Alglat for Modules over FSI Rings and Reflexivity

Thesis submitted for the degree of

Doctor of Philosophy

at the University of Leicester

by

Nicole Jane Snashall BA (Oxon)

Department of Mathematics

University of Leicester

October 1990

Alglat for Modules over FSI Rings and Reflexivity Nicole Jane Snashall

ABSTRACT

For a bimodule ${}_R M_\Delta$ where R and Δ are rings with unity, alglat ${}_R M_\Delta$ is the ring of all Δ–endomorphisms of M leaving invariant every R–submodule of M. The bimodule is said to be reflexive if the elements of alglat ${}_R M_\Delta$ are precisely the left scalar multiplications by elements of R.

For most of the thesis $\Delta = R$, a commutative ring with unity. However, in the early work, some results on the general structure of alglat are obtained, and in particular, Theorem 1.9 shows that it is an inverse limit.

The next section of the thesis is concerned with reflexivity, and considers rings R for which all non-torsion or all finitely generated R–modules are reflexive. Theorem 3.4 gives eight equivalent conditions on an h-local domain R to the assertion that every finitely generated R–module is reflexive, that is R is scalar-reflexive. A local version of this property is introduced, and it is shown in Theorem 2.17 that a locally scalar-reflexive ring is scalar-reflexive.

The remainder of this thesis considers alglat for all modules over an FSI ring. The local FSI rings are precisely the almost maximal valuation rings, and this is the first case to be settled. More details are then given of the structure of FSI rings and related rings. A completion is introduced in 6.4 to enable alglat to be determined for certain torsion modules over an indecomposable FSI ring. Theorem 7.3, in summarising the work of the last two chapters of the thesis, gives a complete characterisation of alglat for all modules over an FSI ring.

Contents

Introduction

This thesis looks at $\text{alglat}_R M_\Delta$ given rings R and Δ with unity and an R–Δ–bimodule M. The definition of alglat for a bimodule was made by Fuller, Nicholson and Watters in [5], where $\text{alglat}_R M_\Delta$ is defined to be the ring of all endomorphisms of M_Δ leaving invariant every R–submodule of M_Δ. The bimodule is said to be reflexive if the elements of $\text{alglat}_R M_\Delta$ are precisely the left scalar multiplications by elements of R. In the majority of the work $\Delta = R$ with R being a commutative ring with unity.

There are three distinct but related parts to this thesis. The first chapter looks at decomposition theorems which help to determine the structure of alglat. After this the ring R is always commutative and any R–module M may then be considered as an R–R–bimodule. Chapters 2 and 3 look at the idea of reflexivity and in particular the case when all finitely generated modules over a given ring are reflexive. Chapter 4 looks at local rings and characterises alglat for all modules over an almost maximal valuation ring. The final three chapters extend Chapter 4 and determine alglat for all modules over an FSI ring.

Before describing this work in more detail some background information on reflexivity and alglat is given.

Halmos, in his paper [10], considered a complex Hilbert space $\mathcal{H}$. To each set of (bounded) operators $\mathcal{A}$ on $\mathcal{H}$ he defined $\text{Lat}\,\mathcal{A}$ to be the set of all (closed) subspaces of $\mathcal{H}$ invariant under every operator in $\mathcal{A}$. Dually to each set $\mathcal{L}$ of (closed) subspaces there corresponds the set $\text{Alg}\,\mathcal{L}$ of those (bounded) operators that leave invariant each element of $\mathcal{L}$. Thus $\text{Lat}\,\mathcal{A} = \{X \mid X \text{ subspace of } \mathcal{H}, AX \subseteq X \text{ for all } A \in \mathcal{A}\}$ and $\text{Alg}\,\mathcal{L} = \{A \mid A \text{ operator on } \mathcal{H}, AL \subseteq L \text{ for all } L \in \mathcal{L}\}$. Then it is clear that $\mathcal{A} \subseteq \text{AlgLat}\,\mathcal{A}$ and $\mathcal{L} \subseteq \text{LatAlg}\,\mathcal{L}$. Halmos defines a lattice $\mathcal{L}$ to be reflexive if $\mathcal{L} = \text{LatAlg}\,\mathcal{L}$. Similarly, an algebra $\mathcal{A}$ is said to be reflexive if $\mathcal{A} = \text{AlgLat}\,\mathcal{A}$.

Following on from this definition Hadwin, in his paper [7] considered a vector space X over a field F and a single linear transformation T. He defined the notion of algebraic reflexivity. In Halmos' terminology this amounts to: a linear transformation T is algebraically reflexive if and only if the algebra generated by 1 and T is reflexive.

This notion of reflexivity is extended to bimodules ${}_RM_\Delta$ in the paper by Fuller, Nicholson and Watters ([5]). They observe that if V is a finite-dimensional vector space over a field K and R is a subalgebra of $\operatorname{End} V_K$, then V is an R–K–bimodule. Moreover AlgLat R consists of those endomorphisms in $\operatorname{End} V_K$ which leave invariant every R–submodule of ${}_RV$. Thus AlgLat R is determined by the R–K–bimodule ${}_RV_K$. For a bimodule ${}_RM_\Delta$ where R and Δ are rings with unity, they define $\operatorname{alglat}{}_RM_\Delta$ to be the ring of all endomorphisms of M_Δ leaving invariant every R–submodule of M_Δ. Defining the map $\lambda : R \rightarrow \operatorname{alglat}{}_RM_\Delta$ by $\lambda(r)$ acts on M as left multiplication by r, $\lambda(R)$ is always contained in $\operatorname{alglat}{}_RM_\Delta$. When equality holds, that is $\operatorname{alglat}{}_RM_\Delta = \lambda(R)$, the bimodule is said to be reflexive. Much of their work considers the case where M is a left R–module and $\Delta = \operatorname{End}{}_RM$; for then M has the structure of an R–Δ–bimodule.

Hadwin and Kerr studied reflexive modules in [8] and [9] where all rings are commutative with unity. Their work centred on whether or not a module is reflexive, and has not discussed alglat M when the module M is not reflexive. In [8], Hadwin and Kerr defined a ring R to be strongly scalar-reflexive if every R–module is reflexive and strictly scalar-reflexive if every finitely generated R–module is reflexive. They omitted the word "strictly" in [9], calling a ring R scalar-reflexive if every finitely generated R–module is reflexive. Hadwin and Kerr considered this change in terminology to be appropriate since they completely characterised all strongly scalar-reflexive rings in [9]. Throughout this thesis a ring in which every finitely generated module is reflexive will be called scalar-reflexive, following [9]. Properties of these rings taken from [8] will also be used.

Chapter 1 looks at the general structure of alglat. The first decomposition result involves inverse limits and may be applied to any R–S–bimodule where R and S are rings with unity (not necessarily commutative). It is known that any module is the direct limit of its finitely generated submodules and that the index set is directed. Using the category equivalence of R–S–bimodules and left $R\otimes S^{op}$–modules, any R–S–bimodule M is the direct limit of its finitely generated R–S–sub-bimodules $\{M_k \mid k \in K\}$ and again the index set K is directed. Both direct and inverse limits are used in Theorem 1.9 to prove that, in this case, $\operatorname{alglat}_R(\varinjlim M_k)_S \cong \varprojlim (\operatorname{alglat}_R {M_k}_S)$. Thus given any R–S–bimodule M, $\operatorname{alglat}_R M_S$ is always expressible as an inverse limit. This result is not used directly later in the thesis but motivated the theorems involving completions, as every topological completion is an inverse limit.

Two more specific decompositions are also given in Chapter 1, which are used in later work in determining alglat for modules over an FSI ring. The first of these is applicable when the ring R is a finite direct sum of rings, $R = \bigoplus_{i=1}^{n} R_i$. Then any R–module T may also be expressed as a direct sum with $T = \bigoplus_{i=1}^{n} T_i$ and each T_i is an R_i–module. The result given in Theorem 1.10 uses this known decomposition of T to prove that $\operatorname{alglat} T \cong \bigoplus_{i=1}^{n} \operatorname{alglat} T_i$. This theorem is useful when a ring is a finite direct sum of indecomposable rings, the structures of which are known.

The third decomposition applies to h-local domains. In [15] (1.11) Matlis defined an h-local domain to be an integral domain such that (i) each non-zero prime ideal is contained in a unique maximal ideal, and (ii) each non-zero element is contained in only finitely many maximal ideals. Matlis showed in [15] (1.12) that any torsion module T over an h-local domain R may be written as a direct sum $T \cong \bigoplus_{M} T_M$ where M ranges over all maximal ideals of R and T_M is the corresponding localisation. Theorem 1.18 uses this decomposition to show, for any torsion module T over an h-local domain R, that $\operatorname{alglat} T \cong \prod_{M} \operatorname{alglat} T_M$ where M ranges over all maximal ideals of R. This result is extended in Theorem 1.21 to the case where P is a prime ideal of R such that R/P is an h-local domain and T is a torsion

R–module with $P \subseteq \operatorname{Ann} T$ such that T is also a torsion R/P–module. It is in this form that the decomposition is used in Chapter 7.

Chapters 2 and 3 are concerned with reflexive modules, leaving a discussion of alglat in non-reflexive cases until the later chapters. Chapter 2 is particularly concerned with reflexive non-torsion modules. In [9; Proposition 7], Hadwin and Kerr proved that for a commutative domain R every non-torsion R–module is reflexive. This chapter looks at classes of rings for which it is true that every non-torsion module is reflexive. The main result of this chapter shows that it is sufficient to have this condition for 2–generated non-torsion modules. This result, given in Theorem 2.2, states that for a commutative ring R, the condition that every non-torsion R–module is reflexive, is equivalent to every 2–generated non-torsion R–module being reflexive. The result of Hadwin and Kerr concerning domains can be derived from this result and is given as Corollary 2.3. In [9] Hadwin and Kerr defined a ring R to be scalar-reflexive if every finitely generated R–module is reflexive. Another corollary (which appears as Corollary 2.5 and is not proven by Hadwin and Kerr) shows that every non-torsion module over a scalar-reflexive ring is reflexive.

Hadwin and Kerr raise various questions about the property of scalar-reflexivity in [8]. In particular, having stated that scalar-reflexive rings are closed under direct sums and under quotients, they ask what happens under localisations. In their second paper [9], Hadwin and Kerr gave equivalent conditions for a local ring to the ring being scalar-reflexive (see Theorem 2.11). In particular a local ring is scalar-reflexive if and only if it is an almost maximal valuation ring. This motivates the definition in 2.12 where a ring is defined to be locally scalar-reflexive if every localisation at a maximal ideal is scalar-reflexive. Thus a ring is locally scalar-reflexive if each localisation at a maximal ideal is an almost maximal valuation ring. Theorem 2.17 provides a link between the two concepts of being scalar-reflexive and locally scalar-reflexive, proving that every locally scalar-reflexive ring is scalar-reflexive. Thus if every localisation of a ring R is scalar-reflexive then every localisation at a maximal ideal is scalar-reflexive and so R is scalar-reflexive. It is still an open question as to whether or not the converse is true.

A theorem of Hadwin and Kerr concerning scalar-reflexivity is given in Theorem 2.18. This result shows that if R is an h-local domain with R_M an almost maximal valuation ring for all maximal ideals M then R is scalar-reflexive. Thus the hypotheses require R to be locally scalar-reflexive and an h-local domain. Thus using Theorem 2.17, the condition that R be an h-local domain is redundant, and Theorem 2.17 is seen to be an extension of this theorem of Hadwin and Kerr.

Returning to non-torsion modules at the end of the chapter, Corollary 2.19 shows that every non-torsion module over a locally scalar-reflexive ring is reflexive. This plays an important part in the discussion of alglat for non-torsion modules in the later chapters.

Chapter 3 continues the theme of scalar-reflexive and locally scalar-reflexive rings. The main result is Theorem 3.4, which gives eight properties of an h-local domain which are equivalent to the condition that the ring is locally scalar-reflexive. One of these equivalent properties is that every 2–generated torsion module is a direct sum of cyclic modules. This links the study of alglat and reflexivity with the structure and decomposition of modules. Conditions on finitely generated modules also appear in Theorem 3.4.

The main part of the proof of Theorem 3.4 is to show that, for an h-local domain R with every 2–generated R–module reflexive, then R is locally scalar-reflexive. This result appears in Theorem 3.1. It is worth remarking that Theorem 3.1 shows that an h-local domain is scalar-reflexive if and only if it is locally scalar-reflexive. This provides a partial converse to the result that every locally scalar-reflexive ring is scalar-reflexive.

Chapter 3 finishes with an example of a domain which is locally scalar-reflexive and thus scalar-reflexive but is not an h-local domain. This answers a question of Hadwin and Kerr posed in [9; p12] in the negative and leaves the scalar-reflexive domains as yet unclassified. The scalar-reflexive h-local domains are classified here in a variety of ways.

The remaining chapters are concerned with determining alglat for all modules over particular classes of rings. Chapter 4 looks at alglat for all modules over an almost maximal valuation ring. These rings are scalar-reflexive and so all finitely generated and all non-torsion modules over an almost maximal valuation ring are reflexive. The study of torsion modules divides into two cases, considering faithful and non-faithful modules. Theorem 4.3 shows that, for any almost maximal valuation ring R and R–module T which is not faithful, T is reflexive.

Results of Gill and of Hadwin and Kerr reduce the study to the case of a faithful torsion module over an almost maximal valuation domain. This is where the non-reflexive cases arise. In view of the decomposition of Theorem 1.9 a completion is an obvious choice of candidate for alglat in these cases, and it is the R-completion which is used. This is defined for an integral domain which is not a field and is discussed by Matlis in [15; §6] (4.4). This topology takes the non-zero principal ideals of R to form a subbase for the open neighbourhoods of 0 in R. A domain R is Hausdorff in this topology so R embeds in its completion. Theorem 4.9 shows that, for a faithful torsion R–module T over an almost maximal valuation domain R, alglat T is isomorphic to the R-completion of R. The results of Chapter 4 are summarised in Theorem 4.10 which shows that the only modules which are not reflexive over an almost maximal valuation ring R are those which are faithful and torsion when R is not maximal.

The aim of Chapter 5 is to provide information on FSI rings and on related rings. Much of this will be used in Chapters 6 and 7 to determine alglat for all modules over an FSI ring. A large part of this material is in the literature. In

[25], Vámos defined a ring to be fractionally self-injective (FSI) if for each ideal I of R the classical ring of quotients of R/I is self-injective. As well as studying FSI rings, Vámos also studied FGC rings. Other work on FGC rings by Shores and R. Wiegand includes a study of CF rings.

Structure theorems are given for all three types of ring. These three classes of rings are related in that all FGC rings are FSI rings and all FSI rings are CF rings. Examples are also given in this chapter to show that the classes of rings are distinct.

It is known that the local FGC rings are the almost maximal valuation rings and as such are scalar-reflexive. Vámos proved that the local FSI rings are also precisely the almost maximal valuation rings. Thus every FSI ring is locally scalar-reflexive. Every FSI ring is a finite direct sum of indecomposable FSI rings. The indecomposable FSI rings are the almost maximal valuation rings, the locally almost maximal h-local domains and a third type, the locally almost maximal torch rings. (Torch rings are not domains and are discussed in Chapter 5.) Note that a ring is locally almost maximal if each localisation at a maximal ideal is an almost maximal valuation ring.

The obvious generalisation of alglat for modules over an FSI ring is to determine alglat for all modules over a CF ring. Every valuation ring is a CF ring, and in view of Theorem 2.11, arbitrary valuation rings are not reflexive. Thus not every CF ring is scalar-reflexive. An example is given at the end of Chapter 5 which determines alglat for a specific 2–generated module over a valuation ring which is not almost maximal. The nature of the work in Chapters 6 and 7 together with this example indicates that any characterisation of alglat for modules over CF rings will not be a simple extension of the results for modules over FSI rings. However it is hoped that a study of examples such as this will help determine the structure of alglat for a larger class of rings than FSI rings.

Chapter 6 outlines the strategy to characterise alglat for modules over FSI rings and does most of the work to reach this end. It was remarked in the comments about Chapter 5 that every FSI ring is locally scalar-reflexive. Thus all non-torsion modules over an FSI ring are reflexive. This leaves the study of torsion modules. Every FSI ring is a finite direct sum of indecomposable FSI rings. Using the second decomposition theorem of Chapter 1, this may be reduced to the study of torsion modules over an indecomposable FSI ring.

Any indecomposable FSI ring R has a unique minimal prime ideal P which is comparable to every ideal of R and such that the ideals of R contained in P form a chain. For a torsion module over an indecomposable FSI ring R either $\operatorname{Ann} T \subset P$ or $P \subseteq \operatorname{Ann} T$. Chapter 6 studies the case where $\operatorname{Ann} T \subset P$. Then $P \neq 0$ and so R is not a domain. From the work in Chapter 4 on almost maximal valuation rings, this reduces the case $\operatorname{Ann} T \subset P$ to the study of alglat where R is a locally almost maximal torch ring. This is further reduced to the study of alglat for a faithful torsion module over a locally almost maximal torch ring.

To characterise alglat in this case a completion is introduced in 6.4. In order to describe alglat in terms of a completion, the ring must be Hausdorff in the topology. This ensures that there is an embedding of the ring in its completion. The unique minimal prime ideal P of a locally almost maximal torch ring R is comparable to every ideal of R and the ideals of R contained in P form a chain. It is shown that $\bigcap_{0 \neq p \in P} Rp \subseteq \operatorname{Ann} T$ for all torsion R–modules T. So if T is a faithful torsion R–module, then $\operatorname{Ann} T = 0$ and hence $\bigcap_{0 \neq p \in P} Rp = 0$. The P-topology (defined in 6.4) takes the non-zero principal ideals of R contained in P to form a sub-base for the open neighbourhoods of 0 in R. Thus if R has a faithful torsion module then R is Hausdorff in this topology. In determining alglat, the results proved are more general than those required for this particular case, but they are included as they may be of independent interest. Corollary 6.8 states that, for a locally almost maximal torch ring with unique minimal prime ideal P and faithful torsion R–module T, alglat T is isomorphic to the completion of R in the

P-topology. This completes the case where $\operatorname{Ann} T \subset P$. A summary of these results is given in Theorem 6.10 to bring all the results of Chapter 6 together.

Chapter 7 discusses the case where $P \subseteq \operatorname{Ann} T$ and T is a torsion R/P–module. (The non-torsion case has been dealt with in Chapter 6.) Since R is an indecomposable FSI ring the factor ring R/P is an h-local domain. Theorem 7.1 uses the decomposition result of Theorem 1.21 to write $\operatorname{alglat}_R T_R \cong \prod_M \operatorname{alglat} T_M$ where M ranges over all maximal ideals of R and T_M is an R_M–module. Each localised ring R_M is an almost maximal valuation ring, and has R_M-completion $\widetilde{R_M}$ whenever R_M is in addition a domain. The results of Chapter 4 are used to show that $\operatorname{alglat}_R T_R \cong (\prod_{M \in X} \lambda(\widetilde{R_M})) \oplus (\prod_{M \in Y} \lambda(R_M))$, where $X = \{M \mid M \text{ is a maximal ideal of } R, R_M \text{ is not } R_M\text{-complete}, \operatorname{Ann} T_M = 0\}$ and $Y = \{M \mid M \text{ is a maximal ideal of } R, M \notin X\}$. (It is noted in Theorem 7.1 that if $M \in X$ then R_M is indeed a domain and so the completion $\widetilde{R_M}$ exists.)

The final theorem of Chapter 7 combines this result with those of Chapter 6 to give a complete characterisation of alglat for all modules over an FSI ring. The chapter ends with an illustration indicating the nature of alglat for any module over an FSI ring.

This thesis discusses the general structure of alglat, showing it to be an inverse limit, as well as the more specific case of determining alglat for all modules over an FSI ring. The work on reflexivity and on scalar-reflexive rings extends that known previously from the literature.

Notation

All the rings considered are rings with unity.

Let R be a commutative ring with unity and let T be an R–module. Then the bimodule structure of T is always that of an R–R–bimodule with rm = mr for all r in R and m in M. Where it will not cause confusion alglat T is written for alglat$_R T_R$.

For a ring R with maximal ideal M and R–module T, R_M is the ring localised at M and T_M is the module localised at M. Then T_M is an R_M–module with the obvious product.

The notation "Ann" is used to indicate the annihilator of a module. For any S–module M, Ann M is taken to be the annihilator of M in the ring S. In particular, for finitely generated modules, Ann Rx denotes the annihilator of the R–module Rx in R whereas Ann $R_M y$ is used for the annihilator of the R_M–module $R_M y$ in R_M.

The notation $\subset$ is always used to denote a strict inclusion. The symbol $\subseteq$ is used to indicate an inclusion which is not necessarily strict.

All maps are written on the left of the elements upon which they act.

Acknowledgements

I wish to express my thanks to my supervisor Dr J. F. Watters for all his help and encouragement and to SERC for their financial support.

Chapter 1 Decomposition theorems for alglat

This chapter examines the structure of alglat and gives three decomposition theorems for alglat. The definitions of alglat${}_R M_\Delta$ and reflexivity for an R–Δ–bimodule M are given in 1.1. (Recall that all the rings considered in this thesis have a 1.) The first decomposition theorem of alglat${}_R M_\Delta$ applies in the general case where R and Δ are any rings with unity and M is any R–Δ–bimodule. The remaining two decompositions concern alglat${}_R M_R$ where R is a commutative ring with unity and M is an R–R–bimodule. These results will be used in later chapters to characterise alglat for all modules over particular classes of rings.

Theorem 1.9 gives the first decomposition and uses direct and inverse limits to describe the structure of alglat. It is known that any R–S–bimodule M can be expressed as the direct limit of its finitely generated R–S–sub-bimodules $\{M_k \mid k \in K\}$ where R and S are any rings with 1, not necessarily commutative. For reference this result is included as Theorem 1.6. Theorem 1.9 uses this description of M to give a characterisation of alglat M in terms of the inverse limit of the family of rings $\{\text{alglat}\, M_k \mid k \in K\}$. This decomposition of alglat M in terms of an inverse limit may be applied to any R–S–bimodule M.

Theorem 1.9 is not used directly in any later results but has motivated the theorems which involve completions, since every completion is an inverse limit. Topological completions play an important role in determining alglat${}_R M_R$ where R is a commutative ring and M is not reflexive.

The other two decompositions given in the chapter are not so general but do provide useful information in determining the structure of alglat. For these two results, and indeed for the remainder of the thesis after Theorem 1.9, it will be assumed that R is a commutative ring with 1 and that M is an R–module with the bimodule structure as described in 1.2(b).

The second decomposition result of this chapter can be used when the ring R (commutative with 1) is a finite direct sum of rings R_i, $i = 1, \ldots, n$. Then there are idempotent elements e_i in R_i with $1 = e_1 + ... + e_n$ and $R_i = e_iR$. In this case an R–module M has a decomposition as the direct sum $\bigoplus_{i=1}^{n} e_iM$. Writing $M_i = e_iM$, each M_i can be considered as an R_i–R_i–bimodule. Theorem 1.10 shows that in this case $\operatorname{alglat} M \cong \bigoplus_{i=1}^{n} \operatorname{alglat} M_i$. This theorem is used in Chapter 6 to determine the structure of alglat for all modules over FSI rings, since every FSI ring is a finite direct sum of indecomposable FSI rings, the structure of which are known.

The third decomposition applies to torsion modules over h-local domains. Again the decomposition of alglat T is motivated by a known decomposition for the module T. Matlis defined an h-local domain in [15] and showed in the same paper that any torsion module T over an h-local domain R may be expressed as a direct sum $T \cong \bigoplus_{M} T_M$ where M ranges over all maximal ideals of R and T_M is the corresponding localisation. Theorem 1.18 gives a decomposition for alglat in this case, showing that $\operatorname{alglat} T \cong \prod_{M} \operatorname{alglat} T_M$ where M ranges over all maximal ideals of R and T_M is considered as an R_M–R_M–bimodule. This result is extended in Theorem 1.21 to the situation where P is a prime ideal of R such that R/P is an h-local domain and T is a torsion R–module with $P \subseteq \operatorname{Ann} T$ such that T is also a torsion R/P–module. Theorem 1.21 is the third decomposition theorem of this chapter and is used in Chapter 7.

The first section gives the definitions of $\operatorname{alglat}_R M_\Delta$ and reflexivity as made in [5] by Fuller, Nicholson and Watters.

1.1 Definitions of $\operatorname{alglat}_R M_\Delta$ and reflexivity ([5])

Let M be an R–Δ–bimodule where R and Δ are rings with unity. The ring $\operatorname{alglat}_R M_\Delta$ is the ring of all endomorphisms of M_Δ which leave invariant every R–submodule of M. Thus $\operatorname{alglat}_R M_\Delta = \{\phi \in \operatorname{End} M_\Delta \mid \phi N \subseteq N \text{ for all } {}_RN \leq {}_RM\}$ $= \{\phi \in \operatorname{End} M_\Delta \mid \phi m \in Rm \text{ for all } m \in M\}$.

Let λ be the map defined by $\lambda : R \to \operatorname{alglat}_R M_\Delta$, $\lambda(r) : M \to M$, $m \mapsto rm$. Then it is always the case that $\lambda(R) \subseteq \operatorname{alglat}_R M_\Delta$. The bimodule is said to be reflexive if there is equality, that is if $\lambda(R) = \text{alglat}_R M_\Delta$. Thus the module is reflexive if the elements of the ring $\text{alglat}_R M_\Delta$ are precisely the left scalar multiplications by elements of R.

The map λ gives rise to a map from R to $\lambda(R)$ defined by $r \mapsto \lambda(r)$. This is always a surjective ring homomorphism and has kernel AnnM. Thus $\lambda(R) \cong R/\text{Ann}M$. In particular, if M is a faithful R–module then $R \cong \lambda(R)$.

1.2 Examples of bimodules

(a) Let M be a left R–module where R is any ring with 1 (not necessarily commutative) and let $S = \text{End}_R M$. Then M is an R–S–bimodule. Any module can be considered as a bimodule in this way.

(b) Let R be a commutative ring with 1 and let M be a left R–module. Then M has an R–R–bimodule structure. This is given by defining a right R–module structure on M by $mr := rm$ for all $m \in M$ and $r \in R$.

(c) As an illustration of (b) let M be an abelian group. It is well-known that M can be considered as a $\mathbb{Z}$–module. Thus M can be given the structure of a $\mathbb{Z}$–$\mathbb{Z}$–bimodule.

Throughout this thesis, where M is any module over a commutative ring R, the bimodule structure of M is always the R–R–bimodule structure defined in 1.2(b).

Before proving the first decomposition theorem, the next few sections give some background information about direct and inverse limits from category theory. The definitions and notation used here follow the approach of Rowen ([21]). The motivation for the decomposition of alglat is the known result that an R–S–bimodule M is the direct limit, over a directed index set, of its finitely generated R–S–sub-bimodules. The only direct limits that are needed in the proof of Theorem 1.9 are those over a directed index set and this is taken into account in the definition of a direct limit given in 1.4.

1.3 Definition of a directed set

A directed set is a partially ordered set K, with partial order $\leq$, such that for any i, j $\in$ K there exists k $\in$ K with $i \leq k$ and $j \leq k$.

1.4 Definition of a direct limit over a directed index set.

Let $\{A_k \mid k \in K\}$ be a set of R–S–bimodules indexed by a directed set K (with partial order $\leq$) and suppose that there are R–S–homomorphisms $\theta_{ji} : A_j \to A_i$ whenever $j \leq i$, satisfying

(i) for all $k \in K$, $\theta_{kk} : A_k \to A_k$ is id_{A_k}, and

(ii) for $k \leq j \leq i$ (so θ_{ji}, θ_{kj} and θ_{ki} are defined) $\theta_{ki} = \theta_{ji}\theta_{kj}$.

Then the direct limit $\varinjlim A_k$ is an R–S–bimodule together with a set of R–S–homomorphisms $\alpha_k : A_k \to \varinjlim A_k$ satisfying $\alpha_k = \alpha_j\theta_{kj}$ whenever $k \leq j$, such that, given any R–S–bimodule X and R–S–homomorphisms $\beta_k : A_k \to X$ satisfying $\beta_k = \beta_j\theta_{kj}$ whenever $k \leq j$, there is a unique R–S–homomorphism $\beta : \varinjlim A_k \to X$ with $\beta\alpha_k = \beta_k$ for each k.

Thus $\varinjlim A_k$ is a quotient of the direct sum of the A_k, namely $(\oplus A_k)/N$ where N is the sub-bimodule of $\oplus A_k$ generated by all the elements $\theta_{kj}a_k - a_k$, $(a_k \in A_k)$ whenever $k \leq j$. The maps $\alpha_k : A_k \to \varinjlim A_k$ are just $a_k \mapsto a_k + N$. Whenever K is a directed set and $\{A_k \mid k \in K\}$ is a family of R–S–bimodules, then the direct limit always exists ([21; Theorem 1.8.7 p113]).

The following result which shows that any module can be expressed as a direct limit is included, without proof, for completeness (see [21; Example 1.8.9 p114]).

1.5 Proposition

Every R–module is the direct limit (over a directed index set) of its finitely generated R–submodules.

For R–S–bimodules M and N, the map $\phi : M \to N$ is an R–S–homomorphism if, for all elements m_1, m_2, m in M, r in R and s in S, (i) $\phi(m_1 + m_2) = \phi m_1 + \phi m_2$, (ii) $\phi(rm) = r(\phi m)$ and (iii) $\phi(ms) = (\phi m)s$. The class of all R–S–bimodules together with R–S–homomorphisms is a category. There is a category equivalence between the category of R–S–bimodules with R–S–homomorphisms and the category $R\otimes S^{op}$–mod (of left $R\otimes S^{op}$–modules with $R\otimes S^{op}$–homomorphisms). The following result is an immediate consequence of Proposition 1.5.

<u>1.6 Theorem</u>

Every R–S–bimodule is the direct limit (over a directed index set) of its finitely generated R–S–sub-bimodules.

Two further properties of a direct limit over a directed index set are required before looking at inverse limits. The proofs follow Rotman in [20; pp31-32].

<u>1.7 Proposition</u>

Let K be a directed set, with partial order $\leq$, and let $\{A_k \mid k \in K\}$ be a family of R–S–bimodules. Write $\varinjlim A_k = (\oplus A_k)/N$ with the notation of 1.4. Then

(i) for any $x \in \varinjlim A_k$ there is an index i and some a_i in A_i with $x = a_i + N$, and

(ii) for $a_k \in A_k$ with $a_k + N = 0$ in $\varinjlim A_k$ there is some index t with $k \leq t$ and $\theta_{kt}a_k = 0$.

<u>Proof</u>

(i) Let $x \in \varinjlim A_k$. Then $x = y + N$ where $y \in \oplus A_k$. Write $y = \sum_{j=1}^{n} a_{k_j}$. Since K is a directed set there is an index i with $k_j \leq i$ for all j (j = 1, ..., n). Let $z = \sum_{j=1}^{n} \theta_{k_j i} a_{k_j}$ so that z is an element of A_i. Then $\theta_{k_j i} a_{k_j} - a_{k_j} \in N$ and it follows that $z - y$ is in N. Thus $x = z + N$ with $z \in A_i$ for some index i.

(ii) Suppose that $a_k \in A_k$ with $a_k + N = 0$ in $\varinjlim A_k$. The elements of N are the finite sums $\sum (\theta_{ji}a_j - a_j)$ with a_j in A_j, so let $a_k = \sum (\theta_{ji}a_j - a_j)$. The set K is directed so there is an index t with $k \leq t$ and $j \leq i \leq t$ for each i, j occurring in this sum. Then $\theta_{kt}a_k = (\theta_{kt}a_k - a_k) + a_k = (\theta_{kt}a_k - a_k) + \sum (\theta_{ji}a_j - a_j)$. Each term $(\theta_{ji}a_j - a_j)$ can be rewritten, with the second index as t, in the form $\theta_{ji}a_j - a_j = (\theta_{jt}a_j - a_j) + \left[(\theta_{it}(-\theta_{ji}a_j) - (-\theta_{ji}a_j)\right]$.

Thus $\theta_{kt}a_k = (\theta_{kt}a_k - a_k) + \sum \left[(\theta_{jt}a_j - a_j) + \left((\theta_{it}(-\theta_{ji}a_j) - (-\theta_{ji}a_j)\right)\right]$. Combining all terms with the same first index l gives $\theta_{kt}a_k = \sum_l (\theta_{lt}b_l - b_l)$ with b_l in A_l. Since the sum $\oplus A_k$ is direct, if $l \neq t$ then $b_l = 0$. But also $\theta_{tt}b_t - b_t = 0$ and so every term in the summation over l is 0. Hence $\theta_{kt}a_k = 0$ as required. □

The dual notion to a direct limit of an inverse limit is now introduced. The definition given is for the category of rings with ring homomorphisms. It is in this form that it will be used in Theorem 1.9.

1.8 Definition of an inverse limit

Let $\{L_k \mid k \in K\}$ be a set of rings indexed by a partially ordered set K (with partial order $\leq$) and suppose that there are ring homomorphisms $\phi_{ij} : L_i \to L_j$ whenever $j \leq i$, satisfying

(i) for all $k \in K$, $\phi_{kk} : L_k \to L_k$ is id_{L_k}, and

(ii) for $k \leq j \leq i$ (so ϕ_{ij}, ϕ_{jk} and ϕ_{ik} are defined) $\phi_{ik} = \phi_{jk}\phi_{ij}$.

Then the inverse limit $\varprojlim L_k$ is a ring together with a set of ring homomorphisms $\eta_k : \varprojlim L_k \to L_k$ satisfying $\eta_j = \phi_{kj}\eta_k$ whenever $j \leq k$, such that, given any ring X and ring homomorphisms $\xi_k : X \to L_k$ satisfying $\xi_j = \phi_{kj}\xi_k$ whenever $j \leq k$, there is a unique ring homomorphism $\xi : X \to \varprojlim L_k$ with $\eta_k\xi = \xi_k$ for each k.

Thus $\varprojlim L_k$ is the subring of $\prod L_k$ consisting of all (l_k) for which $\phi_{kj}l_k = l_j$ whenever $j \leq k$. The maps η_k are just the projections.

The next result is the first decomposition theorem for alglat. Theorem 1.9 characterises $\text{alglat}_R M_S$ using inverse limits where M is expressed as a direct limit over a directed index set. Theorem 1.6 may be used to write M as a direct limit of its finitely generated R–S–sub-bimodules. In this way, Theorem 1.9 gives a decomposition for $\text{alglat}_R M_S$ for all R–S–bimodules M.

1.9 Theorem

Let K be a directed index set, with partial order $\leq$, and let $\{A_k \mid k \in K\}$ be a set of R–S–bimodules where R and S are arbitrary rings with unity. Let $\theta_{ji} : A_j \to A_i$ be monic R–S–homomorphisms whenever $j \leq i$, with θ_{ji} satisfying the following two conditions:

(i) for all $k \in K$, $\theta_{kk} : A_k \to A_k$ is id_{A_k}, and

(ii) if $k \leq j \leq i$ then $\theta_{ki} = \theta_{ji}\theta_{kj}$.

Then $\text{alglat}_R(\varinjlim A_k)_S \cong \varprojlim (\text{alglat}_R A_{k\,S})$.

Proof

Let $L_k = \text{alglat}_R(A_k)_S$ for $k \in K$ so that $\{L_k \mid k \in K\}$ is a family of rings indexed by the directed set K. The maps θ_{ji} are monic so there are inverse R–S–homomorphisms $(\theta_{ji})^{-1} : \text{im}(\theta_{ji}) \to A_j$. Then there is a set of ring homomorphisms $\phi_{ij} : L_i \to L_j$ defined by $\phi_{ij} : \psi \mapsto (\theta_{ji})^{-1}\psi(\theta_{ji})$ whenever $j \leq i$.

Let $\psi \in L_i$. Since ψ preserves the lattice of R–S–sub-bimodules of A_i, $\psi : \text{im}\,\theta_{ji} \to \text{im}\,\theta_{ji}$. Thus $\phi_{ij}\psi$ is well-defined.

$\phi_{ij} : L_i \to L_j$ is a ring homomorphism

Each of the maps $(\theta_{ji})^{-1}$, ψ and θ_{ji} is a right S–homomorphism so that $\phi_{ij}\psi \in \text{End}(A_j)_S$. Let $a_j \in A_j$. There is an element r in R with $\psi(\theta_{ji}a_j) = r(\theta_{ji}a_j)$. Then $(\phi_{ij}\psi)(a_j) = (\theta_{ji})^{-1}(\theta_{ji}(ra_j)) = ra_j \in Ra_j$. Thus $\phi_{ij}\psi \in L_j$ and so $\text{im}\,\phi_{ij} \subseteq L_j$.

Let ψ, ψ' be elements of L_i. Then $\phi_{ij}(\psi + \psi') = (\theta_{ji})^{-1}(\psi + \psi')(\theta_{ji}) =$

$(\theta_{ji})^{-1}(\psi\theta_{ji} + \psi'\theta_{ji}) = (\theta_{ji})^{-1}\psi(\theta_{ji}) + (\theta_{ji})^{-1}\psi'(\theta_{ji}) = \phi_{ij}\psi + \phi_{ij}\psi'$. Also $\phi_{ij}(\psi\psi') = (\theta_{ji})^{-1}(\psi\psi')(\theta_{ji}) = (\theta_{ji})^{-1}\psi[(\theta_{ji})(\theta_{ji})^{-1}]\psi'(\theta_{ji}) = (\phi_{ij}\psi)(\phi_{ij}\psi')$. (The introduction of $(\theta_{ji})(\theta_{ji})^{-1}$, the identity on $\operatorname{im}\theta_{ji}$, is valid since ψ' preserves the lattice of R–S–sub-bimodules of A_i and thus, in particular, $\psi' : \operatorname{im}\theta_{ji} \to \operatorname{im}\theta_{ji}$.) Let 1_{L_k} be the identity element in the ring L_k for all $k \in K$. Then $\phi_{ij}(1_{L_i}) = (\theta_{ji})^{-1}1_{L_i}(\theta_{ji}) = 1_{L_j}$. Thus ϕ_{ij} is a well-defined ring homomorphism.

Moreover, these ring homomorphisms satisfy $\phi_{kk} = \mathrm{id}_{L_k}$ for all k and $\phi_{jk}\phi_{ij} = \phi_{ik}$ whenever $k \leq j \leq i$.

<u>$\phi_{kk} = \mathrm{id}_{L_k}$ for all k and $\phi_{jk}\phi_{ij} = \phi_{ik}$ whenever $k \leq j \leq i$.</u>

Let $\psi \in L_k$. Then $\phi_{kk} : L_k \to L_k$ and $\phi_{kk}\psi = (\theta_{kk})^{-1}\psi(\theta_{kk}) = (\mathrm{id}_{A_k})^{-1}\psi(\mathrm{id}_{A_k}) = \psi$. Thus $\phi_{kk} = \mathrm{id}_{L_k}$.

Suppose that $k \leq j \leq i$. Then the ring homomorphisms $\phi_{jk}, \phi_{ij}, \phi_{ik}$ all exist and $\phi_{jk}\phi_{ij} : L_i \to L_k$. Let $\psi \in L_i$. Then $(\phi_{jk}\phi_{ij})(\psi) = \phi_{jk}[(\theta_{ji})^{-1}\psi(\theta_{ji})] = (\theta_{kj})^{-1}(\theta_{ji})^{-1}\psi(\theta_{ji})(\theta_{kj}) = (\theta_{ji}\theta_{kj})^{-1}\psi(\theta_{ji}\theta_{kj}) = (\theta_{ki})^{-1}\psi(\theta_{ki}) = \phi_{ik}\psi$. Thus $\phi_{jk}\phi_{ij} = \phi_{ik}$.

The direct limit of the R–S–bimodules A_k exists since K is a directed index set. Let $D = \varinjlim A_k$, so that D is a quotient of $\oplus A_k$, namely $D = (\oplus A_k)/N$ where N is the sub-bimodule of $\oplus A_k$ generated by all the elements $\theta_{kj}a_k - a_k$ ($a_k \in A_k$) whenever $k \leq j$. Then the R–S–homomorphisms $\alpha_k : A_k \to D$ satisfying $\alpha_k = \alpha_j\theta_{kj}$ whenever $k \leq j$ are given by $\alpha_k : a_k \mapsto a_k + N$ (1.4).

For each $k \in K$, the R–S–homomorphism $\alpha_k : A_k \to D$ is monic. So there are inverse R–S–homomorphisms $(\alpha_k)^{-1} : \operatorname{im}(\alpha_k) \to A_k$ for each k.

<u>α_k is monic</u>

Let $a_k \in \ker\alpha_k$ so that $a_k + N = 0$ in D. Then there is an index t in K with $k \leq t$ and $\theta_{kt}a_k = 0$ (Proposition 1.7). But θ_{kt} is monic and so $a_k = 0$. Hence

α_k is monic.

For each $k \in K$, define $\eta_k : \mathrm{alglat}_R D_S \to L_k$ by $\psi \mapsto (\alpha_k)^{-1}\psi(\alpha_k)$. Then each η_k is a ring homomorphism.

Let $\psi \in \mathrm{alglat}_R D_S$. Since ψ preserves the lattice of R–S–sub-bimodules of D, $\psi : \mathrm{im}\,\alpha_k \to \mathrm{im}\,\alpha_k$. Thus $\eta_k\psi$ is well-defined.

<u>$\eta_k : \mathrm{alglat}_R D_S \to L_k$ is a ring homomorphism</u>

Each of the maps $(\alpha_k)^{-1}$, ψ and α_k is a right S–homomorphism so that $\eta_k\psi \in \mathrm{End}(A_k)_S$. Let $a_k \in A_k$. There is an element r in R with $\psi(\alpha_k a_k) = r(\alpha_k a_k)$. Then $(\eta_k\psi)(a_k) = (\alpha_k)^{-1}(\alpha_k(ra_k)) = ra_k \in Ra_k$. Thus $\eta_k\psi \in L_k$ and so $\mathrm{im}\,\eta_k \subseteq L_k$.

Let ψ, ψ' be elements of $\mathrm{alglat}_R D_S$. Then $\eta_k(\psi + \psi') = (\alpha_k)^{-1}(\psi + \psi')(\alpha_k) = (\alpha_k)^{-1}(\psi\alpha_k + \psi'\alpha_k) = (\alpha_k)^{-1}\psi(\alpha_k) + (\alpha_k)^{-1}\psi'(\alpha_k) = \eta_k\psi + \eta_k\psi'$. Also $\eta_k(\psi\psi') = (\alpha_k)^{-1}(\psi\psi')(\alpha_k) = (\alpha_k)^{-1}\psi[(\alpha_k)(\alpha_k)^{-1}]\psi'(\alpha_k) = (\eta_k\psi)(\eta_k\psi')$. (The introduction of $(\alpha_k)(\alpha_k)^{-1}$, the identity on $\mathrm{im}\,\alpha_k$, is valid since ψ' preserves the lattice of R–S–sub-bimodules of D and thus, in particular, $\psi' : \mathrm{im}\,\alpha_k \to \mathrm{im}\,\alpha_k$.) Let 1_D be the identity element in the ring $\mathrm{alglat}_R D_S$. Then $\eta_k(1_D) = (\alpha_k)^{-1}1_D(\alpha_k) = 1_{L_k}$, the identity element in the ring L_k. Thus η_k is a well-defined ring homomorphism.

Moreover these ring homomorphisms satisfy $\phi_{kj}\eta_k = \eta_j$ whenever $j \leq k$.

<u>$\phi_{kj}\eta_k = \eta_j$ whenever $j \leq k$</u>

Suppose that $j \leq k$. The ring homomorphism ϕ_{kj} exists and $\phi_{kj}\eta_k : \mathrm{alglat}_R D_S \to L_j$. Let $\psi \in \mathrm{alglat}_R D_S$. Then $(\phi_{kj}\eta_k)(\psi) = \phi_{kj}[(\alpha_k)^{-1}\psi(\alpha_k)] = (\theta_{jk})^{-1}(\alpha_k)^{-1}\psi(\alpha_k)(\theta_{jk}) = (\alpha_k\theta_{jk})^{-1}\psi(\alpha_k\theta_{jk}) = (\alpha_j)^{-1}\psi(\alpha_j) = \eta_j\psi$. Thus $\phi_{kj}\eta_k = \eta_j$.

Let X be a ring with ring homomorphisms $\xi_k : X \to L_k$ satisfying $\phi_{kj}\xi_k = \xi_j$ whenever $j \leq k$. Then there is a unique ring homomorphism $\xi : X \to \text{alglat}_R D_S$ such that $\eta_k \xi = \xi_k$ for each k in K. Moreover ξ is defined by $\xi : x \mapsto \psi_x$ where $x \in X$ and $\psi_x \in \text{alglat}_R D_S$ is given by $\psi_x : a_k + N \mapsto (\xi_k x)(a_k) + N$ for a_k in A_k.

Note that for $j \leq k$, it follows that $\xi_j x = (\phi_{kj}\xi_k)(x) = (\phi_{kj})(\xi_k x) = (\theta_{jk})^{-1}(\xi_k x)(\theta_{jk})$. Then $(\theta_{jk})(\xi_j x) = (\xi_k x)(\theta_{jk})$. This will be used in the following proofs.

<u>ψ_x is well-defined</u>

Suppose that an element in D has two representations $a_i + N$ and $a_j + N$ with $i, j \in K$ (Proposition 1.7). Then $a_i - a_j + N = 0$ in D. The index set K is directed so there is an index k with $i \leq k$ and $j \leq k$. Then both $\theta_{jk}a_j - a_j$ and $\theta_{ik}a_i - a_i$ are in N (1.4), and so $(\theta_{jk}a_j - \theta_{ik}a_i) + N = 0$ in D. From Proposition 1.7, there is an index t with $k \leq t$ and $\theta_{kt}(\theta_{jk}a_j - \theta_{ik}a_i) = 0$. But θ_{kt} is monic and so $(\theta_{jk}a_j - \theta_{ik}a_i) = 0$. Thus $\theta_{jk}a_j = \theta_{ik}a_i$.

Write $\widetilde{a_i} = (\xi_i x)(a_i)$ and $\widetilde{a_j} = (\xi_j x)(a_j)$ so that $\widetilde{a_i} \in A_i$ and $\widetilde{a_j} \in A_j$. The elements $\theta_{ik}\widetilde{a_i} - \widetilde{a_i}$ and $\theta_{jk}\widetilde{a_j} - \widetilde{a_j}$ are both in N so that $\widetilde{a_i} - \widetilde{a_j} + N = \theta_{ik}\widetilde{a_i} - \theta_{jk}\widetilde{a_j} + N$. Using the above results, $\theta_{ik}\widetilde{a_i} = (\theta_{ik})(\xi_i x)(a_i) = (\xi_k x)(\theta_{ik})(a_i) = (\xi_k x)(\theta_{jk})(a_j) = (\theta_{jk})(\xi_j x)(a_j) = \theta_{jk}\widetilde{a_j}$. Hence $\widetilde{a_i} - \widetilde{a_j} + N = 0$ in D. Thus $\widetilde{a_i} + N = \widetilde{a_j} + N$ so that $(\xi_i x)(a_i) + N = (\xi_j x)(a_j) + N$. Hence $\psi_x(a_i + N) = \psi_x(a_j + N)$ and so ψ_x is well-defined.

<u>$\text{im}\,\xi \subseteq \text{alglat}_R D_S$</u>

Let $x \in X$ with image ψ_x under ξ. The map $\psi_x : D \to D$ is given by $a_k + N \mapsto (\xi_k x)(a_k) + N$ for a_k in A_k and is well-defined.

Let $a_i + N$, $a_j + N$ be elements of D. Since K is a directed index set, there is an index k in K with $i \leq k$ and $j \leq k$. Then $a_i + a_j + N = a_k + N$ where $a_k = \theta_{ik}a_i + \theta_{jk}a_j$ in A_k. So $\psi_x(a_i + N + a_j + N) = \psi_x(a_k + N) = (\xi_k x)(a_k) + N =$

$(\xi_k x)(\theta_{ik})(a_i) + (\xi_k x)(\theta_{jk})(a_j) + N = (\theta_{ik})(\xi_i x)(a_i) + (\theta_{jk})(\xi_j x)(a_j) + N$. The elements $(\theta_{ik})(\xi_i x)(a_i) - (\xi_i x)(a_i)$ and $(\theta_{jk})(\xi_j x)(a_j) - (\xi_j x)(a_j)$ are in N. Thus $\psi_x(a_i + N + a_j + N) = (\xi_i x)(a_i) + (\xi_j x)(a_j) + N = \psi_x(a_i + N) + \psi_x(a_j + N)$. Let $a_k + N$ be an element of D and let s be an element of S. Then $\left[\psi_x(a_k + N)\right]s = \left[(\xi_k x)(a_k) + N\right]s = (\xi_k x)(a_k)s + N = (\xi_k x)(a_k s) + N = \psi_x(a_k s + N) = \psi_x((a_k + N)s)$. Thus $\psi_x \in \operatorname{End} D_S$.

Let $a_k + N$ be an element of D. Since $\xi_k x \in L_k$, there is an element r in R with $(\xi_k x)(a_k) = ra_k$. Thus $\psi_x(a_k + N) = ra_k + N = r(a_k + N) \in R(a_k + N)$. Hence $\psi_x \in \operatorname{alglat}_R D_S$. Thus $\operatorname{im}\xi \subseteq \operatorname{alglat}_R D_S$.

ξ is a ring homomorphism

The map ξ, defined by $\xi : X \to \operatorname{alglat}_R D_S$, $x \mapsto \psi_x$, is clearly well-defined. Recall that for each k, $\xi_k : X \to L_k$ is a ring homomorphism.

Let x, x′ be elements of X. Let $a_k + N$ be an element of D. Let $y = x + x'$ so that $\psi_y = \xi(x + x')$. Then $\psi_y(a_k + N) = (\xi_k y)(a_k) + N = \left[\xi_k(x + x')\right](a_k) + N = \left[(\xi_k x) + (\xi_k x')\right](a_k) + N = (\xi_k x)(a_k) + (\xi_k x')(a_k) + N = \psi_x(a_k + N) + \psi_{x'}(a_k + N)$. Thus $\xi(x + x') = \psi_y = \psi_x + \psi_{x'} = \xi(x) + \xi(x')$. Let $z = xx'$ so that $\psi_z = \xi(xx')$. Then $\psi_z(a_k + N) = (\xi_k z)(a_k) + N = \left[\xi_k(xx')\right](a_k) + N = \left[(\xi_k x)(\xi_k x')\right](a_k) + N = (\xi_k x)\left[(\xi_k x')(a_k)\right] + N = \psi_x\left[(\xi_k x')(a_k) + N\right] = \psi_x\left[\psi_{x'}(a_k + N)\right] = (\psi_x \psi_{x'})(a_k + N)$. Thus $\xi(xx') = \psi_z = \psi_x\psi_{x'} = \xi(x)\xi(x')$.

Let 1_X be the identity element in the ring X. Then $\psi_{1_X} = \xi(1_X)$. Let $a_k + N$ be an element of D. Then $\psi_{1_X}(a_k + N) = (\xi_k 1_X)(a_k) + N = 1_{L_k} a_k + N = a_k + N$. Thus $\xi(1_X) = \psi_{1_X} = 1_D$, the identity element in the ring $\operatorname{alglat}_R D_S$. Thus ξ is a well-defined ring homomorphism.

$\eta_k \xi = \xi_k$ for each k

Let x be an element of X and let $k \in K$. Then $(\eta_k \xi)(x) = \eta_k \psi_x = (\alpha_k)^{-1}\psi_x(\alpha_k)$. For any element a_k of A_k, $\left[(\alpha_k)^{-1}\psi_x(\alpha_k)\right](a_k) = \left[(\alpha_k)^{-1}\psi_x\right](a_k + N) = (\alpha_k)^{-1}\left[(\xi_k x)(a_k) + N\right] = (\xi_k x)(a_k)$. Thus $(\alpha_k)^{-1}\psi_x(\alpha_k) = \xi_k x$. So $(\eta_k \xi)(x) = \xi_k x$

for all x in X. Hence $\eta_k\xi = \xi_k$.

ξ is unique

Suppose that $\gamma : X \rightarrow \mathrm{alglat}_R D_S$ is a ring homomorphism with $\xi_k = \eta_k\gamma$ for all $k \in K$. Then $\eta_k\gamma = \eta_k\xi$ so that $\eta_k(\gamma - \xi) = 0$ for all $k \in K$. Let x be an element of X and let $a_j + N$ be an element of D. Then $0 = [\eta_j(\gamma - \xi)](x) = \eta_j(\gamma x - \xi x) = (\alpha_j)^{-1}(\gamma x - \xi x)(\alpha_j)$. Since $(\alpha_j)^{-1}$ is monic, $(\gamma x - \xi x)(\alpha_j) = 0$. Thus $(\gamma x)(a_j + N) = [(\gamma x)(\alpha_j)](a_j) = [(\xi x)(\alpha_j)](a_j) = (\xi x)(a_j + N)$. Hence $\gamma x = \xi x$ for all $x \in X$. Thus $\gamma = \xi$ and ξ is unique.

Hence $\varprojlim L_k \cong \mathrm{alglat}_R D_S$. Thus $\mathrm{alglat}_R(\varinjlim A_k)_S \cong \varprojlim (\mathrm{alglat}_R(A_k)_S)$. □

Throughout the rest of this thesis, R is a commutative ring with 1 and M is an R–module. Then M is given the R–R–bimodule structure from Example 1.2(b). From 1.1, $\mathrm{alglat}_R M_R = \{\phi \in \mathrm{End}\, M_R \mid \phi m \in Rm \text{ for all } m \in M\}$ and M is reflexive when $\lambda(R) = \mathrm{alglat}_R M_R$.

The second decomposition of $\mathrm{alglat}_R M_R$ arises when R is decomposable into a direct sum of finitely many commutative rings R_i, $i = 1, \ldots, n$. The following results about R are well-known and the approach taken here is that of Lambek ([13; pp17-19]).

Let the commutative ring R have a finite direct sum decomposition $R = \bigoplus_{i=1}^{n} R_i$. Then there are idempotent elements e_i in R_i with $1 = e_1 + e_2 + \cdots + e_n$ for $i = 1, \ldots, n$. Then for any i, $e_i = e_ie_1 + \cdots + e_i^2 + \cdots + e_ie_n$. The sum is direct so that $e_ie_j = \begin{cases} 0 & \text{if } i \neq j \\ e_i & \text{if } i = j \end{cases}$. Let r_j be an element of $R_j \subseteq R$. Then $r_j = e_1r_j + \cdots + e_nr_j$. The sum is direct so $e_ir_j = \begin{cases} 0 & \text{if } i \neq j \\ r_j & \text{if } i = j \end{cases}$. So any element r_i in R_i can be written $r_i = e_ir_i$ and thus $R_i \subseteq e_iR$ for each i. To show the reverse inclusion let r be an element of R so that $e_ir \in e_iR \subseteq R$. From the decomposition of R, e_ir can be expressed as $e_ir = s_1 + \cdots + s_n$ with s_j in R_j. Then $e_ir = e_i^2r =$

$e_is_1 + \cdots + e_is_n = s_i$ so that $e_ir \in R_i$ and thus $e_iR \subseteq R_i$. Hence $R_i = e_iR$.

Let M be an R–module. Then, for any element m in M, $m = e_1m + \cdots + e_nm$ so that $M = \sum_{i=1}^{n} e_iM$. Moreover the sum is direct. For suppose that there is some $m \in e_1M \cap (\sum_{i=2}^{n} e_iM)$. Then $m = e_1x_1 = e_2x_2 + \cdots + e_nx_n$ with x_i in M for all i. Multiplying through by e_1 gives $e_1e_2x_2 + \cdots + e_1e_nx_n = e_1^2x_1 = e_1x_1$ and thus $m = 0$. Hence $M = \bigoplus_{i=1}^{n} e_iM$. Thus M has a decomposition into a finite direct sum of R–modules. Let $M_i = e_iM$. Then each M_i can be considered as an R_i–module and hence as an R_i–R_i–bimodule.

These finite direct sum decompositions of R and M are used in the following theorem to give the second decomposition of $\operatorname{alglat}_R M_R$.

1.10 Theorem

Let R be a commutative ring with 1 with a decomposition into a finite direct sum of rings, $R = \bigoplus_{i=1}^{n} R_i$, and let M be any R–module. There exist elements e_i in R such that $R_i = e_iR$ for $i = 1, \ldots, n$. Let $M_i = e_iM$ so that $M = \bigoplus_{i=1}^{n} M_i$. Then $\operatorname{alglat} M \cong \bigoplus_{i=1}^{n} \operatorname{alglat} M_i$ where each M_i is considered as an R_i–R_i–bimodule.

Proof

Let $\lambda_i : M \to M$ be the R–endomorphism $\lambda(e_i)$ (left scalar multiplication by e_i) for $i = 1, \ldots, n$. Then $\lambda_i = \lambda_i^2$ since $e_i = e_i^2$. Let ϕ be any element of $\operatorname{alglat}_R M_R$. Then ϕ is in $\operatorname{End} M_R$ and so both $\phi\lambda_i$ and $\lambda_i\phi$ are in $\operatorname{End} M_R$. So, given any element m of M, $(\phi\lambda_i)m = \phi(e_im) = \phi(me_i) = (\phi m)e_i = e_i(\phi m) = (\lambda_i\phi)m$. Thus $\phi\lambda_i = \lambda_i\phi$ for $i = 1, \ldots, n$.

Define $\sigma : \operatorname{alglat}_R M_R \to \bigoplus_{i=1}^{n} \operatorname{alglat} M_i$ by $\phi \mapsto (\phi\lambda_1, \ldots, \phi\lambda_n)$ where each M_i is considered as an R_i–R_i–bimodule.

$\underline{\operatorname{im}\sigma \subseteq \bigoplus_{i=1}^{n} \operatorname{alglat} M_i}$

Let ϕ be an element of $\operatorname{alglat}_R M_R$. Then $\phi\lambda_i \in \operatorname{End} M_i$ since $R_i \subseteq R$ and $M_i \subseteq$

M. Let $e_i m$ be an element of M_i. Then there is some r in R with $\phi(e_i m) = r(e_i m)$ and so $(\phi\lambda_i)(e_i m) = \phi(e_i^2 m) = \phi(e_i m) = r(e_i m) = (re_i)(e_i m) \in R_i(e_i m)$. Thus $\phi\lambda_i \in$ $\text{alglat}\, M_i$ for all $i = 1, \ldots, n$. Hence $\text{im}\,\sigma \subseteq \bigoplus_{i=1}^{n} \text{alglat}\, M_i$.

<u>σ is a ring homomorphism</u>

For each i, $\phi\lambda_i$ is well-defined and so σ is well-defined.

Let ϕ, θ be in $\text{alglat}_R M_R$. Then $\sigma\phi + \sigma\theta = (\phi\lambda_1, \ldots, \phi\lambda_n) + (\theta\lambda_1, \ldots, \theta\lambda_n) =$ $((\phi + \theta)\lambda_1, \ldots, (\phi + \theta)\lambda_n) = \sigma(\phi + \theta)$. Also $(\sigma\phi)(\sigma\theta) = (\phi\lambda_1, \ldots, \phi\lambda_n)(\theta\lambda_1, \ldots, \theta\lambda_n)$ $= (\phi\theta\lambda_1^2, \ldots, \phi\theta\lambda_n^2) = ((\phi\theta)\lambda_1, \ldots, (\phi\theta)\lambda_n) = \sigma(\phi\theta)$. Let 1 be the identity element in $\text{alglat}_R M_R$ and let 1_i be the identity element in $\text{alglat}\, M_i$ for $i = 1, \ldots, n$. For each i and each element $e_i m$ in M_i, $(1\lambda_i)(e_i m) = 1(\lambda_i(e_i m)) = e_i^2 m = e_i m$ and so $1\lambda_i$ is the identity element in $\text{alglat}\, M_i$. Thus $1\lambda_i = 1_i$. So $\sigma 1 = (1_1, \ldots, 1_n)$, the identity element in $\bigoplus_{i=1}^{n} \text{alglat}\, M_i$. Thus σ is a well-defined ring homomorphism.

<u>σ is a monomorphism</u>

Suppose that $\phi \in \ker\sigma$. Then $(\phi\lambda_1, \ldots, \phi\lambda_n) = (0, \ldots, 0)$ so that $\phi\lambda_i = 0$ for $i = 1, \ldots, n$. Let m be any element of M. Then $\phi m = \phi(e_1 m + \cdots + e_n m) =$ $(\phi\lambda_1)m + \cdots + (\phi\lambda_n)m = 0$. Thus $\phi = 0$ and so σ is a monomorphism.

<u>σ is an epimorphism</u>

Let $(\theta_1, \ldots, \theta_n)$ be in $\bigoplus_{i=1}^{n} \text{alglat}\, M_i$. Define a map $\theta : M \to M$ by $m \mapsto \sum_{i=1}^{n} \theta_i(\lambda_i m)$.

For each i, $\lambda_i(M) = e_i M = M_i$ and so $\theta_i(\lambda_i m)$ is well-defined. Thus θ is well-defined.

Let m and m′ be elements of M and r an element of R. Then $\theta_i(\lambda_i(m + m')) =$ $\theta_i(\lambda_i m) + \theta_i(\lambda_i m')$ and so $\theta(m + m') = \sum_{i=1}^{n} \theta_i(\lambda_i(m + m')) = \sum_{i=1}^{n} \left[\theta_i(\lambda_i m) + \theta_i(\lambda_i m')\right]$ $= \sum_{i=1}^{n} \theta_i(\lambda_i m) + \sum_{i=1}^{n} \theta_i(\lambda_i m') = \theta m + \theta m'$. Also $\theta_i(\lambda_i(mr)) = \theta_i(e_i mr) = \theta_i(e_i m e_i r) =$ $\left[\theta_i(e_i m)\right](e_i r) = \left(\left[\theta_i(e_i m)\right]e_i\right)r = \left[\theta_i(e_i m e_i)\right]r = \left[\theta_i(e_i m)\right]r = (\theta_i(\lambda_i m))r$. Then $\theta(mr) =$ $\sum_{i=1}^{n} \theta_i(\lambda_i(mr)) = \sum_{i=1}^{n} (\theta_i(\lambda_i m))r = \left[\sum_{i=1}^{n} \theta_i(\lambda_i m)\right]r = (\theta m)r$. Thus $\theta \in \text{End}\, M_R$.

Let m be any element of M. There is some s_i in R_i with $\theta_i(e_i m) = s_i e_i m$ for each $i = 1, \ldots, n$. Then $\theta m = \sum_{i=1}^{n} \theta_i(\lambda_i m) = \sum_{i=1}^{n} \theta_i(e_i m) = \sum_{i=1}^{n} (s_i e_i m) = \left[\sum_{i=1}^{n} s_i e_i\right] m \in Rm$. Thus $\theta \in \text{alglat}_R M_R$.

For any j, $(\theta\lambda_j)(e_j m) = \theta(e_j^2 m) = \theta(e_j m) = \sum_{i=1}^{n} \theta_i(\lambda_i(e_j m)) = \sum_{i=1}^{n} \theta_i(e_i e_j m) = \theta_j(e_j m)$ and thus $\theta\lambda_j = \theta_j$ on M_j. Hence $\sigma\theta = (\theta\lambda_1, \ldots, \theta\lambda_n) = (\theta_1, \ldots, \theta_n)$ and so σ is an epimorphism.

Thus σ is a ring isomorphism and $\text{alglat}_R M_R \cong \bigoplus_{i=1}^{n} \text{alglat} M_i$ where each M_i is considered as an R_i–R_i–bimodule. □

The third decomposition of alglat applies to torsion modules over h-local domains. Matlis gave the definition of an h-local domain in [15; §8] and then showed that any torsion module over an h-local domain has a decomposition as the direct sum of the localised modules T_M where M ranges over all maximal ideals of R. In order to use this to give a decomposition of alglat, several results concerning h-local domains are required.

1.11 Definition of an h-local domain (Matlis)

An h-local domain is an integral domain which satisfies the following two conditions:

(i) each non-zero prime ideal is contained in a unique maximal ideal, and

(ii) each non-zero element is contained in only finitely many maximal ideals.

Thus an integral domain is h-local if and only if modulo any non-zero prime ideal it is a local ring and modulo any non-zero ideal at all it is a semilocal ring. Local domains and Dedekind domains are examples of h-local domains.

The following theorem of Matlis is from [15; Corollary 8.6] and gives a decomposition for any torsion module over an h-local domain. In fact Matlis proved in [16; Theorem 3.1] that for an integral domain R, the statement that R is an h-local domain is equivalent to the condition that $T \cong \bigoplus_M T_M$ for all torsion R–modules T, where M ranges over all maximal ideals of R.

<u>1.12 Theorem</u> (Matlis)

Let R be an h-local domain and let T be a torsion R–module. Then the localisation T_M is a torsion R_M–module and $T \cong \bigoplus_M T_M$ where M ranges over all maximal ideals of R.

The isomorphism in Theorem 1.12 is given by $\tau : t \mapsto \left(\frac{t}{1}\right)$. For each non-zero element t of T, Ann(t) is a non-zero ideal and is thus contained in only finitely many maximal ideals of R. If M is a maximal ideal of R not containing Ann(t) then $\frac{t}{1} = \frac{0}{1}$ in T_M. So only finitely many entries in the image $\left(\frac{t}{1}\right)$ of t are non-zero. Thus the image of T under τ does indeed lie in the direct sum $\bigoplus_M T_M$.

The definition of a colocal ideal is given next (from [16]). This is followed by a characterisation by Matlis ([16; Theorem 2.3]) of h-local domains using colocal ideals.

<u>1.13 Definition of a colocal ideal</u> (Matlis)

An ideal of an integral domain R is said to be colocal if it is contained in only one maximal ideal of R.

<u>1.14 Proposition</u> (Matlis)

Let R be an integral domain. Then R is an h-local domain if and only if every non-zero ideal of R is a finite intersection of colocal ideals.

Matlis gives various elementary properties of colocal ideals in [16; p148], some of which appear in Proposition 1.16. These use the following definition of his, of a normal decomposition of an ideal into a finite intersection of colocal ideals.

1.15 Definition of a normal decomposition (Matlis)

Let $I = \bigcap_{i=1}^{n} I_i$ be a finite intersection of ideals, where I_i is a colocal ideal belonging to a maximal ideal M_i. This decomposition is said to be normal if $M_j \neq M_k$ for $j \neq k$.

1.16 Proposition (Matlis)

Let R be an h-local domain with I, J, I_i non-zero ideals of R and M, M_i maximal ideals of R ($i = 1, \ldots, n$). Then

(i) if I and J are colocal in M then $I \cap J$ is colocal in M,

(ii) if I is colocal in M and $v \notin M$ then $I + Rv = R$,

(iii) if $\bigcap_{i=1}^{n} I_i$ is a normal decomposition with I_i colocal in M_i then $I_1 + \prod_{i=2}^{n} I_i = R$,

(iv) if $J = \bigcap_{i=1}^{n} I_i$ is a normal decomposition with I_i colocal in M_i and if $J \subseteq M$ then $M \in \{M_1, \ldots, M_n\}$.

Proof

(i) It is clear that $I \cap J \subseteq M$. Suppose for contradiction that $I \cap J$ is not colocal in M. Then there is a maximal ideal N of R, distinct from M, with $I \cap J \subseteq N$. Since $I \not\subseteq N$ and $J \not\subseteq N$ there are elements $i \in I$, $i \notin N$ and $j \in J$, $j \notin N$. But then $ij \notin N$ which contradicts $I \cap J \subseteq N$. Thus $I \cap J$ is colocal in M.

(ii) Suppose for contradiction that $I + Rv \subset R$. Then there is a maximal ideal N of R with $I + Rv \subseteq N$ and so $I \subseteq N$. But I is colocal in M and thus $N = M$. This gives $v \in M$ which is the required contradiction.

(iii) From property (ii), $I_1 + I_2 = R$. For the induction hypothesis assume that $I_1 + \prod_{i=2}^{n-1} I_i = R$. Again from (ii), $I_1 + I_n = R$. Then $R = (I_1 + \prod_{i=2}^{n-1} I_i)(I_1 + I_n) = (I_1^2 + I_1 \prod_{i=2}^{n-1} I_i + I_1 I_n) + (\prod_{i=2}^{n} I_i) \subseteq I_1 + \prod_{i=2}^{n} I_i$. Thus $R = I_1 + \prod_{i=2}^{n} I_i$. This completes the proof by induction.

(iv) Since $J = \bigcap_{i=1}^{n} I_i \subseteq M$, a maximal ideal, there is some j with $I_j \subseteq M$. The ideal I_j is colocal in M_j and so $M = M_j$. Thus $M \in \{M_1, \ldots, M_n\}$. □

Remarks

(a) It is clear from property (i) that every finite intersection of colocal ideals can be normalised.

(b) It follows immediately from property (ii) that if I and J are colocal ideals belonging to different maximal ideals then $I + J = R$. This is used in the proof of (iii).

The next lemma, which does not appear to be in the literature, gives a form of "Chinese Remainder Theorem" for h-local domains and uses these properties of colocal ideals. This will enable the decomposition for alglat of Theorem 1.18 to be proved.

1.17 Lemma

Let R be an h-local domain and let T be a torsion R–module. Let t be a non-zero element of T with $M_1, \ldots, M_n$ the distinct maximal ideals of R containing Ann(t). Let $\frac{a_i}{s_i}$ be any elements of R_{M_i} $(i = 1, \ldots, n)$. Then there is an element r of R with $\frac{rt}{1} = \frac{a_it}{s_i}$ in T_{M_i} for $i = 1, \ldots, n$.

Proof

Since T is a torsion module, $\mathrm{Ann}(t) \neq 0$. From Proposition 1.14, there is a normal decomposition $\mathrm{Ann}(t) = \bigcap_{i=1}^{n} I_i$ with I_i colocal in M_i for $i = 1, \ldots, n$.

Consider the maximal ideal M_1. Since I_1 is colocal in M_1 and $s_1 \notin M_1$, Proposition 1.16 gives $I_1 + Rs_1 = R$. Then there are elements $u_1 \in I_1$ and $v_1 \in R$ with $u_1 + v_1s_1 = 1$. Again from Proposition 1.16, $I_1 + \prod_{i=2}^{n} I_i = R$. This gives elements $b_1 \in I_1$ and $c_1 \in \prod_{i=2}^{n} I_i$ with $b_1 + c_1 = 1$. Let $r_1 = a_1c_1v_1$. Then in T_{M_1}, $\frac{r_1t}{1} = \frac{a_1c_1v_1t}{1} = \frac{a_1c_1v_1s_1t}{s_1} = \frac{a_1c_1(1-u_1)t}{s_1}$. Since $c_1u_1 \in \mathrm{Ann}(t)$, $\frac{r_1t}{1} = \frac{a_1c_1t}{s_1}$. So $\frac{r_1t}{1} = \frac{a_1(1-b_1)t}{s_1} = \frac{a_1t}{s_1} - \frac{a_1b_1c_1t}{s_1c_1} = \frac{a_1t}{s_1}$, noting that $c_1 \notin M_1$ and $b_1c_1 \in \mathrm{Ann}(t)$. For $i = 2, \ldots, n$, $b_1 \notin M_i$ and so in T_{M_i}, $\frac{r_1t}{1} = \frac{a_1c_1v_1t}{1} = \frac{a_1b_1c_1v_1t}{b_1} = \frac{0}{1}$.

By considering each maximal ideal M_i, $i = 1, \ldots, n$, there are elements r_i in R such that $\frac{r_i t}{1} = \frac{a_i t}{s_i}$ in T_{M_i} and $\frac{r_i t}{1} = \frac{0}{1}$ in T_{M_j} for $j \neq i$. Let $r = \sum_{i=1}^{n} r_i$. Then r is the required element, since it now follows that, in T_{M_i}, $\frac{rt}{1} = \frac{(r_1 + \ldots + r_n)t}{1} = \frac{r_i t}{1} = \frac{a_i t}{s_i}$ for $i = 1, \ldots, n$. □

The next result is the decomposition theorem anticipated from Theorem 1.12.

1.18 Theorem

Let R be an h-local domain and let T be a torsion R–module so that $T \cong \bigoplus_M T_M$ where M ranges over all maximal ideals of R. Then $\operatorname{alglat}_R T_R \cong \prod_M \operatorname{alglat} T_M$ where M ranges over all maximal ideals of R and each T_M is an R_M–R_M–bimodule.

Proof

Let $N = \prod_M \operatorname{alglat} T_M$ where each T_M is an R_M–R_M–bimodule. Let τ be the R–isomorphism introduced in Theorem 1.12, $\tau : t \mapsto \left(\frac{t}{1}\right)$ from T into $\bigoplus_M T_M$.

Define $\alpha : N \to \operatorname{alglat}_R T_R$ by $(\theta_M) \mapsto \theta$ where $\theta_M \in \operatorname{alglat} T_M$ and $\theta t = \tau^{-1}\left(\theta_M \frac{t}{1}\right)$.

$\operatorname{im}\alpha \subseteq \operatorname{alglat}_R T_R$

Let $t_1, t_2, t \in T$ and $r \in R$. Then $\theta t_1 + \theta t_2 = \tau^{-1}\left(\theta_M \frac{t_1}{1}\right) + \tau^{-1}\left(\theta_M \frac{t_2}{1}\right) = \tau^{-1}\left(\theta_M\left(\frac{t_1}{1} + \frac{t_2}{1}\right)\right) = \tau^{-1}\left(\theta_M\left(\frac{t_1 + t_2}{1}\right)\right) = \theta(t_1 + t_2)$, and $\theta(tr) = \tau^{-1}\left(\theta_M \frac{tr}{1}\right) = \tau^{-1}\left(\left(\theta_M \frac{t}{1}\right)\frac{r}{1}\right) = \tau^{-1}\left(\left(\theta_M \frac{t}{1}\right)r\right) = \left[\tau^{-1}\left(\theta_M \frac{t}{1}\right)\right]r = (\theta t)r$. Hence $\theta \in \operatorname{End} T_R$.

Let t be a non-zero element of the torsion module T. Then Ann(t) is non-zero and is thus contained in only finitely many maximal ideals of R, $M_1, \ldots, M_n$. If $M \notin \{M_1, \ldots, M_n\}$ then $\operatorname{Ann}(t) \not\subseteq M$ and so $\frac{t}{1} = \frac{0}{1}$ in R_M. For $i = 1, \ldots, n$, θ_{M_i} is in $\operatorname{alglat} T_{M_i}$ (where T_{M_i} is an R_{M_i}–module) and so there are elements $\frac{a_i}{s_i}$ in R_{M_i} with $\theta_{M_i}\left(\frac{t}{1}\right) = \frac{a_i t}{s_i}$. Then, from Lemma 1.17, there is an element r in R with $\frac{a_i t}{s_i} = \frac{rt}{1}$ for $i = 1, \ldots, n$. Hence $\theta t = \tau^{-1}\left(\theta_M \frac{t}{1}\right) = \tau^{-1}\left(\frac{rt}{1}\right) = rt$. Thus $\theta \in \operatorname{alglat}_R T_R$. Hence $\operatorname{im}\alpha \subseteq \operatorname{alglat}_R T_R$.

α is a ring homomorphism

Let (θ_M) and (ϕ_M) be in N with images θ and ϕ under α respectively. Let ψ be the image of $(\theta_M + \phi_M)$ under α. Then $\psi t = \tau^{-1}\left[(\theta_M + \phi_M)\frac{t}{1}\right] = \tau^{-1}\left[\theta_M\frac{t}{1} + \phi_M\frac{t}{1}\right] = \tau^{-1}\left[\theta_M\frac{t}{1}\right] + \tau^{-1}\left[\phi_M\frac{t}{1}\right] = \theta t + \phi t = (\theta + \phi)t$ for all t in T. So $\psi = \theta + \phi$ and thus $\alpha(\theta_M) + \alpha(\phi_M) = \alpha(\theta_M + \phi_M)$. Let ξ be the image of $(\theta_M\phi_M)$ under α. For t in T, write $\phi t = t_1$ in T so that $\phi_M\frac{t}{1} = \frac{t_1}{1}$ in T_M for all maximal ideals M. Then $\xi t = \tau^{-1}\left[(\theta_M\phi_M)\frac{t}{1}\right] = \tau^{-1}\left[\theta_M(\phi_M\frac{t}{1})\right] = \tau^{-1}\left[\theta_M\frac{t_1}{1}\right] = \theta t_1 = \theta(\phi t) = (\theta\phi)t$ for all t in T and so $\xi = \theta\phi$. Hence $\alpha(\theta_M)\alpha(\phi_M) = \alpha(\theta_M\phi_M)$.

Let 1_M be the unit element in alglat T_M for all maximal ideals M and let 1 be the unit element in $\text{alglat}_R T_R$. Then (1_M) is the unit element in N. If $\alpha(1_M) = \theta$ then $\theta t = \tau^{-1}\left[1_M\frac{t}{1}\right] = \tau^{-1}\left[\frac{t}{1}\right] = t$ and so $\theta = 1$. Thus $\alpha(1_M) = 1$. Hence α is a well-defined ring homomorphism.

α is a monomorphism

Suppose $(\theta_M) \in \ker\alpha$ with image θ under α so that $\theta = 0$.

Let M′ be any maximal ideal of R and consider an element $\frac{t}{s}$ in $T_{M'}$. There is an element t_1 in T with $\tau t_1 = \left[\frac{0}{1}, \ldots, \frac{0}{1}, \frac{t}{s}, \frac{0}{1}, \ldots\right]$ so that $\frac{t_1}{1} = \frac{t}{s}$ if $M = M'$ and $\frac{t_1}{1} = \frac{0}{1}$ otherwise. Then $\theta_{M'}(\frac{t_1}{1}) = \theta_{M'}(\frac{t}{s})$ and, for all $M \neq M'$, $\theta_M\frac{t_1}{1} = \frac{0}{1}$. So $0 = \theta t_1 = \tau^{-1}\left[\theta_M\frac{t_1}{1}\right] = \tau^{-1}\left[\frac{0}{1}, \ldots, \frac{0}{1}, \theta_{M'}(\frac{t}{s}), \frac{0}{1}, \ldots\right]$. Thus $\theta_{M'}(\frac{t}{s}) = \frac{0}{1}$ for all $\frac{t}{s}$ in $T_{M'}$ so $\theta_{M'} = 0$.

Thus $\theta_M = 0$ for all maximal ideals M of R. Hence $\ker\alpha = 0$ and α is a monomorphism.

α is an epimorphism

For each maximal ideal M of R, let $\pi_M : \bigoplus_M T_M \to T_M$ be the canonical projection map and $\mu_M : T_M \to \bigoplus_M T_M$ be the canonical injection map. Then π_M and μ_M are R–homomorphisms and $\pi_M\mu_M$ is the identity map on T_M for all maximal ideals of R.

Define $\beta : \text{alglat}_R T_R \to N$ by $\theta \mapsto (\theta_M)$ where $\theta_M = \pi_M\tau\theta\tau^{-1}\mu_M$.

$\operatorname{im}\beta \subseteq N$

Let M be any maximal ideal of R. Let $\frac{t_1}{s_1}$, $\frac{t_2}{s_2}$ and $\frac{t}{s}$ be elements of T_M and $\frac{r}{u}$ an element of R_M. Then θ_M is an R–endomorphism of T_M, being a product of R–homomorphisms. So $\theta_M(\frac{t_1}{s_1} + \frac{t_2}{s_2}) = \theta_M(\frac{t_1}{s_1}) + \theta_M(\frac{t_2}{s_2})$, and $\left[\theta_M(\frac{t\,r}{s\,u})\right]u = \theta_M(\frac{tr}{s}) = \left[\theta_M(\frac{t}{s})\right]r$ so that $\theta_M(\frac{t\,r}{s\,u}) = \left[\theta_M(\frac{t}{s})\right]\frac{r}{u}$. Thus $\theta_M \in \operatorname{End} {T_M}_{R_M}$.

For $\frac{t}{s}$ in T_M, there is some element a in R with $\theta\tau^{-1}\mu_M(\frac{t}{s}) = a\tau^{-1}\mu_M(\frac{t}{s})$. Then $\theta_M\frac{t}{s} = (\pi_M\tau\theta\tau^{-1}\mu_M)(\frac{t}{s}) = (\pi_M\tau a\tau^{-1}\mu_M)(\frac{t}{s}) = (\pi_M\tau\tau^{-1}\mu_M)(\frac{at}{s}) = \frac{at}{s} = \frac{a}{1}\cdot\frac{t}{s}$. So $\theta_M\frac{t}{s} \in R_M\frac{t}{s}$. Hence $\theta_M \in \operatorname{alglat} T_M$ (where T_M is an R_M–module). Thus $\operatorname{im}\beta \subseteq N$.

β is a ring homomorphism

Let θ and ϕ belong to $\operatorname{alglat}_R T_R$. Then $\beta(\theta + \phi) = (\pi_M\tau(\theta + \phi)\tau^{-1}\mu_M) = (\pi_M\tau\theta\tau^{-1}\mu_M + \pi_M\tau\phi\tau^{-1}\mu_M) = (\pi_M\tau\theta\tau^{-1}\mu_M) + (\pi_M\tau\phi\tau^{-1}\mu_M) = \beta\theta + \beta\phi$. Also $(\beta\theta)(\beta\phi) = (\pi_M\tau\theta\tau^{-1}\mu_M)(\pi_M\tau\phi\tau^{-1}\mu_M) = (\pi_M\tau\theta\tau^{-1}\mu_M\pi_M\tau\phi\tau^{-1}\mu_M)$. Let $\frac{t}{s}$ be in T_M for some maximal ideal M. Then there is an element a in R with $\phi\tau^{-1}\mu_M(\frac{t}{s}) = a\tau^{-1}\mu_M(\frac{t}{s})$. So $(\pi_M\tau\theta\tau^{-1}\mu_M\pi_M\tau\phi\tau^{-1}\mu_M)(\frac{t}{s}) = (\pi_M\tau\theta\tau^{-1}\mu_M\pi_M\tau)(a\tau^{-1}\mu_M)(\frac{t}{s}) = (\pi_M\tau\theta\tau^{-1}\mu_M\pi_M\tau)(\tau^{-1}\mu_M)(\frac{at}{s}) = (\pi_M\tau\theta\tau^{-1}\mu_M)(\pi_M\tau\tau^{-1}\mu_M)(\frac{at}{s}) = (\pi_M\tau\theta\tau^{-1}\mu_M)(\frac{at}{s}) = (\pi_M\tau\theta)(a\tau^{-1}\mu_M)(\frac{t}{s}) = (\pi_M\tau\theta)(\phi\tau^{-1}\mu_M)(\frac{t}{s}) = (\pi_M\tau\theta\phi\tau^{-1}\mu_M)(\frac{t}{s})$. Thus $\pi_M\tau\theta\tau^{-1}\mu_M\pi_M\tau\phi\tau^{-1}\mu_M = \pi_M\tau\theta\phi\tau^{-1}\mu_M$. Hence $(\beta\theta)(\beta\phi) = (\pi_M\tau\theta\phi\tau^{-1}\mu_M) = \beta(\theta\phi)$.

Using the notation for unit elements as above, let (θ_M) is the image of 1 under β. Then $\theta_M = \pi_M\tau 1\tau^{-1}\mu_M = \pi_M\tau\tau^{-1}\mu_M = 1_M$. So $\beta 1 = (1_M)$, the unit element in N. Thus β is a well-defined ring homomorphism.

$\alpha\beta$ acts as the identity on $\operatorname{alglat}_R T_R$

Let θ be an element of $\operatorname{alglat}_R T_R$ and let ϕ be the image of θ under $\alpha\beta$. Then $\phi = \alpha(\beta\theta) = \alpha(\theta_M)$ where $\theta_M = \pi_M\tau\theta\tau^{-1}\mu_M$. Thus $\phi t = \tau^{-1}\left(\theta_M\frac{t}{1}\right)$ for all t in T.

Let t be a non-zero element of T so that Ann(t) is contained in only finitely many maximal ideals of R, $M_1, \ldots, M_n$. If M is a maximal ideal of R which does not contain Ann(t) then $\frac{t}{1} = \frac{0}{1}$ in T_M and so $\theta_M\frac{t}{1} = \frac{0}{1}$. Then $\tau t = \left(\frac{t}{1}\right) =$

$\sum_{i=1}^{n} \mu_{M_i}\frac{t}{1}$ and $\left(\theta_M\frac{t}{1}\right) = \sum_{i=1}^{n} \mu_{M_i}\theta_{M_i}\frac{t}{1}$. There are elements a_i in R, for $i = 1, \ldots, n$, with $\theta\tau^{-1}\mu_{M_i}\frac{t}{1} = a_i\tau^{-1}\mu_{M_i}\frac{t}{1}$. Then $\theta t = \sum_{i=1}^{n} \theta\tau^{-1}\mu_{M_i}\frac{t}{1} = \sum_{i=1}^{n} a_i\tau^{-1}\mu_{M_i}\frac{t}{1} = \sum_{i=1}^{n} a_i\tau^{-1}\mu_{M_i}(\pi_{M_i}\tau\tau^{-1}\mu_{M_i})\frac{t}{1} = \sum_{i=1}^{n} \tau^{-1}\mu_{M_i}(\pi_{M_i}\tau a_i\tau^{-1}\mu_{M_i})\frac{t}{1} = \tau^{-1}(\sum_{i=1}^{n} \mu_{M_i}\theta_{M_i}\frac{t}{1}) = \tau^{-1}\left(\theta_M\frac{t}{1}\right) = \phi t$.

Thus $\theta = \phi$ and so $\alpha\beta(\theta) = \theta$ and $\alpha\beta$ acts as the identity element on $\text{alglat}_R T_R$.

So, given θ in $\text{alglat}_R T_R$, $\beta\theta$ is in N and $\alpha(\beta\theta) = \theta$. Hence α is onto.

Thus α is a ring isomorphism and $\text{alglat}_R T_R \cong \prod_M \text{alglat}\, T_M$ where each T_M is an R_M–module and M ranges over all maximal ideals of R. □

The third decomposition theorem for alglat, Theorem 1.21, is an extension of this result. Theorem 1.21 deals with the case where R is a commutative ring with a prime ideal P such that R/P is an h-local domain and T is a torsion R/P–module. Theorem 1.18 is used in the proof. The following two propositions are also required, the first of which relates $\text{alglat}_R T_R$ to both $\text{alglat}_A T_A$ where $A = R/P$ and $\text{alglat}_B T_B$ where $B \cong R$. The second proposition is well-known and can be found in [19; p23].

1.19 Proposition

Let R be a commutative ring and let T be an R–module.

(i) If $I \subseteq \text{Ann}\,T$ for some ideal I of R, then T has an R/I–module structure given by $(r + I)t = rt$ for all $r \in R$, $t \in T$. Then $\text{alglat}_R T_R = \text{alglat}_A T_A$ where $A = R/I$.

(ii) If $\psi : B \to R$ is a ring isomorphism then T has a B–module structure given by $bt = (\psi b)t$ for all $b \in B$, $t \in T$. Then $\text{alglat}_R T_R = \text{alglat}_B T_B$.

Proof

Let t_1, t_2, t be elements of T and let r be an element of R.

(i) Let $\bar{r}$ be the coset r + I in A.

Let $\phi \in \text{alglat}_R T_R$. Then $\phi(t_1 + t_2) = \phi t_1 + \phi t_2$, $\phi(t\bar{r}) = \phi(tr) = (\phi t)r = (\phi t)\bar{r}$, and for each $t \in T$ there is some $s \in R$ with $\phi t = st$, so that $\phi t = \bar{s}t$ with $\bar{s} \in A$. Thus $\phi \in \text{alglat}_A T_A$.

Let $\phi \in \text{alglat}_A T_A$. Then $\phi(t_1 + t_2) = \phi t_1 + \phi t_2$, $\phi(tr) = \phi(t\bar{r}) = (\phi t)\bar{r} = (\phi t)r$, and for each $t \in T$ there is some $\bar{s} \in A$ with $\phi t = \bar{s}t$, so that $\phi t = st$ with $s \in R$. Thus $\phi \in \text{alglat}_R T_R$.

Hence $\text{alglat}_R T_R = \text{alglat}_A T_A$.

(ii) Let b be the element of B with $\psi b = r$.

Let $\phi \in \text{alglat}_R T_R$. Then $\phi(t_1 + t_2) = \phi t_1 + \phi t_2$, $\phi(tb) = \phi(t(\psi b)) = (\phi t)(\psi b) = (\phi t)b$, and for each $t \in T$ there is some $u \in R$ with $\phi t = ut$ and $u = \psi v$, so that $\phi t = (\psi v)t = vt$, $v \in B$. Thus $\phi \in \text{alglat}_B T_B$.

Let $\phi \in \text{alglat}_B T_B$. Then $\phi(t_1 + t_2) = \phi t_1 + \phi t_2$, $\phi(tr) = \phi(t(\psi b)) = \phi(tb) = (\phi t)b = (\phi t)(\psi b) = (\phi t)r$, and for each $t \in T$ there is some $v \in B$ with $\phi t = vt$, so that $\phi t = (\psi v)t$, $\psi v \in R$. Thus $\phi \in \text{alglat}_R T_R$.

Hence $\text{alglat}_R T_R = \text{alglat}_B T_B$. □

1.20 Proposition

Let R be a commutative ring with P a prime ideal of R and N a maximal ideal of R containing P. Then $(R/P)_{N/P} \cong R_N/P_N$.

The ring isomorphism in Proposition 1.20 is given by $\gamma : (R/P)_{N/P} \rightarrow R_N/P_N$, $\dfrac{r + P}{s + P} \mapsto \dfrac{r}{s} + P_N$.

The next result is the third decomposition theorem of Chapter 1. This will be used in Chapter 7 in determining the structure of alglat for modules over an FSI domain.

1.21 Theorem

Let R be a commutative ring with a prime ideal P such that R/P is an h-local domain. Let T be an R–module with $P \subseteq \mathrm{Ann}\,T$ and such that T is a torsion R/P–module. Then $\mathrm{alglat}_R T_R \cong \prod_N \mathrm{alglat}\, T_N$ where T_N is an R_N–module and N ranges over all maximal ideals of R containing P.

Proof

Write $S = R/P$. There is a 1-1 correspondence between the maximal ideals of R containing P and the maximal ideals of S.

Let N be any maximal ideal of R containing P and let $M = N/P$ so that M is a maximal ideal of S. Then $P_N \subseteq \mathrm{Ann}\,T_N$ (T_N is an R_N–module) so T_N can be considered as an R_N/P_N–module via $\left(\frac{r}{s} + P_N\right)\frac{t}{u} = \frac{rt}{su}$. From Proposition 1.19, $\mathrm{alglat}_{R_N} T_{N R_N} = \mathrm{alglat}_{(R_N/P_N)} T_{N(R_N/P_N)}$. Using the ring isomorphism γ in Proposition 1.20, T_N has an S_M–module structure defined by $\left(\frac{r+P}{s+P}\right)\frac{t}{u} = \left[\gamma\left(\frac{r+P}{s+P}\right)\right]\frac{t}{u} = \left(\frac{r}{s} + P_N\right)\frac{t}{u}$. Then $\mathrm{alglat}_{S_M} T_{N S_M} = \mathrm{alglat}_{(R_N/P_N)} T_{N(R_N/P_N)}$, again from Proposition 1.19. The map between T_N and T_M given by $\delta_N : T_N \to T_M$, $\frac{t}{s} \mapsto \frac{t}{s+P}$ is well-defined and is an S_M–module isomorphism. This gives rise to a ring isomorphism $\epsilon_N : \mathrm{alglat}_{S_M} T_{N S_M} \to \mathrm{alglat}_{S_M} T_{M S_M}$, $\phi_N \mapsto \delta_N \phi_N \delta_N^{-1}$. Combining these results gives a ring isomorphism $\epsilon_N : \mathrm{alglat}_{R_N} T_{N R_N} \to \mathrm{alglat}_{S_M} T_{M S_M}$, $\phi_N \mapsto \delta_N \phi_N \delta_N^{-1}$. Then there is a ring isomorphism defined by $\epsilon : \prod_N \mathrm{alglat}_{R_N} T_{N R_N} \to \prod_M \mathrm{alglat}_{S_M} T_{M S_M}$, $(\phi_N) \mapsto (\epsilon_N \phi_N) = (\delta_N \phi_N \delta_N^{-1})$, where N ranges over all maximal ideals of R containing P and M ranges over all maximal ideals of S.

By hypothesis S is an h-local domain and T is a torsion S–module. So $\mathrm{alglat}_S T_S = \alpha\left(\prod_M \mathrm{alglat}_{S_M} T_{M S_M}\right)$ where α is the ring isomorphism in Theorem 1.18 and M ranges over all maximal ideals of S. Using Proposition 1.19 again, $\mathrm{alglat}_R T_R = \mathrm{alglat}_S T_S$ since $P \subseteq \mathrm{Ann}\,T$.

Thus $\mathrm{alglat}_R T_R = \mathrm{alglat}_S T_S = \alpha\left(\prod_M \mathrm{alglat}_{S_M} T_{M S_M}\right) = \alpha\epsilon\left(\prod_N \mathrm{alglat}_{R_N} T_{N R_N}\right)$. Hence $\mathrm{alglat}_R T_R \cong \prod_N \mathrm{alglat}\, T_N$ where T_N is an R_N–module and N ranges over all maximal ideals of R containing P. □

Remark

If $P = 0$ then R is an h-local domain, $M = N$ and $S = R$ and so $S_M = R_N$. Thus $T_M = T_N$ so δ_N is the identity on T_N and ϵ_N is the identity on $\mathrm{alglat}_{R_N} T_{N R_N}$. The map ϵ acts on $\prod_N \mathrm{alglat}_{R_N} T_{N R_N}$ as the identity and so $\mathrm{alglat}_R T_R = \alpha\left(\prod_N \mathrm{alglat}_{R_N} T_{N R_N}\right)$ as given in Theorem 1.18.

A discussion of the structure of alglat for all modules over particular classes of rings will be given in the later chapters, when these decomposition theorems will be used. The next two chapters are concerned with reflexive modules and rings over which every finitely generated module is reflexive.

Chapter 2 Non-torsion modules, scalar-reflexive and locally scalar-reflexive rings

For a ring R, the characterisation of alglat$_R M_R$ for all R–modules M falls broadly into the discussion of two cases, where M is torsion and where M is non-torsion. This chapter looks at results for non-torsion modules and gives conditions on a ring for all non-torsion modules to be reflexive. These results are used in later chapters to reduce the study of alglat for all modules over particular classes of rings to the consideration of torsion modules.

The main result of this chapter is Theorem 2.2 which gives a condition on a ring R that is equivalent to every non-torsion R–module being reflexive. It is shown that it is sufficient to have all 2–generated non-torsion modules reflexive. Hadwin and Kerr have proved in [9; Proposition 7] that every non-torsion module over a domain is reflexive. This result can now be derived from Theorem 2.2 and is given in Corollary 2.3.

In [9] Hadwin and Kerr defined a ring to be scalar-reflexive if every finitely generated module is reflexive (2.4). Another corollary of Theorem 2.2 (which is not proven by Hadwin and Kerr) shows that every non-torsion module over a scalar-reflexive ring is reflexive. The work of Hadwin and Kerr on finitely generated modules and reflexivity in [8] and [9] means that the property that every 2–generated non-torsion module is reflexive is a useful equivalent to the condition that every non-torsion module be reflexive.

The results proved in this chapter concerning non-torsion modules have a greater degree of generality than any results as yet obtained for torsion modules. In particular the corresponding result to Theorem 2.2 does not hold for torsion modules. An example of a local scalar-reflexive ring illustrating this is discussed following Theorem 2.11.

Theorem 2.11 is a result of Hadwin and Kerr and gives equivalent conditions, for a local ring, to the ring being scalar-reflexive. In particular a local ring is scalar-reflexive if and only if it is an almost maximal valuation ring. This motivates the definition in 2.12 where a ring is defined to be locally scalar-reflexive if every localisation at a maximal ideal is scalar-reflexive. Hadwin and Kerr ask in [8] whether the property of being scalar-reflexive is closed under localisations. This is an open question. However a converse is proved in Theorem 2.17 which shows that any locally scalar-reflexive ring is scalar-reflexive.

The final result of the chapter returns to non-torsion modules. This is a further corollary to Theorem 2.2 and shows that every non-torsion module over a locally scalar-reflexive ring is reflexive.

The first result, which was noted by Hadwin and Kerr in [8; p3], is well-known and will be frequently used.

2.1 Proposition

Let R be a commutative ring. Then every finite direct sum of cyclic R-modules is reflexive.

Proof

Let $M = \bigoplus_{i=1}^{n} Rm_i$, a finite direct sum of cyclic R-modules, and let ϕ be in $\mathrm{alglat}_R M_R$. Then there are elements r_i and r in R with $\phi m_i = r_i m_i$ $(i = 1, \ldots, n)$ and $\phi(m_1 + m_2 + \cdots + m_n) = r(m_1 + m_2 + \cdots + m_n)$. So $r(m_1 + m_2 + \cdots + m_n) = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n$. The sum is direct and so $rm_i = r_i m_i$ for all i. It follows that $\phi = \lambda(r)$. Hence $\mathrm{alglat}_R M_R = \lambda(R)$ and M is reflexive. $\square$

The next theorem is the main result of the chapter. It limits the study of non-torsion modules not only to the finitely generated case but to the consideration of 2-generated modules. This helps in the construction of examples when looking at non-torsion modules. Both finitely generated and 2-generated conditions play an important part in this thesis, especially in this chapter and in Chapter 3.

2.2 Theorem

Let R be a commutative ring. Then the following are equivalent:

(i) every non-torsion R–module is reflexive,

(ii) every 2–generated non-torsion R–module is reflexive.

Proof

The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i)

Let M be a non-torsion R–module and let ϕ be any element of $\mathrm{alglat}\,{}_RM_R$. Then there is an element m in M with $\mathrm{Ann}(m) = 0$. The R–module Rm is reflexive and ϕ is in $\mathrm{alglat}\,{}_RRm_R$. Thus there is some r in R with $\phi = \lambda(r)$ on Rm.

Let x be any element of the module M and consider $N = Rm + Rx$. Then ϕ is in $\mathrm{alglat}\,{}_RN_R$. The R–module N is non-torsion and 2–generated and so, by hypothesis, is reflexive. So there is an element s in R with $\phi = \lambda(s)$ on N. Since m is in N, $\phi m = rm = sm$ and so $r = s$. Then $\phi = \lambda(r)$ on N and so $\phi x = rx$.

Thus $\phi = \lambda(r)$ and $\mathrm{alglat}\,{}_RM_R = \lambda(R)$. Hence M is reflexive as required. □

The following corollary to this theorem, which was mentioned above and is proved by Hadwin and Kerr in [9; Proposition 7], can now be derived from this theorem.

2.3 Corollary (Hadwin and Kerr)

Let R be a commutative domain. Then every non-torsion R–module is reflexive.

Proof

Let $M = Rx + Ry$ be a non-torsion 2–generated R–module. If $Rx \cap Ry = 0$ then $M = Rx \oplus Ry$ which is reflexive (Proposition 2.1). So suppose $Rx \cap Ry \neq$

0. Then there are elements a, b in R with $0 \neq ax = by$. Let ϕ be an element of $\mathrm{alglat}_R M_R$. There are elements r, s in R with $\phi x = rx$ and $\phi y = sy$. Then $rby = rax = \phi(ax) = \phi(by) = sby = sax$. So $b(r - s) \in \mathrm{Ann}(Ry)$ and $a(r - s) \in \mathrm{Ann}(Rx)$. Since the module M is faithful, $ab(r - s) \in \mathrm{Ann}(Rx) \cap \mathrm{Ann}(Ry) = 0$. The elements a, b are non-zero and R is a domain so $r - s = 0$. Then $\phi x = rx$ and $\phi y = ry$ and so $\phi = \lambda(r)$. Hence $\mathrm{alglat}_R M_R = \lambda(R)$.

Thus every non-torsion 2–generated R–module is reflexive. The result follows from Theorem 2.2. □

In [8] and [9], Hadwin and Kerr studied rings in which every finitely generated module is reflexive, making the following definition in [9] (see comments in the Introduction).

2.4 Definition of a scalar-reflexive ring (Hadwin and Kerr)

A ring R is said to be scalar-reflexive if every finitely generated R–module is reflexive.

The ring of integers, $\mathbb{Z}$, is an example of a scalar-reflexive ring. For it is known from abelian group theory that every finitely generated $\mathbb{Z}$–module can be expressed as finite direct sum of cyclic modules. From Proposition 2.1, every finite direct sum of cyclic $\mathbb{Z}$–modules is reflexive. Thus every finitely generated $\mathbb{Z}$–module is reflexive and so $\mathbb{Z}$ is scalar-reflexive.

From the definition of a scalar-reflexive ring it is clear that every 2–generated module over a scalar-reflexive ring is reflexive. This gives the following corollary to Theorem 2.2.

2.5 Corollary

Let R be a scalar-reflexive ring. Then every non-torsion R–module is reflexive.

The definition of an FGC ring, made in [23], is given next and provides a class of rings that are scalar-reflexive. This was noted in [8; Proposition 4] and is immediate from Proposition 2.1.

2.6 Definition of an FGC ring (Shores and R. Wiegand)

A ring is an FGC ring if every finitely generated module over the ring is a direct sum of cyclic submodules.

These rings have been studied and characterised by Brandal, Shores, Vámos, R. Wiegand and S. Wiegand in [2], [3], [23], [25], [26]. The structure theorems for FGC rings are given in Chapter 5. Examples of FGC rings are provided by the principal ideal domains. Moreover the local FGC rings are precisely the almost maximal valuation rings ([6; Main Theorem]).

The following definitions made in [6] are generalisations of those of maximal and almost maximal valuation domains made by Kaplansky in [12; p336].

2.7 Definitions of a maximal and an almost maximal valuation ring

A valuation ring R is maximal if every system of pairwise soluble congruences of the form $\{x \equiv x_\alpha \bmod I_\alpha\}$ has a simultaneous solution in R, where $x_\alpha \in R$, I_α is an ideal of R and α is in some index set J.

A valuation ring R is almost maximal if the above congruences have a simultaneous solution whenever $\bigcap_{\alpha \in J} I_\alpha \neq 0$.

Thus every maximal valuation ring is almost maximal. An equivalent definition of an almost maximal valuation ring (AMVR), given in [23], is that R is an AMVR if R/I is maximal for every non-zero ideal I of R.

The following sections give some examples of these types of rings.

2.8 Examples of almost maximal valuation rings

(a) Every discrete (noetherian) valuation domain is an almost maximal valuation ring ([12; p336]).

(b) The power series ring in one indeterminate over a field is a discrete valuation domain and hence is almost maximal. In addition it is complete and so is a maximal valuation domain ([12; p336], [16; p160]). As an example, $\mathbb{C}[[x]]$ is a maximal valuation domain. (A discussion of maximal valuation domains and completions will be in section 4.4 and following.) Then the quotient $\mathbb{C}[[x]]/(x^2)$ is also a maximal valuation ring, but is not a domain since the ideal (x^2) is not prime in $\mathbb{C}[[x]]$.

(c) The localisation of $\mathbb{Z}$ at a non-zero prime ideal $P = (p)$, denoted $\mathbb{Z}_P$, is a discrete valuation domain. Its proper non-zero ideals are precisely those generated by $\frac{p^n}{1}$ for $n \geq 1$. Then $\mathbb{Z}_P$ is an almost maximal valuation ring but is not maximal.

2.9 Examples of valuation rings that are not almost maximal

(a) This first example looks at subvaluation domains of "long power series" rings and was communicated to me by Vámos. More details and proofs are given in [3], [22] and [24].

Let Γ ($\neq \mathbb{Z}$) be a totally ordered abelian group and let F be a field. Then Γ^+ denotes the positive cone of Γ, $\Gamma^+ = \{g \in \Gamma \mid g \geq 0\}$. For a function $f : \Gamma \rightarrow F$ define the support of f by $\sup f = \{a \in \Gamma \mid f(a) \neq 0\}$. Let $F((\Gamma)) = \{f \in F^{\Gamma} \mid \sup f$ is well-ordered$\}$. Addition and multiplication are defined in $F((\Gamma))$ by

$(f + g)(\alpha) = f(\alpha) + g(\alpha)$,

$(fg)(\alpha) = \sum_{\beta+\gamma=\alpha} f(\beta)g(\gamma)$ where $f, g \in F((\Gamma))$ and $\alpha, \beta, \gamma \in \Gamma$.

These operations are well-defined and give $F((\Gamma))$ the structure of a field. There is a maximal valuation v on $F((\Gamma))$ given by $v : F((\Gamma)) \rightarrow \Gamma \cup \{\infty\}$, $v(f) = \min \operatorname{sup} f$. The valuation ring of v is $F[[\Gamma]]$, the "long power series" ring relative to F and Γ. Thus $F[[\Gamma]] = \{f \in F((\Gamma)) \mid \operatorname{sup} f \subseteq \Gamma^+\}$. The residue field of the valuation ring is F. Hence $F[[\Gamma]]$ is a maximal valuation domain.

Now suppose $\Gamma \subseteq \mathbb{R}$ ($\Gamma \neq \mathbb{Z}$). Call a subset S of Γ^+ almost finite if the set $\{\alpha \in S \mid \alpha \leq \gamma\}$ is finite for all $\gamma \in \Gamma^+$. It is clear that an almost finite set is partially well-ordered. Let $R = \{f \in F[[\Gamma]] \mid \operatorname{sup} f \text{ is almost finite}\}$. Then R is a valuation domain under the valuation v above (with v restricted to the field of fractions of R). Thus R is a sub-valuation domain of $F[[\Gamma]]$. Moreover R has the same value group Γ and residue field F as $F[[\Gamma]]$.

Let A and A′ be valuation domains and let $A \rightarrow A'$ be an embedding. Then A′ is an immediate extension of A if the value groups and residue fields of A and A′ are isomorphic via this embedding. Thus $F[[\Gamma]]$ is an immediate extension of R. A valuation domain is said to be maximally complete if it has no proper immediate extensions. Thus R is not maximally complete. Moreover a valuation domain is maximally complete if and only if it is a maximal valuation domain. Hence R is not a maximal valuation domain.

A valuation domain A is complete in the A–topology if the embedding $\phi : A \rightarrow \varprojlim A/Ar$, $a \mapsto (a + Ar)$ (indexed by $0 \neq r \in A$) is an isomorphism (see Definition 4.4). To show that R is complete in the R–topology it is thus sufficient to show that ϕ is onto. Let $(f_r + Rr) \in \varprojlim R/Rr$. Define a map $f : \Gamma \rightarrow F$ by

$$f : a \mapsto \begin{cases} f_s(a) & \text{if there is some } s\ (0 \neq s \in R) \text{ with } a \in \operatorname{sup} f_s,\ a \leq v(s),\ a \neq v(s), \\ 0 & \text{otherwise.} \end{cases}$$

Then f is indeed an element of R and $\phi f = (f_r + Rr)$. Thus ϕ is onto and so the ring R is indeed complete in the R–topology. Hence R is not an almost maximal valuation domain ([16; p160]).

Then the sub-valuation ring R of $F[[\Gamma]]$ is a valuation ring which is not almost maximal.

(b) Let S be the ring $\mathbb{C}[X_0, X_1, X_2, \dots]$ and I the ideal of S generated by all the elements $X_n^2 - X_{n-1}$ for $n \geq 1$. Let A be the quotient ring S/I, so that $A = \mathbb{C}[x_0, x_1, x_2, \dots]$ where $x_i = X_i + I$, and let M be the maximal ideal $(x_0, x_1, x_2, \dots)$ of A. Let $R = A_M$. Then R is a valuation domain. For ease of notation write r for the element $\frac{r}{1}$ in R since A embeds in R (via $a \mapsto \frac{a}{1}$). (Further information on the ring A may be found in [11; 39].)

However R is not an almost maximal valuation domain. Take $I_n = (x_n^{2^n-1})$ as the family of ideals in R ($n \geq 1$) and consider the system of congruences $\{a_n \bmod I_n\}$ where $a_1 = 1$ and $a_n = a_{n-1} + x_{n+1}^{(2^{n+1}-3)}$ for $n \geq 2$. Then $a_n - a_{n-1} \in I_{n-1} \setminus I_n$ for $n \geq 2$ and $0 \neq (x_0) \subseteq \bigcap I_n$. This system of congruences is pairwise soluble but there is no simultaneous solution in R.

Hence R is a valuation ring that is not almost maximal.

2.10 Examples of local rings that are not valuation rings

The following two examples are of power series rings which are local rings but not valuation rings.

(a) The ring $K[[x^2, x^3]]$ where K is a field is a 1-dimensional local domain which is not integrally closed ([11; 11]). But valuation domains are integrally closed (see [1; Proposition 5.18]). Thus $K[[x^2, x^3]]$ is a local domain which is not a valuation ring.

(b) The ring $\mathbb{C}[[x_1, x_2, \dots]]$ with infinitely many indeterminates has a unique maximal ideal $(x_1, x_2, \dots)$ so is a local ring, but is not a valuation ring since the indeterminates are not comparable.

It has been seen that all FGC rings are scalar-reflexive and that the local FGC rings are just the almost maximal valuation rings. The next theorem is a result of Hadwin and Kerr ([9; Theorem 6]) which shows that these three conditions are equivalent for local rings.

2.11 Theorem (Hadwin and Kerr)

Let R be a local ring. Then the following are equivalent:

(i) R is scalar-reflexive,

(ii) R is an FGC ring,

(iii) R is an almost maximal valuation ring.

This theorem can be used to give an example of a ring R which shows that Theorem 2.2 cannot be generalised to apply to torsion modules. Let $R = \mathbb{Z}_P$, where $P = (p)$ is any non-zero prime ideal of $\mathbb{Z}$. Then, from Example 2.8(c), R is an almost maximal valuation domain which is not maximal. From Theorem 2.11, R is scalar-reflexive. Hence all 2–generated modules and all non-torsion modules over R are reflexive (Corollary 2.5).

Let T be the R–module $<\frac{c_1}{1}, \frac{c_2}{1}, \ldots \mid \frac{pc_1}{1} = \frac{0}{1}, \frac{pc_n}{1} = \frac{c_{n-1}}{1}>$. Then T is a faithful torsion R–module. The results in Chapter 4 show that T is not reflexive. Thus it is not the case that every torsion R–module is reflexive. (For the above, Theorem 4.9 shows that $\text{alglat}_R T_R$ is isomorphic to $\tilde{R}$, the completion of R in the R-topology. Since R is a domain and is not maximal, R is not complete in the R-topology (see after 4.4). Thus R is strictly embedded in $\tilde{R}$ and hence T is not reflexive. Note that T is the localisation of the $\mathbb{Z}$–module $\mathbb{Z}_{p^\infty}$ at the prime P. It can be shown that the ring homomorphism in Lemma 2.16 is an isomorphism in this case. Then $\text{alglat}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})_{\mathbb{Z}} \cong \text{alglat}_R T_R$. Indeed, $\text{alglat}\,\mathbb{Z}_{p^\infty}$ is precisely the ring of left scalar multiplications by elements of the p-adic completion of $\mathbb{Z}$.)

Hadwin and Kerr remark in [9; p8] that their proof of Theorem 2.11 shows that these three conditions are equivalent, for a local ring R, to a fourth condition:

(iv) every 2–generated R–module is reflexive.

It will be shown later in Chapter 5 (following Theorem 5.17) that, although conditions (i) and (ii) are equivalent for a local ring, they are not equivalent in general. Theorem 2.11 characterises local rings that are scalar-reflexive. Thus it can be used to provide information on rings whose localisations are scalar-reflexive. This motivates the following definition.

2.12 Definition of a locally scalar-reflexive ring

A ring is locally scalar-reflexive if every localisation at a maximal ideal is scalar-reflexive.

Any FGC ring R is a locally scalar-reflexive ring. For every localisation of R at a maximal ideal is a local FGC ring and hence, from Theorem 2.11, is scalar-reflexive. Thus R is locally scalar-reflexive. From Theorem 2.11, every almost maximal valuation ring is an FGC ring and is thus locally scalar-reflexive. Hence every local ring which is scalar-reflexive is also locally scalar-reflexive.

The last result of this chapter is a corollary to Theorem 2.2 for locally scalar-reflexive rings. Before giving Corollary 2.19, locally scalar-reflexive rings together with some of their properties are discussed. In particular Theorem 2.17 shows that every locally scalar-reflexive ring is scalar-reflexive.

First an equivalent definition of a locally scalar-reflexive ring is given. This makes use of the following well-known definition.

2.13 Definition of an arithmetical ring

A ring is arithmetical if every localisation at a maximal ideal is a valuation ring.

An alternative definition is that a ring is arithmetical if every localisation at a prime ideal is a valuation ring. Thus the arithmetical domains are just the Prüfer domains.

In [25], Vámos introduced the terminology of a locally almost maximal arithmetical ring. This is a ring in which every localisation at a prime ideal is an almost maximal valuation ring. From Theorem 2.11 these are the rings in which every localisation at a prime ideal is scalar-reflexive. Proposition 2.15 shows that the locally almost maximal arithmetical rings (those satisfying (i) in 2.15) are precisely the locally scalar-reflexive rings (those satisfying (iv) in 2.15). The following lemma was proved by Gill in [6; Lemma 2] and is used in Proposition 2.15.

2.14 Lemma (Gill)

Let R be a valuation ring and let P be a prime ideal of R. Then R is maximal (almost maximal) $\Rightarrow$ R_P is maximal (almost maximal).

2.15 Proposition

Let R be a commutative ring. Then the following are equivalent:

(i) R_P is scalar-reflexive for all prime ideals P of R,

(ii) Every 2–generated R_P–module is reflexive for all prime ideals P of R,

(iii) Every 2–generated R_M–module is reflexive for all maximal ideals M of R,

(iv) R_M is scalar-reflexive for all maximal ideals M of R.

Proof

The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial and (iii) $\Rightarrow$ (iv) follows from the remark after Theorem 2.11.

(iv) $\Rightarrow$ (i)

Let P be a prime ideal of R and let M be a maximal ideal of R containing P. Then $R_P \cong (R_M)_{P_M}$ (see [19; p24]). By hypothesis R_M is scalar-reflexive, and so from Theorem 2.11, R_M is an almost maximal valuation ring. The ideal P_M is prime in R_M and thus the localised ring $(R_M)_{P_M}$ is also an almost maximal valuation ring (Lemma 2.14). Hence R_P is an almost maximal valuation ring and thus R_P is scalar-reflexive (Theorem 2.11). $\square$

Hadwin and Kerr asked in [8; p7] whether the class of scalar-reflexive rings is closed under localisations. A particular case of this, when the localisations are at the maximal ideals, asks whether every scalar-reflexive ring is locally scalar-reflexive. These remain open questions, with the only known examples of locally scalar-reflexive rings being provided by the scalar-reflexive rings, and vice versa. There is nevertheless a connection between scalar-reflexive rings and locally scalar-reflexive rings. In Theorem 2.17 it will be proved that every locally scalar-reflexive ring is scalar-reflexive. Then a local ring is scalar-reflexive if and only if it is locally scalar-reflexive. The following lemma is used in the proof of Theorem 2.17.

<u>2.16 Lemma</u>

Let M be a maximal ideal of a ring R and let T be an R–module with localisation T_M. Then there is a ring homomorphism from $\text{alglat}\,{}_RT_R$ to $\text{alglat}\,T_M$ where T_M is an R_M–module.

<u>Proof</u>

Define a map $\alpha : \text{alglat}\,{}_RT_R \to \text{alglat}\,T_M$ by $\phi \mapsto \phi_M$ where $\phi_M : \frac{t}{s} \mapsto \frac{\phi t}{s}$.

<u>$\text{im}\,\alpha \subseteq \text{alglat}\,T_M$</u>

Let $\phi \in \text{alglat}\,{}_RT_R$ so that $\phi_M \in \text{im}\,\alpha$ and $\phi_M : T_M \to T_M$.

Suppose that $\frac{t_1}{s_1} = \frac{t_2}{s_2}$ in T_M so that there is some $u \notin M$ with $(t_1s_2 - t_2s_1)u = 0$. Then $[(\phi t_1)s_2 - (\phi t_2)s_1]u = [\phi(t_1s_2) - \phi(t_2s_1)]u = \phi((t_1s_2 - t_2s_1)u) = 0$ and so $\frac{\phi t_1}{s_1} = \frac{\phi t_2}{s_2}$. Thus ϕ_M is well-defined.

Let $\frac{t_1}{s_1}, \frac{t_2}{s_2}, \frac{t}{s} \in T_M$ and $\frac{r}{u} \in R_M$. Then $\phi_M\left(\frac{t_1}{s_1} + \frac{t_2}{s_2}\right) = \phi_M\left(\frac{t_1s_2 + t_2s_1}{s_1s_2}\right) = \frac{\phi(t_1s_2 + t_2s_1)}{s_1s_2} = \frac{(\phi t_1)s_2 + (\phi t_2)s_1}{s_1s_2} = \frac{\phi t_1}{s_1} + \frac{\phi t_2}{s_2} = \phi_M\left(\frac{t_1}{s_1}\right) + \phi_M\left(\frac{t_2}{s_2}\right)$. Also $\phi_M\left(\frac{t}{s}\cdot\frac{r}{u}\right) = \frac{\phi(tr)}{su} = \frac{(\phi t)r}{su} = \frac{\phi t}{s}\cdot\frac{r}{u} = \left[\phi_M\left(\frac{t}{s}\right)\right]\frac{r}{u}$. Hence $\phi_M \in \text{End}\,T_{M\,R_M}$.

Let $\frac{t}{s} \in T_M$. There is some element a of R with $\phi t = at$. Then $\phi_M\left(\frac{t}{s}\right) = \frac{\phi t}{s} = \frac{at}{s} = \frac{a}{1}\cdot\frac{t}{s} \in R_M\frac{t}{s}$. Hence $\phi_M \in \operatorname{alglat} T_M$ (where T_M is an R_M–module).

Thus $\operatorname{im}\alpha \subseteq \operatorname{alglat} T_M$.

α is a ring homomorphism

Let ϕ and θ be elements of $\operatorname{alglat}_R T_R$. Let $\chi = \phi + \theta$ so that χ_M is the image of $\phi + \theta$ under α. Then $\chi_M\left(\frac{t}{s}\right) = \frac{\chi t}{s} = \frac{(\phi + \theta)t}{s} = \frac{\phi t}{s} + \frac{\theta t}{s} = \phi_M\left(\frac{t}{s}\right) + \theta_M\left(\frac{t}{s}\right)$ and so $\chi_M = \phi_M + \theta_M$. Thus $\alpha(\phi + \theta) = \alpha\phi + \alpha\theta$. Let $\psi = \phi\theta$ so that ψ_M is the image of $\phi\theta$ under α. Then $\psi_M\left(\frac{t}{s}\right) = \frac{\psi t}{s} = \frac{(\phi\theta)t}{s} = \frac{\phi(\theta t)}{s} = \phi_M\left(\frac{\theta t}{s}\right) = \phi_M\left(\theta_M\frac{t}{s}\right)$ and so $\psi_M = \phi_M\theta_M$. Thus $\alpha(\phi\theta) = (\alpha\phi)(\alpha\theta)$. Let 1 be the identity in $\operatorname{alglat}_R T_R$ so that 1_M is the image of 1 under α. For $\frac{t}{s}$ in T_M, $1_M\left(\frac{t}{s}\right) = \frac{1t}{s} = \frac{t}{s}$. Hence $\alpha(1)$ is indeed the identity element in $\operatorname{alglat} T_M$. Thus α is a well-defined ring homomorphism. □

2.17 Theorem

Every locally scalar-reflexive ring is scalar-reflexive.

Proof

Let R be a locally scalar-reflexive ring and let $\{M_i \mid i \in I\}$ be the set of all maximal ideals of R. Let $T = Rx_1 + Rx_2 + \cdots + Rx_n$ be a finitely generated R–module and let $\phi \in \operatorname{alglat}_R T_R$.

Let M_i be any maximal ideal of R. Then there is a map ϕ_{M_i} in $\operatorname{alglat} T_{M_i}$ given by $\frac{t}{s} \mapsto \frac{\phi t}{s}$ (Lemma 2.16). The R_{M_i}–module T_{M_i} is finitely generated and is therefore reflexive (by hypothesis). So there is an element $\frac{a_i}{u_i}$ in R_{M_i} with $\phi_{M_i} = \lambda(\frac{a_i}{u_i})$. Then $\phi_{M_i}\left(\frac{x_j}{1}\right) = \frac{a_i x_j}{u_i} = \frac{\phi x_j}{1}$ for each $j = 1, \ldots, n$, so there are elements $s_{ij} \in R\backslash M_i$ with $(a_i x_j - u_i(\phi x_j))s_{ij} = 0$. Let $s_i = \prod_{j=1}^{n} s_{ij}$ so then $s_i \notin M_i$. Then $(a_i x_j - u_i(\phi x_j))s_i = 0$ for all $j = 1, \ldots, n$. So $u_i, s_i \notin M_i$ and, for $j = 1, \ldots, n$, $u_i s_i(\phi x_j) = a_i s_i x_j$.

The sum $\sum_{i \in I} Ru_i s_i = R$, for otherwise there is some maximal ideal N with $\sum_{i \in I} Ru_i s_i \subseteq N$. But $N = M_k$ for some $k \in I$ and $u_k s_k \notin M_k$ giving the required contradiction. So there is a finite subset K of I with $1 = \sum_{k \in K} r_k u_k s_k$ and $r_k \in R$. Then, for $j = 1, \ldots, n$, $\phi x_j = \sum_{k \in K} r_k u_k s_k(\phi x_j) = \sum_{k \in K} r_k a_k s_k x_j$. Let $r = \sum_{k \in K} r_k a_k s_k$ so that $\phi x_j = r x_j$ for $j = 1, \ldots, n$ and $r \in R$. Then $\phi = \lambda(r)$ and so T is reflexive. Hence R is scalar-reflexive. □

The next few comments relate Theorem 2.17 to another result in the literature. In [9; Theorem 12], Hadwin and Kerr proved the following theorem.

2.18 Theorem (Hadwin and Kerr)

Let R be an h-local domain with R_M an almost maximal valuation ring for all maximal ideals M of R. Then R is scalar-reflexive.

The hypotheses of this theorem may be rewritten, requiring R to be an h-local domain which is locally scalar-reflexive. From Theorem 2.17 it is clear that the condition that R be an h-local domain is redundant.

The third and final corollary to Theorem 2.2 now follows. The proof is immediate from Corollary 2.5 and Theorem 2.17.

2.19 Corollary

Let R be a locally scalar-reflexive ring. Then every non-torsion R–module is reflexive.

Scalar-reflexive rings, locally scalar-reflexive rings and 2–generator conditions will be studied further in Chapter 3.

Chapter 3 Reflexivity and local properties

This chapter builds on the work done in Chapters 1 and 2 on h-local domains, scalar-reflexive rings and locally scalar-reflexive rings and looks at some local properties for an h-local domain. Recall from 2.4 and 2.12 that a ring R is scalar-reflexive if every finitely generated R–module is reflexive and that R is locally scalar-reflexive if every localisation R_M at a maximal ideal M of R is scalar-reflexive. In addition every locally scalar-reflexive ring is scalar-reflexive (Theorem 2.17).

The main result of the chapter is Theorem 3.4 which gives eight properties of an h-local domain which are equivalent to the condition that the ring is locally scalar-reflexive. The principal component of Theorem 3.4 is the proof that, for an h-local domain R with every 2–generated R–module being reflexive, then R is locally scalar-reflexive. This result may be of independent interest and as such it appears in Theorem 3.1. In addition Theorem 3.1 provides a partial converse to Theorem 2.17, showing for an h-local domain, that being scalar-reflexive is equivalent to being locally scalar-reflexive.

Theorem 3.4 links the structure and decomposition of modules with the study of reflexivity. In particular it is shown that an h-local domain is locally scalar-reflexive if and only if every 2–generated torsion module is a direct sum of cyclic modules. The corresponding statement for finitely generated torsion modules also appears as one of the nine equivalent properties of Theorem 3.4.

3.1 Theorem

Let R be an h-local domain. Then the following are equivalent:

(i) R is locally scalar-reflexive,

(ii) R is scalar-reflexive,

(iii) Every 2–generated R–module is reflexive.

Proof

The implication (i) $\Rightarrow$ (ii) is given in Theorem 2.17 and (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i)

It is sufficient to show, for all maximal ideals M of R, that every 2–generated R_M–module is reflexive. For then R_M is scalar-reflexive for all maximal ideals M (Theorem 2.11) and so R is locally scalar-reflexive.

Let M be a maximal ideal of R and let $T = R_M x + R_M y$ be a 2–generated R_M–module. Suppose that both $R_M x$ and $R_M y$ are non-zero. For if not then T is cyclic and thus reflexive (Proposition 2.1). Note that the ring R is a domain and so $R \subseteq R_M$ via the embedding $r \mapsto \frac{r}{1}$. Then, for any element t of T, there is a well-defined R–module structure on t given by $rt := \frac{r}{1}t$.

There are two cases to consider.

Case i) T non-torsion

The local ring R_M is a domain. It then follows from Corollary 2.3 that T is reflexive.

Case ii) T torsion

The first step is to find new generators x′ and y′ for T so that the ideal $\mathrm{Ann}(Rx' + Ry')$ is colocal in M. The construction begins by showing that the ideal $\mathrm{Ann}\,Rx$ is non-zero and is contained in M. The module T is torsion so let $\frac{r}{u}$ be a non-zero element of $\mathrm{Ann}(R_M x)$. Then $r \neq 0$, $\frac{r}{1} \in \mathrm{Ann}(R_M x)$ and $rx = \frac{r}{1}x = 0$. Thus $\mathrm{Ann}\,Rx \neq 0$. Suppose $\mathrm{Ann}\,Rx \not\subseteq M$ and let s be an element in $\mathrm{Ann}\,Rx$ with $s \notin M$. Then, since $R_M = R_M\frac{1}{s}$, it follows that $R_M x = R_M sx = 0$, a contradiction. Thus $\mathrm{Ann}\,Rx \subseteq M$.

So $\mathrm{Ann}\,Rx$ has a normal decomposition $\mathrm{Ann}\,Rx = \bigcap_{i=1}^{n} I_i$ with I_i colocal in M_i. From Proposition 1.16, $M \in \{M_1, \ldots, M_n\}$. To ease notation suppose $M = M_1$. Then, also from Proposition 1.16, $\bigcap_{i=2}^{n} I_i + I_1 = R$. This gives elements a in $\bigcap_{i=2}^{n} I_i$

and b in I_1 with $1 = a + b$. Then $\operatorname{Ann} Rax = I_1$. For if $r \in \operatorname{Ann} Rax$, then $rax = 0$ and so $ra \in \operatorname{Ann} Rx \subseteq I_1$. Thus $r = ra + rb \in I_1$. For the reverse inclusion let $r \in I_1$ so that $ra \in \bigcap_{i=1}^{n} I_i = \operatorname{Ann} Rx$. Then $rax = 0$ and so $r \in \operatorname{Ann} Rax$. Thus $\operatorname{Ann} Rax = I_1$ which is colocal in $M_1 = M$. It is also clear that a is not an element of M and so $R_M ax = R_M x$.

In the same way there is an element c in R and a colocal ideal J of M with $R_M cy = R_M y$ and $\operatorname{Ann} Rcy = J$. Let $x' = ax$ and $y' = cy$. Then $T = R_M x' + R_M y'$ and $\operatorname{Ann}(Rx' + Ry') = \operatorname{Ann} Rx' \cap \operatorname{Ann} Ry' = I_1 \cap J$ which is colocal in M (Proposition 1.16). Thus x' and y' are the new generators.

The next step is to prove that $Rt = R_M t$ for all t in T. Let $t \in T$. Then clearly $Rt \subseteq R_M t$. For the reverse inclusion let $\frac{r}{u} \in R_M$ so that $\frac{r}{u} t \in R_M t$. Then, since $\operatorname{Ann}(Rx' + Ry')$ is colocal in M and $u \notin M$, it follows from Proposition 1.16 that $Ru + \operatorname{Ann}(Rx' + Ry') = R$. Thus there are elements f in R and g in $\operatorname{Ann}(Rx' + Ry')$ with $1 = fu + g$. Then $\frac{1}{u} = \frac{f}{1} + \frac{g}{u}$ giving $\frac{r}{u} t = \frac{fr}{1} t + \frac{r}{u} gt$. But $gt = 0$. (Write $t = \frac{d_1}{s_1} x' + \frac{d_2}{s_2} y' = \frac{1}{s_1 s_2} \left(\frac{d_1 s_2}{1} x' + \frac{d_2 s_1}{1} y'\right)$. Thus $gt = \frac{g}{1} t = \frac{g}{s_1 s_2} (d_1 s_2 x' + d_2 s_1 y') = 0$.) So $\frac{r}{u} t = \frac{fr}{1} t = frt \in Rt$. Thus $R_M t \subseteq Rt$. Hence $R_M t = Rt$ for all t in T. In particular $T = R_M x' + R_M y' = Rx' + Ry'$ since x' and y' are in T.

The final step is to show that T is a reflexive R_M–module. Considering T as an R–module, $T = Rx' + Ry'$ is 2–generated and so is reflexive by hypothesis. Thus $\operatorname{alglat}_R T_R = \lambda(R) \subseteq \lambda(R_M) \subseteq \operatorname{alglat}_{R_M} T_{R_M}$. But $R \subseteq R_M$ and so $\operatorname{End} T_{R_M} \subseteq \operatorname{End} T_R$. Then $\operatorname{alglat}_{R_M} T_{R_M} = \{\phi \in \operatorname{End} T_{R_M} \mid \phi t \in R_M t \text{ for all } t \in T\} \subseteq \{\phi \in \operatorname{End} T_R \mid \phi t \in R_M t \text{ for all } t \in T\} = \{\phi \in \operatorname{End} T_R \mid \phi t \in Rt \text{ for all } t \in T\} = \operatorname{alglat}_R T_R$. Thus $\operatorname{alglat}_{R_M} T_{R_M} = \lambda(R_M)$ and hence T is a reflexive R_M–module. $\square$

The next two sections concern local properties and will be used in Theorem 3.4. The following definition of a local property may be found with examples in [1; pp40-41].

3.2 Definition of a local property

A property Q of a ring R is a local property if the following are equivalent:

(i) R has property Q,

(ii) R_M has property Q for all maximal ideals M of R,

(iii) R_P has property Q for all prime ideals P of R.

3.3 Proposition

Suppose that, for h-local domains, property Q is equivalent to property Q′ and that property Q is a local property. Then property Q′ is also a local property for h-local domains.

Proof

The proof is an easy consequence of the fact that, for any prime ideal P of an h-local domain R, R_P is a local domain and hence an h-local domain. □

The next theorem is the main result of the chapter.

3.4 Theorem

Let R be an h-local domain. Then the following are equivalent local properties:

(1) R is scalar-reflexive,

(2) every finitely generated torsion R–module is reflexive,

(3) every finitely generated torsion R–module is a direct sum of cyclic modules,

(4) every 2–generated R–module is reflexive,

(5) every 2–generated torsion R–module is reflexive,

(6) every 2–generated torsion R–module is a direct sum of cyclic modules,

(7) R is a Prüfer domain and Q/R is injective, where Q is the quotient field of R,

(8) R_M is an almost maximal valuation ring for every maximal ideal M of R,

(9) R is locally scalar-reflexive.

Note

The quotient Q/R is always considered as an R–module. An R–module M is injective if, for any injective map $f : N' \to N$ and any map $g : N' \to M$, there is a map $h : N \to M$ with $g = hf$ (where N, N′ are R–modules and all the maps are R–homomorphisms).

Proof

It follows from Proposition 2.15 and Theorem 3.1 that (1) is a local property. It is thus sufficient to prove that these nine properties are equivalent, since it is then immediate from Proposition 3.3 that all nine properties are local.

The proof of (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (9) has already been given in Theorem 3.1, and the results (3) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) are proved by Matlis in [16; Theorem 5.7]. It follows from the definition of a locally scalar-reflexive ring and from Theorem 2.11 that (8) $\Leftrightarrow$ (9). The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (5) and (3) $\Rightarrow$ (6) $\Rightarrow$ (5) are trivial consequences of Proposition 2.1. Finally the implication (5) $\Rightarrow$ (4) follows from Corollary 2.3 since R is a domain. Thus all nine properties are equivalent. This completes the proof of the theorem. □

The next part of this chapter uses this theorem to answer a question raised by Hadwin and Kerr in [9; p12]. Hadwin and Kerr ask whether every scalar-reflexive domain is h-local. An example is given in 3.6 which answers this question in the negative. This example uses rings of type I, which were first defined by Matlis in [17]. The definition is given below.

3.5 Definition of a ring of type I (Matlis)

A ring R is of type I if R is an integral domain with exactly two maximal ideals M_1 and M_2 such that R_{M_1} and R_{M_2} are maximal valuation rings and there is no non-zero prime ideal contained in $M_1 \cap M_2$.

An example of a ring of type I is given by Osofsky in [17]. Brandal includes this same ring in his paper [3; Example 14.1] with more accompanying detail. The following example (3.6) was given by Matlis in [18; Example 2] to show that there is a ring which is the intersection of two dependent maximal valuation rings, but is not a ring of type I. This latter ring is now shown to be a locally scalar-reflexive (and hence scalar-reflexive) domain which is not an h-local domain.

3.6 Example of a scalar-reflexive domain which is not h-local

Let A be a ring of type I with two maximal ideals M_1 and M_2. Let B be the field of fractions of A. Let R be the ring of formal power series in an indeterminate X with coefficients in B but with constant term in A so that R = $\{\sum_0^\infty b_i X^i \mid b_0 \in A,\ b_i \in B \text{ for } i > 0\}$. Then R is a domain. Let P be the prime ideal of R consisting of power series with constant term $b_0 = 0$. Then R has precisely two maximal ideals $N_1 = M_1 + P$ and $N_2 = M_2 + P$. The prime ideal P satisfies $0 \neq P \subseteq N_1 \cap N_2$. Thus R is not an h-local domain.

The power series in the domain R_{N_1} have constant term in A_{M_1}. Then R_{N_1} is a valuation ring since A_{M_1} is a valuation ring. It is known that $(R_{N_1})_P \cong R_P$ (see [19; p24]). Since $R_P \cong B[[X]]$, a maximal valuation ring, the ring $(R_{N_1})_P$ is also a maximal valuation ring (see Example 2.8(b)). The quotient ring R_{N_1}/P is a maximal valuation ring too, being isomorphic to A_{M_1}. For a valuation domain S and prime ideal Q of S, S is a maximal valuation ring if and only if both S_Q and S/Q are maximal valuation rings (a proof can be found in [18; Corollary 2]). Thus R_{N_1} is a maximal valuation ring. Similarly R_{N_2} is a maximal valuation ring. Thus R is locally scalar-reflexive.

Hence R is a scalar-reflexive domain which is not an h-local domain.

A second related question posed by Hadwin and Kerr in [9; p12] asks what are the scalar-reflexive domains. This question remains open in view of Example 3.6. However, Theorem 3.4 has classified all the scalar-reflexive h-local domains.

Finally recall the result of Hadwin and Kerr given in Theorem 2.18 which proves, for an h-local domain R with R_M an almost maximal valuation ring for all maximal ideals M of R, that R is scalar-reflexive. The equivalence of properties (1) and (8) of Theorem 3.4 for h-local domains provides a converse to this result. This gives a second generalisation of Theorem 2.18 (see comments following Theorem 2.18).

The next chapters work towards a characterisation in Chapter 7 of alglat $_RM_R$ for all modules M over fractionally self-injective (FSI) rings. In discussing these FSI rings in Chapter 5 another characterisation of scalar-reflexive h-local domains will be given with Theorem 5.19 proving that the scalar-reflexive h-local domains are precisely the FSI domains. First Chapter 4 looks at the local case and determines alglat for all modules over an almost maximal valuation ring.

Chapter 4 Modules over almost maximal valuation rings

The rest of this thesis builds towards the results in Chapter 7 which characterise alglat for all modules over fractionally self-injective rings. Chapter 5 looks at the structure of FSI rings, with Chapters 6 and 7 providing a characterisation of alglat for modules over an FSI ring. The aim of this chapter is to determine alglat for all modules over the local FSI rings. As well as being of independent interest, these results will be used in Chapter 7.

It is known that an FGC ring is fractionally self-injective and this result will be found in Theorem 5.12. Theorem 2.11 showed that a local FGC ring is an almost maximal valuation ring. It will be seen in Proposition 5.16 that the local FSI rings are also precisely the almost maximal valuation rings. Thus this chapter aims to characterise alglat for all modules over an almost maximal valuation ring. Recall from 2.7 the definitions of a maximal and an almost maximal valuation ring. If R is an almost maximal valuation ring then R is scalar-reflexive (Theorem 2.11). Thus all finitely generated and all non-torsion R–modules are reflexive (Corollary 2.5).

The first two results concern maximal valuation rings. The first theorem, by Hadwin and Kerr, is part of [9; Theorem 5] and shows that all modules over a maximal valuation ring are reflexive. The subsequent proposition was proved by Gill in [6; Proposition 1] and gives a condition for an almost maximal valuation ring to be maximal.

4.1 Theorem (Hadwin and Kerr)

Let R be a maximal valuation ring. Then every R–module is reflexive.

4.2 Proposition (Gill)

Let R be a valuation ring which is not a domain. Then R is almost maximal if and only if R is maximal.

Let R be an almost maximal valuation ring and T any R–module. If R is maximal then T is reflexive by Theorem 4.1. If R is not maximal then from Proposition 4.2, R is a domain. So to find $\mathrm{alglat}_R T_R$, R may be taken to be an almost maximal valuation domain. Since all non-torsion modules over an almost maximal valuation ring are reflexive, it is sufficient to consider torsion modules over almost maximal valuation domains. The study of $\mathrm{alglat}_R T_R$ then splits into two sections according as to whether the torsion R–module T is faithful or not. The next theorem concerns modules which are not faithful and two alternative proofs are given. The first proof uses Theorem 4.1, whereas the second proof is more direct.

4.3 Theorem

Let R be an almost maximal valuation ring and let T be an R–module which is not faithful. Then T is reflexive.

Proof 1

Let $I = \mathrm{Ann}\,T$ so that I is a non-zero ideal of R. Let $A = R/I$ so then A is a maximal valuation ring. From Theorem 4.1, T is reflexive as an A–module and hence (from Proposition 1.19) T is reflexive as an R–module. □

Proof 2

Let $\phi \in \mathrm{alglat}_R T_R$. For each t in T there is an element r_t of R with $\phi t = r_t t$. Let $I_t = \mathrm{Ann}(t)$ so that $0 \neq \mathrm{Ann}\,T = \bigcap I_t$. Then $\{I_t \mid t \in T\}$ is a family of ideals of R with non-zero intersection.

Consider the system of congruences $\{r \equiv r_t \bmod I_t\}$.

Let x, y be any elements of T and let $N = Rx + Ry$. From Theorem 2.11, R is scalar-reflexive and so N is reflexive. Thus there is an element a in R with $\phi = \lambda(a)$ on N. Then $ax = \phi x = r_x x$ and $ay = \phi y = r_y y$. So $a - r_x \in I_x$ and $a - r_y \in I_y$. Thus $a \equiv r_x \bmod I_x$ and $a \equiv r_y \bmod I_y$. Hence the system of congruences is pairwise soluble.

The ring R is an almost maximal valuation ring so there is a solution r in R to the system of congruences. Then, for any t in T, $r - r_t \in I_t$ and so $\phi t = r_t t = rt$. Thus $\phi = \lambda(r)$. Hence $\text{alglat}_R T_R = \lambda(R)$ and so T is reflexive. □

Before determining $\text{alglat}_R T_R$ for faithful torsion R–modules T over an almost maximal valuation domain R, it is necessary to look at the completion of a ring R in the R-topology. The definition of the R-topology was given by Matlis in [15; §6] and more details can be found there.

<u>4.4 Definition of the R-topology</u>

Let R be an integral domain (not a field). A topology, called the R-topology, is defined on R by letting the non-zero ideals of R form a sub-base for the open neighbourhoods of 0 in R. The same topology is given to R by letting the non-zero principal ideals of R form a sub-base for the open neighbourhoods of 0 in R. The R-topology on R makes R into a topological ring. The intersection $\bigcap_{0 \neq r \in R} Rr$ is the closure of 0 in R, and R is Hausdorff if and only if the closure of 0 in R is zero.

The ring R is Hausdorff in this topology. To prove this, let $A = \bigcap_{0 \neq r \in R} Rr$ and suppose for contradiction that there is an element $0 \neq a \in A$. Then $a^2 \neq 0$ (R is a domain) and $a \in A \subseteq Ra^2$. So there is some b in R with $a = ba^2$. Thus $ab = 1$ and hence $A = R$. Then for any non-zero element r of R, $Rr = R$ and so r is a unit in R. Thus R is a field, giving the required contradiction. Hence $A = 0$. Thus R is Hausdorff in the R-topology.

The inverse limit $\varprojlim R/I$ exists and is isomorphic to $\varprojlim R/Rr$ ($0 \neq I \lhd R$, $0 \neq r \in R$). Let $\tilde{R} = \varprojlim R/Rr$ ($0 \neq r \in R$), so that $\tilde{R}$ is the completion of R in the R-topology ([15; Proposition 6.1]). Then $(a_r + Rr)$ is an element of $\tilde{R}$ if and only if for any non-zero elements u, v in R, $a_u - a_{uv} \in Ru$ (if and only if for any non-zero elements c, d in R with $Rc \subseteq Rd$, $a_c - a_d \in Rd$). There is a canonical ring homomorphism $\phi : R \to \tilde{R}$ given by $\phi : a \mapsto (a + Rr)$ for a in R. This gives $\ker\phi = \bigcap_{0 \neq r \in R} Rr$. Since R is Hausdorff, $\ker\phi = 0$ and so ϕ embeds R in $\tilde{R}$. The domain R is complete in the R-topology if and only if the homomorphism $\phi : R \to \tilde{R}$ is an isomorphism.

There are alternative definitions of maximal and almost maximal valuation domains to those given in 2.7 which include this idea of completeness in the R-topology (see Example 2.8(b)). These were given by Matlis in [16; pp159-160]. These definitions are a result of proving, for a valuation domain R with quotient field Q, that R is almost maximal if and only if the R–module Q/R is injective, and that R is maximal if and only if it is both almost maximal and complete in the R-topology (see [14; Theorems 4, 9]).

The next result was proved by Matlis as part of [15; Theorem 8.5] and will be used in Corollary 7.2. It is included here with the discussion of the R-topology for convenience.

<u>4.5 Lemma</u> (Matlis)

Let R be an h-local domain. Let $\tilde{R}$ be the R-completion of R and let $\widetilde{R_M}$ be the R_M-completion of R_M where M is a maximal ideal of R. Then $\tilde{R} \cong \prod_M \widetilde{R_M}$ where M ranges over all maximal ideals of R.

The isomorphism in Lemma 4.5 is given by $\gamma : (a_r + Rr) \mapsto \left((\frac{a_r}{1} + R_M\frac{r}{1})\right)$. Suppose that $(a_r + Rr) \in \tilde{R}$ and let M be any maximal ideal of R. For an element $s \notin M$, $R_M\frac{1}{s} = R_M$, and so any principal ideal of R_M can be expressed in the form $R_M\frac{r}{1}$. Suppose $\frac{u}{1}$, $\frac{v}{1}$ are non-zero elements of R_M. Then u, v are non-zero elements of R and so there is some b in R with $a_u - a_{uv} = bu \in Ru$. Thus $\frac{a_u}{1} - \frac{a_{uv}}{1} = \frac{bu}{1} \in R_M\frac{u}{1}$. Hence the element $(\frac{a_r}{1} + R_M\frac{r}{1})$ is indeed in $\widetilde{R_M}$ and so the image of $\tilde{R}$ under γ does lie in $\prod_M \widetilde{R_M}$.

In order to characterise alglat for a faithful torsion R–module T over an almost maximal valuation domain R, it has to be shown that T has an $\tilde{R}$–module structure. The following proposition is more general than is required, showing that any torsion R–module over a domain R has an $\tilde{R}$–module structure (see [4; Exercise 6 p101]). Note that if R is a domain and T is a torsion R–module then R cannot be a field. For if $0 \neq t \in T$ then Ann(t) is a non-zero ideal of R which is strictly contained in R.

4.6 Proposition

Let R be a domain with R-completion $\tilde{R}$. Let T be a torsion R–module. Then T has an $\tilde{R}$–module structure given by $(a_r + Rr)t = a_s t$ where $(a_r + Rr) \in \tilde{R}$, $t \in T$ and $0 \neq Rs \subseteq \mathrm{Ann}(t)$. Moreover $\mathrm{alglat}_R T_R = \mathrm{alglat}_{\tilde{R}} T_{\tilde{R}}$.

Proof

For any element t of T, $\mathrm{Ann}(t) \neq 0$ since T is torsion.

Suppose $Ru \subseteq \mathrm{Ann}(t)$ and $Rv \subseteq \mathrm{Ann}(t)$ for $0 \neq u, v \in R$. Then $a_u - a_{uv} \in Ru \subseteq \mathrm{Ann}(t)$ and $a_v - a_{uv} \in Rv \subseteq \mathrm{Ann}(t)$ so that $a_u - a_v \in \mathrm{Ann}(t)$. Thus $a_u t = a_v t$.

Suppose $(a_r + Rr) = (b_r + Rr)$ and that $Rs \subseteq \mathrm{Ann}(t)$. Then $a_r - b_r \in Rr$ for all $0 \neq r \in R$. In particular $a_s - b_s \in Rs$ and so $a_s t = b_s t$.

Let $\tilde{a} = (a_r + Rr)$, $\tilde{b} = (b_r + Rr)$ be elements of $\tilde{R}$ and let t, t_1, t_2 be elements of T. Then $\mathrm{Ann}(Rt_1 + Rt_2) \neq 0$ so there is a non-zero element u in R with $Ru \subseteq \mathrm{Ann}(t_1)$, $Ru \subseteq \mathrm{Ann}(t_2)$ and $Ru \subseteq \mathrm{Ann}(t_1 + t_2)$. So $\tilde{a}(t_1 + t_2) = a_u(t_1 + t_2) = a_u t_1 + a_u t_2 = \tilde{a}t_1 + \tilde{a}t_2$. Let $0 \neq Rv \subseteq \mathrm{Ann}(t)$. Now $\tilde{a} + \tilde{b} = (a_r + b_r + Rr)$ and $\tilde{a}\tilde{b} = (a_r b_r + Rr)$. So $(\tilde{a} + \tilde{b})t = (a_v + b_v)t = a_v t + b_v t = \tilde{a}t + \tilde{b}t$. Since $\mathrm{Ann}(t) \subseteq \mathrm{Ann}(b_v t)$, $Rv \subseteq \mathrm{Ann}(b_v t)$. Thus $\tilde{a}(\tilde{b}t) = \tilde{a}(b_v t) = a_v(b_v t) = (a_v b_v)t = (\tilde{a}\tilde{b})t$. The identity element of $\tilde{R}$ is $(1 + Rr)$, and $(1 + Rr)t = 1t = t$.

Thus T is an $\tilde{R}$–module under this product.

From 4.4, $R \subseteq \tilde{R}$ so that $\mathrm{End}\, T_{\tilde{R}} \subseteq \mathrm{End}\, T_R$. Let $\phi \in \mathrm{alglat}_{\tilde{R}} T_{\tilde{R}}$. Let $t \in T$.

There is an element $\tilde{a} \in \tilde{R}$ with $\phi t = \tilde{a}t$. Then, from the $\tilde{R}$–module structure of T, $\phi t \in Rt$. Hence $\phi \in \mathrm{alglat}_R T_R$ and so $\mathrm{alglat}_{\tilde{R}} T_{\tilde{R}} \subseteq \mathrm{alglat}_R T_R$.

Let $\phi \in \mathrm{alglat}_R T_R$. Then $\phi(t_1 + t_2) = \phi t_1 + \phi t_2$ for t_1, t_2 in T. Let $\tilde{a} = (a_r + Rr) \in \tilde{R}$ and let $t \in T$ with $0 \neq Rs \subseteq \mathrm{Ann}(t)$. Then there is some b in R with $\phi t = bt$ and so $Rs \subseteq \mathrm{Ann}(t) \subseteq \mathrm{Ann}(bt)$. Thus $\phi(t\tilde{a}) = \phi(ta_s) = (\phi t)a_s = (bt)a_s = (bt)\tilde{a} = (\phi t)\tilde{a}$. So $\phi \in \mathrm{End}\, T_{\tilde{R}}$. Let t be any element of T. Then, since $\phi \in \mathrm{alglat}_R T_R$ and $R \subseteq \tilde{R}$, $\phi t \in \tilde{R}t$. Hence $\phi \in \mathrm{alglat}_{\tilde{R}} T_{\tilde{R}}$ and so $\mathrm{alglat}_R T_R \subseteq \mathrm{alglat}_{\tilde{R}} T_{\tilde{R}}$. Thus $\mathrm{alglat}_R T_R = \mathrm{alglat}_{\tilde{R}} T_{\tilde{R}}$. □

In the light of this result $\lambda(\tilde{R})$ can be considered as a subring of $\mathrm{alglat}_R T_R$ whenever T is a torsion module over a domain R. This will be used without further comment.

The two subsequent results are used in Theorem 4.9 to prove, for a faithful torsion module T over an almost maximal valuation domain R, that $\mathrm{alglat}_R T_R$ is isomorphic to $\tilde{R}$. Theorem 4.7 uses the $\tilde{R}$–module structure on T from Proposition 4.6, and shows that, for a valuation domain R and faithful torsion R–module T, it is always the case that $\tilde{R} \cong \lambda(\tilde{R}) \subseteq \mathrm{alglat}_R T_R$. The proof depends on showing that T is a faithful $\tilde{R}$–module. Thus if R is not complete in the R-topology then $R \neq \tilde{R}$ and so $\lambda(R)$ is strictly contained in $\lambda(\tilde{R})$. In this case T will not be reflexive. A general characterisation of alglat with the hypotheses of Theorem 4.7 is not known. However the additional requirement in Theorem 4.9 that R be almost maximal enables $\mathrm{alglat}_R T_R$ to be determined.

4.7 Theorem

Let R be a valuation domain with R-completion $\tilde{R}$. Let T be a faithful torsion R–module. Then T is a faithful $\tilde{R}$–module and $\tilde{R} \cong \lambda(\tilde{R}) \subseteq \mathrm{alglat}_R T_R$.

Proof

From Proposition 4.6, $\lambda(\tilde{R}) \subseteq \mathrm{alglat}_R T_R$. The main part of the proof is to show that T is faithful as an $\tilde{R}$–module. It then follows from the discussion in 1.1 that $\lambda(\tilde{R}) \cong \tilde{R}$. The first step is to show that there is a sequence of elements (t_c) in T with $\mathrm{Ann}(t_c) \subseteq Rc$ for each $0 \neq c \in R$. Suppose for contradiction that there is a non-zero element s of R with $\mathrm{Ann}(t) \not\subseteq Rs$ for all t in T. Then, since R is a valuation ring, $Rs \subseteq \mathrm{Ann}(t)$ for all t in T. So $0 \neq Rs \subseteq \bigcap_{t \in T} \mathrm{Ann}(t) = \mathrm{Ann}\,T$. This contradicts the statement that T is a faithful R–module. So for each $0 \neq c \in R$ there is an element t_c in T with $\mathrm{Ann}(t_c) \subseteq Rc$. For each t_c $(0 \neq c \in R)$ there is an element d_c in R with $0 \neq Rd_c \subseteq \mathrm{Ann}(t_c)$.

Let $\tilde{a} = (a_r + Rr)$ be an element of $\mathrm{Ann}_{\tilde{R}} T$ and let $0 \neq c \in R$. Then $0 = \tilde{a}t_c = a_{d_c}t_c$ and so $a_{d_c} \in \mathrm{Ann}(t_c) \subseteq Rc$. However $Rd_c \subseteq Rc$ so $a_{d_c} - a_c \in Rc$. Thus $a_c \in Rc$ and so $a_c + Rc = 0 + Rc$. Hence $\tilde{a} = (0 + Rr)$, the zero element of $\tilde{R}$. Thus T is a faithful $\tilde{R}$–module and $\tilde{R} \cong \lambda(\tilde{R})$, completing the proof. □

This next result, proved by Matlis in [16; Proposition 4.7], is used with Theorem 4.7 to determine alglat T for a faithful torsion module T over an almost maximal valuation domain.

4.8 Proposition (Matlis)

Let R be an integral domain and $\tilde{R}$ its completion in the R-topology. Then R is an almost maximal valuation ring if and only if $\tilde{R}$ is a maximal valuation ring.

4.9 Theorem

Let R be an almost maximal valuation domain with R-completion $\tilde{R}$. Let T be a faithful torsion R–module. Then $\tilde{R} \cong \lambda(\tilde{R}) = \mathrm{alglat}_R T_R$.

Proof

From Proposition 4.6 and Theorem 4.7, $\tilde{R} \cong \lambda(\tilde{R})$ and $\lambda(\tilde{R}) \subseteq \mathrm{alglat}_R T_R =$

$\text{alglat}_{\tilde{R}}T_{\tilde{R}}$. The valuation domain R is almost maximal so, from Proposition 4.8, the completion $\tilde{R}$ is a maximal valuation ring. Then T is a reflexive $\tilde{R}$-module (Theorem 4.1) and so $\text{alglat}_{\tilde{R}}T_{\tilde{R}} = \lambda(\tilde{R})$. Hence there is equality and so $\lambda(\tilde{R}) = \text{alglat}_R T_R$. □

Remark

This theorem holds irrespective of whether or not the domain R is complete. For, if R is complete then $R = \tilde{R}$ and R is maximal (see comments following 4.4), so T is reflexive as already shown by Theorem 4.1. However if R is not complete then, since T is a faithful $\tilde{R}$-module, $\lambda(R)$ is strictly contained in $\lambda(\tilde{R})$ and so T is not a reflexive R-module (see discussion before Theorem 4.7).

The results of this chapter are summarised in Theorem 4.10. This gives the structure of alglat for any module over an almost maximal valuation ring. An illustration of the nature of alglat completes the chapter.

4.10 Theorem

Let R be an almost maximal valuation ring and let T be an R-module. If R is not maximal and if T is faithful and torsion, then $\text{alglat}_R T_R = \lambda(\tilde{R}) \cong \tilde{R}$, and T is not reflexive. In all other cases T is reflexive.

4.11 Illustration of the nature of alglat

R is an almost maximal valuation ring

T is an R–module

T is non-torsion

Then T is reflexive

(2.5, 2.11)

T is torsion

T is not faithful

Then T is reflexive (4.3)

T is faithful

R is maximal

Then T is reflexive (4.1)

R is not maximal

Then R is a domain (4.2) and

R is not complete (after 4.4)

So $\tilde{R} \cong \lambda(\tilde{R}) = \text{alglat}_R T_R$

and T is not reflexive (4.9)

Chapter 5 The structure of FGC, FSI and CF rings

Vámos defined a ring to be fractionally self-injective or FSI if for each ideal I of R the classical ring of quotients of R/I is self-injective (see Definition 5.1). This chapter is concerned with the structure of FSI rings. There are also results on FGC and CF rings, both classes having similar properties to FSI rings. No results are given here for alglat for modules over FSI rings. Instead, this chapter includes all the information on FSI rings which is needed to determine alglat and thus provides the necessary background to the study of alglat for modules over FSI rings in the next two chapters.

In [25], Vámos studied FGC rings as well as FSI rings. (The definition of an FGC ring was given in 2.6.) Brandal, Shores, R. Wiegand and S. Wiegand also studied FGC rings in [2], [23] and [26]. In [23] Shores and R. Wiegand introduced and studied CF rings (see Definition 5.3). All three types of rings have been characterised in terms of their indecomposable rings and the structure theorems are given in this chapter. Vámos proved (Theorem 5.12) that every FGC ring is an FSI ring and Theorem 5.15 shows that every FSI ring is a CF ring. A description of local FSI rings and FSI domains is also included. Specific examples are then given to illustrate all these relationships and to show that the classes of rings are distinct.

It has already been seen that every FGC ring is scalar-reflexive (see note prior to 2.6) and it will be seen in Theorem 5.17 that every FSI ring is locally scalar-reflexive and hence also scalar-reflexive (Theorem 2.17). This chapter ends with a discussion of alglat for a specific module over a valuation ring which is not almost maximal, recalling from Theorem 2.11 the fact that arbitrary valuation rings are not scalar-reflexive. It is remarked (before Proposition 5.5) that every valuation ring is a CF ring, and thus not every CF ring is scalar-reflexive. The study in Chapters 6 and 7 of alglat for modules over FSI rings also determines alglat for modules over FGC rings. The next obvious generalisation of these results is to

CF rings. The nature of the work in Chapters 6 and 7 together with Example 5.22, indicates that any characterisation of alglat for modules over the more general CF rings will not be a simple extension of the results for modules over FSI rings.

The chapter begins with the definitions of FSI, CF and FGCF rings. (Recall that a ring R is self-injective if R is an injective R–module.)

5.1 Definition of an FSI ring (Vámos [25])

A ring R is a fractionally self-injective ring or an FSI ring if for each ideal I of R the classical ring of quotients of R/I is self-injective.

The concept of a canonical form or canonical decomposition for a module was defined by Shores and R. Wiegand in [23] and was used there to make the definition of a CF ring. The two definitions of FGC and CF rings were then combined in [23] to form a class of rings called FGCF rings and this definition is given in 5.4.

5.2 Definition of a canonical form

A canonical form for an R–module M is a decomposition $M \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n$ where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subset R$.

5.3 Definition of a CF ring

A ring R is said to be a CF ring if every direct sum of finitely many cyclic R–modules has a canonical form.

5.4 Definition of an FGCF ring

A ring is an FGCF ring if it is both an FGC ring and a CF ring.

Shores and R. Wiegand characterised all CF rings and all FGCF rings in [23]. In [25], Vámos gives a complete description of all FSI rings and, independent of [23], of all FGCF rings. Vámos indicated in [25; Remark p217] that a proof that every FGC ring has only finitely many minimal prime ideals is sufficient to show that every FGC ring is an FGCF ring. This proof (that an FGC ring has only finitely many minimal primes) was given by Brandal and R. Wiegand in [2; Lemma 3]. These results then show that the definitions of FGC and FGCF rings are equivalent and hence determine the structure of all FGC rings. A detailed account of these proofs and the structure of FGC rings was given by R. Wiegand and S. Wiegand in the expository article [26] and later by Brandal in [3].

The next part of this chapter is concerned with the structure theorems. The first type of rings to be studied are the CF rings, and three decomposition theorems for a CF ring will be given. Proposition 5.5 was proved by Shores and R. Wiegand in [23; Corollary 1.7] and provides examples of CF rings. It was noted in [23], in the proof of this result, that a valuation ring is a CF ring.

5.5 Proposition ([23])

Every h-local Prüfer domain is a CF ring.

The study of CF rings introduces another class of rings which were called ?-rings in [23] by Shores and R. Wiegand. The definition follows in 5.6. Shores and R. Wiegand gave an example of a ?-ring in [23; Example 3.13].

5.6 Definition of a ?-ring ([23])

A ring is a ?-ring if it is an indecomposable CF ring that is neither a valuation ring nor an h-local domain.

There is now sufficient information to present the structure theorems for CF rings. The following three theorems of Shores and R. Wiegand are from [23; Theorems 3.10, 3.11 and 3.12] and characterise CF rings and ?-rings.

5.7 Theorem ([23])

Every CF ring is a finite direct sum of indecomposable CF rings. The indecomposable CF rings are precisely the rings R such that (i) R is arithmetical, (ii) R has a unique minimal prime P, (iii) R/P is an h-local domain, and (iv) every ideal contained in P is comparable with every ideal of R.

5.8 Theorem ([23])

Every ?-ring R has the following properties (in addition to (i) - (iv) of Theorem 5.7): (v) $P \neq P^2 = 0$, (vi) R has at least two maximal ideals, (vii) P is an indecomposable, torsion, divisible R/P–module, (viii) $P = P_M$ for a unique maximal ideal M, (ix) $P_N = 0$ for every maximal ideal $N \neq M$. Conversely every ideal satisfying (i) - (vi) is a ?-ring.

5.9 Theorem ([23])

A ring is a CF ring if and only if it is a finite direct sum of valuation rings, h-local Prüfer domains, and ?-rings.

Thus the indecomposable CF rings are precisely the valuation rings, the h-local Prüfer domains and the ?-rings. Note that every CF ring is arithmetical; it is not only the indecomposable CF rings that are arithmetical ([23; Proposition 1.10]). In [25], Vámos gave the ?-rings the name of torch rings, this name being suggested by the shape of the ideal lattice of these rings. His definition of a torch ring follows in 5.10. (Note that a module is uniserial if all its submodules are totally ordered.)

5.10 Definition of a torch ring ([25])

A ring R is a torch ring if the following conditions are satisfied:

(1) R is an arithmetical ring with at least two maximal ideals, and

(2) R has a unique minimal prime ideal P such that R/P is an h-local domain, P is uniserial and $P \neq 0$.

Note that a definition of a torch ring is given in [26] which does not require R/P to be an h-local domain. However if R is a torch ring in the sense of [25] (5.10) then it is clear that R is a torch ring in the sense of [26]. In particular results from [26] on torch rings may be used.

Vámos stated in [25] that the definitions of a torch ring and a ?-ring are indeed equivalent. A proof is included in Lemma 5.11 for completeness.

5.11 Lemma

The definitions of a torch ring and a ?-ring are equivalent.

Proof

Let R be a ?-ring. Properties (i) and (vi) from Theorem 5.8 give condition (1) for a torch ring. From (iv) it can be seen that the ideals of R contained in P form a chain, and so P is uniserial. Then conditions (ii), (iii), (iv) and (v) ensure that R satisfies condition (2). Thus R is a torch ring.

Let R be a torch ring with unique minimal prime ideal P. It is clear from the definition that R satisfies (i), (ii), (iii) and (vi). From [23; Lemma 3.1] P is comparable to every ideal of R. Then, since P is uniserial, condition (iv) holds. From [26; Lemma 18], $P^2 = 0$. But $P \neq 0$ and so condition (v) is satisfied. Thus R is a ?-ring. □

The name torch ring will be used from now on for this class of rings.

Now that CF rings have been characterised, the next part of the chapter is concerned with the decomposition of FSI and FGC rings. Results are also given which show that every FGC ring is FSI and every FSI ring is CF. The next theorem begins this classification by relating the structure of an FGC ring to that of an FSI ring. This was proved by Vámos in [25; Theorem A].

5.12 Theorem ([25])

For a ring R the following are equivalent:

(i) R is an FGC ring,

(ii) R is an FSI Bézout ring.

Hence every FGC ring is an FSI ring. So this theorem provides examples of FSI rings. Moreover the class of FGC rings is strictly contained in the class of FSI rings. This will be shown in Example 5.14 by using the structure theorem for FSI rings to give an FSI ring which is not an FGC ring. The next result, Theorem 5.13, is the structure theorem for FSI rings and was proved by Vámos in [25; Theorem B].

5.13 Theorem ([25])

Every FSI ring is the finite direct sum of indecomposable FSI rings. The indecomposable FSI rings are precisely the almost maximal valuation rings, the locally almost maximal h-local domains and the locally almost maximal torch rings.

From [25; Lemma 6], every FSI ring is arithmetical. So, in particular, these locally almost maximal h-local domains, being FSI rings, are arithmetical. From Theorem 5.13, any indecomposable FSI ring R has a unique minimal prime ideal P which is uniserial, that is the ideals of R contained in P form a chain (see also [25; Lemmas 5 and 8]), and the ideal P is comparable to every ideal of R ([23; Lemma 3.1]).

5.14 Example of an FSI ring that is not an FGC ring

Let R be a Dedekind domain which is not a PID. Then R is an h-local domain and every localisation of R at a maximal ideal is an almost maximal valuation domain. Thus R is a locally almost maximal h-local domain and so, from Theorem 5.13, R is an FSI ring. However, since R is not a PID, there are ideals which are finitely generated but not principal. Thus R is not a Bézout ring. Hence, from Theorem 5.12, R is not an FGC ring. So every Dedekind domain is an FSI ring, and those Dedekind domains which are not PIDs are FSI rings but not FGC rings. Hence the class of FGC rings is strictly contained in the class of FSI rings.

Theorems 5.12 and 5.13 together characterise all FGC rings in terms of the indecomposable FGC rings. This same characterisation was proved independently by Shores and R. Wiegand in [23; Corollary 4.2]. Another decomposition theorem for FGC rings was proved by R. Wiegand and S. Wiegand in [26; Theorem 5] which classifies the FGC rings in terms of the properties of the indecomposable FGC rings.

So FSI and FGC rings have all been characterised, and it has been shown that all FGC rings are FSI. Theorem 5.15, which does not appear to be in the literature, follows from the structure theorems for CF and FSI rings and shows that every FSI ring is a CF ring.

5.15 Theorem

For a ring R the following are equivalent:

(i) R is an FSI ring,

(ii) R is a locally almost maximal CF ring.

Proof

(i) $\Rightarrow$ (ii)

Let R be an FSI ring. Then R is a finite direct sum of indecomposable FSI rings (Theorem 5.13). Each indecomposable FSI ring is an indecomposable CF ring (Theorem 5.13 and comments after Theorem 5.9). Thus R is a finite direct sum of valuation rings, h-local Prüfer domains and torch rings. Hence R is a CF ring (Theorem 5.9). From [25; Lemma 6], R is a locally almost maximal arithmetical ring.

Thus R is a locally almost maximal CF ring.

(ii) $\Rightarrow$ (i)

Let R be a locally almost maximal CF ring. Then R is a finite direct sum of indecomposable CF rings (Theorem 5.7), $R = \bigoplus_{i=1}^{n} R_i$. Then, since R is a locally almost maximal arithmetical ring, it can be seen that each of the rings R_i is also a locally almost maximal arithmetical ring. Thus each R_i is an indecomposable locally almost maximal CF ring. From Theorems 5.9 and 5.13 each R_i is an indecomposable FSI ring, and so R is a finite direct sum of FSI rings. Thus R is an FSI ring ([25; Lemma 1]). $\square$

Thus every FSI ring is a CF ring. Proposition 5.16 and the ensuing comments will show that these classes of rings are distinct, completing the presentation of the structure theorems for CF, FSI and FGC rings. The next results of this chapter determine the local FSI rings and the FSI domains. In Proposition 5.16, the equivalence of conditions (i), (ii) and (iii) was proved by Vámos in [25; Lemma 5] and condition (iv) follows immediately from Theorem 2.11.

5.16 Proposition

Let R be a local ring. Then the following are equivalent:

(i) R is an almost maximal valuation ring,

(ii) R is an FGC ring,

(iii) R is an FSI ring, and

(iv) R is scalar-reflexive.

Examples were given in 2.9 of valuation rings which are not almost maximal. These rings are local CF rings but are not FSI rings (see note before 5.5 and Proposition 5.16). So the class of FSI rings is strictly contained in the class of CF rings.

An easy consequence of Proposition 5.16 was given by Vámos in [25; Lemma 6] and states that if R is an FSI ring, then R is a locally almost maximal arithmetical ring. Recall that the locally almost maximal arithmetical rings are precisely the locally scalar-reflexive rings (discussion after 2.13). From these remarks it is clear that every FSI ring is locally scalar-reflexive and this result is given in Theorem 5.17. Theorem 5.17 will be used at the beginning of Chapter 6 to determine alglat for non-torsion modules over FSI rings.

5.17 Theorem

Let R be an FSI ring. Then R is locally scalar-reflexive.

It was remarked in Chapter 2 following Theorem 2.11 that, for a local ring, the conditions of being scalar-reflexive and of being an FGC ring are equivalent, but that these conditions are not equivalent in general. It has also been noted that every FGC ring is scalar-reflexive. Example 5.14 provides a ring R which is an FSI ring but not an FGC ring. From Theorems 2.17 and 5.17, R is scalar-reflexive. Thus not every scalar-reflexive ring is an FGC ring.

Theorem 5.18, which does not appear to be in the literature, characterises all FSI domains and follows from Theorem 5.13.

5.18 Theorem

A ring is an FSI domain if and only if it is a locally almost maximal h-local domain.

Proof

Suppose that R is an FSI domain. Then R is an indecomposable FSI ring. From the definition in 5.10, it is clear that R is not a torch ring. Moreover an almost maximal valuation domain is a locally almost maximal h-local domain. Thus R is a locally almost maximal h-local domain (Theorem 5.13).

The converse is immediate from Theorem 5.13. □

Thus all local FSI rings and all FSI domains are characterised. The next theorem gives an alternative description of FSI domains. This result was stated at the end of Chapter 3 and provides another classification of all scalar-reflexive h-local domains.

5.19 Theorem

Let R be a commutative ring. Then the following are equivalent:

(i) R is a scalar-reflexive h-local domain, and

(ii) R is an FSI domain.

Proof

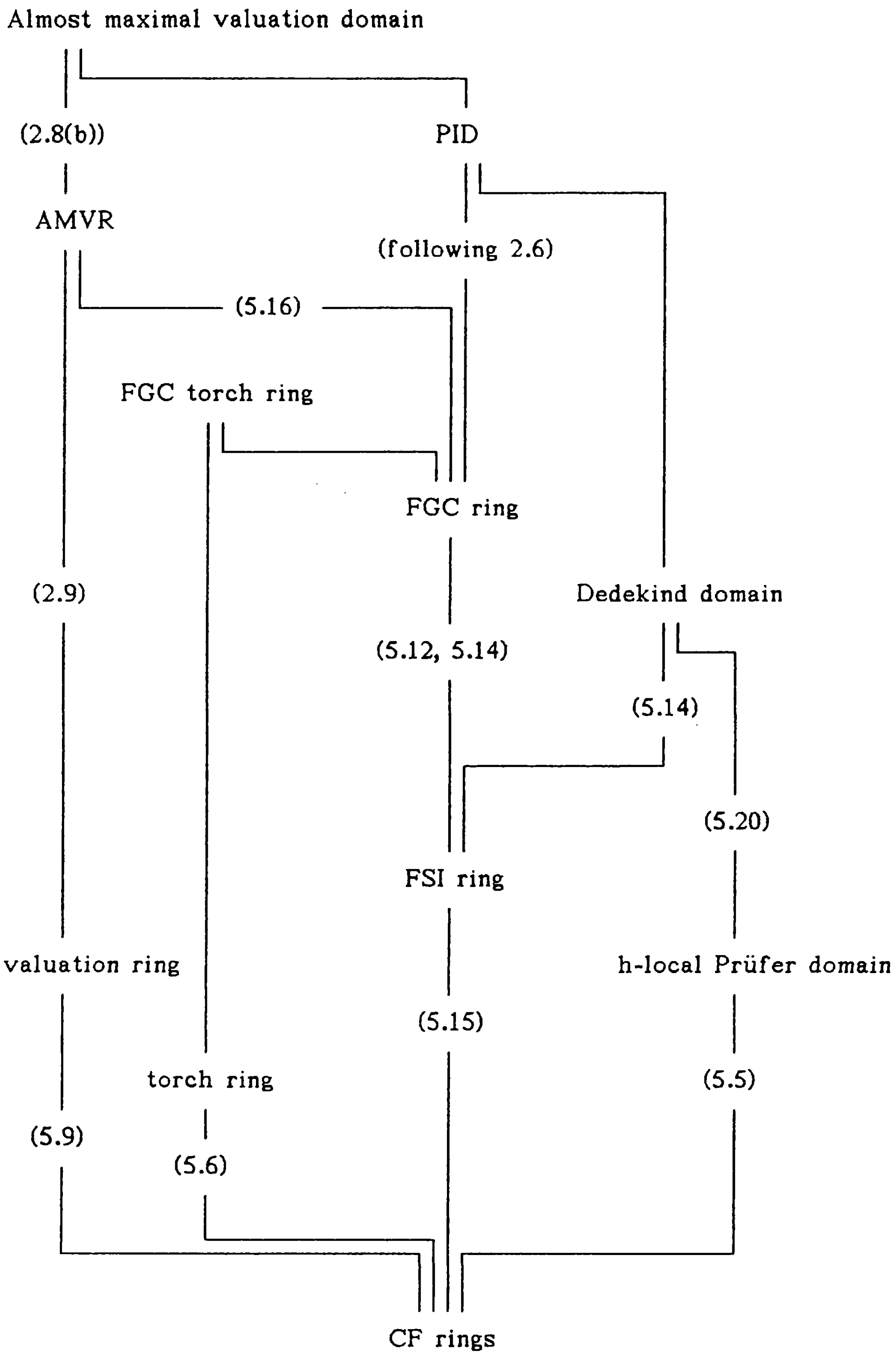
From Theorem 3.1, condition (i) is equivalent to the assertion that R is a locally scalar-reflexive h-local domain. But the locally scalar-reflexive h-local domains are precisely the locally almost maximal h-local domains (see after 2.13). The result then follows from Theorem 5.18. □

The next part of the chapter summarises the relationships between the classes of rings discussed in this chapter. The example in 5.20 of an h-local Prüfer domain that is not a Dedekind domain will be required. The examples of these rings are illustrated below.

5.20 Example of an h-local Prüfer domain that is not Dedekind

Note that all Dedekind domains and all valuation domains are h-local Prüfer domains. From Example 2.9 there is a valuation domain R which is not almost maximal. Then R is an h-local Prüfer domain which is not noetherian (see Example 2.8(a)). Since every Dedekind domain is noetherian, R is not a Dedekind domain.

5.21 Illustration of FGC, FSI and CF rings



All the classes of rings shown here are distinct.

The final part of this chapter looks at alglat for a specific 2–generated module M over the valuation ring R in 2.9(b). It will be helpful for the study of arbitrary modules over valuation rings to obtain some information from examples. This particular construction of a module M over a valuation ring which is not almost maximal was used by Hadwin and Kerr in the proof of Theorem 2.11 ([9; Theorem 6]).

Theorem 2.11 states that, for a local ring, being scalar-reflexive is equivalent to being an almost maximal valuation ring. Thus any valuation ring R which is not almost maximal has finitely generated R–modules which are not reflexive. In order to prove that a local scalar-reflexive ring is an almost maximal valuation ring, Hadwin and Kerr used [9; Lemma 4] which shows that a local scalar-reflexive ring is a valuation ring. They then assumed for contradiction that R was a scalar-reflexive valuation ring but not almost maximal. A 2–generated R–module M was constructed which was not reflexive, thus contradicting the assertion that R is scalar-reflexive. Their proof did not determine alglat M, but provided a single map in alglat M which is not in $\lambda(R)$.

Example 5.22 concludes this chapter by using this construction to give a 2–generated module M over the valuation ring in 2.9(b) which is not reflexive. The ring alglat M is then fully determined. The module M can also be considered as an S–module where $S = R/\operatorname{Ann} M$ and then $\operatorname{alglat}_R M_R = \operatorname{alglat}_S M_S$ (Proposition 1.19). Then M is not reflexive as an S–module. It will be shown that M is a non-torsion module over the valuation ring S and thus it is not even the case that every non-torsion module over an arbitrary valuation ring is reflexive.

The proofs within Example 5.22 are outlined with some of the details being omitted. The reader is also referred to the proof of Theorem 2.11 in [8; Theorem 6]. It is hoped that this will open up the discussion and solution of similar problems.

5.22 Example of a 2-generated module over a valuation ring which is not reflexive

Let R be the valuation ring of Example 2.9(b). Then R is a domain and $R = A_M$ where $A = \mathbb{C}[x_0, x_1, x_2, \dots]$, $x_n^2 = x_{n-1}$ and M is the maximal ideal $(x_0, x_1, x_2, \dots)$. For ease of notation write r for the element $\frac{r}{1}$ in R. Let $a_1 = 1$ and, for $n \geq 2$, $a_n = a_{n-1} + x_{n+1}^{(2^{n+1}-3)}$ be elements of R and let $I_n = (x_n^{2^n-1})$ be a family of ideals in R $(n \geq 1)$. Then the system of congruences $\{a_n \bmod I_n\}$ is pairwise soluble but has no simultaneous solution in R. Let $b_n = a_n - a_{n-1}$ so that $b_n \in I_{n-1} \setminus I_n$ and $b_{n+1}J(R) \subsetneq I_n \subsetneq b_nJ(R)$ (note that R is local so J(R) is the unique maximal ideal MR of R). Then the system of congruences $\{a_n \bmod b_nJ(R)\}$ is also pairwise soluble with no simultaneous solution in R. The intersection of these ideals $\bigcap b_nJ(R) = \bigcap I_n = Rx_0$.

Let $M = (R \oplus R)/K$ where K is the R-submodule generated by $x_0J(R) \oplus x_0J(R)$ and $\{(x_n^3, -x_n^3a_{n-1}) : n \geq 2\}$. Then it can be shown that an arbitrary element of K has the form $s(x_n^3, -x_n^3a_{n-1}) + (x_0g, x_0h)$ where $s \in R$ and $g, h \in J(R)$. (Note that $x_{n+1}^3b_n = x_{n+1}^3x_{n+1}^{(2^{n+1}-3)} = x_{n+1}^{2^{n+1}} = x_0$. So comparing with the notation of the proof of Theorem 2.11, $c = x_0$ and for each b_n, $w_{b_n} = x_{n+1}^3$. Thus $b_nw_{b_n} = c$.) Define a map $\phi : M \to M$ by $(u, v) + K \mapsto (x_0u, 0) + K$. Then $\phi \in \operatorname{alglat}_R M_R$ and $\phi \notin \lambda(R)$ (the proof is identical to that of Theorem 2.11). Note that $\phi^2((u, v) + K) = (x_0^2u, 0) + K = (0, 0) + K$ and so $\phi^2 = 0$. Then M is not reflexive and $\lambda(R) + \lambda(R)\phi \subseteq \operatorname{alglat}_R M_R$.

To prove the reverse inclusion let $\theta \in \operatorname{alglat}_R M_R$. Then there are elements r, s, t in R with $\theta((1, 0) + K) = r((1, 0) + K)$, $\theta((0, 1) + K) = s((0, 1) + K)$ and $\theta((1, 1) + K) = t((1, 1) + K)$. So $(r - t, s - t) \in K$ and thus there are elements $f \in R$ and $g \in J(R)$ with $r - s = fx_n^3(1 + a_{n-1}) + x_0g$. Then $\theta((u, v) + K) = s((u, v) + K) + ((r - s)u, 0) + K = s((u, v) + K) + (fx_n^3(1 + a_{n-1})u, 0) + K$ since $x_0g \in x_0J(R)$.

The next step is to show that $fx_n^3(1 + a_{n-1}) \in Rx_0$. For any $m \geq 2$, $\theta((0, 0) + K) = \theta((x_m^3, -x_m^3a_{m-1}) + K) = s((x_m^3, -x_m^3a_{m-1}) + K) + (fx_n^3(1 + a_{n-1})x_m^3, 0) + K$. Thus $(fx_n^3(1 + a_{n-1})x_m^3, 0) \in K$. It follows that

$fx_n^3(1 + a_{n-1})x_m^3 \in x_0J(R)$ and so $fx_n^3(1 + a_{n-1}) \in b_mJ(R)$ for all $m \geq 2$. Hence $fx_n^3(1 + a_{n-1}) \in \bigcap b_mJ(R) = Rx_0$.

So there is some element d of R with $fx_n^3(1 + a_{n-1}) = dx_0$. Then $\theta((u, v) + K) = s((u, v) + K) + (dx_0u, 0) + K = s((u, v) + K) + d\phi((u, v) + K)$. Hence $\theta = \lambda(s) + \lambda(d)\phi$ and so $\text{alglat}_R M_R \subseteq \lambda(R) + \lambda(R)\phi$. Thus $\text{alglat}_R M_R = \lambda(R) + \lambda(R)\phi$.

It is clear that $x_0J(R) \subseteq \text{Ann}\,M$. Let $r \in \text{Ann}((1, 0) + K)$ so then $(r, 0) \in K$. Noting that each a_n is a unit in R, it follows that $r \in x_0J(R)$. So $x_0J(R) = \text{Ann}((1, 0) + K) = \text{Ann}\,M$. Then M is a 2–generated torsion module over a valuation domain which is not reflexive.

Let $S = R/\text{Ann}\,M$ so that S is a valuation ring (not a domain). Consider M as an S–module in the natural way. Then $\text{alglat}_R M_R = \text{alglat}_S M_S$ and M is not reflexive as an S–module. However $\text{Ann}_S((1, 0) + K) = 0$ and so M is a non-torsion S–module. Thus there are 2–generated non-torsion modules over valuation rings that are not reflexive.

The next two chapters will use the results about FSI rings given in this chapter to characterise alglat for all modules over an FSI ring.

Chapter 6 Study of alglat for modules over FSI rings - part 1

The study of alglat for a module over an FSI ring divides into the two cases of considering non-torsion modules and then torsion modules. The first result of the chapter deals with the non-torsion case and proves that every non-torsion module over an FSI ring is reflexive. This follows from the structure of an FSI ring, in that every FSI ring is locally scalar-reflexive, and from the work in Chapter 2 on non-torsion modules over locally scalar-reflexive rings.

Having dealt with the non-torsion case, the remainder of the chapter begins the characterisation of alglat for torsion modules. First it is seen that the torsion case can be reduced to the study of a torsion module over an indecomposable FSI ring. This uses the decomposition for alglat in Theorem 1.10.

There are three types of indecomposable FSI rings, namely almost maximal valuation rings, locally almost maximal h-local domains and locally almost maximal torch rings (Theorem 5.13). Any indecomposable FSI ring R has a unique minimal prime ideal P which is comparable to every ideal of R (see after Theorem 5.13). Then for a torsion R–module T, either $P \subseteq \mathrm{Ann}\,T$ or $\mathrm{Ann}\,T \subset P$. This chapter looks at the case where $\mathrm{Ann}\,T \subset P$. Then $P \neq 0$ and thus R is not a domain. Modules over almost maximal valuation rings were studied in Chapter 4. So this chapter is concerned only with locally almost maximal torch rings.

In order to study and characterise alglat for torsion modules over locally almost maximal torch rings, a completion will be introduced in 6.4. This completion is defined for any ring R with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R. The unique minimal prime ideal P of a locally almost maximal torch ring satisfies these conditions with $P = I$. In this topology (of 6.4) the non-zero principal ideals of R contained in I form a sub-base for the open neighbourhoods of 0 in R. This is called the I-topology of R.

The main theorem in this section is Theorem 6.7 which uses the completion of

6.4. Let R be a scalar-reflexive ring with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R; then Theorem 6.7 proves that for a faithful torsion R–module T, alglat T is isomorphic to the I-completion of R. Corollary 6.8 follows easily from this more general theorem and determines alglat for a faithful torsion module over a locally almost maximal torch ring.

Chapter 7 begins by studying the case where $P \subsetneq \operatorname{Ann} T$. The results of these two chapters are then brought together to give a complete characterisation of alglat for modules over FSI rings.

Theorem 6.1 determines alglat for non-torsion modules over FSI rings. The proof is immediate from Theorem 5.17 which shows that an FSI ring is locally scalar-reflexive, and from Corollary 2.19 which deals with non-torsion modules.

6.1 Theorem

Let R be an FSI ring. Then every non-torsion R–module is reflexive.

The rest of this chapter looks at the torsion case and begins by reducing this to the study of alglat for a torsion module over an indecomposable FSI ring. This is done by observing that there is a decomposition for alglat using Theorem 1.10 from Chapter 1.

Let T be any torsion module over an FSI ring R. The structure theorem for FSI rings (Theorem 5.13) shows that R is a finite direct sum of indecomposable FSI rings R_i for $i = 1, \ldots, n$. From the discussion in Chapter 1 prior to Theorem 1.10, T can also be decomposed as a direct sum with $T = \bigoplus_{i=1}^{n} T_i$ where each T_i is an R_i–R_i–bimodule. Then, from Theorem 1.10, $\operatorname{alglat} T \cong \bigoplus_{i=1}^{n} \operatorname{alglat} T_i$ and the T_i are modules over the indecomposable FSI rings R_i $(i = 1, \ldots, n)$. Since T is a torsion R–module, at least one of the T_i is a torsion R_i–module. For otherwise, each module T_i has a non-torsion element t_i $(i = 1, \ldots, n)$. Then the sum of these elements $t = t_1 + t_2 + \cdots + t_n$ is a non-torsion element of T, contradicting T torsion. From Theorem 6.1 every non-torsion module over an FSI ring is reflexive. Thus it is sufficient to determine alglat for torsion modules over indecomposable FSI rings.

Let R be an indecomposable FSI ring and let T be a torsion R–module. Then R has a unique minimal prime ideal P which is comparable to every ideal of R. So either $P \subseteq \operatorname{Ann} T$ or $\operatorname{Ann} T \subset P$. This chapter looks at the second case and determines $\operatorname{alglat} T$ where $\operatorname{Ann} T \subset P$. Since $P \neq 0$, R must be either an almost maximal valuation ring or a locally almost maximal torch ring. As recalled above, alglat was determined for modules over almost maximal valuation rings in Chapter 4. Thus it is sufficient to consider R as a locally almost maximal torch ring.

So let R be a locally almost maximal torch ring and let T be a torsion R–module with $\operatorname{Ann} T \subset P$. Let $S = R/\operatorname{Ann} T$ so that S is an FSI ring ([25; Lemma 1]). Then T is a faithful S–module and, from Proposition 1.19, $\operatorname{alglat}_R T_R = \operatorname{alglat}_S T_S$. If T is a non-torsion S–module then T is reflexive as an S–module (Theorem 6.1) and hence reflexive as an R–module. Thus the only case to consider is where T is a faithful torsion S–module. Vámos states in the proof of [25; Theorem B] that a factor ring of a locally almost maximal torch ring is either a ring of the same type or a locally almost maximal h-local domain or factor rings of this latter ring. The next proposition is part of the preceeding statement and shows that, for $S = R/\operatorname{Ann} T$ with $\operatorname{Ann} T \subset P$, the factor ring S is a locally almost maximal torch ring. A proof is included here for completeness.

6.2 Proposition

Let R be a locally almost maximal torch ring with unique minimal prime ideal P. Let $I \subset P$ and write $S = R/I$. Then S is a locally almost maximal torch ring.

Proof

The ring R is an FSI ring and so from [25; Lemma 1] the factor ring S is also an FSI ring. The first step is to show that the prime ideal $Q = P/I$ is the unique minimal prime ideal of S. Suppose Q_1 is a prime ideal of S contained in Q. Then $Q_1 = P_1/I$ where P_1 is a prime ideal of R contained in P. So by minimality of P, $P_1 = P$ and thus $Q_1 = Q$. Hence Q is a minimal prime ideal of S. Suppose Q_2 is a minimal prime ideal of S. Then $Q_2 = P_2/I$ where P_2 is a prime ideal of R. Now P is comparable to every ideal of R and so either $P_2 \subseteq P$ or $P \subset P_2$. If $P \subset P_2$ then $Q \subset Q_2$ which contradicts the minimality of Q_2. So $P_2 \subseteq P$. Then by minimality of P, $P_2 = P$ and so $Q_2 = Q$. Hence Q is unique and is therefore the unique minimal prime ideal of S.

It now follows from [25; Lemma 8] that S is an indecomposable FSI ring. The ideal I is strictly contained in P and so is not a prime ideal of R. Thus S is not a domain. Moreover R has at least two maximal ideals M and N. Then M/I and N/I are distinct maximal ideals of S and so S is not a local FSI ring. Hence, from Theorem 5.13, S is a locally almost maximal torch ring. □

Thus to characterise $\operatorname{alglat} T$ for a torsion R–module T where R is an indecomposable FSI ring and $\operatorname{Ann} T \subset P$, it is enough to determine alglat for a faithful torsion module over a locally almost maximal torch ring. The rest of this chapter works towards the structure theorem for alglat in this case which is given in Corollary 6.8. Then Theorem 6.10 summarises the information in this chapter to give a concise characterisation of $\operatorname{alglat} T$ where T is an R–module over an indecomposable FSI ring R with $\operatorname{Ann} T \subset P$.

As already discussed in Chapters 1 and 4, the structure of alglat is closely associated with inverse limits and topological completions. The R-topology of a domain R was defined in 4.4, and this topology takes the non-zero principal ideals of R as a sub-base for the open neighbourhoods of 0 in R. A completion will be introduced in 6.4 to study faithful torsion modules over locally almost maximal torch rings.

In order to describe alglat in terms of a completion, the ring must be Hausdorff in this new topology. This ensures that there is an embedding of the ring in its completion. The unique minimal prime ideal P of a locally almost maximal torch ring R is comparable to every ideal of R and the ideals of R contained in P form a chain. The next lemma shows that $\bigcap_{0 \neq p \in P} Rp \subseteq \operatorname{Ann} T$ for all torsion R–modules T (note that $P \neq 0$). So if R has a faithful torsion module T then $\operatorname{Ann} T = 0$ and hence $\bigcap_{0 \neq p \in P} Rp = 0$. Thus if the non-zero principal ideals of R contained in P are taken to form a sub-base for the open neighbourhoods of 0 in R and if R has a faithful torsion module then R is Hausdorff in this topology. This motivates the definition of the I-topology given in 6.4.

6.3 Lemma

Let R be a commutative ring with a non-zero ideal I such that I is comparable to every ideal of R. Let T be any torsion R–module. Then $\bigcap_{0 \neq x \in I} Rx \subseteq \operatorname{Ann} T$.

Proof

Let $a \in \bigcap_{0 \neq x \in I} Rx$ so then $a \in I$ and let $t \in T$. I is comparable to every ideal of R so either $I \subseteq \operatorname{Ann}(t)$ or $\operatorname{Ann}(t) \subseteq I$. If $I \subseteq \operatorname{Ann}(t)$ then clearly $a \in \operatorname{Ann}(T)$. Now suppose $\operatorname{Ann}(t) \subseteq I$. Then since T is torsion there is some non-zero element r of R with $r \in \operatorname{Ann}(t)$ and so $0 \neq r \in I$. Then $a \in Rr$ and so $a \in \operatorname{Ann}(t)$. So for all $t \in T$, $a \in \operatorname{Ann}(t)$ and thus $a \in \operatorname{Ann} T$. Hence $\bigcap_{0 \neq x \in I} Rx \subseteq \operatorname{Ann} T$. □

Thus any indecomposable FSI ring R with unique minimal prime ideal $P \neq 0$ satisfies the hypotheses of Lemma 6.3 (with $I = P$). It is also the case that the ideals of R contained in P form a chain. Although these rings were the motivation for the I-topology, the definition in 6.4 is valid for a larger class of rings.

6.4 Definition of the I-topology

Let R be a commutative ring with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R. Define a topology on R called the I-topology by letting the non-zero ideals of R contained in I form a sub-base for the open neighbourhoods of 0 in R. The same topology is given to R by letting the non-zero principal ideals of R contained in I form a sub-base for the open neighbourhoods of 0 in R. This topology on R makes R into a topological ring and R is Hausdorff if and only if $\bigcap_{0 \neq x \in I} Rx = 0$. From the preceeding lemma and comments it can be seen that if R has a faithful torsion module then R is Hausdorff in the I-topology.

Suppose that R is Hausdorff in the I-topology. Then the inverse limit $\varprojlim R/J$ exists and is isomorphic to $\varprojlim R/Rx$ (where the index sets are, respectively, the family of non-zero ideals of R contained in I and the family of non-zero principal ideals of R contained in I and both index sets are ordered by inclusion). Let $R' = \varprojlim R/Rx$ $(0 \neq x \in I)$ so then R' is the completion of R in the I-topology. Then $(a_x + Rx)$ is an element of R' if and only if for any non-zero elements y, z in I with $Ry \subseteq Rz$ then $a_y - a_z \in Rz$ (recall that the ideals of R contained in I form a chain). There is a canonical ring homomorphism $\phi : R \rightarrow R'$ given by $\phi : a \mapsto (a + Rx)$ for $a \in R$ and $0 \neq x \in I$. Then $\ker\phi = \bigcap_{0 \neq x \in I} Rx$. Since R is Hausdorff, $\ker\phi = 0$ and so ϕ embeds R in R'. The ring R is complete in the I-topology if and only if ϕ is an isomorphism.

From 4.4, the R-topology is defined for a domain R. The R-completion of a domain R is $\tilde{R} = \varprojlim R/Rr$ for $0 \neq r \in R$. Let R be a valuation domain. Then the completion of R in the I-topology with $I = R$ is $R' = \varprojlim R/Rx$ for $0 \neq x \in R$. Thus $R' = \tilde{R}$ where $\tilde{R}$ is the completion of R in the R-topology. So these topologies coincide and there is no ambiguity in referring to the "R-topology" and the "R-completion".

More generally, let R be a domain with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R. For example R could be a valuation domain with I as any non-zero ideal of R. Then $\varprojlim R/Rr$ is isomorphic to $\varprojlim R/Rx$ where $0 \neq r \in R$ and $0 \neq x \in I$. That is, R has both the R- and the I-topologies and these topologies coincide.

Let R be a locally almost maximal torch ring with unique minimal prime ideal P ($P \neq 0$) and let T be a faithful torsion R–module. Let R′ be the completion of R in the P-topology. Then R is Hausdorff in the P-topology (see before Lemma 6.3) and so the homomorphism $\phi : R \rightarrow R'$ given by $\phi : a \mapsto (a + Rp)$ for $a \in R$ and $0 \neq p \in P$ embeds R in R′. In order to determine alglat T it has to be shown that T has an R′–module structure and this is the next result (cf Proposition 4.6). As with Proposition 4.6 the result proved here is more general than needed. Proposition 6.5 is then used in Theorem 6.7 to characterise alglat for faithful torsion modules over a class of rings which includes locally almost maximal torch rings. Corollary 6.8 then shows for a faithful torsion module T over a locally almost maximal torch ring R, that alglat T is isomorphic to R′, the completion of R in the P-topology.

Proposition 6.5 shows, for any ring R with the I-topology, that every faithful torsion R–module has an R′–module structure.

6.5 Proposition

Let R be a commutative ring with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R. Let T be a faithful torsion R–module and let R′ be the completion of R in the I-topology. Then T has an R′–module structure given by $(a_x + Rx)t = a_y t$ where $(a_x + Rx) \in R'$, $t \in T$ and $0 \neq Ry \subseteq \mathrm{Ann}(t)$. Moreover $\mathrm{alglat}_R T_R = \mathrm{alglat}_{R'} T_{R'}$.

Proof

Let $t \in T$. Then $\mathrm{Ann}(t) \neq 0$ since T is a torsion module. The ideal I is comparable to every ideal of R so either $I \subseteq \mathrm{Ann}(t)$ or $\mathrm{Ann}(t) \subseteq I$. In both cases there is an element $0 \neq y \in I$ with $Ry \subseteq \mathrm{Ann}(t)$.

Suppose that $Rx \subseteq \mathrm{Ann}(t)$ and $Ry \subseteq \mathrm{Ann}(t)$ for $0 \neq x, y \in I$. The ideals of R contained in I form a chain so suppose $Rx \subseteq Ry$. Then $a_x - a_y \in Ry \subseteq \mathrm{Ann}(t)$. Thus $a_x t = a_y t$.

Suppose $(a_x + Rx) = (b_x + Rx)$ and that $Ry \subseteq \mathrm{Ann}(t)$. Then $a_x - b_x \in Rx$ for all $0 \neq x \in I$. In particular $a_y - b_y \in Ry$ and so $a_y t = b_y t$.

Let $a' = (a_x + Rx)$ be an element of R' and let t_1, t_2 be elements of T. From the hypotheses it is clear that any ideal of R which is contained in I is comparable to every ideal of R. If at least one of $\mathrm{Ann}(t_1)$ and $\mathrm{Ann}(t_2)$ is contained in I then the ideals are comparable so suppose $\mathrm{Ann}(t_1) \subseteq \mathrm{Ann}(t_2)$. Then there is an element $0 \neq y \in I$ with $Ry \subseteq \mathrm{Ann}(t_1) \subseteq \mathrm{Ann}(t_2)$ and then also $Ry \subseteq \mathrm{Ann}(t_1 + t_2)$. On the other hand if neither $\mathrm{Ann}(t_1)$ nor $\mathrm{Ann}(t_2)$ are contained in I then, since I is comparable to every ideal, $I \subseteq \mathrm{Ann}(t_1) \cap \mathrm{Ann}(t_2)$. Thus again there is an element $0 \neq y \in I$ with $Ry \subseteq \mathrm{Ann}(t_1)$, $Ry \subseteq \mathrm{Ann}(t_2)$ and $Ry \subseteq \mathrm{Ann}(t_1 + t_2)$. Then in both cases, $a'(t_1 + t_2) = a_y(t_1 + t_2) = a_y t_1 + a_y t_2 = a't_1 + a't_2$.

Let $a' = (a_x + Rx)$, $b' = (b_x + Rx)$ be elements of R' and let t be an element of T with $0 \neq Ry \subseteq \mathrm{Ann}(t)$. Then $a' + b' = (a_x + b_x + Rx)$ and $a'b' = (a_x b_x + Rx)$. So $(a' + b')t = (a_y + b_y)t = a_y t + b_y t = a't + b't$. Since $\mathrm{Ann}(t) \subseteq \mathrm{Ann}(b_y t)$, $Ry \subseteq \mathrm{Ann}(b_y t)$. Thus $a'(b't) = a'(b_y t) = a_y(b_y t) = (a_y b_y)t = (a'b')t$. The identity element of R' is $(1 + Rx)$, and $(1 + Rx)t = 1t = t$.

Thus T is an R'-module under this product.

From 6.4, R is Hausdorff in the I-topology and so $R \subseteq R'$. Thus $\mathrm{End}\, T_{R'} \subseteq \mathrm{End}\, T_R$. Let $\phi \in \mathrm{alglat}_{R'} T_{R'}$. Let $t \in T$. There is an element $a' \in R'$ with $\phi t = a't$. Then, from the R'-module structure of T, $\phi t \in Rt$. Hence $\phi \in \mathrm{alglat}_R T_R$ and so $\mathrm{alglat}_{R'} T_{R'} \subseteq \mathrm{alglat}_R T_R$.

Let $\phi \in \mathrm{alglat}_R T_R$. Then $\phi(t_1 + t_2) = \phi t_1 + \phi t_2$ for t_1, t_2 in T. Let $a' = (a_x + Rx) \in R'$ and let $t \in T$ with $0 \neq Ry \subseteq \mathrm{Ann}(t)$. Then there is some b in R with $\phi t = bt$ and so $Ry \subseteq \mathrm{Ann}(t) \subseteq \mathrm{Ann}(bt)$. Thus $\phi(ta') = \phi(ta_y) = (\phi t)a_y =$

$(bt)a_y = (bt)a' = (\phi t)a'$. So $\phi \in \operatorname{End} T_{R'}$. Let t be any element of T. Then, since $\phi \in \operatorname{alglat}_R T_R$ and $R \subseteq R'$, $\phi t \in R't$. Hence $\phi \in \operatorname{alglat}_{R'} T_{R'}$ and so $\operatorname{alglat}_R T_R \subseteq \operatorname{alglat}_{R'} T_{R'}$. Thus $\operatorname{alglat}_R T_R = \operatorname{alglat}_{R'} T_{R'}$. $\square$

From this result, $\lambda(R')$ may be considered as a subring of $\operatorname{alglat}_R T_R$ whenever R and T are as given in Proposition 6.5 (compare with the remark following Proposition 4.6). This will be used without further comment.

The next result retains this degree of generality proving, for a faithful torsion R–module T where R satisfies the hypotheses of Proposition 6.5, that it is always the case that $R' \cong \lambda(R') \subseteq \operatorname{alglat}_R T_R$. As with Theorem 4.7, part of the proof consists of showing that T is a faithful R′–module. If R is not complete in the I-topology, $\lambda(R)$ is strictly contained in $\lambda(R')$ and so T is not reflexive.

6.6 Theorem

Let R be a commutative ring with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R. Let T be a faithful torsion R–module and let R′ be the completion of R in the I-topology. Then T is a faithful R′–module and $R' \cong \lambda(R') \subseteq \operatorname{alglat}_R T_R$.

Proof

From Proposition 6.5, $\lambda(R') \subseteq \operatorname{alglat}_R T_R$. The main part of the proof is to show that T is faithful as an R′–module. It then follows from the discussion in 1.1 that $\lambda(R') \cong R'$. The first step is to show that there is a sequence of elements (t_u) in T with $\operatorname{Ann}(t_u) \subseteq Ru$ for each $0 \neq u \in I$. Suppose for contradiction that there is a non-zero element y of I with $\operatorname{Ann}(t) \not\subseteq Ry$ for all t in T. Then, since the ideals of R contained in I are comparable with every ideal of R, $Ry \subseteq \operatorname{Ann}(t)$ for all t in T. So $0 \neq Ry \subseteq \bigcap_{t \in T} \operatorname{Ann}(t) = \operatorname{Ann} T$. This contradicts the statement that T is a faithful R–module. So for each $0 \neq u \in I$

there is an element t_u in T with $Ann(t_u) \subseteq Ru$. For each t_u ($0 \neq u \in I$) there is an element v_u in I with $0 \neq Rv_u \subseteq Ann(t_u)$.

Let $a' = (a_x + Rx)$ be an element of $Ann_{R'}T$ and let $0 \neq u \in I$. Then $0 = a't_u = a_{v_u}t_u$ and so $a_{v_u} \in Ann(t_u) \subseteq Ru$. However $Rv_u \subseteq Ru$ so $a_{v_u} - a_u \in Ru$. Thus $a_u \in Ru$ and so $a_u + Ru = 0 + Ru$. Hence $a' = (0 + Rx)$, the zero element of R′. Thus T is a faithful R′–module and so $R' \cong \lambda(R')$. This completes the proof. □

Theorem 6.7 imposes the additional condition on the hypotheses of Theorem 6.6 that R be a scalar-reflexive ring. Then it is shown that for a faithful torsion R–module T, alglat T is isomorphic to R′, the I-completion of R. So this extra condition is sufficient to give equality and thus $R' \cong \lambda(R') = \text{alglat}_R T_R$.

6.7 Theorem

Let R be a scalar-reflexive ring with a non-zero ideal I such that the ideals of R contained in I form a chain and I is comparable to every ideal of R. Let T be a faithful torsion R–module and let R′ be the completion of R in the I-topology. Then $R' \cong \lambda(R') = \text{alglat}_R T_R$.

Proof

From Theorem 6.6, $R' \cong \lambda(R')$ and $\lambda(R') \subseteq \text{alglat}_R T_R$. Let $\phi \in \text{alglat}_R T_R$. To complete the proof it is sufficient to show that $\phi \in \lambda(R')$. From Proposition 6.5, T has an R′–module structure given by $(a_x + Rx)t = a_y t$ where $t \in T$ and $0 \neq Ry \subseteq Ann(t)$. From the proof of Theorem 6.6, there is a sequence of elements (t_u) in T with $Ann(t_u) \subseteq Ru$ for each $0 \neq u \in I$. Since $\phi \in \text{alglat}_R T_R$, there is an element b_u of R with $\phi t_u = b_u t_u$ for each $0 \neq u \in I$. Let $r' = (b_x + Rx)$ for $0 \neq x \in I$.

The first step is to show that $r' \in R'$. Let y and z be non-zero elements of I with $Ry \subseteq Rz$. Then $Ann(t_y) \subseteq Ry$ and $Ann(t_z) \subseteq Rz$. The ideals of R contained in I form a chain so $\mathrm{Ann}(t_y)$ and $\mathrm{Ann}(t_z)$ are comparable. Suppose that $\mathrm{Ann}(t_y) \subseteq \mathrm{Ann}(t_z)$. The R–module $N = Rt_y + Rt_z$ is reflexive since R is scalar-reflexive. So there is an element b in R with $\phi = \lambda(b)$ on N. Then $b_y t_y = \phi t_y = bt_y$ and $b_z t_z = \phi t_z = bt_z$. Thus $b_y - b \in \mathrm{Ann}(t_y) \subseteq \mathrm{Ann}(t_z)$ and $b_z - b \in \mathrm{Ann}(t_z)$. So $b_y - b_z \in \mathrm{Ann}(t_z) \subseteq Rz$. Thus $r' \in R'$.

The next step is to show that $\phi = \lambda(r')$. Let $t \in T$. Then there is an element $0 \neq y \in I$ with $Ry \subseteq \mathrm{Ann}(t)$ and so $r't = b_y t$. The R–module $N' = Rt + Rt_y$ is reflexive since R is scalar-reflexive. So there is an element c in R with $\phi = \lambda(c)$ on N'. Then $\phi t = ct$ and $b_y t_y = \phi t_y = ct_y$. Thus $b_y - c \in \mathrm{Ann}(t_y)$. But $\mathrm{Ann}(t_y) \subseteq Ry$ and so $b_y - c \in Ry \subseteq \mathrm{Ann}(t)$. So $b_y t = ct$ and thus $\phi t = b_y t = r't$. Hence $\phi = \lambda(r')$ on T.

Thus $\phi \in \lambda(R')$ and so $\mathrm{alglat}_R T_R \subseteq \lambda(R')$. Hence $\mathrm{alglat}_R T_R = \lambda(R')$ and the proof is complete. $\square$

Compare Theorems 6.6 and 6.7 and the corresponding change in hypotheses with Theorems 4.7 and 4.9 on valuation domains. In these latter results, the change from R being a valuation domain to R being an almost maximal valuation domain was sufficient to give equality in Theorem 4.9. Recall that Theorem 4.9 proved, for a faithful torsion module T over an almost maximal valuation domain R, that alglat T is isomorphic to the completion of R in the R-topology. From Theorem 2.11, a local ring is scalar-reflexive if and only if it is an almost maximal valuation ring. Thus to give the equality of Theorem 4.9, the additional condition that R be scalar-reflexive was imposed on the hypotheses of Theorem 4.7. So the similarity in the construction and results of Chapters 4 and 6 is evident throughout. In both cases, the addition of R being scalar-reflexive is sufficient to characterise alglat for faithful torsion modules over the respective types of rings R.

The next result is a corollary of Theorem 6.7 and determines alglat for a faithful torsion module over a locally almost maximal torch ring.

6.8 Corollary

Let R be a locally almost maximal torch ring with unique minimal prime ideal P. Let R′ be the completion of R in the P-topology and let T be a faithful torsion R–module. Then $R' \cong \lambda(R') = \mathrm{alglat}_R T_R$.

Proof

From Theorems 2.17, 5.13 and 5.17, the ring R is scalar-reflexive. The ideal P is non-zero, the ideals of R contained in P form a chain and P is comparable to every ideal of R. The result is then immediate from Theorem 6.7. □

Theorem 6.7 can be applied to almost maximal valuation domains as well as to locally almost maximal torch rings. Suppose that T is a faithful torsion module over an almost maximal valuation domain R. Then R and T satisfy the hypotheses of Theorem 6.7 with I as any non-zero ideal of R. Let R′ be the completion of R in the I-topology. Then (from Theorem 6.7) $R' \cong \lambda(R') = \mathrm{alglat}_R T_R$. This is the same result as Theorem 4.9, since it was noted following the definition of the I-topology in 6.4 that $R' = \tilde{R}$ in this particular case, where $\tilde{R}$ is the completion of R in the R-topology. Thus Theorem 4.9 is shown to be a corollary of Theorem 6.7. Although the result given in Theorem 4.9 could have been omitted from Chapter 4 and introduced for the first time here, Theorem 4.9 motivated the enquiry which led to Theorem 6.7. The direct proof of Theorem 4.9 may also be of independent interest, being of a different nature to that of Theorem 6.7, and using the relationship between a domain R and its R-completion given in Proposition 4.8. Furthermore, the result of Theorem 4.9 was used in Chapter 4 to complete the characterisation of alglat for modules over an almost maximal valuation ring. Theorem 4.9 will be used in Chapter 7 in the proof of Theorem 7.1.

There is now sufficient information to characterise alglat for an R–module T where R is an indecomposable FSI ring and $\mathrm{Ann}\,T \subset P$. Before giving this full characterisation, Proposition 6.9 determines alglat for any module T over a locally almost maximal torch ring R with unique minimal prime ideal P where $\operatorname{Ann} T \subset P$. The proof of this result follows from Proposition 1.19, Theorem 6.1, Proposition 6.2 and Corollary 6.8.

6.9 Proposition

Let R be a locally almost maximal torch ring with unique minimal prime ideal P. Let T be an R–module with $\operatorname{Ann} T \subset P$. Let $S = R/\operatorname{Ann} T$ and $Q = P/\operatorname{Ann} T$. If T is torsion as an S–module then $\operatorname{alglat}_R T_R = \lambda(S') \cong S'$ where S' is the completion of S in the Q-topology. Thus if S is not complete in the Q-topology then T is not reflexive. In all other cases T is reflexive.

The final theorem of this chapter brings all the results together and characterises alglat for a module T over an indecomposable FSI ring R with unique minimal prime ideal P where $\operatorname{Ann} T \subset P$. If R is such a ring and if T is an R–module with $\operatorname{Ann} T \subset P$ then $P \neq 0$. Thus R is either an almost maximal valuation ring (not a domain) or a locally almost maximal torch ring. But an almost maximal valuation ring which is not a domain is maximal (Proposition 4.2) and every module over a maximal valuation ring is reflexive (Theorem 4.1). The proof of Theorem 6.10 is now immediate from Proposition 6.9 and these remarks.

6.10 Theorem

Let R be an indecomposable FSI ring with unique minimal prime ideal P. Let T be an R–module with $\operatorname{Ann} T \subset P$. Let $S = R/\operatorname{Ann} T$ and $Q = P/\operatorname{Ann} T$. If S is a locally almost maximal torch ring and if T is torsion as an S–module then $\operatorname{alglat}_R T_R = \lambda(S') \cong S'$ where S' is the completion of S in the Q-topology. Thus if S is not complete in the Q-topology then T is not reflexive. In all other cases T is reflexive.

Chapter 7 discusses the case where R is an indecomposable FSI ring with unique minimal prime ideal P and T is a torsion R–module with $P \subseteq \mathrm{Ann}\,T$. Using the results of this present chapter, Chapter 7 gives a full characterisation of alglat for a module over an FSI ring. An illustration of all these results is also included.

Chapter 7 Study of alglat for modules over FSI rings - part 2

In Chapter 6 it was shown that every non-torsion module over an FSI ring is reflexive (Theorem 6.1). This reduced the study of alglat to the torsion case. The discussion after Theorem 6.1 then showed that the characterisation of alglat could be reduced to the consideration of alglat for torsion modules over indecomposable FSI rings. Any indecomposable FSI ring R has a unique minimal prime ideal P which is comparable to every ideal of R. Thus for a torsion R–module T, either $P \subseteq \operatorname{Ann} T$ or $\operatorname{Ann} T \subset P$. Chapter 6 dealt with the case where $\operatorname{Ann} T \subset P$.

Theorem 7.1 is the main result of the chapter and characterises alglat for torsion R–modules T such that $P \subseteq \operatorname{Ann} T$. Since R is an FSI ring, the factor ring R/P is an FSI domain ([25; Lemma 1]). Theorem 7.1 uses this property of FSI rings to apply the decomposition for alglat of Theorem 1.21. This completes the individual results needed to describe alglat for modules over FSI rings.

Theorem 7.3 combines the results of Chapter 6 with Theorem 7.1 to give a full characterisation of alglat for all modules over an FSI ring. An illustration indicating the structure of alglat for any module over an FSI ring follows in 7.4.

Let R be an indecomposable FSI ring with unique minimal prime ideal P. Let T be a torsion R–module such that $P \subseteq \operatorname{Ann} T$. Then T can be considered as an R/P–module. If T is a non-torsion R/P–module then T is reflexive as an R/P–module and also as an R–module (Proposition 1.19 and Theorem 6.1). This leaves the case where T is torsion as an R/P–module.

Suppose that T is a torsion R/P–module. The ring R/P is an FSI domain and hence an h-local domain (Theorem 5.18). In order to characterise alglat the decomposition of Theorem 1.21 is used. From this theorem, $\operatorname{alglat}_R T_R \cong \prod_M \operatorname{alglat} T_M$ under the ring isomorphism $\alpha\epsilon$ of 1.21, where T_M is an R_M–module and M ranges over all maximal ideals of R containing P. Given that P is the unique minimal prime ideal of R this means that M ranges over all maximal ideals of R. Thus $\operatorname{alglat}_R T_R \cong \prod_M \operatorname{alglat} T_M$ where M ranges over all maximal ideals of R. Recall from Proposition 5.16 that each of the localised rings R_M is an almost maximal valuation ring.

Theorem 7.1 combines the results of Chapter 4 on alglat for modules over an almost maximal valuation ring with the above remarks. This characterises alglat for a torsion module T over an indecomposable FSI ring with unique minimal prime ideal P where $P \subseteq \operatorname{Ann} T$.

7.1 Theorem

Let R be an indecomposable FSI ring with unique minimal prime ideal P. Let T be a torsion R–module with $P \subseteq \operatorname{Ann}T$. Let $\widetilde{R_M}$ be the R_M-completion of R_M, whenever M is a maximal ideal of R such that R_M is a domain. Let $X = \{M \mid M$ is a maximal ideal of R, R_M is not R_M-complete, $\operatorname{Ann} T_M = 0\}$ and $Y = \{M \mid M$ is a maximal ideal of R, $M \notin X\}$ where T_M is an R_M–module. If T is a torsion R/P–module then $\operatorname{alglat}_R T_R \cong (\prod_{M \in X} \lambda(\widetilde{R_M})) \oplus (\prod_{M \in Y} \lambda(R_M))$. Otherwise T is reflexive.

Note

Suppose that T_M is a faithful R_M–module so that $\operatorname{Ann} T_M = 0$. Since $P \subseteq \operatorname{Ann} T$ it follows that $P_M \subseteq \operatorname{Ann} T_M$ and so $P_M = 0$. Thus R_M is an almost maximal valuation domain. Hence the R_M-completion of R_M exists. In particular, if $M \in X$ then $\widetilde{R_M}$ always exists. Note also that, from Proposition 4.6, $\operatorname{alglat}_{R_M} T_{M\,R_M} = \operatorname{alglat}_{\widetilde{R_M}} T_{M\,\widetilde{R_M}}$.

Proof

The module T has an R/P–module structure. If T is a non-torsion R/P–module then T is reflexive as an R/P–module and thus as an R–module.

Suppose T is a torsion R/P–module. For each maximal ideal M, T_M is an R_M–module. For any $\frac{t}{s} \in T_M$, there is some non-zero $(r + P) \in R/P$ with $rt = (r + P)t = 0$. Thus $\frac{r}{1} \cdot \frac{t}{s} = \frac{0}{1}$ in T_M. Since P is a prime ideal of R, $\frac{r}{1} \neq \frac{0}{1}$ in R_M and so each T_M is a torsion R_M–module. From Theorem 1.21 and the above discussion, $\operatorname{alglat}_R T_R \cong \prod_M \operatorname{alglat} T_M$ where M ranges over all maximal ideals of R. Thus $\operatorname{alglat}_R T_R \cong (\prod_{M \in X} \operatorname{alglat} T_M) \oplus (\prod_{M \in Y} \operatorname{alglat} T_M)$.

Let $M \in X$ so that $\operatorname{Ann} T_M = 0$. Then T_M is a faithful torsion R_M–module and R_M is an almost maximal valuation domain (see note). So from Theorem 4.9 $\operatorname{alglat} T_M = \lambda(\widetilde{R_M}) \cong \widetilde{R_M}$. The ring R_M is not complete in the R_M-topology (since $M \in X$) and thus T_M is not reflexive.

Let $M \in Y$. If T_M is faithful then R_M is an almost maximal valuation domain (see note) and R_M is R_M-complete. So (using Theorem 4.9 again) $\operatorname{alglat} T_M = \lambda(\widetilde{R_M}) = \lambda(R_M)$. If T_M is not faithful then from Theorem 4.3, T_M is reflexive and thus $\operatorname{alglat} T_M = \lambda(R_M)$. The result now follows. $\square$

Suppose, with the notation of Theorem 7.1, that $\operatorname{alglat}_R T_R \cong (\prod_{M \in X} \lambda(\widetilde{R_M})) \oplus (\prod_{M \in Y} \lambda(R_M))$. This description of alglat does not exclude the possibility that T may nevertheless be reflexive.

In particular suppose that T is non-torsion as an R/Ann T–module with $P \subset \operatorname{Ann} T$ (the case $P = \operatorname{Ann} T$ is dealt with in Theorem 7.1). Then T is reflexive (Theorem 6.1). In this case $\operatorname{Ann} T_M \neq 0$ where M ranges over all maximal ideals of R. For if r is a non-zero element of Ann T with $r \notin P$ and M is any maximal ideal of R then $\frac{r}{1}$ is in $\operatorname{Ann} T_M$. Since P is a prime ideal of R, $\frac{r}{1} \neq \frac{0}{1} \in R_M$ and thus $\operatorname{Ann} T_M \neq 0$. Then $M \in Y$ for all maximal ideals M of R and hence, in this case, $X = \varnothing$.

The next result is a special case of Theorem 7.1. This corollary deals with the situation when each localised module T_M is a faithful R_M–module. A description of alglat when $Y = \varnothing$ follows Corollary 7.2.

7.2 Corollary

Let R be an FSI domain and let T be a torsion R–module with T_M faithful as an R_M–module for all maximal ideals M of R. Then T is a faithful R–module and $\mathrm{alglat}_R T_R = \lambda(\tilde{R})$. In particular if R is complete in the R-topology then T is reflexive.

Proof

The first step is to show that T is a faithful R–module. Suppose for contradiction that T is not faithful so that there is a non-zero element a in Ann T. Let M be a maximal ideal of R. Then $\frac{a}{1}$ is a non-zero element of R_M and, for any element $\frac{t}{s}$ in T_M, $\frac{a}{1} \in \mathrm{Ann}(\frac{t}{s})$. Thus $\frac{a}{1} \in \mathrm{Ann}\, T_M$ and so T_M is not a faithful R_M–module. This gives the required contradiction.

From Theorem 7.1 $\mathrm{alglat}_R T_R \cong (\prod_{M \in X} \lambda(\widetilde{R_M})) \oplus (\prod_{M \in Y} \lambda(R_M))$ where $X = \{M \mid M$ is a maximal ideal of R, R_M is not R_M-complete, $\mathrm{Ann}\, T_M = 0\}$ and $Y = \{M \mid M$ is a maximal ideal of R, $M \notin X\}$. (Note that $P = 0$ and thus each R_M is a domain with corresponding completion $\widetilde{R_M}$.) If $M \in Y$ then, since T_M is faithful, R_M is complete in the R_M-topology. Thus $\lambda(R_M) = \lambda(\widetilde{R_M})$. So $\mathrm{alglat}_R T_R \cong \prod_M \lambda(\widetilde{R_M})$ where M ranges over all maximal ideals of R. From Theorem 1.18 and using the notation from there, the ring isomorphism is given by α with $\alpha : \prod_M \lambda(\widetilde{R_M}) \to \mathrm{alglat}_R T_R$, $(\theta_M) \mapsto \theta$ where $\theta t = \tau^{-1}\left(\theta_M \frac{t}{1}\right)$. The ring R is an h-local domain so from Lemma 4.5 and the remarks following, $\tilde{R} \cong \prod_M \widetilde{R_M}$ with the isomorphism given by $\gamma : (a_r + Rr) \mapsto ((\frac{a_r}{1} + R_M\frac{r}{1}))$. Thus $\lambda(\tilde{R}) \cong \prod_M \lambda(\widetilde{R_M})$ under the isomorphism $\delta : \lambda((a_r + Rr)) \mapsto (\lambda(\frac{a_r}{1} + R_M\frac{r}{1}))$. So $\delta(\lambda(\tilde{R})) = \prod_M \lambda(\widetilde{R_M})$. Thus $\mathrm{alglat}_R T_R = \alpha(\prod_M \lambda(\widetilde{R_M})) = \alpha\delta(\lambda(\tilde{R}))$.

The final step of the proof shows that $\alpha\delta(\lambda(\tilde{R})) = \lambda(\tilde{R})$. Let $\tilde{r} = (a_r + Rr) \in \tilde{R}$ and let t be a non-zero element of T. Then there is some non-zero element s of R with $Rs \subseteq \mathrm{Ann}(t)$. Using the $\tilde{R}$–module structure of T, $(\lambda(\tilde{r}))t = \tilde{r}t = a_s t$ (from Proposition 4.6). Then $[\alpha\delta(\lambda(\tilde{r}))]t = [\alpha(\lambda(\frac{a_r}{1} + R_M\frac{r}{1}))]t = \tau^{-1}\left(\lambda(\frac{a_r}{1} + R_M\frac{r}{1})\frac{t}{1}\right)$ $= \tau^{-1}\left((\frac{a_r}{1} + R_M\frac{r}{1})\frac{t}{1}\right)$. Since $0 \neq Rs \subseteq \mathrm{Ann}(t)$, it follows that $0 \neq R_M\frac{s}{1} \subseteq \mathrm{Ann}(\frac{t}{1})$ for all maximal ideals M of R. Thus $[\alpha\delta(\lambda(\tilde{r}))]t = \tau^{-1}\left(\frac{a_s}{1}.\frac{t}{1}\right) = \tau^{-1}\left(\frac{a_s t}{1}\right) = a_s t$. So

$[\alpha\delta(\lambda(\tilde{r}))]t = (\lambda(\tilde{r}))t$ for all t in T. Thus $\alpha\delta(\lambda(\tilde{r})) = \lambda(\tilde{r})$ for all $\tilde{r}$ in $\tilde{R}$ and so $\alpha\delta(\lambda(\tilde{R})) = \lambda(\tilde{R})$ as required. Hence $\mathrm{alglat}_R T_R = \lambda(\tilde{R})$.

If the ring R is complete in the R-topology then $R = \tilde{R}$ and so $\mathrm{alglat}_R T_R = \lambda(R)$. Thus T is a reflexive R–module. □

With the notation of Theorem 7.1, suppose that T is a torsion R/P–module and that $Y = \varnothing$. Then T_M is faithful where M ranges over all maximal ideals of R. Since $P \subseteq \mathrm{Ann}\,T$, it follows that $P_M \subseteq \mathrm{Ann}\,T_M$ for all M. Thus each $P_M = 0$ and hence $P = 0$. The ring R is therefore an FSI domain. From Corollary 7.2, T is a faithful R–module and $\mathrm{alglat}_R T_R = \lambda(\tilde{R})$.

Theorem 7.3 summarises all the results on alglat for modules over an FSI ring. Thus alglat for any module over an FSI ring is completely characterised. Although this theorem provides a full description of alglat, recall in particular (from Theorems 5.17 and 6.1) that all finitely generated and all non-torsion modules over an FSI ring are reflexive. The proof of Theorem 7.3 is immediate from Theorems 6.1, 6.10 and 7.1 and the discussion after Theorem 6.1.

7.3 Theorem

(a) Arbitrary FSI rings

Let R be an FSI ring and let T be an R–module.

If T is non-torsion then T is reflexive.

If T is torsion then $\mathrm{alglat}\,T \cong \bigoplus_{i=1}^{n} \mathrm{alglat}\,T_i$ where $R = \bigoplus_{i=1}^{n} R_i$, each R_i is an indecomposable FSI ring, $T = \bigoplus_{i=1}^{n} T_i$ and each T_i is an R_i–module.

(b) Indecomposable FSI rings

Let R be an indecomposable FSI ring with unique minimal prime ideal P. Let T be an R–module.

If $\mathrm{Ann}\,T \subset P$, $S = R/\mathrm{Ann}\,T$ is a locally almost maximal torch ring and T is a torsion S–module then $\mathrm{alglat}_R T_R = \lambda(S') \cong S'$ where S' is the completion of S in the Q-topology, Q being P/Ann T.

If $P \subseteq \mathrm{Ann}\,T$ and T is a torsion R/P–module then $\mathrm{alglat}_R T_R \cong (\prod_{M \in X} \lambda(\widetilde{R_M})) \oplus (\prod_{M \in Y} \lambda(R_M))$ where $\widetilde{R_M}$ is the completion of R_M in the R_M-topology and X, Y are as defined in Theorem 7.1.

In all other cases T is reflexive.

The chapter ends with an illustration which indicates the nature of alglat for any module over an FSI ring.

7.4 Illustration of the nature of alglat

R is an FSI ring

T is an R–module

T is non-torsion

Then T is reflexive

(6.1)

T is torsion

$R = \bigoplus_{i=1}^{n} R_i$ where each R_i is an indecomposable FSI ring

$T = \bigoplus_{i=1}^{n} T_i$ where each T_i is an R_i–module (at least one T_i torsion)

Then $\mathrm{alglat}\,T \cong \bigoplus_{i=1}^{n} \mathrm{alglat}\,T_i$

(1.10, following 6.1)

(continued ...)

R is an indecomposable FSI ring with unique minimal prime ideal P
T is an R–module

- T is non-torsion
  T is reflexive
  (6.1)
- T is torsion
  - $P \subseteq \operatorname{Ann} T$
    - T non-torsion
      R/P–module
      T is reflexive
      (6.1)
    - T torsion R/P–module
      $\mathrm{alglat}_R T_R \cong (\prod_{M \in X} \lambda(\widetilde{R_M})) \oplus (\prod_{M \in Y} \lambda(R_M))$
      (with notation of 7.1)
  - $\operatorname{Ann} T \subset P$
    - R AMVR not domain
      Then R maximal
      T is reflexive
      (4.1, 4.2)
    - R locally almost maximal
      torch ring
      $S = R/\operatorname{Ann} T$ is a locally
      almost maximal torch ring
      with unique minimal prime
      ideal $Q = P/\operatorname{Ann} T$
      (6.2)
      - T non-torsion
        S–module
        T is reflexive
        (6.1)
      - T torsion S–module
        $\mathrm{alglat}_R T_R = \lambda(S') \cong S'$,
        the completion of S in
        the Q-topology
        (6.9)

References

[1] M. F. Atiyah and I. G. MacDonald, Introduction to commutative algebra (Addison-Wesley, 1969)

[2] W. Brandal and R. Wiegand, Reduced rings whose finitely generated modules decompose, Comm. in Algebra 6 (1978), 195-201

[3] W. Brandal, Commutative rings whose finitely generated modules decompose, Springer Lecture Notes in Mathematics 723 (1979)

[4] L. Fuchs and L. Salce, Modules over valuation domains (Lecture Notes in Pure and Applied Mathematics 97, Marcel Dekker, New York, 1985)

[5] K. R. Fuller, W. K. Nicholson and J. F. Watters, Reflexive bimodules, Can. J. Math. 41 (1989), 592-611

[6] D. T. Gill, Almost maximal valuation rings, J. London Math. Soc. (2), 4 (1971), 140-146

[7] D. Hadwin, Algebraically reflexive linear transformations, Linear and Multilinear Alg. 14 (1983), 225-233

[8] D. Hadwin and J. W. Kerr, Scalar-reflexive rings, Proc. Amer. Math. Soc. 103 (1988), 1-8

[9] D. Hadwin and J. W. Kerr, Scalar-reflexive rings II, preprint (1988)

[10] P. R. Halmos, Reflexive lattices of subspaces, J. London Math. Soc. (2), 4 (1971), 257-263

[11] H. C. Hutchins, Examples of commutative rings (Polygonal, Passaic, NJ, 1981)

[12] I. Kaplansky, Modules over Dedekind rings and valuation rings, Trans. Amer. Math. Soc. 72 (1952), 327-340

[13] J. Lambek, Lectures on rings and modules (Blaisdell, Waltham, MA, 1966)

[14] E. Matlis, Injective modules over Prüfer rings, Nagoya Math. J. 15 (1959), 57-69

[15] E. Matlis, Cotorsion modules, Mem. Amer. Math. Soc. No. 49 (1964)

[16] E. Matlis, Decomposable modules, Trans. Amer. Math. Soc. 125 (1966), 147-179

[17] E. Matlis, Rings of type I, J. Algebra 23 (1972), 76-87

[18] E. Matlis, Rings with property D, Trans. Amer. Math. Soc. 170 (1972), 437-446

[19] H. Matsumura, Commutative ring theory (Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, 1986)

[20] J. J. Rotman, An introduction to homological algebra (Academic Press, 1969)

[21] L. Rowen, Ring theory volume 1 (Pure and Applied Mathematics 127, Academic Press, 1988)

[22] O. F. G. Schilling, The theory of valuations, Amer. Math. Soc. Surveys No. 4, (New York, 1950)

[23] T. S. Shores and R. Wiegand, Rings whose finitely generated modules are direct sums of cyclics, J. Algebra 32 (1974), 152-172

[24] P. Vámos, Multiply maximally complete fields, J. London Math. Soc. (2), 12 (1975), 103-111

[25] P. Vámos, The decomposition of finitely generated modules and fractionally self-injective rings, J. London Math. Soc. (2), 16 (1977), 209-220

[26] R. Wiegand and S. Wiegand, Commutative rings whose finitely generated modules are direct sums of cyclics, Springer Lecture Notes in Mathematics 616 (1977), 406-423