristic function of A_s of appropriate length with each of $T_{e,s+1,1}(0)$, (1). Let x be the least number for which $\sigma_1(x) \neq \sigma_2(x)$. x becomes the follower of 2f and remains so as long as $T_{e,t}(0)$, (1) are equal to $T_{e,s+1,1}(0)$, (1) respectively at stages $t > s+1$.

(iii) $T_{e,s}$ (0), (1) do not satisfy I and $e = 2f + 1$.

If e is satisfied we proceed as in case (ii) except that all mention of the followers is omitted. Otherwise look for $\sigma_1, \sigma_2 \supset T_{e,s}(\phi)$ satisfying (ii) and also such that there exist, an x such that $\sigma_1(x)$, $\sigma_2(x)$ are defined and no string of length $\geqslant x$ has been associated with a string on a tree $T_{e,t}$ for any e or for any $t \leqslant s+1$, and x is not in any requirement of order $< e$. As in subcase (ii) we take the action under II(a) for σ_1, σ_2. Also associate with σ_1 a string of equal length which is an initial segment of the characteristic function of A_s. x becomes the follower of $2f + 1$ and remains so as long as $T_{e,t}(0)$, (1) are equal to $T_{e,s+1,1}(0)$, (1) respectively at stage $t > s+1$.

A follower x of e is said to require attention at step $s + 1$ if $e = 2f$ and $\Phi_{f,s+1}(A_s; x)$ is defined $= \beta_s(x)$, or if $e = 2f + 1$ and

$\Phi_{f,s+1}$ (N; x) is defined and is equal to $\alpha_s(x)$ where $\alpha_s \sim \beta_s$.

Construction of β_{s+1}.

Let e be the least number such that either the follower of e requires attention or $T_{e,s+1,1}(0)$, (1) have been defined other than through sub-case (i) (or both). If the former holds and $T_{e,s+1,1}(k) \subseteq \beta_s$, define $\beta_{s+1} = T_{e,s+1,1}(1-k)$, and if $e = 2f$ appoint a requirement of order e to preserve $\Phi_{f,s+1}$ $(A_s; x)$. If $e = 2f + 1$, e now becomes satisfied and x is enumerated in A_{s+1} and associated with $T_{e,s+1,1}(1-k)$ is an initial segment of the characteristic function of A_{s+1} of appropriate length. If the former does not hold, for $e = 2f$ define $\beta_{s+1} = T_{e,s+1,1}(0)$ and for $e = 2f + 1$ where e has follower x, define $\beta_{s+1} = \sigma_1$ (as defined in case (iii) above) When β_{s+1} has been chosen on $T_{e,s+1,1}$, say, we set $T_{i,s+1} = T_{i,s+1,e+1-i}$ for all $i \leqslant e$, and define $T_{e+1,s+1}(\phi) = \beta_{s+1}$. $T_{g,s+1}$ is undefined everywhere else for $g > e$.

LEMMA 1. If $T_e = \mathrm{Lim}_s\, T_{e,s}$ exists and there is an infinite path on T_e above $T_e(\tau)$, then there exist incompatible strings σ_1, σ_2 and a $t \geqslant 0$ such that for all $s > t$, $T_{e,s}(\tau * \sigma_1)$ $(\tau * \sigma_2)$ are defined and compatible with A_s.

PROOF: Assume that the number of strings $\sigma \subseteq T_e(\tau)$ lying on $T_e(\tau)$ lying on T_e are m in numbers. Let t be such that for all $s \geqslant t,$, $T_{e,s}(\tau) = T_e(\tau)$ and for each $2p + 1 \leqslant e + m$, $2p + 1$ is either satisfied at step t or is never satisfied. At some step $s \geqslant t$ there will be strings $T_{e,s}(\sigma * 0), (\sigma * 1)$ defined for the first time equal to the strings $T_e(\sigma * 0), (\sigma * 1)$ respectively, where $T_e(\sigma)$ lies on an infinite path on T_e above $T_e(\tau)$. In order to contradict the lemma we must have incompatible strings $\alpha_1, \alpha_2 \sim T_e(\sigma * 0), (\sigma * 1)$ respectively at some stage $u \geqslant s$. This can only happen through some $e' > e + m$ becoming satisfied at stage u. But in order for this to happen, we must associate initial segments of the characteristic function of A_u with strings on T_e corresponding to the strings $T_{i,u}(0), (1)$, $e \leqslant i \leqslant e + m$, and since there are only m strings on T_e below $T_e(\tau)$, there must be $T_{e+m,u}(0),$ $(1) = T_{e,u}(\pi * 0), (\pi * 1)$ respectively associated with A_u through case (ii) where $T_e(\tau) \subseteq T_{e,u}(\pi) \subset T_e(\sigma)$. Since whenever we subsequentlu satisfy some $i > e + m$, the follower used will not have been associated with any $T_{j,v}(0), (1)$, $j < i$, at an earlier stage v, and since $T_{e,u}(\pi) \subset T_e(\sigma)$, the lemma follows.

In order that the construction of A,B may be seen to be well-defined, we should note that if at stage s $T_{e,s}(\tau)$ is defined and compatible with A_s, then if $T_{e,s}(\tau * 0), (\tau * 1)$ are defined, then at least one of them is compatible with A_s.

LEMMA 2. $\mathrm{Lim}_s\, T_{e,s}$ exists for all e.

PROOF: Assume that $\mathrm{Lim}_s\, T_{i,s}$ exists for all $i < e$. The existence of $T_e(\phi)$, (0), (1) follows easily and we will only prove the existence of limits of extensions of $T_e(\tau)$, $\tau \supset \phi$. If $T_{e,s}(\tau)$ is undefined for all $s > t$, say, then so are $T_{e,s}(\tau * 0)$, $(\tau * 1)$. Assume that $\mathrm{Lim}\, T_{e,s}(\tau)$ exists, and that $T_{e,s}$ $(\tau * 0)$, $(\tau * 1)$ are defined for some s greater than any given N. Let j be the greater number less than e such that there are (least) incompatible extensions $T_j(\pi * 0)$, $(\pi * 1)$ of T_e (τ) on T_j compatible with A_i such that there is no proper extension of $T_j(\pi)$ on T_i for any i with $j < i < e$. Let t be such that for all $s > t$:
(1) $T_{e,s}(\tau) = T_e(\tau)$, (2) if there is no proper extension of $T_j(\pi)$ on T_i $(i < e)$ then there is no proper extension of $T_j(\pi)$ on $T_{i,s}$, (3) if there is a (least) pair of proper extensions $T_i(\omega * 0)$, $(\omega * 1)$, $i \leqslant j$, of T_j (π) compatible with A, then $T_{i,s}(\omega * 0)$, $(\omega * 1) = T_i(\omega * 0)$, $(\omega * 1)$ respectively and are compatible with A_s. We first show that if a pair of extensions $T_{e,s}$ $(\tau * 0)$, $(\tau * 1)$ of $T_e(\tau)$ change only by stretching at stages $s \geqslant s^*$, say, $> t$, then $T_e(\tau * 0)$, $(\tau * 1)$ exist and are extensions of T_{e,s^*} $(\tau * 0)$, $(\tau * 1)$ respectively. Let m be as in Lemma 1. Then $T_{e,s}(\tau * 0)$ can only be stretched to strings

on trees T_i such that $e < i < e + m$. So if $T_{e,s}(\tau * 0)$ is stretched infinitely often there must be some i, $e < i < e + m$, such that $T_{e,s}(\tau * 0)$ is stretched at infinitely many stages u to a string on $T_{i,u}$, and so there must be some σ, i such that $e < i < e + m$ and at infinitely many stages $u > s^*$ we have $T_{i,u+1}(\sigma) \subseteq T_e(\tau)$ and $T_{i,u+1}(\sigma * 0) \not\supseteq T_{i,u}(\sigma * 0)$. Let k be the least such i, and σ the least such sequence for k. Then we can choose $s' > s^*$ such that $T_{k,u}(\sigma)$ only changes by stretching after stage s' and hence an $\bar{s}$ such that $T_{k,u}(\sigma) = T_k(\sigma)$ for all $u \geqslant \bar{s}$ and such that for no i, $e < i < k$, $u \geqslant \bar{s}$, do we have $T_{i,n+1}(\pi) \subseteq T_e(\tau)$ and $T_{i,u+1}(\pi * 0) \not\supseteq T_{i,u}(\pi * 0)$. For all $u > \bar{s}$ and i, $e \leqslant i < k$, if we choose a new pair $T_{k,u}(\sigma * 0), (\sigma * 1)$ such that $T_{i,u}(\pi) \subseteq T_{k,u}(\sigma * 0), (\sigma * 1)$ we will either stretch extensions of $T_{i,u}(\pi)$ to equal $T_{k,u}(\sigma * 0), (\sigma * 1)$ or no extensions of $T_{i,u}(\pi)$ ever become defined, or say, $T_{i,u}(\pi * 0) \subseteq T_{k,u}(\sigma * 0), (\sigma * 1)$ (and so $T_{i,u}(\pi * 0) \subseteq T_e(\tau)$). Therefore $T_{k,u}(\sigma * 0)$ (and $T_{k,u}(\sigma * 1)$ similarly) will never change except through stretching. By the above, if the limits of $T_{e,s}(\tau * 0), (\tau * 1)$ are not to exist they must change at some stage $s+1 > t$ other than through stretching, and so we define $T_{e,s+1}(\tau * 0), (\tau * 1) = \sigma_1, \sigma_2 \supseteq T_j(\pi * 0), (\pi * 1)$ respectively, and stretch any strings on trees $T_{i,s+1}$, $i < j$, likely to extend incompatibly with σ_1, σ_2. Therefore $T_{e,s'}(\tau * 0)$ $(\tau * 1)$ only change by stretching at stages $s' \geqslant s+1$.

LEMMA 3. A is recursively enumerable and $\underline{b} \leqslant \underline{0}'$.

PROOF: From the construction and lemma 2.

LEMMA 4. $\underline{0} < \underline{a} < \underline{b}$.

PROOF: The steps we take to make $\underline{a} \neq \underline{0}$ and $\underline{b} \not\leqslant \underline{a}$ succeed since T_e exists for all e. To compute A(m) from knowledge of B, carry though the construction until a string β_s is defined which is a beginning of B and has length greater than m. Then there is a string $\alpha_s \sim \beta_s$ of length greater than m and $\alpha_s(m) = A(m)$.

LEMMA 5. For each e, if $\Phi_e(B)$ is total then either $\Phi_e(B)$ is recursive in A or B is recursive in $\Phi_e(B)$.

PROOF: T_e is a splitting tree for e although not necessarily partial recursive. If B lies on T_e we show how to compute B from $\Phi_e(B)$. Let k_i, $i \leqslant m$, be the numbers $< e$ such that B lies on T_{k_i}. Let t be such that for each $i < e$ such that there is an end string $T_i\ (\pi_i)$ which is a beginning of B we have for each $s > t$ $T_i(\pi_i)$ is an end string for $T_{i,s}$. Assume that $T_{e,s}(\tau)$, $s > t$, is known to be a beginning of B where the length of $T_{e,s}(\tau)$ is greater than the maximum of the lengths of the strings $T_i(\pi_i)$. We wait for some stage $s' \geqslant s$ such that $T_{e,s'}(\tau * 0)$, $(\tau * 1)$ are defined and lie on every T_{k_i}, $i \leqslant m$. Then $T_{e,s'}$ $(\tau * 0)$, $(\tau * 1)$ split for e and since they can only change by stretching one of them is a beginning of B, and we can decide this with help from $\Phi_e(B)$. If B does not lie on T_e, then we cannot have a terminal string on T_e as beginning of B, and so we know that there is a beginning β of B such that there is no pair $\sigma_1, \sigma_2 \supset \beta$ compatible with A and compatible with every T_i, $i < e$, such that σ_1, σ_2 split for e. We may take β to be $>$ max $\{$length of $T_i\ (\pi_i)\}$. To compute $\Phi_e\ (B; x)$ for $x >$ length of β, we need only look for a string $T_{e,s}(\tau) \supseteq \beta$, $s > t$, such that $T_{e,s}(\tau)$ lies on every $T_{k_i, s}$, $i \leqslant m$, $T_{e,s}(\tau)$ is compatible with A, and $\Phi_{e,s}\ (T_{e,s}(\tau); x)$ is defined.

There are various ways in which we might try to improve the theorem. A strong result seems to be precluded by recent work which indicates that we cannot make $\underline{a}' = \underline{0}''$.† A starting point would be to show that $\underline{a}$ can be any r.e. degree with jump $< \underline{0}''$. For this we might use the characterization of the Δ_2 sets with jump $< \underline{0}''$ given in [1]. On the other hand, treating such initial segments as interesting objects in themselves, one might, for instance, look for an analogue of the 'nth Chinese lantern' (see [7]).

† Cf conjecture on p.30 of Chapter 2

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The degrees of sets bounded truth-table reducible to creative sets

Introduction

By degree we mean degree of recursive unsolvability K is defined to be the set $\{x \mid (Ey)T((x)_1,(x)_2,y)\}$. Recursively axiomatizable first-order theories tend to be creative, and form a many-one degree determined by K [5]. There is a richness of structure for the degrees below $\underline{0}'$, and in particular for the recursively enumerable (r.e.) degrees, but although all r.e. degrees are degrees of recursively axiomatizable first-order theories [2] or of diophantine sets [4 , 7], we are unable as yet to find such non-recursive decision problems which can be approached other than through a decision procedure for K. We look at the degrees below $\underline{0}'$ through a consideration of the ways in which they can be obtained from K under different reducibilities. A lot is already known about the sets one-one (or many-one) reducible to K, these being just the r.e. sets. Taking K under bounded truth-tables (a notion first studied by Post [6]) we obtain another class of sets uniformly recursive in $\underline{0}'$, namely the Boolean algebra formed from the r.e. sets and their complements (see [8 p.317]). We can characterise the members of this class as the sets corresponding to predicates of the form: these exist exactly m_1 or m_2 or .... or m_t n-tuples $(y_1, \ldots\ldots, y_n)$ for which $R(x,y_1, \ldots\ldots, y_n)$, where R is recursive. It seems possible that we might be able to replace R by a diophantine predicate. We give a constructivised version of a theorem of Shoenfield [9] from which arises a set btt - reducible to K which is not contained in a r.e. degree. The theorem already gives a degree recursive in $\underline{0}'$ containing no set btt - reducible to K. The structural features of this proper extension of the r.e. degrees are very like those of the r.e. degrees. Since the proofs are rather long, we do not give them here. We could have looked at the degrees of sets truth-table reducible to K, but these are not uniformly recursive in $\underline{0}'$, and the structure of their degrees has as many irregularities as do the degrees of the Δ_2 predicates. For instance it is easily seen that both of the methods to appear so far [1 , 10] for deriving sets of minimal degree by recursive approximation give sets tt - reducible to K. Perhaps all the degrees below $\underline{0}'$ are degrees of sets tt - reducible to K?

<u>There are degrees btt - reducible to K other than the r.e. degrees</u>

A <u>string</u> is a restriction, $A[n]$ say, of a characteristic function A to an initial segment of the integers of length $n+1$, some n. We take $\{\Phi_e | e \geqslant 0\}$ to be a standard enumeration of the partial recursive functionals. We may consider Φ_e to consist of a countable collection of equations of the form $\Phi_e(\sigma, x) = y$, σ a string. Then there is a double sequence $\{\Phi_{e,s} | e, s \geqslant 0\}$ of finite sets of equations satisfying : (i) $\{\Phi_{e,s}\}$ is uniformly recursive, (ii) for each s, $\Phi_{e,s}$ is empty for all but a finite number of e's, (iii) $\Phi_{e,s} \subseteq \Phi_{e,s+1}$, each e, each s, (iv) $\Phi_e = \bigcup_{s \geqslant 0} \Phi_{e,s}$. For each e, s define $W_{e,s} = \mathrm{dom}\, \Phi_{e,s}(\Phi)$.

Then $\{W_{e,s} \mid e, s \geqslant 0\}$ is a double sequence of approximations to a standard enumeration $\{W_e \mid e \geqslant 0\}$ of the r.e. sets, satisfying the four conditions obtained from modifying (i) - (iv) above. Finally, we assume the existence of a recursive ordering of the triples (e, f, g) where e, f, g are natural numbers.

<u>THEOREM</u> There is a set btt - reducible to K whose degree is not recursively enumerable.

<u>PROOF</u>: We construct a set $A = B \cap \overline{C}$ where B, C are r.e., satisfying the following (ordered) set of conditions:

$$\{\Phi_g(A) \neq W_e \text{ or } \Phi_f(\Phi_g(A)) \neq A \mid e, f, g \geqslant 0\}.$$

The style of presentation is that of [3].

We define eight recursive objects:

(1) a string α is <u>acceptable at stage $s+1$</u>, i.e., $\mathrm{Acc}(\alpha, s+1)$, if for each x we have

$$\alpha(x) = 0 \rightarrow (t)(t \leqslant s \rightarrow A_s(x) \leqslant A_t(x)),$$

(2) a triple (e, f, g) is said to <u>require attention of the first kind at stage $s+1$</u>, i.e., $\mathrm{Req}^1(e, f, g, s+1)$, if $\mathrm{Foll}(e, f, g, s)$ is undefined,

(3) a triple (e, f, g) is said to <u>require attention of the second kind at stage $s+1$</u>, i.e., $\mathrm{Req}^2(e, f, g, s+1)$, if there is a string α such that $\mathrm{Acc}(\alpha, s+1)$ and for each z for which $\alpha(z)$ is defined, $\alpha(z) \neq A_s(z) \rightarrow z > \mathrm{Pres}(e, f, g, s)$ or $z = \mathrm{Foll}(e, f, g, s)$ and a number y such that $\Phi_{g,s}(\alpha, i)$ is defined for each $i < y$ and

$$\Phi_{f,s}(\Phi_{g,s}(\alpha)[y], \mathrm{Foll}(e, f, g, s)) \text{ is defined and}$$

$$\Phi_{g,s}(\alpha, i) = W_{e,s}(i) \text{ for all } i < y \text{ or for some } i < y,$$

$$\Phi_{g,s}(\alpha, i) = W_e(i) = 0 \text{ and } \mathrm{Sat}(e, f, g, s+1) \text{ does not hold,}$$

(4) (e, f, g) is said to <u>require attention at stage $s+1$</u>, i.e., $\mathrm{Req}(e, f, g, s+1)$ if (e, f, g) is the least triple for which $\mathrm{Req}^i(e, f, g, s+1)$, $i = 1$ or 2,

(5) (e, f, g) is <u>satisfied at step $s+1$</u>, i.e., $\mathrm{Sat}(e, f, g, s+1)$, if $\mathrm{Req}^2(e, f, g, t)$ and $\mathrm{Req}(e, f, g, t)$ hold for some $t < s+1$ and $\mathrm{Foll}(e, f, g, t) = \mathrm{Foll}(e, f, g, s)$ and if (α, y) is the pair for which $\mathrm{Req}^2(e, f, g, t)$ holds then for some $i < y$

$$\Phi_{g,s}(A_s, i) \neq W_{e,s}(i) \quad \text{or}$$

$$\Phi_{f,s}(W_{e,s}[y], \mathrm{Foll}(e, f, g, s)) \neq A_s(\mathrm{Foll}(e, f, g, s)).$$

Lastly we define three recursive functions by induction on S.

Stage 0

Pres $(e,f,g,0) = 0$ each e,f,g (Pres is a preserving function),

Foll $(e,f,g,0)$ is undefined, each (e,f,g),

$$A_0(x) = 1 \quad \text{each } x.$$

Stage s+1

Assume Req $(e,f,g,s+1)$.

For each $(e', f', g') > (e,f,g)$,

Foll $(e', f', g', s+1)$ is undefined.

For each $(e', f', g') < (e,f,g)$,

set Pres $(e',f',g',s+1)$ = Pres (e',f',g',s),

Foll $(e',f',g',s+1)$ = Foll (e',f',g',s).

There are now two cases.

(I) $\mathrm{Req}^1\,(e,f,g,s+1)$.

Define

$$\text{Foll}\,(e,f,g,s+1) = 1 + \max\{\text{Pres}\,(e',f',g',s) \mid \text{any } (e',f',g')\}$$

$$= \text{Pres}\,(e',f',g',s+1) \text{ each } (e',f',g') \geqslant (e,f,g),$$

$$A_{s+1} = A_s.$$

(II) $\mathrm{Req}^2\,(e,f,g,s+1)$.

Define

Foll$(e,f,g,s+1)$ = Foll (e,f,g,s).

sub-case (a) $\Phi_{g,s}(\alpha, i) = W^{i}_{e,s}(i)$ for all $i < y$.

Define

$$A_{s+1}(\mathrm{Foll}(e, f, g, s)) = \begin{cases} 0 \text{ if } \Phi_{f,s}(\Phi_{g,s}(\alpha)[y], \\ \qquad \mathrm{Foll}(e, f, g, s)) = 1, \\ 1 \text{ otherwise.} \end{cases}$$

Otherwise set

$$A_{s+1}(x) = \begin{cases} \alpha(x) & \text{if } \alpha(x) \text{ is defined,} \\ A_s(x) & \text{otherwise.} \end{cases}$$

Let Pres $(e', f', g', s+1)$

$= \max \{\mathrm{Pres}\ (e', f', g', s), \text{any } (e', f', g'),$

$\max \{x \mid \alpha(x) \text{ defined }\}\}$ for each

$(e', f', g') \geqslant (e, f, g)$.

sub-case (b). If $\Phi_{g,s}(\alpha, i) \neq W_{e,s}(i) = 0$ for some $i < y$, set

$A_{s+1}(x) = \alpha(x)$ for each x for which $\alpha(x)$ is defined, $= A_s(x)$ otherwise.

Define Pres$(e', f', g', s+1)$ as in subcase (a) for each $(e', f', g') \geqslant (e, f, g)$.

<u>LEMMA</u> $A = \lim_s A_s$ is btt-reducible to $0'$.

<u>PROOF</u>: Define

$$A^1 = \{x \mid (E s \geqslant 0)\ (x \in A_s)\}$$

$$A^2 = \{x \mid (Es_1, s_2 \geqslant 0)(s_1 < s_2 \text{ and } A_{s_1}(x) < A_{s_2}(x))\}.$$

Then A^1, A^2 are recursively enumerable.

We show that $A = A^1 \cap \overline{A^2}$.

Obviously $A' \cap \overline{A^2} \subseteq A$.

Assume $x \in A$ but $x \notin A^1 \cap \overline{A}^2$.

Since $x \in A'$ we must have $x \in A^2$.

Let s be the least number for which $A_s(x) < A_{s+1}(x)$.

Then at stage s+1, $Req^2(e, f, g, s+1)$ for some (e, f, g).
There are two cases to consider.

(1) $Foll(e, f, g, s) \neq x$.

By definition of Pres (e, f, g, s) in part (I) above and by the conditions put on α in the definition of $Req^2(e, f, g\ s+1)$, $Foll(e', f', g', s) = x$ for no $(e', f', g') < (e, f, g)$.

And for each $(e', f', g') > (e, f, g)$,

$Foll\ (e', f', g', s+1)$ is undefined.

Also $Pres(e, f, g, s+1) \geqslant x$, so by (I) above $Foll(e', f', g', s') = x$ for no (e', f', g'), no $s' > s+1$. Since any α used in (II) at any stage $s' > s+1$ must be acceptable at stage s', we move onto case:

(2) $Foll\ (e, f, g, s) = x$.

Then $x \leqslant Pres\ (e, f, g, s)$.

Say sub-case (b) applies at stage s+1.

Then $A_{s+1}\ (i) = \alpha(i)$ for each $i < lh\,\alpha$, and since

$Pres\ (e, f, g, s+1) \geqslant lh\,\alpha$, we have $\Phi_{g,s'}\ (A_{s'}, j) \neq W_{e,s'}(j) = 0$

and Sat (e, f, g, s') for each $s' > s$ unless $Req\ (e', f', g', w+1)$ some $(e', f', g') < (e, f, g)$, some $w > s$. So for no s' do we have $s' > s$ and $Req\ (e, f, g, s'+1)$ and $x = Foll\ (e, f, g, s')$, so that $x \notin A$, contrary to the assumption.

Now say sub-case (a) applies at stage s+1. Since $A^2_s(x) = 0$, either $x < lh\,\alpha'$ for some α' such that $\mathrm{Req}^2(e', f', g', t+1)$, $\mathrm{Req}\,(e', f', g', t+1)$ hold for some e', f', g', t, $t < s$, through a pair (α', y') and $\alpha'(x) \neq A_t(x) = 1$ or $\mathrm{Req}^2(e, f, g, t+1)$ holds for some $t < s$ where $x = \mathrm{Foll}\,(e, f, g, t)$ and sub-case (a) applies. The latter must hold since if the former holds, $\mathrm{Foll}\,(e, f, g, t) \neq x$ implies that $x = \mathrm{Foll}\,(e, f, g, s) \geqslant \mathrm{Pres}\,(e', f', g', t+1) \geqslant lh\,\alpha'$, or $(e', f', g') < (e, f, g)$ implies that $\mathrm{Foll}\,(e, f, g, t+1)$ is undefined and so $\mathrm{Foll}\,(e, f, g, s) > \mathrm{Foll}\,(e, f, g, t) = x$, or $(e', f', g') > (e, f, g)$ implies that $\mathrm{Pres}\,(e', f', g', t) \geqslant \mathrm{Pres}\,(e, f, g, t) \geqslant \mathrm{Foll}\,(e, f, g, t) = x$, and since $\mathrm{Foll}\,(e', f', g', t) \neq x$, and by definition of Req^2, we have $A_{t+1}(x) = A_t(x) = 1$.

Assume $\mathrm{Req}^2\,(e, f, g, t+1)$ holds through (α, y). Then $\Phi_{g,t}(\alpha, i) = W_{e,t}(i)$ for all $i < y$. By definition of Req^2, $\mathrm{Sat}\,(e, f, g, s+1)$ does not hold. So $\Phi_{g,s}(A_s, i) = W_{e,s}(i)$, each $i < y$, and $\Phi_{f,s}(W_{e,s}[y], x) = A_s(x)$. But by definition of A_{t+1},

$$\Phi_{f,t}(\Phi_{g,t}(\alpha)[y], x) \neq A_{t+1}(x).$$

So $W_{e,s}(i) < W_{e,t}(i)$, some $i < y$, or $A_s(x) \neq A_{t+1}(x)$.

So $A_s(x) \neq A_{t+1}(x)$ or $\Phi_{g,t}(\alpha, i) \neq W_{e,s}(i) = 0$, some $i < y$.

In either case $\mathrm{Req}^2\,(e, f, g, t'+1)$ must hold through sub-case (b) for some t', $t < t' < s$, and as above we cannot have $\mathrm{Req}\,(e, f, g, s+1)$ and $x = \mathrm{Foll}\,(e, f, g, s)$, a contradiction.

It follows from the lemma by a simple induction that for each

(e, f, g) we have $\text{Req}\,(e, f, g, s)$ for only finitely many numbers s. From this it follows that $\Phi_g(A, i) \neq W_e(i)$, some i, or $\Phi_f(W_e\ x) \neq A(x)$, some x, and so $A \not\equiv_T W_e$.

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Minimal upper bounds for ascending sequences of degrees

G.E. Sacks [1] asked whether there is a uniformly enumerable ascending sequence of degrees which has as one of its minimal upper bounds another r.e. degree. We show below that the answer is 'yes'.

Let $\{\Phi_e\}$ be a standard enumeration of the partial recursive functionals. We will need a uniformly recursive double sequence $\{\Phi_{e,s}\}$ of finite approximations to $\{\Phi_e\}$ such that $\Phi_{e,s} \subseteq \Phi_{e,s+1}$ for each e,s and such that for each s $\Phi_{e,s}$ is empty for all but a finite number of e's. Define for each e, $F_e = \Phi_e(\phi)$, $F_{e,s} = \Phi_{e,s}(\phi)$. Then $\{F_e\}$ is a standard list of the partial recursive functions with suitably well-behaved set of approximations $\{F_{e,s}\}$.

Let p be a recursive injection of the pairs of integers into the integers.

THEOREM . There is a sequence of simultaneously r.e. degrees $\{\underline{b}_i\}$ and a r.e. degree $\underline{a}$ such that $\underline{a}$ is recursive in no finite subset of $\{\underline{b}_i\}$, $\underline{b}_i$ is recursive in $\underline{a}$ for each i, and for each $\underline{c} < \underline{a}$ $\underline{c}$ is not an upper bound for $\{\underline{b}_i\}$.

PROOF: We enumerate at stages 0,1,2,..., s,... finite sets A_s, $B_{i,s}$, $i \geqslant 0$, such that, if $A = \bigcup_{s\geqslant 0} A_s$, $B_i = \bigcup_{s\geqslant 0} B_{i,s}$, then the following objectives are satisfied:

$$A \neq \Phi_i \Big(\bigcup_{0 \leqslant k \leqslant j} B_k \Big), \quad i, k \geqslant 0,$$

$$B_i = \Phi_j\,(\Phi_i(A)) \rightarrow A \leqslant_T \Phi_i\,(A).$$

The first group of conditions are the objectives <u>of the first kind</u>, the second those <u>of the second kind</u>. (Since all the sets A, B_i, $i \geqslant 0$, will be disjoint we may use the union in place of the more usual recursive join operations). We will assume some recursive ordering of the objectives so that we may talk during the construction of 'the m^{th} objective'. The m^{th} objective will be said to be <u>satisfied at stage $s+1$</u> if m has required attention at some stage $t+1 < s+1$ and no objective below m has required attention at a stage $t'+1$, $t+1 < t'+1 < s+1$. At stage 0 we have $A_0 = B_{i,o} = \phi$, each $i \geqslant 0$, and there are no <u>obstacles</u> of any order, and no objective has any <u>followers</u>. At the stages $2s+1 > 0$ we are concerned with the objectives of the first kind. m <u>requires attention at stage $2s+1$</u> if m is not satisfied at step $2s+1$ and there exists a number $p(m,n)$, $n \geqslant 0$, which is greater than any obstacle of order less than m and is less than the <u>failure of m at stage $2s+1$</u>, where the failure of m at stage $2s+1$ is defined to be (for appropriate i, j) the least number x for which $A_s(x) \neq \Phi_{i,s}(\bigcup_{0 \leqslant k \leqslant j} B_{k,s}, x)$. The <u>reference of m at stage $2s+1$</u> is the least number z for which $\Phi_{i,s}(\bigcup_{0 \leqslant k \leqslant j} B_{k,s}\,[z], x)$ is defined for each number x less than the failure of m at stage $2s+1$, the reference of m is appointed to be an obstacle of order m, and $p(m,n)$ is enumerated in A_{s+1} or not according as $\Phi_{i,s}(\bigcup_{0 \leqslant k \leqslant j} B_{k,s}, p(m,n)) = 1$ or not.

At stages $2s+2 > 0$ we are concerned with objectives of the second kind. Firstly certain numbers $p(m,n)$ may become followers of such objectives. If m is of the second kind then the failure of m at stage 2t is defined to be the least number x for which $B_{i,t}(x) \neq \Phi_{j,t}(\Phi_{i,t}(A_t), x)$ for appropriate i, j. Assume that $p(m,n)$ is less than the failure of m at some stage $2t < 2s+2$ and that $A_s[p(m,n)] \neq A_{s+1}[p(m,n)]$. Appoint $p(m,n)$ to follow m and associate with $p(m,n)$ an entry condition $(\sigma \subset \Phi_{i,u+1}(A_{u+1}))$ for each σ such that σ is the string of least length for which $\sigma \subset \Phi_{i,t}(A_t)$ at a stage $2t < 2s+2$ for which $\Phi_{j,t}(\Phi_{i,t}(A_t), p(m,n))$ is defined. This entry condition may become satisfied in an obvious sense at a stage $2u+2 > 2t$. We say that m requires attention at stage $2s+2$ if m is not satisfied at stage $2s+2$ and there is a follower $p(m,n)$ of m greater than any obstacle of lesser order one of whose entry conditions is satisfied. Also m is the least such number. Enumerate $p(m,n)$ in $B_{i,s+1}$ or not according as $\Phi_{j,s+1}(\Phi_{i,s+1}(A_{s+1}), p(m,n)) = 1$ or not. Set up an obstacle z of order m where z is the least number for which $\Phi_{j,s+1}(\Phi_{i,s+1}(A_{s+1}[z]), p(m,n))$ is defined.

Lemma 1. Each number m only requires attention finitely often.

PROOF: By induction. If there is a stage t such that no objective less than m requires attention at a stage greater than t, then if m requires attention at a stage $w > t$, m is satisfied at every stage $> w$ and cannot require attention at any stage $> w$.

LEMMA 2. For each $i, k \geq 0$ we have

$$A \neq \Phi_i\left(\bigcup_{0 \leqslant k \leqslant j} B_k\right).$$

PROOF: Let the above condition be the m^{th} objective and let t be as in lemma 1. Let K be the largest of the obstacles of order less than m set up at any stage > 0 (K must exist since t does and since obstacles of order m' are only set up at stages at which m' requires attention). We may assume that m is not satisfied at step $t+1$. Let $p(m,n)$ be the least such number greater than K such that $p(m,n) \notin A_t$. If the failure of m at each stage $> t$ is not greater than $p(m,n)$ then we cannot have $A = \Phi_i\left(\bigcup_{0 \leqslant k \leqslant j} B_k\right)$. Let $2u + 1$ be the least stage greater than t at which the failure of m is greater than $p(m,n)$. m is not satisfied at stage $2u + 1$ and so m requires attention at stage $2u + 1$. At the end of this stage we have

$\Phi_{i,u}\left(\bigcup_{0 \leqslant k \leqslant j} B_{k,s}, p(m,n)\right)$ is defined and is not equal to

$A_{u+1}(p(m,n))$. Since m is satisfied at each stage $> 2u+1$ and since an obstacle greater than the least z for which

$\Phi_{i,u}\left(\bigcup_{0 \leqslant k \leqslant j} B_{k,u}[z], p(m,n)\right)$ is defined is associated with m,

we have $\Phi_i\left(\bigcup_{0 \leqslant k \leqslant i} B_k, p(m,n)\right)$ defined $\neq A(p(m,n))$.

LEMMA 3. For each $i, j \geqslant 0$ if $B_i = \Phi_j(\Phi_i(A))$ then A is recursive in $\Phi_i(A)$.

PROOF: Let $B_i = \Phi_j(\Phi_i(A)) \rightarrow A \leqslant_T \Phi_i(A)$ be the m^{th} objective and let t and K be as in lemma 2. As in lemma 2 we assume that the failure of m is not bounded since otherwise there is nothing to prove. We show how to compute an arbitrary value A(x) using some beginning of $\Phi_i(A)$. As in lemma 2, m can be satisfied at no stage $> t$. Let p(m,n) be greater than x and greater than K and let $p(m,n) \notin B_{i,t}$. Let $2s^* + 2$ be the least such stage greater than t for which p(m,n) is less than the failure of m at stage $2s^*$ and for which, if z is the least number for which $\Phi_{j,s^*}(\Phi_{i,s^*}(A_{s^*})[z], p(m,n)$ is defined we have that $\Phi_{i,s^*}(A_{s^*})[z]$ is a beginning of $\Phi_i(A)$. Now say $A_{s^*}(x) \neq A(x)$. In this case we would have at some stage $w + 1 > s^*$ that $A_w[p(m,n)] \neq A_{w+1}[p(m,n)]$, and at the beginning of stage $2w + 2$ we would appoint p(m,n) as a follower of m with $(\Phi_{i,s^*}(A_{s^*})[z] \subset \Phi_{i,u+1}(A_{u+1}))$ as one of its entry conditions. By definition of s^*, at some stage $u \geqslant w$, this entry condition will be satified, m will require attention at stage $2u + 2$, and we define

$B_{i,u+1}(p(m,n)) \neq \Phi_{j,u+1}(\Phi_{i,u+1}(A_{u+1}), p(m,n))$ and set up an obstacle of order m to protect the right hand side of this definition. This would give $B_i(p(m,n)) \neq \Phi_j(\Phi_i(A), p(m,n))$, contrary to assumption.

LEMMA 4. For each i B_i is recursive in A.

PROOF: It will be convenient to assume that we have included in the construction steps to make $\lim_s \Phi_{i,s}(A_s, x)$ exist for all numbers $i, x \geqslant 0$. We can do this since by lemma 1 our actions with regard to a particular objective only involve finite injury of higher objectives. So there is a number s^* such that for all $s > s^*$, $x \geqslant 0$, if $\Phi_{i,s}(A_s, x)$ is defined then $\Phi_i(A,x)$ is defined. First of all we notice that the only numbers enumerated in B_i are followers of m^{th} objectives of the form

$B_i = \Phi_j(\Phi_i(A)) \to A \leqslant_T \Phi_i(A)$. We compute $B_i(p(m,n))$ with aid from A. Let f be $\Phi_i(A)$ or the longest string $\subseteq \Phi_i(A)$ according as $\Phi_i(A)$ is total or not. We can also assume knowledge of the graph of f. To begin with we look for a stage $u > s^*$ for which we have

$$A_u[p(m,n)] = A[p(m,n)].$$

If $B_{i,u}(p(m,n)) = 0$ then $B_i(p(m,n)) = 0$. If $p(m,n)$ is not a follower of m at stage $2u+2$ and $B_{i,u}(p(m,n)) = 1$, then $B_i(p(m,n)) = 1$. Say that $B_{i,u}(p(m,n)) = 1$ and $p(m,n)$ is a follower of m at stage $2u+2$. Since $p(m,n)$ can acquire no more entry conditions at any stage $s \geqslant 2u + 2$, we look at the entry conditions associated with $p(m,n)$ at stage $2u+2$. We look for a stage w such that if $k+1$ is the lesser of the length of f and of the largest of the lengths of the strings σ involved in entry conditions associated with $p(m,n)$ at stage $2u + 2$, then $\Phi_{i,w}(A_w)[k]$ is well-defined and if z is the least number for which $\Phi_{i,w}(A_w[z])[k]$ is well-defined then $A_w[z] = A[z]$. Obviously $\Phi_{i,v}(A_v)[k]$ is a beginning of f for all $v \geqslant w$. This means that if no entry condition associated with $p(m,n)$ involves a string $\sigma \subseteq f$, then $B_{i,w}(p(m,n)) = B_i(p(m,n))$ since any entry condition associated with $p(m,n)$ must become satisfied before stage $2w+2$. No entry condition involving a string σ of length greater than that of f can become satisfied by our choice of s^*. If there is associated with $p(m,n)$ at stage $2w+2$ an entry condition

$$(\sigma \subset \Phi_{i,u+1}(A_{u+1}))$$

with $\sigma \subseteq f$, then this entry condition is satisfied at each stage $2u + 2 \geqslant 2w + 2$, and if $B_{i,w}(p(m,n)) \neq 0$ then at some stage $2u + 2 \geqslant 2w + 2$ m will require attention and $p(m,n)$ will be enumerated in B_i, unless an obstacle greater than $p(m,n)$ of order less than m is set up in which case $B_i(p(m,n)) = 1$. We can tell which occurs by following through a finite number of stages of the construction.

It seems unlikely that any degree below $\underline{0}'$ is a minimal upper bound for an ascending sequence uniformly recursive in that upper bound.

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