On Finite Groups of p-Local Rank One and a Conjecture of Robinson

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Abstract

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We use the classification of finite simple groups to verify a conjecture of Robinson for finite groups G where $G/O_p(G)$ has trivial intersection Sylow p-subgroups. Groups of this type are said to have p-local rank one, and it is hoped that this invariant will eventually form the basis for inductive arguments, providing reductions for the conjecture, or even a proof using the results presented here as a base. A positive outcome for Robinson's conjecture would imply Alperin's weight conjecture.

It is shown that in proving Robinson's conjecture it suffices to demonstrate only that it holds for finite groups in which $O_p(G)$ is both cyclic and central.

Part of the proof of the former result is used to complete the verification of Dade's inductive conjecture for the Ree groups of type G_2.

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Declaration

I declare that to the best of my knowledge the material contained in this thesis is original unless explicitly stated otherwise

Contents

1 Introduction and notation

We present the beginnings of a possible inductive approach to Alperin's weight conjecture (see Alperin [2]) suggested by Robinson in [47]. There a conjecture equivalent to Alperin's is presented along with an inductive invariant, the p-local rank of a finite group. The hope is that one may reduce to the p-local rank one case (which includes those groups with split BN-pairs of rank one in characteristic p). Here we verify Robinson's refined conjecture (Conjecture 4.4) in just this situation by applying the classification of finite simple groups (CFSG).

Finite groups G of p-local rank one are those in which $G/O_p(G)$ has trivial intersection (TI) Sylow p-subgroups (i.e., any two distinct Sylow p-subgroups of G intersect in $O_p(G)$, the unique maximal normal p-subgroup of G). Finite groups with TI Sylow p-subgroups have been studied to an extent by Blau and Michler in [7], in which the CFSG is used to prove an earlier conjecture of Alperin, and for a large part of our work here we use similar techniques to generalize their results. All groups considered in this thesis are finite.

Our main result is the following. The description of Conjecture 4.4 is delayed until Chapter 4, since the material of Chapter 3 is necessary to understand it fully.

Theorem 4.7 *Conjecture 4.4 holds for every finite group of p-local rank one.*

The first five chapters are general in that the results largely apply to all finite groups.

In Chapter 2 there is a brief review of the material necessary for the later results. This in particular includes a brief introduction to the theory of p-chains, an overview of the properties of groups of p-local rank one and a brief discussion on Schur multipliers. We do not include however the basic results of Clifford theory or block theory. When we refer to 'Clifford theory' we refer to any results found in, say Chapter 6 of [29]. Our omission of any block theoretic background may be a little more controversial, but we feel that what we do use (such as Brauer's main theorems) is familiar enough not to warrant explicit reference. Excellent sources for this material however are [15] and [22].

Chapter 3 contains the Clifford-theoretic reductions we use later in eliminating complicated normal p-subgroups or p'-subgroups. The reductions themselves are well-known, but their compatibility with block theory has only been discussed fully in the setting of twisted group algebras (see [18]). We use notation set up in the first part of the chapter to give a review of some results of Külshammer and Robinson and to sketch their proofs.

Chapter 5 largely consists of a proof that (for finite groups of any p-local rank) in verifying Robinson's conjecture it suffices to consider only those groups in which $O_p(G)$ is both central and cyclic. In doing so we are able to present an apparent refinement of Conjecture 4.4 which is in fact equivalent.

The remainder of the thesis concentrates solely on finite groups of p-local rank one, and consists of the proof of Theorem 4.7.

In Chapter 6 there is a reworking and extension of the reduction used in [7]. Having proved in Chapter 5 that we need only consider finite groups with almost TI Sylow p-subgroups ('almost TI' is defined in Section 2.3), and after some more Clifford theoretic reductions, it is then possible to show that every such group is a central extension of an automorphism group of a nonabelian simple group. By examining the relationship between the Schur multiplier of a simple group and those of its automorphism groups, we are able to reduce to a consideration of automorphism groups of quasisimple groups. Such groups are well-covered in the literature. The efficiency of this reduction owes much to the results of [6] and to the Schur multiplier package in MAGMA. Classification results achieved by Gorenstein and Lyons ($p \neq 2$) and Suzuki ($p = 2$) give a list of all nonabelian simple groups of p-local rank one, allowing us to give a full list of the remaining possibilities for G.

Chapter 7 consists of the verification of Conjecture 4.4 for the automorphism groups of the quasisimple groups listed in Chapter 6.

In Chapter 8 there is a review of the proof of Theorem 4.7.

In Chapter 9 we verify that Dade's final (inductive) conjecture (see [20]) holds for the finite groups ${}^2G_2(3^{2m+1})$ for the prime 3, completing the verification of Dade's conjecture for this class of groups.

Appendix A consists of the information necessary to replicate the calculations performed on MAGMA.

1.1 Notation

G	a finite group
$H < G$	H is a proper subgroup of G
$H \leq G$	H is a subgroup of G
$H \triangleleft G$	H is a normal subgroup of G
$N_G(P)$	the normalizer of P in G
$C_G(P)$	the centralizer of P in G
$Z(G)$	the centre of G
$Irr(G)$	the set of irreducible characters of G
(K, R, k)	a p-modular system
$J(R)$	the Jacobson radical of a ring R
$IBr(G)$	the set of irreducible Brauer characters of G
	with respect to (K, R, k)
$k(G)$	$\lvert Irr(G)\rvert$
$l(G)$	$\lvert IBr(G)\rvert$
$f_0(G)$	the number of p-blocks of defect zero of G
$w(G, Q)$	the number of Q-projective irreducible characters of G
(χ, θ)	for characters χ, θ of G, $(\chi, \theta) = \frac{1}{\lvert G\rvert} \sum_{x \in G} \chi(x)\theta(x^{-1})$
σ	a p-chain of G
$\lvert\sigma\rvert$	the length of σ
$G_\sigma = N_G(\sigma)$	the stabilizer of σ under conjugation in G
V_σ	the initial subgroup of σ
V^σ	the final subgroup of σ
$\mathcal{C}(G)$	the set of all p-chains of G
$\mathcal{X}(G)$	a subset of $\mathcal{C}(G)$
$\mathcal{X}(G)/H$	a set of orbit representatives for the action of H on $\mathcal{X}(G)$
$\mathcal{X}(G\lvert Q)$	$\sigma \in \mathcal{X}(G)$ with $V_\sigma = Q$

$\mathcal{X}_Q(G)$	$\sigma \in \mathcal{X}(G)$ with $Q \leq V_\sigma$
$\mathcal{N}(G)$	$\sigma \in \mathcal{X}(G)$ with $V^\sigma \leq N_G(\sigma)$
$\mathcal{R}(G)$	the radical p-chains of G
$plr(G)$	the p-local rank of G
$O_\pi(G)$	π a set of primes, the unique maximal normal π-subgroup of G
G_π	π a set of primes, the set of elements of G of order divisible only by primes in π. Note that $1 \in G_\pi$
$H^2(G, F^\times)$	the second cohomology group of G with respect to the field F
$H_2(G, \mathbb{Z})$	the second integral homology group of G
$[G, G] = G'$	the derived subgroup of G
$M(G)$	the Schur multiplier of G
$Aut(G)$	the automorphism group of G
$Out(G)$	the outer automorphism group of G
$s(g)$	the G-conjugacy class sum of $g \in G$
w_χ	the central character $Z(G) \to R$ associated to $\chi \in Irr(G)$
$Bl(\chi)$	the p-block of G containing $\chi \in Irr(G)$
ζ^G	the character of G induced from the character ζ of $H \leq G$
$\chi\|_H$	the restriction of the character χ of G to $H \leq G$
$M^\star$	the dual of a module M
$\mathcal{R}_0(G)$	the set of radical p-subgroups of G
x_π	for π a set of primes, the π-part of a group element/integer x
b^G	for a p-block b of $H \leq G$, the Brauer correspondent in G
$I_G(\mu)$	inertial subgroup in G of the character μ of $N \lhd G$
$\mathbb{F}_t$	the field with t elements
$\mathbb{F}_t^\times$	multiplicative group of $\mathbb{F}_t$
$f_S(G)$	the number of fixed points of the G-set S

$irr(G)$	the multiset of irreducible character degrees of G,
	e.g., $irr(G) = \{\ldots, m \times t, \ldots\}$, m characters of degree t
$\Phi(G)$	Frattini subgroup of G
CFSG	Classification of Finite Simple Groups

Define the *defect* of a character $\chi \in Irr(G)$ to be the integer d such that $|G|_p = p^d \chi(1)_p$. Write $Irr_d(G)$ for the subset of $Irr(G)$ of elements of defect d, and $k_d(G)$ for their number. If χ lies in a p-block of defect d' then the *height* of χ is $d' - d$. This definition of $k_d(G)$ is a modern convention - care should be taken when reading references that d does indeed refer to the defect rather than the height.

Denote the number of p-blocks of defect zero of G by $f_0(G)$, and if H is a section of G and B is a block of G, then denote by $f_0^{(B)}(H)$ the number of blocks of defect zero of H corresponding to B under the Brauer correspondence and the natural correspondence with a quotient group.

In general if we are considering an invariant involving blocks of a section H of G that correspond to B, then we write, for example, $k(H, B)$. Further, if we are considering only characters covering a given irreducible character μ of a normal subgroup then write, for example, $k(H, B, \mu)$ (if $T, H \leq G$, then we say that $\chi \in Irr(T)$ *covers*, or *lies over* $\mu \in Irr(H)$ if $(\chi|_{T \cap H}, \mu|_{T \cap H}) \neq 0$).

Notation for specific finite groups is in general as given in [12].

2 Background

2.1 Choice of p-modular system

When working with blocks it is important to choose the ring and fields of definition carefully. We use a p-modular system, as defined for example in [14], but since we make use of the results of [35] (in particular those given in Proposition 3.13) we also take the residue field to be algebraically closed.

Let R be a complete discrete valuation ring with unique maximal ideal $J(R)$ and field of fractions K of characteristic zero. Suppose that the residue field $k = R/J(R)$ is algebraically closed of characteristic p, p a prime. We assume further that R contains all $|G|^3 - th$ roots of unity for whichever finite group G we are considering, i.e., K is a splitting field for G and any of its covers we will be considering (see [14, 15.16]). (K, R, k) form a p-modular system.

2.2 Radical p-chains and the p-local rank of a finite group

Following the reformulation of Alperin's weight conjecture by Knörr and Robinson in [34], most work on the verification of the conjecture has centred around the study of chains of p-groups. Definitions of these *p-chains* vary a little from author to author, mostly in the matter of restrictions of the choice of *initial subgroup.* The stabilizers of these p-chains may be regarded as generalized p-local subgroups.

Definition A *p-chain* σ for G of *length* $|\sigma| = n$ is a non-empty strictly increasing chain

$$Q_0 < \cdots < Q_n$$

of p-subgroups Q_i of G. We call the p-chain

$$\sigma_i : Q_0 < \cdots < Q_i$$

an *initial subchain* of σ, $V^{\sigma} = Q_n$ the *final subgroup* of σ and $V_{\sigma} = Q_0$ the *initial subgroup* of σ. Let $\mathcal{C} = \mathcal{C}(G)$ denote the set of p-chains for G.

Suppose that A is a group acting on G. For $\alpha \in A$ define

$$\sigma^{\alpha} : Q_0^{\alpha} < \cdots < Q_n^{\alpha},$$

and define the chain stabilizer by

$$A_{\sigma} = N_A(\sigma) = N_A(Q_0) \cap \cdots \cap N_A(Q_n).$$

It is clear that length of a p-chain is invariant under this action.

If $\mathcal{X}(G) \subset \mathcal{C}(G)$ and $Q \leq G$, then we write

$$\mathcal{X}(G|Q) = \{\sigma \in \mathcal{X}(G) : V_{\sigma} = Q\},$$

the set of those p-chains *starting* with Q. Write

$$\mathcal{X}_Q(G) = \{\sigma \in \mathcal{X}(G) : V_{\sigma} \geq Q\}$$

for the set of those chains whose initial subgroup *contains* Q.

Denote by $\mathcal{N} = \mathcal{N}(G)$ the set of p-chains whose stabilizer in G contains the final subgroup of the chain, i.e., all terms of the chain are normal in the final subgroup.

An important class of p-chains are the radical p-chains - a generalization of the radical p-subgroups, of which the p-subgroups under consideration in the statement of Alperin's weight conjecture are examples:

Definition Let G be a finite group. We say that a p-subgroup Q of G is *radical* if $Q = O_p(N_G(Q))$. The following lemma lists some important properties of radical p-subgroups:

Lemma 2.1 *(a) Each defect group of any p-block is radical,*

(b) $O_p(G)$ and the Sylow p-subgroups of G are radical,

(c) each radical p-subgroup of G contains $O_p(G)$,

(d) each radical p-subgroup may be regarded as the intersection of a collection of Sylow p-subgroups of G.

Proof For (a)-(c), see, for example [17]. For (d) see [34]. $\square$

Definition A p-chain $\sigma : Q_0 < \cdots < Q_n$ is *radical* if $Q_i = O_p(N_G(\sigma_i))$ for each i. In other words, Q_0 is a radical p-subgroup of G and each Q_{i+1} is a radical p-subgroup of $N_G(\sigma_i)$.

We denote the set of radical p-chains for G by $\mathcal{R} = \mathcal{R}(G)$, and note that when G is a finite group of Lie type (of characteristic p) the normalizers of the radical p-subgroups are nested so that the radical p-chains are just the chains of radical p-subgroups. In this case the radical p-subgroups are the unipotent radicals of the parabolic subgroups.

A treatment of some other families of p-chains may be found in Knörr and Robinson [34]. For example, one may consider the p-chains consisting of elementary abelian p-groups, radical p-groups or indeed intersections of Sylow p-subgroups. However, we shall not be using these here.

The next result is an exercise from Robinson [50], and shows that the three families of p-chains are fairly interchangeable:

Lemma 2.2 *Let G be a finite group and let $f : \mathcal{C}(G) \to \mathbb{Z}$ be a function stable under the conjugation action of G and dependent only on $N_G(\sigma)$ and on the initial subgroup V_σ, for $\sigma \in \mathcal{C}(G)$. Let Q be a p-subgroup of G. Then*

$$\sum_{\sigma \in \mathcal{C}(G|Q)/G} (-1)^{|\sigma|} f(\sigma) = \sum_{\sigma \in \mathcal{N}(G|Q)/G} (-1)^{|\sigma|} f(\sigma) = \sum_{\sigma \in \mathcal{R}(G|Q)/G} (-1)^{|\sigma|} f(\sigma).$$

Proof It follows from the definititions that $\mathcal{R}(G|Q) \subset \mathcal{N}(G|Q)$. The first equality is demonstrated in [50]. The second is proved in [17], although the result stated is weaker than ours. We include a proof here.

Let $\sigma \in \mathcal{N}(G|Q) - \mathcal{R}(G|Q)$, $\sigma : Q_0 < \cdots < Q_n$. Then $Q_i \neq O_p(N_G(\sigma_i))$ for some i. Choose i maximal with this property, and observe that $Q_i < O_p(N_G(\sigma_i))$. Observe that if $i < n$, then $Q_{i+1} = O_p(N_G(\sigma_{i+1})) = O_p(N_{N_G(\sigma_i)}(Q_{i+1})) \geq O_p(N_G(\sigma_i))$ by maximality and by Lemma 2.1. There are two cases to consider:

(i) Suppose that $i < n$ and $Q_{i+1} = O_p(N_G(\sigma_i))$. Then form the chain

$$\overline{\sigma} : Q_0 < \cdots < Q_i < Q_{i+2} < \cdots < Q_n$$

from σ by deleting the term Q_{i+1}. Clearly $\overline{\sigma} \in \mathcal{N}(G|Q) - \mathcal{R}(G|Q)$. We have $N_G(\sigma) = N_G(\overline{\sigma}) \cap N_G(Q_{i+1})$ and $N_G(\overline{\sigma}) \leq N_G(Q_{i+1})$, so $N_G(\overline{\sigma}) = N_G(\sigma)$.

(ii) Suppose that $i = n$ or $Q_{i+1} > O_p(N_G(\sigma_i))$. Then form the chain

$$\overline{\sigma} : Q_0 < \cdots < Q_i < O_p(N_G(\sigma_i)) < Q_{i+1} < \cdots < Q_n$$

from σ by inserting the term $O_p(N_G(\sigma_i))$ between Q_i and Q_{i+1} if $i < n$ or at the end if $i = n$. Again $\overline{\sigma} \in \mathcal{N}(G|Q) - \mathcal{R}(G|Q)$. We have $N_G(\overline{\sigma}) = N_G(\sigma) \cap N_G(O_p(N_G(\sigma_i)))$ and $N_G(\sigma) \leq N_G(O_p(N_G(\sigma_i)))$, so $N_G(\overline{\sigma}) = N_G(\sigma)$.

In each case $V_\sigma = V_{\overline{\sigma}}$ and $\overline{(\sigma^g)} = (\overline{\sigma})^g$ for $g \in G$, and $\overline{\overline{\sigma}} = \sigma$, so the contributions from orbits of chains containing σ and $\overline{\sigma}$ cancel as required, since $(-1)^{|\overline{\sigma}|} = (-1)^{|\sigma|+1}$.□

The analogous result holds for the other families of p-chains in [34], except for the family of chains of elementary abelian p-subgroups (see [50]). Note that the result does hold for the chains of intersections of Sylow p-subgroups.

We include here a result which will be crucial in defining the variants of Alperin's conjecture which involve alternating sums of the type found in Lemma 2.2

Lemma 2.3 (Lemma 3.2 of [34]) *Let $\sigma \in \mathcal{C}(G)$ and suppose that b is a p-block of G_σ. Then b has a Brauer correspondent in G.*

Definition The *p-local rank* of a finite group G, *plr*(G) is the length of the longest radical p-chain of G. Robinson proves some strong results concerning this invariant (see [47]).

A trivial observation however is that $plr(G) = 0$ if and only if G has a normal Sylow p-subgroup. An important and non-trivial property of the p-local rank is that it respects the partial ordering of subgroups:

Lemma 2.4 (Lemma 7.2 of [47]) *Let $H \leq G$. Then $plr(H) \leq plr(G)$.*

2.3 Properties of finite groups of p-local rank one

Definition A subset X of a finite group G is said to be a trivial intersection (TI) set if $X^g \cap X \subset 1$ for each $g \in G$ with $g \notin N_G(X)$. If $P \in Syl_p(G)$ is a TI set and $P \neq O_p(G)$, then we say that G has TI Sylow p-subgroups. If $O_p(G) \leq Z(G)$ is cyclic and $G/O_p(G)$ has TI Sylow p-subgroups, then we say that G has *almost TI Sylow p-subgroups.*

Lemma 2.5 (Lemma 7.1 of [47]) *Let G be a finite group with $plr(G) > 0$. Then $plr(G) = 1$ if and only if $G/O_p(G)$ has TI Sylow p-subgroups.*

We review a few of the properties of finite groups with TI Sylow p-subgroups.

Lemma 2.6 *Suppose that $P \in Syl_p(G)$ is a TI set and $1 \neq Q \leq P$. Then $N_G(Q) \leq N_G(P)$. Further if $g \in G$ and $Q^g \leq P$, then $g \in N_G(P)$.*

Lemma 2.7 *Let P, G be as in the previous lemma and suppose that $M \lhd G$ has order divisible by p. Then $Q = P \cap M \in Syl_p(M)$ and $MN_G(P) = MN_G(Q) = G$.*

Proof This follows by using the Frattini argument and the previous lemma. $\square$

Lemma 2.8 *Suppose that $P \in Syl_p(G)$ is a TI set of p-rank strictly greater than one. Then $O_{p'}(G) \leq C_G(P)$.*

Proof Consider $E_2 \leq P$ elementary abelian of rank 2. Since this is a noncyclic abelian p-group of automorphisms of $O_{p'}(G)$, by Gorenstein [24, 6.2.4] we have

$$O_{p'}(G) = \langle C_{O_{p'}(G)}(\alpha) \mid \alpha \in E_2 - \{1\} \rangle \leq N_G(P).$$

Hence $O_{p'}(G) \leq N_G(P)$, and $C_P(O_{p'}(G)) = P$. □

Finally we prove a special case of a result of Robinson [48].

Lemma 2.9 *Suppose that $O_p(G) \leq Z(G)$ and that $G/O_p(G)$ has TI Sylow p-subgroups. Let B be a block of G with defect group $P \in Syl_p(G)$. Then*

$$k(G,B) - k(N_G(P),B) = |O_p(G)| \left(l(G,B) - l(N_G(P),B) \right).$$

Now let $\overline{B}$ be the unique p-block of $\overline{G} = G/O_p(G)$ corresponding to B, and let $\lambda \in Irr(O_p(G))$. Suppose further that $O_p(G)$ is cyclic. Then

$$k(G,B,\lambda) - k(N_G(P),B,\lambda) = k(\overline{G},\overline{B}) - k(N_{\overline{G}}(\overline{P}),\overline{B}).$$

Proof $\overline{G}$ possesses an unique p-block $\overline{B}$ corresponding to B by [22, V.4.5]. (6D) of Brauer [8] gives us

$$k(G,B) = \sum_{u \in G_p/G} l(C_G(u),B).$$

Choose $P \in Syl_p(G)$. By Lemma 2.6 we may choose G_p/G so that $C_G(u) \leq N_G(P)$ for each $u \in G_p/G - O_p(G)$. Hence we may take $G_p/G - O_p(G) = N_G(P)_p/N_G(P) - O_p(G)$. For each $u \in O_p(G)$ we have $C_G(u) = G$, and so

$$\begin{aligned} k(G,B) - k(N_G(P),B) &= \sum_{u \in G_p/G} l(C_G(u),B) - \sum_{u \in N_G(P)_p/N_G(P)} l(C_{N_G(P)}(u),B) \\ &= |O_p(G)| \left(l(G,B) - l(N_G(P),B) \right). \end{aligned}$$

Noting that $l(G,B) - l(N_G(P),B) = l(\overline{G},\overline{B}) - l(N_{\overline{G}}(\overline{P}),\overline{B})$, we see that

$$k(G,B) - k(N_G(P),B) = |O_p(G)| \left(k(\overline{G},\overline{B}) - k(N_{\overline{G}}(\overline{P}),\overline{B}) \right).$$

To count the irreducible characters lying over λ we use an inductive argument on $|O_p(G)|$.

Suppose first that $|O_p(G)| = p$. Then $k(G, B, 1) - k(N_G(P), B, 1) = k(\overline{G}, \overline{B}) - k(N_{\overline{G}}(\overline{P}), \overline{B})$ trivially and $k(G, B, \lambda) - k(N_G(P), B, \lambda)$ takes identical values on each of the remaining $\lambda \in Irr(O_p(G))$, so the result holds in this case.

Suppose that the result holds for $|O_p(G)| = p, p^2, \ldots, p^{n-1}$ and suppose that $|O_p(G)| = p^n$. Then $k(G, B, \lambda) - k(N_G(P), B, \lambda)$ takes the same value for each λ of order p^n, and the result follows by the inductive hypothesis. $\square$

2.4 The Schur Multiplier

In applying the results of the next chapter we will make extensive use of the Schur multiplier of a group. Although this arises from the study of projective representations (as will be explained in the next chapter), it can also be explained purely in terms of group theory and cohomology/homology. We give a definition of the Schur multiplier and some results which will be of use later on. We direct the reader to [14], [29] or [32] for further detail, although this is by no means a definitive list.

Definition We define the *Schur multiplier* $M(G)$ of a group G to be $H^2(G, \mathbb{C}^\times)$, the second cohomology group of G over $\mathbb{C}^\times$. Equivalently, $M(G)$ is maximal such that there is a central extension H of G with $A \leq Z(H)$, $H/A \cong G$, $A \leq [H, H]$ and $A \cong M(G)$. It is important to note that $H^2(G, K^\times) \cong H^2(G, \mathbb{C}^\times)$ (see, for example the proof of [35, 2.1]).

The following result is well known, and highlights the local nature of the Schur multiplier.

Lemma 2.10 *Let G be a finite group and let $P \in Syl_p(G)$. Then a Sylow p-subgroup of $M(G)$ is isomorphic to a subgroup of $M(P)$.*

Proof This is [28, V.25.1]. The method of proof is similar to that used in the next result. □

The following is a special case of [25, 4.229].

Lemma 2.11 *Let G be a finite group and let $N \leq G$ with $p \not| [G : N] = n$. Suppose also that $p \not| |M(N)|$. Then $p \not| |M(G)|$.*

Proof Suppose that H is a central extension of G with $H/A \cong A$ and $A \leq Z(H) \cap H'$. Write L for the preimage of N in H, and let $V : H \to L/L'$ be the transfer map. Let $z \in O_p(A)$. Then $V(z) = z^n L'$ (see for example [28, IV.2.1]). But $H' \leq ker(V)$, so $V(z) = L'$. Since $(p, n) = 1$, there is an integer x such that $z = z^{nx} \in O_p(A) \cap L' \leq O_p(Z(L)) \cap L'$. So $O_p(A)$ is isomorphic to a subgroup of $O_p(M(N)) = 1$, and the result follows. □

The next result allows us (after some more work) to avoid the difficulties posed by the presence of an outer automorphism group which is a direct product of two cyclic groups.

Lemma 2.12 ([6]) *Let G be a group and let $N \lhd G$, where N is perfect and G/N is cyclic. Then $M(G)$ is a homomorphic image of $M(N)$.*

Proof This follows immediately from Theorem 4.2 of [6], observing that $H^2(G, \mathbb{C}^\times) \cong H_2(G, \mathbb{Z})$ (see [58, p.199]). □

The next result, although not strictly necessary for this thesis (less sophisticated methods would suffice), is worth including given our concentration on finite groups with TI Sylow p-subgroups.

Lemma 2.13 (9.6.3 of [32]) *Let G be a finite group such that $P \in Syl_p(G)$ is a TI set. Then $O_p(M(G)) \cong O_p(M(N_G(P)))$.*

It is convenient to include here some results concerning the Schur multipliers of various p-groups we will encounter.

Lemma 2.14 *If P is a finite p-group which is either (a) cyclic, (b) generalized quaternion, or (c) extraspecial of order p^3 and exponent p^2, then $M(P) = 1$. If P is elementary abelian then $M(P)$ is elementary abelian.*

Proof If P is of the form (a) or (b), then this is [28, V.25.3]. If P is of the form (c), then by [32, 2.4.9]

$$P = \langle x, y \mid x^{p^2} = y^p,\ y^{-1}xy = x^{p+1} \rangle,$$

and the result follows by [28, V.25.2] since P may be defined using two generators and two relations.

The last part is exercise 11.16 of [29]. □

3 Clifford-theoretic reductions

3.1 Introduction

In this chapter we detail block-friendly character correspondences, which we will later apply to demonstrate that a counterexample to Conjecture 4.4 with $[G : Z(G)]$ minimal has only cyclic, central p-subgroups or p'-subgroups.

Paul Fong in the second part of his famous paper [23] introduced two character correspondences, obtained by observing the action of a finite group G on the irreducible characters of a normal p'-subgroup and applying Schur's theory of projective representations. Dubbed the Fong correspondences, the first shows that to some extent we may replace G by the inertial subgroup $I_G(\mu)$ of an irreducible character μ of N. The second is considerably more involved and shows that we can further replace N by a central, cyclic p'-subgroup. Refinements and modifications of Fong's work were studied by Reynolds [46], Cliff [11] and Okuyama and Tsushima [39].

Whereas Fong used elementary character theoretic methods to verify that the correspondences may be refined to correspondences between p-blocks of characters, the subsequent results in [39] use the theory of twisted group algebras to establish that the correspondences are compatible with the Brauer correspondence. Dade in [18] uses similar methods to establish similar correspondences with no restriction on N, but in the (equivalent) setting of blocks of twisted group algebras (of which he gives a comprehensive account).

We work with non-twisted group algebras and use elementary methods to demonstrate correspondences similar to Fong's second correspondence, where we have no restriction on the order of N but a restriction on the degree of μ, and check that the correspondence may be refined to p-blocks and is compatible with the Brauer correspondence (where it exists). Analogies to Fong's first correspondence for p-singular normal subgroups are easy to obtain (blockwise refinements following directly from

Lemma 3.1) and are constructed where needed.

The following lemma will be the key to the Brauer correspondence. Let G be a finite group.

Lemma 3.1 *Let $H \leq G$ and b a p-block of H. Let $\zeta \in Irr(H, b)$ and C be a conjugacy class sum of G. If $\pi_H : RG \to RH$ is the projection map of G onto H, then*

$$\omega_\zeta \circ \pi_H(C) = \frac{\zeta^G(C)}{\zeta^G(1)}.$$

Proof This is [37, 5.3.1]. □

To use this we need the following observation:

Lemma 3.2 *Let $N \lhd G$ and write $\overline{G} = G/N$, indicating objects associated to $\overline{G}$ with a '$^-$'. Let $H \leq G$ with $N \lhd H$. Suppose that $\theta \in Irr(G)$, and that $\zeta \in Irr(H)$ and $\chi \in Irr(\overline{H})$ are characters such that for each $x \in \overline{x} \in \overline{H}$ we have $\zeta(x) = \chi(\overline{x})\theta(x)$. Then $\zeta^G(x) = \chi^{\overline{G}}(\overline{x})\theta(x)$ for each $x \in G$.*

Proof

$$\zeta^G(x) = \frac{1}{|H|} \sum_{g \in G, x^g \in H} \zeta(x^g) = \frac{1}{|H|} \sum_{g \in G, x^g \in H} \chi(\overline{x}^{\overline{g}})\theta(x^g).$$

But $\theta(x^g) = \theta(x)$, and $x^g \in H \Leftrightarrow \overline{x}^{\overline{g}} \in \overline{H}$, so

$$\zeta^G(x) = \frac{\theta(x)}{|H|} \sum_{g \in G, \overline{x}^{\overline{g}} \in \overline{H}} \chi(\overline{x}^{\overline{g}}) = \frac{|N|\theta(x)}{|H|} \sum_{\overline{g} \in \overline{G}, \overline{x}^{\overline{g}} \in \overline{H}} \chi(\overline{x}^{\overline{g}}) = \chi^{\overline{G}}(\overline{x})\theta(x).$$

□

3.2 Character correspondences

Let G be a finite group and $N \lhd G$. Suppose that $\mu \in Irr(N)$ is G-stable. Denote by π the set of primes dividing $|N|$, and write π' for its complement in the set of primes. We construct a finite group $\hat{G}$ with cyclic π-group $\hat{M} \leq Z(\hat{G})$ such that $\hat{G}/\hat{M} \cong G/N$

and there exists a 1-1 correspondence between irreducible characters of G lying over μ and irreducible characters of $\hat{G}$ lying over a given $\hat{\mu} \in Irr(\hat{M})$. This character correspondence is well known, but we include details here in order to establish the notation necessary to check its compatibility with block theory.

We establish the character correspondences by way of a third group $\tilde{G}$ of which both G and $\hat{G}$ are factor groups.

Fix a representation τ affording μ.

Lemma 3.3 *We may choose a projective representation*

$$\rho : G \to GL_{\mu(1)}(R)$$

extending τ such that (i) $\rho(n)\rho(g) = \rho(ng)$ and $\rho(g)\rho(n) = \rho(gn)$ $\forall\ g \in G$, $n \in N$, (ii) $det(\rho(g))$ is a π-power root of unity for each $g \in G$ and (iii) ρ has factor set $\epsilon : G \times G \to R^\times$ of order s dividing $|G|_p\mu(1)$ in $H^2(G, K^\times)$.

Proof This is [35, 2.1] (that there is a complex projective representation extending τ is [29, 11.2]). □

Let $\alpha \in R$ be a primitive s-th root of unity, and define the group

$$\tilde{G} = \{(\alpha^i, g) | 0 \leq i < s,\ g \in G\},$$

with group operation given by

$$(\alpha^{i_1}, g_1)(\alpha^{i_2}, g_2) = (\epsilon(g_1, g_2)\alpha^{i_1+i_2}, g_1g_2).$$

Write $\tilde{M} = \{(\alpha^i, 1) : 0 \leq i < s\} \leq Z(\tilde{G})$, so that we have a short exact sequence

$$1 \to \tilde{M} \to \tilde{G} \to G \to 1.$$

Let $\tilde{N} = \{(\alpha^i, n) : 0 \leq i < s, n \in N\} \lhd \tilde{G}$, and $\overline{N} = \{(1, n) : n \in N\} \lhd \tilde{G}$, so that $\overline{N} \cong N$.

Lemma 3.4 *$\tilde{\rho} : \tilde{G} \to GL_{\mu(1)}(R)$ defined by $\tilde{\rho}(\alpha^i, g) = \alpha^i \rho(g)$ and $\tilde{\tau} : \tilde{N} \to GL_{\mu(1)}(R)$ defined by $\tilde{\tau}(\alpha^i, n) = \alpha^i \tau(n)$ are irreducible representations, and $\tilde{\tau} = \tilde{\rho}|_{\tilde{N}}$, $\tilde{\tau}|_{\overline{N}} = \tau$.*

Proof That $\tilde{\rho}$ and $\tilde{\tau}$ are representations is clear from the definition of a projective representation. It is also clear that $\tilde{\tau} = \tilde{\rho}|_{\tilde{N}}$, $\tilde{\tau}|_{\overline{N}} = \tau$. Now suppose that $\tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2$, then $\tau = \tilde{\rho}|_{\overline{N}} = \tilde{\rho}_1|_{\overline{N}} + \tilde{\rho}_2|_{\overline{N}}$, so that $\tilde{\rho}_1 = 0$ or $\tilde{\rho}_2 = 0$. Similarly $\tilde{\tau}$ is irreducible. $\square$

Denote all objects associated to $\tilde{G}$ with a '$\tilde{\ }$' symbol.

$\epsilon(g, n) = \epsilon(n, g) = 1$ for each $g \in G$, $n \in N$, so we may regard ϵ as a factor set for G/N, i.e., $\epsilon \in H^2(G/N, K^\times)$. Define

$$\hat{G} = \{(\alpha^i, \overline{g}) : 0 \leq i < s,\ \overline{g} \in G/N\} \cong \tilde{G}/\overline{N},$$

with group operation given by

$$(\alpha^{i_1}, \overline{g}_1)(\alpha^{i_2}, \overline{g}_2) = (\epsilon(\overline{g}_1, \overline{g}_2)\alpha^{i_1+i_2}, \overline{g}_1\overline{g}_2).$$

Setting $\hat{M} = \{(\alpha^i, \overline{1}) : 0 \leq i < s\} \leq Z(\hat{G})$, we have a short exact sequence

$$1 \to \hat{M} \to \hat{G} \to G/N \to 1.$$

Let $\hat{\mu} : \hat{M} \to R$ be the linear representation given by $\hat{\mu}(\alpha^i, \overline{1}) = \alpha^{-i}$. Denote all elements, characters and subgroups of $\hat{G}$ with a '$\hat{\ }$' symbol.

Remark 3.5 *$\hat{M} \leq [\hat{G}, \hat{G}]$ by [29, 11.19], so $\hat{M}$ is isomorphic to a subgroup of the Schur multiplier of G/N.*

Write $\tilde{\theta}$ for the irreducible character of $\tilde{G}$ afforded by $\tilde{\rho}$, so that $\tilde{\theta}|_{\overline{N}} = \mu$. Let $\tilde{\mu}$ be the irreducible character of $\tilde{N}$ defined by $\tilde{\mu}(\alpha^i, n) = \mu(n)$. As a consequence of Clifford's theorem, the irreducible characters of $\tilde{G}$ covering $\tilde{\mu}$ are precisely the characters $\tilde{\chi} = \tilde{\theta}\hat{\chi}$ where $\hat{\chi} \in Irr(\hat{G}, \hat{\mu})$. In particular there is a canonical bijection

$$Irr(\tilde{G}, \tilde{\mu}) \leftrightarrow Irr(\hat{G}, \hat{\mu}),$$

since for each $(\alpha^i, n) \in \tilde{N}$,

$$\tilde{\chi}(\alpha^i, n) = \tilde{\theta}(\alpha^i, n)\hat{\chi}(\alpha^i, \bar{1}) = \alpha^i\mu(n)\alpha^{-i}\hat{\chi}(1, \bar{1}) = \mu(n)\hat{\chi}(1, \bar{1}).$$

Summarizing, we have:

Lemma 3.6 *There is a 1-1 correspondence $Irr(G, \mu) \leftrightarrow Irr(\tilde{G}, \tilde{\mu})$ given by $\tilde{\chi} = 1_{\tilde{G}}\chi$, where $\tilde{\chi} \in Irr(\tilde{G}, \tilde{\mu})$ and $\chi \in Irr(G, \mu)$. Note that this correspondence is the natural one obtained by regarding χ as a character of $\tilde{G}$.*

There is a 1-1 correspondence $Irr(\tilde{G}, \tilde{\mu}) \leftrightarrow Irr(\hat{G}, \hat{\mu})$ given by $\tilde{\chi} = \tilde{\theta}\hat{\chi}$, where $\tilde{\chi} \in Irr(\tilde{G}, \tilde{\mu})$ and $\hat{\chi} \in Irr(\hat{G}, \hat{\mu})$.

We examine how the character correspondences of Lemma 3.6 fit with the p-block correspondences between $\tilde{G}$, G and $\hat{G}$.

If $g \in G$, then denote the conjugacy class sum of the class of G containing g by $s(g)$, with similar notation $s(\tilde{g})$ and $s(\hat{g})$ applied to $\tilde{g} \in \tilde{G}$ and $\hat{g} \in \hat{G}$. We denote the coset of $\overline{N}$ in $\tilde{G}$ containing $\tilde{g}$ by $\hat{g}$, and make the appropriate identification with the corresponding element of $\hat{G}$. We use a similar convention for the other factor groups studied here.

Recall that two irreducible characters belong to the same p-block if and only if their central characters agree on the p-block idempotents for the group, and that this occurs precisely when they agree modulo $J(R)$ on the p-regular conjugacy classes (noting that p-block idempotents have p-regular support). Since all the results of the remainder of this section are concerned with values of central characters, we may assume throughout that $\tilde{g} \in \tilde{G}_{p'}$.

Let $\omega_\chi : Z(RG) \to R$ be the central character corresponding to χ, so that $\omega_\chi(s(g)) = \frac{[G:C_G(g)]\chi(g)}{\chi(1)}$. Define $\omega_{\tilde{\chi}}$, $\omega_{\hat{\chi}}$ similarly.

Suppose that $N = U$ is a normal p-subgroup of G. Then we may assume that $\tilde{g} \in C_{\tilde{G}}(\tilde{U})_{p'}$:

Lemma 3.7 *Suppose that $\tilde{g} \in \tilde{G}$ and that $\tilde{g} \notin C_{\tilde{G}}(\tilde{U})$. Then*

$$\omega_{\tilde{\chi}}(s(\tilde{g})) \equiv 0 \ mod \ J(R)$$

for all $\tilde{\chi} \in Irr(\tilde{G})$.

Proof By Isaacs [29, 15.38] $s(\tilde{g})$ is nilpotent and so lies in $J(Z(k\tilde{G}))$, which is contained in the kernel of every central character of $k\tilde{G}$. □

The first correspondence of the lemma behaves well with respect to blocks:

Lemma 3.8 *Let χ_i correspond to $\tilde{\chi}_i$, $i = 1, 2$, via the natural correspondence. Then χ_1, χ_2 both lie in the same p-block of G if and only if $\tilde{\chi}_1$, $\tilde{\chi}_2$ both lie in the same p-block of $\tilde{G}$.*

Proof $\tilde{M} = O_p(\tilde{M}) \times O_{p'}(\tilde{M})$. By [22, V.4.3] there is a 1-1 correspondence between p-blocks of $\tilde{G}$ covering the trivial character of $O_{p'}(\tilde{M})$ (here all irreducible characters belonging to such a block cover the trivial character) and the p-blocks of $\tilde{G}/O_{p'}(\tilde{M})$. By [22, V.4.5] there is a 1-1 correspondence between the p-blocks of $\tilde{G}/O_{p'}(\tilde{M})$ and the p-blocks of $\tilde{G}/\tilde{M} = G$. It suffices to show that if two irreducible characters χ_1, $\chi_2 \in Irr(G, \mu)$ lie in the same p-block of G, then $\tilde{\chi}_1$ and $\tilde{\chi}_2$ lie in the same p-block of $\tilde{G}$. Let $\tilde{g} \in \tilde{G}_{p'}$.

$$\omega_{\tilde{\chi}_i}(s(\tilde{g})) = [\tilde{G} : C_{\tilde{G}}(\tilde{g})]\frac{\tilde{\chi}_i(\tilde{g})}{\tilde{\chi}_i(\tilde{1})} = \frac{[\tilde{G} : C_{\tilde{G}}(\tilde{g})]}{[G : C_G(g)]}[G : C_G(g)]\frac{\chi_i(g)}{\chi_i(1)} = \frac{[\tilde{G} : C_{\tilde{G}}(\tilde{g})]}{[G : C_G(g)]}\omega_{\chi_i}(s(g)).$$

But $[G : C_G(g)] | [\tilde{G} : C_{\tilde{G}}(\tilde{g})]$, so if χ_1, χ lie in the same p-block of G, i.e., $\omega_{\chi_i}(s(g)) - \omega_{\chi_i}(s(g)) \in J(R)$, then $\omega_{\tilde{\chi}_i}(s(\tilde{g})) - \omega_{\tilde{\chi}_i}(s(\tilde{g})) \in J(R)$ and we are done. □

We now examine how the second character correspondence of Lemma 3.6 fits with the p-block correspondence between $\tilde{G}$ and $\hat{G}$.

Lemma 3.9 *Let $\tilde{\chi}_i \in Irr(\tilde{G}, \tilde{\mu})$ correspond to $\hat{\chi}_i \in Irr(\hat{G}, \hat{\mu})$ under the correspondence given in Lemma 3.6, $i = 1, 2$. Suppose that either N is a p-group or $p \not| \mu(1)$. If $\hat{\chi}_1$, $\hat{\chi}_2$ lie in the same p-block of $\hat{G}$ then $\tilde{\chi}_1$, $\tilde{\chi}_2$ lie in the same p-block of $\tilde{G}$.*

Proof Let $\tilde{g} \in \tilde{G}$.

$$\omega_{\tilde{\chi}_i}(s(\tilde{g})) = [\tilde{G} : C_{\tilde{G}}(\tilde{g})]\frac{\tilde{\chi}_i(\tilde{g})}{\tilde{\chi}_i(\tilde{1})} = \frac{[\tilde{G} : C_{\tilde{G}}(\tilde{g})]}{[\hat{G} : C_{\hat{G}}(\hat{g})]}[\hat{G} : C_{\hat{G}}(\hat{g})]\frac{\hat{\chi}_i(\hat{g})\tilde{\theta}(\tilde{g})}{\hat{\chi}_i(\hat{1})\tilde{\theta}(\tilde{1})} = k_{\tilde{g}}\omega_{\hat{\chi}_i}(s(\hat{g})),$$

where

$$k_{\tilde{g}} := \frac{[\tilde{G} : C_{\tilde{G}}(\tilde{g})]\tilde{\theta}(\tilde{g})}{[\hat{G} : C_{\hat{G}}(\hat{g})]\tilde{\theta}(\tilde{1})}.$$

Suppose that $\hat{\chi}_1$, $\hat{\chi}_2 \in Irr(\hat{G}, \hat{\mu})$ lie in the same p-block of $\hat{G}$. Then $\omega_{\hat{\chi}_1}(s(\hat{g})) - \omega_{\hat{\chi}_2}(s(\hat{g})) \in J(R)$ for each $\hat{g} \in \hat{G}_{p'}$. So if $k_{\tilde{g}} \in R$ for each $\tilde{g} \in \tilde{G}_{p'}$, then $\tilde{\chi}_1$, $\tilde{\chi}_2$ must lie in the same p-block of $\tilde{G}$. We claim that this is the case.

Since $[\hat{G} : C_{\hat{G}}(\hat{g})] \mid [\tilde{G} : C_{\tilde{G}}(\tilde{g})]$ and $\tilde{\theta}(\tilde{g}) \in R$ for each $\tilde{g} \in \tilde{G}$, this does indeed occur when $(p, \tilde{\theta}(\tilde{1})) = 1$. Hence we may assume that $N = U$ is a p-group. By Lemma 3.7 we may assume that $\tilde{g} \in C_{\tilde{G}}(\tilde{U})_{p'}$.

Now $\tilde{\rho}|_{\tilde{U}}$ is irreducible and for each $\tilde{g} \in C_{\tilde{G}}(\tilde{U})_{p'}$, $\tilde{\rho}(\tilde{g})$ commutes with all $\tilde{\rho}(\tilde{u})$, $\tilde{u} \in \tilde{U}$, so by Schur's lemma $\tilde{\rho}(\tilde{g})$ must be a scalar matrix, and $\tilde{\theta}(\tilde{g}) = \kappa\tilde{\theta}(\tilde{1})$, where κ is a p' root of unity. But we have chosen $\tilde{\rho}$ so that $(det(\tilde{\theta}))(\tilde{g})$ is a p-power root of unity, so $\kappa = 1$, and $\tilde{\theta}(\tilde{g}) = \tilde{\theta}(\tilde{1})$ for each $\tilde{g} \in C_{\tilde{G}}(\tilde{U})_{p'}$.

Fix $\tilde{g} \in C_{\tilde{G}}(\tilde{U})_{p'}$. We have

$$[\tilde{G} : C_{\tilde{G}}(\tilde{g})] = [\hat{G} : C_{\hat{G}}(\hat{g})]$$

since $\overline{U} \lhd C_{\tilde{G}}(\tilde{g})$ and $|\overline{U}|$ is coprime to the order of $\tilde{g}$ (see for example Dornhoff [21, 64.1]). So $k_{\tilde{g}} = 1$ and if $\hat{\chi}_1$, $\hat{\chi}_2$ lie in the same p-block of $\hat{G}$ then $\tilde{\chi}_1$, $\tilde{\chi}_2$ lie in the same p-block of $\tilde{G}$. □

Proposition 3.10 *Suppose that one of the following occurs:*

(i) $p \nmid \mu(1)$,

(ii) N is a p-group.

Then for each p-block B of G, we have, in the notation of Lemma 3.6, a 1-1 correspondence

$$Irr(G, B, \mu) \leftrightarrow \bigcup_{i=1}^{r} Irr(\hat{G}, \hat{B}_i, \hat{\mu}),$$

where $\hat{B}_1, \ldots, \hat{B}_r$ are p-blocks of $\hat{G}$.

Suppose that $N \leq H \leq G$ and that b is a p-block of H. Then we may apply the same methods to the irreducible characters ζ of H lying over μ. Suppose that $\zeta \in Irr(H, b, \mu)$. Let $\hat{b}$ be the p-block of $\hat{H}$ containing $\hat{\zeta}$. Suppose that the Brauer correspondents of b and $\hat{b}$ in G and $\hat{G}$ are defined, and that $\hat{b}^{\hat{G}}$ is one of the $\hat{B}_i$ defined above. Then $b^G = B$.

Proof It remains to verify that the correspondence is compatible with the Brauer correspondence.

Let π_H denote the projection $G \to H$, where H is a subgroup of G containing N. If b is a block of H, then assume further that $\omega_b \circ \pi_H$ is a central character of B. This does indeed occur when H is a p-chain stabilizer, by Lemma 2.3.

Note that $\tilde{\rho}$ restricted to $\tilde{H}$ is irreducible, so we may use $\tilde{\theta}|_{\tilde{H}}$ to repeat the above process to obtain $\tilde{\zeta} \in Irr(\tilde{H}, \tilde{\mu})$ and $\hat{\zeta} \in Irr(\hat{H}, \hat{\mu})$ corresponding to a given $\zeta \in Irr(H, \mu)$, where $\tilde{H}$ and $\hat{H}$ have the obvious meanings. Denote by $\pi_{\tilde{H}}$ and $\pi_{\hat{H}}$ the respective projections.

Suppose first that $Bl(\zeta)^G = Bl(\chi) = B$, with χ, $\tilde{\chi}$ and $\hat{\chi}$ defined as before. Hence for each $\tilde{g} \in \tilde{G}_{p'}$ we have $\omega_\chi(s(g)) - (\omega_\zeta \circ \pi_H)(s(g)) \in J(R)$. By Lemma 3.1, this is $\omega_\chi(s(g)) - \frac{\zeta^G(s(g))}{\zeta^G(1)}$. So by Lemma 3.1 and Lemma 3.2

$$\begin{aligned}(\omega_{\tilde{\zeta}} \circ \pi_{\tilde{H}})(s(\tilde{g})) &= \frac{\tilde{\zeta}^{\tilde{G}}(s(\tilde{g}))}{\tilde{\zeta}^{\tilde{G}}(\tilde{1})} = [\tilde{G} : C_{\tilde{G}}(\tilde{g})]\frac{\zeta^G(g)}{\zeta^G(1)} = \frac{[\tilde{G}:C_{\tilde{G}}(\tilde{g})]}{[G:C_G(g)]}[G : C_G(g)]\frac{\zeta^G(g)}{\zeta^G(1)} \\ &\equiv \frac{[\tilde{G}:C_{\tilde{G}}(\tilde{g})]}{[G:C_G(g)]}\omega_\chi(s(g)) \ mod\ J(R)\end{aligned}$$

But

$$\frac{[\tilde{G} : C_{\tilde{G}}(\tilde{g})]}{[G : C_G(g)]}\omega_\chi(s(g)) = \omega_{\tilde{\chi}}(s(\tilde{g}))$$

by the proof of Lemma 3.8. Hence the first correspondence of Lemma 3.6 is compatible with the Brauer correspondence.

Now suppose that $Bl(\hat{\zeta})^{\hat{G}} = Bl(\hat{\chi}) = \hat{B}$. Then for each $\tilde{g} \in \tilde{G}_{p'}$ we have $\omega_{\hat{\chi}}(s(\hat{g})) - \frac{\hat{\zeta}^{\hat{G}}(s(\hat{g}))}{\hat{\zeta}^{\hat{G}}(\hat{1})} \in J(R)$.

So

$$(\omega_{\tilde{\zeta}} \circ \pi_{\tilde{H}})(s(\tilde{g})) = \frac{\tilde{\zeta}^{\tilde{G}}(s(\tilde{g}))}{\tilde{\zeta}^{\tilde{G}}(\tilde{1})} = k_{\tilde{g}} \frac{\hat{\zeta}^{\hat{G}}(s(\hat{g}))}{\hat{\zeta}^{\hat{G}}(\hat{1})} \equiv k_{\tilde{g}} \omega_{\hat{\chi}}(s(\hat{g})) \equiv \omega_{\tilde{\chi}}(s(\tilde{g})) \ mod \ J(R)$$

by Lemma 3.1, Lemma 3.2 and the proof of Lemma 3.9, where $k_{\tilde{g}} \in R$ is defined as before. Hence the correspondence is compatible with the Brauer correspondence. □

3.3 Characters afforded by relatively projective modules

Now that we have set up the appropriate notation we outline some results of Külshammer and Robinson [35], and of Robinson [47], who use a similar set-up to determine when an irreducible character of G is afforded by an N-projective RG-module, where $N \lhd G$. We focus on the situation where N is a p-group, although the results of [35] are considerably more general.

The results of the remainder of this chapter are entirely expository.

Definition Let $H \leq G$. We say that $\chi \in Irr(G)$ is H-projective with respect to R if there is some H-projective RG-module M affording χ. If R is given then we just say that χ is H-projective. We say that a p-subgroup of G is a vertex for χ if it is a vertex for some RG-module affording χ.

Write $w(G, H)$ for the number of H-projective irreducible characters of G.

Of course we may assume that H is a p-group since $R/J(R)$ has characteristic p.

Strong results in this direction have been achieved by Knörr in [33], where he proves the following. Note that he does not assume that the subgroup is normal:

Proposition 3.11 *Let $\chi \in Irr(G)$ lie in a p-block B of G with defect group D. If V is a vertex of χ (which we may suppose to be contained in D), then*

$$C_D(V) \leq V.$$

Proof This is [33, 3.7]. □

For the remainder of this chapter suppose that $N \lhd G$ is a p-group and that $\mu \in Irr(N)$. We examine when $\chi \in Irr(G, \mu)$ is N-projective. It is clear from the definitions that there is a bijection

$$Irr(G, \mu) \leftrightarrow Irr(I_G(\mu), \mu)$$

which respects N-projectivity. This bijection also respects p-blocks and the Brauer correspondence by Lemma 3.1, so we may assume that $G = I_G(\mu)$.

Lemma 3.12 (**[35]**) *Apply all the constructions from earlier in the chapter. Then χ is N-projective if and only if $\hat{\chi}$ is $\hat{M}$-projective.*

Sketch proof Suppose that χ is afforded by the N-projective RG-module T and suppose that $\tilde{\theta}$ is afforded by the $R\tilde{G}$-module X. Regard T as an $\tilde{N}$-projective $R\tilde{G}$-module. Then $\hat{\chi}$ is afforded by $(T \otimes X^\star)_{\hat{\mu}}$, the sum of those components of $T \otimes X^\star$ affording characters covering $\hat{\mu}$. $(T \otimes X^\star)_{\hat{\mu}}$ is an $\tilde{N}$-projective $R\tilde{G}$-module, and is $\hat{M}$-projective when regarded as an $R\hat{G}$-module (identifying $\hat{M}$ with $\tilde{M}$). The reverse direction is clear. □

Proposition 3.13 (**[35]**) *The N-projective irreducible characters covering μ are in 1-1 correspondence with the p-blocks of defect zero of $I_G(\mu)/N$. Hence the number of N-projective irreducible characters lying in a given p-block B of G is*

$$\sum_{\mu \in Irr(N)/G} f_0^{(B)}\left(\frac{I_G(\mu)}{N}\right).$$

Sketch proof For the first part fix $\mu \in Irr(N)$. By Proposition 3.10 and the remarks preceding Lemma 3.12, we may assume that $G = I_G(\mu)$ and that N is central in G and cyclic. Let $\chi \in Irr(G, \mu)$ be N-projective. If $g \in G$, then write g_p for the p-part of g.

Define

$$\mu^{-1} \star \chi = \begin{cases} \mu^{-1}(g_p)\chi(g) & if\ g_p \in N \\ 0 & otherwise \end{cases}$$

Then by [35, 2.3], $\mu^{-1} \star \chi$ is N-projective. $\mu^{-1} \star \chi \in Irr(G, 1_N)$ and each N-projective element of $Irr(G, 1_N)$ may be obtained from an unique $\chi \in Irr(G, \mu)$ in this way.

$\mu^{-1} \star \chi$ may be regarded as a character for G/N, afforded by a projective $R(G/N)$-module as required.

Notice that when $g \in G$ is p-regular we have $\mu^{-1} \star \chi(g) = \chi(g)$, so that $\mu^{-1} \star \chi$ and χ lie in the same p-block of G. Hence the correspondence established above respects p-blocks, and the last part of the proposition holds. □

Dade in [18] is concerned with $\chi \in Irr(G, \mu)$ satisfying $\chi(1)_p = [G : N]_p \mu(1)_p$, and calls them weights with respect to N. Robinson in [47] demonstrates that (within our choice of p-modular system) this is just the same as saying that χ is N-projective:

Proposition 3.14 ([47]) *$\chi \in Irr(G, \mu)$ is N-projective if and only if*

$$\chi(1)_p = [G : N]_p \mu(1).$$

Sketch proof As usual we may assume that μ is stable under the action of G. By Lemma 3.12 χ is N-projective if and only if $\hat{\chi}$ is $\hat{M}$-projective. Observe that $\chi(1)_p = [G : N]_p \mu(1)_p$ if and only if $\hat{\chi}(\hat{1})_p = [\hat{G} : \hat{M}]_p \hat{\mu}(\hat{1})_p$ $(= [\hat{G} : \hat{M}]_p)$. Hence we may assume WLOG that N is central in G and cyclic.

Suppose that χ is N-projective. Then by the proof of Proposition 3.13 $\chi(1) = \mu^{-1} \star \chi(1)$ and $\mu^{-1} \star \chi$ may be regarded as lying in a p-block of defect zero of G/N, so $\chi(1)_p = [G : N]_p$, and $\chi(1)_p = [G : N]_p \mu(1)_p$ as required.

Now suppose that $\chi(1)_p = [G : N]_p \mu(1)_p$.

Define $\chi^\star$ by

$$\chi^\star(g) = \begin{cases} |N|\chi(g) & if\ g \in G_{p'} \\ 0 & otherwise \end{cases}$$

Now for each $g \in G$,

$$\sum_{\lambda \in Irr(N)} (\lambda \star \chi)(g) = \begin{cases} \left(\sum_{\lambda \in Irr(N)} \lambda(g_p)\right) \chi(g) & if\ g_p \in N \\ 0 & otherwise \end{cases}.$$

But

$$\sum_{\lambda \in Irr(N)} \lambda(g_p) = \begin{cases} |N| & if\ g_p = 1 \\ 0 & otherwise \end{cases}$$

since N is abelian. So $\chi^\star = \sum_{\lambda \in Irr(N)}(\lambda \star \chi)$.

Recall that each $\lambda \star \chi$ is an irreducible character, and that $\mu^{-1} \star \chi$ and χ lie in the same p-block of G. But $\mu^{-1} \star \chi$ has kernel containing N and defect $log_p(|N|)$, so may be regarded as a p-block of defect zero for G/N. Hence $\mu^{-1} \star \chi$, and so χ, lies in a block of G with defect group N. The result follows. □

A trivial but useful observation following from the results of this chapter will be needed when dealing with chains of p-subgroups:

Corollary 3.15 *Let G be a finite group and $N \lhd G$ be a p-subgroup of G. Let $\mu \in Irr(N)$. If $w(G, \mu, N) \neq 0$, then $N = O_p(I_G(\mu))$.*

4 Conjectures

4.1 Introduction

We outline the various conjectures of local representation theory to which we will refer, highlight the relations between them and give a brief review of the progress made at the time of writing.

Alperin's weight conjecture, announced in the mid eighties, makes sweeping predictions based on the theory of finite groups of Lie type. Almost immediately it was reformulated by Knörr and Robinson based on topological methods. Part of this reformulation put the conjecture in a form compatible with character theory, and Dade used this and the Alperin-McKay conjecture to produce his own refined form of the conjecture. Dade has produced a series of progressively stronger conjectures, with the intention of giving a reduction to simple groups for the last. We are interested in a separate reformulation of Robinson which uses the Knörr-Robinson reformulation in conjunction with his work with Külshammer outlined in the previous chapter. Included in this chapter is a simple demonstration of the equivalence between Robinson's and Dade's projective conjecture organized with use of the p-local rank.

4.2 Alperin's weight conjecture

Alperin's 'weight conjecture' first appeared in his article [2], the proceedings of a symposium held on finite groups in Arcata. It relates the number of simple kG-modules to the number of 'weights' (which are locally defined objects).

Alperin defines a weight to be (the isomorphism class of) a projective simple $k(N_G(Q)/Q)$-module, where Q is a representative of a G-conjugacy class of p-subgroups of G, and allocates the weights to p-blocks in the obvious way. If such a $(kN_G(Q)/Q)$-module exists then Q must necessarily be a radical p-subgroup of G, for otherwise $N_G(Q)/Q$ has a non-trivial normal p-subgroup and $f_0(kN_G(Q)/Q) = 0$. The weight conjecture is then that the number of these weights should be $l(G)$, with a block-wise version saying that the number of weights associated to a p-block B of G should be $l(G,B)$. This is

$$l(G,B) = \sum_{Q} f_0^{(B)} \left(\frac{N_G(Q)}{Q} \right),$$

where the sum runs over the conjugacy classes of radical p-subgroups of G. We may rewrite this in the language of p-chains using Corollary 3.15 to highlight the similarity to the reformulation due to Robinson which we give later:

Conjecture 4.1 (Alperin) *Let B be a p-block of a finite group G, then*

$$l(G,B) = \sum_{\sigma \in \mathcal{R}/G} (-1)^{|\sigma|} f_0^{(B)} \left(\frac{I_{N_G(\sigma)}(1_{V_\sigma})}{V_\sigma} \right).$$

Alperin's weight conjecture is known to hold in the following cases (amongst others):

- Finite groups with split BN-pairs in the defining characteristic (non-blockwise version only). See [9] or [53].
- p-solvable groups (see [38] or [30]).
- Symmetric and general linear groups (see [3]).
- The reduction of Alperin's conjecture to the case $O_p(G) = 1$ is almost trivial. Hence it holds for finite groups of p-local rank one by [7].

4.3 A reformulation of the weight conjecture by Knörr and Robinson

Soon after the publication of [2], Knörr and Robinson presented a reformulation of the weight conjecture in [34], relating it to the work of Quillen (see [43]) and others on simplicial complexes consisting of p-chains. This new conjecture is presented in several (equivalent) forms, varying in the family of p-chains used and in the use of simple kH-modules or simple KH-modules (or indeed the p-rank of the Cartan matrix of a p-block).

The reformulated conjecture states that for each p-block B of positive defect of each finite group G we have

$$\sum_{\sigma \in \mathcal{C}(G|1)/G} (-1)^{|\sigma|} l(G_\sigma, B) = 0.$$

This conjecture is equivalent to the weight conjecture in the sense that if one holds for all finite groups then so does the other (the situation is not quite as bad as this, but the important thing to note is that we do not know that if one conjecture holds for a particular block, then that block must also satisfy the other conjecture).

Knörr and Robinson show in [34] that (amongst other things)

$$\sum_{\sigma \in \mathcal{C}(G|1)/G} (-1)^{|\sigma|} l(G_\sigma, B) = \sum_{\sigma \in \mathcal{C}(G|1)/G} (-1)^{|\sigma|} k(G_\sigma, B).$$

It is proved in [34] that if A is the sum of the p-blocks of positive defect of G, then

$$\sum_{\sigma \in \mathcal{C}(G|1)/G} (-1)^{|\sigma|} l(G_\sigma, A) = 0$$

for finite groups with split BN-pairs in the defining characteristic (the proof uses the fact that Alperin's conjecture holds in this situation). Since the principal p-block is generally the only p-block of positive defect for simple groups of Lie type in the defining characteristic, it should be noted that this suffices to demonstrate the fully blockwise conjecture in many cases.

4.4 The conjectures of Dade

Dade announced the first of his conjectures in [17]. This 'ordinary conjecture' states that for each p-block B of positive defect of each finite group G satisfying $O_p(G) = 1$ and for each integer d we have

$$\sum_{\sigma \in \mathcal{R}(G|1)/G} (-1)^{|\sigma|} k_d(G_\sigma, B) = 0.$$

$\mathcal{R}(G|1)$ may be replaced by $\mathcal{C}(G|1)$ or any other of the families of chains discussed in [34]. Also, being a refinement of the Knörr-Robinson reformulation, if this conjecture is true then so is the weight conjecture.

If one counts only characters of defect equal to that of B, then this conjecture is identical to the Alperin-McKay conjecture for groups with $O_p(G) = 1$. In fact, in this situation the projective form of the conjecture given below is equivalent to the Alperin-McKay conjecture.

Dade has announced a series of progressively stronger conjectures in [20], the strongest of which, the 'inductive conjecture', he claims admits a reduction to finite simple groups. Although the reduction step has yet to be published, work has already begun on verifying the final form of the conjecture for the finite simple groups, with a good portion of the sporadic groups already checked.

We give the 'projective conjecture', which may be found in [18]. It is convenient to give the form in which G is allowed to have any normal p-subgroup, although Dade demonstrates that it suffices to assume that $O_p(G)$ is central. We note that Dade defines $w(G,U)$ to be the number of irreducible characters χ of G such that $\chi(1)_p = [G:U]_p\mu(1)_p$, where $\mu \in Irr(U)$ is covered by χ, and U is a normal p-subgroup of G, but as shown in the previous chapter this is equivalent to our definition of $w(G,U)$.

Conjecture 4.2 (Dade's projective) *Let B be a p-block of a finite group G. Let d be an integer and $\lambda \in Irr(O_p(Z(G)))$. Then*

$$\sum_{\sigma \in \mathcal{R}(G|O_p(G))/G} (-1)^{|\sigma|} k_d(G_\sigma, B, \lambda) = \sum_{\sigma \in \mathcal{R}(G|O_p(G))/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, O_p(G)).$$

Dade also gives a 'weight' form of this conjecture, which, as may be seen later, is essentially Robinson's reformulation.

4.5 Robinson's reformulation

Robinson in [47] uses the results of [35] to present a conjecture equivalent to Alperin's giving the number of irreducible ($K-$)characters in a p-block B of G in terms of the number of p-blocks of defect zero of certain locally-determined sections of G. We give the conjecture as presented in [50].

Conjecture 4.3 (Robinson [47]) *Let B be a p-block of a finite group G. Then*

$$k(G,B) = \sum_{\sigma\in\mathcal{R}/G} (-1)^{|\sigma|} \sum_{\mu\in Irr(V_\sigma)/G_\sigma} f_0^{(B)}\left(\frac{I_{G_\sigma}(\mu)}{V_\sigma}\right).$$

Just as with the conjectures of Dade, this may be refined with the Alperin-McKay conjecture in mind:

Conjecture 4.4 (Robinson [47]) *Let G be a finite group and $\lambda \in Irr(O_p(Z(G)))$. Then for each p-block B of G and each non-negative integer d, we have*

$$k_d(G,B,\lambda) = \sum_{\sigma\in\mathcal{R}/G} (-1)^{|\sigma|} \sum_{\mu\in Irr_d(V_\sigma,\lambda)/G_\sigma} f_0^{(B)}\left(\frac{I_{G_\sigma}(\mu)}{V_\sigma}\right).$$

This refined form is the conjecture which we will be concerned with in this thesis.

The following is effectively proved in [18], and shows how Robinson's conjecture and Dade's projective conjecture are related. Our argument however is organized using the p-local rank.

Proposition 4.5 *Suppose that $plr(G) = n$ and that Robinson's conjecture holds for finite groups of p-local rank strictly less than n. Write $U = O_p(G)$. Then*

$$\sum_{\sigma\in\mathcal{R}(G|U)/G} (-1)^{|\sigma|} k_d(G_\sigma,B,\lambda) = \sum_{\sigma\in\mathcal{R}(G|U)/G} (-1)^{|\sigma|} w_d(G_\sigma,B,\lambda,U)$$

if and only if

$$k_d(G,B,\lambda) = \sum_{\sigma\in\mathcal{R}/G} (-1)^{|\sigma|} w_d(G_\sigma,B,\lambda,V_\sigma).$$

Proof Note that by the inductive hypotheses both equalities hold for all groups of p-local rank less than n.

Suppose that the former holds.

Let $\mathcal{R}_0(G)$ be the set of all radical p-subgroups of G (i.e., terms of radical p-chains of length zero). We have

$$\begin{aligned} & \textstyle\sum_{\sigma\in\mathcal{R}/G}(-1)^{|\sigma|}w_d(G_\sigma,B,\lambda,V_\sigma) \\ = & \textstyle\sum_{Q\in\mathcal{R}_0(G)/G}\left(\sum_{\sigma\in\mathcal{R}(N_G(Q)|Q)/N_G(Q)}(-1)^{|\sigma|}w_d(N_G(Q)_\sigma,B,\lambda,Q)\right) \\ = & \textstyle\sum_{Q\in\mathcal{R}_0(G)/G}\left(\sum_{\sigma\in\mathcal{R}(N_G(Q)|Q)/N_G(Q)}(-1)^{|\sigma|}k_d(N_G(Q)_\sigma,B,\lambda)\right) \\ = & \textstyle\sum_{\sigma\in\mathcal{R}/G}(-1)^{|\sigma|}k_d(G_\sigma,B,\lambda) \end{aligned}$$

We may pair every chain $Q_0<\cdots<Q_r$ in $\mathcal{R}(G)$ in which $Q_0\neq U$ with the chain $U<Q_0\cdots<Q_r$ in $\mathcal{R}(G)$ of length $r+1$. These two chains have the same stabilizer, so their contributions cancel. We may cancel all chains in $\mathcal{R}(G)$ in this way except for the chain U. Hence

$$\sum_{\sigma\in\mathcal{R}/G}(-1)^{|\sigma|}w_d(G_\sigma,B,\lambda,V_\sigma)=k_d(G,B,\lambda)$$

and Robinson's conjecture is satisfied in that case.

Now suppose that B satisfies the second equality. Then

$$\begin{aligned} & k_d(G,B,\lambda) \\ = & \textstyle\sum_{\sigma\in\mathcal{R}/G}(-1)^{|\sigma|}w_d(G_\sigma,B,\lambda,V_\sigma) \\ = & \textstyle\sum_{Q\in\mathcal{R}_0(G)/G}\left(\sum_{\sigma\in\mathcal{R}(N_G(Q)|Q)/N_G(Q)}(-1)^{|\sigma|}w_d(N_G(Q)_\sigma,B,\lambda,Q)\right) \\ = & \textstyle\sum_{\sigma\in\mathcal{R}(G|U)/G}(-1)^{|\sigma|}w_d(G_\sigma,B,\lambda,U) \\ + & \textstyle\sum_{U\neq Q\in\mathcal{R}_0(G)/G}\left(\sum_{\sigma\in\mathcal{R}(N_G(Q)|Q)/N_G(Q)}(-1)^{|\sigma|}k_d(N_G(Q)_\sigma,B,\lambda)\right) \end{aligned}$$

Replacing each chain $Q_0<\cdots<Q_r$ in $\mathcal{R}(G)$ in which $U\neq Q_0$ by $U<Q_0\cdots<Q_r$, we have

$$\sum_{U\neq Q\in\mathcal{R}_0(G)/G}\left(\sum_{\sigma\in\mathcal{R}(N_G(Q)|Q)/N_G(Q)}(-1)^{|\sigma|}k_d(N_G(Q)_\sigma,B,\lambda)\right)$$

$$= \sum_{\sigma\in\mathcal{R}(G|U)/G} (-1)^{|\sigma|+1} k_d(G_\sigma, B, \lambda) - k_d(G, B, \lambda),$$

and so

$$\sum_{\sigma\in\mathcal{R}(G|U)/G} (-1)^{|\sigma|} k_d(G_\sigma, B, \lambda) = \sum_{\sigma\in\mathcal{R}(G|U)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, U)$$

as required. □

Remark Although the conjectures of Dade and Robinson are similar, one should note that they represent very different approaches to Alperin's conjecture. The driving force behind Dade's conjectures is the proof using the CFSG, whereas Robinson's approach is to conserve the information implicit in the action on a normal p-subgroup and to keep the idea of relative projectivity that lies behind Alperin's conjecture.

4.6 Statement of results

Proposition 4.6 *If Conjecture 4.4 holds for all finite groups G of p-local rank n or smaller and with $O_p(G) \leq Z(G)$ cyclic, then the conjecture holds for all finite groups of p-local rank n. In particular in proving the conjecture it suffices to check it just for those groups with $O_p(G) \leq Z(G)$ cyclic.*

Theorem 4.7 *Conjecture 4.4 holds for all finite groups of p-local rank one.*

Corollary 4.8 *Dade's projective conjecture holds for all finite groups with almost TI Sylow p-subgroups. In particular, Dade's inductive conjecture holds for all simple groups with TI Sylow p-subgroups, cyclic Schur multiplier and trivial outer automorphism group.*

Proof By [20] the inductive conjecture is identical to the projective conjecture in the situation given above. □

Proposition 4.9 *Dade's inductive conjecture is satisfied for the Ree groups of type G_2 for the prime $p = 3$.*

Remark Since the notion of relative projectivity is well understood one may identify group theoretic conditions for a p-chain to contribute nothing to the alternating sum of Conjecture 4.4. For example, suppose that $\sigma : Q_0 < \cdots < Q_n$ is a p-chain of G and B a p-block of G_σ with defect group D; if

$$log_p(|Q_0|) < (log_p(|Q_n|) + log_p(|Z(Q_n)|))\,/2$$

or $C_D(Q_0) \not\leq Q_0$, then $w(G_\sigma, B, Q_0) = 0$ (see [49], the proof is an application of the results at the end of Chapter 3). Stronger results of this kind may be obtained if we count only characters covering a given $\theta \in Irr(Q_0)$, although the result as it stands is optimal (see for example $GL_3(q)$ in the defining characteristic).

Clearly results of this kind are not directly applicable to Dade's conjecture (at least in its non-weight form). However, by structuring via the p-local rank we may in some circumstances eliminate certain chains from the calculation of the alternating sums of Dade's projective conjecture (assuming that the conjecture holds for finite groups of smaller p-local rank). As an example, when verifying Dade's projective conjecture, we may eliminate the need for the calculation of p-chains totally when a group has Sylow p-subgroups of order p^2 (as is the case when $p = 3$ and $G = M_{22}$) by making use of Theorem 4.7. Such reductions would be best done on an ad hoc basis. It should be noted that in Dade's papers [17], [18], [20], etc., the residue field considered is not assumed to be algebrically closed, so one must use the fact that both Dade's and our set-up stipulate that K be a splitting field for G in order to apply these kinds of reduction.

5 Finite groups with normal p-subgroups

5.1 Introduction

In this chapter we consider what happens when a finite group G has a normal p-subgroup. Whereas Alperin's weight conjecture reduces quickly to the case where $O_p(G) = 1$, the reduction regarding Robinson's reformulation of the conjecture is somewhat more involved, and not as complete (for reasons which will become obvious). It is worth noting however that a complete reduction of the weaker form of the conjecture (Conjecture 4.3) to the case $O_p(G) = 1$ follows from the results of [48].

We demonstrate that in order to verify the conjecture for finite groups of p-local rank n it suffices to check the conjecture for p-blocks of finite groups of p-local rank at most n and having only cyclic, central normal p-subgroups. This will be applied in later chapters to the p-local rank one case in order to use the classification of finite simple groups.

The reduction involves using methods developed by Robinson in [47] to 'prepare' the formula predicted by Robinson's conjecture, so that we may apply the results of Chapter 3.

It is interesting to note that as a consequence of the results of this chapter we may give another equivalent version of the conjecture which at first sight appears stronger:

Conjecture 5.1 *Let B be a p-block of a finite group G. Then for each $\theta \in O_p(G)$ and for each integer d we have*

$$k_d(G, B, \theta) = \sum_{\sigma \in \mathcal{R}/G} (-1)^{|\sigma|} \sum_{\mu \in Irr_d(V_\sigma, \theta)/G_\sigma} f_0^{(B)} \left(\frac{I_{G_\sigma}(\mu)}{V_\sigma} \right).$$

5.2 Reduction for Robinson's conjecture

Let $\lambda \in Irr(O_p(Z(G)))$. By Clifford's theorem we may divide $Irr_d(G, B, \lambda)$ into sets of characters lying over a given G-orbit of $Irr(O_p(G))$, so that

$$k_d(G, B, \lambda) = \sum_{\mu \in Irr(O_p(G), \lambda)/G} k_d(G, B, \mu).$$

We aim to prove the following result, which then allows us to apply the results of the last chapter:

Proposition 5.2 *Let B be a p-block of a finite group G. Write $U = O_p(G)$. Let $\lambda \in Irr(O_p(Z(G))$ and d be an integer. Then*

$$\sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma)$$

$$= \sum_{\mu \in Irr(U, \lambda)/G} \left(\sum_{\sigma \in \mathcal{R}(I_G(\mu))/I_G(\mu)} (-1)^{|\sigma|} w_d(I_G(\mu)_\sigma, B, \mu, V_\sigma) \right).$$

Recall that $\mathcal{C}_U(G)$ is the set of all p-chains of G whose initial subgroup contains U (not necessarily properly). By Lemma 2.2 we may replace $\mathcal{R}$ in the first alternating sum of Proposition 5.2 with $\mathcal{C}_U$, since the initial subgroup of every radical p-chain of G is a radical p-subgroup of G and so contains U by Lemma 2.1. We obtain

$$\sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma) = \sum_{\sigma \in \mathcal{C}_U(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma).$$

For each $\sigma \in \mathcal{C}_U$,

$$w_d(G_\sigma, B, \lambda, V_\sigma) = \sum_{\mu \in Irr(U, \lambda)/G_\sigma} w_d(G_\sigma, B, \mu, V_\sigma),$$

so

$$\sum_{\sigma \in \mathcal{C}_U(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma) = \sum_{\sigma \in \mathcal{C}_U(G)/G} (-1)^{|\sigma|} \left(\sum_{\mu \in Irr(U, \lambda)/G_\sigma} w_d(G_\sigma, B, \mu, V_\sigma) \right).$$

Lemma 5.3

$$\begin{aligned} & \sum_{\sigma\in\mathcal{C}_U(G)/G}(-1)^{|\sigma|}\left(\sum_{\mu\in Irr(U,\lambda)/G_\sigma} w_d(G_\sigma,B,\mu,V_\sigma)\right) \\ = & \sum_{\mu\in Irr(U,\lambda)/G}\left(\sum_{\sigma\in\mathcal{C}_U(G)/I_G(\mu)}(-1)^{|\sigma|}w_d(G_\sigma,B,\mu,V_\sigma)\right) \end{aligned}$$

Proof Observe that the pair (σ,μ) lies in an orbit of length $[G:G_\sigma][G_\sigma:I_{G_\sigma}(\mu)] = [G:I_{G_\sigma}(\mu)]$ in the first instance and $[G:I_G(\mu)][I_G(\mu):I_{G_\sigma}(\mu)] = [G:I_{G_\sigma}(\mu)]$ in the second, and that $w_d(G_\sigma,B,\mu,V_\sigma)$ is constant under conjugation of (σ,μ) in G. □

We conclude that if

$$k_d(G,B,\mu) = \sum_{\sigma\in\mathcal{C}_U(G)/I_G(\mu)}(-1)^{|\sigma|}w_d(G_\sigma,B,\mu,V_\sigma)$$

for each $\mu\in Irr(U,\lambda)$, then the conjecture holds for that choice of B, d and λ.

We fix $\mu\in Irr(U,\lambda)$ and write $H=I_G(\mu)$. Clifford theory then allows us to move from counting characters of G and G_σ to counting characters of H and H_σ:

Lemma 5.4 $k_d(G,B,\mu) = k_d(H,B,\mu)$ *and*

$$\sum_{\sigma\in\mathcal{C}_U(G)/H}(-1)^{|\sigma|}w_d(G_\sigma,B,\mu,V_\sigma) = \sum_{\sigma\in\mathcal{C}_U(G)/H}(-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma\cap H).$$

Proof For each $\sigma\in\mathcal{C}_U(G)$, $I_{G_\sigma}(\mu) = H_\sigma$, and so Clifford's theorem gives a 1-1 correspondence

$$Irr(H_\sigma,\mu)\leftrightarrow Irr(G_\sigma,\mu)$$

given by induction of characters (see, for example [29, 6.11]), and note that $G=G_\sigma$ for the chain $\sigma=U$ of length 0. This correspondence is clearly defect-preserving. Suppose that $\chi\in Irr(G_\sigma,\mu)$ corresponds to $\eta\in Irr(H_\sigma,\mu)$. We claim that χ is V_σ-projective if and only if η is $V_\sigma\cap H$-projective.

Proof of claim Let $\varphi\in Irr(V_\sigma\cap H,\mu)$ be a character covered by η. Now $I_{V_\sigma}(\mu) = H\cap V_\sigma$, so by Clifford's theorem $\varphi^{V_\sigma}=\theta$ for some $\theta\in Irr(V_\sigma,\mu)$. But

$$(\chi|_{V_\sigma},\theta) = (\chi|_{V_\sigma},\varphi^{V_\sigma}) = (\chi|_{V_\sigma\cap H},\varphi) = (\eta^{G_\sigma}|_{V_\sigma\cap H},\varphi) = (\eta^{G_\sigma}|_{H_\sigma},\varphi^{H_\sigma}).$$

But $(\eta, \eta^{G_\sigma}|_{H_\sigma}) \neq 0$ and $(\eta, \varphi^{H_\sigma}) \neq 0$, so χ covers θ. We have $\chi(1)_p = [G_\sigma : H_\sigma]_p \eta(1)_p$ and

$$[G_\sigma : V_\sigma]_p \theta(1)_p = [G_\sigma : V_\sigma]_p [V_\sigma : V_\sigma \cap H]_p \varphi(1)_p = [G_\sigma : H_\sigma]_p [H_\sigma : V_\sigma \cap H]_p \varphi(1)_p.$$

So $\chi(1)_p = [G_\sigma : V_\sigma]_p \theta(1)_p$ if and only if $\eta(1)_p = [H_\sigma : V_\sigma \cap H]_p \varphi(1)_p$, and the claim follows by Proposition 3.14.

It follows immediately from Lemma 3.1 that every p-block of H_σ containing an irreducible character lying over μ has a Brauer correspondent in G_σ and the character correspondence given by induction respects the Brauer correspondence, i.e., if $\eta \in Irr(H_\sigma, \mu)$, then η^{G_σ} lies in the p-block of G_σ which is a Brauer correspondent of the p-block of H_σ containing η.

Hence

$$w_d(G_\sigma, B, \mu, V_\sigma) = w_d(H_\sigma, B, \mu, V_\sigma \cap H)$$

since the Brauer correspondence is transitive, and the result follows. □

We use the theory of deficient p-chains introduced in [47] in order to overcome the problem of summing over $I_G(\mu)$-orbits of p-chains of G (note in particular that a given $\sigma \in \mathcal{C}_U(G)$ need not lie in the stabilizer of μ). This idea originated in [51], but was defined formally in [47].

Definition Given a subgroup T of G, we say that a p-chain

$$\sigma : Q_0 < \cdots < Q_n \in \mathcal{C}(G)$$

is *T-deficient* if $Q_n \cap T \leq O_p(G)$. Given a p-chain $\sigma \in \mathcal{C}(G)$ we call the longest deficient initial subchain the *T-deficient part*, and denote it by $d_T(\sigma)$. Note that $d_T(\sigma)$ may be empty. As an abuse of notation denote the empty chain by $\emptyset$.

Returning to our original hypotheses, for brevity we write $\mathcal{D}(G) = \mathcal{D}_H(G)$ for the set of non-empty H-deficient chains in $\mathcal{C}_U(G)$. Write $\mathcal{D}^\sharp(G) = \mathcal{D}(G) - \{U\}$ and $d(\sigma) = d_H(\sigma)$.

Observe that we may write

$$\sum_{\sigma\in\mathcal{C}_U(G)/H}(-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma\cap H)$$

$$= \sum_{\tau\in\mathcal{D}(G)/H}\left(\sum_{\sigma\in\mathcal{C}(G|V^\tau)/H_\tau,d(\sigma)=\sigma_0}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)\right)$$
$$+\sum_{\sigma\in\mathcal{C}_U(G)/H,d(\sigma)=\emptyset}(-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma\cap H),$$

since $V_\tau\cap H=U$ when $\tau\in\mathcal{D}(G)$. We show that it suffices to consider only those chains contained in the stabilizer of their deficient part.

Lemma 5.5

$$\sum_{\tau\in\mathcal{D}(G)/H}\left(\sum_{\sigma\in\mathcal{C}(G|V^\tau)/H_\tau,d(\sigma)=\sigma_0}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)\right)$$

$$=\sum_{\tau\in\mathcal{D}(G)/H}\left(\sum_{\sigma\in\mathcal{C}(G_\tau|V^\tau)/H_\tau,d(\sigma)=\sigma_0}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)\right)$$

Proof We wish to cancel the contributions to the alternating sum of those chains $\sigma\in\mathcal{C}_U(G)$ with $\sigma\notin\mathcal{C}_U(N_G(d(\sigma)))$. We do this by pairing such chains with identical stabilizers but with lengths of differing parity. Note that the pairs of chains need not necessarily have the same, or even conjugate deficient part.

Let $\sigma: Q_0<\cdots<Q_n<\cdots<Q_m\in\mathcal{C}_U(G)$ with $d(\sigma)=\sigma_n$ non-empty. Suppose that $\sigma\notin\mathcal{C}_U(N_G(d(\sigma)))$, i.e., $Q_m\not\leq N_G(Q_i)$ for some $i\leq n$. Choose i minimal with this property. Given i, choose j maximal such that $Q_i\lhd Q_j$. So $i\leq j<m$ and $Q_j\leq N_{Q_{j+1}}(Q_i)\neq Q_{j+1}$. There are two cases to consider: (i) $Q_j=N_{Q_{j+1}}(Q_i)$ and (ii) $Q_j\neq N_{Q_{j+1}}(Q_i)$.

(i) Suppose that $Q_j=N_{Q_{j+1}}(Q_i)$. Then form the chain

$$\overline{\sigma}: Q_0<\cdots<Q_{j-1}<Q_{j+1}<\cdots<Q_m$$

from σ by deleting the term Q_j. If $i=j$, then $Q_j\neq N_{Q_{j+1}}(Q_i)$ since Q_{j+1} is a p-group, so in particular we must have $j\neq 0$ for case (i) to occur, and $V_{\overline{\sigma}}=V_\sigma$. Again using the fact that case (i) cannot occur when $i = j$, $N_H(\overline{\sigma}) \leq N_H(Q_j)$ and so $H_\sigma = H_{\overline{\sigma}}$. Also Q_i is still a term of $\overline{\sigma}$ and $j < m$, so $\overline{\sigma} \notin \mathcal{C}_U(N_G(d(\overline{\sigma})))$.

(ii) Suppose that $Q_j \neq N_{Q_{j+1}}(Q_i)$. Then form the chain

$$\overline{\sigma} : Q_0 < \cdots < Q_j < N_{Q_{j+1}}(Q_i) < Q_{j+1} < \cdots < Q_m$$

from σ. Of course $V_{\overline{\sigma}} = V_\sigma$, and it is easy to see that $N_H(\sigma) \leq N_{Q_{j+1}}(Q_i)$ and so $H_\sigma = H_{\overline{\sigma}}$. As above Q_i is still a term of the deficient part of $\overline{\sigma}$ and Q_m is still the final subgroup of $\overline{\sigma}$, so $\overline{\sigma} \notin \mathcal{C}_U(N_G(d(\overline{\sigma})))$.

In both cases $(\overline{\sigma})^g = \overline{(\sigma^g)}$ and $\overline{\overline{\sigma}} = \sigma$, so we may cancel the contributions from orbits of chains containing σ and $\overline{\sigma}$, since $(-1)^{|\overline{\sigma}|} = (-1)^{|\sigma|+1}$. □

The next lemma may seem somewhat technical but it will later allow us to reduce the summation to a consideration of chains of H rather than G, the key to proving Proposition 5.2.

Lemma 5.6 *Given $\tau \in \mathcal{D}(G)$, denote by $\mathcal{C}^\star(G_\tau|V^\tau)$ the set of those chains*

$$\sigma : Q_0 < \cdots < Q_n$$

in $\mathcal{C}(G_\tau|V^\tau)$ satisfying $Q_i = V^\tau(H \cap Q_i)$ for each $i = 0, \ldots, n$. We also allow τ to be the empty chain, in which case we set $G_\tau = G$ and set $\mathcal{C}_U^\star(G) = \mathcal{C}_U(H)$. Then

$$\sum_{\sigma \in \mathcal{C}_U(G)/H, d(\sigma)=\emptyset} (-1)^{|\sigma|} w_d(H_\sigma, B, \mu, V_\sigma \cap H) = \sum_{\sigma \in \mathcal{C}_U^\star(G)/H, d(\sigma)=\emptyset} (-1)^{|\sigma|} w_d(H_\sigma, B, \mu, V_\sigma \cap H)$$

(representing the case τ is empty), and when $\tau \in \mathcal{D}(G)$ we have

$$\sum_{\sigma \in \mathcal{C}(G_\tau|V^\tau)/H_\tau, d(\sigma)=\sigma_0} (-1)^{|\tau|+|\sigma|} w_d((H_\tau)_\sigma, B, \mu, U)$$

$$= \sum_{\sigma \in \mathcal{C}^\star(G_\tau|V^\tau)/H_\tau, d(\sigma)=\sigma_0} (-1)^{|\tau|+|\sigma|} w_d((H_\tau)_\sigma, B, \mu, U).$$

Proof Let $\tau \in \mathcal{D}(G)$, and let $\sigma : Q_0 < \cdots < Q_n \in \mathcal{C}(G_\tau|V^\tau)$ with $d(\sigma) = \sigma_0$ and suppose that $\sigma \notin \mathcal{C}^\star(G_\tau|V^\tau)$, i.e., $Q_i \neq V^\tau(H \cap Q_i)$ for some i. Choose i minimal with this property, noting that $i > 0$. Now

$$Q_i > V^\tau(H \cap Q_i) \geq V^\tau(H \cap Q_{i-1}) = Q_{i-1}$$

by the minimality of i. There are two cases to consider: (i) $V^\tau(H \cap Q_i) = Q_{i-1}$ and (ii) $V^\tau(H \cap Q_i) \neq Q_{i-1}$.

(i) Suppose that $V^\tau(H \cap Q_i) = Q_{i-1}$. Observe that $i > 1$, since if $i = 1$ and case (i) occurs, then $V^\tau(H \cap Q_1) = V^\tau$, and so $H \cap Q_1 = H \cap V^\tau = U$, contradicting our assumption that $d(\sigma) = \sigma_0$. Form the chain

$$\overline{\sigma} : Q_0 < \cdots < Q_{i-2} < Q_i < \cdots < Q_n$$

from σ by deleting the term Q_{i-1}. Since $i > 1$, we must have $V_{\overline{\sigma}} = V_\sigma$, and it is easy to see that $N_H(\overline{\sigma}) \leq N_H(Q_{i-1})$, so $N_H(\overline{\sigma}) = N_H(\sigma)$. Also $\overline{\sigma} \in \mathcal{C}(G_\tau|V^\tau)$, with $d(\overline{\sigma}) = \overline{\sigma}_0$ and $\overline{\sigma} \notin \mathcal{C}^\star(G_\tau|V^\tau)$.

(ii) Suppose that $V^\tau(H \cap Q_i) \neq Q_{i-1}$. Then form the chain

$$\overline{\sigma} : Q_0 < \cdots < Q_{i-1} < V^\tau(H \cap Q_i) < Q_i < \cdots < Q_n$$

from σ. $V_{\overline{\sigma}} = V_\sigma$ since $i > 0$. Clearly $N_H(\sigma) \leq N_H(V^\tau(H \cap Q_i))$, so that $N_H(\overline{\sigma}) = N_H(\sigma)$. Since $V^\tau(H \cap Q_i) \cap H = H \cap Q_i > U$, we have $d(\overline{\sigma}) = \overline{\sigma}_0$, and $\overline{\sigma} \notin \mathcal{C}^\star(G_\tau|V^\tau)$.

In both cases $(\overline{\sigma})^g = \overline{(\sigma^g)}$ and $\overline{\overline{\sigma}} = \sigma$, so we may cancel the contributions from orbits of chains containing σ and $\overline{\sigma}$, since $(-1)^{|\overline{\sigma}|} = (-1)^{|\sigma|+1}$. Hence the second part of the Lemma holds.

To consider those chains with deficient part the empty chain we may use a similar argument to cancel all chains in which $Q_i \neq H \cap Q_i$ for some i. We leave this to the reader. $\square$

Lemma 5.7

$$\sum_{\sigma\in\mathcal{C}^*_U(G)/H,d(\sigma)=\emptyset} (-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma\cap H) = \sum_{\sigma\in\mathcal{C}_U(H)/H,d(\sigma)=\emptyset} (-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma),$$

and if $\tau\in\mathcal{D}(G)$, *then*

$$\sum_{\sigma\in\mathcal{C}^*(G_\tau|V^\tau)/H_\tau,d(\sigma)=\sigma_0} (-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)$$

$$= \sum_{\sigma\in\mathcal{C}(H_\tau|U)/H_\tau} (-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U).$$

Proof Suppose that $\tau\in\mathcal{D}(G)$ and $\sigma\in\{\alpha\in\mathcal{C}^*(G_\tau|V^\tau) : d(\alpha)=\alpha_0\}$, where $\sigma : Q_0<\cdots<Q_n$. Then by definition $Q_i=V^\tau(H\cap Q_i)$ for each $i\geq 0$, and so

$$|Q_i| = \frac{|V^\tau||H\cap Q_i|}{|V^\tau\cap H\cap Q_i|} = \frac{|V^\tau||H\cap Q_i|}{|U|}$$

(recall that $H\cap V^\tau=U$). Since the Q_i's are distinct, the chain $\varphi(\sigma)$ defined to be

$$\varphi(\sigma) : H\cap V^\tau < H\cap Q_1<\cdots<H\cap Q_n$$

has distinct terms. Since $H\cap V^\tau=U$, we have $\varphi(\sigma)\in\mathcal{C}(H_\tau|U)$.

Conversely, if $\theta\in\mathcal{C}(H_\tau|U)$, where $\theta : P_0<\cdots<P_n$, then define $\psi(\theta)$ to be the chain

$$\psi(\theta) : V^\tau P_0<V^\tau P_1<\cdots<V^\tau P_n.$$

Observe that $V^\tau P_i=V^\tau(H\cap V^\tau P_i)$ for each i and that $\psi(\theta)\in\{\mathcal{C}^*(G_\tau|V^\tau) : d(\alpha)=\alpha_0\}$. Hence φ and ψ are inverse maps between $\{\alpha\in\mathcal{C}^*(G_\tau|V^\tau) : d(\alpha)=\alpha_0\}$ and $\mathcal{C}(H_\tau|U)$ which preserve length. It is clear that chains in $\{\alpha\in\mathcal{C}^*(G_\tau|V^\tau) : d(\alpha)=\alpha_0\}$ are conjugate in H_τ if and only if their images in $\mathcal{C}(H_\tau|U)$ are conjugate in H_τ. Hence we have a bijection $\{\alpha\in\mathcal{C}^*(G_\tau|V^\tau)/H_\tau : d(\alpha)=\alpha_0\}\leftrightarrow\mathcal{C}(H_\tau|U)/H_\tau$. It remains to check that $N_{H_\tau}(\varphi(\sigma))=N_{H_\tau}(\sigma)$. But this is clear since $N_{H_\tau}(\sigma)\leq N_{H_\tau}(\varphi(\sigma))$ and (since τ is stabiized by H_τ) $N_{H_\tau}(\varphi(\sigma))\leq N_{H_\tau}(\psi(\varphi(\sigma)))=N_{H_\tau}(\sigma)$.

The first equality of the lemma is immediate since $\mathcal{C}^*_U(G)$ is by definition $\mathcal{C}_U(H)$. $\square$

Proof of Proposition 5.2 We have seen that

$$\sum_{\sigma\in\mathcal{R}(G)/G}(-1)^{|\sigma|}w_d(G_\sigma,B,\lambda,V_\sigma)=\sum_{\mu\in Irr(U,\lambda)/G}\left(\sum_{\sigma\in\mathcal{C}_U(G)/I_G(\mu)}(-1)^{|\sigma|}w_d(G_\sigma,B,\mu,V_\sigma)\right)$$

(see Lemma 5.3 and the discussion preceding it). We fix $\mu\in Irr(U,\lambda)$ and write $H=I_G(\mu)$. By Lemmas 5.4, 5.5, 5.6 and 5.7,

$$\sum_{\sigma\in\mathcal{C}_U(G)/H}(-1)^{|\sigma|}w_d(G_\sigma,B,\mu,V_\sigma)$$

$$=\sum_{\tau\in\mathcal{D}(G)/H}\left(\sum_{\sigma\in\mathcal{C}(H_\tau|U)/H_\tau}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)\right)$$
$$+\sum_{\sigma\in\mathcal{C}_U(H)/H,d(\sigma)=\emptyset}(-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma),$$

where the final alternating sum represents those chains in $\mathcal{C}_U(G)$ with empty deficient part.

Now if we set $\alpha\in\mathcal{D}(G)$ to be the chain $\alpha=U$ (of length zero), then

$$\sum_{\sigma\in\mathcal{C}(H_\alpha|U)/H_\alpha}(-1)^{|\alpha|+|\sigma|}w_d((H_\alpha)_\sigma,B,\mu,U)+\sum_{\sigma\in\mathcal{C}_U(H)/H,d(\sigma)=\emptyset}(-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma)$$
$$=\sum_{\sigma\in\mathcal{C}_U(H)/H}(-1)^{|\sigma|}w_d(H_\sigma,B,\mu,V_\sigma),$$

as we are considering on the one hand chains whose initial subgroup *is* U and on the other chains whose initial subgroup *strictly contains* U. We claim that

$$\sum_{\tau\in\mathcal{D}^\sharp(G)/H}\left(\sum_{\sigma\in\mathcal{C}(H_\tau|U)/H_\tau}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)\right)=0.$$

Proof of claim Suppose that $\tau\in\mathcal{D}^\sharp(G)$, and consider a chain $\sigma\in\mathcal{C}(H_\tau|U)$, $\sigma:Q_0<\cdots<Q_n$, of length $|\sigma|>0$. Then $U<Q_1\lhd N_{H_\tau}(\sigma)=I_{N_{H_\tau}(\sigma)}(\mu)$, i.e., Q_1 is a normal p-subgroup of $(H_\tau)_\sigma$ strictly containing U and stabilizing μ, and so $w_d((H_\tau)_\sigma,B,\mu,U)=0$ by Corollary 3.15. Hence

$$\sum_{\tau\in\mathcal{D}^\sharp(G)/H}\left(\sum_{\sigma\in\mathcal{C}(H_\tau|U)/H_\tau}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma,B,\mu,U)\right)=\sum_{\tau\in\mathcal{D}^\sharp(G)/H}(-1)^{|\tau|}w_d(H_\tau,B,\mu,U).$$

But notice that we may pair each deficient chain $\tau \in \mathcal{D}^\sharp(G)$ satisfying $V_\tau = U$ with another chain in $\mathcal{D}^\sharp(G)$ with initial term strictly containing U, since we have defined $\mathcal{D}^\sharp(G)$ to exclude the chain $\tau = U$. Clearly paired chains lie in H-orbits of the same size, and the lengths of the chains in each pair differ in parity, so we may cancel their contributions to this last alternating sum. This gives $\sum_{\tau\in\mathcal{D}^\sharp(G)/H}(-1)^{|\tau|}w_d(H_\tau, B, \mu, U) = 0$, and the claim follows.

Hence we have

$$\begin{aligned}\sum_{\sigma\in\mathcal{C}_U(G)/H}(-1)^{|\sigma|}w_d(G_\sigma, B, \mu, V_\sigma) &= \sum_{\sigma\in\mathcal{C}_U(H)/H}(-1)^{|\sigma|}w_d(H_\sigma, B, \mu, V_\sigma)\\ &\quad + \sum_{\tau\in\mathcal{D}^\sharp(G)/H}\left(\sum_{\sigma\in\mathcal{C}(H_\tau|U)/H_\tau}(-1)^{|\tau|+|\sigma|}w_d((H_\tau)_\sigma, B, \mu, U)\right)\\ &= \sum_{\sigma\in\mathcal{C}_U(H)/H}(-1)^{|\sigma|}w_d(H_\sigma, B, \mu, V_\sigma) \overset{2.2}{=} \sum_{\sigma\in\mathcal{R}(H)/H}(-1)^{|\sigma|}w_d(H_\sigma, B, \mu, V_\sigma),\end{aligned}$$

completing the proof of the Proposition. □

Proposition 4.6 *If Conjecture 4.4 holds for all finite groups G of p-local rank n or smaller and with $O_p(G) \leq Z(G)$ cyclic, then the conjecture holds for all finite groups of p-local rank n. In particular in proving the conjecture it suffices to check it just for those groups with $O_p(G) \leq Z(G)$ cyclic.*

Proof Let B be a p-block of a finite group G with $plr(G) = n$. We have seen (by Proposition 5.2, its proof and the accompanying discussion) that in order to verify Conjecture 4.4 for B (for a given $\lambda \in Irr(O_p(Z(G)))$ and integer d) it suffices to show that

$$k_d(I_G(\mu), B, \mu) = \sum_{\sigma\in\mathcal{R}(I_G(\mu))/I_G(\mu)}(-1)^{|\sigma|}w_d(I_G(\mu)_\sigma, B, \mu, V_\sigma) \qquad (1)$$

for each $\mu \in Irr(U, \lambda)$. Hence we may assume that $I_G(\mu) = G$.

Now by the results of Chapter 3 we may choose an irreducible extension $\tilde{\theta}$ of μ to a covering group $\tilde{G}$ of G, and we may use restrictions of this extension to examine the characters of each G_σ lying over μ via covering groups of G_σ/U. We use the notation of Proposition 3.10, with $N = U$.

Let $\hat{B}$ be the sum of the p-blocks of $\hat{G}$ corresponding to B in Proposition 3.10 (i.e., the sum of the p-blocks $Bl(\hat{\chi})$, where $\hat{\chi}$ corresponds to some $\chi \in Irr(G, B, \mu)$). It is clear that $\mathcal{R}(\hat{G}/\hat{M})/\hat{G}$ and $\mathcal{R}(G/U)/G$ may be identified, so $\hat{G}$ and G have 'similar' p-local stucture. For each $\sigma \in \mathcal{R}(G)$, define $\hat{\sigma}$ in the obvious way. We may make the identification

$$Irr(G_\sigma, B, \mu, V_\sigma) \leftrightarrow Irr(\hat{G}_{\hat{\sigma}}, \hat{B}, \hat{\mu}, V_{\hat{\sigma}}).$$

Let $a = log_p(|G|_p)$ and $a' = log_p(|\hat{G}|_p)$. In order to count characters of defect d in G, we must count characters of defect $d' = d + a' - a + log_p(\mu(1))$ in $\hat{G}$. Hence the correspondence above may be refined to

$$Irr_d(G_\sigma, B, \mu, V_\sigma) \leftrightarrow Irr_{d'}(\hat{G}_{\hat{\sigma}}, \hat{B}, \hat{\mu}, V_{\hat{\sigma}}).$$

Of course we also have a 1-1 correspondence

$$Irr_d(G, B, \mu) \leftrightarrow Irr_{d'}(\hat{G}, \hat{B}, \hat{\mu}).$$

Hence (1) if and only if

$$k_{d'}(\hat{G}, \hat{B}, \hat{\mu}) = \sum_{\hat{\sigma} \in \mathcal{R}(\hat{G})/\hat{G}} (-1)^{|\hat{\sigma}|} w_{d'}(\hat{G}_{\hat{\sigma}}, \hat{B}, \hat{\mu}, V_{\hat{\sigma}}),$$

since by Proposition 3.10 the correspondences respect the Brauer correspondence. It is clear that $\hat{M} = O_p(\hat{G}) \leq Z(\hat{G})$, and by definition $\hat{M}$ is cyclic. Hence the proposition holds, since by Lemma 2.4 we have $plr(I_G(\mu)) \leq plr(G)$. □

Corollary 5.8 *Conjecture 4.4 is equivalent to Conjecture 5.1.*

Proof Clearly Conjecture 5.1 implies Conjecture 4.4. Suppose that Conjecture 4.4 holds. By the proof of Proposition 4.6 it suffices in proving Conjecture 5.1 to prove it only for finite groups with no non-central normal p-subgroups. But in this case the two conjectures are identical, and the result follows. □

5.3 Groups of p-rank one

We are now in a position to verify Theorem 4.7 for those finite groups G (of p-local rank one) in which $G/O_p(G)$ has p-rank one (i.e., $G/O_p(G)$ has no elementary abelian subgroups of order p^2). It is well-known that a p-group has p-rank one if and only if it is cyclic or generalized quaternion (see for example [24, 5.4.10]).

Proposition 5.9 *Let G be a counterexample to Theorem 4.7. Then $G/O_p(G)$ contains an elementary abelian subgroup of order p^2.*

Proof By Proposition 5.2 and its proof, it suffices to verify the conjecture for central extensions, with centre a cyclic p-group, of each subgroup of $G/O_p(G)$. Since each subgroup of $G/O_p(G)$ is either a p'-group or has p-rank one, it then suffices to assume that G has p-local rank one, $O_p(G) = O_p(Z(G)) \leq G'$ and $G/O_p(G)$ has cyclic or generalized quaternion Sylow p-subgroups. Let B be a p-block of G. We may assume that B has maximal defect.

Suppose first that $G/O_p(G)$ has a cyclic Sylow p-subgroup P. Then by Lemma 2.14 P has trivial Schur multiplier, and so by Lemma 2.10 $O_p(M(G/O_p(G))) = 1$, so that $O_p(G) = 1$. By the results of Dade [16] $k(B) = k(b)$, where b is the unique block of $N_G(P)$ corresponding to B, and all irreducible characters in B and b have height zero. Hence Conjecture 4.4 holds for B. (Note that Dade considers his conjectures for blocks of cyclic defect in [19]).

Now suppose that $p = 2$ and that $G/O_p(G)$ has a generalized quaternion Sylow 2-subgroup P. By Lemma 2.14 $M(P) = 1$, and so by Lemma 2.10 $O_p(M(G/O_p(G))) = 1$, so that $O_p(G) = 1$. It then follows from [56], which in turn relies upon the description of the irreducible characters of such a group given by Olsson in [40], that the conjecture holds for B, and the result follows. $\square$

6 Characterization and classification of finite groups with almost TI Sylow p-subgroups

6.1 Introduction

We have already seen that it suffices to assume that $O_p(G)$ is central in G and cyclic, and that G has p-rank at least two, so from now on we will assume that this is so. As in [7], we demonstrate that it also suffices to assume that $O_{p'}(G)$ is cyclic and central in G. These reductions are crucial in order to make use of the CFSG.

Having first established such groups as automorphism groups of quasi-simple groups, we use the classifications of Gorenstein and Lyons [26] (which assumes that p is odd, $O_{p'}(G) = 1$ and lists simple groups with strongly p-embedded subgroups - a condition similar to TI-ness) and of Suzuki (which assumes $p = 2$ and looks directly at the TI case) to list all such quasisimple groups.

6.2 Finite groups with normal p'-subgroups

Lemma 6.1 *In order to verify that Conjecture 4.4 holds for finite groups with almost TI Sylow p-subgroups of p-rank greater than one, it suffices to verify it just for those with only cyclic, central normal p'-subgroups.*

Proof Let B be a p-block of a finite group G with almost TI Sylow p-subgroups. Let $P \in Syl_p(G)$. In this case, by Corollary 3.15 Conjecture 4.4 states that $k_d(G, B, \lambda) = k_d(N_G(P), B, \lambda)$ for each integer d and each $\lambda \in Irr(O_p(Z(G)))$. Let $K = O_{p'}(G)$. By Lemma 2.8 K centralizes P and so $K \leq N_G(P)$. The p-blocks of K (in this case all are blocks of defect zero) covered by B form a single G-orbit (see [22, V.2.3]), and $Irr_d(G, B, \lambda) = Irr_d(G, B, \mu, \lambda)$ for some $\mu \in Irr(K)$. Clifford theory gives a 1-1 correspondence

$$Irr(G,B,\lambda) = Irr(G,B,\mu,\lambda) \leftrightarrow Irr(I_G(\mu),B,\mu,\lambda)$$

given by induction, which preserves character defects. It is clear that we also have a 1-1 correspondence $Irr(N_G(P),B,\mu,\lambda) \leftrightarrow Irr(I_{N_G(P)}(\mu),B,\mu,\lambda)$. Hence we assume that μ is G-stable.

By Proposition 3.10 there is a finite group $\hat{G}$ with $O_{p'}(\hat{G}) \leq Z(\hat{G})$ cyclic, $\hat{G}/O_{p'}(\hat{G}) \cong G/O_{p'}(G)$, $\hat{\mu} \in Irr(O_{p'}(\hat{G}))$ and a sum of p-blocks $\hat{B}$ of $\hat{G}$ such that

$$k_d(G,B,\lambda) = k_d(G,B,\mu,\lambda) = k_d(\hat{G},\hat{B},\hat{\mu},\lambda) = k_d(\hat{G},\hat{B},\lambda).$$

Clearly $\mathcal{R}(\hat{G}/O_{p'}(\hat{G}))/\hat{G} = \mathcal{R}(G/O_{p'}(G))/G$. By Proposition 3.10 we also have

$$k_d(N_G(P),B,\lambda) = k_d(N_G(P),B,\mu,\lambda) = k_d(N_{\hat{G}}(\hat{P}),\hat{B},\hat{\mu},\lambda) = k_d(N_{\hat{G}}(\hat{P}),\hat{B},\lambda),$$

where $\hat{P}$ is defined in the obvious way. Hence if Conjecture 4.4 holds for $\hat{G}$ then it holds for G, and we are done. $\square$

6.3 Characterization of finite groups with almost TI Sylow p-subgroups

Lemma 6.2 *Let H be a finite group with $plr(H) = 1$ and $O_p(H) = O_{p'}(H) = 1$. Then there is an unique nontrivial minimal normal subgroup N of H. Further, N is nonabelian simple, $plr(N) = 1$ and H is isomorphic to a subgroup of $Aut(N)$, the automorphism group of N.*

Proof Let $P \in Syl_p(H)$. We note that P is not normal in H, since otherwise $plr(H) = 0$ by the remarks preceding Lemma 2.4.

Let N be a (nontrivial) minimal normal subgroup of H. Then $p \mid |N|$, since otherwise $N \leq O_{p'}(H) = 1$.

Now $P \cap N$ is a nontrivial Sylow p-subgroup of N so, by Lemma 2.7 $H = N_H(P)N$. Suppose that N_1 and N_2 are distinct nontrivial minimal normal subgroups of H. By minimality we have $N_1 \cap N_2 = 1$, and $N_1 \leq C_H(N_2) \leq C_H(N_2 \cap P) \leq N_H(N_2 \cap P)$. Since $N_2 \cap P$ is nontrivial, by Lemma 2.6 $N_H(N_2 \cap P) \leq N_H(P)$. But then $H = N_H(P)$, a contradiction. So N is the unique nontrivial minimal normal subgroup of H.

Now by Gorenstein [24, 2.1.5] either N is isomorphic to a direct product $S_1 \times \cdots \times S_r$ of isomorphic nonabelian simple groups or N is elementary abelian. If the latter occurs, then N is a normal p-subgroup of H since $p||N|$, contradicting $O_p(H) = 1$. So we may assume the former. We show that $r = 1$:

Suppose that $r > 1$. Clearly $p \mid |S_1|$, so we have

$$S_2 \times \cdots \times S_r \leq C_H(S_1) \leq C_H(S_1 \cap P) \leq N_H(S_1 \cap P) \leq N_H(P)$$

by Lemma 2.6 since $S_1 \cap P$ is nontrivial. By the same argument each S_i is contained in $N_H(P)$ and $N \leq N_H(P)$, again a contradiction. So $r = 1$ and N is nonabelian simple. It is clear from Lemma 2.4 that $plr(N) = 1$. It remains to show that $C_H(N) = 1$, so that H is isomorphic to a subgroup of $Aut(N)$.

Since $C_H(N) \lhd H$ and N is the unique minimal normal subgroup of H, either $C_H(N) \geq N$ or $C_H(N) = 1$. Since N is nonabelian we must have the latter, and the result follows. □

Now by Proposition 4.6, Proposition 5.9 and Lemma 6.1, in proving Theorem 4.7 we may assume that G has $O_p(G)O_{p'}(G) = Z(G)$ cyclic and is a central extension of $G/Z(G)$ with $Z(G) \leq [G, G]$, and that $G/Z(G)$ has p-rank greater than one. Of course $plr(G) = 1$. By Lemma 6.2, $\overline{G} = G/Z(G)$ is an automorphism group of a finite simple group $\overline{N} = N/Z(G)$ of p-local rank one, where $Z(G) < N$. We use the classification of finite groups with TI Sylow 2-subgroups by Suzuki, and the classification of finite groups with a proper strongly p-embedded subgroup, non-cyclic Sylow p-subgroups and no nontrivial normal p'-subgroup, for p an odd prime, by Gorenstein and Lyons to give a list of possibilities for N.

The following hypotheses hold throughout the remainder of the thesis.

Hypotheses 6.3 *G is a finite group of p-local rank one and p-rank greater than one. $Z(G) \leq G'$, $Z(G) < N \lhd G$ and $Z(G)$ is cyclic. $\overline{N} = N/Z(G)$ is a nonabelian simple group of p-local rank one and $\overline{N} \leq \overline{G} \leq Aut(\overline{N})$. Write $L = N'$.*

Remark 6.4 *It is important to note that $Z(L)$ is central in G. This excludes some automorphism groups of quasisimple groups which do not act trivially on the centre of that quasisimple group (see [12, p.xxii]).*

Definition Let $Q \leq H$ be a p-group, H a finite group, and $k \in \mathbb{N}$. Let $m(X)$ be the p-rank of the group X. Define

$$\Gamma_{Q,k}(H) = \langle N_H(X) \mid X \leq Q, m(X) \geq k \rangle.$$

For $P \in Syl_p(H)$ we call $\Gamma_{P,k}(H)$ the *k-generated p-core* of H.

We call a proper subgroup $M \leq H$ *strongly p-embedded* if $\Gamma_{P,1}(H) \leq M$ for some $P \in Syl_p(H)$.

Lemma 6.5 *Suppose that $O_p(H) = 1$. Then H has a TI Sylow p-subgroup P if and only if $\Gamma_{P,1}(H) = N_H(P)$, i.e., $N_H(P)$ is a proper strongly p-embedded subgroup of H.*

Proof The 'only if' part is clear from Lemma 2.6. Suppose that $1 < Q < P$ is a radical p-subgroup of H and that $\Gamma_{P,1}(H) = N_H(P)$. Then $N_H(Q) \leq \Gamma_{P,1}(H) = N_H(P)$. So $Q < P \cap N_H(Q) \leq O_p(N_H(Q))$, contradicting our assumption that Q is radical. Hence the only radical p-subgroups of H are 1 and the Sylow p-subgroups of H, so that $plr(H) = 1$. The result then follows by Lemma 2.5. □

Lemma 6.6 *Consider Hypotheses 6.3. Then $(p, \overline{N})$ is isomorphic to one of the following:*

(a) $(2, {}^2B_2(2^{2m+1}))$, $m \geq 1$

(b) $(3, {}^2G_2(3^{2m+1}))$, $(3, PSL_3(4))$, $(3, {}^2G_2(3)')$, $(3, M_{11})$, $m \geq 1$

(c) $(5, {}^2B_2(32))$, $(5, {}^2F_4(2)')$, $(5, McL)$

(d) $(11, J_4)$

(e) $(p, PSL_2(p^m))$, $(p, PSU_3(p^m))$, $m \geq 1$.

Proof For $p = 2$ this is Theorem 1 of [55]. Suppose that p is odd. Then $\overline{N}$ has a proper strongly p-embedded subgroup. So by [26, 24.9] $(p, \overline{N})$ is isomorphic to one of the pairs listed or to $(5, M_{22})$ or (p, A_{2p}). By [26, 24.2], $\Gamma_{Q,1}(\overline{N}) \neq N_{\overline{N}}(Q)$ (for $Q \in Syl_p(\overline{N})$) when $(p, \overline{N})$ is isomorphic to $(5, M_{22})$ or (p, A_{2p}), so by Lemma 6.5 in these two cases $\overline{N}$ cannot have TI Sylow p-subgroups. Also by [26, 24.2], $\Gamma_{Q,1}(\overline{N}) = N_{\overline{N}}(Q)$ for each of the remaining cases, so the result holds by Lemma 6.5. □

The argument used in proving the following result is essentially that used in [59].

Lemma 6.7 *Consider Hypotheses 6.3. Then* $(p, [G : N]) = 1$, *except when* $(p, \overline{N}) \cong (3, {}^2G_2(3)')$ *or* $(5, {}^2B_2(32))$.

Proof Suppose first that $p = 2$. Then the result follows from Theorem 2 of [55].

Assume now that $p \neq 2$ and that $(p, \overline{N}) \ncong (3, {}^2G_2(3)')$ or $(5, {}^2B_2(32))$. By [12], $\overline{N}$ has p-regular outer automorphism group except when possibly $(p, N) \cong (p, PSL_2(p^m))$, $(p, PSU_3(p^m))$ or $(3, {}^2G_2(3^{2m+1}))$. In each of these cases an outer automorphism of p-power order must be a field automorphism, i.e., a power of the Frobenius map $\tau : x \to x^p$.

Assuming that $\overline{N}$ is one of these classes of groups, let $P \in Syl_p(\overline{G})$ and $Q = P \cap \overline{N}$. We assume that $p \mid [\overline{G} : \overline{N}]$ and derive a contradiction.

Let $g \in P$ be a field automorphism of order p. Then g fixes a nontrivial proper subfield of $\mathbb{F}_q$, say $\mathbb{F}_t$, where $\mathbb{F}_q$ is the field of definition of $\overline{N}$ as a group of Lie type. Hence g fixes a subgroup T of $\overline{N}$ isomorphic to $PSL_2(\mathbb{F}_t)$, $PSU_3(\mathbb{F}_t)$ or ${}^2G_2(\mathbb{F}_t)$ respectively, so that

$$T \leq C_{\overline{G}}(g) = C_{\overline{G}}(\langle g \rangle) \leq N_{\overline{G}}(\langle g \rangle).$$

But $N_{\overline{G}}(\langle g \rangle) \leq N_{\overline{G}}(P)$ since $\overline{G}$ has TI Sylow p-subgroups, so $T \leq N_{\overline{G}}(P) \cap \overline{N} = N_{\overline{N}}(P)$. But $N_{\overline{N}}(P)$ is of the form PB, where B is abelian, so $N_{\overline{G}}(P)$ is solvable, a contradiction since $N_{\overline{G}}(P)$ contains the nonabelian simple group T and subgroups of solvable groups must be solvable.

When $(p, \overline{N}) \cong (3, {}^2G_2(3)')$ or $(5, {}^2B_2(32))$, by [12] we have $[G : N] = p$. □

In verifying Conjecture 4.4 for the various possibilities for G we rely upon knowledge of the automorphism groups of covers of simple groups. The remainder of this chapter is devoted to overcoming the problem of deriving information concerning covering groups of the automorphism groups. We do this both by determining the Schur multipliers explicitly and by using reduction techniques developed in [7]. The aim is to show that we may assume $L = N$.

We first deal with the cases where $[G : N]$ is divisible by p.

Lemma 6.8 *Consider Hypotheses 6.3. If* $(p, \overline{N}) \cong (3, {}^2G_2(3)')$ *or* $(5, {}^2B_2(32))$, *then* $Z(G) = 1$.

Proof By [12] the Schur multiplier $M(\overline{N})$ is trivial in these cases, and $[\overline{G} : \overline{N}] = p$. Then since $\overline{G}/\overline{N}$ is cyclic, by Lemma 2.12 $M(\overline{G}) = 1$ and the result follows. □

We now assume that $p \nmid [G : N]$.

Lemma 6.9 *Consider Hypotheses 6.3, and suppose also that* $p \nmid [G : N]$. *If* $O_p(Z(G)) \neq 1$, *then* $L = N$.

Proof By Lemma 2.11 if $O_p(M(\overline{N})) = 1$, then $O_p(M(\overline{G})) = 1$, so we need only consider those cases where $O_p(M(\overline{N})) \neq 1$. By [12] this occurs only when $(p, \overline{N}) \cong (2, PSL_2(4))$, $(2, {}^2B_2(8))$, $(3, PSL_2(9))$ or $(3, PSL_3(4))$.

If G/N is cyclic, then by Lemma 2.12 $M(\overline{G})$ is a homomorphic image of $M(\overline{N})$. But $Z(L) \leq Z(G)$, so in this case $Z(L) = Z(G)$, and $L = N$ as required.

Hence we may assume that G/N is noncyclic. This occurs only when $(p, \overline{G}) \cong (3, PSL_2(9))$ or $(3, PSL_3(4))$. We use the MAGMA package to compute the Schur multipliers of the relevant automorphism groups of $PSL_2(9)$ and $PSL_3(4)$. I would like to thank Derek Holt for suggesting a set of generators for $PSL_2(4).2^2$, which we give in Appendix B. $PSL_2(9).2^2$ may be obtained as $P\Gamma L_2(9)$.

$\overline{G}$	$M(\overline{G})$	$\overline{G}$	$M(\overline{G})$
$PSL_2(9)$	C_6	$PSL_3(4)$	$C_4 \times C_4 \times C_3$
$PSL_2(9).2_1$	C_2	$PSL_3(4).2_1$	$C_2 \times C_2 \times C_3$
$PSL_2(9).2_2$	C_2	$PSL_3(4).2_2$	C_4
$PSL_2(9).2_3$	C_3	$PSL_3(4).2_3$	C_4
$PSL_2(9).2^2$	$C_2 \times C_2$	$PSL_3(4).2^2$	$C_2 \times C_2$

Hence when G/N is noncyclic, we have $O_p(Z(G)) = 1$ and we are done. □

The following is essentially Lemma 3.5 of [7]. We leave the precise details of the proof to the reader since most of the methods have been used in Chapter 3.

Lemma 6.10 *Let B be a p-block of a finite group X with $Z(X)$ p-regular. Let V be a cyclic p'-group containing $Z(X)$ and let $A = X \star V$ be a central product with $V \cap X = Z(X)$. Suppose that B_1 is a p-block of A covering B. Then Conjecture 4.4 holds for B if and only if it holds for B_1.*

Proof Fix an integer d. The irreducible characters in B_1 all lie over a single $\mu \in Irr(V/Z(X))$ by [22, V.2.3]

Since $V \leq Z(A)$ and $(p, [A : X]) = 1$ we may identify $\mathcal{R}(X)/X$ and $\mathcal{R}(A)/A$, and $N_A(\sigma) = N_X(\sigma) \star V$ for each $\sigma \in \mathcal{R}(X)$, with $N_X(\sigma) \cap V = Z(X)$.

Fix $\sigma \in \mathcal{R}(X)$. Since $N_A(\sigma)/N_X(\sigma)$ is cyclic and each $\chi \in Irr(N_X(\sigma))$ is $N_A(\sigma)$-stable, by Clifford's Theorem we have a 1-1 correspondence

$$Irr_d(N_A(\sigma), \mu) \leftrightarrow Irr_d(N_X(\sigma)).$$

By Lemma 3.14 this takes V_σ-projective characters to V_σ-projective characters. It is clear from the proof of Proposition 3.10 that this correspondence is compatible with p-blocks and with the Brauer correspondence. Hence we have a 1-1 correspondence

$$Irr_d(N_A(\sigma), B_1) \leftrightarrow Irr_d(N_X(\sigma), B)$$

and $w_d(N_A(\sigma), B_1, V_\sigma) = w_d(N_X(\sigma), B, V_\sigma)$. The result follows. □

Suppose that G satisfies Hypotheses 6.3 and that $(p, [G : N]) = 1$, $O_p(Z(G)) = 1$. By [12] G/N is a product of two cyclic p'-groups. Let $N \lhd T \lhd G$, where T/N and G/T are cyclic. Write W for the central extension of $T/Z(G)$ by $T' \cap Z(G)$ such that $W \leq T$. We have $Z(L) \leq Z(W) \leq Z(G)$.

Lemma 6.11 *Let G be as above. Then there is a finite group H satisfying*

(i) $Z(H) \leq L \lhd H$,

(ii) $O_{p'}(H) = Z(H) \leq H'$, $Z(H)$ is cyclic and $O_p(H) = 1$,

(iii) $C_H(L) = Z(H) = Z(L)$,

(iv) H/L is a p'-group,

(v) Conjecture 4.4 holds for G if and only if it holds for H.

Proof This is based upon [7, 3.6].

Choose $u \in G$ such that $uWZ(G) = uT$ generates the cyclic p'-group $G/WZ(G) = G/T$ of order m. Hence $u^m \in WZ(G)$, say $u^m = wz$ where $w \in W$, $z \in Z(G)$.

Let V be a cyclic p'-group such that $Z(G) \leq V$, and containing an element $v \in V$ such that $v^m = z$. Write $A = G \star V$, where we identify $Z(G)$ with $G \cap V$.

Setting $h = uv^{-1} \in A$, define $H = \langle W, h \rangle \lhd A$. Now

$$\frac{A}{WV} = \frac{GV}{WV} \cong \frac{G}{G \cap WV} = \frac{G}{WZ(G)}$$

and hWV has order m, so hWV generates A/WV. Hence $A = H \star V$ with $Z(H)$ and $V \cap H$ identified.

Clearly $L \lhd W$. Since $h = uv^{-1}$ and $v \in Z(A)$, conjugation by h induces the same (outer) automorphism on L as does u. Hence

$$C_H(L) = C_W(L) = Z(W) = W \cap Z(G) = W \cap (G \cap V) = W \cap V = H \cap V = Z(H).$$

Now by Lemma 2.12 $Z(W) = Z(L)$, so $Z(H) = Z(L)$ as required. It is clear that H/L is a p'-group, $O_{p'}(H) = Z(H) = H \cap V$ is cyclic and $O_p(H) = 1$.

Hence the result follows from Lemma 6.10. □

To summarize, in order to prove Theorem 4.7, it suffices to assume Hypotheses 6.3 and $L = N$, i.e., G is an automorphism group of the quasisimple group N:

Proposition 6.12 *Suppose that G is as in Hypotheses 6.3 and $L = N$. Then one of the following possibilities for (p, N) occurs:*

(a) $(p, PSL_2(p^m))$, $(p, SL_2(p^m))$, $(p, PSU_3(p^m))$, $(p, SU_3(p^m))$, $m > 1$

(b) $(2, {}^2B_2(2^{2m+1}))$, $m \geq 1$, $(2, 2.\,{}^2B_2(2^3))$, $(2, 2.PSL_2(4))$.

(c) $(3, {}^2G_2(3^m))$, $m > 1$ *odd,* $(3, {}^2G_2(3)')$, $(3, PSL_3(4))$, $(3, 2.PSL_3(4))$, $(3, 3.PSL_3(4))$, $(3, 4_1.PSL_3(4))$, $(3, 4_2.PSL_3(4))$, $(3, 6.PSL_3(4))$, $(3, 12_1.PSL_3(4))$, $(3, 12_2.PSL_3(4))$, $(3, 3.PSL_2(9))$, $(3, 6.PSL_2(9))$ *or* $(3, M_{11})$.

(d) $(5, {}^2B_2(2^5))$, $(5, {}^2F_4(2)')$, $(5, McL)$ *or* $(5, 3.McL)$.

(e) $(11, J_4)$.

Further, $p \mid [G : N]$ only when $(p, N) \cong (3, {}^2G_2(3)')$ or $(5, {}^2B_2(2^5))$.

7 Checking the conjecture

7.1 Introduction

Over this chapter we verify the conjecture for the groups listed in Proposition 6.12.

Since some of the calculations have been performed in part or in full by Blau and Michler in [7], we give a summary of their results, and list the cases where Robinson's conjecture follows directly from their work.

We assume Hypotheses 6.3 and assume (as we may) that N is perfect. Let B be a p-block of G with defect groups the Sylow p-subgroups of G (otherwise the conjecture holds trivially).

7.2 Some results of Blau and Michler

Although we endeavour to cite any results of [7] used as fully as possible, we feel that due to the closeness of the two sets of calculations some lapse is inevitable and so make our apologies in advance.

Proposition 7.1 (Blau and Michler [7]) *Let G be a finite group with TI Sylow p-subgroups. Let $a = log_p(|G|_p)$ and let B be a p-block of G of defect a. Let $P \in Syl_p(G)$. Then the following are true:*

(a) $k(G, B) = k(N_G(P), B)$.

(b) $k_a(G, B) = k_a(N_G(P), B)$.

(c) $l(G, B) = l(N_G(P), B)$.

Corollary 7.2 *Let B be a p-block of a finite group G with TI Sylow p-subgroups. If B has abelian defect groups then B satisfies Conjecture 4.4.*

Proof By Ito's theorem and Proposition 7.1 $k_a(G, B) = k_a(N_G(P), B) = k(N_G(P), B) = k(G, B)$. $\square$

Much of [7] is devoted to the calculation of the fixed points of the action of outer automorphisms on conjugacy classes of groups of Lie type. We summarize one of the main results here.

Lemma 7.3 ([7]) *Let $N = {}^2G_2(3^{2m+1})$, ${}^2B_2(2^{2m+1})$ or $PSU_3(p^m)$, where $m > 0$. Then the (outer) field automorphisms of N consist of automorphisms of the form τ^r, where τ is the Frobenius automorphism $x \to x^p$. Suppose that $r|2m+1$ in the first two cases or $r|2m$ if $N = PSU_3(p^m)$. Then τ^r fixes precisely p^r p-regular conjugacy classes of N.*

In their proof of Proposition 7.1 Blau and Michler use some neat combinatorial results for dealing with the action of the outer automorphisms on the irreducible (ordinary or Brauer) characters of the group. We give these here for convenience, and make the trivial observation that the results can be extended to counting characters of any given height (given that the index is coprime to p).

Lemma 7.4 (Lemma 2.2 of [7]) *Let A be a finite group acting on two sets S_1 and S_2 as a permutation group. For $H \leq A$, write $f_{S_i}(H) = |\{s \in S_i | s^a = s \ \forall \ a \in H\}|$ and $m_{S_i}(H) = |\{s \in S_i | C_A(s) = H\}|$.*

If $f_{S_1}(H) = f_{S_2}(H)$ for all $H \leq A$ then $m_{S_1}(H) = m_{S_2}(H)$ for all $H \leq A$.

Proof This follows by an easy inductive argument on $[A : H]$, given in [7]. □

Lemma 7.5 (Lemma 2.3 of [7]) *Let $H \leq G \triangleleft E$, where $H \triangleleft E$ and E/G is cyclic and p-regular. Suppose further that for a generator $y \in E$ of E/G we have $[E : G] = [H\langle y \rangle : H]$. Let B be a p-block of G and b a p-block of H, with both B and b are stable under the action of y. Let d be an integer. Then if (in the notation of the previous lemma) $f_{Irr_d(G,B)}(C) = f_{Irr_d(H,b)}(C)$ for each subgroup C of $\langle y \rangle$, then $k_d(E, B_1) = k_d(H\langle y \rangle, b_1)$, where B_1, b_1 is the collection of p-blocks of E, $H\langle y \rangle$ covering B, b respectively.*

The analagous result holds when we consider Brauer characters instead of ordinary characters.

Proof This is a standard application of Clifford's theorem using Lemma 7.4, and is given in [7]. □

7.3 Sporadic simple groups of p-local rank one

Proposition 7.6 *Conjecture 4.4 holds for G when* $(p, \overline{N}) \cong (3, M_{11})$, $(5, McL)$, *and* $(11, J_4)$.

Proof If $(p, \overline{N}) \cong (3, M_{11})$, then $G = \overline{N}$ since $M(M_{11}) = Out(M_{11}) = 1$. M_{11} has abelian Sylow 3-subgroups , so Conjecture 4.4 holds in this case by Corollary 7.2.

If $(p, \overline{N}) \cong (11, J_4)$, then again we have $G = \overline{N}$. By Ostermann [41] the principal 11-block B_0 is the only 11-block of J_4 of positive defect. Let $P \in Syl_p(G)$, so $|P| = 11^3$. By Ostermann's character table for $N_G(P)$ ([41, pp69]) $k_3(N_G(P), B_0) = 42$, $k_2(N_G(P), B_0) = 7$ and $k(N_G(P), B_0) = 49$. By [12, pp.188] $k_3(G, B_0) = 42$, $k_2(G, B_0) = 7$ and $k(G, B_0) = 49$. So Conjecture 4.4 holds in this case.

Now suppose that $(p, \overline{N}) \cong (5, McL)$. The conjecture has been verified independently in this case by Murray [36]. By [12] the outer automorphisms of McL do not act trivially on the centre of $3.McL$, so by Remark 6.4 we do not have to consider the group $3.McL.2$. Hence $G \cong McL$, $3.McL$ or $McL.2$. First suppose that $G \cong McL$. By [41] McL has just one p-block B_0 of positive defect. It follows from the character tables given in [41] and [12] that for $P \in Syl_5(G)$ we have $k_3(G, B_0) = 13 = k_3(N_G(P), B_0)$ and $k_2(G, B_0) = 6 = k_2(N_G(P), B_0)$ (this accounting for all the irreducible characters of B_0 and its Brauer correspondent in $N_G(P)$).

Suppose that $G \cong 3.McL$. Then $N_G(P)$ has the form $3.5_+^{1+2} : 3 : 8$, where 5_+^{1+2} is the non-abelian group of order 5^3 and exponent 25. Now $C_G(P) \cong 3.5_+^{1+2}$, so G has three 3-blocks B_0, B_1, B_2 of positive defect, each lying over a distinct $\lambda \in Irr(Z(G))$, B_0 the principal 3-block. By [12, p.101] $k_3(G, B_i) = k_3(G, B_0) = 13$, $k_2(G, B_i) = k_2(G, B_0) = 6$ and $k(G, B_i) = k(G, B_0) = 19$ for each i. By Lemma 2.10 and Lemma

2.14 we have $M(N_G(P)/Z(G)) = 1$, so each $\lambda \in Irr(Z(G))$ extends to an irreducible character of $N_G(P)$, and $irr(N_G(P), B_i)$ is the same for each i. Since $irr(N_G(P), B_0) = irr(N_G(P)/Z(G), B_0)$ and $irr(G, B_0) = irr(G/Z(G), B_0)$, it follows that Conjecture 4.4 holds each each B_i.

It remains to demonstrate the conjecture for $G \cong McL.2$. This has just one 5-block B_0 of positive defect. The character degrees of $N_G(P)$ are given in full in [7] ($irr(N_G(P), B_0) = \{8 \times 1, 10 \times 2, 4 \times 20, 2 \times 24, 2 \times 40\}$). Comparing these to the character degrees given for G in [12] gives the required result. □

7.4 Groups of Lie type of p-local rank one

Proposition 7.7 *G satisfies Conjecture 4.4 when* $(p, \overline{N}) \cong (3, {}^2G_2(3)')$ *or* $(5, {}^2B_2(2^5))$.

Proof If $(p, \overline{N}) \cong (5, {}^2B_2(2^5)$, then $G \cong {}^2B_2(2^5)$ or ${}^2B_2(2^5).5$. Now ${}^2B_2(2^5)$ has cyclic Sylow 5-subgroups, so the conjecture holds when $G \cong {}^2B_2(2^5)$ by Proposition 5.9. Let $G \equiv {}^2B_2(2^5).5$ and $P \in Syl_5(G)$. By [12, p.77] $N_G(P)$ has the form $C_{25} : C_{20}$ and C_{20} acts on C_{25} with orbits of length 1, 4 and 20. Hence by Brauer's theorem ([14, 11.9]) C_{20} acts on $Irr(C_{25})$ with three orbits, which by counting must have lengths 1, 4 and 20. Hence by Clifford's Theorem $irr(N_G(P)) = \{20 \times 1, 5 \times 4, 1 \times 20\}$. We have $C_G(P) \leq P$, so G has only one 5-block B_0 of positive defect. By the character table in [12] we get $k_3(G, B_0) = 25$, $k_2(G, B_0) = 1$ and $k(G, B_0) = 26$, so Conjecture 4.4 holds for G in this case.

If $(p, \overline{N}) \cong (3, {}^2G_2(3)')$, then $G \cong {}^2G_2(3)'$ or ${}^2G_2(3)$. Now ${}^2G_2(3)'$ has cyclic Sylow 3-subgroups, so the 3-blocks of ${}^2G_2(3)'$ satisfy Conjecture 4.4. Let $G \equiv {}^2G_2(3)$ and $P \in Syl_3(G)$. By [12, p.6] $N_G(P)$ has the form $C_9 : C_6$ and C_6 acts on C_9 with orbits of length 1, 2 and 6. Again by Brauer's theorem C_6 acts on $Irr(C_9)$ with three orbits, which by counting must have lengths 1, 2 and 6. Hence by Clifford's Theorem $irr(N_G(P)) = \{6 \times 1, 3 \times 2, 1 \times 6\}$. We have $C_G(P) \leq P$, so G has only one 3-block B_0

with defect group P. By the character table in [12] we get $k_3(G, B_0) = 9$, $k_2(G, B_0) = 1$ and $k(G, B_0) = 10$, so Conjecture 4.4 holds for G in this case. $\square$

Proposition 7.8 *Conjecture 4.4 holds for G when* $(p, \overline{N}) \cong (2, {}^2B_2(2^{2m+1}))$, $m \geq 1$.

Proof In this case $G \cong {}^2B_2(2^{2m+1}).C_t$, where $t|2m+1$, or $G \cong 2.\ {}^2B_2(2^3)$ (there is no finite group of the form $2.\ {}^2B_2(2^3).3$). Write $q = 2^{2m+1}$. The first part of the proof is [7, 6.3].

First suppose that $G \cong {}^2B_2(2^{2m+1}).C_t$, and let $P \in Syl_2(G)$. By [54, p.107] and since the outer automorphisms are field automorphisms, $C_G(P) \leq P$ and so G has only one 2-block B_0 of positive defect. By [54, pp.126,142], $irr(N_N(P)) = \{q-1 \times 1, 1 \times q-1, 2 \times 2^m(q-1)\}$, and by [54, p.141] $irr(N) = \{1 \times 1, 1 \times q^2, q/2-1 \times q^2+1, (q+2^{m+1})/4 \times (q-2^{m+1}+1)(q-1), (q-2^{m+1})/4 \times (q+2^{m+1}+1)(q-1), 2 \times 2^m(q-1)\}$ (the only 2-block of defect zero being that consisting of the Steinberg character).

Write $\tau : x \to x^2$ for the Frobenius map, and let $r|2m+1$ so that τ^r generates a subgroup of C_t. Note that every 2-regular outer automorphism of N can be described in such a way. By Lemma 7.5 it suffices to show that $f_{Irr_d(N,B_0)}(\langle\tau^r\rangle) = f_{Irr_d(N_N(P),B_0)}(\langle\tau^r\rangle)$ for each choice of τ^r.

By Lemma 7.3 τ^r fixes 2^r 2-regular conjugacy classes of N. By examination of the character table τ^r fixes the 2-singular conjugacy classes of N. Hence τ^r fixes 2^r+3 conjugacy classes of N, and so 2^r+3 irreducible characters (including the Steinberg character) by Brauer's theorem. Hence $f_{Irr_{4m+2}(N,B_0)}(\langle\tau^r\rangle) = 2^r$, since τ^r must fix the three irreducible characters of N of even degree.

Now $N_N(P) = PW$ where $W \cong \mathbb{F}_q^\times$. Hence τ^r fixes 2^r-1 of the $q-1$ 2-regular conjugacy classes of $N_N(P)$, and so 2^r-1 Brauer characters of $N_N(P)$. But τ^r fixes 2^r-1 Brauer characters of N (excluding the Steinberg character), so $f_{IBr(N,B_0)}(\langle\tau^r\rangle) = f_{IBr(N_N(P),B_0)}(\langle\tau^r\rangle)$ and by Lemma 7.5 $l(G, B_0) = l(N_G(P), B_0)$. Hence by Lemma 2.9 $k(G, B_0) = k(N_G(P), B_0)$. Since all but one of the irreducible characters of height one of $N_N(P)$ are linear, it follows that $f_{Irr_{4m+2}(N_N(P),B_0)}(\langle\tau^r\rangle) = 2^r$. So by Lemma 7.5 $k_{4m+2}(G,B_0) = k_{4m+2}(N_G(P),B_0)$. The result follows in this case since all positive height charaters of N, $N_N(P)$ have the same defect.

Finally suppose that $G \cong 2.\ {}^2B_2(2^3)$. Again we have just one 2-block B_0 of positive defect. Let $Irr(Z(G)) = \{1,\lambda\}$ and $P \in Syl_2(G)$. We have already shown that Conjecture 4.4 holds for B_0 and the character $1 \in Irr(Z(G))$. It remains to verify that $k_d(G,B_0,\lambda) = k_d(N_G(P),B_0,\lambda)$ for each d. By [12, p.28] $N_G(P)$ has the structure $2.2^{3+3}:7$. Using the MAGMA package we may construct the character tables of P and $P/Z(G)$ (see Appendix B for details), to obtain $irr(P,\lambda) = \{8\}$. By Clifford's theorem this gives $irr(N_G(P),B_0,\lambda) = \{7\times 8\}$. By [12] $k_4(G,B_0,\lambda) = 7 = k(G,B_0,\lambda)$ and we are done. □

Proposition 7.9 *G satisfies Conjecture 4.4 when $(p,\overline{N}) \cong (2,PSL_2(4))$, $(2,PSL_2(9))$ and $(3,PSL_3(4))$.*

Proof In each of these cases $P \in Syl_p(G)$ has order p^3 and $|O_p(Z(G))| = p$, so that P is extraspecial. Let B be a p-block of G with P as a defect group. Write $\overline{G} = G/O_p(Z(G))$ and let $\overline{B}$ be the block of $\overline{G}$ corresponding to B. Since $P/O_p(Z(G))$ is abelian, Conjecture 4.4 holds for $\overline{B}$ by Corollary 7.2. Hence by Lemma 2.9, $k(G,B,\lambda) = k(N_G(P),B,\lambda)$ for each $\lambda \in Irr(O_p(Z(G)))$. Now

$$irr(P,\lambda) = \begin{cases} \{p^2\times 1\} & if\ \lambda = 1 \\ \{p\} & if\ \lambda \neq 1 \end{cases}$$

so by Clifford's theorem $k(N_G(P),\lambda) = k_3(N_G(P),\lambda)$ if $\lambda = 1$ and $k(N_G(P),\lambda) = k_2(N_G(P),\lambda)$ if $\lambda \neq 1$.

From the character tables at [12, pp.2,4,24] we see that $k(G,B,\lambda) = k_3(G,B,\lambda)$ if $\lambda = 1$ and $k(G,B,\lambda) = k_2(G,B,\lambda)$ if $\lambda \neq 1$. Hence $k_3(G,B,\lambda) = k_3(N_G(P),B,\lambda)$ and $k_2(G,B,\lambda) = k_2(N_G(P),B,\lambda)$ for each $\lambda \in Irr(O_p(Z(G)))$ as required. □

Proposition 7.10 *G satisfies Conjecture 4.4 when $(p, \overline{N}) \cong (p, PSL_2(p^m))$ or $(5, {}^2F_4(2)')$, $m > 1$.*

Proof In each case G has abelian Sylow p-subgroups and the result follows by Lemma 7.2. $\square$

7.5 Ree groups of type G_2

Our aim is to prove the following:

Proposition 7.11 *Every 3-block of G satisfies Conjecture 4.4 when $(p, \overline{N}) \cong (3, {}^2G_2(3^{2m+1}))$, $m > 0$.*

Note that $\overline{N} = N$. Blau and Michler use Ward [57] in their proof [7] of Proposition 7.1. However, since 3-blocks (of positive defect) of N contain irreducible characters of three distinct heights we need more exact information on the character degrees. To do this we return to Ree's original description of N as a twisted Chevalley group.

We follow Carter [10] in our desciption of $N = {}^2G_2(q)$, where $q = 3^{2m+1}$, $m > 0$.

Lemma 7.12 *Let $P \in Syl_3(N)$. Consider the automorphism of $\mathbb{F}_q$ defined by $\lambda \mapsto \lambda^f$, where $f = 3^m$.*

P has order q^3, and we may label its elements as $x(t, u, v)$, where $t, u, v \in \mathbb{F}_q$, with multiplication given by

$$x(t_1, u_1, v_1)x(t_2, u_2, v_2) = x(t_1 + t_2, u_1 + u_2 - t_1 t_2^{3f}, v_1 + v_2 - t_2 u_1 + t_1 t_2^{3f+1} - t_1^2 t_2^{3f}).$$

$Z(P) = \{x(0, 0, v) \mid v \in \mathbb{F}_q\}$ and $|Z(P)| = q$. The subgroup

$$P_1 = \{x(0, u, v) \in P \mid u, v \in \mathbb{F}_q\}$$

of order q^2 is normal in P, P/P_1 is elementary abelian and $P_1 = P'$.

Proof The description of the elements and their multiplication follows from [10]. This gives $x(t,u,v)^{-1} = x(-t, -u-t^{3f+1}, -v-tu+t^{3f+2})\ \forall\ t,u,v \in \mathbb{F}_q$, so we see that conjugation in P is given by

$$x(t_1,u_1,v_1)x(t_2,u_2,v_2)x(t_1,u_1,v_1)^{-1}$$

$$= x(t_2, u_2 - t_1t_2^{3f} + t_2t_1^{3f}, v_2 - t_2u_1 + t_1u_2 + t_1t_2^{3f+1} + t_1^{3f+1}t_2 + t_1^2t_2^{3f} + t_1^{3f}t_2^2).$$

Suppose that $x(t_2,u_2,v_2) \in Z(P)$. Then

$$-t_1t_2^{3f} + t_2t_1^{3f} = 0,$$

$$-t_2u_1 + t_1u_2 + t_1t_2^{3f+1} + t_1^{3f+1}t_2 + t_1^2t_2^{3f} + t_1^{3f}t_2^2 = 0$$

$\forall\ x(t_1,u_1,v_1) \in P$, so $t_2 = u_2 = 0$. Hence

$$Z(P) = \{x(0,0,v) \mid v \in \mathbb{F}_q\}$$

and $|Z(P)| = q$ as required. The last part is clear. □

Lemma 7.13 *P has q conjugacy classes of length 1, $q-1$ of length q and $3(q-1)$ of length $\frac{1}{3}q^2$.*

Proof

$$x(t_1,u_1,v_1)x(0,u_2,0)x(t_1,u_1,v_1)^{-1} = x(0,u_2,t_1u_2),$$

so for each $u_2 \in \mathbb{F}_q^\times$, $x(0,u_2,0)$ is conjugate to all elements $x(0,u_2,v) \in P$ where $v \in \mathbb{F}_q$. In this way we obtain $q-1$ conjugacy classes of length q, each corresponding to a distinct value of u_2 in $\mathbb{F}_q^\times$. It is clear that we have q conjugacy classes of length one, corresponding to the elements of $Z(P)$. We have thus accounted for every element of P_1.

Suppose now that $x(t_2,u_2,v_2) \in P$ with $t_2 \neq 0$. If $x(t_1,u_1,v_1) \in C_P(x(t_2,u_2,v_2))$, then $t_2t_1^{3f} = t_1t_2^{3f}$. Let $\phi : \mathbb{F}_q \to \mathbb{F}_q$ be the additive endomorphism sending λ to

$t_2\lambda^{3f} - \lambda t_2^{3f}$. If $0 \neq \lambda \in ker(\phi)$, then $\lambda^{3f} = \lambda t_2^{3f-1}$. Without loss of generality, assume $t_2^{3f-1} = 1$, so $\lambda^{3f} = \lambda$. Hence

$$\lambda^{3^{m+1}-1} = 1 = \lambda^{q-1} = \lambda^{3^m(3^{m+1}-1)+3^m-1}$$

and

$$1 = \lambda^{q-1} = \lambda^{3(3^m-1)(3^m+1)+2} = \lambda^2,$$

so $|ker(\phi)| = 3$. This gives $|C_P(x(t_2, u_2, v_2))| = 3q$ since we can choose u_1 to give t_2u_1 any value in $\mathbb{F}_q$. Hence we have found $3(q-1)$ conjugacy classes of length $\frac{1}{3}q^2$. This accounts for all the elements of P. □

We use the action of P on $Irr(P_1)$ to calculate the nature and degrees of the irreducible characters of P.

Lemma 7.14 $irr(P) = \{q \times 1, (q-1) \times q, 3(q-1) \times 3^m\}$.

Proof Considering $\mathbb{F}_q$ as an $n = (2m+1)$-dimensional vector space over $\mathbb{F}_3$, write $\zeta \in \mathbb{F}_q$ as $\zeta = (\zeta_1, \ldots, \zeta_n)$. Write the elements of $Irr(P_1)$ as $\chi_{x,y}$, $x, y \in \mathbb{F}_q$, where, for $x(0, u, v) \in P_1$,

$$\chi_{x,y}(x(0,u,v)) = \prod_{k=1}^{n} e^{\frac{2u_k x_k \pi i}{3}} \prod_{j=1}^{n} e^{\frac{2v_j y_j \pi i}{3}}.$$

Recall that

$$x(t_1, u_1, v_1)x(0, u_2, 0)x(t_1, u_1, v_1)^{-1} = x(0, u_2, t_1u_2)\ \forall\ t_1, u_1, v_1, u_2 \in \mathbb{F}_q.$$

From this we see that for any $x \in \mathbb{F}_q$, we have $I_P(\chi_{x,0}) = P$ and that for any $x \in \mathbb{F}_q$, $y \in \mathbb{F}_q^{\times}$, we have $I_P(\chi_{x,y}) = P_1$. This gives us q orbits of length one and $q-1$ orbits of length q. Representatives of the long orbits are $\chi_{0,y}$, $y \in \mathbb{F}_q^{\times}$.

By Clifford theory representatives of the long orbits induce to give distinct irreducible characters of P of degree q. $\chi_{0,0} = 1_{P_1}$ extends to P in q distinct ways to give q linear characters of q. Since P_1 is the derived subgroup of P, every linear character of P is an extension of $\chi_{0,0}$.

Now consider the characters $\chi_{x,0}$, where $x \in \mathbb{F}_q^\times$. Although these characters are stable under the action of P, they do not extend to irreducible characters of P. We show that all irreducible characters of P covering such characters of P_1 have the same degree.

Fix $\mu = \chi_{x,0}$, $x \in \mathbb{F}_q^\times$. Since $I_P(\chi_{x,0}) = P$, by Lemma 3.6 there is a degree preserving 1-1 correspondence $Irr(P, \mu) \leftrightarrow Irr(\widehat{(P/P_1)}, \widehat{\mu})$, where $\widehat{(P/P_1)}$ is a central extension of P/P_1 and $\widehat{\mu} \in Irr(Z(\widehat{(P/P_1)}))$.

Now by Lemma 3.6 we may take $Z(\widehat{(P/P_1)})$ to be a cyclic p-group. However, by Lemma 2.14 the Schur multiplier of an elementary abelian group is itself elementary abelian and so we have $|Z(\widehat{(P/P_1)})| = 3$. Note that $\widehat{\mu}$ is non-trivial since we may not extend μ to P. This demonstrates that $irr(P, \chi_{x,0})$ is independent of the choice of $x \in \mathbb{F}_q^\times$.

Since P possesses $5q-4$ conjugacy classes, we must have $|irr(P, \chi_{x,0})| = 3$, and all irreducible characters of P lying over $\chi_{x,0}$ have the same degree, which must then be 3^m, as required. $\square$

We examine the action of $N_N(P)$ on the irreducible characters of P to calculate the irreducible character degrees of $N_N(P)$. It is interesting to note that by using Ward's character table for G (see [57]) and the fact that the non-blockwise version of the Knörr-Robinson reformulation of Alperin's weight conjecture is known to hold for finite groups with split BN-pairs in the defining characteristic, we may already observe that $N_N(P)$ possesses $q+7$ conjugacy classes.

Lemma 7.15 *$N_N(P)$ has irreducible characters of degrees $1, q-1, 3^m(q-1)/2, 3^m(q-1), q(q-1)$ with multiplicity $q-1, 1, 4, 2, 1$ respectively.*

Proof As noted at [57, p.63], $N_N(P) = PW$, where W is cyclic of order $q-1$ and acts fixed-point freely on P/P_1. Hence W acts transitively on the $q-1$ non-trivial linear characters of P, giving $q-1$ linear characters of $N_N(P)$ and one irreducible character of degree $q-1$. The irreducible characters of degree q of P are also permuted transitively by W, giving one irreducible character of degree $q(q-1)$. Since $k(N_N(P)) = q+7$ we are left with $k_{5m+3}(N_N(P)) = 6$. By Ree [44], W possesses an unique involution, h_0, and $C_P(h_0) = \{x(0,u,0)|u \in \mathbb{F}_q\}$. W acts transitively on $Z(P)$ and P/P_1, and $W/ < h_0 >$ acts on $P_1/Z(P)$ with 2 orbits of length $(q-1)/2$ and one of length one. Recall that each $\chi_{x,0}$ with $x \in \mathbb{F}_q^\times$ is covered by three irreducible characters of P. It follows that the irreducible characters of $N_N(P)$ of height 3^m consist of two of degree $3^m(q-1)$ and four of degree $3^m(q-1)/2$. □

Proof of Proposition 7.11

By Ward [57, p.63], $C_N(P) \leq P$. Since the outer automorphisms of N are field automorphisms, $C_G(P) \leq P$. So G possesses only one 3-block B_0 of positive defect.

Ward [57, p.85] gives $k(N) = q+8$ and from the character table at [57, p.87], we have $k_{6m+3}(N, B_0) = q$, $k_{5m+3}(N, B_0) = 6$ and $k_{4m+2}(N, B_0) = 1$

Let τ be the Frobenius map as in Lemma 7.3 and let r be an integer dividing $2m+1$ such that τ^r generates a subgroup G/N. We calculate the number of characters of N, $N_N(P)$ of each height fixed by τ^r. Note that we are *not* assuming that τ^r is 3-regular (this will be important in Chapter 9). By Lemma 7.3 τ^r fixes 3^r 3-regular conjugacy classes of N. Examination of the character table given in [57] reveals that τ^r fixes every 3-singular conjugacy class of N, hence τ^r fixes 3^r+8 conjugacy classes in total. So $f_{Irr(N,B_0)}(\langle\tau^r\rangle) = 3^r+7$ by Brauer's theorem. Further examination of the character table and using the fact that τ^r has odd order reveals that τ fixes every irreducible character of positive height, and so 3^r irreducible characters of height zero (excluding the Steinberg character), i.e., $f_{Irr_{6m+3}(N,B_0)}(\langle\tau^r\rangle) = 3^r$, $f_{Irr_{5m+3}(N,B_0)}(\langle\tau^r\rangle) = 6$ and $f_{Irr_{4m+2}(N,B_0)}(\langle\tau^r\rangle) = 1$.

Now consider the action of τ^r on $Irr(N_N(P))$. The 3-regular conjugacy classes of $N_N(P) = PW$ are represented by the elements of W, and so τ^r fixes 3^r-1 3-regular conjugacy classes of $N_N(P)$. So $f_{IBr(N_N(P),B_0)}(\langle\tau^r\rangle) = 3^r - 1$ and τ^r fixes $3^r - 1$ linear characters of $N_G(P)$. Clearly τ^r fixes the unique irreducible character of degree $q-1$, so $f_{Irr_{6m+3}(N_N(P),B_0)}(\langle\tau^r\rangle) = 3^r$.

It remains to examine the action on the characters of $N_N(P)$ of positive height. Clearly the unique irreducible character of degree $q(q-1)$ is fixed by τ^r, as are the two of degree $3^m(q-1)$ since τ^r has odd order. Finally it is clear from the construction of the remaining characters that τ^r cannot act on the four irreducible characters of degree $3^m(q-1)/2$ with an orbit of length three, and so τ^r fixes these also. The result then follows from Lemma 7.5. □

7.6 The unitary groups of degree 3 in the defining characteristic

Our aim is to prove the following:

Proposition 7.16 *Every p-block of G satisfies Conjecture 4.4 when* $\overline{N} \cong PSU_3(p^m)$, $m > 1$.

Our approach is similar to that used in verifying the conjecture for the Ree groups in defining characteristic. Again we use Carter [10] to view N as a twisted Chevalley group and so calculate the irreducible character degrees for the normalizer for a Sylow p-subgroup of N, and compare the numbers obtained with those from the character tables for N found in Simpson and Frame [52].

Let $q = p^m$, $m > 1$.

Lemma 7.17 *Let* $P \in Syl_p(N)$. *Consider the automorphism of* $\mathbb{F}_{q^2}$ *defined by* $\lambda \mapsto \lambda^f$, *where* $f = q$.

P has order q^3, *and we may label its elements as* $x(t,u)$, *where* $t, u \in \mathbb{F}_{q^2}$ *and* $u + u^f = t^{f+1}$, *with multiplication given by*

$$x(t_1,u_1)x(t_2,u_2) = x(t_1+t_2, u_1+u_2 - t_1^f t_2).$$

$Z(P) = \{x(0,u) \mid u^f = -u\}$ *and* $|Z(P)| = q$.

Proof Note that $N \cong {}^2A_2(q^2)$, and construct G as a twisted $A_2(q^2)$ following Carter [10].

The characterization and multiplaction of the elements of P follow from [10], so that

$$x(t,u)^{-1} = x(-t, -u - t^{f+1}) \ \forall \ x(t,u) \in P,$$

and conjugation in P is given by

$$x(t_1,u_1)x(t_2,u_2)x(t_1,u_1)^{-1} = x(t_2, u_2 + t_2^f t_1 - t_1^f t_2).$$

Denote by $\phi : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ the additive homomorphism given by

$$\phi(\lambda) = \lambda + \lambda^f.$$

$|ker(\phi)| = q$, so $|P| = q^3$.

Suppose that $x(t_2,u_2) \in Z(P)$. Then $t_2^f t_1 = t_1^f t_2$ for all $t_1 \in \mathbb{F}_{q^2}$. So $t_2 = 0$ and $u_2 \in ker(\phi)$. Hence $|Z(P)| = q$. □

Lemma 7.18 *P has q conjugacy classes of length one and $q^2 - 1$ of length q.*

Proof In Lemma 7.17 we saw that $Z(P)$ has order q and consists of elements of the form $x(0,u)$, where $u^f = -u$. This gives us the q conjugacy classes of length one.

Now suppose that $x(t_2,u_2) \in P$, $t_2 \neq 0$. Suppose that $x(t_1,u_1) \in C_P(x(t_2,u_2))$. Then $t_2^f t_1 = t_1^f t_2$. Assume that $t_1 \neq 0$, then we want $t_1^{f-1} = t_2^{f-1}$. This has $q-1$ solutions for each t_2, and so t_1 can take q values in $\mathbb{F}_{q^2}$. So $|C_P(x(t_2,u_2))| = q^2$, and we have found the remaining $q^2 - 1$ conjugacy classes. □

We use the action of P on $Irr(Z(P))$ to calculate the degrees of the irreducible characters of P.

Lemma 7.19 $irr(P) = \{q^2 \times 1, (q-1) \times q\}$

Proof Consider $\mathbb{F}_{q^2}$ as an $n = 2m$-dimensional vector space over $\mathbb{F}_p$, and write $\zeta \in \mathbb{F}_{q^2}$ as $\zeta = (\zeta_1, \ldots, \zeta_n)$. Denote the elements of $Irr(Z(P))$ as χ_y, $y \in \mathbb{F}_{q^2}$, where, for $x(0,u) \in Z(P)$ we have

$$\chi_y(x(0,u)) = \prod_{k=1}^{n} e^{\frac{2u_k y_k \pi i}{p}}.$$

Each χ_y is stable under the action of P.

Note that for all $x(t_1, u_1)$, $x(t_2, u_2) \in P$, we have

$$[x(t_1, u_1), x(t_2, u_2)] = x(0, t_2^f t_1 - t_1^f t_2),$$

so $P' = Z(P)$. Hence P has q^2 linear characters, namely those which are extensions of χ_0.

Fix $\mu = \chi_y$, $y \in \mathbb{F}_{q^2}^{\times}$. Since $I_P(\chi_y) = P$, by Lemma 3.6 there is a degree-preserving 1-1 correspondence $Irr(P, \mu) \leftrightarrow Irr((\widehat{P/Z(P)}), \hat{\mu})$, where $(\widehat{P/Z(P)})$ is a central extension of $P/Z(P)$, $\hat{\mu} \in Irr(Z((\widehat{P/Z(P)})))$ and $Z((\widehat{P/Z(P)}))$ is a cyclic p-group. As in Lemma 7.14, $P/Z(P)$ is elementary abelian and so has elementary abelian Schur multiplier by Lemma 2.14. Hence $|Z((\widehat{P/Z(P)}))| = p$. Again since we cannot extend μ to an irreducible character of P, we have $\hat{\mu} \neq 1$ and so $irr(P, \chi_y)$ is independent of the choice of $y \in \mathbb{F}_{q^2}^{\times}$. Hence all the non-linear irreducible characters of P have the same degree and the result follows. □

Let $d = (3, q+1)$. If $N = PSU_3(q)$ then by [52, p.487] $C_N(P) \leq P$, and N has just one p-block of positive defect. $Out(N)$ consists of diagonal and field automorphisms. By [7, p.436] and since a nontrivial field automorphism cannot centralize P, $C_G(P) \leq P$ and G has just one p-block of positive defect. If $N = SU_3(q)$, then G has d p-blocks of positive defect since $|Z(G)| = d$, which is coprime to p. Each p-block of positive defect consists only of irreducible characters lying over a distinct irreducible character of $Z(G)$. So the principal p-block of $G/Z(G)$ possesses precisely the same irreducible characters as the principal p-block of G (after factoring out the kernel of course), and similarly for $N_{G/Z(G)}(Z(G)P/Z(G))$ and $N_G(P)$. Hence it suffices in proving Proposition 7.16 to assume that $N = SU_3(q)$, which we do.

Let B_λ be the p-block of G containing only irreducible characters lying over $\lambda \in Irr(Z(G))$.

Lemma 7.20 *Each p-block of $N_N(P)$ possesses irreducible characters of degrees $1, (q^2-1)/d, q(q-1)$ with multiplicities $(q^2-1)/d, d, (q+1)/d$ respectively.*

Proof Write $\overline{N} = N/Z(N)$. By [28, II.10.12], $N_{\overline{N}}(\overline{P}) = \overline{N_N(P)} = W\overline{P}$, where W is cyclic of order $(q^2-1)/d$. Also by [28, II.10.12], W acts on the linear characters of $\overline{P}$ with one orbit of length one and d orbits of length $(q^2-1)/d$, and transitively on the remaining $q-1$ irreducible characters. Hence by Clifford theory $irr(\overline{N_N(P)}) = \{(q^2-1)/d \times 1, d \times (q^2-1)/d, (q+1)/d \times q(q-1)\}$. By Lemmas 2.13, 2.10 and 2.14 $M(N_N(P)) = 1$, so each irreducible character of $Z(N)$ extends uniquely to an irreducible character of $N_N(P)$ and the result follows. □

Blau and Michler in [7] give a full account of the action of the diagonal and field automorphisms on the conjugacy classes of N and $N_N(P)$. Let δ be an outer diagonal automorphism such that $\langle \delta N \rangle$ has order d. Write $E =< \delta > N$ and note that δ stabilizes P.

Using Lemma 7.20, [7, 4.5] and the character tables given in [52], we see that all irreducible characters have defects $3m$, $2m$ or zero, and

$$k_{3m}(N_N(P), B_\lambda) = k_{3m}(N, B_\lambda) = (q^2-1)/d + d,$$

$$k_{2m}(N_N(P), B_\lambda) = k_{2m}(N, B_\lambda) = (q+1)/d,$$

$$k_{3m}(N_E(P), B_\lambda) = k_{3m}(E, B_\lambda) = q^2,$$

$$k_{2m}(N_E(P), B_\lambda) = k_{2m}(E, B_\lambda) = q+1$$

for each $\lambda \in Irr(Z(G))$.

Let τ^r be a field automorphism as in Lemma 7.3 generating a subgroup of G/E, so $r|2m$. By [7, 4.5],

$$
\begin{aligned}
f_{Irr_{3m}(N_N(P),B_\lambda)}(\langle\tau^r\rangle) &= f_{Irr_{3m}(N,B_\lambda)}(\langle\tau^r\rangle) \\
&= \begin{cases} p^m & if\ r|m \\ p^m+2 & if\ r \not| m\ and\ 3|(2m/r) \\ (p^m-1)/3+3 & if\ r \not| m\ and\ 3 \not|(2m/r) \end{cases},
\end{aligned}
$$

$$
\begin{aligned}
f_{Irr_{2m}(N_N(P),B_\lambda)}(\langle\tau^r\rangle) &= f_{Irr_{2m}(N,B_\lambda)}(\langle\tau^r\rangle) \\
&= \begin{cases} (2,p-1) & if\ r|m \\ p^{r/2}+1 & if\ r \not| m\ and\ 3|(2m/r) \\ (p^{r/2}+1)/3 & if\ r \not| m\ and\ 3 \not|(2m/r) \end{cases},
\end{aligned}
$$

$$
f_{Irr_{3m}(N_E(P),B_\lambda)}(\langle\tau^r\rangle) = f_{Irr_{3m}(E,B_\lambda)}(\langle\tau^r\rangle) = p^m
$$

and

$$
\begin{aligned}
f_{Irr_{2m}(N_E(P),B_\lambda)}(\langle\tau^r\rangle) &= f_{Irr_{2m}(E,B_\lambda)}(\langle\tau^r\rangle) \\
&= \begin{cases} (2,p-1) & if\ r|m \\ p^{r/2}+1 & if\ r \not| m \end{cases}.
\end{aligned}
$$

The result follows by Lemma 7.5. □

8 Proof of Theorem 4.7

Theorem 4.7 *Conjecture 4.4 holds for every finite group of p-local rank one.*

Proof By Proposition 4.6 it suffices to verify the conjecture for p-blocks B of finite groups G with $O_p(G) \leq Z(G)$, $O_p(G) \leq G'$, $O_p(G)$ cyclic and $plr(G) \leq 1$. If $plr(G) = 0$, then Conjecture 4.4 states that $k_d(G, B, \lambda) = w_d(G, B, \lambda, O_p(G))$ for each integer d and each $\lambda \in Irr(O_p(Z(G)))$. But in this case $O_p(G) \in Syl_p(G)$, so the conjecture holds trivially. Hence we may assume that G has p-local rank one, and by Lemma 2.5 G has almost TI Sylow p-subgroups. In this case Conjecture 4.4 states that $k_d(G, B, \lambda) = w_d(G, B, \lambda, O_p(G)) - w_d(N_G(P), B, \lambda, O_p(G)) + w_d(N_G(P), B, \lambda, P)$, where $P \in Syl_p(G)$. If B has defect group $O_p(G)$, then the conjecture predicts that $k_d(G, B, \lambda) = w_d(G, B, \lambda, O_p(G))$, which holds trivially. Hence we may assume that B is a p-block of maximal defect, and so it has $P \in Syl_p(G)$ as a defect group (since defect groups are radical p-subgroups and the only radical p-subgroups of G are $O_p(G)$ and the Sylow p-subgroups). Now $w_d(N_G(P), B, \lambda, P) = k_d(N_G(P), B, \lambda)$ and by Corollary 3.15 $w_d(G, B, \lambda, O_p(G)) = w_d(N_G(P), B, \lambda, O_p(G)) = 0$, so we have only to check that $k_d(G, B, \lambda) = k_d(N_G(P), B, \lambda)$.

By Proposition 5.9, Conjecture 4.4 holds for every such p-block when $P/O_p(G)$ has p-rank one. Hence we may assume that $P/O_p(G)$ has p-rank strictly greater than one, and so $O_{p'}(G) \leq N_G(P)$ by Lemma 2.8. By Lemma 6.1 we may then assume that $O_{p'}(G) \leq Z(G)$ is cyclic and $Z(G) \leq G'$. So $Z(G) = O_p(G)O_{p'}(G)$ is cyclic.

By Lemma 6.2 there is a unique normal subgroup N of G minimal such that it strictly contains $Z(G)$. The factor group $N/Z(G)$ has TI Sylow p-subgroups and is nonabelian simple. Further $N/Z(G) \leq G/Z(G) \leq Aut(N/Z(G))$. Lemma 6.12 gives a list of the possibilities for $N/Z(G)$ using the classification of finite simple groups. By Lemma 6.7, Lemma 6.8 and Proposition 7.7 we may assume that $(p, [G : N]) = 1$.

Next we show that we may assume that N is quasisimple, i.e., that G is an automorphism group of a quasisimple group: (a) Lemma 6.9 tells us that if $O_p(Z(G)) > 1$, then N is quasisimple; (b) Lemma 6.11 tells us that we may assume that N is quasisimple when $O_p(Z(G)) = 1$.

The results of Chapter 7 give a case-by-case verification of Conjecture 4.4 for each of the possibilities for G where $N/Z(G)$ is one of the nonabelian simple groups listed in Proposition 6.12, and the result follows. □

9 Dade's inductive conjecture for the Ree groups of type G_2

As an aside and whilst we have introduced the notation necessary to study the Ree groups of type G_2, we go on to demonstrate the inductive form of Dade's conjecture for these groups for the prime $p = 3$. Jianbei An has achieved a verification for all primes but $p = 3$ in [4], and so the results of this section will complete the verification for this class of groups.

Since $Out(^2G_2(q))$ is cyclic and the Schur multiplier of $^2G_2(q)$ is trivial, it follows from the remarks at the end of Chapter 3 of [20] that in order to verify the inductive conjecture for this class of groups it suffices to check only that they satisfy the invariant conjecture. We state the invariant conjecture here and direct the reader to [20] for a statement of the inductive conjecture.

Conjecture 9.1 (Dade's invariant) *Let B be a p-block of a finite group G satisfying $O_p(G) = 1$, and suppose that $G \lhd E$. If $G \lhd H \leq E$ and $\sigma \in \mathcal{R}(G)$, then denote by $Irr_d(G_\sigma, B, H)$ the set of those characters in $Irr_d(G_\sigma, B)$ with inertial subgroup $N_H(\sigma)$ in $N_E(\sigma)$. Write $k_d(G_\sigma, B, H) = |Irr_d(G_\sigma, B, H)|$. Then*

$$\sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} k_d(G_\sigma, B, H) = 0.$$

Proposition 4.9 *Dade's inductive conjecture is satisfied for the Ree groups of type G_2 for the prime $p = 3$.*

Proof Let $G = {}^2G_2(q)$, where $q = 3^{2m+1}$. Let $G \lhd E$ and let B_0 denote the unique 3-block of G of positive defect. Choose $t | 2m + 1$ such that τ^t generates E/G, where τ is as in Lemma 7.3. Suppose that $G \lhd H \leq E$, and choose $r | 2m + 1$ such that τ^r generates H/G. Note that we may now have $3 | [E : G]$. Let $P \in Syl_p(G)$.

From the proof of Proposition 7.11, $f_{Irr_{6m+3}(G,B_0)}(\langle \tau^r \rangle) = 3^r$, $f_{Irr_{5m+3}(G,B_0)}(\langle \tau^r \rangle) = 6$ and $f_{Irr_{4m+2}(G,B_0)}(\langle \tau^r \rangle) = 1$, this accounting for all the irreducible characters of

B_0. Also from the proof of Proposition 7.11, we have $f_{Irr_{6m+3}(N_G(P),B_0)}(\langle\tau^r\rangle) = 3^r$, $f_{Irr_{5m+3}(N_G(P),B_0)}(\langle\tau^r\rangle) = 6$ and $f_{Irr_{4m+2}(N_G(P),B_0)}(\langle\tau^r\rangle) = 1$, this accounting for all the irreducible characters of $N_G(P)$. The result follows by Lemma 7.4. $\square$

10 Appendix A

10.1 Generators for $PSL_3(4).2^2$

Regarded as a permutation group acting on a set of 56 elements, a system of generators for $PSL_3(4).2^2$ is $\langle a, b, c, d\rangle$, where

$a = (2,6,22,31,8)(3,9,32,39,12)(4,14,42,46,16)(5,18,38,51,21)(7,26,49,52,27)$
$(10,19,23,54,35)(11,36,50,33,37)(13,40,20,45,41)(15,30,55,44,28)$
$(17,47,29,24,48)(25,43,56,53,34)$,

$b = (1,2,5)(3,10,13)(4,15,17)(6,23,25)(7,20,19)(8,28,32)(9,33,21)$
$(11,30,38)(12,18,14)(16,45,40)(22,24,53)(26,48,37)(27,39,43)(29,47,46)$
$(31,44,41)(34,49,55)(35,42,51)(50,52,54)$

$c = (1,3)(4,11)(5,19)(6,24)(8,29)(9,32)(10,34)(12,39)(14,43)(15,23)(16,44)$
$(20,49)(21,50)(25,42)(27,45)(30,36)(33,56)(38,53)(40,41)(46,54)(47,48)$

$d = (1,4)(2,7)(3,11)(5,20)(6,24)(8,30)(9,34)(10,32)(12,39)(13,35)(14,33)$
$(15,44)(16,23)(17,18)(19,49)(21,46)(22,52)(25,53)(26,55)(27,41)(28,51)(29,36)$
$(31,37)(38,42)(40,45)(43,56)(47,48)(50,54)$

It can easily be checked using MAGMA that $PSL_3(4).2^2 \cong \langle a, b, c, d\rangle$ by computing the centre of $PSL_3(4).2^2$, the centralizer of $PSL_3(4)$ and making use of the CompositionFactors command, or simply by using the CharacterTable command and comparing with that given in [12].

10.2 Generators for 2. ${}^2B_2(8)$

Following [12, pp.28] 2. ${}^2B_2(8)$ may be regarded as the subgroup of $GL_8(5)$ generated by the matrices a,b and c given below:

$$a=\begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&4&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&4\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix},\ b=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0 \end{pmatrix}.$$

$$c=\begin{pmatrix} 2&1&1&1&1&1&1&1\\ 1&4&3&3&4&1&0&3\\ 1&3&3&4&1&0&3&4\\ 1&3&4&1&0&3&4&3\\ 1&4&1&0&3&4&3&3\\ 1&1&0&3&4&3&3&4\\ 1&0&3&4&3&3&4&1\\ 1&3&4&3&3&4&1&0 \end{pmatrix}$$

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