Polynomial programming approach to weak approximation of Lévy-driven stochastic differential equations with application to option pricing

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Abstract: We propose an optimization approach to weak approximation of Lévy-driven stochastic differential equations. We employ a mathematical programming framework to obtain numerically upper and lower bound estimates of the target expectation, where the optimization procedure ends up with a polynomial programming problem. An advantage of our approach is that all we need is a closed form of the Lévy measure, not the exact simulation knowledge of the increments or of a shot noise representation for the time discretization approximation. We present numerical examples of the computation of the moments, as well as the European call option premium, of the Doléans-Dade exponential model.

Keywords: Lévy process, Stochastic approximation, Polynomial optimization, Option pricing.

1. INTRODUCTION

Stochastic differential equations have long been used to build realistic models in economics, finance, biology, the social sciences, chemistry, physics and other fields. In most active fields of application, dynamics with possible sudden shift have become more and more important. To model such shifts, one would like to employ stochastic differential equations where the underlying randomness contains jumps. For this purpose, the well-known Wiener process (diffusion) is not sufficient since its sample paths are almost surely continuous. On the other hand, Lévy-driven stochastic differential equations, which contain diffusion as a special case, can formulate stochastic behavior with jumps. Regardless of its practical importance, however, the theory and the computational techniques of the Lévy processes have not been developed thoroughly as in the diffusion case. As nice references on the subject, we refer to Applebaum [1] and Bass [2].

From a practical point of view, the sample paths approximation of stochastic differential equations has been a central issue for the purpose of numerical evaluation and simulation on the computer. There are two notions of the approximation; strong and weak approximations. The strong approximation schemes provide pathwise approximations which can be employed in scenario analysis, filtering or hedge simulation. For applications such as derivative pricing, the computation of moments or expected utilities, the so-called weak approximations are sufficient, that is, we need to estimate the expected value of a function. Other applications of the weak approximation include the computation of functional integrals, invariant measures, and Lyapunov exponents.

The theoretical properties of time discretization schemes are mostly studied for the diffusion case. See [10, 12, 13, 16, 17] for detailed investigation. For the Lévy-driven stochastic differential equations, the weak rate of convergence of the Euler scheme is studied, for example, [7, 13, 16, 21]. The jump-adapted discretization is investigated, for example, in [6, 9, 17], while the jump adaptation is only valid in the compound Poisson framework.

The main purpose of this paper is to propose a new approach to weak approximation of Lévy-driven stochastic differential equations. Unlike Monte Carlo simulation with the time discretization approximation of sample paths, we employ a mathematical programming framework to obtain numerically upper and lower bounds of the target expectation, where the optimization procedure ends up with a polynomial programming. To this end, we follow the idea in [4, 8, 15, 23]. Note that these existing results have dealt only with the pure diffusion case (i.e., without jump component) for which standard Monte Carlo methods are sufficient. In this sense, it should be emphasized that our result is not a trivial extension. The main drawback is the complexity of the Ito formula for general Lévy processes. As such, we need to carefully examine whether or not the resulting optimization problems are practically solvable. Fortunately, as we show in the following sections, our approach covers various practically important Lévy processes.

The rest of this paper is organized as follows. Section 2 gives mathematical definition of Lévy-driven stochaptic differential equations and weak approximation problem. Section 3 introduces and studies our optimization approach to the weak approximation with a numerical example. In Section 4, we apply the proposed method to the barrier option pricing problem. applies the proposed method to discusses a way to improve the approximation accuracy. Finally, Section 5 concludes this paper.

2. PROBLEM SETTING

Let us begin this section with general notations which will be used throughout the text. For $k \in \mathbb{N}$, ∂_k indicates the partial derivative with respect to k-th argument. We denote by C^{k_1,k_2} the class of continuous functions with continuous differentiability of k_1-time for the first argu-
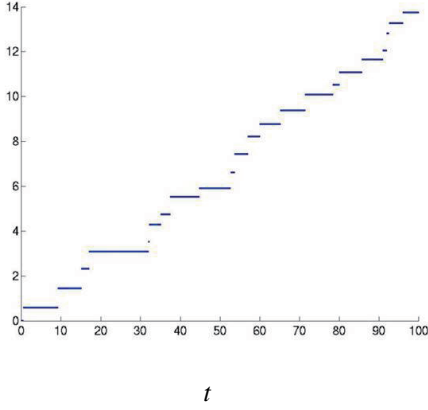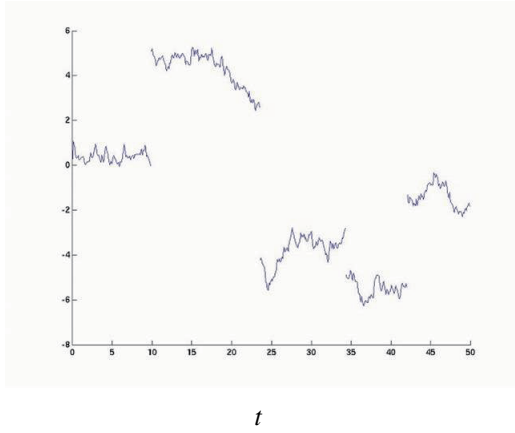
Fig. 1 Typical sample path of Lévy process

Fig. 2 Typical sample path of Lévy driven stochastic process

ment and of k_2-time for the second argument.

Let X_0 be given in $\mathbb{R}$ and let $T > 0$. Consider a one-dimensional Lévy-driven stochastic differential equation

$$dX_t = a_0(X_t)dt + a_1(X_t)dW_t \qquad (1)$$
$$+ \int_{\mathbb{R}_0} zb(X_{t-})(\mu - \nu)(dz,dt), \quad t \in [0,T],$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion and where μ is a Poisson random measure on $\mathbb{R}_0$ whose compensator is given by the Lévy measure ν. In order for the solution of (1) to be well defined, we impose the usual Lipschitz conditions and linear growth conditions on a_0, a_1 and b.

Typical sample paths of Lévy process (jump component) and Lévy-driven stochastic system are illustrated Figs. 1 and 2, respectively. For an example to provide a concrete image of Lévy processes, see the Doléans-Dade exponential in Appendix.

Our interest throughout this study is in approximating the expectation

$$\mathbb{E}[V(X_T)], \qquad (2)$$

for $V : [0,+\infty) \times \mathbb{R} \mapsto \mathbb{R}$ such that $\mathbb{E}[|V(X_T)|] < +\infty$. This class of V covers most of prictically interesting case. We

here show 2 direct application of this problem. If we take

$$V(x) := \begin{cases} 0, & x < \eta \\ 1, & x \geq \eta, \end{cases}$$

for $\eta > 0$, then (2) is the tail probability of X_T:

$$\mathbb{E}[V(X_T)] = \mathbb{P}[X_T \geq \eta].$$

If we take

$$V(x) := (x - K)_+ := \max\{x - K, 0\}$$

for $K > 0$, then (2) is the European call option premium with the strike price K at period T.

3. MAIN RESULT

3.1 Lévy-Ito formula

We are now in a position to introduce our optimization approach to the weak approximation. Let $\mathscr{X}(\subseteq \mathbb{R})$ be a support of $\{X_t : t \in [0,T]\}$ defined in (1). For $f \in C^{1,2}([0,T] \times \mathscr{X}; \mathbb{R})$, the Ito formula yields

$$df(t,X_t) = \mathscr{A}f(t,X_t)dt + \partial_2 f(t,X_t)a_1(X_t)dW_t$$
$$+ \int_{\mathbb{R}_0} B_z f(t,X_{t-})(\mu - \nu)(dz,dt), \quad a.s.,$$

where the infinitesimal generator is given by

$$\mathscr{A}f(t,x) := \partial_1 f(t,x) + \partial_2 f(t,x)a_0(x)$$
$$+ \frac{1}{2}\partial_2^2 f(t,x)a_1(x)^2$$
$$+ \int_{\mathbb{R}_0}(B_z f(t,x) - \partial_2 f(t,x)b(x,z))\nu(dz),$$

and for $z \in \mathbb{R}_0$,

$$B_z f(t,x) := f(t,x+zb(x)) - f(t,x).$$

We can then derive the corresponding Dynkin formula, for $T \geq 0$,

$$\mathbb{E}[f(T,X_T)] - f(0,X_0) = \mathbb{E}\left[\int_0^T \mathscr{A}f(t,X_t)dt\right].$$

Therefore, as soon as one finds an $f \in C^{1,2}([0,T] \times \mathscr{X}; \mathbb{R})$ such that $\mathscr{A}f(t,x) \leq 0$ for $(t,x) \in [0,T] \times \mathscr{X}$, and that $f(T,x) \geq V(x)$ for $x \in \mathscr{X}$, we get

$$\mathbb{E}[V(X_T)] \leq \mathbb{E}[f(T,X_T)] \leq f(0,X_0).$$

Clearly, $f(0,X_0)$ serves as an upper bound of $\mathbb{E}[V(X_T)]$. To minimize the upper bound $f(0,X_0)$, we now turn to the optimization problem

$$\begin{vmatrix} \min & f(0,X_0) \\ \text{s.t.} & f(T,x) \geq V(x), \, x \in \mathscr{X}, \\ & \mathscr{A}f(t,x) \leq 0, \, (t,x) \in [0,T] \times \mathscr{X}, \\ & f \in C^{1,2}([0,T] \times \mathscr{X}; \mathbb{R}). \end{vmatrix}$$

3.2 Main result

This optimization problem is very difficult to deal with since the class definitions of the functions f and V are too broad. To ease the above optimization problem, we restrict the class of the function f to be a polynomial both in t and x, that is, in the form

$$f(t,x) = \sum_{\{0 \leq k_1 \leq K_1, 0 \leq k_2 \leq K_2\}} c_{k_1,k_2} t^{k_1} x^{k_2}, \qquad (3)$$

for some natural numbers K_1 and K_2 and for a sequence $\{c_{k_1,k_2}\}_{k_1 \leq K_1, k_2 \leq K_2}$ of constants. For convenience in notation, we henceforth denote by C_p the class of polynomial functions in the form (3). We also need to set V to be a *piecewise* polynomial both in t and x. Moreover, we assume that both a_0 and a_1 are polynomials. We are then instead to solve the following optimization problem

$$
\begin{aligned}
\min \quad & f(0, X_0) \\
\text{s.t.} \quad & f(T,x) \geq V(x), \; x \in \mathscr{X}, \\
& \mathscr{A} f(t,x) \leq 0, \; (t,x) \in [0,T] \times \mathscr{X}, \\
& f \in C_p.
\end{aligned} \qquad (4)
$$

For the purpose of comparison, suppose that there is no jump in (1), that is, $b \equiv 0$ as in [20]. This assumption clearly makes $\mathscr{A} f$ a polynomial, and consequently (4) is a polynomial optimization problem. This is the main reason that the pure diffusion case is easier to deal with in this framework. In general, polynomial optimization problems are still NP hard. However, if the degrees of f are fixed, sums of squares relaxation enables us to solve the problem efficiently. For details, we refer to Parrilo [18]. On the other hand, this technique is not directly applicable to the model with general stochastic jumps. This is because $\mathscr{A} f$ is not necessarily a polynomial due to the additional integral term.

To make this problem numerically tractable, we decompose the function b as follows:

Assumption 1: Functions a_0, a_1 and b are polynomials and Lévy measure satisfies

$$\int_{\mathbb{R}_0} |z|^k \nu(dz) < +\infty, \quad k = 2, \ldots, K_2.$$

□

The next assumption is on the existence of the moments.

Assumption 2: For any $t \in [0,T]$ there exist integer-order moments of X_t:

$$\mathbb{E}[X_t^k] < \infty, \quad k = 1, \ldots, K_2.$$

□

Theorem 1: Under Assumptions 1 and 2, for any polynomial $f(t,x)$, $\mathscr{A} f(t,x)$ is also a polynomial whose coefficients are affine with respect to those of f.

Proof: A simple algebra yields

$$
\begin{aligned}
\mathscr{A} f(t,x) = {} & \partial_1 f(t,x) + \partial_2 f(t,x) a_0(t,x) \\
& + \frac{1}{2} \partial_2^2 f(t,x) a_1(t,x)^2 \\
& + \sum_{\{0 \leq k_1 \leq K_1, 2 \leq k_2 \leq K_2\}} c_{k_1,k_2} t^{k_1} \times \\
& \sum_{k=0}^{k_2-2} {}_{k_2} C_k x^k b_1(t,x)^{k_2-k} M_{k_2-k}
\end{aligned}
$$

where

$$M_l := \int_{\mathbb{R}_0} z^l \nu(dz), \; l = 2, \ldots, K_2.$$

This completes the proof. □

Clearly, the optimization (4) is now a polynomial programming problem. To be more precise, this problem is numerically tractable for any piecewise polynomial V. Finally, to obtain a lower bound for $\mathbb{E}[V(X_T)]$, we are to find a $g \in C_p$ via the polynomial programming

$$
\begin{aligned}
\max \quad & g(0, X_0) \\
\text{s.t.} \quad & g(T,x) \leq V(x), \; x \in \mathscr{X}, \\
& \mathscr{A} g(t,x) \geq 0, \; (t,x) \in [0,T] \times \mathscr{X}, \\
& g \in C_p.
\end{aligned} \qquad (5)
$$

Notice that our optimization approach does not require the sample paths simulation at all for the computation of the expectation $\mathbb{E}[V(X_T)]$. It is a great advantage of our approach that all we need is the Lévy measure in closed form, not the exact knowledge of the increments or of a shot noise representation for sample paths simulation for the weak approximation with the sample paths discretization.

Remark 1: The weak approximation problem is well-posed whenever $\mathbb{E}[V(X_t)]$ is finite. On the contrary, most of existig numerical approaches listed above impose Assumption 2. However, this assumption rules out some interesting processes. For example, the *stable subordinator*

$$X_t = \int_0^t \int_{\mathbb{R}_+} z d\mu(dt, dz)$$

with

$$\nu(dz) = \frac{1}{z^{1+\alpha}} dz, \quad z \in \mathbb{R}_+, \qquad (6)$$

and $\alpha \in (0,1)$, or even *Ornstein-Uhlenbeck process* driven by the stable subordinator

$$X_t = -\lambda X_t + \int_{\mathbb{R}_+} z d\mu(dt, dz), \; \lambda > 0$$

have no (finite) interger order moments. In addition, even in the case where these moments exist, it is not necessarily trivial to prove it especially in the case of stochastic differential equations with jumps. *Exponential tempering technique*, proposed by the authors in the preceding research [11] enables to circumvent this issue, along with some accuracy improvement. □

3.3 Numerical example (Moment estimation)

In this section, we test numerically our method on a Doléans-Dade stochastic exponential without the diffusion component, that is, $a_1(t,x) \equiv 0$. We here consider the polynomial moment of those processes, which can be derived in closed form. By obtaining tight upper and lower bounds for those moments, we illustrate that our optimization approach is able to capture the distributional transition of Lévy-driven stochastic differential equations.

Set $X_0 > 0$, $a_0(t,x) = a_1(t,x) = 0$, $b_1(t,x) = x$, $b_2(z) = z$, and

$$\nu(dz) = a\frac{e^{-bz}}{z}dz, \quad z > 0,$$

for $a > 0$ and $b > 0$, that is a gamma Lévy measure. In this setting, (1) reduces to a Doléans-Dade stochastic exponential driven by gamma process

$$dX_t = X_{t-}\int_{\mathbb{R}_0} z(\mu - \nu)(dz,dt), \quad X_0 > 0, \quad (7)$$

which is a martingale with respect to its natural filtration. It is clear that $\mathbb{E}[X_T] = X_0$. Moreover, we have $\mathbb{E}[X_t^2] = X_0^2 e^{\frac{a}{b^2}t}$, since by the Ito-Wiener isometry,

$$\mathbb{E}\left[X_T^2\right] = X_0^2 + \int_{\mathbb{R}_+} z^2\nu(dz)\mathbb{E}\left[\int_0^T X_t^2 dt\right]$$
$$= X_0^2 + \frac{a}{b^2}\int_0^T \mathbb{E}\left[X_t^2\right]dt,$$

where the interchange of the integrals holds by the Fubini theorem with the almost sure nonnegativity of X_t^2.

Here, we test our optimization approach on $\mathbb{E}[X_t] = X_0$ and on $\mathbb{E}[X_t^2] = X_0^2 e^{\frac{a}{b^2}t}$. Noting that $\mathscr{X} = \mathbb{R}_+$ and that

$$\int_{\mathbb{R}_+} z^k\nu(dz) = \frac{a(k-1)!}{b^k} \text{ for } k = 2,3,\ldots,$$

where we used the formula

$$\int_0^\infty z^{n-1}e^{-z}dz = (n-1)!, \quad n = 1,2,\cdots.$$

Then, we have for $f \in C_p([0,T]\times\mathbb{R}_+;\mathbb{R})$,

$$\mathscr{A}f(t,x) = \sum_{\{1\leq k_1\leq K_1, 0\leq k_2\leq K_2\}} c_{k_1,k_2}k_1 t^{k_1-1}x^{k_2}$$
$$+ \sum_{\{0\leq k_1\leq K_1, 2\leq k_2\leq K_2\}} c_{k_1,k_2}t^{k_1}x^{k_2}\sum_{k=0}^{k_2-2} {}_{k_2}C_k\frac{a(k_2-k-1)!}{b^{k_2-k}}.$$

Let us consider the p-th moment estimation to compute $\mathbb{E}[x_T^p]$ by taking

$$V(x) := x^p.$$

We present numerical results in Table 1. Note that, when X is unbounded, we must choose $K_2 \geq p$ for the estimation of the p-th moment because of the constraint $f(T,x) \geq x^p$ for $x \in \mathscr{X}$. In view of this, we choose the minimal degree $K_2 = p$. We set $K_2 = K_1$. In the numerical examples presented hereafter, we utilized MATLAB SOSTOOLS combined with SeDuMi [19, 22], using a computer with a Pentium 4 3.2GHz processor and 2 GB memory. It took at most 1 second to obtain a bound. We can see from Table 1 that even such a low degree polynomial f can achieve the tight upper and lower bounds.

4. APPLICATION TO OPTION PRICING

4.1 Weak approximation formulated by using stopping time

In this section we generalize the procedure in the previous section to some extent.

First, fix $B_u > X_0$ and define sets

$$E_0 := [0,T]\times[0,B_u],$$
$$E_u := [0,T]\times[U,+\infty),$$
$$E_r := \{T\}\times[0,B_u].$$

Theorem 2: Let τ be the $(\mathscr{F}_t)_{t\in[0,T]}$-stopping time defined by

$$\tau := \inf\{t \geq 0 : X_t \notin E_0\}\wedge T,$$

that is, the first exit time of $\{X_t : t \in [0,T]\}$ out of E_0. Suppose f satisfies

$$f(t,x) \geq V(t,x), \ (t,x) \in E_1 := E_u \cup E_r \quad (8)$$
$$\mathscr{A}f(t,x) \leq 0, \ (t,x) \in E_0. \quad (9)$$

Then,

$$f(0,X_0) \geq \mathbb{E}[V(\tau,X_\tau)].$$

Proof: Define the exit location measure by

$$\nu_0(B) := \mathbb{P}((\tau,X_\tau)\in B), \ B \in \mathscr{B}(E_1)$$

and the expected occupation measure up to time τ by

$$\nu_1(B) := \mathbb{E}\left[\int_0^\tau \chi_{(t,X_t)\in B}dt\right], \ B \in \mathscr{B}(E_0).$$

Then, we derive the so-called basic adjoint equation

$$\mathbb{E}[f(\tau,X_\tau)] = f(0,X_0) + \mathbb{E}\left[\int_{E_0}\mathscr{A}f(t,x)\nu_1(dt,dx)\right].$$

Combining this equation with

$$\mathbb{E}[f(\tau,X_\tau)] \geq \mathbb{E}[V(\tau,X_\tau)],$$

we obtain the desired inequality. □

This theorem means that we can compute the upper and lower bounds for

$$\mathbb{E}[V(\tau,X_\tau)] \quad (10)$$

by solving the following optimization problems:

$$\begin{aligned}
\max \quad & f(0,X_0) \\
\text{s.t.} \quad & f(T,x) \geq V(t,x), \ (t,x) \in E_1, \\
& \mathscr{A}f(t,x) \leq 0, \ (t,x) \in E_0, \\
& f \in C_p.
\end{aligned} \quad (11)$$

Table 1 Moment transition with $X_0 = 1$ and $(a,b) = (0.1, 1.0)$. The numbers in parentheses indicate theoretical value.

	$t = 0.8$	$t = 1.0$	$t = 0.2$
$\mathbb{E}[X_t]$	1.0000000 – 1.0000000 (1)	1.0000000 – 1.0000000 (1)	1.0000000 – 1.0000000 (1)
$\mathbb{E}[X_t^2]$	1.0832865 – 1.0832889 (1.0832871)	1.1051694 – 1.1051754 (1.1051709)	1.1274938 – 1.1275064 (1.1274969)
$\mathbb{E}[X_t^3]$	1.4922061 – 1.4914382	1.6489387 – 1.6484948	1.8252420 – 1.8164289
$\mathbb{E}[X_t^4]$	4.9367416 – 5.1987506	7.3128369 – 7.9377967	11.3286418 – 11.3566778

$$\begin{vmatrix} \max & g(0, X_0) \\ \text{s.t.} & g(T, x) \leq V(t, x), \ (t, x) \in E_1, \\ & \mathscr{A} g(t, x) \geq 0, \ (t, x) \in E_0, \\ & g \in C_p. \end{vmatrix} \quad (12)$$

Similarly to the previous section, these problems are numerically solvable if $V(t, x)$ defined in E_1 is a piecewise polynomial (with respect to t and x).

We here list 2 examples for this problem. If we take

$$V(t, x) := \begin{cases} 1, & \text{in } E_r \\ 0, & \text{in } E_u, \end{cases}$$

then (10) is the survival probability of X_t in E_0, i.e., the probality that X_t does not touch the upper boundary E_u until $t = T$:

$$\mathbb{E}[V(\tau, X_\tau)] = \mathbb{P}[X_\tau \in E_r].$$

If we take

$$V(t, x) := \begin{cases} 0, & \text{in } E_r \\ t, & \text{in } E_u, \end{cases}$$

then (10) gives the average exit time of X_t from E_0:

$$\mathbb{E}[V(\tau, X_\tau)] = \mathbb{E}\left[\int_{E_u} t \nu_1(dt, dx) \right].$$

Another exmaple is illustrated in the next section.

4.2 Numerical example (Barrier option pricing)

In this section, we take

$$V(t, x) := \begin{cases} (x - K)_+, & \text{in } E_r \\ 0, & \text{in } E_u. \end{cases}$$

Then, (10) gives the European call option premium with barrier B_u and strike price K at time T:

$$\mathbb{E}[V(\tau, X_\tau)] = \mathbb{E}\left[\int_{E_r} (x - K)_+ \nu_1(dt, dx) \right].$$

We consider the same stochastic process as the previous section. Table 2 shows obtained lower and upper bounds for the barrier options, where $B_u = 5$.

5. CONCLUSION

In this paper, we have developed a new approach to the weak approximation of Lévy-driven stochastic differential equations via an optimization problem yielding upper and lower bounds on the target expectation. The advantage of our approach is that all we need is the Lévy measure in closed form. We need neither the exact knowledge of the increments nor a shot noise representation for sample path simulation for the weak approximation with the sample path discretization. We have already confirmed [10] that our method is able to adequately capture the distributional characteristics of other Lévy processes. The most important remaining work is the improvement of the approximation accuracy. Other currently ongoing work is

- application to calibration in finance,
- construction of (infinite-time/finite-horizon) control theory for systems described by Lévy driven stochastic differential equations.

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Table 2 Barrior option price transition with $X_0 = 1$ and $(a, b) = (0.1, 1.0)$.

	$t = 0.8$	$t = 1.0$	$t = 1.2$
$K = 0.9S_0$	0.0968273 – 0.1073178	0.0924480 – 0.1191860	0.0982955 – 0.1221464
$K = 1.0S_0$	0.0276832 – 0.0646183	0.0474994 – 0.0733355	0.0560411 – 0.0612659
$K = 1.1S_0$	0.0259302 – 0.0439486	0.0328449 – 0.0551848	0.0481350 – 0.0490982

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APPENDIX

A DOLÉANS-DADE EXPONENTIAL

It is elementary to yield the canonical form

$$
\begin{aligned}
X_t \;=\;\; & X_0 \exp\left[-t \int_{(-1,+\infty)} z\nu\,(dz) \right. \hspace{1cm} (13)\\
& \left. + \int_0^t \int_{(-1,+\infty)} \ln(1+z)\,\mu\,(dz,ds) \right].
\end{aligned}
$$

It follows from (13) that $X_t > 0$, *a.s.* For the computation of $\mathbb{E}[\mathbb{V}(\mathbb{X}_\mathbb{T})]$, a standard technique is the Monte Carlo simulation with the sample generation of the marginal X_T. To point out some difficulties arising in the approximation of Lévy-driven stochastic differential equations in a simple setting, we briefly introduce 2 typical procedures.

The Euler-Maruyama scheme does not guarantee the non-negativity of the sample paths. To illustrate this, let $N \in \mathbb{N}$, let $\Delta := T/N$, and consider the equidistant time discretization approximation of $\{X_t : t \in [0,T]\}$, ending up with

$$
\frac{X_{k\Delta}}{X_{(k-1)\Delta}} = 1 + \gamma_k(a\Delta,b) - a\Delta/b,
$$

where $\{\gamma_k(a,b)\}_{k\in\mathbb{N}}$ is a sequence of iid gamma random variables with the common distribution $b^a/\Gamma(a)y^{a-1}e^{-by}dy$ on $y \in (0,+\infty)$. With a choice of a, b, and Δ such that $1 - a\Delta/b < 0$, the discretized sample paths may drop below zero. Such numerical experiments provide us with a misleading result of $\mathbb{E}[X_T] \ll X_0$. In view of (13), those phenomena are inappropriate.

On the other hand, based upon the canonical form (13) with a series representation due to Bondesson [5], the sample paths can be simulated as

$$
X_t = X_0 \exp\left[-t\frac{a}{b} + \sum_{k=1}^{+\infty} \ln\left(1 + e^{-\Gamma_k/a}\frac{V_k}{b}\right) \chi_{(T_k \le t)} \right],
$$
$$
t \in [0,T],
$$

where $\{\Gamma_k\}_{k\in\mathbb{N}}$ is a sequence of arrival times of a standard Poisson process, where $\{V_k\}_{k\in\mathbb{N}}$ is a sequence of iid exponential random variables with unit parameter, and where $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of iid uniform random variables on $[0,T]$. It is obviously expensive to generate the infinite sum for a single path.