The kite family and other animals: Does a dragging utilisation scheme generate only shapes or can it also generate mathematical meanings?

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Abstract

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This thesis is about development of students' geometrical reasoning, in particular of inclusive relations between 2 dimensional shapes, e.g. the rhombus as a special case of the kite. Students in the study worked with a dynamic figure constructed using Dynamic Geometry Software. The figure is a quadrilateral whose diagonals are constructed so that they are of fixed length and perpendicular.

All students in the study were observed to use a strategy of 'dragging' to keep one diagonal as the perpendicular bisector of the other. This generated a 'family of shapes' which was comprised of an infinite number of kites, arrowheads (i.e. concave kites), one rhombus and two isosceles triangles. I have called this strategy 'Dragging Maintaining Symmetry' (DMS) and I claim it has the potential to mediate the understanding of the rhombus as a special case of the kites (and the isosceles triangle in the context of dynamic geometry).

However students in the study typically perceived the shapes, generated using DMS, according to a partitional view i.e. as different shapes, albeit with common properties such as line symmetry. When asked how many kites it would be possible to make by dragging the figure some students reported that there were four kites (one typical kite in each of four relative positions). It appears that they perceived the dragging activity as a journey to a discrete end position rather than as an action that resulted in a continuously changing figure. To address this problem I showed the students an animation of the figure under DMS. This proved to be the catalyst which moved their reasoning towards perceiving inclusive relations between the rhombus and kite.

Acknowledgements

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1 Introduction

This thesis is about students learning geometry with specialised computer software. In particular it is concerned with students' understanding of shapes and their properties and the development of analytical reasoning where they begin to consider that some shapes are special cases of other shapes. As an example, kites are quadrilaterals with two pairs of adjacent congruent sides. Other shapes such as the squares and rhombuses are considered by mathematicians to be special cases of the kites because the definition still holds true but with additional properties. Another way of explaining this is to say that the set of rhombuses (or squares) is included in the set of kites. The value of including squares and rhombuses as special cases of the kites is that if a geometric theorem can be proved for kites then it has automatically been proved to be true for squares and rhombuses. Furthermore, the ability to reason analytically is an important foundation stage for students who later go on to use deduction and proof in geometry.

This thesis is also concerned with how students learn using Dynamic Geometry Software (DGS) and in particular the Geometers Sketchpad™ version 4.05 (Jackiw, 2001) which is the specific commercial version of DGS used in the study. DGS programs are designed to be dynamic versions of Euclidean Geometry in that figures are constructed from basic points, lines and circles. The dynamic nature of the figures constructed using DGS has implications for the way students think and reason about the figures which is qualitatively different from the way they think and reason about static figures constructed on paper. Researchers often talk about these two different situations as the Dynamic Geometry Environment and the pencil and paper environment. More will be said about DGS programs and the Dynamic Geometry Environment in section 1.3 and in chapter three.

When I began the work in preparation for this thesis I wanted to explore how students learn within a Dynamic Geometry Environment. How do students develop mathematical concepts in this environment and how is this mediated through the dynamic aspect of the software? As a consequence there are two aspects to the work in this thesis which are intertwined: how students used the dynamic nature of the software to carry out a task involving a geometric figure, and how students reasoned about the properties of the geometric figure which was displayed on the computer screen. This thesis is therefore written in the interpretative paradigm using qualitative methods as the best way to answer such questions as ‘how’.

I came to this study with twenty one years’ experience as a secondary classroom teacher and I wanted to develop an intervention which could be used as a pedagogical tool in the classroom and which would be effective in helping children learn about an aspect of geometry. I have used a Design Based Research approach to develop and trial an intervention with pairs of students over several iterations of evaluation and improvement.

There is no perfect intervention for developing students’ geometrical reasoning. Learners construct their own meanings (in mathematics and in all other aspects of their lives) and it is important to give them the opportunity to engage with the mathematical topic and with educational resources. Concepts need to be developed carefully if they are to become internalised and this process does not happen overnight. What I hope is that I have developed an intervention which will be a useful tool in developing students’ conceptual understanding about the inclusive classification of triangles and quadrilaterals, a subject about which a great deal has been written.

Students also learn in a social context and construct meaning as a result of social interactions with their peers and their teacher. In the final iteration I was able to use the intervention, the task and computer files which were developed over the course of the study, to deliver a series of lessons to a class of 12-13 year old students and to observe how their discussion about 2-dimensional shapes developed in the light of the intervention.

1.1The value of learning geometry

Sir Michael Attiyah stated :

> “Our brains have been constructed in such a way that they are extremely concerned with vision. Vision, I understand from friends who work in neurophysiology, uses up something like 80 or 90 percent of the cortex of

the brain...

Understanding, and making sense of, the world that we see is a very important part of our evolution. Therefore spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...

I think it is very fundamental that the human mind has evolved with this enormous capacity to absorb a vast amount of information, by instantaneous visual action, and mathematics takes that and perfects it".

Attiyah (2001)

Attiyah has made the point that geometry is a valuable part of mathematics because it allows us to use our spatial intelligence, gleaned from our visual experience of the world, to make connections with mathematical concepts. Geometry affords students the opportunity to form reasoned arguments based on geometrical figures that they can see (and perhaps touch in the case of 3D figures) to convince themselves and others. This can eventually lead to the skills of conjecture and deductive reasoning.

Geometry is increasingly useful as visual information becomes more ubiquitous, through digital technologies in robotics, computer generated imagery, mapping the human genome and much more (Royal Society, 2001, Watson, Jones and Pratt, 2013). Given the importance of geometry as a discipline and as a tool in the every-day world, it is perhaps surprising that, in the UK at least, geometry has a low profile compared to other aspects of the mathematics curriculum (Royal Society, 2001).

1.2 The curriculum for geometry in England

Before 1970 the geometry curriculum in the UK was based on Euclidean geometry (Jones 2002). School students of secondary school age (11-16 years) were expected to learn deductive proofs and many did this by learning the proofs by rote which eventually became to be seen as inappropriate (Kuchemann, 1981). After 1970 Euclidean geometry was replaced by transformation geometry (which in the school curriculum deals with transforming shapes using reflections, rotations, translations and shears) and analytical geometry (where, in the school curriculum, geometric shapes plotted on a Cartesian frame are analysed algebraically) although some aspects of Euclidean geometry remained in the curriculum (Jones, 2002). Transformation and analytic geometry can both be aligned with algebraic methods and this may be why geometry has since lost much of its status as a topic in its own right.

1.2.1 Geometry in the National Curriculum

In 1988 a National Curriculum for England, Wales and Northern Ireland was introduced and then revised in 1995 and 1999. Whilst 'Shape, Space and Measures' was one of four attainment targets set down in the National Curriculum, it has never been allocated 25% of the teaching time in mathematics (Royal Society, 2001). Geometry appears to be the poor relation in the UK National Curriculum with little space allocated to the explicit development of spatial awareness and visualisation skills (Jones, 2002). The situation for mathematics of 16-19 year olds is even more serious with a minimal amount of geometry covered in the Advanced level mathematics curriculum (Royal Society, 2001, Jones, 2002). The situation has not changed in the interim period since those reports were written.

At the time of writing (December 2013) the incumbent coalition government have prepared to introduce a new English National Curriculum for ages 5-14 (published in July 2013) to be implemented in September 2014. The National Curriculum for ages 14-16 will be implemented from September 2015. An important difference between the new National Curriculum and the one it will replace is that a number of topics which used to be covered at Key Stage Three will now be covered in primary school. On reading the new programme of study in geometry for students from ages 5 to 14 (DfE, 2013) the learning objectives are articulated as 'recognise', 'identify and describe', 'classify', 'know and use' , 'apply', 'derive and illustrate'. The learning objectives do not appear to encourage the development of visualisation skills to support geometrical reasoning but skilled and creative teachers will have scope to develop these skills within the framework of the new National Curriculum, as indeed they had in previous National Curricula.

Both the 2007 and the 2014/15 versions of the National Curriculum for Mathematics include some element of constructing formal geometric proofs using deductive reasoning for the highest attaining students by the age of sixteen. However, a trawl of the most recent examination papers for the General Certificate in Secondary Education in Mathematics (taken by all students at age sixteen) reveals that deductive reasoning is allocated very few marks and is usually tested in the topic of Circle Theorems. The implication is that classroom teachers are unlikely to spend much classroom time on developing geometrical reasoning in their students.

The next sections describe two reports, published in 2001 and 2004, on the status of geometry teaching and learning in the UK. Section 1.2.4 presents student answers to geometry questions taken as part of national tests at age 14 years.

1.2.2 Teaching and Learning Geometry 11-19

Following concern regarding the standards of attainment in geometry, the UK government of the time commissioned a report into the teaching and learning of geometry from ages 11 to 19 (Royal Society, 2001). The working group that produced the report concluded that the current National Curriculum and Framework for Teaching Mathematics had sufficient geometrical content but they identified a need for geometry to be taught in a more engaging way, including approaches using ICT. In particular the working group recommended that geometry should be "taught in such a way as to achieve the following objectives:

a) Develop spatial awareness, geometrical intuition and the ability to visualise
b) Provide a breadth of geometrical experiences in 2- and 3-dimensions
c) Develop knowledge and understanding of and the ability to use geometrical properties and theorems
d) Encourage the development and use of conjecture, deductive reasoning and proof
e) Develop skills of applying geometry through problem solving and modelling in real world contexts
f) Develop useful ICT skills in specifically geometrical contexts

g) Engender a positive attitude to mathematics
h) Develop an awareness of the historical and cultural heritage of geometry in society, and the contemporary applications of geometry.

(Royal Society 2001, p. xii)

Among the several key principles listed in the report, the committee argued that geometry needs to be given a higher status within the UK mathematics curriculum and that it be allocated more teaching time (25-30%) than was presently the case. The report also recommended that the assessment weighting for geometry in examinations such as the SATs and GCSEs reflects this. The working group noted that the difficulty of assessing work done in 3D geometry and using ICT in geometry has resulted in a low profile in the teaching of geometry. Within a climate of targets it is only natural that teachers focus on what will be tested. If geometrical reasoning is to be developed then appropriate questions which test for reasoning need to appear in national tests.

1.2.3 Developing reasoning through algebra and geometry

In 2004 the then Qualifications and Curriculum Authority (QCA) published the result of a project on the development of geometrical reasoning undertaken by six working groups made up of teachers, academics and education consultants (QCA, 2004). The report was written for mathematics teachers in schools with the intention of helping them implement the new approaches to geometry (and algebra) in the revised National Curriculum. The reporting group made a number of observations and recommendations.

- Geometry gives the opportunity to develop skills in mathematical reasoning which can be built up by working on mental imagery and visualisation.
- Students often have problems associated with failing to appreciate that geometrical statements do not only refer to the particular diagram on the page. They can also frequently suffer from misconceptions that arise because they see figures drawn in specific orientations and decide that the orientation is a feature of the shape (eg not recognising squares when they are drawn at 45 degrees).

- Dynamic Geometry Software (DGS) is a helpful tool for observing which properties of a shape remain constant as it is changed by dragging. DGS can also allow students to switch between using measurements in their observations and explanations to a more formal geometrical line of reasoning.
- Geometry affords students the opportunity to develop informal reasoning through giving verbal explanations. It is important to develop students' mathematical explanations from spoken explanations to written and symbolic forms.

These two reports make the case for the study of geometry in school and the rich experience which it offers students. The tone of the reports is optimistic and hopeful yet, since they were published, (from my own observation) there is little evidence of change or improvement in the teaching and learning of geometry.

1.2.4 English 14 year olds' attempts at geometry questions in the Key Stage 3 SATs

This resource is based on analysis, carried out by the then Qualifications and Curriculum Development Authority (QCDA), of the way 14 year old students answered questions which required them to articulate the definition of 2D shapes. The questions appeared on the Key stage 3 Standard Assessment Tests (SATs) papers for 14 year olds in 2004 (QCDA, 2004). An account of the responses of students who achieved National Curriculum levels 5 and 6 (the expected levels of attainment for students aged 14) are described below in figure 1.1 to show the range of understanding in relation to the necessary properties needed to define a kite.

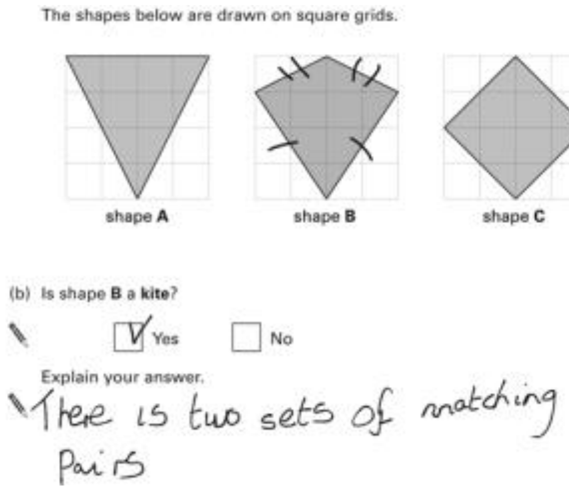
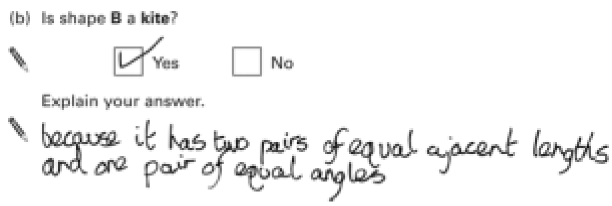
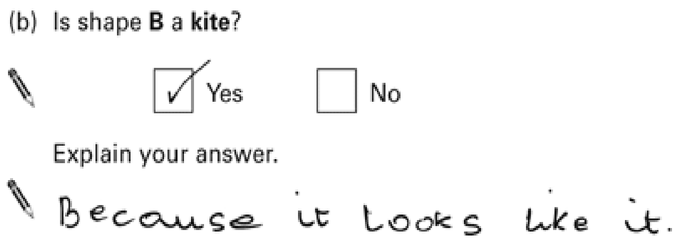
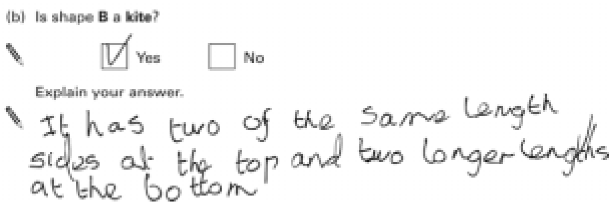
Example 1 	8 % of level 5 students, 18% of level 6 students and only 30% of level 7 (high attainers) achieved marks for a correct answer to this question. Over half the students recognised the shape as a kite but did not give a complete explanation. This response was a correct one since the annotations on the diagram showed exactly which sides were the matching pairs. Many students find it hard to articulate the properties of a kite yet if allowed to demonstrate on the diagram can indicate which sides are equal.
Example 2 	This definition describes a kite although the properties of the angles are extra to the minimal conditions needed for the definition.
Example 3 	The student who gave this response may not have understood what was expected of them in terms of precisely defining a kite.
Example 4 	Unfortunately this definition is not sufficiently precise to gain the mark for the question although they may actually understand that the two longer lengths are equal but have not specifically stated this.

Figure 1.1 Examples of student solutions to a geometry question

The low percentage of students who answered these in sufficient detail indicate that most had a poor understanding of what is required to define the shape correctly. This may arise from the lack of class time given to the topic of shape properties. A contributing factor could be that the students have learned lists of properties but have not had experience of developing the concept of shape properties for themselves and consequently have no deep understanding of the necessary properties needed in a definition of a specific shape such as a kite (Freudenthal, 1971, De Villiers, 1998, Battista, 2002).

Students of around 13 years of age are commonly expected to know and use properties of 2 dimensional shapes, in particular triangles and quadrilaterals. This topic appeared in the list of test items in the Trends in International Mathematics and Science Study (TIMSS, 2007, 2011) and most participating countries reported that the topic is a regular part of their school curriculum for this age group. My experience of school text books and test questions in the UK is that these require simple recall of facts rather than deeper learning.

1.3 Dynamic Geometry Software (DGS)

The QCA report described in 1.2.3 (QCA, 2004) stated the effectiveness of DGS in developing students' ability to observe and measure towards geometrical reasoning. My study focuses on developing an intervention using DGS, in particular the Geometers Sketchpad version 4 (Jackiw, 2001). Like other DGS programs it has been designed to embody Euclidean geometry and as such it is based on primitive objects such as points, lines and circles and constructions performed on these objects (such as mid-point, perpendicular through a point to a line, etc).

1.3.1 A DGS figure as a prototype

Figures in dynamic geometry are made by connecting components, for example, a triangle is made by connecting three line segments as displayed in figure 1.2. The tools are displayed on the left hand side of the screen and menus across the top. The triangle was drawn using the line segment tool. The corner labels were added using the text tool, **A**. The measurements of the angles and the angle sum have been displayed**.** Interestingly, the sum of angles in figure 1.2, measured to 2 decimal places, give 180.01 due to a rounding error. This occurrence needs to be taken into account when using DGS and can provide an opportunity to talk to students about the effects of rounding to specific degrees of accuracy.

The Arrow tool is used to select objects and also to **drag** them across the screen. In figure 1.2, if the corner marked A were to be picked up and dragged, then the lengths of the lines AC and AB would change and the measurements shown on the screen would continuously update to take account of this (Olivero and Robutti, 2007, Hollebrands, 2007). The **Measure** menu and the **Dragging** mode are important affordances of DGS for the purposes of this study. More will be said about the affordances of dragging and measuring in chapter three. Using the drag mode allows the user to transform the triangle on the screen to demonstrate a possibly infinite number of triangles. The dynamic triangle is a prototype for all possible triangles, not one static example as would be the case for a triangle drawn on paper (Olive, 2000).

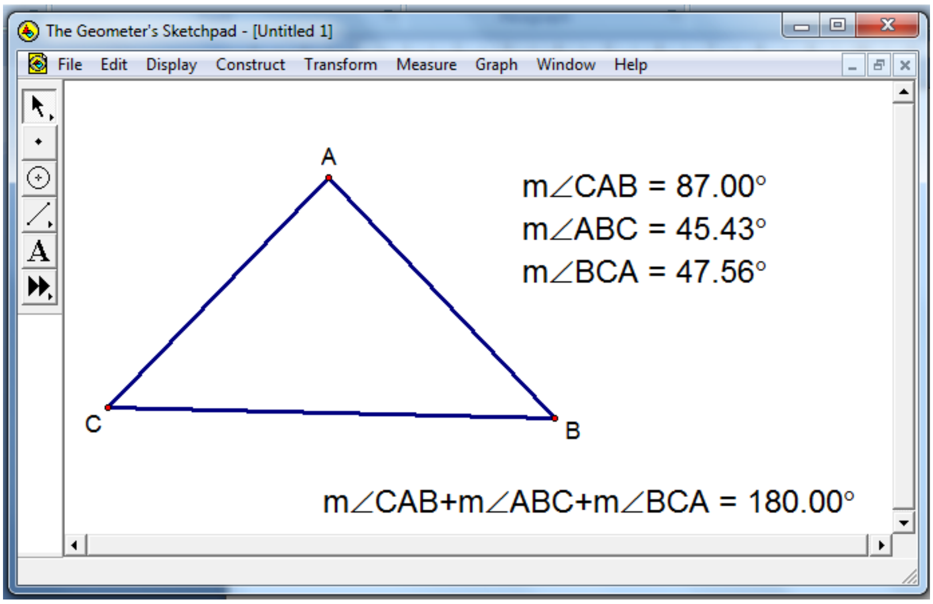


Figure 1.2 Screen shot displaying a triangle constructed by joining three line segments

1.3.2 Dragging in DGS and functional dependency

The drag mode allows students to manipulate primitive objects (lines, points, circles) which are part of a figure or sketch, whilst keeping constructed properties constant and thus make the link between the conceptual properties of a figure and the geometric construction itself (Laborde, 1993). The dragging mode thus acts to mediate the concept of geometrical relationships and helps students to see the relationships between objects rather than focusing on the objects themselves (Holzl et al, 1994). Dragging is the affordance which gives DGS its power and enhances the complexity of the learning situation (Holzl, 1996). To illustrate this point, consider the screen shot in figure 1.3 which shows one method of constructing an isosceles triangle in the Geometers Sketchpad.

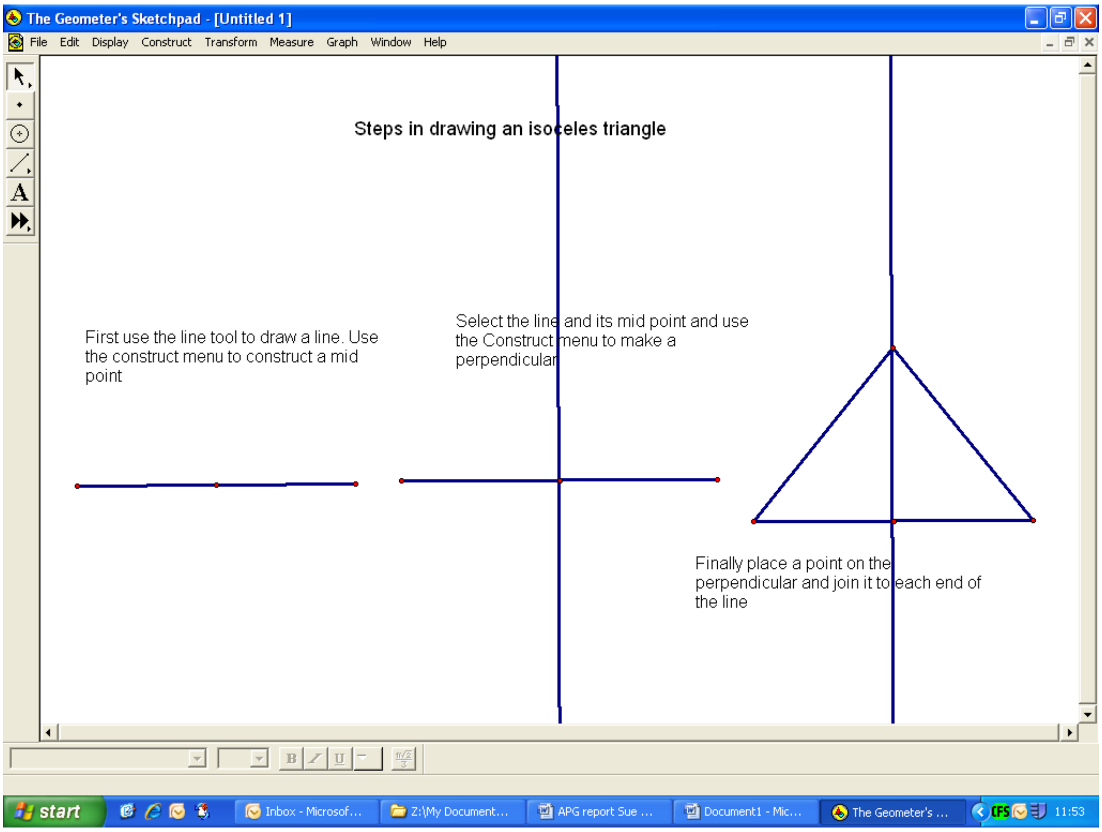


Figure 1.3 Screen shot which demonstrates a construction method for an isosceles triangle

In this construction the original line segment is a primitive object, all other points and lines have been constructed. The first construction is the mid-point of the primitive line.

Next a perpendicular to the primitive line segment through the mid-point is constructed. Then a point is placed on the perpendicular line (which is an infinite line rather than a line segment). The line segments are drawn from the placed point to the ends of the original primitive line segment. When this resulting figure is dragged the constructions will be maintained. So that if one end of the primitive line segment were dragged (eg. shortened, lengthened or rotated) the mid-point would adjust itself to keep being the mid-point, the perpendicular would adjust its position in response, the point on the perpendicular would maintain its relationship to it and so would the two line segments which comprise the other two sides of the equilateral triangle. Thus each constructed object is dependent on the previous constructions, forming a hierarchy of dependency. When the figure is dragged, its behaviour will be influenced by the functional dependencies of the primitive line segment and subsequent constructed objects which comprise the isosceles triangle (Holzl, 1996). It is in observing this behaviour that students can learn more about the relationships between objects within the figure and connect it to properties of the figure. In this way DGS has accomplished the link between the experimental field of geometric constructions and theoretical geometry (Laborde, 1993). As will be noted in chapter two the ability to move between experimental and theoretical geometry is important for students' development of theoretical understanding of geometry.

1.3.3 A new pedagogical environment

DGS has opened up a new pedagogical environment for teachers and students to explore geometry across both the experimental and theoretical fields. In particular the drag mode allows students to explore and experiment with dynamic figures which can lead to the generation of conjectures (Leung, 2011). The drag mode has even made possible new ways of thinking and learning about geometry, including a new discourse, which provides an alternative to the traditional Euclidean way of deductive reasoning (ibid).

1.4 Aims of the research and my position on the learning of mathematics

The aims of this research are to explore how students work on a task that requires them to interact with a figure constructed in a Dynamic Geometry Environment, and asks how their conceptual understanding of the properties of 2D shapes might develop as a result. At this point it is necessary to describe the research method I have used since it has implications for how I have structured the thesis. I have chosen to use the Design Based Research method which is based on a design experiment (the task) and uses an iterative process. Each iteration can be considered to be a small piece of research which builds on the findings which have emerged from the previous iteration. The DBR methodology is described in detail in chapter four: each iteration will be dealt with in a separate chapter with a retrospective analysis conducted at the end, after the final iteration, which will draw together the findings from the whole study.

In devising the research study I have naturally drawn on my own philosophy about how we learn which is based on the theory of constructivism originally developed by Jean Piaget (Wood, 1998). Human beings construct interior models of how the world works, and of how mathematics works, adjusting their mental models as they learn. Advocates of constructivism, in particular Papert (1980), developed the theory of constructionism whereby learning opportunities are set up for students to construct their own mathematical understandings by interacting with artifacts such as computers.

Human beings also learn in a social context as described by the theory of Socio-Constructivism (Vygotsky, 1978). Vygotsky considered that humans actively construct their own knowledge of the world but that this is done, not just through one's own exploration of the physical world, but also in interaction with other human beings which may be parents, teachers or more knowledgeable peers. Vygotsky identified the zone of proximal development (ZPD) as the difference between what one can achieve alone and what can be achieved with the guidance of others (e.g. more able peers). Vygotsky claimed that it is within the ZPD that developmental learning occurs.

Most learning of mathematics takes place in the classroom. Brousseau (1997) described the didactic contract whereby the teacher's role is to design a learning situation for their students so that they are allowed to experience discovering mathematics themselves.

Freudenthal (1971) also held the view that we should allow students to make their own journey of discovery and that we do students a disservice if we simply transmit information to them. This does not mean that students are expected to discover all of mathematics from first principles, which has taken centuries to develop. Rather the teacher should guide the students by choice of tasks and activities so that they are able to experience the joy of discovery. Another model to describe the classroom contract is to view it as an expert – apprentice model whereby the teacher apprentices the students into school mathematics (Lave and Wenger, 1991), working with them so that they learn how to become mathematicians by observing and working with the expert. I can testify to the power of this as someone who experienced as a sixth form pupil, with my Applied Mathematics A level teacher, the value of working with a more experienced adult to differentiate from first principles and to derive all the equations we needed for Mechanics problems.

It is of the utmost importance that learners develop a deep conceptual understanding of mathematics as a foundation on which to build future learning. Skemp (1973) described deep understanding of mathematical concepts which he named relational understanding in opposition to instrumental understanding, where students view mathematics as a series of procedures that are learned without understanding how and why they work. My aim, therefore, in any task which I give to students is that it should facilitate them in constructing their own mathematical meanings by investigating and discovering mathematics for themselves but in a social context with peers and teacher expert and with the intention that deep relational learning should result. However, before arriving at this juncture it was necessary to design a task and to test it.

1.5 The dynamic figure

The idea for the figure on which the task is based comes from a simple toy kite (which could be flown on a windy day). The rigidity of the kite rests on the two bars which cross each other so that one is the perpendicular bisector of the other. The bars are covered in a fabric which is stretched over the bars and forms the shape of a mathematical kite as shown in figure 1.4.

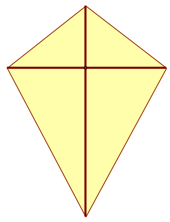


Figure 1.4 A typical kite shape

Suppose that the bars which give the toy kite its structure can be moved and suppose the covering fabric is sufficiently elastic then many different shapes are possible. This would be difficult to achieve in a practical sense but it could be done in the imagination and by using dynamic geometry software. The dynamic figure in the task, constructed using the Geometers Sketchpad version 4.05 (Jackiw, 2001), has been based around two fixed length perpendicular bars. The bars are constructed separately by placing a point on the screen, translating that point using a vector and using a line segment to join the points. This method ensures that the bars are unchangeable in length and orientation. One bar is moved over to cross the other and the ends are joined using line segments to create a perpendicular quadrilateral (a four sided shape whose diagonals are perpendicular). The interior is constructed which fills the shape with colour.

The difference in how I use the words ‘figure’ and ‘shape’ should be explained. Most of the time I have endeavoured to use the term ‘figure’ to describe a general geometrical object which possesses a minimum number of properties. The figure may be a representative of a set of geometrical objects which contains subsets of specific shapes that are particular members of the larger set. In the Dynamic Geometry Environment the dynamic figure is the perpendicular quadrilateral which can be dragged to generate specific shapes (such as kites and isosceles triangles).

If the bars are moved inside the dynamic figure (dynamic perpendicular quadrilateral) then shapes can be generated such as those shown in figure 1.5. (I hope the reader does not become confused between the use of the word ‘figure’ as a mathematical object and figure denoting an illustration included in the thesis.)

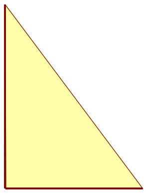
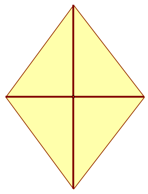
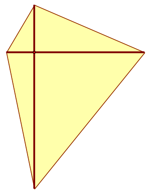
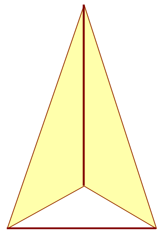
			
Right angled triangle	rhombus	Perpendicular quadrilateral	Concave kite

Figure 1.5 Shapes which can be generated by moving the bars inside the dynamic figure.

All the shapes in figure 1.5 are particular cases of a perpendicular quadrilateral since they were generated by moving (dragging) the bars inside a perpendicular quadrilateral. This raises questions about the right angled triangle which only has three sides and no diagonals. It could be argued that it is a perpendicular quadrilateral in the context of dynamic geometry but this will be addressed later.

The dynamic figure could have been modified so that the bars could be rotated as well as moved around inside the shape. Indeed I have created files where the bars do this and have recorded students working with them. However, for two reasons I have chosen to focus on perpendicular bars: first the methods of constructing rotating bars can result in problematic figures which cannot be dragged as easily. Secondly most of the interesting shapes can be generated using perpendicular bars. Most quadrilaterals that we name have perpendicular diagonals (kite, rhombus, square, concave kite) and the others (rectangle, parallelogram) have diagonals which are not perpendicular but they do intersect at their mid points. The following table illustrates this:

Table 1.1 Shapes which can be constructed using two bars

	Perpendicular diagonals		Non-perpendicular diagonals	
	One diagonal bisects the other	Both diagonals bisect each other	One diagonal bisects the other	Both diagonals bisect each other
Equal length diagonals	Kite Concave kite (isosceles triangle)	square		rectangle
Non-equal length diagonals	Kite Concave kite (isosceles triangle)	rhombus		parallelogram

Each empty cell has an irregular quadrilateral in it. The interesting thing (to someone such as myself who had never thought about this despite studying degree level mathematics) is that all the named shapes have symmetry; either line symmetry or rotational symmetry or both. From their symmetry the properties of the shapes such as equal sides and angles, perpendicular sides and parallel sides can be deduced. However, in school, in England, children are most often taught about the properties of shapes and about their symmetries as if they were two unconnected aspects.

The dynamic perpendicular quadrilateral forms the basis of a task using the Geometers Sketchpad, where students are asked to investigate what shapes they can make by dragging the bars. When they claim to have made a specific shape, such as a kite, they are asked to identify the properties of the shape and to use the measurements menu to check if they can make the shape accurate as indicated by the necessary equal sides and angles within the figure. Using this task the students can demonstrate the knowledge they already have regarding the properties of the shape, review their knowledge of shape properties and sometimes identify properties they did not previously know (for example the properties of the diagonals). However the main focus of this study is to ascertain whether the task could be instrumental in encouraging the development of students' conceptual understanding of shapes and their properties and in particular if the students can develop inclusive relations between shapes. The dynamic perpendicular quadrilateral can act as a *microworld* (Papert, 1980) which allows students to explore and investigate mathematical principles in a specific situation. The term *microworld* will discussed in chapter three.

1.5.1 Constructive classification of shapes generated from the dynamic figure.

The dynamic figure is a perpendicular quadrilateral which is the general form of a set of shapes with four sides whose diagonals are perpendicular. If the figure is dragged such that extra properties are added to the figure, for example if one bar is dragged so that it bisects the other bar, then a subset of the original perpendicular quadrilaterals is generated. De Villiers (1994) refered to this as *constructive classification*.
In the constructive definition, generalisations or specialisations are used to produce a new shape either by relaxing properties to move to the more general shape, or by adding extra properties to move to the more specific shape. For example a square can generate a rectangle if we relax the requirement for equal sides but keep the requirement for equal angles. A parallelogram can generate a rhombus if we add the requirement that all sides must be equal.

Figure 1.6 illustrates the sets and subsets of shapes which can be generated by the gradual adding of constraints to the perpendicular quadrilateral. Adding the constraint that one diagonal must be the bisector of the other generates the kites. Adding another constraint that both diagonals must bisect each other generates the rhombus. Adding yet a further constraint that both diagonals must be equal length generates the squares. The diagram illustrates the meaning of the term inclusive when applied to these subsets of the perpendicular quadrilaterals. It can be seen how each subset of shapes is included in the sets that surround it.

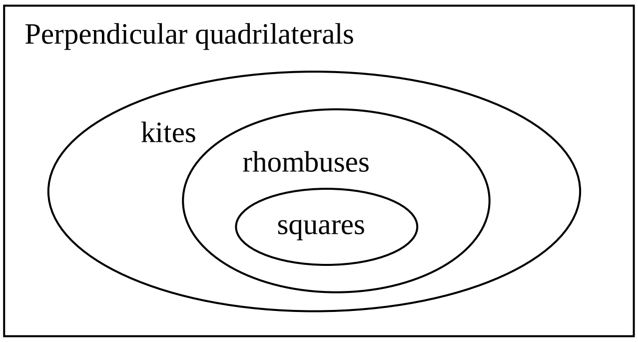


Figure 1.6 Venn diagram showing inclusive relationships between quadrilaterals which can be generated from a dynamic perpendicular quadrilateral

1.6 A synopsis of what follows in the thesis

Chapters 2 and 3 contain the review of the literature which I originally carried out before starting the research and which is pertinent for the whole study. This has been updated as the study progressed but in order not to spoil the anticipation of the findings which emerged as the research went through the various iterations I will add further sections of literature review at later parts of the thesis.

- Chapter 2 addresses geometrical reasoning

- Chapter 3 addresses issues of learning geometry through Dynamic Geometry Software.

- Chapter 4 - I describe the design based research methodology and relate it to my own work. This is followed by the iterations in the research process:

- Chapter 5: Iteration 0 – the pilot study, pairs of students worked with the dynamic figure and then went on to construct geometric figures starting with a blank screen,

- Chapter 6: Iteration 1- students worked with the dynamic figure.

- Chapter 7: Iteration 2 - Students worked with the dynamic figure in a whole class context, followed by pairs of students working with the dynamic figure constructed so that it was oriented at an angle.

- Chapter 8: Iteration 3 - Pairs of students worked with the dynamic figure oriented at an angle, followed by the introduction of the animated dynamic figure

- Chapter 9: Iteration 4 – A class of students undertook three lessons working with the concept of constructive classification, pairs working with the dynamic figure, whole class discussion with the dynamic figure projected for the whole class.

- Chapter 10 – A retrospective analysis of all the data using Duval's theory of Cognitive Apprehensions (Duval, 1995) which is described in chapter three.

- Chapter 11 - The discussion chapter where I have considered the efficacy of the task acting as a microworld and address the learning made by the students participating in the study.

- Chapter 12 - The conclusions chapter includes a consideration of how the design based research method facilitated the development of the task using the dynamic figure as a research tool and intervention for the teaching and learning of shapes and inclusive relations between them.

Appendices for iterations one to four contain tables of episodes from the recordings as well as other examples of students' work.
Appendix five includes sample letters to parents / carers, etc.

2 The development of geometrical reasoning

In this chapter I present literature which describes how humans perceive shapes within a horizontal and vertical frame and the importance of symmetry. I also address three important models of how humans conceptualise geometrical figures and consider the practical experimental and the theoretical aspects of geometry.

2.1 Perception of shape from birth

The study of geometry has its foundation in the student's experience of the environment in which they live. More than any other area of mathematics we learn about shape and space in an intuitive way through our experience. By the time children go to school they have already learned much about shapes and have developed spatial perception in a natural way by interacting with their environment. They are able to recognise geometric shapes such as square, triangles and circles and can distinguish between them in an implicit way (Bryant, 2009). It would seem appropriate that any approach to learning about the geometry of shape must build on the intuitive understanding which is gleaned from our lived experience and Vygotsky (1978) emphasised the importance of the child's pre-school learning as forming the basis on which school learning should build. Nevertheless, even though children come to school with an impressive understanding of shape and space pertaining to their environment, most encounter difficulties when learning about geometry (Bryant, 2009).

2.1.1 Frames of reference and orientation

Children first meet geometrical shapes as young babies exploring the world about them in 3 dimensions. Experience of 2 dimensional shapes will be encountered as the surfaces of 3D solids. Doors and windows (rectangle shapes) are orientated in the vertical / horizontal frame. Perceptions of the material world are affected by gravity and privilege upright shapes with their base on a level with the horizontal (Piaget and Inhelder, 1956).

When we look at an object or a shape we view it within an environment which may be the world about us or it may be on paper or on the computer screen. We view what we see in terms of a frame of reference. This frame of reference can be described as a container which is independent of the objects inside it. The most natural frame of reference is to use the vertical and horizontal axes as they are derived from the physical world (Piaget and Inhelder, 1956). The paradox is that, due to the curvature of the planet earth's surface, verticals are not all parallel and surfaces of liquids are actually curved. However humans behave as if we live within a vertical and horizontal framework.

2.1.2 Symmetry as part of the intuitive way that humans perceive shape.

In his famous book 'The Descent of Man' and as part of a discussion on the sense of beauty Charles Darwin commented:

> "The eye prefers symmetry or figures with some regular occurrence".
> (Darwin, 1887, p. 93).

Our preference for symmetry seems to develop early in life. Bornstein et al (1981) used experimental methods to demonstrate that babies of four months habituated most easily to patterns with vertical symmetry (whilst not showing a preference between vertical and horizontal symmetry) and that by the age of twelve months babies actually showed a preference for patterns with vertical symmetry. It is thought that the preference for vertical symmetry arises from the experience of living in an environment where objects possessing vertical symmetry are more common or by an appreciation that it is more efficient to process information by considering vertical symmetry (Bornstein et al, 1978, cited in Bornstein et al 1981). It seems that humans focus on one line of symmetry at a time (Attneave, 1968). This is demonstrated by the way that equilateral triangles may be perceived as isosceles triangles in one of three different orientations.

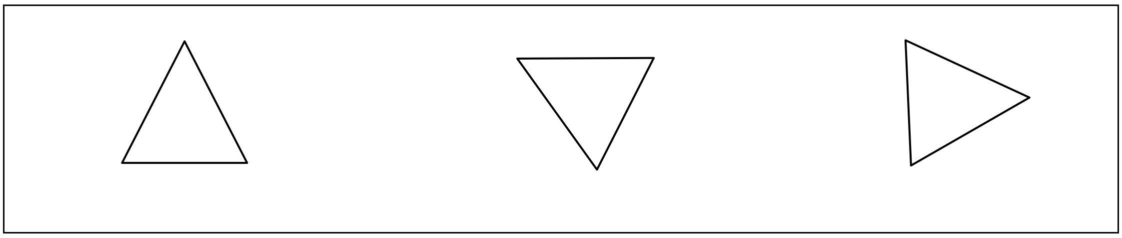


Figure 2.1 Equilateral triangles in different orientations may be perceived as isosceles

Attneave also noted that isosceles triangles are only perceived as such if their line of symmetry is at or close to the horizontal or vertical. Otherwise they may be perceived as scalene triangles (see figure 2.2).

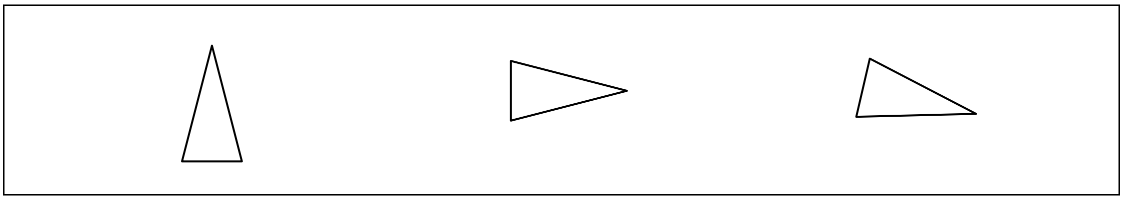


Figure 2.2 Isosceles triangles may be perceived as scalene if line symmetry is not close to the vertical or horizontal

In mathematics the definition for symmetry is precise, eg a 2 dimensional object is symmetrical if it is invariant under a reflection about a line, named the axis or line of symmetry. Although the material world hosts a great number of examples of symmetry, the exact mathematical definition of symmetry is not adequate to describe most of them because the symmetry is not usually exact (Zabrodsky et al, 1992). Humans are sensitive to approximate symmetry an example of which is the human face or body (Palmer, 1985). It appears that humans perceive objects as having degrees of approximation to symmetry. However in mathematics an object either has true symmetry or not (Shepard, 1994).

2.1.3 A connection between a vertical axis of symmetry and orientation

Vertical symmetry and a sense of orientation are closely connected. We can talk of a global or field axis of symmetry (in the environment) and a local or figural axis of symmetry (of a specific figure). This can affect how we perceive shapes.

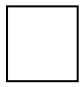


Figure 2.3a

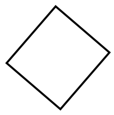


Figure 2.3b

Although figure 2.3a and figure 2.3b are congruent squares they are commonly perceived to be different. Figure ‘a’ is usually thought of as being a square whilst figure ‘b’ is perceived as a ‘diamond’. This is because we tend to use the vertical axis, of the page, as our frame of reference so that the sides of the upright square in figure 2.3a would be described as being parallel and perpendicular to the reference orientation whereas the sides of the tilted square in figure 2.3b would be described as being at a slant. However if we used another local frame of reference for example by planting the shapes inside an oblique rectangle at 45 degrees to the vertical, we may see figure ‘2.4a’ as being the diamond and figure ‘2.4b’ as being the square (Palmer, 1985).

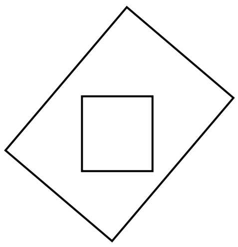


Figure 2.4a

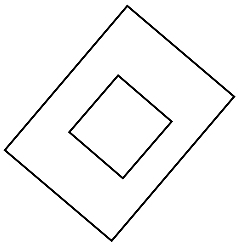


Figure 2.4b

Hence shapes are perceived within a perceptual reference frame. There is a strong tendency to choose a reference frame orientated along an axis of reflection symmetry and overall there is a preference towards vertical and horizontal reference frames (Palmer, 1985). How the local reference frame sits in the larger environmental frame (perhaps of the paper or computer screen or the world about us) leads to the concept of orientation. Or, to explain this in another way, the relationship between the figural vertical axis and the field vertical axis constitutes the concept of orientation (Attneave, 1968).

Since there appears to be an innate human tendency to prefer shapes which sit within a vertical and horizontal framework, it is perhaps natural that we should present shapes (square, rectangles, triangles, etc) to children in this way. It is common for children in early years schooling to be presented with shapes which are upright, in the case of squares and rectangles, and triangles with a horizontal base, etc (Kerslake, 1979). However even children who have been given ‘slanted’ shapes to work with have still shown a preference for upright figures (Fisher, 1978) which suggests this may be due to the human tendency to prefer the vertical / horizontal framework. Given these considerations perhaps it is not surprising that learners include the orientation of a shape among its properties, even if this is in an informal unstated way.

2.1.4 Line symmetry as a rotation in the plane

It is thought, by some cognitive psychologists, that transformations of one shape into another are conceptualised as rotations and that a reflection in the plane is conceptualised as a 180 degree rotation out of the plane with the line of symmetry being the axis of rotation (Shepard, 1994). Translations are conceptualised as a rotation where the centre is at infinity.
In this way a simple 2 dimensional rectangle is mapped onto itself by 180 degree rotation about its centre and also by 180 degree rotations perpendicular to the plane along the vertical and horizontal axes through its centre.

2.2 Models of geometrical reasoning

At first glance geometry appears to be a practical area of mathematics but it is actually very abstract. Consider a 2D shape such as a kite like the one in the SATs question in figure 1.1. Researchers have suggested that problems in geometry arise because students find it hard to appreciate the difference between the actual figure (of say, a kite) on paper and the theoretical object that it represents (Battista, 2007). Three important models which can provide theoretical frameworks to describe students’ geometrical reasoning (Jones, 1998) are described below. Fischbein (1993) described how, in geometry, we work with a mental picture of a figure which is determined by both its figural and conceptual nature. Van Hiele (1986) provided a hierarchical structure to analyse the development of students’ geometrical thinking. Duval (1995, 1998) described the complexity of geometrical representations (usually on paper) and deconstructed the ways in which we understand and use them to develop geometrical concepts.

2.3 Fischbein's theory of the 'figural concept'

Fischbein (1993) said that figures in geometry are made from points, line segments, angles, plane shapes, solid shapes, etc and these have conceptual aspects; humans have a mental idea of what these objects are. Drawings or models of geometrical objects can only be considered as representations of the theoretical objects since they can never be constructed to perfection. In order to be able to work with a drawing of a geometrical figure humans have to deal with a representation of it which exists in the three dimensional world. For example a point, which in its abstract form has no dimensions only position, is usually represented as a little circle in order that it can be seen. Similarly a line segment is theoretically one–dimensional but in a drawing a line has width as dictated by the drawing instrument. Even three-dimensional models of shapes can never be perfect. Only in a theoretical conceptual manner can perfect geometrical entities be operated on.

When mathematicians work on problems in geometry their mental construct of a geometrical figure is of a generalised figure and even its material representation (perhaps a pencil and paper drawing) is taken to be an example of a class of figures. For example a drawing of a kite represents the class of all kites not just the particular example on paper. A class of figures is defined according to geometrical axioms and this gives rise to their properties. The definitions may be inclusive and hierarchical. For example Fischbein (1993) gave the example of a square being a rectangle with equal sides from which other properties could be deduced (equal angles, equal length diagonals).

Fischbein described a geometrical figure as having conceptual properties which is the definition that generates the properties of the figure, along with its figural nature which is the visual image of its spatial qualities (the square-ness of a square, the circular nature of a circle). The fusion of these two is what Fischbein called the *figural concept*. When we investigate or solve problems in geometry we work with the figural concept.

As Fischbein stated we need to distinguish between the theoretical domain where we work with a figure by referring to its definitions based on axioms, and the empirical domain where we may investigate particular examples of the figure. According to Fischbein it takes an intellectual effort to understand that we are working with abstract theoretical figures when we solve problems in geometry.

The figural concept is a mathematical meaning (in the way that words in speech convey meaning) which includes the spatial aspects of the figure as an intrinsic property and whose behaviour is controlled by its conceptual aspects (its definition). Fischbein said that the problem many children have with accepting the figural concept is that they are influenced by their experiences of working with particular representations (which sometimes include properties such as colour and orientation). In other words the figural aspect dominates over the conceptual definition. One example of this is when having proved a theorem for a class of figures, the result is also true for any subclass of those figures. However students for whom the figural aspect of a figure dominates over its conceptual definition may insist on reproving the theorem for every one of the subclasses.

> "This difficulty in manipulating figural concepts, that is the tendency to neglect the definition under the pressure of figural constraints, represents a major obstacle in geometrical reasoning"
> (Fischbein, 1993, p. 155).

Thus Fischbein recommended that the figural concept should be specifically addressed in the teaching and learning of geometry rather than expecting students to spontaneously develop it.

2.3.1 Concept image and concept definition

Children come to the study of geometry with prior experience of shape and space, so they tend to have developed concepts for shapes which may include other aspects (such as orientation or preferred proportions for shapes) which may arise from prototypes they have previously met. The figural features of a shape may thus interfere with the mathematical definition and cause problems with the fusion of the figural and conceptual features of the figural concept. Fischbein noted that Tall and Vinner (1981) had expressed this issue when they described the 'concept image' and 'concept definition'. As Tall and Vinner (1981) stated, we all have experience of mathematical concepts (such as shape and space) before we formally learn about them in mathematics and so have already developed a mental structure of these concepts. Each concept is built upon and develops as the individual gains further experience of it. Tall and Vinner referred to this as the 'concept image,' which includes all aspects of the individual's mental picture. In the case of geometric shapes this may include prototypical representations and common orientations. On the other hand, the 'concept definition' is a description of the concept in words. Students might hold a personal concept definition using their own words to describe it or they might use a formal concept definition which is the official definition used by the mathematics community. In the case of shapes and their definitions, Fujita and Jones (2007) referred to the 'personal figural concepts' and 'formal figural concepts'. If, for example, we hold a personal figural concept of a rhombus as 'a squashed square which consequently has four equal sides', and a kite as 'a shape which must have two smaller sides at the top and two longer sides at the bottom' then we may have difficulty accepting that a rhombus could be a special case of a kite.

2.3.2 The Dynamic Geometry Software figure as a mediator for the figural concept

Laborde (1993) explained that Dynamic Geometry Software enables us to redefine the distinction between the theoretical object and its material representation. There is now a figure on the screen which is a new kind of mediator for the theoretical object: it is different from a paper drawing in that it is dynamic and can be dragged on the screen and additionally its behaviour when dragged is determined by the method used to construct it (that is the geometrical properties designed into its construction).

Mariotti (1995) extended this point by claiming that drawings act as mediators between concrete and theoretical objects: screen images represent the external version of the figural concept. The conceptual and figural aspects must be made explicit in the process of constructing an object in a dynamic geometry environment and develops the correct interaction between the figural and conceptual aspects of geometrical reasoning. The internal logic of the geometrical figure becomes apparent when it is dragged since the geometrical relationships by which it has been defined remain constant under dragging.

2.4 The Van Hiele model of development in geometrical reasoning.

Pierre Van Hiele and Dinah Geldof, (1955 cited in Van Hiele, 1986) studied the development of mathematical reasoning in the students they taught and devised a set of levels to describe this development which are known as the Van Hiele levels. One of the examples given was the structure of reasoning about two dimensional shapes in geometry. Van Hiele (1986) described how learners progress through levels of thinking and how at each level they build a mental structure of the concept (for example of two dimensional shapes). In order to progress to the next level it is necessary to learn how to use symbols, usually word symbols, associated with the current level. When learners have attained the language associated with that level they can then go through a period of development which culminates in their reaching the next level. A learner's level is related to their ability for discursive argument at that level. Van Hiele believed that being able to use language to describe a mathematical situation is the key to further development.

The first three levels related to two dimensional shapes are described below. I will only concern myself with these since the students participating in my study will be shown to be working just within these three levels. When the levels were first devised in the fifties they were labelled from zero. Later Van Hiele labelled them from one in the 1986 publication. In subsequent literature concerning the Van Hiele theory some researchers label the levels zero to four and others from one to five. I will use the convention of labelling the levels from one to five.

Level one

This is the **visual level** at which learners recognise shapes in an intuitive fashion. They may not use any language to explain why they recognise a shape. They may just say "this is a square because it looks like it".

Developing the language to describe what is seen at level one leads to level two.

Level two

Level two is the **descriptive analytic level**. Level two thinking allows learners to think about the shapes whose structure they observed in level one thinking. Learners are able to describe a shape as having a collection of properties. For example they understand that the diagonals of a square are equal and bisect each other at right angles and are able to describe this in words.

Once a thorough understanding of level two has been achieved learners begin to develop a network of relations and this leads them to level three.

Level three

Level three is the **abstract / relational level**. Reasoning in the third level deals with the structure of the second level. Learners understand how one property follows from another. For example, knowing that a rhombus is a quadrilateral whose diagonals bisect each other at right angles, implies four equal sides which in turn implies that the square is a special case of a rhombus (when the diagonals are equal). Thus inclusive classification of shapes is also part of level three reasoning.

Level four is the **formal deduction level** where students can use an axiomatic system and use this to develop proofs.

Level five is the **level of rigour** where students can reason formally within and between different systems in geometry.

2.4.1 Progress through the levels

Van Hiele viewed the levels as being hierarchical with each level built on the foundations of the previous one. He maintained that the levels are discrete and that the associated discursive reasoning employed at one level is different to that employed at another level. At level one a learner might say ‘that is a rhombus’ or ‘that is a square’ based on their holistic perception of the shape. In the period leading to level two a learner might argue ‘that is not a rhombus because it is a square’. Learners at level two are able to provide a reason why a particular shape is a member of a class due to its properties, eg ‘that is a rhombus because it has four equal sides’. In moving to level three learners begin to make connections between the network of relations at level two. At level three they are able to perceive that a square belongs to the class of rhombuses because it has all the properties of a rhombus plus a few more.

Van Hiele was absolutely clear that progress from one level to the next develops in the periods between levels, not as a natural process, but as a result of a teaching / learning programme which facilitates the development of understanding (as opposed to learners being told about the structures of each level which is not in the least effective). He differed in his view from Piaget who considered that children move through developmental stages as they mature. Van Hiele said that learners simply need time to work through the necessary learning processes and that their development is dependent on experience of learning of the concepts in an educational context (Van Hiele, 1999).

Development is important in terms of the learning and can be divided into periods. Van Hiele described how, in period one (between levels one and two), symbols (perhaps the material representation of a shape such as a square or the label ‘square’) begin to act as signals by which they become imbued with properties of the shape. As the shapes become associated with their properties then the images of the shapes fall into the background and the properties become foremost.

In the second period (between levels two and three) the properties begin to be ordered, the learner develops an understanding of how properties are related to each other and the concept of implication develops. For example a geometric shape can be defined by a smaller set of properties than the list of properties which learners identify at level two. The understanding that a shape can be defined using a minimal set of properties (necessary and sufficient) and that those properties can be used to imply the other properties is characteristic of level three reasoning. Again Van Hiele stressed the importance of learners developing reasoning at level three as the result of a carefully designed pedagogical experience. He advised that to foster development towards the next level learning activities need to be devised which gradually build concepts and the related language and encourage students to integrate what they have learned into their existing knowledge and understanding (Van Hiele, 1999).

Since my research study is concerned with the inclusive relations between 2-dimensional shapes which involves development of reasoning from level two to level three, this second period will be the focus of the rest of the thesis. In particular I propose a learning task which has been designed to encourage such development towards van Hiele level three reasoning.

2.4.2 Other interpretations of how the levels can be used

The Van Hiele model has been the basis of much research, either to test the theory as an accurate description of learners' development in geometrical reasoning or to use the Van Hiele levels to assess geometrical reasoning which has been observed in research studies (Battista, 2007). However it has been noted that, rather than progressing from one level to the next, learners appear to use different Van Hiele levels in different situations, oscillating from one level to another during the same task and even to regress to earlier levels (Burger and Shaughnessy, 1986). Guttierez et al (1991) tried to address this by devising a method of assessing degrees of acquisition of several levels for individual learners working on one task. This indicates that to some degree they accepted that the levels are not discrete and that students can reason at more than one level simultaneously. In another attempt to improve the Van Hiele levels towards a working model for assessing students' geometrical reasoning, Battista (2007) has elaborated the original levels by giving a full description of each level using sublevels where learners' geometrical reasoning is defined using incremental steps rather than big jumps from one level to the next.

There is a question over whether the Van Hiele levels should be considered to be levels or stages (Battista, 2007). A level is a period of time where qualitatively different reasoning is demonstrated within a specific domain. A stage is a longer period of time characterised by qualitatively different reasoning over several domains. However there is another point of view elaborated by Papademetri-Kachrimani (2012) which is that it is more helpful to view the Van Hiele model as describing modes of thinking rather than hierarchical levels. Papademetri-Kachrani argued that the tendency to talk about levels has led to putting young children's thinking into boxes. She claimed that, whilst young children may not have the language skills to describe shapes (level two reasoning), they can understand and reason about shape properties. She also took Van Hiele researchers to task over their failure to pay enough attention to what Van Hiele said about intuition, an important form of (often non-verbal) reasoning and to the importance he gave to the learning periods. I have found Papademetri-Kachrani's suggestion that we consider the Van Hiele model as modes of thinking to be most helpful when analysing the data from my study and so propose to use the levels in this way. I will later show that the students participating in my research have used all of the first three levels of thinking while they worked on the task and that each level of thinking has been valuable to the students when working with the dynamic figure. At the same time, through the iterations in the study, I have modified the task to ascertain whether it could encourage development of analytic thinking at the third level.

I hold the view that a learner operates up to and including the level at which they would be assessed. So a learner who would be assessed as being at the analytical stage will also use the visual and descriptive modes of thinking depending on what suits them best in any particular context. If I consider how I (as an adult who has studied mathematics at university level) would analyse a geometric figure from its material representation, I would first use visual reasoning to recognise the shape. Then I would use descriptive reasoning to assess my decision. Finally I would only use analytical reasoning if this were helpful in solving a specific problem.

2.4.3 Understanding shape properties and styles of classification

This section addresses the way that shapes can be classified according to their properties, because this forms the basis for deductive reasoning in geometry, whether formally or informally. Shapes and their properties are a focus of the task in my study. Children learn about shapes as the surfaces of objects in our three dimensional world and so they develop an intuitive knowledge of shapes through their lived experience. Children learn to recognise classes of shapes (such as squares, rectangles and circles), and eventually they learn the language to describe the properties of these shapes. Learning to describe shapes, places children into period one of the development between the visual stage and the descriptive stage of reasoning (Van Hiele, 1986). It is common for students at this level to classify shapes into discrete classes as opposed to developing inclusive relations between sets (De Villiers, 1994). For example, squares and rectangles are considered to be two different shapes and this is known as **partitional classification**. The move from level two to level three (period two) will entail students in moving from a partitional classification to an inclusive **hierarchical classification** (De Villiers, 1994), from descriptive to informal deductive reasoning (Van Hiele, 1999), from empirical to logical reasoning (Okazaki, 2013). By the end of period two, on reaching level three, students have developed a more sophisticated level of geometrical reasoning and are ready to learn how to use deductive reasoning and proof in geometry.

2.4.4 Development during period two; moving from partitional to hierarchical classification.

A common aspect of learning about shapes in early childhood is the partitioning of shapes into discrete classes. For example, a square may be defined as a figure with 4 right angles and 4 equal sides whereas a rectangle may be defined as a figure with 4 right angles and 2 *different* pairs of equal sides. A partitional classification view can be held very strongly since the child has developed it from an early age, possibly as a result of having been presented with exclusive definitions in early schooling (De Villiers et al, 2009, Okazaki, 2009). So when in early secondary school children are presented with the idea of including some classes of shapes within others (eg squares as a special case of rectangles) they often find this difficult to accept.

It is perhaps more natural to use a partitional classification of shapes for example to differentiate a square from a rectangle or to partition convex (kites) from concave (arrowhead kites). Even Euclid partitioned quadrilaterals into five mutually exclusive categories (De Villiers, et al 2009). Partitional classifications are not incorrect, and are sometimes more convenient, but there are reasons why mathematicians use a hierarchical classification.

The way a shape is defined will depend on whether the student is using a hierarchical or partitional classification system. A hierarchical classification allows us to see that certain shapes are included as subsets of a more general shape. In a hierarchical system the definitions will be more inclusive whereas in a partitional system the definitions have to exclude other shapes. De Villiers (1994) listed a number of reasons why hierarchical classifications are more useful than partitional classifications, most of which refer to the ease of use when proving or solving problems in geometry. If this can be done in the general case then it will also be true for specific cases. In this way a hierarchical classification results in an economical deductive system.

Van Hiele (1999) referred to level three reasoning as the informal deductive stage. Indeed if students are to progress to formal deduction and proof then it is necessary that they are able to use hierarchical classification and this is why it is important to develop level three reasoning (Fujita and Jones, 2007, Okazaki, 2013).

2.4.5 Difficulties are experienced in progressing between levels two and three

The progression from level two to three is difficult (Fujita and Jones, 2007), and it is thought that students' personal figural concepts, particularly when they are based on definitions which use a partitional classification, can be a strong confounding factor. In order to move from classifying shapes in a partitional manner to a hierarchical one students need to re-construct how they categorise shapes (Tall et al, 2001). This requires students to work on the conceptual nature of the shapes and to allow the figural aspect less importance but students' personal figural concepts are so influential that they dominate the way the student defines the properties of shapes (Jones and Fujita, 2007, Okazaki, 2009).

Students find it easier to accept some inclusions than others. Fujita and Jones (2007) suggested that there could be a hierarchy of difficulties among the acceptance of inclusive relations. They observed that students in their study, who were student teachers of about 18 years of age, found it easier to accept the inclusion of rhombuses in parallelograms than the inclusion of squares in rectangles. Okazaki (2009) observed that fifth grade students (9-10 years) agreed with rhombuses being included in parallelograms whilst they disagreed that rectangles are included in parallelograms. Okazaki found that students recognised tacit properties of rectangles and squares which included the 90 degree angles and that this property was held so strongly that it precluded the inclusion properties of rectangles and squares in any shape where the angles were not right angles.

In a study by Haj Yahya and Hershkowitz (2013) tenth grade students (14-15 years old) were shown different numbered shapes on a worksheet which included a right square, a tilted square, a rectangle, a tilted rectangle, kites and parallelograms. The students were asked to list by number all the shapes which were parallelograms, rectangles, rhombuses, squares and kites. On analysing the student responses the tilted square was found to be 50% more likely to be included in the set of rhombuses than was the upright square. It seems that how the shape looked, its figural attributes, had an influence on the inclusion of the square into the set of rhombuses. A tilted square, rather than an upright square, looks more like a rhombus and perhaps that is why more students included the tilted square as a rhombus. In the same study very few students listed the square as belonging to the set of kites. If one thinks of the transformations needed to turn a square into a kite (rotate onto its vertex, shear into a rhombus, stretch two adjacent sides) then it does seem to be the shape which is least like a square.

As De Villiers (1998) commented, if we wish to increase students' understanding of geometric properties and concepts then it is necessary for the students to be involved in the process of defining properties. De Villiers (1994) makes the point that students are unlikely to see the need to use the more sophisticated hierarchical classification (indicating reasoning at Van Hiele level 3) unless they can appreciate its functionality. This requires the teacher to devise tasks where using a hierarchical classification would be more efficient and helpful than using a partitional classification. He suggests that the use of computer programs such as DGS offer the potential for students to accept hierarchical systems when they experience dragging of a general shape into a specific shape.

Finally Van Hiele maintained that, as students' geometrical reasoning matured, the figural aspects of shape move to the background and the definition becomes uppermost in the student's understanding. However, Mariotti and Fischbein (1997) argued that the relationship between the figural and conceptual aspects of shape will change as the student develops more sophisticated reasoning but figures will always have an important role to play in the student's understanding of shape. They asserted that there may be conflicting relationships between the figural and conceptual constraints whenever an individual's conceptual understanding does not match the formal definition and it is the role of the teacher to stimulate thinking which will produce cognitive conflict aimed at getting the student to revise their definition of the shape.

It is interesting to note the range of ages of students, in the studies quoted in this section, who have been confronted with the concept of inclusive relations. From 9-18 years of age similar concepts have been introduced to the students. As Van Hiele thought, this does suggest that progress in geometric reasoning depends on the learning experiences of the students and may not develop independently. Certainly it may not be age dependent apart from the consideration of the length of time needed to develop progressively more sophisticated levels of reasoning. As a summary for future reference, table 2.1 indicates how students whose reasoning is at level two and three may describe the rhombuses and kites.

Table 2.1 Common descriptions at levels two and three of the rhombuses and kites

level two	level three
Only uses a partitional classification	Can also use a hierarchical classification and include rhombuses in the kites
Describes a rhombus as a quadrilateral with 4 equal sides, 2 lines of symmetry, 2 pairs of opposite equal angles	Defines a rhombus as a quadrilateral which has 4 equal sides *or* whose diagonals are both lines of symmetry *or* which has 2 pairs of opposite equal angles and a line of symmetry
Describes a kite as a quadrilateral with 2 pairs of adjacent equal sides, 1 pair of opposite angles, 1 line of symmetry	Defines a kite as a quadrilateral which has 2 pairs of adjacent equal sides *or* which has a diagonal as a line of symmetry *or* which has a line of symmetry and a pair of opposite equal angles

2.5 How students use diagrams as representations of geometric figures

Laborde (1995) wrote of space and geometry as being two separate domains. The spatial domain is that which humans experience in the real world whereas geometry is a theoretical domain which models space.

2.5.1 The spatio- graphical field and the theoretical field

Diagrams in mathematics have a theoretical meaning and their constructions are connected to theory (Mariotti, 2000). In geometry, diagrams have a particular importance for when we look at a geometrical object, it may look as if it is a particular instance of the object but it represents a class of figures which share certain properties (Skemp, 1971, Laborde, 2004). The advantage of the diagram is that it embodies the properties of the object (or class of objects) such as its regularity or symmetry. Laborde (2004) referred to these aspects of the diagram as its spatio-graphical properties which students can investigate empirically, for example by measuring sides and angles.

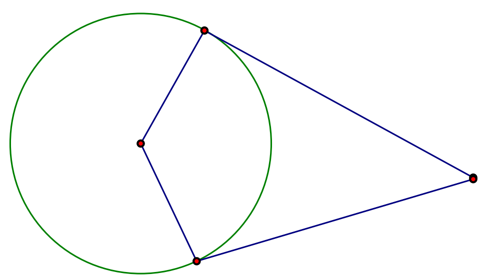


Figure 2.5 Circle and two of its tangents which meet at a common point

This is especially useful when we represent several objects in a sketch, for example if we have a circle showing two tangents drawn from one point and the radii drawn from the intersections of the circle with its tangents as shown in figure 2.5.

When investigating this diagram students may observe that the angle between the tangent and the radius is a right angle and that the length of the tangents between the point and the intersections of the circle is equal. In this way students can learn about the theoretical figure by investigating in the spatio-graphical field hence moving between the spatio-graphical field and the theoretical field. Laborde (2004) maintained that students move between the spatio-graphical and theoretical fields a number of times when solving any problem in geometry using the spatio-graphical realm to help them access geometrical theory.

A figure constructed using Dynamic Geometry Software embodies the properties which were used to construct it and when the DGS figure is dragged it behaves according to those properties. Laborde (2004) showed how grade 7 students (11-12 years) in her study used a DGS figure to support them in moving back and forth between the socio graphical aspects of the figure and the class of theoretical figures of which the screen figure was a representative.

Laborde showed that students working with a dynamic figure were able to use its spatio graphical aspects to help them learn about its theoretical aspects. When working with DGS figures students sought reasons for what they observed when the figure was dragged and this seemed to lead to their development of theoretical understanding. If students in Laborde's study made a prediction of a property of the figure they were able to test the property by dragging and measuring in the figure. In contrast, students working in the pencil and paper environment often tended to become stuck in the spatio graphical realm.

Laborde (2004) claimed that DGS helps to mediate students' development between empirical and theoretical geometry. Logical deductive reasoning may evolve out of empirical investigation in the spatio-graphical field.

2.5.2 Duval's framework of perceptual apprehensions and cognitive processes.

Duval (1995) described a geometric object as a theoretical object which embodies specific properties (so agreeing with Fischbein). A figure is a representation of the geometric object constructed so that it has the properties of the geometric object. Here the way Duval used the word 'figure' appears to mean the physical representation and the words 'geometrical object' to be the abstract, possibly the figural concept. Duval argued that geometric figures have a heuristic role, i.e. they are used by students to discover geometrical theorems and to solve problems. To understand how a figure functions heuristically we need to consider its underlying cognitive complexity. Duval described four aspects of a figure which he named cognitive apprehensions. An apprehension is a way of looking at and understanding a figure.

2.5.2.1 Conceptual apprehensions

First of all a drawing evokes *perceptual apprehension*. This is the recognition of the shape of the figure whether it represents an object in 2 dimensions or in 3 dimensions. It may also involve recognition of sub figures within the figure. To function as a geometrical figure a drawing must first evoke perceptual apprehension and at least one of the other three.

Sequential apprehension is an understanding of how the figure is constructed. This is dependent on the tools used in its construction. Duval suggested that computer software supports the development of sequential apprehension. The commands in the construction menu act simultaneously as instrumental constraints thus embodying the properties of the figure and in providing a narrative of the construction process which is related to discursive apprehension.

Gomes and Vergnaud (2004) showed that students who constructed a geometrical object on paper with ruler and compasses used a different set of geometric properties than when they constructed it using Dynamic Geometry Software (DGS). They concluded that the learning and conceptualisation of shapes is enhanced by students experiencing use of different tools and methods for construction to develop a broader understanding.

Discursive apprehension involves verbalising and reasoning about the figure and its properties. It can also be said that the definition of a geometrical object and a description of its construction are part of discursive apprehension. Discursive apprehension is important for mental organisation of understandings of the geometric object.

Operative apprehension relates to physically or mentally operating on the figure in order to learn more about it. This could be splitting the figure into subfigures, changing its position or orientation or performing a transformation. Operative apprehension does not work independently of the others. It especially requires the use of discursive apprehension. Duval argued that students specifically need to learn how to use operative apprehension and that this is very useful training in helping them solve geometric problems. In particular it is useful to teach students how to split figures into subfigures including the use of overlapping subfigures. However operative apprehension is sometimes confused with pictorial support of discursive apprehension. Duval considered that computers might foster the development of operative apprehension were the software to be designed to allow this to happen.

2.5.2.2 Cognitive processes

Duval (1998) described three kinds of cognitive processes in geometry which fulfil practical functions.

Visualisation helps the student to form an overall impression of the shape of a figure and to explore visually how it is made up. A figure is a configuration of several constituent gestalts such as line segments and angles and which have been put together according to geometrical relationships. Hence there is a discursive element to visualisation as a description of the figure and its properties. Visualisation also helps students to solve problems geometrically using some of the gestalts to operate on the figure, e.g. to split it into sub-figures, change the position of some of the sub-figures or transform the figure in some way. This is use of operative apprehension which Duval described as figural change and which is used to reorganise how the figure is viewed in order to solve a problem. Visualisation thus requires perceptual, discursive and operative apprehensions, but not necessarily sequential. Visualisation plays a basic heuristic role, through operative apprehension, and can provide a basis for making a convincing argument in a problem solving process.

Construction is the use of tools in the process of constructing the diagram (representation) of the figure which works like a model of the object. The student learns about the figure through constructing it.

Reasoning uses the discursive process (including the verbalisation of shape properties, geometrical theorems, etc) to extend student knowledge and understanding. In geometry information is often provided through a diagram and the student organises their mental picture of this diagram at a representative and at a symbolic level.
We may use a natural discursive process which entails using natural (not specifically mathematical) speech to describe the shape and explain reasons for identifying particular aspects of it. Also there is a theoretical discursive process which entails formal deductive reasoning.

In reasoning about a geometrical figure we may use a purely configural process, which is operative apprehension. Duval stated that operative apprehension is a visual process and is independent of discursive processes. Visualisation can be embedded in a natural discursive process (using everyday natural language) when a student describes what they are seeing and uses this as part of their reasoning process. However a purely configural process cannot be embedded in theoretical discourse (formal deductive reasoning). Duval warned that it is important for teachers to realise that there is a gap between the natural discursive process and the theoretical discursive process, resulting in a didactical problem of encouraging students to move from one to the other.

Duval stated that proficiency in geometry requires an understanding of how the three cognitive processes work together. However he also said that each process should be developed separately and only then will students be able to co-ordinate all of the three processes. He was clear that there is no developmental hierarchy between different cognitive activities which goes from concrete to most abstract. Duval also argued that there is no point in expecting most students to learn formal geometry and that doing so forces them to give up their natural ability to use reasoning and justification. It is beneficial for most students to experience shape and space in a less formal way and only to teach formal geometry to those students with a particular interest in it. Duval was writing in the context of French education where geometry is more formally taught than in the UK. The UK student's experience will be mainly developing reasoning and justification to form convincing arguments to solve problems with shape and space whilst in compulsory and post 16 schooling. More formal geometry generally is covered at undergraduate level.

2.5.2.3 Using computer geometry

Finally Duval talked of how computer geometry can help students to explore geometric situations by manipulating figures on the screen. He stated that much of the work undertaken using Dynamic Geometry Software focuses on construction which privileges discursive and sequential apprehension and does not develop operative apprehension. Later on I will show that with the right kind of task, operative apprehension does come into play in a DGS environment.

2.6 Conclusion

This chapter has addressed the importance of symmetry and orientation as factors in the way students perceive 2 dimensional shapes. Later, in the account of the iterations, I show how symmetry and orientation were significant factors in the way that students participating in the study perceived the dynamic figure on the computer screen.

I have described three important models of geometrical reasoning. The figural concept (Fischbein, 1993) is an important idea, in that the dynamic figure is itself a mediator between the shape which the students view on the screen and the theoretical object represented by that shape. As has already been noted Mariotti (1995) claimed that the screen image represents the external version of the figural concept.

The Van Hiele model of development in geometrical reasoning (Van Hiele, 1986) is used throughout the study to analyse the levels of reasoning demonstrated by the participating students as indicated by the dialogue and on-screen activity. Table 2.1 illustrates the types of descriptions of kites and rhombuses common at levels two and three. During the study I will ascertain whether working on the task using the dynamic figure supported students in developing higher level reasoning, in particular whether they perceived the inclusive relationship between the rhombus and kites.

Duval's framework of perceptual apprehensions (Duval, 1995, 1998) is used, with reference to the cognitive processes, to analyse the data overall in the retrospective analysis, chapter ten. As a comprehensive framework describing how students interact with figures when reasoning and problem solving it should cast light onto how students perceive the figure and the shapes generated from it.

3 Working with figures in Dynamic geometry Software

In this chapter I describe two important affordances of Dynamic Geometry Software (DGS), task design within DGS and how DGS acts as a microworld (Papert, 1993) and as a tool for semiotic mediation.

3.1 The affordances of dragging and measuring

Affordance is an important idea. It refers to those qualities of the learning situation and context that allow students to develop knowledge and understanding (Leung, 2011). Dragging and measuring are two important affordances of DGS (Hollebrands, 2007). The use of dragging with measuring is powerful because it allows students to investigate empirically in the spatio-graphical domain and to use measurements to support them in moving towards theoretical reasoning. For example Hollebrands (2007) observed that 15 year old students used evidence from measuring and reasoning together in their explanations of why their constructions worked. It will be seen later in this thesis, in the description of the iterations, that dragging and measuring are both important aspects of the participating students' work with the dynamic figure.

3.1.1 Dragging strategies and how they are aligned to cognitive behaviour

The drag mode is a powerful affordance which is responsible for the visual dynamic nature of DGS figures and effectively provides the main vehicle for students to interact with the figures on the screen (Leung and Lopez-Real, 2002). It would therefore be very useful to consider some of the different ways the drag mode can be used.

Arzarello et al (2002) writing of how they had observed different dragging strategies or 'modalities' being used by students of about 15 years of age (in a number of their earlier studies from 1998 to 2001) described a hierarchy of dragging strategies which they classified according to whether these strategies allow students to move from the practical geometry of the figure on the computer screen to theoretical geometry (which they called an ascending process) or from the theoretical to the practical domain of geometry (which they called a descending process). In other words Arzarello et al claimed that dragging in DGS helps students to move between the spatio graphical and theoretical fields of geometry (Laborde 2004)). Arzarello et al worked with Cabri Geometry™ which is another DGS program, similar to the Geometers Sketchpad but with a slightly different functionality. As such some of the dragging modes that were described in their paper refer to styles of dragging that would not work in the Geometers Sketchpad. I will therefore only describe the dragging modes which are pertinent to the Geometers Sketchpad.

- *Wandering dragging* is used by students to explore a figure in order to discover its properties and so allows students to move towards theoretical geometry.
- *Guided dragging* is used by students to drag the figure into a particular configuration and is an example of moving from theoretical to practical geometry.
- *Dragging test*, which is used to drag objects and check that the figure maintains its constructed properties.
- *Dummy locus dragging* (moving an object on a figure so that the figure keeps a specific property, the movement of the object traces an invisible path, hence 'dummy locus; but this behaviour is not intentional).

Further work on dragging strategies has produced an extrapolation of the dummy locus dragging mode. When the student **intentionally** drags an object to keep a certain property constant, Baccaglini-Frank and Mariotti (2010) refer to it as *maintaining dragging* which they describe as dragging (a base point) so that the (DGS) figure maintains a certain property. (They observed 16-17 year old students in Italian High Schools who were specifically taught the maintaining dragging strategy as a tool they could use to solve problems in geometry. The students' work was analysed to ascertain whether use of maintaining dragging described the students' problem solving activity).

"Maintaining dragging involves the recognition of a particular configuration as interesting, and the user's attempt to induce the particular property to become an invariant under dragging." (ibid, p. 230).

The maintaining dragging strategy is used when a conjecture is formed that connects the path travelled by the object to the property the student is trying to keep invariant. Both dummy locus dragging and maintaining dragging are examples of when the student moves from the spatio-graphical field to the theoretical field.

3.1.1.1 Three functions of dragging

Other researchers have described dragging strategies in broader terms. Lopez-Real and Leung (2006) classified dragging strategies into three functions: *confirmatory* (eg the drag test which confirms that the figure has been constructed to keep certain properties constant) and *exploratory* (students drag objects on a constructed figure to investigate invariant properties of the figure, which appears to be similar to wandering dragging). A third function arose from their observation of students solving a geometric problem using dragging by trial and improvement which they called *drag to fit.* This function seems to include guided dragging, dummy locus dragging and maintaining dragging as in each case an object on a figure is being dragged in order to give the figure a particular configuration or property. Lopez-Real and Leung commented that the function is related to the problem task and how students choose to solve the problem.

3.1.1.2 Robust and soft constructions

The dragging modalities and functions described above can be seen to refer to two categories of dynamic figure; *robust* constructions and *soft* constructions. Healy (2000) made the distinction between these two types of constructions in the following way. A robust construction will maintain its constructed properties, no matter how it is dragged on the computer screen. In other words it cannot be 'messed up'. A robust construction affords the use of the confirmatory and exploratory functions, which appear to fit with the dragging test. On the other hand a soft construction is a figure such that one of the desired properties must be carefully maintained through the student dragging an object 'by eye'. This allows students to explore the desired property particularly to investigate all the cases where the property holds true (which may be a locus of points, hence bringing into play dummy locus dragging or maintaining dragging). Soft constructions afford the use of the drag to fit strategy.

3.1.2 Measuring whilst dragging.

The measurement tool in DGS allows for lengths, angles, areas, etc to be measured and displayed on the computer screen. As the figure is dragged the measurements continuously update. Olivero and Robutti (2007) conducted a study of three pairs of 15-16 year old students from two different High Schools in Italy and observed that they used the Drag mode and the Measuring facility to move between the spatio-graphical field, where they felt most comfortable, and the theoretical field where they needed to develop more sophisticated levels of geometrical reasoning.

Olivero and Robutti (2007) described different strategies of using dragging and measuring and how they enable students to move between the spatio-graphical field of geometry and the theoretical field. As part of the team working with Arzarello it is not surprising that Olivero and Robutti connect dragging and measuring modalities to the dragging modalities identified by Arzarello et al (2002). They identified three modalities for measuring for discovery and conjecturing:

- **Wandering measuring** –students drag figures and watch the measures change. They do this randomly and to see what they can notice, which may be developed into a conjecture.
- **Guided measuring** – students drag generic figures into particular figures by reference to the changing measurements.
- **Perceptual measuring** – students use the measures to test a perceived relationship. This may result in the students forming a conjecture in which case the students may be able to move from the spatio-graphical field to the theoretical field.

They identified two modalities for measuring for validating a conjecture:

- **Validation measuring** – students use measures to check a conjecture they have made works for a figure on the screen. This may allow the students to link the theoretical field with the spatio-graphical field.
- **Proof measuring** – after constructing a figure, students may test it in DGS in order to understand its properties by measuring sides and angles in the figure.

Hollebrands (2007) conducted a study with a whole class, of 15 and 16 year old students in the USA, working with transformations in geometry using the Geometers Sketchpad. She observed how the students used dragging and measuring to explore relationships, create and verify conjectures and check the correctness of constructions. She described two ways that students use the two affordances of dragging and measuring together in order to do this. When students drag in a fairly random fashion in order to see what happens and when their decision of what to do next is based on the results of the previous action then the students are using *reactive* strategies. When the students develop their understanding of both the mathematical concepts and how the technology works then they are able to predict the outcome of their actions and become *proactive* in their strategies. An example here would be if the students predicted that placing the diagonals so that they bisected at right angles would result in the quadrilateral being a rhombus. Encouraging students to use strategies that are more proactive may be achieved by asking students to explain and justify what happens on the computer screen in terms of geometrical properties. However Hollebrands found that the dragging mode is only useful for learning if students know and understand how to use it to investigate the variance and invariance in geometrical objects constructed in DGS. Hollebrands' findings provide further evidence that working in DGS allows students to move back and forth between experimental and theoretical geometry which Laborde (1993) maintained is needed for students to progress towards theoretical geometry (see chapter one, section 1.3.1).

3.1.3 An important consideration of the accuracy of measurements in DGS

Measuring with dragging can be seen to be important in helping students, who are working at Van Hiele levels 2 and 3 (see chapter two, section 2.4) to work between the spatio-graphical field of geometry, where they feel secure, and the theoretical field of geometry entailing higher order geometrical reasoning. However it is important to be aware of the constraints of using measuring in DGS programs. It is easy to imagine that the computer measures line segments, angles, etc with absolute accuracy but this is not the case. Despite the software simulating a geometrical environment the accuracy of measurements are dependent on pixel size, the algorithm designed to carry out measurements and the degree of accuracy chosen for the computer file. This can result in measurements which should be equal being displayed as unequal measurements and vice versa. These contradictions are a consequence of the software acting as a mediator between a quantity and the result of its measurement displayed on the computer screen (Olivero and Robutti 2007).

3.2 Considerations when designing a pedagogical task using Dynamic geometry Software

The computer offers ways of working that help students to access approaches and solutions which would not be available to them using pencil and paper (Hoyles and Noss, 1992). However students will not necessarily appreciate the intended mathematical ideas just because they are interacting in a particular computer environment. Tasks need to be designed with the pedagogical principles built into them. As Hoyles and Noss point out, carefully constructed activities reveal and develop students' intuitive ideas and develop the use of language to represent these ideas.

A pedagogical task is based on activity in context designed to help students form generalised abstract concepts. It is designed to be a tool which brings about learning and should be situated in the setting in which it will be carried out taking into account the teachers, students, classroom environment, tools used for teaching and learning and pedagogical approaches (Leung, 2011).

A pedagogical task is most effective if it provides students with a sense of purpose and an appreciation of the utility of the mathematical concepts being used. Technology allows us to create purposeful tasks where the pupils work with mathematical concepts thus developing an understanding of these concepts through use (Ainley et al, 2006). Technology allows students to do and see what they could not without the technology and can amplify the ability of students to explore, reconstruct and explain mathematical concepts using embedded tools (Leung, 2011).

However it is important that the teacher carefully designs the learning situation and the task for the students when working with computers in classroom situations (Sutherland, 2004). How the computer software is incorporated into the learning activities is crucial to students' potential learning. The concept of the *microworld* will now be described as a means to address the issues listed in this section.

3.3 Microworlds

A microworld can be described as a computer environment which embodies an area of mathematics or science (Edwards, 1995) where students can explore objects and discover the relationships between them (Ainley et al, 2006) and which has been designed from a pedagogical perspective (Noss and Hoyles, 1996). The term microworld is attributed to Seymour Papert who was the developer of LOGO, a system using a mechanical 'turtle' which moved on the floor in response to programming commands or its on-screen version. Papert observed children working and solving problems in LOGO using intuitive ways of thinking and problem solving that often involved finding approximate solutions and then refining these to get to the optimal solution. It was a different way to do and learn mathematics which appeared to give children access to more sophisticated mathematical ideas (Papert, 1993). Noss and Hoyles (1996) found that students can learn fairly sophisticated and abstract mathematical ideas if they are given the opportunity to explore in a microworld environment.

We usually refer to computer based microworlds although a microworld is not necessarily 'virtual' as with the mechanical 'turtle' in LOGO. When working in a microworld, students manipulate objects on the computer screen in order to explore how these objects behave and discover the underlying rules for their behaviour. Thus students receive instant feedback and are able to use this to correct wrong assumptions they may have made and effectively 'de-bug' their understanding of the domain (Edwards, 1995). Learning results from adapting to the microworld and making sense of the feedback. Microworlds can help students access sophisticated mathematical concepts via the concrete nature and manipulability of computer representations (Balacheff and Kaput, 1997, Ainley et al, 2006).

However working in a microworld does not guarantee that specific learning will occur or that students will focus on the desired aspects of mathematics which the software designer or teacher intended (Balacheff and Kaput, 1997, Pratt and Ainley, 1997, De Villiers, 2007). Tasks which are given to students working in a microworld need to be carefully designed but when this is effective it is possible that students create their own mathematics and discover their own theorems (Olive, 2000).

Dynamic Geometry Software is an example of a microworld which embodies the attributes of Euclidean Geometry (Balacheff and Kaput, 1997, Mariotti, 2000, Lopez-Real and Leung, 2006). DGS provides interactive ways for exploring geometrical concepts in visual and dynamic ways and has the potential for allowing students to develop mathematical meanings (Leung, 2008). An important aspect of DGS is how it allows students the opportunity to investigate and conceptualise the invariant properties of figures, which is a powerful way to learn in mathematics (ibid).

"What makes a dynamic geometry environment a powerful mathematical knowledge acquisition microworld is its ability to make visually explicit the implicit dynamism of thinking about mathematical geometrical concepts." (Leung 2008, p.1)

Whilst working in DGS is considered a useful support to learning Euclidean geometry some researchers are beginning to suggest that DGS is a new field of geometry that belongs to the computer environment. This offers the exciting possibility that the experience of Euclidean geometry on paper can be fused with the experience of geometry in a DGE to form a more rounded network of geometrical concepts (Lopez Real and Leung, 2006).

"In a dynamic geometry environment traditional Euclidean geometry may be transformed into a new situated geometry that might not just change the rules of the game, but the game itself." (Lopez Real and Leung 2006, p.667).

3.3.1 Webs of meaning

Webbing is an important idea in the context of the microworld. A web of meaning is a structure which supports the learner whilst giving them the autonomy to construct new knowledge and understanding (Noss and Hoyles, 1996). The web itself is situated in the domain of knowledge, in the setting within which the learner is working, resources, the symbol systems used to express the ideas and access to the help of the teacher. Learners may construct explanations and /or proofs entirely within the context of a microworld but it is not always certain that they can transfer these to another context, e.g. the paper and pencil environment. This is where the role of the teacher is important to promote shared meanings and to guide the students in abstracting the mathematics from meanings which have been developed within the computer environment (Mariotti, 2009).

3.4 Situated abstractions and situated proofs in Dynamic Geometry

When working in a microworld students need to be given the opportunity to discover mathematics which is new to them and to develop their own concepts of how that mathematics works within the context of the microworld. When people make (mathematical) sense of the results of their actions in specific environments then they are forming *situated abstractions* (Hoyles and Noss, 1992). An important point about situated abstractions is that they are developed by students and their meaning is controlled by the students (Noss and Hoyles, 1995). This is why the objects in the microworld need to represent powerful mathematical ideas. That is to say when designing a microworld it is necessary to create a domain where students can learn about mathematics through forming situated abstractions (ibid).

Explorations made in the medium of the computer are situated in that medium and so are the corresponding theorems and proofs. For example students may be able to construct hypotheses which are understood specifically in the context of the Dynamic geometry environment and use dragging to test them. This can lead to *situated proofs* when the students are able to show that the theorem holds in this environment (Armell and Sriraman, 2005). This type of proof tends to be inductive rather than a formalised deductive proof.

As an example of this, Leung (2008) described an activity he undertook where he attempted to make one angle in a figure to be twice the magnitude of the other by dragging points so that the angle measurements indicated a ratio of 2:1. He did this by first dragging to get the ratio close to 2:1 and observing the area of dragging where this happened. He used his observations to try and get the ratio closer to 2:1 by what he called ‘refining wandering dragging’. Next, he went into more detail and developed a colour coded map of areas where each point on the figure could be dragged to maintain the ratio between the angles close to 2:1. When this ratio was 2:1 within one degree he had a ring of colour which led Leung to conjecture that the locus of points he was seeking was in fact a circle. This work could not easily have been carried out in anything other than a DGS environment and this demonstrates a situated conjecture and proof in the context of dynamic geometry.

Leung, as an expert mathematician, was able to transfer the situated proof from the domain of the DGS microworld into the domain of formal mathematics. However, as was stated in section 3.3.1. for school students who are not so expert, the role of the teacher is very important in guiding them to make the necessary links between what they have discovered while working with a task in DGS and what this implies for formal mathematics using pencil and paper.

3.5 DGS as a tool of semiotic mediation

3.5.1 Signs and semiotic systems

Vygotsky (1978) showed that when artifacts are used in social contexts then shared, signs are generated which may be word or symbols. Signs are psychological tools (ibid). The term ‘semiotic system’ refers to the signs and the logical structures which govern their use. Duval (2006) wrote that mathematics as an intellectual domain has a large number of semiotic systems (more than any other school subject) which are specific to mathematics. In geometry at least two semiotic systems are needed to represent geometrical figures; one for the verbal expression of properties or magnitude, and the other for visualisation (ibid). A geometric figure always associates discursive and visual representations and students are expected to move between the two, which is cognitively complex since it goes against the common association between words and shapes (ibid).

3.5.2 Tools and utilisation schemes

All human action is mediated by tools (Vygotsky, 1978). In schools this may include a wide range of artifacts (including pencil and paper, text books and computers) and semiotic systems which together make up tools of which the most important is language (Drijvers et al, 2010). The drag mode in DGS can also be considered to be a tool because it arises from the DGS environment (the artifact) and it has its own grammar of use linked to students' mathematical meanings (Lopez-Real and Leung, 2006). When a particular dragging strategy is intentionally used by a student according to an underlying mathematical logic in order to construct new mathematical meanings then the strategy becomes a utilisation scheme (Verillon and Rabardel, 1995). Baccaglini-Frank and Mariotti (2010) claimed the maintaining dragging strategy to be an example of a 'dragging utilisation' scheme, because it is connected to a mental construct (the intention to keep a geometric property constant) concerning the figure under dragging using this strategy.

3.5.3 Tools of semiotic mediation

Bartolini Bussi and Mariotti (2008) described how artifacts which have been designed to support the learning of mathematical structures, such as the abacus and DGS, act to mediate the learning of mathematics and the related semiotic systems. As such they are known as tools of semiotic mediation (ibid). In particular and pertinent to this thesis, the affordance of dragging can be considered as a process of making new mathematical meanings about geometry and geometric figures; a kind of temporal-dynamic semiotic mediation instrument (Mariotti, 2000, Lopez Real and Leung, 2006). It may be that the kinaesthetic aspect of dragging, whilst observing visual changes to the dynamic figure, facilitates the use of perceptual apprehension and operative apprehension and when students verbally describe these changes, giving reasons for what they have observed, they are using discursive apprehension (Duval, 1995, Leung, 2011).

We have already seen how a number of dragging styles have been described and related to cognitive activity (Arzarello et al, 2002, Baccaglini Frank and Mariotti, 2010). Dragging takes on the meaning of a check of the invariant properties in the construction and thus has become a 'sign' in the Vygotskyan sense (Mariotti, 2000, Drijvers et al, 2010). Dragging is a tool which enables students to create new mathematical meanings in geometry as they use the information they absorb from the computer screen (Lopez-Real and Leung, 2006, Bartolini Bussi and Mariotti, 2008). Drijvers et al (2010) stress the importance of the role of the teacher in making the artefact (such as DGS) function as a tool for semiotic mediation in helping the students to become aware of the mathematical meanings which can be drawn from it. In mathematical activity there exists a link between the artefact being used (in this case, dragging in a DGS environment), the task which students work on and the mathematical meanings which they develop (Bartolini Bussi and Mariotti, 2008).

3.6 The development of discourse whilst working in a DGS environment

The development of the ability to use mathematical language is important in the development of students' mathematical reasoning and certain words or expressions in mathematics convey a complex web of ideas which form a mathematical concept and help to support students' formation of concepts in mathematics (Lee, 2006). As students discuss geometrical principles using geometrical vocabulary this supports them in developing their skills of explanation and justification.
Working in DGS can support the development of geometrical language as Jones (2000) observed in a study with thirteen year old students in England. In the earlier part of the study the students were able to describe what was happening on the computer screen but without the use of precise mathematical language. As the study progressed they became more able to give mathematically precise explanations although these tended to be mediated by the nature of the language used by the DGS program. By the end of the study students had started to give explanations relating to the mathematical content using more mathematical terminology.

Working with computer software can thus mediate learning through the language and notational system that is designed into the program (Hollebrands, 2007). When the teacher and student interact while using DGS they both adopt the language of the software, a language they then use to communicate with each other. They can also communicate mathematical ideas through discussing the visual images on the screen.

DGS uses geometrical vocabulary and thus can help to develop students' use of this important part of the mathematics register.

Students' use of language about their mathematical experience in DGS reflects the dynamics of the drag mode. Situated descriptions, abstractions or theorems tend to be expressed with active verbs especially those of movement (Holzl, 1996).

3.6.1 Reasoning and discourse specifically aligned to the dynamic nature of DGS

When mathematicians work with geometrical objects in a static environment, they may mentally animate the figures in order to perceive the variants and invariants (Leung, 2008, Sinclair et al, 2009). A simple example could be thinking about the similarities and differences between arrowheads and kites by mentally moving the shorter congruent sides as indicated in figure 3.1.

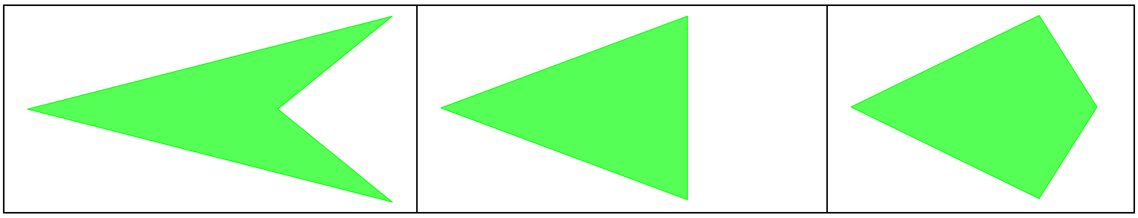


Figure 3.1 From arrowhead to kite by moving the shorter congruent sides

In DGS this mental animation can be actualised on the computer screen in a highly visual manner. This means that the computer makes visible a mental activity which is intuitive to expert mathematicians making it accessible even for students who may find this kind of imagery difficult to imagine. It seems obvious that reasoning about a static geometric figure is different to reasoning about a dynamic figure although they may be representations of the same theoretical object.

Two important contributions that the dynamic nature of DGS makes to students' learning and reasoning about geometric figures are:

> "the powerful, temporalised representation of continuity and continuous change (dynamism's mathematical aspect), and the sensory immediacy of direct interaction with mathematical representations (dynamism's pedagogic aspect)".
> (Jackiw and Sinclair, 2009, p. 413)

When students engage with a DGS figure by dragging they observe it morphing through a possibly infinite number of versions of itself and this experience has an effect on how they perceive the figure and thus how they reason and talk about it (Jackiw and Sinclair, 2009, Sinclair et al, 2009). An important aspect of the discourse, when students work with dynamic figures, is the forming of a narrative to describe the sequence of events unfolding on the computer screen and to interpret these events (Sinclair et al, 2009). When working with DGS figures students observe what happens under dragging, noting variant and invariant aspects of the figure. Constructing a narrative of what they have observed can help students to use reasoning about the mathematical relationships which are evident amongst the objects in the DGS figure (ibid).

3.7 The research questions arising from the review of the literature

I postulate that there may be other important factors at work, alongside the development of a narrative way of reasoning, when students work with dynamic figures which are connected to their perceptual apprehension (Duval, 1995) of the figure while it is being dragged. The work undertaken for this thesis will attempt to address this and to **ascertain those aspects of the dynamic nature of DGS which impact on students' reasoning about geometrical figures in 2 dimensions**.

In addition, another research question developed during the progress of the research: of several dragging strategies which participating students were observed to use, one has been studied in greater depth to **ascertain whether it acts as a dragging utilisation scheme to mediate the meaning of inclusive relations between shapes generated from the dynamic perpendicular quadrilateral, in particular between the rhombus and kites.**

4 Methodology and Methods

At the end of chapter three I stated two research questions concerning the task using the dynamic figure in the context of the DGS environment. These concerned how students worked on the task and interacted with the dynamic figure and whether this had an impact on their ability to reason about 2 dimensional shapes in geometry, in particular whether they could start to appreciate inclusive relations between the rhombus and kites. I expected that the task would need to be modified through a number of iterative changes by the time the research was completed. For this reason I have taken a Design Based approach in how I conduct the research. The task using the dynamic perpendicular quadrilateral constitutes the design experiment.

My study focuses on how 12-13 year old students' geometrical reasoning developed as they work on this task. I chose this age group because the task fitted well with the learning objectives for Geometry and Measures in the English National curriculum for Mathematics 2007. These learning objectives are given in chapter five, section 5.3.2 as the instructional starting points for the pilot study. The research has developed through several iterations as I have used my observations and conclusions from each stage of the study to modify the task to improve it and to test mini theory which has emerged.

4.1 What is Design Based research?

Design Based Research (DBR) as a methodology emerged at the beginning of the 1990s, with the work of Brown (1992) and Collins (1992), as an approach to the study of learning through the design of teaching and learning interventions (Swan, 2006). It was developed due to a realisation that laboratory-type experiments which seek to control for variables were not suitable for describing the complexity of the classroom context where there are myriad variables (Brown, 1992) and that there was a need for research which would be relevant for the classroom, teachers and students (Swan, 2006). The typical classroom is a complex environment and it is not possible to isolate one factor in this situation without it affecting all the other factors (Brown, 1992). DBR works with this complexity leading to a greater understanding of the complex learning environment (Cobb et al, 2003).

DBR has two important goals; to advance new theory of how students learn and to use this theory to develop educational interventions which can be used in the classroom to facilitate that learning (Lamberg and Middleton, 2009). A distinctive feature of DBR methodology is that it enables the researcher to deepen their understanding of learning in the context of the experiment while it is in progress (Cobb et al, 2003, Design Based Research Collective, 2003). It also facilitates innovation, as the Design Experiment can be used to design new learning activities and resources which can be tested and revised thus leading to improvements in teaching and learning (Design Based Collective, 2003). DBR allows emerging theory to shed light on learning in a specific educational setting and this can be applied and have relevance to other contexts outside of the intended original (Design Based Collective, 2003, Barab and Squire, 2004).

DBR develops theory of learning in a specific educational setting and works with all the factors which will affect learning, for example, resources, artifacts, teaching practices, classroom culture, etc (Cobb et al, 2003). The research uses an iterative cycle of designing the intervention, implementing the intervention, analysing the learning that took place developing the theory of this learning, and then modifying the design to test an emergent theory (Design Based Research Collective, 2003). This modification of the intervention during the research study has led some to question the rigour of DBR (Hoadley, 2004). However the strength of DBR is in how interventions are connected directly to outcomes leading to better alignment of intervention and theory in a complex setting (ibid).

4.1.1 The process of Design Based Research and how this is addressed in my own study

In Table 4.1 I have listed the processes involved in design Based Research and indicated how they worked in this particular study.

Table 4.1 The process of DBR and how this is addressed in this study

What the literature says	**How this worked in my study**
Middleton et al (2008) state that, in a DBR experiment the specific pedagogic issue must be established, providing the rationale for the research and leading to the development of a hypothesis. The intervention (task or tool) which will address this issue becomes the focus of the design experiment.	The pedagogic issue lies with students' geometrical reasoning and in their engagement with knowledge and understanding of shapes and their properties. A higher level of sophisticated reasoning is evidenced when students are able to use an inclusive classification of shapes according to their properties and the design experiment has been designed to test whether using the dynamic perpendicular quadrilateral can be effective in developing such an understanding.
When preparing to undertake a Design Experiment the researcher needs to ascertain the students' prior learning and to form a conjecture about the outcome of the experiment in terms of learning gains (Brown, 1992, Cobb and Gravemeijer, 2008). The experiment should then serve to test these conjectures and study how students' reasoning develops over the intervention (Cobb et al, 2003).	The students in the study showed through questioning that they were able to recognise shapes and list their properties indicating reasoning at van Hiele level two. My conjecture was that, working through the task using the dynamic perpendicular quadrilateral, students could begin to develop a more in depth understanding of shape properties and an inclusive classification of shapes according to their properties.

Questions should be designed to probe students' understanding and provide insight into students developing understanding throughout the intervention (Brown, 1992).	Designing questions which probed understanding and also prompted reasoning has been important to this study. I aimed to use questions to guide the students towards considering a family of shapes without blatantly suggesting the idea to them before they were ready. As Ribbins (2007. P. 208) articulates "the purpose of interviewing is to find out what is in somebody else's mind but not to put things there".
Finally the intervention should be capable of being transported into other educational settings and of contributing to the original educational theory on which the design experiment is based (Middleton et al, 2008).	I have used the task in whole class situations at two stages in the research. The task has had to be modified in order to be used by whole classes instead of pairs but the evidence has shown that it was successful. An important addition to working at the computer task has been class discussion of what students observed.

Many DBR studies described in the literature (e.g. Brown and Campione, 1994, Middleton et al, 2008) are large studies involving multiple researchers and practitioners with multiple classes of students. However large studies often begin as a series of clinical interviews with small numbers of students in order to ascertain their knowledge and understanding of an aspect of mathematics and a potential learning trajectory. This is considered important to give a solid empirical basis to any developing theory and other findings of the research which will eventually underpin the development of the intervention on a larger scale in the classroom (Cobb and Gravemeijer, 2008, Lamberg and Middleton, 2009). However my study is a small one and as such it takes the form of a series of clinical interviews (the clinical interview is described in section 4.3.1) to ascertain the knowledge and understanding of small numbers of students and a potential learning trajectory. This small DBR study constitutes the first part of the sequence of a larger study.

In the case of research with a small grain size, such as this study, the goal is to "develop a psychological model of the process by which students develop a deeper understanding of particular mathematical ideas, together with the types of tasks and teacher practices that can support that learning" (Cobb et al, 2003). This statement provides the aim for my research in ascertaining whether the task involving the dynamic figure can support the development of students' geometrical reasoning.

4.1.2 The stages in the process

Using the recommendations of Cobb and Gravemeijer (2008) a three- stage process is applied to a design experiment:

- Decide on and make clear the instructional goals (what I intend the students will learn)
- Document the instructional starting points (prior knowledge of students and theories of learning)
- Propose a learning trajectory which will describe how and what I hope the students will learn (involving instructional sequences and resources)

At this point a series of iterative design cycles take place. In this study there are five iterations (including the pilot study) and for each iteration, the experimental design is revised in the light of emerging theory, further trials are conducted, data is analysed and adds to or revises theory on students' learning.

Finally a retrospective analysis is carried out which looks over all the data collected across the design cycles and puts it in a broad theoretical context. A *domain specific theory* is developed which provides a rationale for the outcome of the research in terms of instructional sequences and resources designed to enhance student learning (Cobb and Gravemeijer, 2008). It is also important to keep a log of ongoing analyses and design decisions to help when conducting the retrospective analysis (ibid).

4.2 Issues of validity and reliability in Design Based research

In DBR interventions need to be correctly aligned with theory in order for the research to have validity. Emerging theories should provide impact in the specific context and also have relevance in the larger educational context, adding to what we know of teaching and learning (Barab and Squire, 2004). This is important in order to provide evidence for the validity of the theory (ibid). Validity is important if research is to have any worth as a body of work (Cohen et al, 2003). In qualitative research such as DBR, validity is approached through the objectiveness of the researcher, and the care taken with the design of the study (ibid). There are a number of types of validity which we must strive to achieve, in particular:

4.2.1 Construct validity

Construct validity is concerned with how the experiments are designed to be suitable for the study, careful storage and use of data, and the use of the evidence the data provides in establishing a chain of evidence. Does my research instrument (the design experiment) do a good job of testing the theory or hypothesis I wish to test? To establish construct validity I need to know that my understanding of the theory is grounded in the supporting literature. In this case I have studied the literature on geometrical reasoning, and the cognitive aspects of using the drag mode in dynamic geometry environments. This provides the foundation for the explanation of the results and development of theory which arises from my research.

In DBR the theoretical underpinnings of the design experiment are paramount. Middleton et al (2008) make it clear that the design experiment and the theory which underpins it are intimately connected. It is important that emerging theory is tested in subsequent iterations of the study, in order that the theory should be rigorous (Cobb et al, 2003). For this process to be rigorous I need to research the structure and theoretical underpinning of the educational intervention and to present these very clearly and explicitly (Middleton et al, 2008). I have attempted to do this in the chapters which describe the iterations and by maintaining a common structure in the reports.

4.2.2 Internal validity

Internal validity is concerned with using the results from the data to make a case for the findings which emerge from the study. If internal validity is present then the data supports any findings or explanations being made by the researcher (Cohen et al, 2003). For an explanatory or causal study, such as this one, this involves seeking to establish causal relationships, for example; how does dragging and measuring support students' reasoning?

As the researcher I was involved with the intervention and am one of the many factors in the experiment. Since I am involved in the undertaking of the intervention I need to try to minimise subjectivity and to recognise if I have a bias towards any particular interpretation of the findings from the study. For example it would be tempting to select samples of data which support the researcher's theory or hypothesis (Brown 1992). Cobb and Gravemeijer (2008) argue that, when the data in a design experiment is used to make claims and inferences that these claims must be clearly articulated and documented. This meant I needed to check emerging findings against earlier analyses and test them in later iterations.

4.2.3 External validity

External validity refers to how well the findings of the study could be replicated and observed in other situations than this study (Cohen et al, 2003). If another researcher were to carry out the experiment using the same task with the same computer files and with student participants of the same age then external validity would be indicated if they were to find the same results. This means that it is important that I describe how I carried out the experiment so that someone else could attempt to replicate this study. In each chapter describing the iterations, I have therefore included a section entitled **process for iteration ...** which sets out the rationale and design decisions, and describes the task and the computer file and the context in which the iteration took place. External validity can be problematic for DBR studies. DBR takes place in complex learning situations such as the classroom and it is impossible to control one factor without affecting the others (Barab and Squire, 2004). Thus it is generally not possible for a Design Experiment to be exactly replicated and it is vital that the researcher provides a complete description of the context of the intervention and a rational for the modifications (Barab and Squire, 2004, Hoadley, 2004). However Cobb and Gravemeijer (2008) argue that other researchers should be able to use the findings of a design experiment and to adjust it for use in other contexts as long as the domain specific theory and learning trajectories can be used.

4.2.4 Reliability

Reliability relates to consistency. If I repeated the study using the same experimental design would this produce the same results? This may be addressed by undertaking the research sessions with several pairs of students. In this study I worked with twelve pairs of students (either boys or girls) in two different schools and a whole class. I have included an overview of the iterations at the end of this chapter with a description of the schools which the students attended.

Triangulation can provide for a greater degree of reliability. One form of triangulation relates to the use of more than one method or source of data collection in order to provide evidence for emerging findings (Thomas, 2009). In this study there is mainly one source of data gathering which is the dialogue and on screen recording from the sessions with pairs of students but there is also data from a series of lessons with one mathematics class working with the task, which includes, written work, dialogue, on-screen recording and posters made by the students of what they had learned in the lessons.

Another form of triangulation involves the use of more than one theoretical framework when analysing the data. In this study I will use two theoretical frameworks: Van Hiele (1986) and Duval's theory of cognitive apprehensions (Duval, 1995) which will give me two different viewpoints on the data. A further method of triangulation involves having more than one person involved in interpretation and analysis of the data (Thomas, 2009) and whilst I have been the main analyst, the work has also been discussed with my main supervisor and second supervisor. My work has also been presented at national and international conferences and I have taken account of the feedback. For example, the idea of modifying the figure so that the bars were oriented at an angle came after a discussion with an experienced researcher regarding students dragging the vertical and horizontal bars whilst maintaining symmetry.

4.3 The Practicalities

The research took place in two different schools. School A, whose students I worked with from June 2009 until June 2011 is situated in a large village ten miles outside a city in the English East Midlands. Its intake is mainly white British from skilled working class / lower middle class backgrounds. I approached this school because I had a contact in the mathematics department and we had both been members of a local authority working group on ICT in mathematics teaching and learning. This contact agreed to organise for me to work with pairs of students. The students were chosen by the contact teacher who told me that he asked students who had worked really well in mathematics lessons during the year. Apparently working with me was seen as a treat! I asked him to choose students who would be of average attainment and who would be happy to work with an adult they did not know.

School B, whose students I worked with from June 2011 to June 2013 is situated in a suburb of the same city in the East Midlands. Its intake is a mixture of children from White British and British Asian (mostly second or third generation) families in a 60% to 40% ratio and from skilled working class / lower middle class backgrounds. I approached this school because I used to work there as the Head of Mathematics and I continue to be on good terms with the present Head of Mathematics who was my contact. The students were chosen by the contact teacher as being of average attainment and who would happily work with an adult they had not met before.

Ethical considerations, including how I obtained consent to work with these students is included in section 4.4. I worked with equal numbers of boys and girls. In school B I worked with an equal number of students from White British background and British Asian background. There was no intention to compare groups, simply to show that the findings were consistent for all students.

4.3.1 Working with pairs of students

The sessions could be described as semi structured clinical interviews which were originally developed by Piaget in the 1920s (Ginsburg, 1997). Clinical interviews generally begin by the researcher giving the student subjects an open ended task to work with. Further questioning of students is contingent on their responses to the task and later questions.

> "Researchers in mathematics education ask questions, get answers and then engage in attempts to analyse these answers"
> (Zazkis and Hazzan, 1999. p. 429).

The aim is to encourage the students to reflect on their thoughts and to explain how they worked something out and why they may have used a particular process. Since the clinical interview relies on discussion to shed light on students' conceptualisation (in mathematics) it does rely on students being able to describe their thought processes which some may find difficult. Hence the researcher must choose questions carefully to be able to probe students' understanding in a way that helps them to be able to articulate this. The clinical interview technique is defined by flexibility and the asking of questions contingent on student subject responses but even so some variables can be kept constant and a degree of standardisation is possible (Ginsburg, 1997). The beginning task and the focus on certain aspects of the task and the mathematical content can provide some consistency in each experimental session.
The sessions took the form of task-based interviews, between pairs of students (two girls or two boys) and myself. Task-based interviews entail two or more students working together on a mathematical task with the possibility that they may discuss the problem together and make further progress than each might have made on their own (Evens and Houssart, 2007). However this method ascertains the knowledge and understanding of the pair or group rather than that of individuals.

The sessions took place in a small quiet room in the school and lasted about forty minutes. The students sat in front of the computer and one of them had control of the mouse. Halfway through the session the students swapped positions. I sat on a chair behind them. I chose to work with pairs of students for several reasons. The students had not met me previously and it seemed kinder to let them work with a friend rather than being on their own with a strange adult. I could also encourage them to discuss with each other what they were observing on the screen and why this might be. Some pairs of students were good at this and bounced ideas off each other. There were other pairs of students when the student in charge of the computer mouse would respond to my questions and the other just seemed to listen. However the main value of working with pairs of students is that I was able to study in depth their responses to the dynamic figure and the questions I asked them. The recordings of the on-screen activity were vital to ascertain exactly what the students were doing, which was sometimes only observed when I later played back the recordings.

Whilst trying not to impose on the way the students worked on the task, I nevertheless will have influenced the course of the sessions. This is inevitable because the students had to be introduced to the software in order to carry out the task and there was also a need to ensure they reflected and focused on the desired mathematical learning as Hoyles and Noss (1992) discuss. The fact that I as the researcher was very involved in the implementation of the task will obviously impact on the validity (Barab and Squire, 2004). However I could not remain aloof to the teaching experiment and my interventions can be useful to develop further ideas and to spot small scale theory emerging from the findings if done carefully. An objective look at the transcriptions of the audio data may provide a check as to whether the interventions were leading the students down a particular track rather than allowing them to investigate freely.

In the final iteration of the study I was able to work on the task with a full class of 13 year old students and to record some of the dialogue and photograph examples of students' work.

4.3.2 Data collection and analysis

A video of the on screen activity and dialogue between the students and myself was recorded using image capture software (Camtasia ™). After the session this was imported into Transana ™ software, (Faasnacht and Woods, 2010) which is a transcription and video tool (see figure 4.1 for a screen shot of the Transana window).

Transana allowed me to play the video (with sound) and to type the transcript of the video straight into the text box. I could then add time codes throughout the text which linked it to the video so that the transcription of the dialogue rolled through concurrently with the video and sound recording. Another facility of Transana is that video clips can be saved and coded under themes which are displayed in the visualisation box.

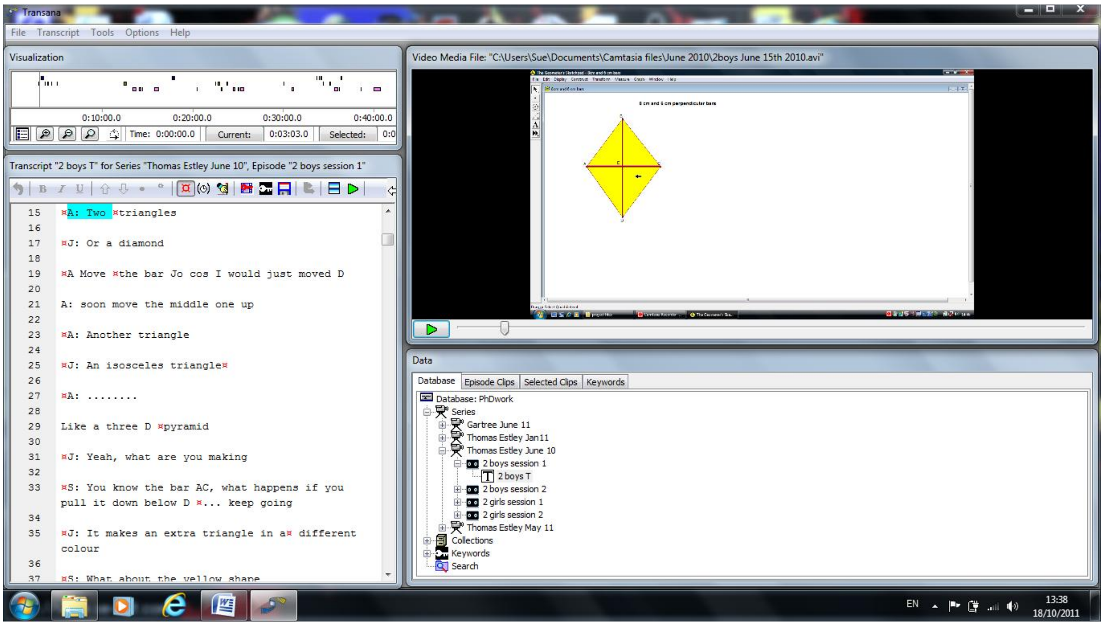


Figure 4.1 The Transana screen shows video, transcript, data and visualisation windows

Once I had completed the transcription in the Transana window, I copied it into a word processing document. I then played the video clip over and over with the aim of describing the on-screen activity concurrent with the dialogue. The on-screen activity generally consisted of students dragging figures on the screen, pointing to objects with the cursor, or clicking on tools and menus in the software. After this I wrote a narrative to combine the dialogue and the on-screen activity. The three different accounts of the sessions were written into three columns of the document so that they describe the same event as can be seen in figure 4.2.

108. A: The bar AC		
109. S: OK then, do you want to have a go? He's been very careful hasn't he? Can you describe to me how you decided to drag that? What have you been able to do?	9.49-9.56 DMS He drags bar AC up so that it continues to be perpendicularly bisected by bar BD 9.59-10.07 RD He drags AC some more to try and get the kite looking better	
110. A: I was watching the measurements on the side	10.15-10.17 GD He drags the constructed mid point of bar AC onto bar BD	
111. S: What did you just do there?		
112. A: I just put the mid point, the middle, on there	The cursor points at the constructed mid point of AC and then points along bar AC either side of the mid pint.	The boys have realised that the constructed mid point of AC needs to lie on bar BD in order to get the best kite.
113. S: OK, so that's really good isn't it. Could you make a kite in another position?		
114. A: Yeah		

Figure 4.2 An excerpt from a description of a recording from one of the sessions which includes dialogue, on screen activity and narrative.

This whole process entailed listening to and watching the session recordings several times. It was an important but difficult decision to make regarding how and what I would code from the recordings. After discussion with my supervisor I decided to code episodes where the on-screen activity and dialogue appeared to address some aspect or aspects of students' understanding about the geometric properties of the figure, known as a *natural unit of meaning* (Cohen et al, 2003). It became clear from repeatedly re-playing the recordings that the on-screen activity was as important as the dialogue for shedding light on students' conceptions of geometrical figures, particularly the dragging strategies which the students were observed to use.

Data from interviews, including clinical interviews, is mainly qualitative and as such the analysis of the data is based on interpretation and reflection. Cohen et al (2003, p. 282) give four generalised stages of analysis:

- Generating natural units of meaning *(the episodes in my study)*
- Classifying, categorising and ordering those units of meaning *(deciding on sub - themes which describe the episodes, grouping these sub-themes into larger themes)*
- Structuring narratives to describe the interview contents *(a description of the on-screen activity, students' conceptualisation of the dynamic figure under dragging)*

- Interpreting the interview data *(attempting to see the overall picture of how the students conceptualise shapes and their properties in the DGS environment).*

Analyses of qualitative data often begin with coding units of data. However, the units of analysis in my study are often short rich sections of on-screen activity and dialogue which indicate a theme or aspect of 'seeing' mathematical concepts. As such I believe that breaking them down into smaller units which can be coded would render them more opaque for the purposes of analysis rather than shedding light into students' mathematical understandings which can be ascertained from the episodes. Hence I have stayed with coding episodes according to themes. As these emerged during the iterations, I will describe the themes as they developed in the chapters on each iteration.

Finally, it is important to keep in mind that carrying out the data analysis in a design based experiment has to be done carefully as Brown (1992) noted. There are so many different ways that the data could be read. It is necessary to do this objectively and the researcher must ensure that all data is examined with equal weighting. It would be detrimental to the study if only the data which supports my preconceived ideas and hopes were to be presented and analysed. All the data is safely stored and it is available for future analysis after the end of the study.

4.4 Ethical considerations

When undertaking research it is necessary to consider the ethical implications and especially in educational research where the participants are often children, to whom adults (including researchers) have a duty of care and responsibility. The British Educational Research Association (BERA) set out guidelines with the aim:

> "to enable educational researchers to weigh up all aspects of the process of conducting educational research within any given context" and "to reach an ethically acceptable position in which their actions are considered justifiable and sound" .
>
> (BERA, 2011, p.4)

Sitting within the BERA guidelines (2011) my own university has developed a set of principles for undertaking research which are listed in their Research Ethics booklet. In undertaking this research I have worked within these guidelines. I sought and was granted ethical approval, to carry out the research. In synthesising these two sets of guidelines I refer to the methodology for ethical analysis proposed by Stutchbury and Fox (2009) which provides a list of considerations under four main headings (external, consequential / utilitarian, deontological and relational / individual). The authors claim that the framework allows the researcher to consider the ethical implications of their research in a logical structured manner.

4.4.1 External issues

The first considerations within this section refer to the awareness by the researcher of the norms and expectations of the institution where the research is carried out (the school in this case). Further considerations in this section address legal requirements of adults working with children and a consideration of risk.

Before undertaking my current role as lecturer and researcher at a university I had been a classroom teacher. As a previous head of a mathematics department I was considerate of the needs of the mathematics departments where I was undertaking the research, to treat the personnel respectfully and to ensure that my activity did not add significantly to their workload.

UK schools are cautious when allowing adults to have access to students and most insist on such persons holding a Criminal Records Bureau (CRB) disclosure form which I already hold as part of the requirements for my role in Initial Teacher Education. Before I began working with students in School A I was asked to attend a meeting with the school's safety officer to show them my enhanced CRB disclosure form and to talk with the safety officer about safeguarding matters. School B did not request such explicit detail but they knew me as someone who had taught there for several years previously, and has been head of the mathematics department.

4.4.2 Consequential / utilitarian issues

This section addresses potential benefits for the participants, the mathematics department and the school, benefits for society and for me as the researcher and considers the need to avoid causing harm.

The benefits to the individual participants (the students who worked with the dynamic figure), was to experience a novel way of working with geometric shapes in a computer based environment. The pairs of students had the attention of one teacher to two students and the chance to work with me was a treat the contact teacher offered to students who had worked hard during the year! The students in both schools seemed to enjoy working on the task as evidenced by their engagement with the tasks. On the other hand the rest of the students in their mathematics class did not this opportunity, thus privileging the participating students. However, if the research can be shown to have a positive impact on how geometric shapes are taught in school then more students could potentially gain from the results of the research in the future.

In iteration two I worked with four whole classes of students prior to recording pairs working with the dynamic figure; two classes in school A and two classes in school B. Part of the reason for working with whole classes was to ascertain whether the task could be modified so that it could be used in the classroom, where most teaching happens. I did not record data from the class lessons but used the opportunity for the pairs of students to work with the upright figure before meeting the figure in different orientations when I recorded them.

In iteration four I collected data from a class of thirty-one students in school B while they worked for three lessons on the modified task. Again, it could be argued that some classes have been privileged over others but the response has to be that if the research is valuable then it will ultimately benefit many more students. Since the whole class lessons were also attended by the regular class teachers (and in one case by a student teacher) there is benefit in that these teachers have observed the use of the task, in the computer environment and can elect to use or modify it themselves. In most cases the teachers had not used DGS in lessons before and so were interested to observe it being used.

As the researcher, benefits include the collection of data from which I hope to produce academic papers and a thesis. I have learned about how students reason geometrically.
The study does not include sensitive issues in the personal sense but I was aware of the need to develop students' confidence when working with mathematics and not to undermine it. For each iteration, I took the role of a teacher / researcher. As an adult this put me in a

position of authority over the students and it was important that I not abuse this position. The students clearly cast me in the role of a teacher and I was aware that they might give me the answers they thought I wanted, so created an environment where they would feel comfortable to say what they thought.

4.4.3 Deontological issues (e.g. truth, honesty, fairness)

This section is concerned with how the research is conducted while ensuring that the researcher is open and honest about the ways in which the research will impact on the participants and then on how the findings will be used to benefit the participants and wider society. Partly, these issues overlap with the next section on relational issues where I have indicated how I informed the students about the research process as it affected them and how I sought their permission and that of their parents / carers.

In the sessions I always worked with pairs of students, or whole classes because I did not want the students to be put in a situation where they might feel uncomfortable in the presence of an unknown adult without the reassurance of having at least one of their peers with them. The contact teachers always made sure that they chose pairs of students who knew each other well and were used to working together. Working with two students, I ensured they each had equal time on the computer. It was also useful to the study when they discussed the task together as I could find out more about their reasoning.

In the sessions I recorded the dialogue and the on-screen activity only and so there was no video data of the students. This may have meant that parents / carers worried less about the data since their son / daughter could not easily be identified as participants. I have stored the data on a computer which needs code access, and the students' names have been changed so that the data is anonymised. The data has been backed up to avoid the risk of losing it.

In order to disseminate the results of my findings, alongside writing my thesis based on the findings I intend to write two articles for academic journals with an international reputation. This was made clear to the school and to the student participants at the time when I sought permission to work with students and to record their work.

4.4.4 Relational / individual issues

This section addresses collaborating with key people in the research, respecting all persons involved and earning their trust, and ensuring validity and reliability in the research

When I planned this study I approached two schools to ask if they would let me work with their students. I was fortunate in having a key contact in each school which made it easier as a level of trust was already established. Having ascertained that these key persons would be happy for me to come into their departments and work with students I then gained the permission of the head teachers of the schools by writing to them and explaining the goals of the research (letter included in appendix 5).

I gained informed consent from potential participants. Informed consent relates to the study participants knowing what is involved in the research such that they can make an informed choice over whether or not to take part in the study. In language they can understand participants need to know the reason for the study and how it will be carried out, information about confidentiality of data and that their contribution will be anonymous, (Thomas, 2009). As my research was carried out with children aged 12-13 years I also obtained the consent of their parents. The students participated with the understanding that they could withdraw this permission at any time. The letters to the children and their parents/carers are included in appendix 5

Confidentiality was ensured by changing the names of the students and omitting the names of the schools. Validity and reliability have been addressed in section 4.2.

4.5 Overview of iterations 0 to 4

In this chapter I have addressed the Design Based research methodology and how this informed my research. I have considered issues of validity and reliability and ethical concerns. Table 4.2 provides an overview of the next section of the thesis.

Table 4.2 An overview of the iterations

		Participating students (pseudonyms)	**Tasks based on the dynamic perpendicular quadrilateral**	**Themes which emerged from analysis of the recordings**	**Conclusions and forward plans**
Iteration 0	June 2009	Mike and Luke Ruth and Rita	Vertical and horizontal bars, Construct drag proof shapes	Dragging strategies	Drop the plan for students to make drag proof shapes. Concentrate on dragging strategies.
Iteration 1	June 2010 – Jan 2011	Tilly and Alice Adam and Jack Colin and Terry Gill and Sara	Vertical and horizontal bars. Students asked to describe relative position of bars inside the shape.	Dragging strategies formally described, symmetry, reviewing of shape properties, rhombus and square. Students do not classify shapes hierarchically.	Change settings on the computer files so that measurements given to nearest tenth of cm and whole degree. Check whether dragging for symmetry is used when the axis of symmetry is not vertical.
Iteration 2	June 2011	Dave and Evan Tara and Ruth Kate and Jane Aftab and Rupen	Whole class lesson with vertical and horizontal bars followed by recorded sessions of pairs working with bars in different orientations	Same dragging strategies used when the bars are oriented at an angle to the vertical. Students appear to turn their head or mentally rotate the figure.	In next iteration give the perpendicular bars oriented away from the vertical straight away. Students identify a discrete number of kites rather than the potentially infinite number. This may explain why they do not perceive the rhombus as a special kite.
Iteration 3	June 2012	Stan and Eric Hemma and Seema	Perpendicular bars oriented at an angle, Introduced to the animated dynamic quadrilateral	In the first session students only meet the perpendicular bars oriented at an angle. They use the same dragging strategies as other students. In the second session students are shown the animation of the dragging maintaining symmetry strategy.	Some complaints about the bars not being the 'right way up'. After being shown the animation students talk about many kites belonging to the same family.
Iteration 4	June 2013	Year 8 class	3 class lessons working on adding constraints and finishing with the animated dynamic quadrilateral	Practical session where students move plastic bars and imagine shapes which could be made after constraints are added to the bar properties. Students work with dynamic quadrilateral. Class watches animation together and discuss. Students make posters of what they have learnt in previous two lessons.	Student talk and discussion proves to be a valuable factor in developing awareness of shapes and inclusive relations. Animation has an effect on some students' perception of an infinite number of kites with the rhombus and isosceles triangles as special cases.

5 Iteration zero; the pilot study

At the beginning of the research I planned to use the dynamic figure only for the purpose of students observing how dragging the bars into different positions would generate specific triangles and quadrilaterals. My intention was that they would use what they had learned to create efficient drag-proof constructions of specific triangles and quadrilaterals. (A drag–proof construction maintains its intended properties no matter how it is dragged on the screen). For example, an efficient way to construct a square in DGS is by rotating a line segment ninety degrees about its mid-point, and then joining the ends of the line segment and its image under rotation. This method used the property that a square's diagonals are of equal length and bisect at ninety degrees.

This pilot study took place in school A with Mike and Luke, and Ruth and Rita.

5.1 Objectives for the pilot study

In the pilot study the main objective was for the participating students to construct 'drag-proof' figures using the Geometers Sketchpad (GSP). In the first session the students would be given the dynamic figure to explore with the expectation that they would notice the position of the diagonals in quadrilaterals and base and height in triangles. In the second session the students would be encouraged to use what they had noticed to construct drag-proof figures starting with a blank screen. The intention was that students would learn about the necessary and sufficient conditions required to construct a specific shape and that by dragging and observing variance and invariance they would learn about inclusive relations between triangles and quadrilaterals. As an example I hoped the students would observe that the rhombus is a special case of the kite through dragging a kite into a rhombus. This would indicate evidence of developing geometrical reasoning at Van Hiele level three since this level includes an appreciation of inclusivity of certain sets of shapes as subsets of others.

5.2 Theoretical Underpinning

This section explores some of the literature on which I based the work in the pilot study.

5.2.1 Previous research into drag-proof constructions

A number of researchers have described studies where students have attempted to construct drag-proof figures and have moved closer to understanding the conceptual aspects of the figures which they have constructed. Some examples are given below.

Healy et al (1994) designed a task using DGS as a vehicle for introducing geometrical constructions. The students, who were 13 years old, were allowed to create a picture of their choice on the screen but this had to be resistant to dragging or 'messing up.' In later sessions the students were asked to draw two dimensional shapes such as rectangles which were resistant to 'messing up'. The students found the concept of 'messing up' to be a meaningful idea and it gave an acceptable form of validation for constructions.

Pratt and Ainley (1997) undertook a study with primary school children in the UK with a particular emphasis on the nature of the task which was given to the children. It is important that a task be carefully designed for it carries the potential for the student to explore and discover mathematical structures and concepts (ibid). The task is accompanied by internal and external resources, which may include a student's understandings and mental picture of (in this case) geometrical shapes and their properties, other people such as peers and teachers and practical resources such as a computer uploaded with DGS. Together this forms a webbing process which can support students in developing mathematical meaning (Noss and Hoyles, 1996, Pratt and Ainley, 1997).

Pratt and Ainley (1997) found that giving students a task to create a drawing kit of shapes for younger children provided a sense of purpose for the students undertaking the task and an appreciation of the utility of the concepts of geometrical construction of the properties of the shapes in the drawing kit.

Jones (2000) studied 12 year old students constructing quadrilaterals with the aim of developing a task which would encourage the students to form a hierarchical classification of quadrilaterals. The tasks were sequenced so as to allow the students to build on their knowledge as they progressed from one task to another and the drag test was introduced as a way for the students to check the geometrical properties of their constructions. Jones noted the importance of socio cultural aspects of the classroom environment to the students' learning such as the careful design of the tasks, teacher input and supportive classroom environment in encouraging the making of conjectures and the development of mathematical explanations.

Gomes and Vergnaud (2004) observed 12 year old students using DGS to construct figures such as the isosceles triangle, particularly noticing the specific geometric relations which were used in the construction of the figure and comparing this to the pencil and paper method of construction. The authors expected that the pencil and paper method which the students learnt was likely to have affected the strategies the students used in DGS. However, they observed that different properties of similar mathematical concepts were used in the DGS constructions than in the pencil and paper constructions. Gomes and Vergnaud (2004) concluded that learning is most effective if students experience learning in a number of different environments.

In summary the studies described above indicate the need for students to be given a reason or purpose to construct drag-proof shapes using DGS and that the property of being drag-proof itself provided a rationale for students to be careful about designing their constructions. Constructing figures in DGS uses a different set of properties of the shapes than those used in paper and pencil constructions.

5.2.2 Construction of drag proof figures

Creating drag proof figures in DGS appears to provide a meaningful task for students working in a Dynamic Geometry environment and can lead to effective learning of geometry and development of mathematical language, both of which are precursors to learning more formal deductive geometry. Even so it is important to be aware of issues which are specific to students working in DGS environments. One issue is the necessity of students appreciating the value to them of constructing drag proof shapes (Pratt and Ainley, 1997). It is important for students to appreciate that dragging is a test of the robustness or validity of a constructed figure in that it allows us to check whether we have programmed the required geometrical properties into the figure (Jones 2000).

5.2.3 Familiarity with the software

Another important issue is how the students become familiar with operating the software and using the commands in the menus. DGS deals with objects differently depending on how they have been constructed. For example objects such as points behave differently depending on their status as basic points, points on objects or constructed points such as intersections or mid-points (Holzl et al, 1994). If a point has been constructed as the mid-point of the line and that line is dragged, the mid-point moves with it (Pratt and Ainley, 1997). Constructed figures in DGS behave according to the hierarchies of dependence of objects which are nested in the sequence of constructions which made them (Holzl et al, 1994, Pratt and Ainley, 1997). Thus the process of learning how to use the software as a tool for learning mathematics (Guin and Trouche, 1999) may be quite complicated in DGS.

5.3 Process for iteration zero

5.3.1 Instructional goals

I wanted to encourage the students in my study to think about constructing the shapes in a different way than had been suggested in previous studies by considering the interior structure of the shape, such as diagonals of a quadrilateral or base and height of a triangle. That gave students a different method of constructing shapes which might allow them more success than other accounts in the literature where students attempted to construct shapes using congruent and perpendicular sides (for example, Straesser (2001). First the students were asked to consider the positions of the diagonals (or base and height) and to do this I alighted on the idea of a toy kite in its typical form and then imagined what would happen if we could move the diagonals (or wooden poles in the toy version of a flying kite). I modelled this situation in a Sketchpad file where I constructed the diagonals (or 'bars' as I named them to the students) so that they would be rigid 8 cm vertical and 6 cm horizontal line segments. In the first session I gave the students the file containing the bars, asked them to drag one bar over the other and complete the shape, then drag the bars to investigate the shapes they could make thereby learning about the properties of the diagonals of quadrilaterals and the base and height of triangles. I did not give the students the completed figure because I wanted them to learn how to use the drawing tools to make shapes. In the second session the students were asked to use what they had observed about the properties of the bars in the first session and start with a blank screen to construct drag proof shapes from the ‘inside out’.

5.3.2 Instructional starting points

The students in the study were in year eight (12-13 years of age) in the English system and were assessed by their mathematics teachers as achieving at level six of the National Curriculum for England and Wales (QCA, 2007) which was current at the time when I undertook the research. Their school worked from the Framework for Teaching Mathematics at Key Stage 3 which states that in year eight students should be able to:

> “Solve geometrical problems using side and angle properties of equilateral, isosceles and right angled triangles and special quadrilaterals, explaining reasoning with diagrams and text and classify quadrilaterals by their geometrical properties.”
>
> (DCSF, 2007)

Having recently taught students of that age and having worked within the National Curriculum and Framework I knew the prior knowledge the students would have. Furthermore I was able to informally assess the students’ knowledge and skills in geometrical reasoning whilst observing them working on the tasks. I am therefore confident that the students could recognise shapes and identify their properties but that they were not able to give a necessary and sufficient definition or to feel completely comfortable with a hierarchical classification of shapes. This indicates reasoning at level two with a readiness to begin to move up to level three with the appropriate learning activity. Van Hiele (1986) considered that students moved up the levels as a result of a learning experience rather than as a natural process of maturation. Therefore it is necessary to design an appropriate task to encourage development of geometrical reasoning.

5.3.3 Learning trajectory

In the first of two sessions the students were presented with a Sketchpad file designed to model the situation of the kite with movable bars. It contained a vertical 8 cm bar (BD) and a horizontal 6 cm bar (AC). The students were asked to place one bar over the other (as shown in figure 5.1), to use the line tool to join the ends of the bars and then to use the Construct menu to fill the resulting shape (ABCD) with colour.

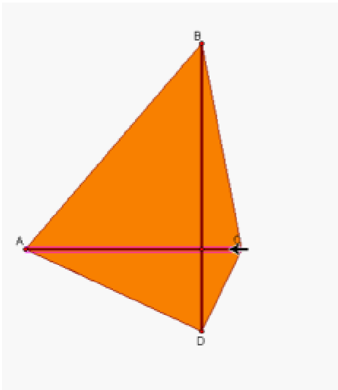


Figure 5.1 Screen shot showing the completed figure

Constructing the interior of the shape made it more visible especially when it was dragged into a concave quadrilateral, which otherwise might have been seen as a triangle with the diagonal providing the base line.

The task had been designed so that the students, who had not used the software before, did not need to learn many of its features. This was important because of the time constraints and the relatively little use of DGS in the UK curriculum. The students were asked to **"drag the bars and investigate what shapes you can make".** When the students made a particular shape, they were asked to give its properties and then to use the Measure menu to check side and angle properties as shown in figure 5.2. They were also asked **"how are the bars positioned with each other?"** to encourage them to observe the properties of the diagonals (in a quadrilateral) and the base and height (for a triangle). I also hoped that the students might link the properties of equal sides and angles with the properties of the diagonals and thus develop their geometrical reasoning towards Van hiele level three. Carefully constructed activities reveal and develop students' intuitive ideas and develop their use of language to represent these ideas (Hoyles and Noss, 1992).

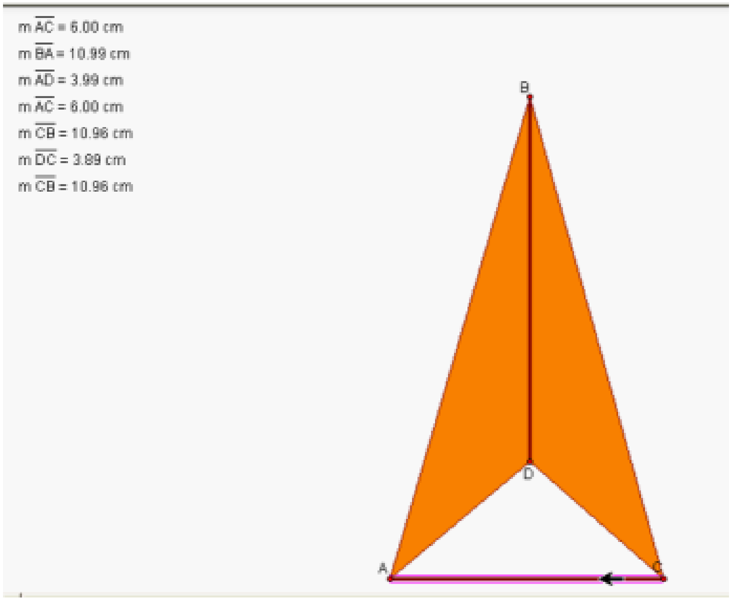


Figure 5.2 Screenshot showing measurements of sides

This task is an example of an open problem which may encourage students to develop meaningful concepts in geometry. Mogetta et al (1999 a) described an open problem as a short statement where students are asked to explore connections between elements of a figure (in this case: “what shapes can you make by dragging the bars to different positions?”). Open problems do not lend themselves to learned procedures; students have to decide for themselves how to explore the problem.

In the second session the students were encouraged to remember the positions of the bars needed to generate any particular shape in order, given a blank screen, to construct bars in those positions and thus build up a shape around them which was ‘drag proof’.

5.4 Results and analysis of session one: working with the dynamic figure

I worked with two pairs of students; Mike and Luke, and Ruth and Rita, in Summer 2009. After constructing the sides and interior of the shape the students would typically drag the figure into a number of different shapes. Usually these were shapes with recognisable names (eg kite, isosceles triangle, rhombus, arrowhead) but some (though not many) irregular shapes were also generated.

Next I suggested that they choose a shape and follow this by asking about the properties of the shape in general and properties of the bars needed to generate the shape. When the students identified side and angle properties, either they asked if there was a way to measure and check or I suggested they find the measurements. (There is a Measures menu in the Geometers Sketchpad and I showed the students how to use it). Having generated a specific shape the expected equal sides and angles would not be exactly equal. In this case the students used very small adjustments of the bars to make those measurements of sides and angles equal. This action was sometimes carried out on the students' initiative and sometimes I suggested they try to get the measurements exact.

When asked how the bars were positioned relative to each other the students were able to give explanations in everyday language eg "in the middle of each other", cutting the other one in half", etc. After they had spent time generating shapes using the 8 cm and 6 cm bars the students spent some time working with equal length bars, with which it is possible to generate a square. Several themes emerged from the recordings of the students' work which are illustrated below.

5.4.1 Initial placing of the bars in an almost symmetrical configuration

When, at the beginning of the session, the students were asked to place one bar over the other, they tended to place the bars in a near symmetrical position as shown in figure 5.3. The cursor can be seen in the screenshot on the left hand side.

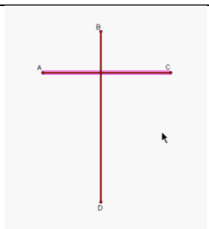
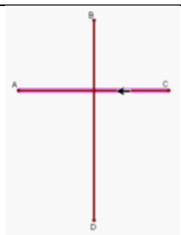
<table>
<tr><td>
Mike and Luke</td><td>
Ruth and Rita</td></tr>
</table>

Figure 5.3 The initial positions in which the students positioned the bars

5.4.2 Exploratory dragging to investigate the shapes which can be generated from the dynamic figure

At first the students dragged in a random fashion to see what shapes they could generate as shown in figure 5.4.

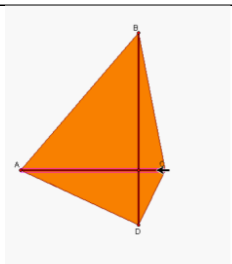
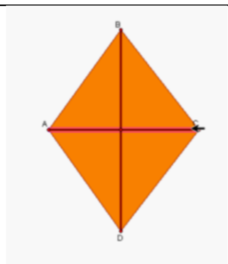
On screen activity	Screenshots	
They dragged the bars to make a kite with horizontal symmetry, an arrowhead kite, an isosceles triangle and a rhombus. This activity usually took place at the beginning of the session when the students were unfamiliar with the dynamic figure.		

Figure 5.4 Mike and Luke generated a number of different shapes

5.4.3 Moving the bars straight into position to generate a desired shape

Once they became familiar with dragging the dynamic figure, which usually took less than five minutes, the students dragged the bars straight away into the position which would generate a desired shape as shown in figure 5.5. The students chose which shape they would make.

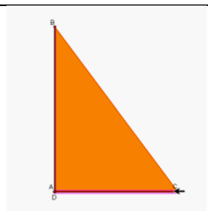
On screen activity	Screenshot
They dragged the bars straight into position to make four different orientations of a right angled triangle. Only one of these right angled triangles is shown in the screenshot.	

Figure 5.5 Mike and Luke dragged the bars to make four right angled triangles.

5.4.4 Fine adjustments to the position of the bars in order to get expected equal measurements to be equal.

After they had made a shape the students checked the measurements of sides and angles. Typically the properties of expected equal sides and angles would not be indicated in the measurements and so the students made fine adjustments to the positions of the bars in order to make the shape more accurately mirror its properties. Two such episodes are shown in figures 5.6 and 5.7.

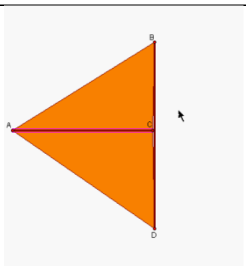
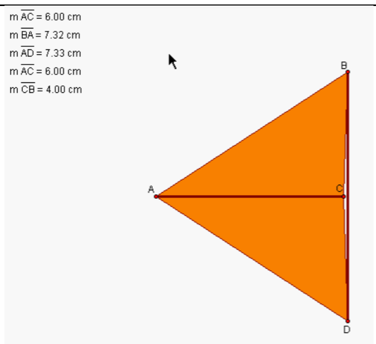
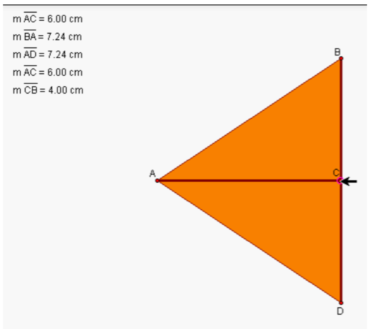
Dialogue	**Screenshots**	**Narrative**
Mike : *It's really quite hard to get them* Luke **:** *No because you need these sides to be the same. That's why it's quite hard to get them?* Susan: *OK* Luke **:** *Is there a measuring tool?* Susan: *There is a measuring tool.* Mike : *There you go there's Measure.* Susan**:** *OK we're going to use Measure to see if you're right.*		Then they make an isosceles triangle with horizontal symmetry
Mike : *This one is slightly longer. I think you have to move one of those. Now try it again.* Susan: *Close* Mike : *What you could do is move this one slightly that way.* Susan: *Yes but look. That bit is AC and that's CB so you're actually measuring two things which aren't a straight line together.* Luke **:** *Ah like that* Susan: *And you've got a gap actually. You might want to move it totally out of the way. Ah, that's better isn't it.* Luke : *And they're actually the same.* Susan: *They are.*	 	Mike tells Luke to make slight adjustments to the positions of the bars. At first the figure is a quadrilateral rather than a triangle. The measures indicate an accurate isosceles triangle. Episodes of this kind of activity lasted up to one minute.

Figure 5.6 Mike and Luke make fine adjustments of the bars to make the shape an accurate isosceles triangle.

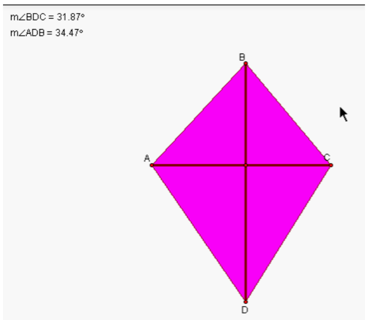
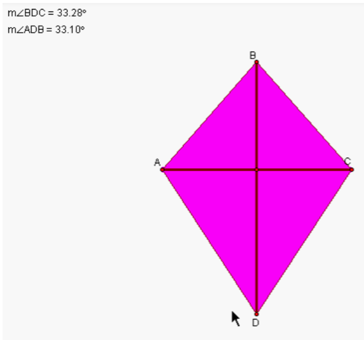
Dialogue	Screenshots	Narrative
Susan: *They're quite close aren't they. Cause after all, you've done that by eye, So do you think you could make it so that they were the same?* Ruth : *Could you just move that one?* Susan: *You could try couldn't you till you get them as close as possible. It might not be absolutely easy to. We're so close there aren't we. There's not much in between them.* Susan: *Once you move one thing the other goes off doesn't it. We might just have to say that that's as near as we can get it. It's close.* Ruth: *It's about there.* Susan: *It's about there isn't it. I mean, there's not much difference between those two angles, is there really.*	  	The girls had generated a shape which was fairly close to being symmetrical. They identified that the kite has a line of symmetry and measured the two angles either side of the vertical bar at the bottom corner of the shape. The girl with the mouse made tiny adjustments to the position of the bars over a 33 second period and got the angles within 0.18 degrees of each other.

Figure 5.7 Ruth and Rita make tiny adjustments to make an accurate kite.

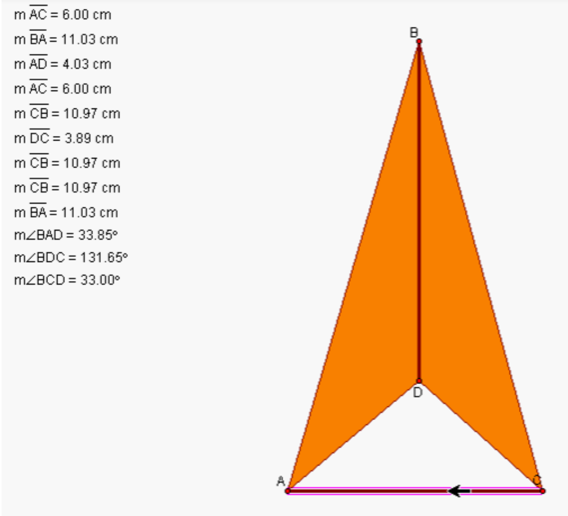
5.4.5 Dragging one bar intentionally through the middle of the other bar

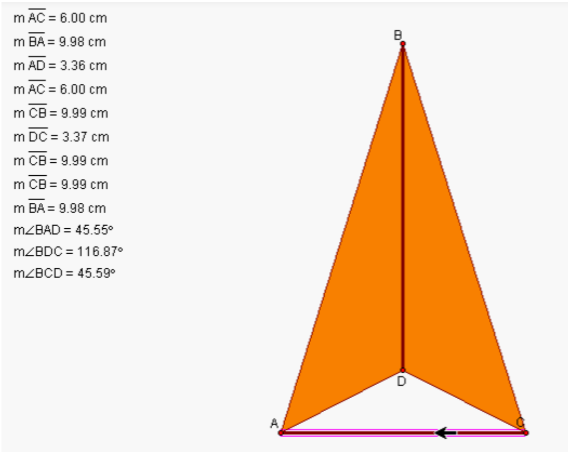
When moving between shapes such as the kite, rhombus and isosceles triangle the students would drag one bar through the centre of the shape as shown in figure 5.8 and 5.9.

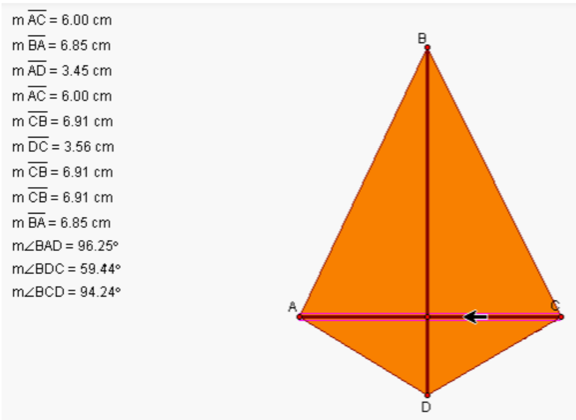
<table>
<tr><th>Description of the measurements under dragging</th><th>Screenshots</th></tr>
<tr><td>
They dragged the bar AC up keeping the bar BD as the line of symmetry.

Episodes of this kind of activity usually lasted just a few seconds. This particular episode was 7 seconds long.

If we look at the measurements of pairs such as AB and BC, they never differ by more than 0.07 cm.

AD and DC never differ by more than 0.14 cm.
Angles BAD and BCD never differ by more than 2.01 degrees.
</td><td>






</td></tr>
</table>

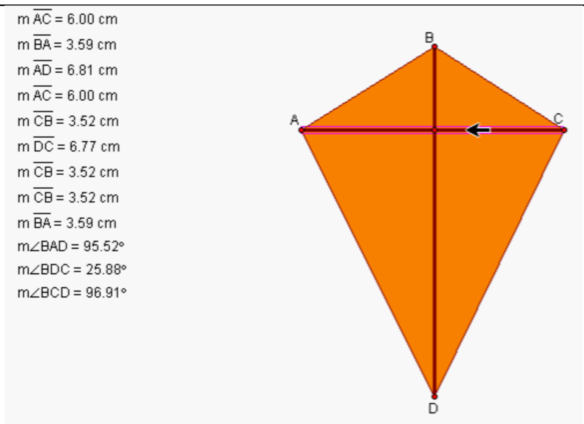
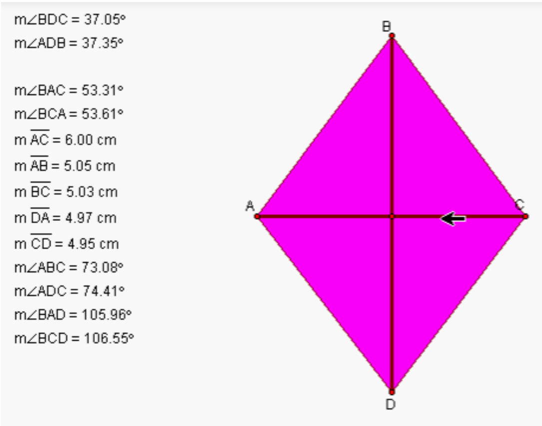
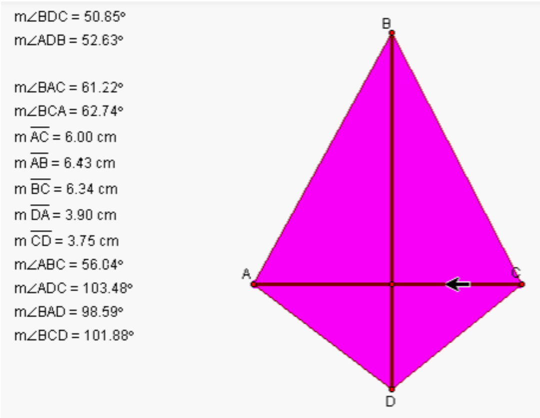


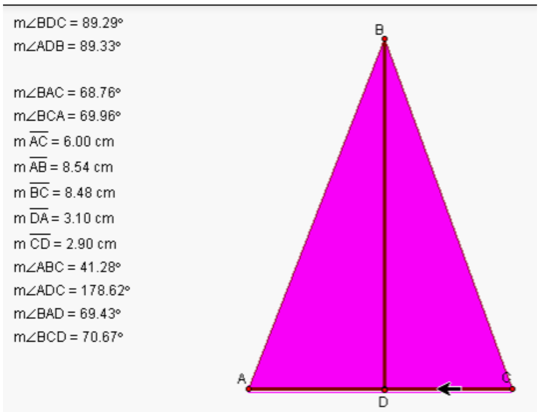
Figure 5.8 Mike and Luke drag the vertical bar through the middle of the figure keeping the shape close to symmetrical.

<table>
<tr><th>Description of the measurements under dragging</th><th>Screenshots</th></tr>
<tr><td>This episode took 7 seconds.

If we look at the measurements of pairs such as AB and BC we can see that they never differ by more than 0.09 cm.
AD and DC never differ by more than 0.2 cm.
Angles BAD and BCD never differ by more than 4 degrees</td><td>







</td></tr>
</table>

Figure 5.9 Ruth and Rita drag the vertical bar through the middle of the figure keeping close symmetry

5.4.6 Splitting the figure into sub figures

Mike and Luke talked about sub figures within the figure as shown in figure 5.10

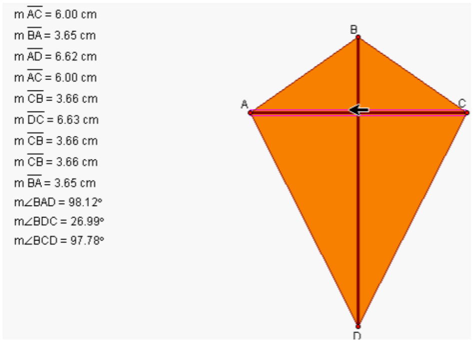
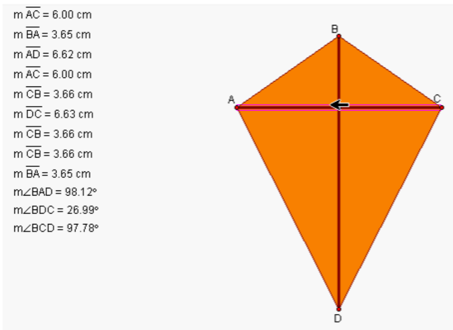
Dialogue	**Screenshots**	**Narrative**
Susan: *Oh well, that's really close isn't it. So what's special about a kite then? So you've got those two sides the same.* Luke : *It's got four right angles* (he is looking at the intersection of the diagonals). Susan: *You have?* Luke : *Cause you've got one there, one there, one there and one there.* Susan: *That's true isn't it?* Mike : *But if you take these away* (pointing to the diagonals) Susan: *He's right though isn't he. It looks like a kite is made up of..,* Mike : *Four sectors* Susan: *Yes, and those four sectors are made up of…* Mike : *Four right angles*		He pointed to each right angle in turn.
Luke : *And it's got an isosceles if you put these two together* (pointing at 2 congruent sectors). Susan: *Oh yes. Is that borne out by the measurements? Have you got the measurement for AD? You've got AD haven't you?* Mike : *A to D equals 6.2* Luke : *C to D, A to C 6.6*		He used the cursor to point to the bottom two triangles

Figure 5.10 Splitting the figure into four right angled triangles or two isosceles triangles

5.4.7 Making connections between the rhombus and the square

Mike and Luke made a shape which was very close to a rhombus. When I pointed out that the sides were very close 'just a little tiny bit either side of five' and asked what shape they had, they replied 'square' and 'parallelogram' respectively. The boy who suggested the square then said that it could not really be the square because 'that isn't the same as that one' (pointing to the bars). The other boy said you could make it into a square by lengthening one bar so that it was the same size as the other bar. These comments suggest that the boys were reasoning geometrically but there was no evidence that they understood why the bars needed to be equal to generate a square.

5.4.8 Discussion of the first sessions

There appeared to be four distinct dragging strategies:

- Exploratory dragging to investigate the shapes which can be generated from the dynamic figure.
- Moving the bars straight into position to generate a desired shape (including the initial dragging of one bar over the other).
- Fine adjustments to the position of the bars in order to get expected equal measurements to be equal.
- Dragging one bar intentionally through the middle of the other bar.

By dragging the bars inside the dynamic figure the students generated isosceles triangles, right angled triangles, kites, squares and rhombuses and some irregular shapes. After having generated a specific shape they would measure sides and angles and drag one of the bars to adjust their shape so that the measurements indicated it to be accurate (or very close to being accurate). Olivero and Robutti (2007) have described *guided measuring* where children use measurements to obtain a particular figure from a generic one. However the evidence from my recordings suggests that the participating students dragged to generate the shape first and then adjusted to make measurements accurate which is slightly different to what Olivero and Robutti described.

The students also mentally split the figure into subfigures and recognised that to generate a square the bars needed to be of equal length. The students knew the common properties of sides and angles for any given shape but they had no understanding of the minimum required properties in order to prove they had the shape. I hoped that the activity of the second session would help the students to develop an understanding of minimum requirements to construct a shape.

5.5. Results and analysis of session 2: constructing drag-proof shapes

In the second session I hoped that the students would be able to use the positions of the diagonals of quadrilaterals or base and height of triangles to construct drag-proof shapes. In this my intention was that they would remember what they had learned about the positions of the bars in each shape they generated in the previous session.

The students were presented with a Geometers Sketchpad file which had nothing on the screen except for the menus and tools. I told the students that I would like them to construct shapes like an isosceles triangle or a square which would remain an isosceles triangle or square even when it was dragged. I encouraged them to remember the positions of the bars for each shape in the first session and this did seem to help them in starting to construct their shapes. I realised before the session that they would need to construct perpendiculars, parallel lines, rotations etc so I made some ‘help’ cards which explained how to use the Construct and Transform menus to make various constructions.

There now follows a description of the constructions which the students made. However, first I need to clarify some of the terms which are used in the description. In this account of the student’s work using the Geometers Sketchpad if they used the Construct or Transform menus to create part of a sketch I write that they constructed it. If they used one of the tools: point, line or circle tool, then I write that they drew that part of the sketch. Laborde (1993) refers to these two ways of creating features on the sketch respectively as a primitive based on geometrical properties and a primitive based on pure drawing. The sketch refers to what they created on the screen and is usually a polygon of some description for the purposes of this project. I have not referred to any difference between the geometrical object being created (such as an isosceles triangle) and its Dynamic Geometry version. Nor did I make any difference between these two when talking with the students participating in the project. However the sketches did behave as DGS sketches do in that they could be changed and manipulated using the dragging mode. They were not single static examples of a type such as sketches created on paper (Olive, 2000).

5.5.1 Ruth and Rita construct an isosceles triangle

The first shape I asked the students to make was the isosceles triangle because I thought it would be the easiest for them to construct. There are a number of ways to construct the isosceles triangle in DGS using its symmetry and the fact that the top corner lies on the perpendicular bisector of the base. I started by asking each pair of students how they would draw an isosceles triangle on paper and then suggested that they try to do something similar on the computer. Both pairs drew the base on paper and then placed the top corner by eye so that it was opposite the mid-point of the base, then they joined up the sides. This seemed like a good method to adapt for use on the computer and mirrored the position of the bars needed to generate the isosceles triangle in the first session.

Ruth and Rita drew a horizontal line segment, highlighted it and constructed the mid-point. (I will refer to line segments as lines from here on and emphasise the occasions when a complete infinite line has been constructed). They drew a vertical line up from the mid-point, using their judgment to keep the line going straight up from the mid-point and then drew the sides of the triangle from the top of the vertical line to the ends of the horizontal line as shown in Figure 5.11.

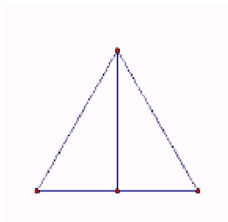


Figure 5.11 Screenshot of Ruth and Rita’s first attempt to construct an isosceles triangle by placing the top corner ‘by eye’.

The shape looked like an isosceles triangle. The girls measured the base angles and found they were equal. When they dragged the top point upwards the lengths of the sides changed. Then they positioned the top point so that the lengths were the same. After this I told them that I wanted them to make an isosceles triangle that could not be dragged out of one. The girls then tried to add things to their sketch which would hold the top point in place. They tried a number of ways to do this but were unable to make the top point stay where they wanted it to. Their attempts are shown in figures 5.12 and 5.13.

First they tried to hold the top vertex down by constructing a perpendicular line through the mid-point of the base over the top of it.

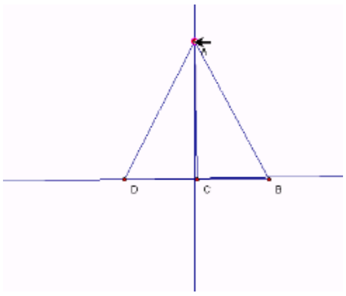


Figure 5.12 Ruth and Rita try to hold the top vertex in place by constructing a perpendicular line over the top of the vertical line they had placed ‘by eye’.

That did not work so they tried to hold the triangle between two parallel lines which were perpendiculars through the ends of the base.

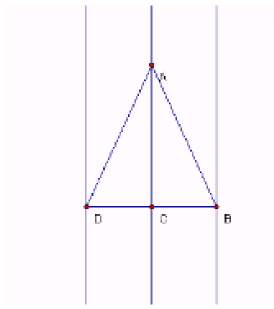


Figure 5.13 Rita and Ruth tried to encase the triangle between 3 lines which they constructed perpendicular to the base through the mid point of the base and the ends of the base.

Eventually the girls gave up trying to hold their triangle in the isosceles position and tried a completely different approach. They drew a vertical line and an oblique line which were joined at their top ends. This is shown in figure 5.14

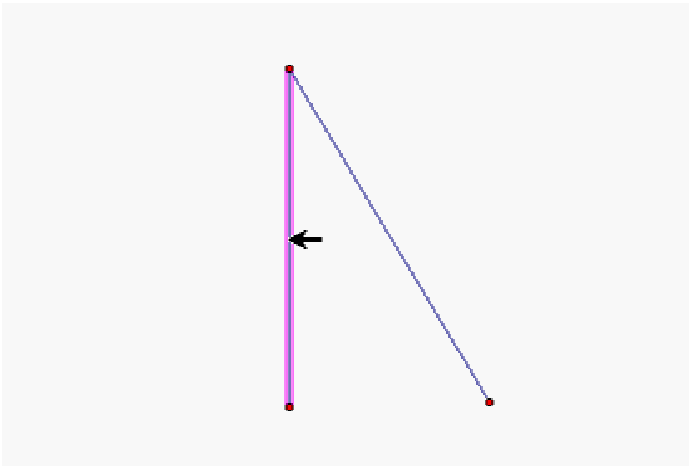


Figure 5.14 A new method to construct an isosceles triangle, part one

After some thought, they marked the vertical line as a mirror. They highlighted the oblique line and reflected it in the vertical line. Then they joined up the ends to make a triangle. The vertical line hung down a little bit below the base of the triangle and this bothered the girls. However they decided to ignore this and checked the base angles by measuring them. They found these were equal. The girls dragged the sketch around and the angles stayed equal. Success!

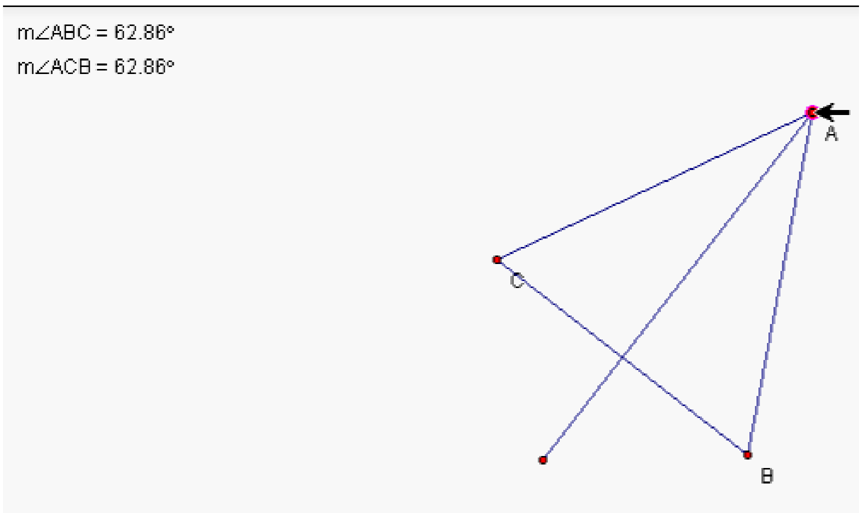
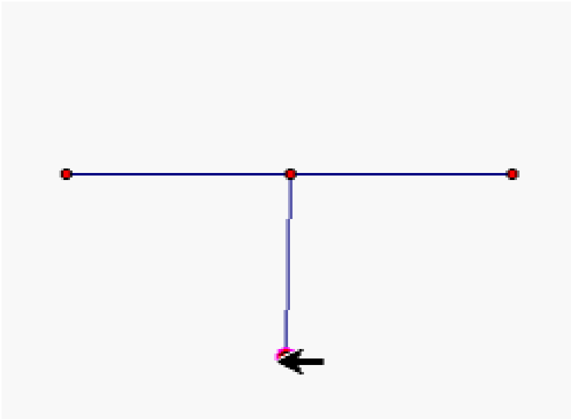
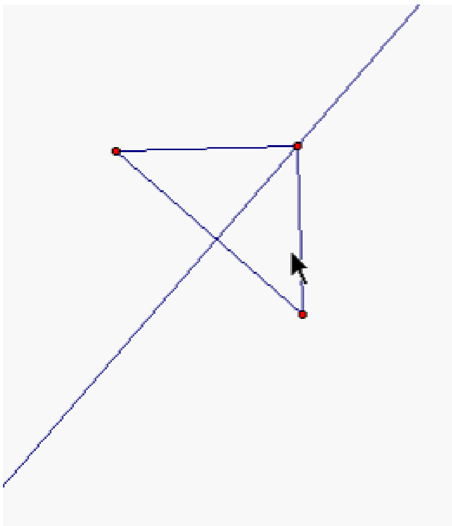


Figure 5.15 Part two, the successful construction of an isosceles triangle

Ruth and Rita tried to make an isosceles triangle by drawing a horizontal line and a vertical line going up from halfway along it, using what they had learned about the position of the base and height from the previous session. They were not successful when using this method in making a drag-proof isosceles triangle because they placed the apex of the triangle on a line which they had *drawn* from the mid-point of the base rather than on the *constructed perpendicular* from the mid-point of the base. The drag mode by its nature disqualifies creating a geometric object by just using the line, point and circle tools (Laborde, 1993). Eventually their successful construction used the symmetry of the isosceles triangle. However I am not convinced that Ruth and Rita understood why this was a successful method.

5.5.2 Mike and Luke construct a square

I suggested to the students that they construct a square by starting with the diagonals in the position as the bars had been put in for the previous session. This is actually an easier method than creating a square from its outer edges. For example, Straesser (2001) described the attempts of some students who were asked to construct a square using Dynamic Geometry. When asked the properties of a square, they responded with congruent sides before right angles. Most of them went on to construct perpendicular lines as part of their attempts at a square and only half of them were able to secure congruent sides in their sketches. I would suggest that this is because the initial questioning had focused their minds on congruent sides and right angles and so they tried to construct squares by starting with those properties. It is more efficient to construct squares using their diagonals (which bisect at right angles and are congruent) when using DGS. Since my students had been encouraged to consider diagonals from their first session that was the method they used when starting with a blank screen. Figure 5.16 shows the steps the boys took.

Description of construction	**Screenshot**
The boys drew a horizontal line and constructed the mid-point. They drew a line underneath the horizontal line going up to the mid-point. They judged how to make it vertical. They tested it by dragging and it did not stay vertical. They measured both lines. Then they copied both lines and deleted the vertical line leaving them with two congruent horizontal lines. They rotated one line and placed it half way along the other to make a T shape. Then they deleted everything.	
Next the boys drew a slanting line and a point not on the line. They constructed a perpendicular to the line from the point. They tried to join up the ends of the line to the perpendicular to make a square, realised this was not successful then deleted everything but the original oblique line. They constructed the mid point of the oblique line and then constructed its perpendicular through the mid point. They tried to join the ends of the oblique line to the perpendicular line by eye.	
Next, Mike and Luke drew an oblique line, constructed its mid-point and a perpendicular line through the mid-point. The boys now had a problem because the perpendicular was an infinite line and they did not know the accurate positions for the vertices of the square. Susan: *What would you really like to do if you wanted to finish off that square?* Mike : *Make the points on there so it's*	

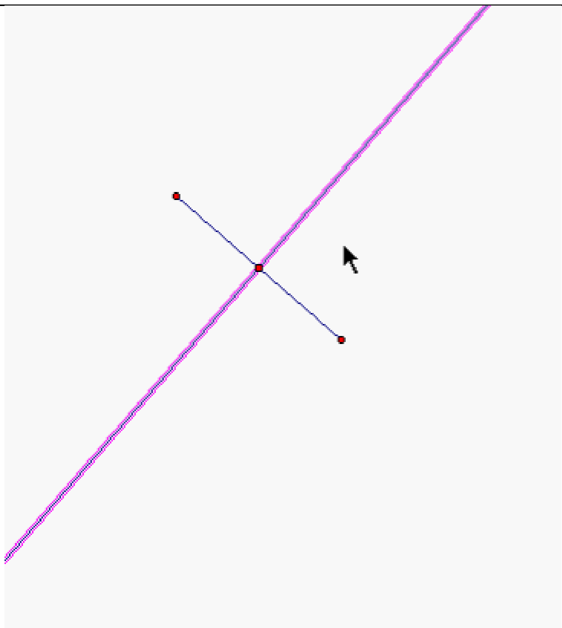
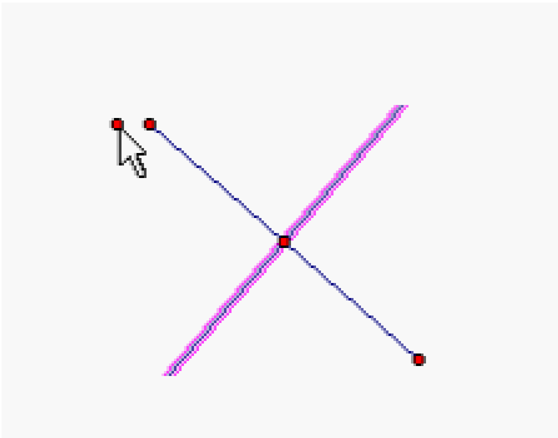
the same length. Susan: *Right, so any ideas how you might actually get that in the right place?* Luke : *Measure that line and then put the measurement on.* Mike : *No but then we're still doing the eye aren't we.* Susan: *If you can do things by eye you can drag it off, you see. So think of another way that you would be able to get that exactly … get these distances exactly the same. There's all sorts of things you could do.*	
Then Mike had a brainwave when he realised that they could find out where the ends of the line segment should be Mike : *Rotate it* Susan: *You could try and rotate it. Now if you did that …* Mike : *Rotate it round the point* Susan: *OK* Mike : *So let's just .. Click on that then Transform* Luke : *Rotate*	

Figure 5.16 Mike and Luke attempt to construct a square and are finally successful by rotating one bar ninety degrees about the mid-point of the other bar

Unfortunately Mike and Luke had highlighted and rotated the lines but not the points on the ends. This meant that there were no points at the ends of the rotated lines which the boys could snap the sides onto. The finished shape was not exactly accurate as a result and the angles were slightly off ninety degrees. The lengths of the sides were equal so the boys had a rhombus. However, on my advice, the boys remade the square in the same way as before but making sure they highlighted and rotated the end points of the lines. This model worked and they were convinced by measuring sides and angles and dragging the shape round to make sure it stayed a square (figure 5.17).

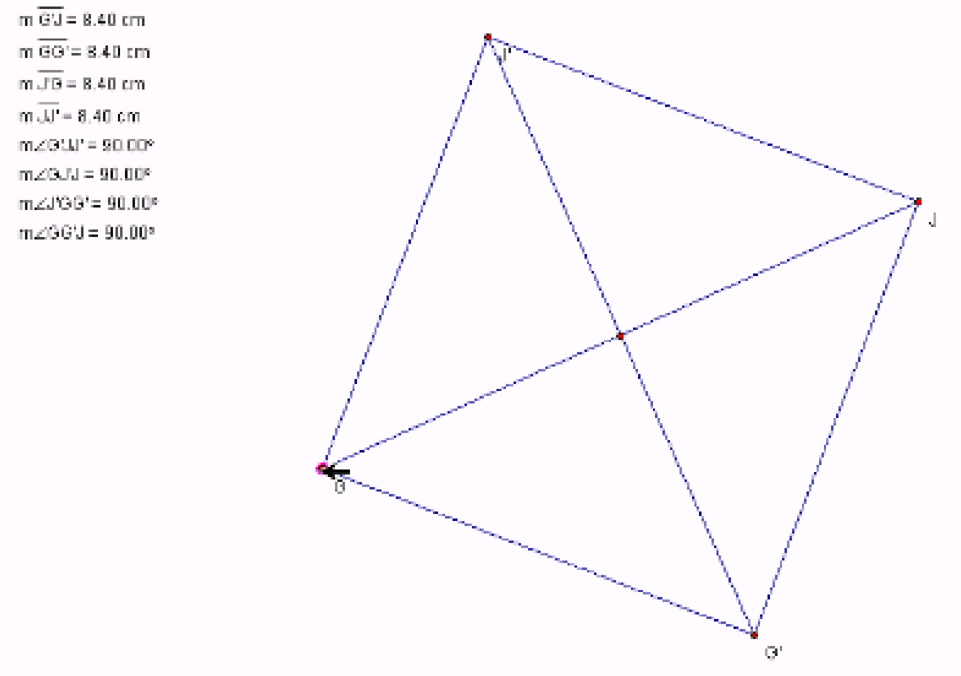


Figure 5.17 The completed square

5.5.3 Discussion of the constructions

The feature of DGS programs is that figures are drawn by a construction process using the primitives (line, point and circle tools) along with the operations (tools in the menus). So the figure behaves according to the way it was constructed and the geometric properties used to define it (Laborde, 1993). The value of using the Dynamic Geometry Software lay in the students having to formalise their methods of construction with regards to the properties of the desired shape.

When I asked the students to prove to me that they really did have a specific shape on the screen they used measuring to validate the properties of their shape, dragging the shape round the screen to look at the object in different orientations (Laborde, 1993) This is an example of ‘validation measuring’ where the students checked that they had indeed produced the shape they were intending to make (Olivero and Robutti, 2007). For the students this seemed to provide sufficient evidence that they had created a specific shape but it is not a formal deductive proof. Using measures in DGS is akin to experimental measuring of a paper object to check its properties. In this way the spatio-graphical way of thinking meets the theoretical considerations of geometry (Olivero and Robutti, 2007).

Although the students had experienced success in constructing some drag-proof shapes I was concerned about the value of this part of the activity because of the necessary investment of time to enable the students to learn how to use the software sufficiently well to learn from it. The problem Mike and Luke had with the square construction due to their not having highlighted the points on the ends of a line segment exemplifies the general difficulties in using DGS when there is not time to become sufficiently familiar with the software. In English classrooms my experience tells me that DGS is likely to be used fairly infrequently and hence students are unlikely to have the time to learn how to use the software well enough to experience real learning gains.

5.6 Conclusion

When working with the dynamic figure the students quickly became familiar with the drag mode and used it to position the bars inside the figure in order to generate different shapes. They were able to generate specific shapes such as the rhombus and isosceles triangle whose name and properties the students clearly already knew from what they had learned in mathematics classes. The students all clearly showed reasoning at Van Hiele level two which requires recognition of shapes as being defined by their conceptual properties (four pairs of equal sides in a rhombus, a line of symmetry in an isosceles triangle, etc).

On analysing the data from session one I had identified several themes which may provide insight into the way the students conceptualised the shapes generated by dragging the dynamic figure. The dragging strategies used were connected to the intentions of the students, e.g. dragging bars straight into position to make a desired shapes and then making fine adjustments so that expected equal side and angle measurements indicated the properties of the shape. The students also talked of the shape as made of smaller triangles. They understood that to make a rhombus into a square they needed equal length bars.

When I reflected on the data from session one I realised that studying how the students dragged the dynamic figure into different shapes could provide some rich data to analyse. Furthermore, my objective of providing support for the students to develop inclusive relations between the rhombus and kite could be achieved through the task using the dynamic figure. I decided to ascertain whether the students would be able to classify the shapes by the relative positions of the two bars leading to a kind of hierarchical classification of the shapes which would suggest a development in their reasoning according to the Van Hiele levels. For example, when one bar in the figure bisects the other the shape generated is a kite. If the bisecting bar is moved so that it is bisected by the other bar then the figure changes into a rhombus.

In summary, I decided to focus on the task using the dynamic figure in subsequent iterations of the study.

6 Iteration one: vertical and horizontal bars

Iteration one took place in June 2010 with Tilly and Alice, and Adam and Jack, and in January 2011 with Colin and Terry and Gill and Sara. The students worked with the dynamic perpendicular quadrilateral which was based on 8cm vertical and 6 cm horizontal bars. In this iteration I focussed on the dragging strategies which the students used when generating different shapes. Emerging themes were:

- The four dragging strategies observed in the pilot study and which will be formally described (and labelled) in this chapter
- The connection between the dragging strategies and cognitive activity in the students, particularly perceptual properties of shape and intuitive use of symmetry
- Preference for a vertical axis of symmetry.

6.1 Objectives for iteration one

By the end of the pilot study I had decided that my research would focus on students working with the task using the dynamic perpendicular quadrilateral as shown in figure 6.1. The figure is based on an 8 cm vertical bar and a 6 cm horizontal bar which can be dragged inside the figure to generate different triangles and quadrilaterals.

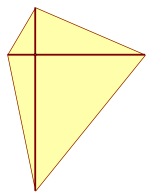


Figure 6.1 the perpendicular quadrilateral

In the pilot study the students had been observed to use four distinct dragging strategies when they worked with the dynamic figure. The objective for this iteration was to ascertain whether different pairs of students would also use these strategies and whether these strategies were the only ones which were observed or whether other strategies might also be used.

- Exploratory dragging to find out which shapes can be made by dragging the dynamic figure.
- Moving the bars straight into position to generate a desired shape.
- Fine adjustments to the position of the bars in order to get expected equal measurements to be equal.
- Dragging one bar intentionally through the middle of the other bar.

6.2 Theoretical background: Drag mode and dragging strategies

The drag mode acts as a mediator between the perceptual aspects (how students visualise a drawing and also how they perceive it through discussion and through use of artifacts) and theoretical aspects of geometrical figures (Arzarello et al, 2002). Thus dragging supports the movement between thinking at the perceptual level and the theoretical level. There are two main types of cognitive activity supported by dragging; ascending processes describe the movement from the spatio-graphical domain to the theoretical domain while descending processes describe the movement from the theoretical to the spatio-graphical domains (Laborde, 1999, Arzarello et al, 2002, Olivero and Robutti, 2007). Perceptual aspects have a connection with the spatio-graphical domain in that a drawing of a figure (representing the theoretical object) may be perceived by a student who also operates on the drawing using measuring instruments or dynamic computer software.

In chapter three, section 3.1.1, I described dragging modalities identified by Arzarello et al (2002). Here I make connections between some of those dragging modalities (wandering dragging, guided dragging, dummy locus dragging) and the four distinct dragging strategies which the students in this study have been observed to use.

- Exploratory dragging to see what shapes can be generated from the dynamic figure is akin to wandering dragging (WD) and I will refer to it as that from now on.
- Moving the bars straight into position to generate a desired shape is akin to guided dragging and it will henceforth be referred to as guided dragging (GD).

- Making fine adjustments to the position of the bars does not have a connection to the strategies listed by Arzarello et al. However, this dragging strategy is usually carried out with reference to the measures of sides and angles which are displayed on the screen. In this case it is necessary to refer to the work of Olivero and Robutti (2007) which describes strategies which students use when measuring whilst dragging. They described an example of guided measuring as dragging a generic quadrilateral into a parallelogram by looking at the measurements of lines or angles which ought to be equal. However, guided measuring is not exactly the same as making fine adjustments to the bars in the dynamic figure. In this case the students have generated a shape which is already very close to being accurate. They have dragged the bars to refine their positions so that the measurements indicate that the properties of the shape are upheld (or are so close that we might agree to ignore the difference). I have named the strategy where students make fine adjustments to the bars as refinement dragging (RD).
- The fourth strategy maintains the property of symmetry of the dynamic figure. Later I will demonstrate from analysis of the recordings how students appeared to use a sense of symmetry when keeping one bar in the middle of the figure. Hence I have named this strategy dragging maintaining symmetry (DMS). When using this strategy the students were observed to drag the vertical bar through the middle of the shape so that it was the perpendicular bisector of the other bar. This strategy could be seen to be an example of dummy locus dragging because an object (a bar) is being dragged along a path or locus. However, Arzarello et al (2002) maintain that students do not usually realise that they are dragging along a locus and this is usually because the locus and the property are not obviously connected. In the dynamic figure dragging one bar along the perpendicular bisector of the other bar, results in the locus and the property being almost the same.

Baccaglini-Frank and Mariotti (2010) provided a description of maintaining dragging see chapter three, section 3.1.1) which seems to be a closer fit to the fourth category observed with the dynamic figure. *Maintaining dragging* is the dragging of an object so that the DGS figure maintains a certain property and is undertaken by the student with a specific intention to maintain that property. Hence there is more of the student's intention in maintaining dragging. It seems therefore that the fourth category has more in common with maintaining dragging but there are differences too. In maintaining dragging the property and the locus of dragging are not necessarily the same whereas they are in DMS.

6.3 Process for iteration one

6.3.1 Instructional goals

As in iteration zero the students were given the GSP file containing the 8 cm vertical bar and the 6 cm horizontal bar and were asked to drag one bar over the other. I then instructed the students on how to join the ends of the bars and construct the interior of the shape thus filling it with colour. The task was to drag the bars to see what shapes they could make, identify the properties of the shape and then use the Measure menu to check side and angle properties. Another file contained equal length vertical and horizontal bars which the students used at the end of the session. With this file it is possible to make the same kinds of shapes as with unequal length bars, but the shape generated when the bars bisect each other is a square rather than a rhombus.

6.3.2 Instructional starting points.

The first two pairs of students who participated in iteration one were in year eight in the English school system and were assessed by their mathematics teachers to be of average attainment and achieving at five/ six of the National Curriculum for England and Wales (QCA, 2007). These students took part in the study in June 2010 and were consequently at the same point of their education as the students in the pilot study. The second two pairs of students, also in year eight and of similar attainment, took part in January 2011 and hence were five months behind in schooling compared to the others. The task had been designed so that the students, who had not used the software before, did not need to learn many of its features. They were asked to drag the bars and investigate what shapes they could make. When the students made a particular shape, they were asked to give their properties and then to check these using measurements of sides and angles. In effect the task which the students undertook in iteration one was the same as that in the first session of the pilot study.

6.3.3 Learning trajectory

In this iteration I set up the task with the objectives that students would:

- Drag the bars to generate different shapes.
- Test the shape properties using the measurements of sides and angles.
- Describe the relative positions of the bars in order to generate the shapes.
- Suggest different sets of bars (lengths and angle between them) required to generate other shapes.

I hoped to observe the students achieve the following:

- Identify the properties of the diagonals needed to generate each shape which would correspond to reasoning at Van Hiele level two.
- Recognise similarities within these properties and use these to devise classifications for groups of shapes, thus developing their reasoning towards Van Hiele level three. An example of this, given that the bars are already perpendicular, is that if one bar is the bisector of the other bar (in student speak this would be ‘if one bar cuts the other in half’) then the generated figure is a kite. If both bars bisect each other then the generated figure is a rhombus and hence the rhombus is a special case of the kite family.

6.4 Results and analysis of the sessions in iteration one

Four pairs of students have undertaken this work. There were common themes throughout each of the sessions although the recordings from June 2010 were more helpful because there was a good dialogue between students and between students and researcher. The students from January 2011 were much quieter and the girls were so quiet that I was unable to transcribe their dialogue, although I was able to observe the episodes of dragging.

6.4.1 Coding the sessions using Transana

After the pilot study I became more systematic in the way in which I coded and analysed the data. After each session the recording was imported into the Transana transcript and analysis software. I transcribed the dialogue and the on-screen activity to create a narrative of the session in a word processed document. I then created clips in Transana of every dragging episode. The clips were coded in Transana according to the dragging strategy employed, and each episode was represented in the visualisation window which can be used to give an overall picture of which strategies are used, when and for how long (see figure 6.3). The original narratives of each recording can be found on the included disk but the dragging episodes for Tilly and Alice's session are included below as an example.

Table 6.2 indicates the use of dragging during the session with timings and a description of the girls' onscreen activity.

Key (using the descriptions of dragging strategies given in section 6.2)

WD Wandering dragging

GD Guided dragging

RD Refinement dragging

DMS Dragging to maintain symmetry

Table 6.1 Dragging strategies used by Tilly and Alice during their session June 2010.

section	**time interval**	**time**	**dragging strategy**	**narrative description**
General 0.00-4.16	00.42-00.48	6 seconds	GD	Drags BD over AC and achieves near symmetry
	3.15-4.10	55 seconds	WD	Students investigated what happened when they dragged the bars
Arrowhead kite 4.16-15.50	4.18-4.22	6 seconds	RD	The students made an arrowhead The students attended to the measurements of the sides which changed as they dragged the bars around. They eventually realised that they needed to drag the vertical bar up through the middle of the shape. They dragged the vertical bar down to an isosceles triangle position then dragged
	10.04-10.27	23 seconds	RD	
	10.34-10.42	8 seconds	RD	
	11.01-12.41	100 seconds	RD	
	12.45-12.47	2 seconds	DMS	
	12.50-12.58	8 seconds	RD	
	13.10-13.20	10 seconds	RD	
	13.20-13.24	4 seconds	DMS	

	13.24-13.45	21 seconds	RD	down and then up, maintaining symmetry.
	13.45-13.50	5 seconds	DMS	
	13.50-13.52	2 seconds	RD	
	15.14-15.17	3 seconds	GD	
	15.28-15.37	9 seconds	GD	
Isosceles triangle 15.50-20.48	15.58-16.01	3 seconds	DMS	They dragged the bars into isosceles triangle position They dragged to adjust the side measurements
	16.41-16.47	6 seconds	RD	
Right angled triangle 20.48-26.11	21.00-21.04	4 seconds	GD	They dragged to make a right angled triangle. They dragged the vertical bar to position the bars more accurately. Then later they dragged to get the angle to be closer to 90 degrees.
	21.17-21.20	3 seconds	RD	
	22.04-22.37	33 seconds	RD	
Rhombus 26.11-37.30	26.30-26.34	4 seconds	WD	The students said “that’s a diamond and that’s a kite” as the vertical bar was moved up and down. First the students got the sides as close to being equal as they could. Later they would try to get all four angles equal but found that it could not be done. However they discovered that they could get two pairs of angles where each pair was close to being equal sizes.
	26.34-26.37	3 seconds	DMS	
	26.37-26.41	4 seconds	GD	
	26.46-26.56	10 seconds	RD	
	26.59-27.09	10 seconds	RD	
	28.07-28.11	4 seconds	RD	
	28.17-28.20	3 seconds	RD	
	28.41-28.54	13 seconds	RD	
	29.00-29.11	11 seconds	RD	
	29.30-29.56	26 seconds	RD	
	32.41-32.48	7 seconds	RD	
	32.53-33.16	23 seconds	RD	
	33.52-33.59	7 seconds	RD	
	34.08-34.35	27 seconds	RD	
Kite 37.30-42.41	37.41-37.47	6 seconds	DMS	They were asked to make a kite and dragged the vertical bar straight down in order to do this. They followed it by refinement dragging to get the required measurements equal.
	37.47-37.50	3 seconds	RD	
	40.02-40.18	16 seconds	RD	
	41.15-41.23	8 seconds	DMS	
	41.23-41.32	9 seconds	RD	

Alongside dragging activity other on screen activities included the cursor pointing to the object that the student was talking about. For example, if the student was talking about equal lengths the cursor would often hover over the two line segments in question. Another screen activity was clicking on points to highlight them so that the Measures tool could be used to generate the measurement.

6.4.2 Visualisation window

Figure 6.3 shows the visualisation window in the Transana file for the Tilly and Alice 2010 recording.

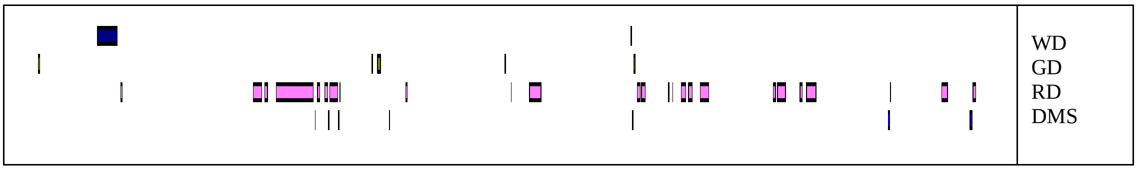


Figure 6.2 The visualisation window showing the dragging episodes for the recording of the session with Tilly and Alice

The top line indicates wandering dragging which occurred twice, at the beginning of the session for 55 seconds when Tilly and Alice were investigating which shapes they could make, and again for 4 seconds when they were investigating the rhombus.

The next line down indicates the episodes of guided dragging which occurred five times throughout the session and lasted from 3 to 9 seconds. Tilly and Alice were deemed to have used guided dragging when they moved the bars into place straightaway in order to generate a specific shape.

The third line shows the episodes of refinement dragging. This was the most common type of dragging activity and individual episodes of refinement dragging lasted for longer than any other type of dragging, from 2 to 99 seconds. However the shorter episodes of refinement dragging usually occurred within a time interval of bursts of refinement dragging so the student with the mouse was likely to be resting between short intervals in a longer session of refinement dragging. Since the students were trying to make the shapes they had generated to be close to accurate it is not surprising that they would spend time using this strategy. Often the student who had control of the mouse would use refinement dragging whilst we were in conversation and I only noticed the use of guided dragging during these conversations when I played back the recording.

The fourth line shows the episodes of dragging maintaining symmetry. There were seven such episodes and they lasted from 2 to 8 seconds. At the beginning I coded most of these episodes as guided dragging and only used the DMS code when the dialogue suggested that the students were deliberately trying to keep one bar in the middle of the other bar. On reflection I decided that the movement itself indicated a desire to keep symmetry a constant. Later in the project I observed the degree of accuracy which was maintained during the dragging maintaining symmetry episodes by looking at the displayed measurements on the screen at intervals during the dragging. They often indicated very fine levels of accuracy (e.g. expected equal measurements being within 0.2 centimetres of each other) and all of the students participating in the study showed at least two episodes of very accurate DMS.

The tables of dragging episodes and visualisation windows for the recordings from Jack and Adam (June 2010) and Colin and Terry (Jan 2011) have been included in appendix 1.1 and 1.2. No patterns were observed in the tables of dragging strategies or the visualisation windows and for this reason I did not produce visualisation windows in later iterations. The use of dragging strategies clearly depends on the nature of the task which in this case is concerned with dragging the bars inside the dynamic figure. However it can be noted that students spend more time using refinement dragging, which they undertake whilst attending to the measurements of sides and angles displayed on the screen. Episodes of guided dragging and dragging maintaining symmetry occur in short bursts and are usually followed by episodes of RD. It seems that students attend to the positions of the bars or the holistic shape of the figure during GD and DMS. Wandering dragging has a different role as it is an exploratory activity which takes place at the beginning of the session or if a new file has been introduced.

6.5 Themes which emerged from the data.

A number of themes emerged from the data. These have been described below and illustrated with examples from the data. Appendices 1.5-1.7 list some interesting episodes from the recordings which can be accessed from the session narratives and recordings included on the accompanying disk.

6.5.1 The use of RD and DMS to generate a specific shape.

There is a particularly interesting episode from 11.01-13.52 where Tilly and Alice made an arrowhead. The section from the visualisation window from this section has been enlarged and it can be seen that there were two short bursts of refinement dragging followed by a long interval of refinement dragging. Next, episodes of dragging maintaining symmetry were interspersed by refinement dragging.

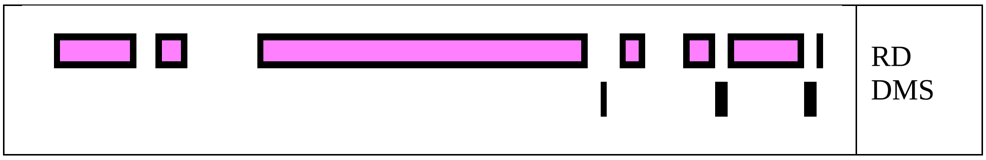
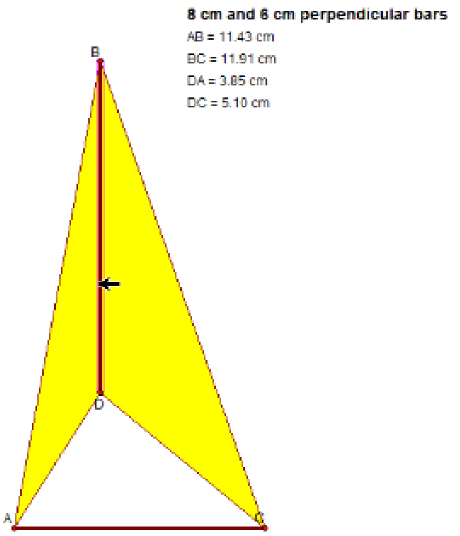


Figure 6.3 A section from the visualisation window from figure 6.3 showing an episode between 11.01-13.52 indicating DMS interspersed with GD

The first excerpt from the narrative (see figure 6.5) is taken from the hundred second time interval of refinement dragging illustrated by the longest purple bar in the visualisation window (figure 6.4). At first Alice, who had control of the mouse, lost sight of symmetry while attending to the Measures of the sides. Tilly gave her instructions to try and get the arrowhead to be more symmetrical.

<table>
<tr><th>Dialogue</th><th>Screenshots</th></tr>
<tr><td>
Alice: I don't think it's, Cos the line is like on a slant .... do you get what I mean. It's a bit...
Tilly: Oh that was right a minute ago. Cause you've gone to that side more. You need to be in the middle and then move up. That's still a bit that side I think
Alice: It still looks wonky to me
Tilly: So move that way a bit, no the other way. Nearly got them two. We're try and aim for D and A first then B and A and BC. If you move that a tiny bit, the other way
Susan: So what are you trying to do, how are you trying to position BD?
Tilly: Yeah we're trying to get BD in the middle of the shape
Alice: A and D and then D and C, more and like the same measurement.
</td><td>


</td></tr>
</table>

Figure 6. Tilly and Alice use refinement dragging to make an arrowhead

The girls appeared to be using symmetry in order to make the shape more accurate. Tilly's comment about getting BD (the vertical bar) in the middle of the shape is followed by Alice's comment that the measurements of AD and DC will be more equal. Following this discussion, the girls decided to drag the vertical bar BD so that its end, point D, touched the mid-point of the horizontal bar. Then they dragged the vertical bar up and down from that position. So they used DMS in short spurts to move the vertical bar down to sit on the horizontal bar, followed by RD to get the bar in the exact middle, then more DMS to move the vertical bar down and then up to the arrowhead position with more RD afterwards. The on screen activity during this episode has been captured in figure 6.6.

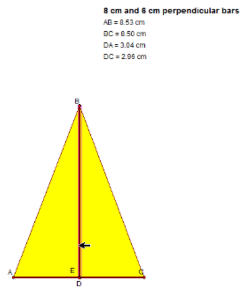
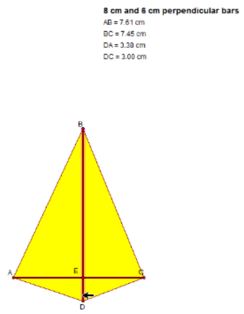
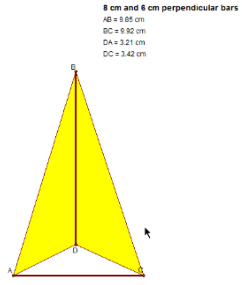
Description of activity	Screenshots
They drag the vertical line so that its bottom point sits on the middle of the horizontal bar. They then adjust by fine dragging movements to get DA and DC as close as possible. In other words they are trying to put D in the middle of the horizontal bar AC.	
They drag the vertical bar down and adjust while attending to the measures of DA and DC	
They drag the vertical bar up	

Figure 6.5 Tilly and Alice drag the vertical bar up and down, dragging maintaining symmetry to move from an isosceles triangle, through a kite, to an arrowhead.

Moving from isosceles triangle to arrowhead, back to isosceles triangle to kite took 9 seconds. It seems from the dialogue that the focus of their attention was first on the figure and then on the measurements. This episode was the one which first alerted me to the possibility that the girls were using symmetry to guide them in positioning the bars. They have positioned the vertical bar with its end on the centre of the horizontal bar and moved it up, trying the keep it central. The focus is on the relative positions of the bars at this point, rather than on the measurements displayed on the screen.

6.5.2 The use of displayed measurements and refinement dragging

With the measurements displayed on the screen, continually updating while the figure was dragged, the students modified or refined the figure so that the expected equal sides and angles were achieved (or very close). When the students corrected the figure in this way they typically used small movements so as to make the displayed measurements indicate, for example, the perfect kite. Hence the students appeared to be refining the particular shape they had generated and so I have called this strategy *refinement dragging*. Refinement dragging is an activity which bridges the gap between experimental geometry and theoretical geometry allowing students to check and review their knowledge of shape properties and is connected to reasoning at Van Hiele level two. Refinement dragging often occurred after students had used GD or DMS so that the bars were already in an approximate position needed to generate the shape. For this reason refinement dragging is different from guided measuring described by Olivero and Robutti (2007) which is the attention to displayed measurements in order to guide the students in dragging a generic quadrilateral into a specific shape.

When using RD the students were not always able to make expected equal sides and angles to display exact equal measurements. However the measurements tended to be close; within a 0.1 cm difference for expected lengths or within 2 degrees difference for expected equal angles. The students appeared to be happy that they had created a shape which was close to the one they were trying to make. It is important to note that I encouraged the students to accept close measurements as indicating that it should be possible to generate the perfect shape. It is quite difficult to use the computer mouse to adjust the position of the bars in the figure in order to get the measurements exact. In iteration one the computer files were set up so that measurements were given to 2 decimal places. This degree of accuracy proved too much for the students whose understanding of place value was not sufficiently developed for them to understand that a difference of, say, 0.08 cm is negligible. In the example below which is taken from the recording of Tilly and Alice, Tilly read 8.60 and 8.52 as having a difference of 8, (rather than 0.08) as it would be a difference of 8 pence if it were money instead of length. We had previously had a discussion of how to work out the differences after the decimal point which us why Tilly said ‘keep on doing that’ when I pointed out that a difference of 0.08 cm is tiny.

Susan: *So which sides should be the same?*

Tilly: *Erm BA and BC*

Susan: *Are they the same?*

Tilly: *Yeah only eight out*

Susan: *... is that 8.60 and 8.52 so it's actually*

Tilly: *eight*

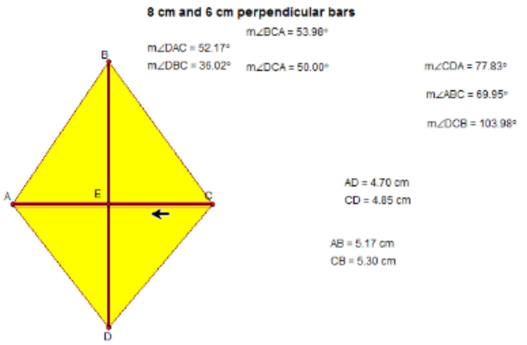
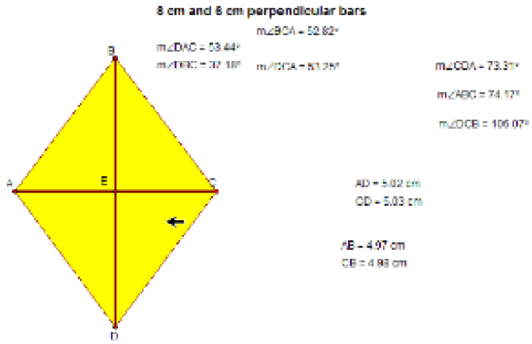
Susan: *point 08 so that's even better*

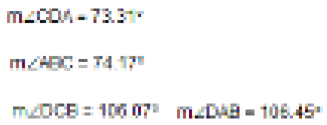
Tilly: *yeah keep on doing that*

In later iterations I changed the degrees of accuracy in the computer files to 0.1 cm for length and 1 degree for angles in order to address this difficulty.

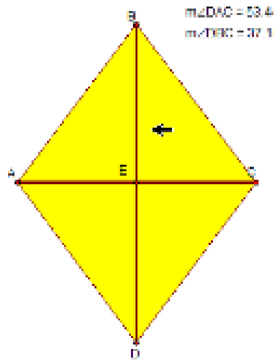
6.5.3 Refinement dragging sometimes helped students to review their knowledge and understanding of shape properties

In the following excerpt Adam and Jack generated a rhombus and then tested their understanding of its properties. It appeared that when they were in primary school they had been taught that the rhombus is a ‘squashed square’. The result of this analogy was that students commonly thought that all the properties of a square would be maintained in the rhombus including the property of equal angles. Here Adam and Jack confirmed that they could generate a rhombus which has equal sides (or at least sides of 5 ± 0.03 cm which I encouraged them to consider as close as makes no matter). Next they tried to get four equal angles but readjusted their perceptual understanding of rhombus properties when they found they could make two pairs of opposite equal angles.

Dialogue	Narrative and screenshots
Susan: *Can you move the bars so that you can make a rhombus.* Jack: *Just move the AC bar into the middle* Adam: *OK* Susan*: The middle of what?* Jack: *The middle of the er the B and D bar* Susan: *So you've made a rhombus how will you check that you've actually made one?* Jack: *All the lines around the outside are the same* Susan: *So have we actually got four lines the same length?* Adam: *No, they're slightly off cause I'm looking at them four and* Jack*: I think the line of AC needs to be moved further up the line BD. Keep it a bit, erm, go further. As when the er AB erm measurement goes down the CB measurement goes down as well.* Adam: *I think I've got it closer* Jack: *Yeah* Susan: *They're very close to five* Jack: *When you look at the measurements of the AD and the CD they're exactly point zero one away. The same as the AB and CB are; point zero one* Susan: *Yeah that's really good OK so you've got four sides which we'll call the same because they're extremely close aren't they.* Susan: *Anything else about a*	Adam dragged the bar AC up to the middle of the bar BD. He paused and then dragged to adjust the position to get it more accurate. I cannot be sure but he may have attended to the measures while doing this.  In between these two screen shots there were four bursts of refinement dragging activity lasting a total of 58 seconds as Jack gave Adam instructions on how to move the bars!  The boys had to measure one more angle to complete the four, which was angle DAB and they put the measurement next to that of angle DCB which they had identified as being equal.

<table>
<tr>
<td>
rhombus that's true that we can check?
Adam: All angles are the same
Susan: Have you got some angles you've already measured on there?
Adam: Yeah
Susan: What about the angle at the top, the angle at B. What's the three letters that would give the angle?
Adam: ABC
Susan: What does that measure?
Adam and J: 74.17
Susan: What would be the angle that should be the same as ABC?
Jack: Be CDA
Adam: Yeah that's right. It's close
Susan: It's close. Right so you've said they should all be the same so are the other two seventy something as well?
Jack: No the other one would be the C measurement which is DCB is one hundred and six
Adam: It would be about the same as angle A
Jack: Quite close, both those are
Adam: Opposite angles need to be the same
</td>
<td>

m<COA=73.31
m<ABC=74.17
m<DCB=106.07   m<DAB=106.45

The screen shot of the angle measurements are rather fuzzy so I have copied it above. It can be seen that the two opposite obtuse angles DCB and DAB are within one degree of difference and the two opposite acute angles CDA and ABC are within one degree of difference.



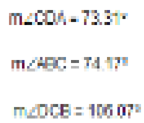

m<COA=73.31
m<ABC=74.17
m<DCB=106.07
</td>
</tr>
</table>

Figure 6.6 Adam and Jack review the side and angle properties of a rhombus

Adam and Jack were able to get the sides of the rhombus to be very close, within 0.03 cm of the 5 cm which would indicate an accurate rhombus. It can be seen that I encouraged them to accept this as being close enough and the boys did not appear to worry about not being spot on with the measurements of the sides. It is as if they were prepared to accept that being close indicated it would be possible to make an accurate rhombus in a perfect world.

When the boys considered the angles, they at first expected all four of them to be equal. However when they tried to make the angle measures equal they were unable to do so. This caused them to rethink their assumptions and they were able to make opposite pairs of angles (close to) equal and to revise their understanding of the properties of the angles in a rhombus. Hence the use of refinement dragging in this case confirmed the property of equal sides but caused Adam and Jack to revise their perceptual understanding of the angle properties of a rhombus. Refinement dragging thus appears to support and strengthen reasoning at Van Hiele level two which deals with the understanding of shape as having a collection of properties.

6.5.4 An episode of guided measuring

Whilst I have asserted that the students participating in iteration one used the four dragging strategies described there was an episode where Adam and Jack appeared to be using guided dragging in the way in which Olivero and Robutti (2007) describe it, i.e. dragging an object whilst attending to the measurements. Adam and Jack decided they would try to make an equilateral triangle which is impossible using the perpendicular 8 cm and 6 cm bars. The excerpt from the recording with commentary shows what they tried to do. The screen shots indicate how the boys used guided measuring to move bar AC down bar BD maintaining a triangle shape in their quest to find the equilateral triangle position.

<table>
<tr><th>Dialogue</th><th>Screenshots</th></tr>
<tr><td>
Jack: Yeah, maybe if you move the middle line a bit further down

Adam: I think a bit further up would do it.

Jack: Change the angle.

Adam: Where, you need to go down. We need em to all be the same, all need to be sixty. I don't think that's gonna happen because they're gonna get to sixty before the other one.

Adam: Keep dragging that down until it gets to D

Jack: If you drag that one down the ABC one is going down, because the angle on that, and then CDA is going up

Adam: Every time you drag that down that gets smaller but that gets smaller faster

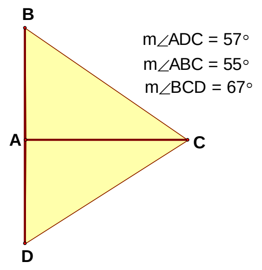
Susan: Which one's getting smaller faster then?

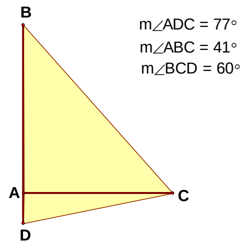
Adam: The ABC

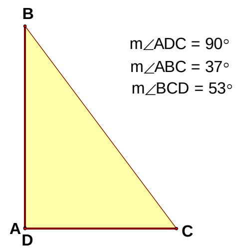
Susan: Mmm, 41 now isn't it

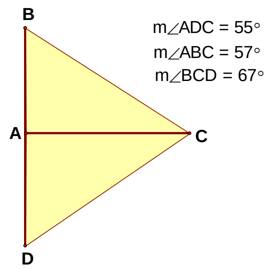
Adam: How about, Jack, try and move it up, see if you can get them all the same angle. See if that will work. Don't think it will

Jack: No because the DCB is going up as well as the ABC

Adam: Oh yeah. I don't think we can make a equilateral.
</td><td>








(These screenshots are not the originals, which were too fuzzy to be presented clearly).
</td></tr>
</table>

Figure 6.7 An example of guided measuring and situated proof that an equilateral triangle could not be made from the figure

In lines 6 to 13 Adam and Jack were reasoning about the possibility of generating an equilateral triangle whilst attending to the measurements of the angles. They realised that when angle B increases, angle D decreases and vice versa. This was noted by Jack in line 6 when he said that dragging the bar down meant angle ABC decreased as angle CDA increased. Angles B and C increase or decrease together when bar AC is below the mid-point of BD with angle B changing at a faster rate. (Angles D and C increase or decrease together when bar AC is above the mid-point of BD). This is what Adam referred to in line 7 when he said “every time you drag that down that gets smaller but that gets smaller faster”. Reasoning about the figure while dragging, and focusing on the displayed measurements, the boys had devised an informal situated proof of why an equilateral triangle cannot be made using the 8 centimetre and 6 centimetre perpendicular bars. By looking at the measurements of the angles they could see that there would never be a situation when all three angle measurements would be 60 degrees simultaneously. In this way the task and the software have allowed them to use practical experimental geometry to devise some theory, albeit situated in the task.

6.5.5 The evidence for Dragging maintaining symmetry

When, at first the students dragged the dynamic figure between shapes it appeared that they were simply using an efficient form of guided dragging (eg to turn a kite into a rhombus) by dragging one bar through the middle of the other. Sometimes this dragging strategy was accompanied by the students explaining what they were trying to do, eg “we're trying to keep BD in the middle of the shape” (see Tilly’s comment in figure 6.5). This kind of dialogue suggests that the students were dragging with the purpose of keeping one bar as the bisector of the other and also that they were aware that symmetry was being maintained.

A close look at the measurements which were displayed on the screen during these particular dragging episodes revealed further evidence which supports the hypothesis that the students were purposefully maintaining symmetry while dragging. Stopping the on-screen recording at intervals and checking measurements of sides and angles which could be expected to be equal (if the dynamic figure maintained its symmetry) revealed a high degree of accuracy which indicates that the dragging action was keeping (near) symmetry constant. In the first iteration of the study four pairs of students worked with vertical and horizontal bars. Each pair were observed to use DMS during their recorded sessions and in each session there were at least two episodes where there is sufficient evidence to make a case that DMS was indeed occurring. These episodes are listed in appendix 1.4 and can be accessed from the original transcripts included on the accompanying disk. What becomes apparent from these data excerpts is that the students developed the symmetrical dragging as they gained experience working with the figure. The best examples are usually towards the end of the sessions. An example is given below (figure 6.7).

Gill and Sara were the students who I interviewed in January 2011 and who were so quiet that I gave up trying to transcribe their session. However, the recording showed their dragging activity and this good example of DMS. Gill and Sara had a kite on the screen (see Figure 6.7 for a series of screenshots taken during a six second episode of DMS). They changed the figure from a kite through a rhombus and an inverted kite to an arrowhead. The congruent sides were with 0.2 cm of each other and the congruent angles (A and C) were almost always within 2 degrees of each other. These measurements show a strong indication that these students were demonstrating DMS.

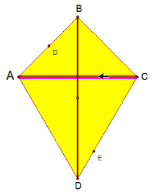
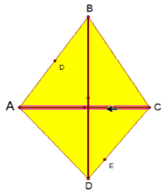
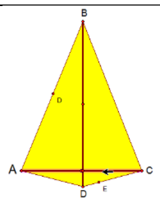
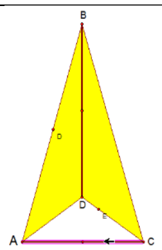
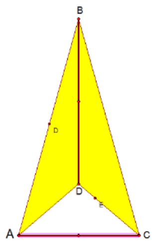
	m∠ABC = 91° m∠BCD = 104° m∠CDA = 62° m∠DAB = 104° CB = 4.2 cm AB = 4.2 cm DC = 5.9 cm DA = 5.9 cm DB = 8.0 cm
	m∠ABC = 68° m∠BCD = 109° m∠CDA = 81° m∠DAB = 103° CB = 5.3 cm AB = 5.5 cm DC = 4.5 cm DA = 4.7 cm DB = 8.0 cm
	m∠ABC = 45° m∠BCD = 83° m∠CDA = 151° m∠DAB = 81° CB = 7.8 cm AB = 7.8 cm DC = 3.0 cm DA = 3.2 cm DB = 8.0 cm
	m∠ABC = 33° m∠BCD = 39° m∠CDA = 111° m∠DAB = 39° CB = 10.5 cm AB = 10.5 cm DC = 3.7 cm DA = 3.6 cm DB = 8.0 cm
	m∠ABC = 32° m∠BCD = 35° m∠CDA = 101° m∠DAB = 34° CB = 10.9 cm AB = 10.9 cm DC = 3.9 cm DA = 3.9 cm DB = 8.0 cm

Figure 6.8: Gill and Sara use DMS with high levels of accuracy

A further DMS episode taken from the recording of Adam and Jack has been included in appendix 6.3.

A combination of student comments about keeping one bar in the middle of the shape and the accuracy with which students drag one bar through the path of the perpendicular bisector of the other bar (when moving between kites, rhombus, isosceles triangles and arrowheads) has led me to hypothesise that students use their holistic understanding of the general shape and its symmetry when using this dragging strategy. Another (rather subjective) source of evidence comes from my own perception of shapes and the way in which I see the symmetry of the whole shape. Although it is possible that I am untypical of the human race in this respect, I suspect that humans do have an instinctive perception of symmetry and this appears to be confirmed by the literature (Darwin, 1887, Bornstein et al, 1981, Palmer, 1985).

6.5.6 Symmetry

The students had a working concept of symmetry as a visualisation of folding one half of the shape over the other half. In the excerpt below Alice and Tilly explain how they understand symmetry,

Alice: *I've done it before but, I'm trying to think. Did we trace it and then checked if it there were lines of symmetry?*

Susan: *When you traced it and you were checking lines of symmetry what did you do?*

Tilly: *Traced half of it and folded it over.*

Susan: *Oh*

Alice: *And then the lines were like you could see whether the lines were the same*

Susan: *Mmm. So what did that prove then when you folded it in half?*

Tilly: *That it had a line of symmetry because it was the same.*

In the following excerpt from the dialogue Jack and Adam discuss the symmetry of the rhombus.

Susan: *So how many lines of symmetry does that shape have?*

Jack: *Is it just two*

Adam: *No it aint .... None*

Jack: *It has two if you split it down from BD line left and right*

Adam: *But if you look at it the other way, so if you like switch the shape round and you try and turn it that way then it won't work. That that way I don't think it'd work.*

Jack: *But if you folded B down to D, that would have a line of symmetry. If you fold it C to A it would be the other line of symmetry*

Adam: *Oh yeah*

The students appeared to use symmetry to decide which side lengths and angles in the figure were equal. Although they have been taught side and angle properties of quadrilaterals and triangles, they often picked out some angles inside the shape as being equal, which would not have been taught to them. This suggests they have used symmetry to identify them as being equal.

In the dialogue below Jack was able to say which angles should be equal if the two triangles either side of the line were equal and the cursor pointed to both of the angles at point C (i.e. angles BCA and DCA). An understanding that AC needed to be the line of symmetry must have underpinned this. Although he would have been taught that isosceles triangles have two equal sides and two equal angles, it is unusual (from my own experience in the classroom) to hear year 8 students engage in reasoning about any other congruent sides or angles inside an isosceles triangle.

<table>
<tr><th>Dialogue</th><th>Screenshot</th></tr>
<tr><td>Susan: Is it possible to, to make that into a symmetrical triangle so that the two triangles either side of the line are the same? What would have to be true for that to happen?

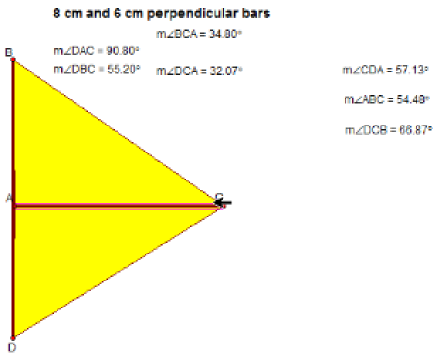
Jack: These two angles, the angle on the opposite side of the line C, would have to be the same</td><td>
</td></tr>
</table>

Figure 6.19 Jack uses symmetry to identify two equal angles

6.5.7 A preference for a vertical axis of symmetry

Another observation was that the students seemed to have a preference for the orientation of the shapes which privileged vertical symmetry above other considerations. This may have been because, from primary school, they had been introduced to shapes drawn in a common orientation (ie squares sitting on one side, triangles sitting on the base, etc). On the other hand it may be (as Pinker, 1997 suggests) that a preference for vertical symmetry is a human trait and therefore we prefer to orient shapes so that they have vertical symmetry. We do, after all, live in a world where vertical symmetry is dominant in objects such as door frames, windows, arches, etc. It cannot be discounted that the vertical bar was also the longer of the two which may be a reason why it was more dominant. This would be tested in iteration two by giving the students a longer horizontal bar and a shorter vertical bar.
It could be argued that the students were able to use symmetry to position the bars because the figure contained vertical and horizontal bars and it may be natural to drag vertically and horizontally. In iteration two I decided to test the notion of symmetry and frame of reference by designing files containing bars which remain perpendicular but at an angle to the vertical.

6.5.8 Properties of shapes and inclusive classifications

During the sessions it was clear that the students had a secure knowledge of the properties of 2D shapes although they did not demonstrate an understanding of the minimum properties required to define any shape. There was no evidence that the students were able to describe an inclusive classification of the shapes. This indicates that they were able to reason at Van Hiele level two but not yet at level three. However there was some evidence that they could see a connection between a rhombus and a square. As in iteration zero they were able to suggest that to make a square they needed equal length bars which crossed at their mid-points. This might indicate that the students had the potential to move towards level three reasoning (Van Hiele, 1986).

6.6 Conclusions

6.6.1 Connecting the observed dragging strategies to students' geometrical reasoning.

By the end of the analysis for iteration one I felt confident that I had observed four dragging strategies which were consistently used by all the students who had so far worked on the task. These dragging strategies also appeared to be aligned to the way students reasoned about the shapes they generated from the dynamic figure.

WD is an exploratory activity when students use random dragging to see what happens. This was an infrequently used strategy and I have not studied it in any depth.

GD tended to be used to put the bars straight into a specific orientation and required a holistic perception of the shape in order to do this. Some students dragged the separate bars into a symmetrical position at the beginning of the session when there was no 'covering' on the shape. This would seem to indicate a preference for symmetrical arrangements. At this stage in the research I began to consider that GD uses reasoning at Van Hiele level one. This is not to suggest that GD is a less sophisticated dragging strategy than the others. Rather GD appears to use a holistic perception of the shape of the figure and the position of the bars needed to generate the shape.

RD was the strategy which was used more than the others and for longer periods. Clearly this strategy arises from the nature of the task but it was surprising how good the students were at using RD and how patient. They were also happy to accept a shape whose expected equal side and angle measurements were close enough even if not exact. It has to be said that I encouraged them to be happy with a close enough shape (albeit very close since the human eye could not have appreciated the difference). RD activity helps students to revise and review their knowledge of shape properties and so is connected to Van Hiele level two reasoning.

DMS was the strategy which I found most intriguing and, at first, I was cautious about my interpretation of the intuitive use of symmetry when dragging. The best way to appreciate this strategy is to view the recordings and some useful episodes are indicated in appendix 1.3. For the reasons given earlier in this chapter I am convinced that students do use a sense of symmetry whilst using this dragging strategy.

6.6.2 Could the use of DMS lead to the concept of the 'dragging family'?

As the students appeared to use DMS to purposefully drag one bar so that "it crossed the other one in the middle" the figure moved between kites, rhombus, isosceles triangles and concave kites (which the students knew as arrowheads). These shapes could be said to form a 'dragging family' whose common property is that one diagonal is the perpendicular bisector of the other (the isosceles triangle can only be said to be a member of this family in the context of Dynamic Geometry, not in a static geometry environment). I wondered if the use of DMS could facilitate a development in students' understanding of inclusive relations based on the dynamic figure (an example of situated abstraction (Noss and Hoyles, 1996)) given the right task and the right questions. If so then this could be the trigger to move students' reasoning towards Van Hiele level three. In the next iteration I explore whether the task could be modified to encourage the students to develop inclusive classifications thus indicating development towards Van Hiele level three. I also test the questions regarding a preference for vertical symmetry and whether working with a figure in a different orientation would affect the outcomes.

7 Iteration two: perpendicular bars at a different orientation

In iteration one I confirmed the observation of the four distinct dragging strategies which participating students used when working with the dynamic figure and connected these to the first three Van Hiele levels. I postulated that use of DMS has the potential to develop students' reasoning at Van Hiele level three and in particular the development of inclusive relations between the rhombus and the kites which can be generated from the dynamic figure. However I had seen no evidence that students were developing the concept of inclusive relations. Since the aim of the research study was to identify an educational intervention which can encourage the development of a hierarchical classification of 2D shapes, the remainder of the study focused on the use of DMS. However I also looked for other themes observed from the data in iteration one, in particular the way in which students in the study described and used symmetry and whether the students would notice similarities in the positions of the bars needed to generate the shapes with special properties.

7.1 Objectives for iteration two.

Iteration two specifically addresses the issue of whether students would continue to use all four dragging strategies, but especially DMS, when the bars were in a different orientation on the computer screen, i.e. a longer horizontal bar and shorter vertical bar, and perpendicular bars oriented at an angle to the vertical.

In iteration two I also tested whether the task using the dynamic figure could be used with a whole class. I took the opportunity to use the computer files containing the original vertical and horizontal bars with four different classes; two from School A and two from School B (a brief account is included in appendix 2.1) and then worked with a pair of students from each class with the bars oriented at an angle. However I did not make any recordings from the whole class lessons in iteration two. In hindsight this may have been a missed opportunity but at the time I was keen to interview pairs of students from these classes who had experience with the vertical and horizontal bars before they worked with the bars in a different orientation.

7.2 Theoretical background: Symmetry and frames of reference

The computer screen has its own frame of reference: vertical and horizontal axes parallel to the edges of the screen. Within this frame of reference the students had hitherto appeared to prefer shapes with vertical symmetry and to a lesser degree shapes with horizontal symmetry. In iteration two this would be tested by giving the student subjects a file where the horizontal bar is longer than the vertical bar (see figure 7.1). This would test whether the preference was for the vertical axis or the axis with the longer length bar.

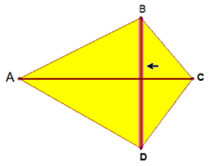


Figure 7.1 The figure with an 8 cm horizontal bar and 6 cm vertical bar

The second task file had oblique perpendicular bars to test whether the students used symmetry to help them position the bars when they could not use the vertical to guide them (see figure 7.2). My hypothesis was that the students would continue to use the concept of symmetry to help them position the bars inside the shape.

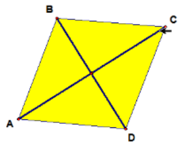


Figure 7.2 The figure with perpendicular 8 cm and 6 cm bars at an angle to the vertical.

Pinker (1997) suggests that when humans look at objects which are orientated away from the vertical axis then they use a frame of reference which is local to the object. This is actually very helpful to us in recognising a multitude of objects. If we had to memorise each object in all possible orientations it would require a large memory store. Instead we store a memory of each object in a typical orientation and when we encounter that object we are able to recognise it by mentally rotating the object into its typical orientation (Pinker, 1997). This is only a simplification of what humans do when we encounter and recognise objects which may occur in different sizes and proportions (Pinker gave the example of a suitcase which could have different lengths, widths and heights, rounded ends or square ends etc).

Cooper and Shepard (1986, cited in Pinker, 1997) showed that when subjects were shown letters from the alphabet, which had been rotated from their typical orientation, they took longer to recognise letters which were rotated furthest from the upright position. It could be surmised from this that humans do rotate mentally until they perceive the object in the typical orientation and that this may be adding to the cognitive load (accounting for the longer time taken to recognise the letters in Cooper and Shepard's experiment).

If students in my study did use DMS when the bars were oriented at an angle I was interested to see whether this dragging would be as accurate as it had been with the vertical and horizontal bars given that they may experience a greater cognitive load with the task.

7.3 Process for iteration two

7.3.1 Instructional goals

The students worked with two Geometers Sketchpad files. One file contained the 6 cm vertical bar and the 8 cm horizontal bar, so the figure would be at a ninety degree rotation from the figure which the students had worked with in the whole class lesson. Figure 7.3 shows a rhombus using this file.

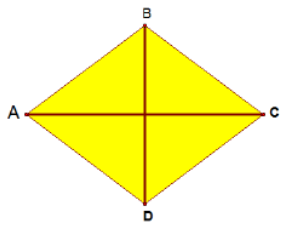


Figure 7.3 A rhombus generated from the horizontal bar and the vertical bar

The second file contained the 8 cm bar oriented at 60 degrees to the vertical and the 6 cm bar oriented at 30 degrees to the vertical. Figure 7.4 shows a kite using this file.

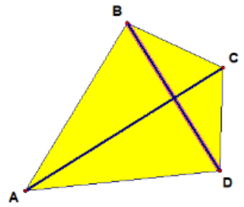


Figure 7.4 A kite generated from perpendicular bars oriented at an angle to the vertical

As in iterations zero and one the students were given the computer file with the bars separately and were asked to drag one bar over the other. I then instructed the students on how to join the ends of the bars and construct the interior of the shape thus filling it with colour. The task was to drag the bars to see what shapes they could make, identify the properties of the shape and then use the Measure menu to check side and angle properties. I also asked the students to describe how the bars were positioned in the shape. I anticipated that the students might begin to develop their reasoning towards Van Hiele level three in particular developing an understanding of inclusive relations between kites and rhombus and being able to deduce some properties from others.

<u>7.3.1.1 Developing inclusive relations</u>

I hoped that the students would notice that there was a common property of the bars in the kite, isosceles triangle and arrowhead in that one bar always crossed the other at its mid-point and that the bars in the rhombus crossed at both their mid-points. Making these observations might lead the students to notice the family of shapes where one bar crosses the other at its mid-point and that the rhombus is a special member of this family.

Another question I asked was “how many kites is it possible to make?” I wanted to know whether students could identify that there should be an infinite number of kites in which case I expected that the suggestion that the one position within the kites, which generates a rhombus, would lead them to consider the rhombus as a special kite.

<u>7.3.1.1 Deducing some shape properties from others.</u>

When students generated shapes which had symmetry I asked questions such as “if you imagine folding the shape along the line of symmetry, which side would fold onto

(AD)?" In asking this type of question I was trying to ascertain whether the students would connect line symmetry with properties of equal sides and angles.

7.3.2 Instructional starting points.

The four pairs of students who participated in iteration two of the study in June 2011 were assessed by their mathematics teachers as achieving at levels five/ six of the National Curriculum for England and Wales (QCA, 2007). Two pairs from School A, Tara and Ruth, and Dave and Evan, were in a class for high attaining students in year seven and two pairs from School B, Kate and Jane, and Aftab and Rupen, were in a class for middle attaining students in year eight. The year seven students clearly had one less year of schooling than their year eight counterparts which may have meant that they had learned less about shapes and their properties. However they were in a class which had been identified as one where the students would be challenged to make faster progress than their peers. I had not chosen to work with year 7 students since it would have been more consistent to work with students from year eight whose teachers identified them as having average attainment. However, this was a case of taking whichever opportunity was offered as the mathematics department in the School A was keen for me to work with their year seven 'top sets'.

Each pair of students had already worked with the dynamic figure in a whole class setting using the 8 cm vertical and 6 cm horizontal bars. This meant that they were already familiar with the task and the shapes which could be generated using the dynamic figure. In this iteration they would work with bars oriented differently so that I could test whether the orientation of the figure had any effect on the students' activity during the time they worked on the task. In the whole class lessons I had observed that students were able to name triangles and quadrilaterals and list their properties and I felt confident that they would be able to manage the task in pairs.

7.3.3 Learning trajectory

In this iteration I set up the task with the objectives that students would:

- Drag the bars to generate different shapes.
- Test the shape properties using the measurements of sides and angles.
- Describe the relative positions of the bars in order to generate the shapes.
- Observe that when if one bar is dragged so that it always crosses the mid-point of the other bar the default shape is a kite and that there are special positions along this dragging journey which generate the rhombus and isosceles triangles.

I hoped to observe the students achieve the following:

- Identify the properties of the diagonals needed to generate each shape which would correspond to reasoning at Van Hiele level two..
- Recognise that it is possible to generate an infinite number of kites, that the arrowheads are kites, and that the rhombus is a special version of a kite made when both bars cross at their mid-points. These last two points would indicate development towards reasoning at Van Hiele level three.

7.4 Results and analysis of the sessions in iteration two

Each of the four sessions was recorded and imported into Transana. The sessions were transcribed into a word processing document containing three columns: the dialogue between the students and myself, a description of the on-screen activity and thirdly a narrative account giving an overview of the sessions. The original transcription/descriptions can be accessed from the accompanying disk. A table of dragging activity and a table of episodes for each recording are included in appendix 2. A number of themes emerged from the data. These have been described below and illustrated with examples from the data.

7.4.1 The use of Guided Dragging, Refinement Dragging and Dragging Maintaining Symmetry to generate a specific shape.

As can be seen from the tables of dragging activities in appendix 7.2, the students in iteration two used the same dragging strategies that the students in earlier iterations had used. This was as true for the perpendicular bars oriented at an angle as it was for the horizontal and vertical bars.

As in previous iterations students dragged the bars using GD or DMS to make a specific shape such as a rhombus or kite and then used RD to adjust the bars whilst focusing on the displayed measurements of sides and angles.

7.4.2 The evidence for Dragging maintaining symmetry

Appendix 2.6 shows a table of DMS episodes in both computer files and the degrees of accuracy within which it was used. Some students, Tara and Ruth, and Aftab and Rupen were less accurate in their use of DMS when the bars were at an angle. Dave and Evan, and Kate and Jane achieved better accuracy as they became more familiar with the bars at an angle.

7.4.3 Symmetry when placing the bars together

At the beginning of their work with each computer file the students were presented with 2 bars which were separate from each other and were asked to drag one bar over the other. In each of the four pairs the student with the mouse dragged the bars so that the arrangement was reasonably symmetrical. By using the term ‘reasonably symmetrical’ I wish to imply that the bars looked as if they had been arranged symmetrically but they were not exactly accurate in this regard (if sides and angles were to be measured). The students arranged the bars as for two kites (year 8 boys and girls) and two rhombuses (year 7 boys and girls). (The year 7 girls even used refinement dragging to make their bars spot on by reference to the angle measurements which were displayed on the screen, see figure 7.5). Of course, it could be argued that they had learned to expect symmetrical shapes from their work in the previous sessions. However, all students were observed to place the bars reasonably symmetrically in the other iterations.

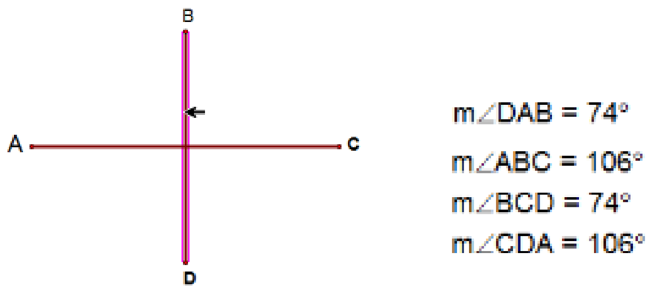


Figure 7.5 Kate and Jane dragged one bar over the other and referred to displayed angle measurements to make the bars symmetrical. (The arrow indicated the cursor on the screen).

7.4.4 A preference for shapes with a vertical axis of symmetry

The students generated more shapes with vertical symmetry than horizontal symmetry in the file with the 8 cm horizontal and 6 cm vertical bars (see transcripts for iteration two on the accompanying disk). An example is shown in figure 7.6. This would indicate that the vertical axis is preferred even when it is shorter than the horizontal axis and resulted in squat shapes such as the kite below.

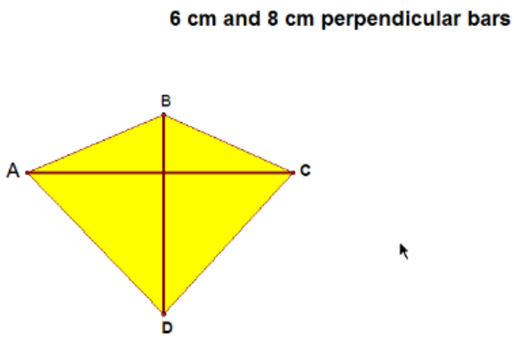


Figure 7.6 A kite with vertical symmetry.

7.4.5 Using symmetry to deduce other properties of the shape.

When trying to make a symmetrical shape students often talked about symmetry as a folding action and cutting a shape in half (e.g. section 6.5.6). I tried to use this to see if Tara and Ruth would deduce equal sides and angles by imagining the effects of folding a shape in half. In the following excerpt Tara was able to identify which side-lengths and angles in the rhombus would sit together if the rhombus were folded along a line of symmetry, although she needed a lot of input from me.

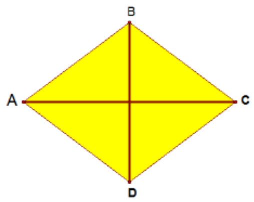
Dialogue	On-screen activity
Susan: *Can you tell me anything about symmetry of a rhombus?* Tara: *It's got two lines of symmetry*. Susan: *So which are they, can you label the lines of symmetry?*	 The girls see that the bars are the lines of symmetry. I try and work with this to see if they can use reasoning about symmetry to identify equal sides and angles.
Tara: *This one* Susan: *That's AC isn't it.* Tara: *Yeah and BD*	She clicks on and off the bar AC She clicks on and off the bar BD
Susan*: So if AC's the line of symmetry which line would sit on AD if we folded it?* Tara*: If we folded AC?* Susan*: If you fold it over AC what line would sit on AD?* Tara: *Erm, AB*	I used the analogy of folding because other students at this school have thought of line symmetry through folding one half of a shape on the other. Here I try to get the girls to think why symmetry of a rhombus leads to equal sides.
Susan: *So, they'd have to be the same size wouldn't they.* Tara: *Oh yeah* Susan*: And what, can you tell me two other sides which sit on each other?* Tara: *BC and CD* Susan: *And if the line AC was the fold line which angles have to be equal?* Tara: *Er, B and D* Susan: *OK, and are B and D equal?* Tara: *It looks like it, yes.* Susan: *What are they?* Tara: *Seventy four, seventy four*	The girls can clearly visualise the results of a folding.

Figure 7.7 I encourage Tara to use symmetry to identify equal sides and angles

It can be seen from the aforementioned excerpt that Tara had not previously linked symmetry to the properties of equal sides and angles although, when prompted, she could identify sides and angles which would sit together when folded.

7.4.6 Deducing that a square needs equal length bars

As in other iterations the students in iteration two realised that they would need equal length perpendicular bars in order to make a square. However there was no evidence that they could explain why this was. Kate and Jane told me that they could not make a square with the 8 cm and 6 cm perpendicular bars because the bars were not the same length (lines 113-117 in the transcription). Later, when Kate and Jane worked with the unequal bars at an adjustable angle they reported that they could not make a rectangle because they would need equal length bars (lines 273-274 in the transcription).

7.4.7 Properties of shapes and inclusive classifications

The kite is the default shape which is generated when one bar is the perpendicular bisector of the other bar. I wanted to ascertain whether the students could appreciate that most of the time they were generating kites, with the rhombus and isosceles triangles as special cases, and hence develop an inclusive classification where the rhombus (and isosceles triangle!) is seen as a special kite. Of course arrowheads are concave kites but the students tended to see arrowheads and kites as two different shapes because they looked different (suggesting that a holistic visual perception overruled a consideration of the shape as given by its properties). I also decided that it would be interesting to see whether the students could draw the conclusion that arrowheads are special types of kites.

The kite proved to be an interesting example. Despite the infinite number of possible kites the students generally made what I have called the ‘three quarter’ kite because the cross bar is approximately three quarters of the way up the bar acting as an axis of symmetry. This seemed to be the preferred version of a kite. The year 7 students in particular had a preference for kites in this proportion with the result that when I asked them how many different kites they could make they answered four (shown below).

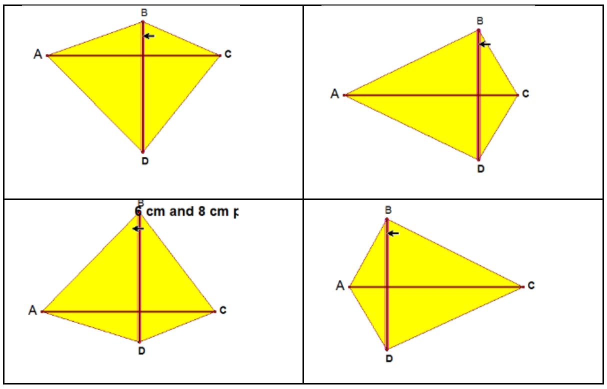


Figure 7.8 kites in the four main orientations generated with vertical and horizontal bars

If I suggested that they move the cross bar a little bit to make different kites the year seven students revised their estimation of the possible number of kites to eight or twelve. They seemed to not perceive that the cross bar could be moved in very small amounts which would produce an infinite number of kites and arrowheads.
However the year 8 students told me that they could make millions of kites by sliding the bar a little bit and a little bit. This kind of reasoning may be the beginning of the realisation that there is a family of kites which have in common that one bar crosses the other bar in the middle and is at right angles to it (perpendicular bisector).

In the bars at a slant file the students tended to generate the ‘three quarter kites’ as before but when generating an arrowhead they were more likely to make these with the cross bar in different positions. This may be because they had less experience of seeing these shapes on paper so were more open to different versions of the arrowhead.

7.4.8 Orientation and its effect on how students perceive shapes

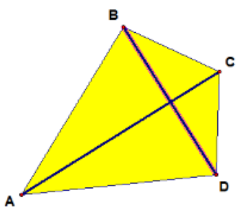


Figure 7.9 An ‘angled kite’

When working with the perpendicular bars oriented an angle the students talked about tilting their head, or they physically tilted their head. An example of this is seen in lines 265-271 appendix 2.5b when Aftab and Rupen turned their heads to look at the figure. Aftab said of the shape “you can tell it’s like a kite this way”. This would indicate that there is a preferred orientation which relates to the frame of reference of our physical environment. If something appears to us to be at an angle to the vertical axis we try to mentally turn it back the right way up (Pinker, 1997). Ruth bemoaned the orientation of the kite which caused her to feel less capable of doing the task. “Erm, it's a bit of an angled kite. I'm rubbish at this.” She would prefer the vertical orientation, and may have been experiencing a greater cognitive load due to having to mentally rotate the kite to the preferred orientation.

7.4.9 The effect of orientation on holistic perception

Tara and Ruth had generated the rhombus using the bars oriented at an angle to the vertical as shown in figure 7.9.

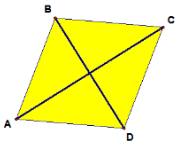


Figure 7.10 Rhombus with bars oriented at an angle to the vertical

The following dialogue (Figure 7.10) indicates that Ruth perceived the rhombus to be a parallelogram (which of course it is if one accepts inclusive relations) and this may have been because it was oriented as the typical parallelogram.

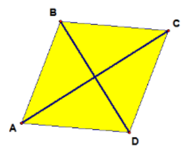
Dialogue	Screen shot
Susan: *Have you made the rhombus you thought you'd made?* Ruth: *It looks like a parallelogram actually* Tara: *It’s definitely a rhombus* Susan: *So why is it definitely a rhombus?* Tara: *Because, erm, I don't really know yet because we haven't really measured it.*	

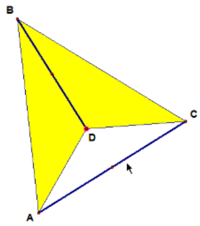
Figure 7.11 Ruth perceives a rhombus oriented at an angle to be a parallelogram

Even though Tara thought the shape was a rhombus she still wanted to measure sides and angles to make sure. In this way she needed the reassurance of displayed measurements when she asserted the shape was a rhombus. This is an example of spatio-graphical geometry supporting the move to theoretical geometry.

7.4.10 Dragging maintaining symmetry with the bars oriented at an angle to the vertical

Dragging deliberately to maintain symmetry when they moved between kites, rhombus, isosceles triangles and arrowheads may have been an expected result in the first computer files when dragging to maintain symmetry meant dragging up and down the computer screen. However when the bars were in a different orientation, dragging to maintain symmetry was not achieved by dragging up or down the computer screen. The students had to drag one bar along the path of the perpendicular bisector of the other bar and this was at 30 degrees or 60 degrees to the vertical depending on which bar was being bisected.

When dragging the bars to generate shapes in the file with the bars oriented at an angle to the vertical the students often talked about dragging ‘up’ or ‘down’ when this actually referred to dragging obliquely towards the top right hand corner of the screen for example. It may be that the students were mentally rotating the figure or screen so that they were dragging up in their own minds. Some students may have found it slightly harder to use dragging to maintain symmetry when they were dragging at an angle to the vertical which is indicated by the measures on the screen during the recording and the longer episodes of dragging due to the students going more slowly, although later episodes of DMS tended to be shorter and more accurate as the students got used to dragging at an angle. However it was clear from my observations of these dragging episodes that the intention of the students was to drag maintaining symmetry as the most efficient way to go from kite to rhombus, for example. An excerpt from the recording of Kate and Jane serves to illustrate these points (Figure 7.11).

<table>
<tr><th>Dialogue</th><th>On-screen activity</th></tr>
<tr><td>Susan: OK so what shapes do you think we're going to be able to make in this file?
Jane: Kite, triangle, arrowhead
Susan: OK, do you think you could do that then? That's an arrowhead there, OK
Jane: It's close.
Susan: It's close isn't it. It's only point one out. (Here we are referring to the closeness of the measurements).
Jane: Triangle
Susan: When you made the triangle, you moved the bars in a particular way. What were you trying to do with that bar?
Kate: That I kept it in the middle (cursor moves along bar BD) so it wouldn't mess around with them (cursor points to the measurements of equal sides lengths).
Susan: So the bar was kept in the middle of what?
Kate: In the middle of A and C.</td><td>She realises that this file will generate the same figures as the first file.
She drags bar BD.
5 seconds of DMS
14 seconds of RD to get it perfect


3 seconds of DMS
She drags the figure into an isosceles triangle (perfect specimen)
She must have dragged really carefully to get it perfect without needing any refinement. The cursor traces down bar BD then traces down the measurements of the sides of the triangle.
She describes DMS as being careful to keep BD in the middle of AC (ie bisecting AC).</td></tr>
</table>

Figure 7.12 Kate and Jane drag one bar through the other to maintain the near symmetry of the shape.

In the following screen shots it can be seen from the position of the cursor at point D how Kate dragged bar BD through the middle of the shape to generate the arrowhead, isosceles triangle and kite.

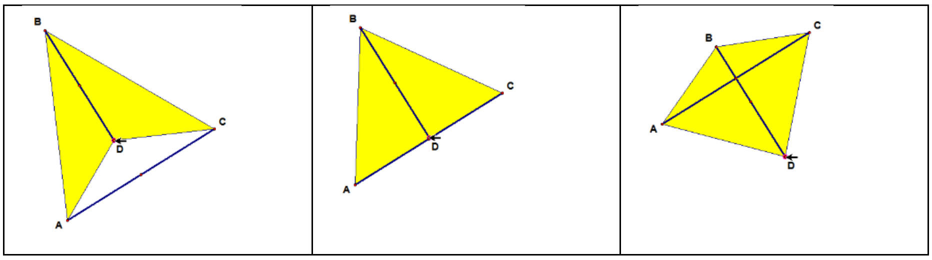


Figure 7.13 Kate and Jane use dragging maintaining symmetry when the bars are at an angle

When Kate said that she kept bar BD in the middle of bar AC during dragging so that "it wouldn't mess around with them" it suggests that she was relating the symmetry of the bars with the properties of equal sides. It seems that she has attended to both the holistic shape and the displayed measurements in order to keep symmetry constant.

7.5 Discussion

In iteration two my observation of the four dragging strategies was confirmed and the students used these strategies when the bars were vertical and horizontal (with a preference for a vertical axis of symmetry) as well as when they were oriented at an angle to the vertical. It appeared that the students could separate the orientation of the figure from the recognition of the figure when they considered the shape properties, i.e. when they used reasoning at Van Hiele level two. However if they used holistic reasoning, which is categorised at Van Hiele level one, then they sometimes failed to recognise the figure, e.g. when naming the rhombus a parallelogram when it appeared with one of its sides near to the horizontal. It must be said that the same students were capable of both types of reasoning and usually did employ both during the sessions.

The year seven students identified a discrete number of kites which could be generated whereas the year eight students told me that it would be possible to make many kites by moving the bar by a small amount each time. The two groups of students differed by school as well as age so it would not be sensible to say definitively that age was a factor in this.

7.5.1 Could DMS mediate reasoning at Van Hiele level three?

It was clear from the recordings that all students were confident in their knowledge of shapes and their properties, being able to recognise shapes and identify their side and angle properties, parallel lines and axes of symmetry. In this they displayed reasoning at Van Hiele level two.

When employing DMS the students generated shapes whose default shape is the kite. The rhombus, isosceles triangles and arrowheads could be considered to be special cases of the kites leading to an inclusive classification. However, whilst the students were able to list the shapes which can be made while dragging the bars using DMS, and have identified that there is a common property ("one bar crosses the other bar in the middle") they did not show an appreciation that this property means the shapes belong to a specific family (or *dragging family*). Furthermore some students working with the figure have typically identified that four kites can be made rather than an infinite number of kites. One explanation may be that they held a concept image of the kite (Tall and Vinner, 1981) in which the cross bar is approximately three quarters along the length of the axis of symmetry which would lead to a rejection of kites in less stereotypical proportions. Another explanation could be that the students viewed the dragging process as a journey to a discrete end point (so that they would accept kites in different proportions) rather than a continuous changing of the figure through an infinite number of possible shapes (in this case an infinite number of kites). In both cases, if the students identified a discrete number of kites which could be generated then other shapes such as the rhombus, isosceles triangle and discrete arrowheads would be likely to be seen as being different shapes albeit with some common properties.

It is possible that the students in my study had visualised the figure under dragging as changing from one discrete shape to another. I decided that if I could find a way to help them perceive the figure as continuously changing they might be able to revise their thinking towards a more inclusive classification of the shapes generated by the dynamic figure.

While reflecting on the results from the analysis of iteration two I realised that the students were likely focusing on their own activity of dragging a bar along the computer screen. I wondered whether if the students were able to sit back and observe the changes in the figure during DMS activity they might be able to visualise the figure as continuously changing through kites, with isosceles triangles and rhombus at specific points during the process. I decided therefore to create an animation of the figure under DMS which I would show to the students in later iterations, after they had worked manually to generate different shapes from the figure.

7.6 Conclusion

By the end of iteration two I concluded that the students engaged in the same dragging and measuring activities which had been used in the previous iterations, whatever the orientation of the perpendicular bars. Working with bars oriented at an angle to the vertical may have caused a greater cognitive load as the students often referred to the shapes as being at an angle (an angled kite) and the DMS strategy took longer and was less accurate in some cases.

The part of the study undertaken during iteration two involved trying a few different ideas to see which way the research should proceed. I discovered that the task could be modified for whole classes but I did not feel that I had yet got the task to the point where it effectively mediated the understanding of inclusivity of shape properties The file with perpendicular bars oriented at an angle to the vertical had been used to demonstrate that the dragging activity observed in iterations zero and one was not dependent on the figure being ‘upright’. In the next two iterations the focus would be on modifying the task to see whether students’ understanding of shapes might be moved towards inclusive relations of the arrowheads and the rhombus as kites.

8 Iteration three: Perpendicular bars oriented at an angle to the vertical and the animation of DMS

So far in the study I had ascertained that when the students dragged the bars inside the dynamic figure they were able to move the bars into place intentionally, indicating that they had an intuitive feel for the bar positions needed to generate each shape. Most of the shapes which have special properties and are given names aside from 'triangle' and 'quadrilateral' are those shapes which have an axis of symmetry. When the students moved between the shapes with symmetry they often used an efficient dragging strategy which I have named *dragging maintaining symmetry* (DMS) which appears to be a specialised form of *maintaining dragging* described by Baccaglini Frank and Mariotti (2010).

The shapes formed during DMS could be considered to be a 'dragging family' of shapes whose common property is that one bar is the perpendicular bisector of the other bar. Most of the time during DMS the figure takes the shape of a kite, including the concave kites which the participating students knew as arrowheads and which they did not in general include as kites. At special points along the dragging journey the figure passes through the rhombus (when both bars are the perpendicular bisector of each other) and the isosceles triangles (when the end of one bar sits on the mid-point of the other bar). The rhombus being a special case of a kite is mirrored in static geometry but the isosceles triangle is only considered to be a special kite in dynamic geometry.
The participating students had not shown any signs of developing reasoning towards this concept of the dragging family and I wondered whether it was because they viewed the dragging action as resulting in a discrete shape, not as a possible continuous action leading to a continually morphing figure.

An important point to note here is that, so far, I had not specifically talked to the students about 'families of shapes' or about inclusive relations such as a rhombus being a special case of a kite. I had tried to ascertain whether working on the task might encourage these concepts but realistically the students had no reason to think in terms of certain shapes being special cases of other shapes as there was no advantage to them in doing so.

8.1 Objectives for iteration three.

Iteration three had three objectives.

a) In session one the students (who had no previous experience of working with the dynamic figure in an 'upright position') would work with the bars (oriented at an angle) to ascertain whether they would use the same dragging strategies as students in iteration two.

b) I would introduce the concept of the rhombus being a special kind of kite to the students and gauge their reaction

c) Then in session two I would show them the animated figure to see whether it could be the catalyst for the development of the concept of the dragging family.

8.2 Theoretical background: Discrete shapes versus a continuously changing figure

What could have hindered the students from visualising the dynamic figure as continuously changing through an infinite number of kites? Work carried out by Mamon Erez and Yerushalmy (2007) may shed some light on this question. They conducted clinical interviews with 11-12 year old students in an Israeli school working at a task in a DGE designed to mediate the understanding of inclusive relations between quadrilaterals. Participating students dragged properly constructed figures e.g. a parallelogram, and were asked to give reasons why it could be dragged into other specific shapes (such as a rhombus). An ability to do this, referring to a hierarchical classification of shapes, was taken as demonstrating reasoning at Van Hiele level three. Other students were deemed to display Van Hiele level two reasoning which was demonstrated by their use of partitional classification e.g. saying that a rhombus cannot be a parallelogram because it has all four sides equal whereas a parallelogram has two long sides and two short sides, even though the constructed parallelogram on the computer screen could be dragged into a rhombus.

Mamon Erez and Yerushalmy (2007) observed that students who conceptualised 2D shapes according to a hierarchical classification tended to visualise the figure under dragging as moving continuously through different cases of the same figure and continuing to embody the invariant properties. Hence when the figure is at a stage where it displays a particular shape these students viewed it as being a special instance of the general figure. Students who conceptualised 2D shapes according to a partitional classification tended to visualise the figure under dragging as changing from one discrete shape to another. These students had difficulty appreciating the geometric logic underpinning the dragging mode (ibid).

This work seemed to explain why I did not observe the students in my study develop inclusive relations between the kites and the other figures (arrowheads, rhombus, isosceles triangles) which could lead to a hierarchical classification. In iteration three I would test an intervention (the animation of DMS) to ascertain whether the students would develop Van Hiele level three reasoning if they could be encouraged to observe the figure as continuously changing between the shapes. Mamon Erez and Yerushalmy (2007) suggested that if students could appreciate the logic behind the dragging of a figure then they could progress to level three reasoning. In the dynamic figure in the present study this would mean an appreciation that when one bar is dragged along the perpendicular bisector of the other bar then a family of shapes is generated with the default shape being a kite.

8.3 Process for iteration three

8.3.1 Instructional goals

Stan and Eric, and Hemma and Seema, from School B participated in iteration three for two sessions. In session one they worked with a file containing the 8 cm bar oriented at 60 degrees to the vertical and the 6 cm bar oriented at 30 degrees to the vertical. Figure 8.1 shows a kite using this file.

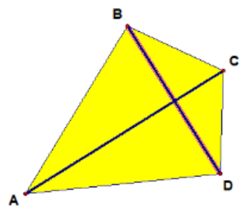


Figure 8.1 A kite generated from perpendicular bars oriented at an angle to the vertical

As before the students were given the computer file with the tilted bars separate and were asked to drag one bar over the other and I instructed the students on how to join the ends of the bars and construct the interior of the shape thus filling it with colour. The first part of the task was to drag the bars to see what shapes they could make, identify the properties of the shape and then use the Measure menu to check side and angle properties. In this iteration I made a point of asking the students to generate shapes which had a line of symmetry, to drag from one of these shapes to another and to describe their dragging activity. In this way I hoped the students would observe the invariant aspects in their dragging activity, the properties of the bars and the shapes themselves.

In the second session the students were shown the animation of the figure as if it were being dragged using the DMS strategy, i.e. so that one bar was always the perpendicular bisector of the other bar. I allowed them to talk about what they were seeing and then used questioning to guide them to consider common properties. For example, I asked them to give the properties of kites and arrowheads and when the students related the same properties I asked them whether the kites and arrowheads might actually be the same shape. I asked the students to tell me how many rhombuses could be made and how many kites and suggested that the rhombus might be a special kite because it was made at one position in between many kites. In this way I was trying to encourage the students to think about inclusive relations and whether the shapes created during the animation show were all members of one family of shapes.

8.3.2 Instructional starting points.

The two pairs of students who participated in iteration three of the study in June 2012 were assessed by their mathematics teachers as achieving at levels five/ six of the National Curriculum for England and Wales (QCA 2007). They had not encountered the computer files containing the vertical and horizontal bars.

8.3.3 Learning trajectory

In this iteration I set up the task with similar objectives as for iteration two, i.e. that students would:

- Drag the bars to generate different shapes.
- Revise and review shape properties using the measurements of sides and angles.
- Describe the relative positions of the bars in order to generate the shapes.
- Investigate the shapes which could be made using one of the bars as a line of symmetry. Move between symmetrical shapes and describe the dragging activity.
- Watch the animation of DMS and describe what happens to the figure.

I hoped to observe the students achieve the following which would indicate that they were beginning to use Van Hiele level three reasoning:

- Observe that most of the time DMS results in kite shapes, including the arrowheads. Conclude that kites and arrowheads are the same shape.
- Consider the rhombus and isosceles triangles as discrete positions along the dragging journey and conclude that this means they are special cases of the kite.

8.4 Results and data analysis of the sessions in iteration three

Each of the four sessions was recorded and imported into Transana. The first sessions were transcribed into a word processing document containing three columns: the dialogue between the students and myself, a description of the on-screen activity and thirdly a narrative account giving an overview of the sessions. The original transcription/descriptions can be accessed from the accompanying disk. A table of dragging activity and a table of episodes for each recording are included in appendix 3. A transcription of the dialogue during the animation was also made.

8.4.1 Orientation

In the recordings of the first sessions with the two pairs of students, the issue of orientation was a strong focus. At the beginning of the session Stan described the bars as being “slanted that way” and did not believe that AC and BD were at ninety degrees.

Figure 8.2 shows the isosceles triangle which was on the screen at the time of this discussion.

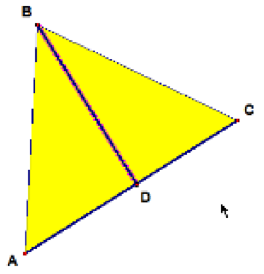
Dialogue	Screenshot
Stan: *The lines, the angles are a bit, the lines are a bit slanted that way. That line doesn't look like a right angle* Susan: *OK. What's that not a right angle with?* Stan: *The A and C* Susan: *You don't think that's at right angles?* Stan: *It's at a weird angle so you can't really tell.*	

Figure 8.2 An oblique isosceles triangle at an angle causes consternation to Stan who finds it difficult to recognise a right angle

It appears that Stan had difficulties recognising a right angle which was not ‘upright’. Later in the session, Stan and Eric both reported finding it difficult to tell which of the two bars was the longer one because “they’re both at an angle”.

At another point in the session the rhombus was on the screen (see figure 8.3) Eric stated that the angle between the bars did not look like ninety degrees because it was not straight. He said that if the figure were turned round then it might look like a ninety degree angle.

After this I asked the boys to use the measuring tool to measure the angles at the intersection of the bars and all the measurements were ninety degrees. This did not convince Eric who thought there might be a rounding error to explain his perception. He made an intriguing comment that the problem they had seeing the angles as right angles might be due to the lengths of the bars being different. “If they’re both eight or both six then it would be all equal so then it would work”.

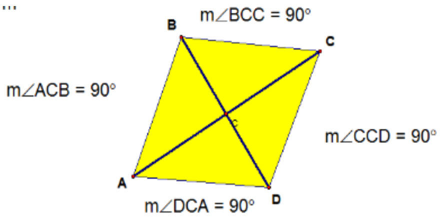
Dialogue	Screenshots
Eric: *It doesn't look like a right angle.* Susan: *Why don't you think it looks like a right angle?* Eric: *Because it's not straight. If it got turned that way and then the cross got turned that way it might. It's cause it's slanted it doesn't look very well.* Susan: *Because the computer's measured it as ninety shall we believe the computer?* Eric: *No because it might be a rounding error.* Susan: *What if we measured one of the others then?* Eric: *Ninety.* Susan: *They're both ninety together.* Stan: *Maybe those two aren't ninety. It might be opposite angles so* Eric: *Yeah it might be* Stan: *They're both ninety and they're both ninety. Yeah, they're opposite angles and that one will be ninety as well.* Eric: *Yeah, all ninety.* Susan: *So do you think it's a rounding error?* Eric: *It might not be cause, if you look at that, they look something like. Look at BCD, that one looks like, that's weird, that one looks acute. And that one says it's BCD.* Stan: *Where's BCD?* Eric: *There. It's to do with the eight and the six centimetres. If they're both eight or both six then it would be all equal so then it would work.*	Due to its orientation? …  Maybe if it was turned the right way up I could see it! They still find it difficult to believe the angle between the bars is a right angle.

Figure 8.3 Stan and Eric find it difficult to recognise right angles in the rhombus made from unequal length slanted perpendicular bars

Hemma and Seema appeared to cope with the slanted figure much better but even so the issue of orientation did affect the way they viewed the figure. The following dialogue occurred while the girls were looking at figure 3.4 on the computer screen.

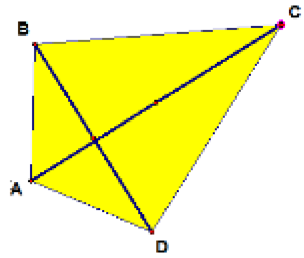
Dialogue	Screenshot
Hemma: *It's a sort of kite* Seema: *It's a kite with the bottom end* Hemma: *Yeah, that bit's more shrivelled down* Seema: *An obtuse triangle at the top and then you've got one isosceles triangle on the bottom* Susan: *So can you give me the 3 letters of the corners of the isosceles triangle* Seema: *C B and D* Susan: *OK then. What about the other triangle?* Seema: *D A and B, the obtuse triangle*	

Figure 8.4 Hemma and Seema perceive the kite as consisting of an obtuse angled triangle on the top and the isosceles triangle on the bottom

This part of the dialogue appears to suggest that the girls had mentally rotated the kite so that they perceived it as the typical upright kite. i.e. the top of the kite is ABD the obtuse angled (isosceles) triangle and the bottom of the kite is CBD the isosceles triangle.

A little later in the session the girls, who had labelled the shape in figure 8.4 as a kite, forgot what it was and suggested 3D pyramid and trapezium as names for it. Orientation did seem to affect their recognition of shapes. The girls also expressed a preference for the bars to be 'straight' as the boys had done. When I asked the girls what sort of bars they would like if they were to make a square Seema replied that she wanted two straight bars, not diagonal. She agreed that the bars did not have to be straight but that "it would make things easier".

Overall, working with oriented bars did appear to make the recognition of objects such as right angles and shapes harder to recognise, though not impossible for the students. When it came to dragging maintaining symmetry, both boys and girls had some episodes of it which were reasonably accurate as shown in appendix 3.3. In the girls' session, Seema had control of the mouse first and used DMS much later during her turn. Hemma used DMS almost straight away. The boys made use of DMS all the way through their session. So students have been able to drag along a perpendicular bisector which is oriented at an angle and with reasonable accuracy. In session one the students' activity and reasoning were comparable to those of the students in iteration two, even though their first introduction to the dynamic figure was with the bars oriented at an angle to the vertical.

8.4.2 Using measures and accepting 'close enough' measures.

As in previous iterations the students were observed to use Guided Dragging or Dragging Maintaining Symmetry to place the bars in order to generate a desired shape and then use Refinement Dragging to make the measurements of expected equal sides and angles to be as close as possible. Often the students got the measurements to be exactly equal and at other times the measurements differed by one degree or 0.1 centimetres. As in previous iterations the students in iteration three accepted that getting measures almost the same was good enough. Seema referred to lines which differed by 0.2 cm as 'similar'. Hemma described angles in a rhombus which were the 'same' (74 degrees) and angles which were 'similar' (105 degrees and 108 degrees). However, Stan and Eric were concerned that the sum of the displayed angle measurements in a triangle was 179 or, after dragging, 181 which led to a discussion about the errors which can be caused by rounding measurements (see appendix 3.2 and lines 113-131 in the transcript).

In Figure 8.5 Stan and Eric thought they had a square because all four sides were equal.

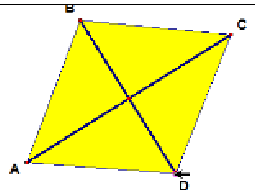
Dialogue	On screen activity
Eric : *Kind of a diamond.* Susan: *Rhombus shape isn't it. What happens if you keep pulling?* Eric: *Oh yes it's a square. All angles equal* Susan: *All sides are equal aren't they?* Eric: *Sides are equal* Susan: *Is that a square?* Stan: *There's no ninety degree angles. So it's not a square* Eric: *They're equal and those two are equal.* Susan: *Actually you've got the almost perfect rhombus there.* Eric: *Just a few off.*	 m∠DAB = 74° m∠ABC = 105° m∠BCD = 74° m∠CDA = 107° In the rhombus position the boys decide they have got a square as they have attended to the measurements of the sides and these are all equal. However Stan notices that the angles are not all ninety degrees and then Eric notices the two pairs of equal angles.

Figure 8.5 Is it a rhombus or a square?

This excerpt demonstrates how using the displayed measurements helped the students to review their understanding of shape properties.

8.4.3 The typical kite shape

It seems that the students had a preference for the way that the kite shape was presented. Hemma and Seema particularly preferred kites made using the 8cm and 6 cm bars in the three quarter position with the two smaller congruent sides at the top of the shape.

The shape in figure 8.6 was on the screen and the following conversation ensued. This kite had been generated using the two 8 cm bars.

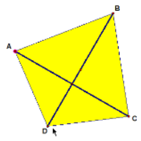
Dialogue	On-screen activity
Susan: *Can you make any kites out of that one?*	
Hemma: *I think we could but it wouldn't be the same as the other ones. They would look very different. You could still class that as a kite.*	She drags the figure into a kite 41.22-41.24 DMS
Susan: *What has to be true for it to be a kite?*	41.25-41.27 RD
Hemma: *Two sides the same small length and then two sides the same long length.*	41.31-41.35 DMS
Seema: *And they're both isosceles.*	
Susan: *What's both isosceles?*	
Seema: *Both, triangles*	
Susan: *So even if it looks a slightly different kite, it can still be called a kite?*	
Seema: *Yeah*	
Hemma: *And it has to have a line of symmetry*	
Susan: *It does doesn't it. So can we make all the shapes we made before even if are slightly different versions.*	Hemma recognises that to be a kite a figure has to have the correct properties no matter how it looks. So she is moving away from the perception of the typical shape.
Seema: *You can but it wouldn't look as accurate because the centimetre bars is different.*	
Hemma: *They're the same, exactly the same. So if that one was smaller or that one was taller it would look more like a kite. It does look like a kite now but before it looked.*	
Seema: *Because the two of them will be different. So when you make a kite because two of them are the same size, same centimetres and the other two are the same centimetres but they have different features but they would look more accurate because that's how a kite made up with two shorter ones and two longer ones.*	equal length bars appear to generate kites which the girls do not feel comfortable calling kites.
Susan: *But that's got two shorter ones which are close aren't they and two longer ones.*	She prefers kites in the typical proportion.
Seema: *I think the other ones are easier to make a kite with.*	
Susan: *In the first file the eight and the six?*	43.12-43.20 DMS
Hemma: *It still could make a kite it would just look more different. Cause that is still a kite because that's erm, it's still got a line of symmetry and that's smaller than them two and like you could still do that as a kite.*	She dragged the bar backwards and forwards to demonstrate multiple different kites.
Seema: *But it wouldn't look the same because*	This is Hemma's definition of a kite
Hemma: *Those just need to be a bit smaller*	

Figure 8.6 A discussion of kites in preferred positions and orientation

The girls used the properties of two pairs of congruent sides to qualify their decision that the figure was a kite but it was clear that they would prefer the kite to be made from the two unequal length bars and to be in an upright position. The kite in figure 3.4 was perhaps too small and squat. Stan and Eric made a right angled isosceles triangle by planning the two 8 cm bars end to end. Eric commented “isosceles don't always have to be big ones. They can be small and fat. This one's a bit chubby”.

Examples like this one indicate that the students had moved on from a purely holistic perception of shapes to being able to reason using the perceptual aspect of the shape properties. Hemma and Seema were able to recognise that the shape was a kite, even when it was in an unfamiliar presentation, by referring to its properties. Stan and Eric accepted an isosceles triangle even though it looked different (short and fat) to isosceles triangles which are typically presented in text books. This aspect of being able to recognise shapes, even when their appearances differ from the norm, because they have the required properties is evidence that the students are being analytical.

8.4.4 Partitional classification

The students held very strong perceptual understanding of shape properties in the form of partitional classification. Figure 8.7 shows how the girls partitioned kites (made of two different sized triangles) from a rhombus (made of two same sized triangles).

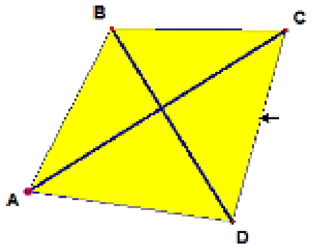
Dialogue	On screen activity
Susan: *Can you move it a little bit more and it still be a kite?* Seema: *No because always one side has to be longer than the top half.* Hemma: *Otherwise it's a diamond.* Susan: *What's the mathematical name for a diamond?* Hemma: *Oh that's it, a rhombus.* .... Hemma: *Yeah I mean like it like it's a triangle at the top and a triangle at the bottom.* Susan: *That's true isn't it. Is that not true for a kite as well? What have you got there?* Seema: *But the triangle at the bottom they're not* Hemma: *They're different. That's an equilateral triangle. It looks like one.*	The bar BD is moved round a bit to generate different kites the girls appear to use a partitional classification and to prefer a typical kite 

Figure 8.7 Partitioning a kite and a rhombus

Seema's comment that "one side has to be longer than the top half" and Hemma's comment that "they're different" shows that they thought of kites as being comprised of two different sized triangles "otherwise it's a diamond" (rhombus).

Stan said described a square and a rectangle in these terms:

"The square has four sides that look the same. A rectangle's longer with two sides that you can do like that"

In these cases the rhombus was partitioned from the kites and the square was partitioned from the rectangles. Partitional classification has been the norm with all pairs of students who participated in the study.

8.4.6 How the students perceived the figure under dragging

Hemma and Seema were an interesting pair of students who interacted well whilst working on the task. Seema appeared to use a holistic perception when recognising shapes. This became apparent when she made comments such as describing the rhombus as “a kind of diagonal square” Later Seema explained how she dragged the bars to make a kite as shown in figure 8. 8.

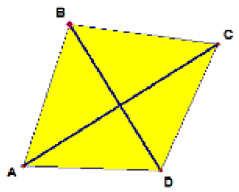
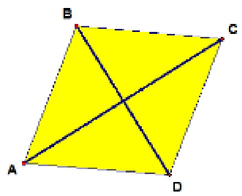
Dialogue	On screen activity
Susan: *So where do those bars cross each other?* Seema: *Half way* Susan: *OK so what would you like to know so you can put them half way* Seema: *You know how that one's eight centimetres, where four centimetres is*. Susan: *Right so you really want to know where the middle of the line is.* Seema: *I wasn't actually looking at that when I was changing it.* Susan: *So what were you looking at then?* Seema: *I was just looking at A and C wanting to know how far it would be until I get like A yeah, diagonal but then kite looking.* Susan: *OK so what would you say you were actually looking at, cause there's two things to look at, there's the measurements* Seema: *I was looking at the area of the shape and then I was wondering how far the lines had to go between*	from kite  to rhombus  Holistic perception of the figure By ‘area’ Seema may be referring to the inside of the shape. She seems to view the shape holistically.

Figure 8.8 Seema describes how she positioned the bars

It does seem as if Seema used the shape of the figure to guide her as she moved the bars. This might also explain why she did not use DMS at the beginning of her session controlling the computer mouse. She used more Guided Dragging which may have been a response to the changing figure and adapting the positions of the bars to gain the desired shape. She always got to the desired shape but not always through keeping symmetry constant. In this Seema was unusual among the participating students.

On the other hand Hemma considered the properties when identifying shapes. This was shown by her willingness to accept shapes as being representative of their class even when they were presented in an atypical form. For example, Hemma led the discussion of whether the kite made with two 8 cm bars was in fact a kite. She said it could still be classed as a kite and quoted the properties of a kite; “Two sides the same small length and then two sides the same long length”. Hemma was also keen to say that a shape had line symmetry. Since she considered the properties of a shape as an identifier of the shape she was more willing to identify that multiple kites and arrowheads could be generated by moving one of the bars by very small amounts. She and Seema had three conversations where Hemma demonstrated that she could make a number of kites or arrowheads and she used the DMS strategy whilst doing this. This was the strongest indication of a student using DMS and linking it to a possibly infinite set of kites. She even managed to convince Seema that there are a number of kites which can be made (they settled on twenty six). This particular conversation is shown below in figure 8.9.

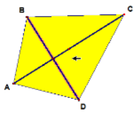
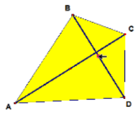
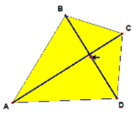
Dialogue	***dragging activity***	***narrative description***
Susan: *They are close enough aren't they. OK so how many kites do you think you could make?* Seema: *About two.* Susan: *You think about two?* Hemma: *I think a bit more because if I do that that's still a kite* Susan: *That's true.* Hemma: *And if I do that it's still a kite. Cause look they're the same and they're closer. Then if I also do that then that's still a kite and that's still a kite.* Susan: *How many kites do you think you could make then.* Hemma: *I think about six or seven.* Susan: *Could you not keep on moving it a little bit all the time and it still be a kite?* Hemma: *Like a millimetre* Seema: *It could go on for ages couldn't it.* Susan: *So how many do you think it could be then if you can go on for ages?* Seema: *About twenty.* Hemma: *Twenty five or twenty six*	 27.45-27.47 DMS 27.48-27.49 DMS 27.55-27.57 DMS 27.58-28.00 DMS   28.10-28.11 DMS 14-28.15 DMS	Hemma keeps moving bar BD a little bit to demonstrate that there are multiple kites which can be made with AC as a line of symmetry.

Figure 8.9 Hemma demonstrates a number of kites

On the other hand Stan and Eric dragged through kites but did not usually stay on a kite. They were more interested in isosceles triangles and arrowheads. They decided that there were four possible arrowheads with the bars in each of the four relative positions. See figure 8.10.

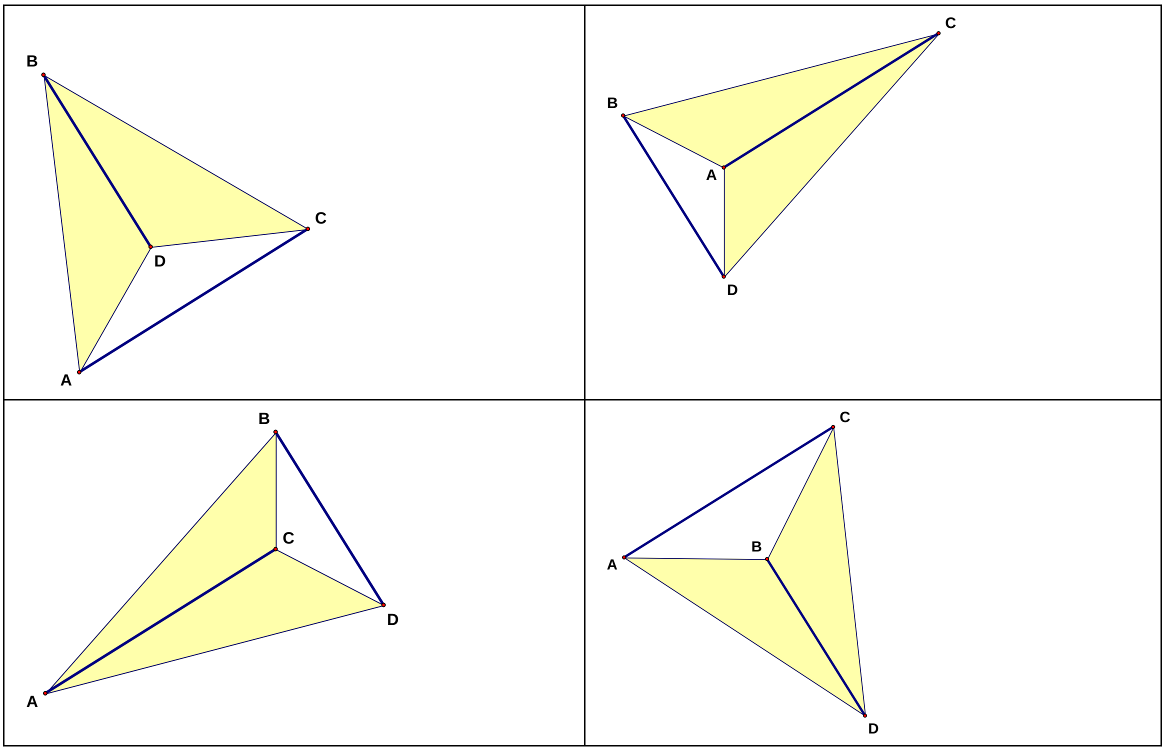


Figure 8.10 The four relative positions of arrowheads.

When I suggested that they move the bar just a little to generate another arrowhead Sam said "You can make loads if you do it slowly. One, two, three". (Meanwhile he dragged the bar AC using small movements to demonstrate more arrowheads).
This has a faint suggestion that Sam had started to view the figure as changing continuously although he only identified a discrete number of arrowheads.

In order to find out whether the students might see that dragging to keep one bar as a line of symmetry would generate a 'family of shapes' I introduced the term in the session. Stan and Eric identified all the different shapes that can be made keeping BD as the line of symmetry but they did not make any use of the 'family of shapes' concept themselves. Similarly I introduced the term to Hemma and Seema but they did not make anything of the term or use it themselves.

8.5 Discussion on session one.

By the end of session one, the students had made all the shapes which students in the other iterations had made, and they used all four of the observed dragging strategies.

I decided at this point that presenting students with bars oriented at an angle had served its purpose. It had demonstrated that the students in the study could recognise and work with the figure when oriented away from the vertical and they used the same dragging strategies as those students who had worked with vertical and horizontal bars. It also showed that students did not only use DMS when the figure was upright thus indicating that they were dragging to keep symmetry a constant rather than just dragging down the computer screen.

All students in the study, including Stan and Eric and Hemma and Seema, demonstrated that they recognised common triangles and quadrilaterals and could list their properties. Sometimes they revised and reviewed this knowledge through the actions of dragging, particularly refinement dragging and attending to the displayed measurements of sides and angles.

I had observed that a notion of symmetry often seemed to be used when students dragged bars to generate a particular shape. When Hemma used short bursts of DMS to demonstrate different positions for kites or arrowheads she gave the clearest indication, thus far in the study, that the DMS strategy could be linked to the concept of a dragging family (all the shapes which can be generated by dragging one bar along the perpendicular bisector of the other bar).

In session one of iteration three I had deliberately asked the students to drag through shapes which had a common line of symmetry and the students had identified a discrete number of shapes which could be generated. It appeared that they viewed the dragging process as a journey towards an end (a discrete shape) and not as a continuous process. I had identified this at the end of iteration two and in section 8.2 I cited the work of Mamon Erez and Yerushalmy (2007) who had observed that students who work at Van Hiele level two have a tendency to perceive a dynamic figure as changing into discrete shapes under dragging, as the students in this study have done. I wondered then, if students could be helped to visualise the figure as continuously changing during dragging, whether this might be the catalyst for development of an inclusive relationship between the shapes which are generated by a DMS action, a dragging journey. My hypothesis was that if they could sit back and watch this DMS action, they might be able to recognise the figure as continuously changing between an infinite number of kites with the rhombus as one discrete point. Therefore, in session two the students were shown an animation of the DMS strategy with horizontal line symmetry.

8.6 Session two: the animation

Session 2 was designed to ascertain whether the students could be encouraged to visualise all the shapes which the figure moves through by animating the dragging maintaining strategy. What follows is an account of the session with Stan and Eric and some points from the recording with Hemma and Seema.

8.6.1 Stan and Eric watch the animation of the figure under DMS

In the second session I introduced the students to the new file and asked them what they could see on the screen. The figure on the screen was in the shape of a kite (see figure 8.8).

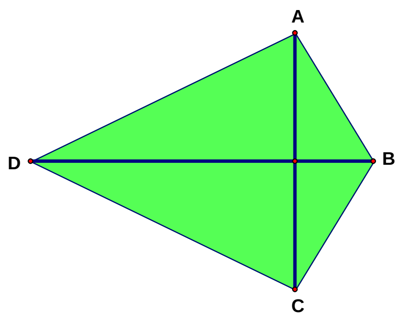


Figure 8.11 The figure before it was animated

Stan said "I see a kite". When asked why he thought it was a kite Eric replied "Because it's shaped like one". This suggests that Eric first looked at the shape in a holistic way recognising it as a kite by its general shape.
Next I asked the boys to tell me the properties of the kite. Eric said:

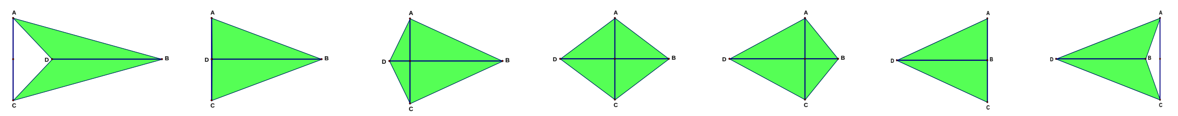
"Two pairs of equal sides and four sides".

Whilst this statement is not sufficient to provide the definition of the kite, since Eric did not say which pairs of sides were equal, he moved the cursor over the sides AD and DC and then moved the cursor to trace over the sides AB and BC. This activity indicates he did know which sides made an equal pair.

When asked about the properties of the bars AC and BD:

Eric: *AC and BD, they're not the same length*
Susan: *What about the angle between them?*
Eric: *It's a right angle*
Susan: *And where do they cross each other?*
Eric: *Mmm, here (the cursor pointed to the intersection of the bars).*
Susan: *How would you describe that?*
Eric: *Not in the middle but near the top.*

As can be seen from Figure 8.8 "not in the middle but near the top" suggests that Eric considered that the point B was at the top of the kite. Was he mentally rotating the kite ninety degrees so that he visualised it in the typical upright orientation? Alternatively he may have been thinking of a toy kite which would fly with B at the top.



arrowhead	Isosceles triangle	kite	rhombus	kite	Isosceles triangle	arrowhead

Figure 8.12 Seven positions of the animated figure

After we had discussed the kite I pointed out the Animate Bar button on the screen. I asked the boys to click on the button and see what happened. The button animates the bar AC so that it moves along the line (which is slightly longer than BD on both sides). As bar AC moves along this line the figure changes shape in a continuous fashion through kites, the rhombus, isosceles triangles and the arrowheads (see figure 8.12). Eric and Stan had previously generated these shapes in session 1 when they used the Dragging Maintaining Symmetry strategy. The animation was designed to mimic what they were doing allowing them to observe the figure continuously changing. The intention was that they would notice many more positions for the kite and arrowhead and that this might lead to a concept of a dragging family

It is difficult to demonstrate the fluid movement as the figure changes between shapes but Figure 8.12 attempts to give an idea of what this looks like. As soon as the animation started the boys began to relate which shapes they were seeing.

Eric: *Triangle, arrowhead*

Stan: *Arrowhead. Now it's gone back to a kite.*

Eric: *It's showing you all the shapes it can be. Oh rhombus, back to a kite.*

Stan: *Isosceles triangle. It's going to another arrowhead.*

Susan: *What do you think is actually happening?*

Eric: *Well you know when we moved the line last week, it's showing the shapes you can make in between them.*

Stan: *Apart from it's a bit slower, so you can see what's happening. It kind of shows you the line of symmetry.*

Susan: *Which one's the line of symmetry?*

Stan: *You've got BD so it shows all the shapes you can make with one line of symmetry.*

Eric: *The arrowheads are changing.*

In this excerpt from the dialogue it appears that Eric and Stan noticed two different aspects of the animated figure. Eric noticed that there were more shapes in between those that he had dragged the figure to in the previous week. The other shapes had been the many kites and arrowheads between the discrete kites and arrowheads he had made by dragging. I had hoped that the animated figure might enable the boys to visualise the large number of kites and arrowheads which can be generated when the bar which is being bisected at right angles is moved along the other bar. In theory there should be an infinite number of kites and arrowheads as the bar would be moved by myriad infinitesimal amounts. In practice, the constraints of the computer software probably mean that the bar makes small discrete movements leading to a finite number of possible shapes.

Stan, on the other hand, had noticed that all the shapes being generated had the bar BD as a line of symmetry. The common property of the symmetry of the shapes made by the animated figure might lead to the concept of a dragging family. Next, I asked some questions to ascertain whether the boys would be able to appreciate the significance of this.

Susan: *How many kites do you think you actually see when you animate the bar?*

Eric: *Two cause you see the ones*

Stan: *You actually see loads cause each angle changes*

Eric: *You see lots of different one's cause that's still a kite. Kite, kite, kite, rhombus, er kite*

At this point Eric began to see that many different kites can be generated from the figure. I got the boys to stop the animation when the shape was an arrowhead.

Dialogue	Commentary
Stan: It has two pairs the same side lengths. It has two angles that are the same. AD and CD,and AB and BC.	Properties of an arrowhead i.e. AD and CD
Susan: Right. and where are the two equal angles?	i.e. AB and BC
Stan: DAB and BCD.	
Eric: It's split in half, it's the same shapes so that's why it's symmetrical.	two equal and opposite angles
Susan: That's right. You know when you described a kite weren't you saying the same things?	I suggested that they told me the same for the properties of a kite
Eric: Yes I suppose they're like brothers.	
Susan: I like that, they're part of the same family?	Eric saw a connection between kites and arrowheads
Stan: The same family yeah.	
Susan: OK so the arrowhead and the kite might be in the same family, brothers together.	We discussed why the kites and arrowheads might be in the same family and this time Stan and Eric ran with the idea, adding the rhombus into the family.
Eric: Same size, two same size angles and one line of symmetry, two pairs of equal sides.	
Stan: That's why, same as a rhombus as well	
Eric: I suppose that's why it does that so you can see that that, all those are in the same family.	They added the isosceles triangle into the family
Stan: And the triangle, and the rhombus.	

Figure 8.13 Arrowheads and kites

At this stage in the session it did look as if the boys had grasped the idea of the dragging family. The boys had noted that the kites and the arrowheads shared the same properties and therefore could be related (although they were not able to see that they might actually be the same shape). Next they suggested that these properties held true for the rhombus and isosceles triangle too. However as the boys thought about this again they reverted to a partitional classification by considering that a rhombus has two pairs of equal angles whereas the kite has one pair of equal angles.

Susan: *Why do you say that the rhombus is in the family of kites?*

Stan: *Because it's an 8 by 6 bar so you've got two different line sizes obviously. So you have two different lengths.*

Susan: *Do you think the rhombus might be a member of that family as well?*

Stan: *Well I suppose it kind of is. But it has two sets of equal angles. No it won't because it has two sets of equal angles.*

Next I suggested that the properties which hold for a kite are also true for a rhombus:

Susan: *If you say that BD is a line of symmetry and AD has to be equal to DC and AB has to be equal to BC, that's what you are saying for the kite. Is that also true for a rhombus?*

Eric: *I think so*

Susan: *So maybe you could say that a rhombus is a special member of the family.*

Eric: *Yeah, maybe it's the dad.*

After a discussion about the properties of the isosceles triangle the boys decided that the isosceles triangles were the uncles in the family. In summary they decided that the rhombus is the father, isosceles triangles the uncles and the arrowheads and kites the children!

8.6.2 Stan and Eric onstruct their own kite

With time left in the session and in order to try and build on Stan and Eric's notion of the family of shapes I asked them to construct their own kite figure. Some hints and prompts from me were needed (mainly suggesting they tried to mimic the conditions of the bars needed to generate a kite) but otherwise the method was the boys' own idea. The boys constructed a kite by drawing a line, constructing its midpoint and then constructing a perpendicular to the line through its midpoint. In DGS this results in an infinite perpendicular so, at my suggestion, the boys drew a line over the perpendicular which crossed their first line. Then they hid the perpendicular line. Figure 8.14 shows their progress so far.

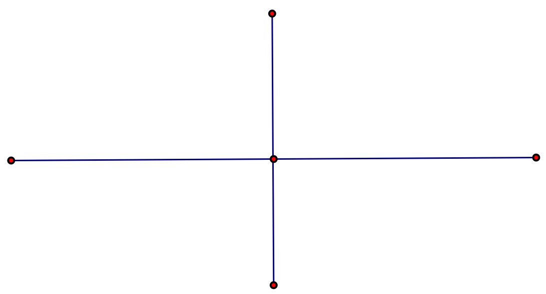


Figure 8.14 perpendicular lines

Next they joined the ends of the lines and constructed the interior of the figure which was close to a rhombus because of the way that the boys had positioned the perpendicular line symmetrical to the original line (see Figure 8.15).

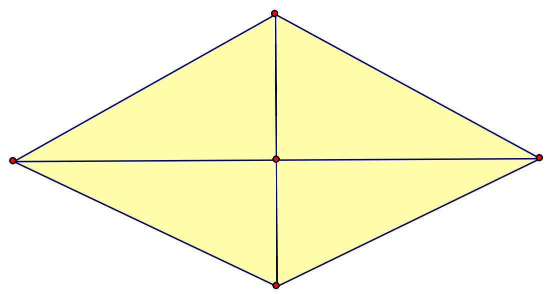


Figure 8.15 a kite figure which is close to a rhombus

The boys dragged the top point and the bottom point and were able to generate kites and arrowheads and isosceles triangles as they had done with the project files. The points at the end of the vertical line are constrained on the perpendicular bisector of the horizontal line. The boys were very pleased that the shapes were the typical orientation, i.e. using the vertical – horizontal frame of reference.

Stan: *It's an awesome kite. It looks better cause it's like upwards.*

I suggested to the boys that since they had constructed a kite then the shapes they generated must be versions of a kite.

Susan: *Is that the kite family because that's what you set out to make isn't it.*
Stan: *It's the rhombus family*
Eric: *Cause the rhombus is the middle one.*
Susan: *But what is it most of the time? Because the rhombus is the middle one doesn't that mean there's only one position you can get the rhombus in?*
Stan: *Yeah all the others, you can make two of them (moving back to thinking about discrete positions of kites)*
Susan: *Mmmmm*
Stan: *Maybe it's the arrowhead*
Eric: *Yeah cause that's like the end product.*

After some more shapes had been generated and the boys had turned the figure round they decided they could make a square if they made the lines inside the figure the same lengths. In doing this they were building on what they had learned about the square in session one.

8.6.3 Excerpts from the recording with Hemma and Seema

Hemma and Seema were looking at the figure in the position below in figure 8.16. This is very close to the isosceles triangle but Hemma realised that it was not exactly in the correct place for a triangle.

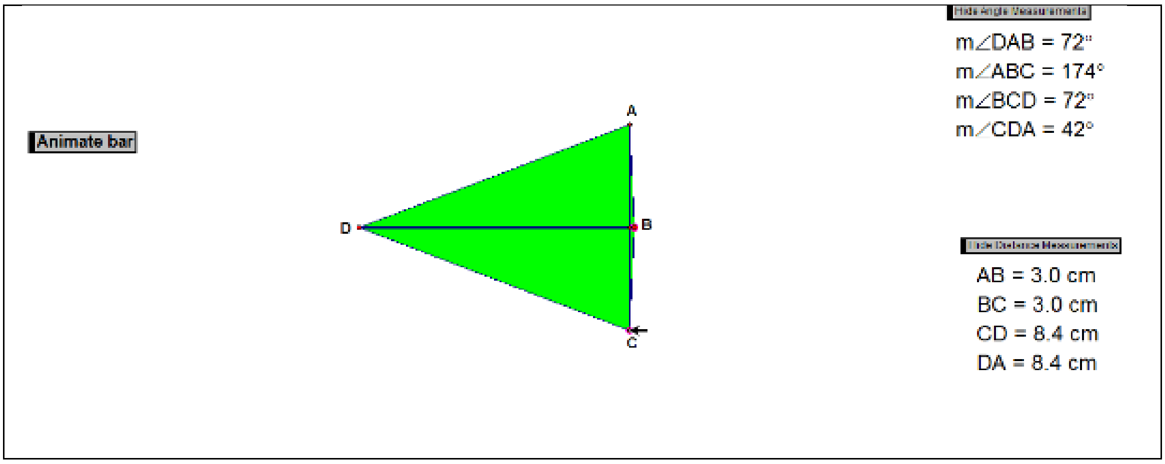


Figure 8.16 A kite near to the isosceles triangle position looks like a n isosceles triangle, but Hemma used its properties to deduce that it was a kite

Hemma made the following observations:

"*I actually think it might be a kite. But it's just a very odd kite. Cause do you know how we were saying that you can make loads and loads and loads of kites, I suppose it is still a kite. It's just got a very small top bit*".

Another interesting comment by Seema was to say that bar BD split the figure (seen in figure 8.17) in half so anything which happened to that side (here the cursor pointed to the top triangle ABD) also happened to that side (then the cursor pointed to the bottom triangle DBC). This is a nice observation of the effects of symmetry in a dynamic environment.

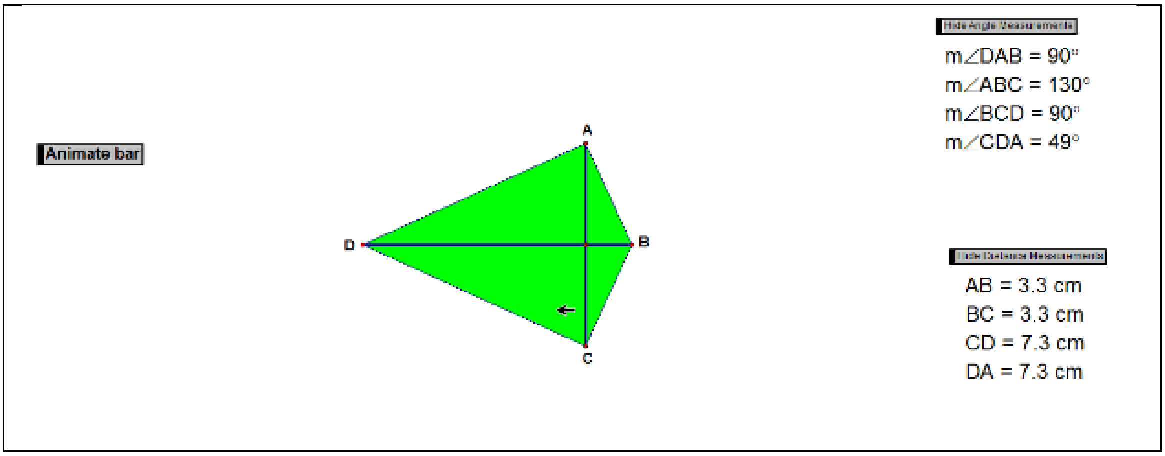


Figure 8.17 The kite can be split into two equal halves along bar BD and dragging affects the two halves in the same way.

Later in the session I asked the girls if kites and arrowheads could be in the same family since they had the same properties. “They might be” came the reply “but normally you wouldn’t see it because they look so different”. When I suggested that the arrowheads and kites might just be different versions of the same shape I got a very unconvinced “mmmm”! When we looked at the rhombus and discussed if this could be a member of the family the girls started to look at the similarities between the kites and rhombus, particularly that AB = BC and AD=DC. When the girls turned their attention to the isosceles triangle they noticed that there were still two pairs of equal sides (just that one pair were on the same line). The dialogue suggested that the girls were beginning to think in a more inclusive way about the shapes which were generated by the animation of the dynamic figure, beginning to notice the similarities between the shapes whilst still being aware of the differences.

8.7 Discussion on session 2.

The animation was designed to allow the students to focus on what happens when one bar is dragged through the figure using a DMS strategy and it certainly did give the students the opportunity to notice the figure as continuously changing through many more shapes than they had identified in session one. When they viewed the animation the boys noticed that there were many more kites generated between the rhombus and isosceles triangle for example. The girls noticed that many shapes were generated which had something in common (BD as a line of symmetry which did not move) and that the changing position of AC changed the lengths of sides and the angles in the figure. That the dynamic figure morphed into kites, arrowheads, isosceles triangle and a rhombus seemed to suggest to the students that they had more in common than they realised at first. The students did seem to be accepting of the idea that kites and arrowheads were the same shape, or that a rhombus was a special instance of a kite (when the bar AC was at the mid-point of bar BD). However a partitional classification view of shapes can be strongly held and not easily given up in favour of a hierarchical classification. As De Villiers (1994) said, students need to appreciate the functionality of a hierarchical classification if they are to use one and even students he interviewed who competently used a hierarchical classification preferred to use a partitional classification.

When Stan and Eric constructed a kite from an empty screen and then dragged it they again saw that they could generate a rhombus, isosceles triangles and arrowheads. When I suggested that these shapes might be special cases of the kites since they had constructed a kite, the boys suggested that it might be a family of shapes, but thought it would be a rhombus family or even an arrowhead family because the arrowhead is the end point. This seems to be the opposite of constructive classification; the boys had started with the most highly defined object and relaxed the rules to generate its family.

However, both Stan and Eric and Hemma and Seema were beginning to accept the idea that shapes can be related if they have a common property which would indicate a (small) movement towards Van Hiele level three reasoning.

My premise in creating the animation was that, if the students saw the rhombus as a discrete position among an infinite number of kites, then they might come to see that the rhombus is a special case of a kite. In order to accept the concept of an infinite number of kites it is necessary to accept unusual looking kites into the set of kites. Hemma showed that she was able to do this by accepting the shape in figure 8.12 as being a kite, if an odd one. Of all the students who took part in iteration three I feel that Hemma was the one who was furthest along the road to reasoning at Van Hiele level three. She was able to recognise kites in all manner of positions and used the properties of the kite in order to do this. She had begun to use analytic reasoning which characterises Van Hiele level three reasoning.

8.8 Conclusion

In order to form a concept of inclusive relations of the rhombus into the family of kites I maintained at this point in the study that it is necessary to view all figures whose properties indicate a kite shape into the family of kites. This includes all the shapes which look very close to the rhombus and those which look very close to isosceles triangles. Students learn about kite shapes in quite typical orientations where the top two congruent sides are equal and the bottom two congruent sides are equal and where the proportions look something like the shape in figure 8.18.

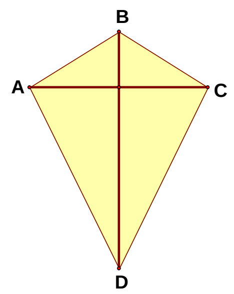


Figure 8.18 The typical kite

This appears to lead to the development of a concept image of the kite. When students are asked to consider kites which are presented in different orientations and proportions they may not wish to label them as kites. However, as can be seen from the data in my study, students often deal with this by labelling such kites as having some peculiarity eg an angled kite, a bit of an odd kite, a short chubby kite. The factor which enables students to accept unusual representations of kites into the kite family is an understanding that if the properties indicate a kite (usually the properties used are two pairs of adjacent equal sides) then the shape is a kite. This is reasoning at Van Hiele level two.

In order to progress to an understanding of the inclusive relations of the rhombus into the family of kites using a hierarchical classification, students need to accept all the kites, no matter their proportions, into the kite family. This is the case for the kite presented in a dynamic form at any rate and I suspect that it may be the case even for the static version of a kite. Thus it is clear that secure Van Hiele level two reasoning is necessary before development of Van Hiele level three reasoning can begin as Van Hiele himself maintained.

At this point in the study I was fairly confident that I had found the intervention which could act as a catalyst for this development. This would be the working with the dynamic perpendicular quadrilateral followed by the animation. In iteration four I would trial this intervention with a whole class of students over three lessons. This time the bars would be vertical and horizontal because I had shown that the students used the same activities and made the same progress with their work on the task whether they worked with the upright figure or the oblique figure.

9 Iteration 4

In iteration three I tested a new hypothesis: that if students could perceive the dynamic figure under DMS as continuously morphing through an infinite number of kites, then this could be the catalyst for raising their reasoning to Van Hiele level three. This entailed students using the perceptual properties of the kite to identify the dynamic figure as a ‘kite’ even when the figure was in an untypical proportion such as when it was close to a rhombus or isosceles triangle. I had found that, observing an animation of the figure which mimicked the DMS strategy horizontally, had helped the students, in iteration three, to perceive that an infinite number of kites could be generated, and that the rhombus was generated at one position in the animation while the isosceles triangles were generated at two positions. The arrowheads, of which there are an infinite number, had the same properties of a kite and so could be the same shape (as long as the convex angle was not included in the list of properties). The dialogue from the recordings showed that the students moved between accepting the rhombus as being a special kite and insisting the rhombus and kite were separate shapes. The animation seemed to convince them but, when they considered the properties of kites and rhombus, they fell back on partitional classification of the shapes.

9.1 Objectives for iteration four

In iteration four I decided to test whether working with the dynamic perpendicular quadrilateral could be an effective intervention to develop the concept of inclusive relations, particularly for the rhombus as a special case of the kites. This iteration was conducted within a whole class context and I modified the task so that it could be used with a class of year eight students over three lessons. This meant I could include the activities of working with the computer files and watching the animation within a pedagogical sequence of activities designed to support the development of the concept of inclusive relations. An important part of this pedagogical sequence was interactive whole class discussion bringing into play the socio-cultural aspects of learning which are discussed in section 9.2.

9.2 Theoretical background: Socio-cultural aspects of learning

"The most powerful way of introducing students to new mathematical ideas is to work creatively with a whole class so that students become collectively aware of the potential of new mathematical tools: new mathematical knowledge" (Sutherland, 2007, p.43).

Vygotsky (1978) said that children actively construct their own knowledge and understanding of the world by building on the knowledge they already have and in the cultural context in which they live. For most of the world's children, and certainly for children in the UK, they learn about formal mathematics in the school classroom. The school mathematics class is a small mathematical community with the teacher as expert. It is the teacher's role to introduce their students to new mathematical concepts, and the tools which would support the learning of new concepts which may be mathematical notions such as mathematical notation, the Cartesian frame or calculators and computers (Sutherland, 2007).

Teachers also plan lessons and design tasks for their students. These tasks are mediated by particular resources and tools which propagate the mathematical ideas the teacher intends the students to learn (ibid). However, the small community in the mathematics classroom provides an additional valuable resource in the teacher and students who make up that community. Human beings are social animals who create meanings and construct knowledge as part of a cognitive community (Donald, 2001). Interactive discussion in the classroom between teacher and students, and between students, is therefore an effective way to share and develop mathematical ideas and concept development in students as individuals. A description of a research project undertaken by Sinclair and Moss (2012) exemplifies this.

Sinclair and Moss found that children, aged four and five years old, who participated in a class lesson on geometric shapes, were initially unwilling to accept three sided shapes as triangles whose proportion and orientation were not that of an upright equilateral triangle. Sinclair used a DGS triangle which she dragged into different versions of a triangle including a long skinny version and recorded the dialogue between her and the children. One particular child in the group perceived the shape to be a triangle by referring to the property of having three corners no matter what the proportion and orientation of the shape, demonstrating level two reasoning. The comments from this child to the class that the shape was a triangle because it had three corners did appear to affect the way the other children viewed the shape, many of whom had originally dismissed the label of ‘triangle’ for this shape. Having responded to questions from Sinclair, listened to the comments of the first child, and viewed the triangle figure under dragging, most children in the group had begun to move towards accepting long skinny triangles in the family of triangles. During the session some of the children oscillated between level one (non-acceptance of an atypical triangle) and level two, acceptance due to its properties. Oscillation between two levels of reasoning was exactly what I had observed in iteration three (although my older student participants oscillated between Van Hiele levels two and three). Their thinking was clearly perturbed by the dichotomy between their old understanding of triangles and the new understanding introduced by their teacher and more knowledgeable peers. The importance of dialogue between teacher and students and between students can therefore have an important part to play in the development of their geometrical reasoning.

9.3 Process for iteration four

9.3.1 Instructional goals

In the whole class lesson my aim was to introduce the concept of inclusive relations to the class through working with the dynamic figure, discussion of the properties of the shapes generated from it and by watching the animation. In particular I wanted to present the idea that the rhombus, isosceles triangles, kites and arrowheads were all members of the same family generated when one bar in the dynamic figure is dragged along the perpendicular bisector of the other. This idea is a form of situated abstraction (Noss and Hoyles, 1996) of the concept of inclusive relations in the context of the dynamic perpendicular quadrilateral. I expected that the class would include some students who were more advanced in their geometrical reasoning than others and whose contributions to class discussion might act as a catalyst for the development of the reasoning of their peers.

The lesson plans are included in appendix 4.1. Activities in the lessons were:

- Students used two geo strips of different lengths as concrete representations of the bars. Keeping the geo strips at right angles students put one over the other and imagined what shapes they could make if they joined the ends of the bars. They were then asked to add the stipulation that one bar now acted as the perpendicular bisector of the other bar and to imagine what shapes could be made. Students sketched some of these shapes on mini white boards. After this I added the stipulation that both bars were the perpendicular bisector of the other. Only the rhombus results from this final stipulation when starting with unequal bars. The activities with the geo-strips were designed to mimic constructive classification of shapes (De Villiers, 1994).

- Students worked in pairs on laptop computers with the computer file containing the dynamic figure constructed around 8 cm vertical and 6 cm horizontal bars. They were asked to investigate what shapes they could generate and to use the displayed measurements to try and make the shapes as accurate as possible.

- For activities with both the geo-strips and the dynamic computer figure students revised and reviewed shape properties using the measurements of sides and angles and described the relative positions of the bars in order to generate the shapes.

- The dynamic figure was projected from my laptop computer onto a whiteboard. Using a radio mouse, volunteer students dragged the figure into different shapes. During this activity a whole class discussion ensued where I posed questions such as

    - What has to be true to make that shape an accurate arrowhead?
    - How could we make that an accurate kite?
    - Are arrowheads and kites the same thing (in response to one pupil suggesting an arrowhead is a concave kite)?

- The students watched the animation of DMS and were asked to describe what happened to the figure.

- Class discussions explored whether shapes generated whilst dragging to keep one bar as the perpendicular bisector of the other might form a family of shapes.

- Finally in the third lesson the students were asked to each make a poster illustrating what they had learned from the previous two lessons.

9.3.2 Instructional starting points.

The class of thirty-one year eight students who participated in iteration four of the study in June 2013 were assessed by their mathematics teacher as achieving at levels six/seven of the National Curriculum for England and Wales (QCA 2007). It would have been more consistent with the rest of the study if I could have worked with a class set of average attainers who in this school would have been working at National Curriculum levels five/six. However, access to a mathematics class was contingent on the availability of a class whose teacher was happy to work with me. In the event I was able to collect useful data whilst working with this class. This class had not previously encountered the computer files containing the vertical and horizontal bars, nor had they previously worked with a DGS program.

9.3.3 Learning trajectory

As can be seen in the lesson plans in appendix 4.1, I decided to change the instructions I gave to the students when they worked on the task in order to develop the concept of a constructive definition of shapes by adding constraints on a figure as described by De Villiers (1994). The properties of the perpendicular quadrilateral are that the diagonals intersect at right angles. Adding the property that one diagonal must bisect the other, generates the kites which are a subset of the perpendicular quadrilaterals. A further property; that both diagonals bisect each other, generates the rhombus, which is a subset of the kites. Using a constructive definition of shapes it may be possible to develop the concept of inclusive relations and a hierarchical classification of the shapes generated using the dynamic figure. I hoped that the sequence of instructions would encourage the students to notice common properties of shapes and to begin to appreciate that some classes of shapes are subsets of others.

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9.4 Results and data analysis of the sessions in iteration four

Data which I collected in iteration four comprised my recollections written straight after the lessons, photographs of students' work, the on-screen recording and dialogue from the whole class session based on the dynamic figure and the posters which each student made.

9.4.1 Lesson one

In lesson one I wrote my impressions of the lesson straight afterwards (given in the rest of this section) and photographed some student work from the mini whiteboards (some examples are shown in appendix 4.2).

The students used the geo strips to find shapes which could be made whilst keeping the diagonals perpendicular and then later keeping one diagonal as the perpendicular bisector of the other. They found it hard to articulate how the bars were positioned with each other and preferred to show me visually how they were positioned. When they worked with the perpendicular quadrilateral on the computer they were also able to generate the arrowheads (which had not occurred to them whilst using the geo-strips). Overall the students were able to identify all the shapes which can be made when one bar is the perpendicular bisector of the other. Between them, the class identified one rhombus, several isosceles triangles and an infinite number of kites and arrowheads.
In their description of the properties of the bars and the other properties of the shapes, students often referred to the diagram they had drawn. The figural aspect and its particular representation were clearly dominant. The student who had drawn these two shapes (Figure 9.1) on the mini whiteboard described the shapes in the following ways:

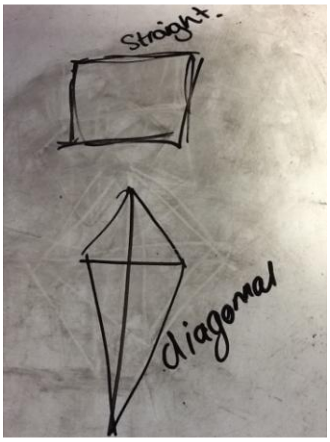


Figure 9.1 photo of sketch of figures drawn on mini whiteboard

To make a kite:

Student: *So you get the 5 centimetre bar and you put it anywhere on the 7 centimetre line but then the 7 centimetre line has to be in the middle of the 5 centimetre line.*

Susan: *What's the difference between a square and a kite?*

Student: *A square is regular because all the sides are the same length and the sides are going either sideways or up and down. And a kite is more irregular with diagonal lines and all the lines are the same length.*

Hence the kite has been described by the process of placing the diagonals (bars) and the square has been described by the property of equal sides and with reference to its typical orientation (from a holistic perception of the square). These descriptions are suggestive of a mixture of level one and level two reasoning. In this way these students were very like the others who had participated in the study and who had been classed as average attainers by their teachers.

In lesson one the students had the opportunity to become familiar with the dynamic figure and to describe the positions of the bars and the properties of the shape. There was a wide range in the sophistication of geometrical reasoning amongst the students in this class. Some students identified that there was an infinite number of kites and of arrowheads which indicates level two with possible level three reasoning. On the other hand some students such as the one who drew the shapes in figure 9.1 described a square according to the orientation of its sides which is indicative of reasoning at level one and level two (since this student also said the square has sides the same length).

9.4.2 Lesson two

In lesson two I used the image capture software to record the on screen activity on my laptop computer and the dialogue between me and the class (transcript in appendix 4.5). I also used a digital recorder to record some of the comments made by individual students.

I decided that the students needed some support to articulate the positions of the bars and the properties of the shapes. I projected the perpendicular quadrilateral onto the white board through my laptop computer and asked student volunteers to generate specific shapes by dragging the figure using a radio mouse. I then encouraged the students to articulate properties of the bars and the properties of the sides and angles in the shape. It can be seen in the dialogue below that the use of mathematical language by me and some students in the class became a catalyst for the improved use of mathematical language by the other students.

Susan: *You're going to make a shape so that one bar bisects the other.*

(The student made the arrowhead kite in figure 9.2 by dragging bar AC to the right of bar BD).

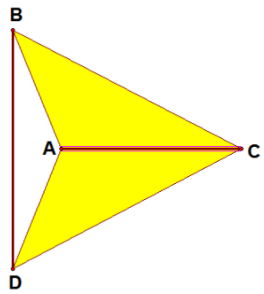


Figure 9.2 The arrowhead

Susan: *OK, thank you very much. Do people agree? Which bar bisects the other?*

Pupil 1: *AC bisects BD*

Susan: *Is that an accurate arrowhead? What has to be true to make that an accurate arrowhead?*

Pupil 5: *The AB length the same as the AD length. The CD length the same as the BC length.*

In this excerpt pupil 1 can be seen to have mirrored my use of the word 'bisect' to tell me which bar bisected the other. Pupil 5 described the side properties of a kite by referring to specific sides in the displayed kite.

Pupil 1 showed that he was already reasoning at Van Hiele level three and like the small child in Sinclair and Moss's study his contributions to the class discussions may have helped other students to begin to use level three reasoning. Pupil 1 asked of the kite displayed in figure 9.2

Pupil 1: *Isn't it a concave kite?*

Susan: *That's an interesting thought. Go on then.*

Pupil 1: *Erm, because it's like a kite but where the A is, if you pulled it out it would be a kite. If you move the line (he meant bar) across.*

He took the radio mouse and demonstrated what he meant by dragging AC to the left and right as shown in figure 9.3.

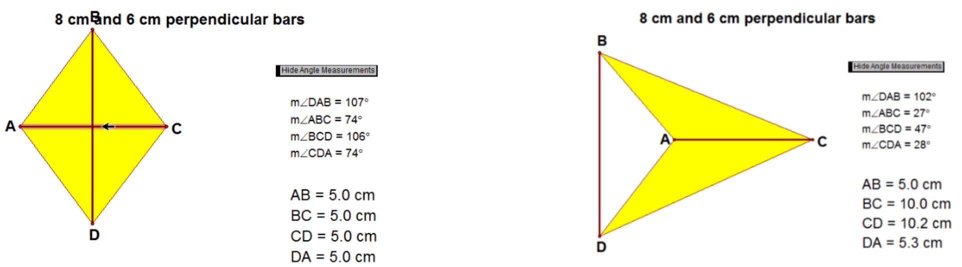


Figure 9.3 bar AC was moved left and right to demonstrate a kite (although the end point was a rhombus) and back to the arrowhead.

This movement suggested that this student conceptualised the arrowhead as a kite which had been dynamically changed by dragging one bar symmetrically through the shape.

The discussion on the shape continued and one of the students displayed a typical view of a kite indicating level two reasoning:

Pupil 10: *Kites are like, shorter at the top and longer at the bottom*.

Since some students were talking of the kite as having two pairs of equal sides I reminded them that the rectangle also has two pairs of equal sides. Pupil 13 clarified:

Pupil 13: *The difference between a kite and a rectangle is the kite has two the same size lengths at a corner, meeting up at a corner.*

From the arrowhead kite we moved on to the isosceles triangle which was displayed with horizontal symmetry as shown in figure 9.4. The move from the arrowhead to the isosceles triangle entailed DMS for 4 seconds followed by RD for 11 seconds. I observed episodes of GD, RD and DMS in this session with the whole class with different students having control of the radio mouse. This demonstrated how those dragging strategies are strongly situated within this task.

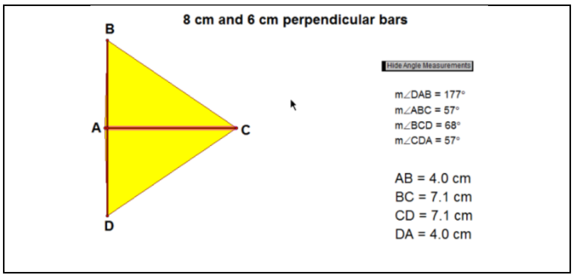


Figure 9.4 The isosceles triangle with horizontal symmetry

When I asked what had to be true to make that an isosceles triangle:

Pupil 17: *The two sides have got to be the same. Not the base line.*

This is a description of the properties of an isosceles triangle in its typical orientation. Perhaps pupil 17 mentally rotated the triangle as the base line would be BD which is not horizontal in this figure.

I asked the class whether the triangle was a perpendicular quadrilateral since we made it from one.

Pupil 18: *A quadrilateral has four sides*

Pupil 19: *When you crossed them over there were four points because there was one of them sticking out. When the point is on the same line, the A point is on the same line as BD, that means there are only three points.*

Pupil 19 had given the class a good explanation of why the figure was now a triangle instead of a quadrilateral. Reaching the conclusion that an isosceles triangle might be a special case of a dynamic kite is, after all, a difficult notion even for most adults.

Next we looked at the rhombus which some students had decided was a parallelogram. We had a discussion on which of those the figure might be. Pupil 1, who was possibly the most advanced in geometrical reasoning described the parallelogram as a pushed over rectangle; another example of a concept formed from a concrete process.

I decided to ask the students to consider the properties of the sides and angles of the figure and suggested that the rhombus might be a special parallelogram. Pupil 33 made a suggestion:

Pupil 33: *Doesn't a parallelogram have no lines of symmetry and that does?*

This excerpt shows that the classification is affected by how a shape is defined. The typical parallelogram presented to students e.g. through textbooks, shows a shape with two longer parallel sides and two shorter parallel sides. This shape has no line symmetry, only rotational symmetry and students will have learnt this when studying symmetry in shapes. A rhombus, which has two lines of symmetry, therefore cannot be a special parallelogram if a parallelogram is defined as having no line symmetry.

Overall all students in the class displayed an understanding of the perceptual nature of shapes by their properties and some students held to a typical view of shapes in specific proportions and orientations. In this they appeared to have a similar level of reasoning as other students in previous iterations. Some students such as pupil 1 appeared to be closer to level three reasoning which was shown by his willingness to consider the arrowheads as special kites.

After the whole class discussion about the dynamic figure under dragging using the radio mouse, the students were asked to fill in a worksheet by sketching and labelling the shapes they could make from the dynamic figure, the properties of the bars and the properties of the shape (some examples of completed sheets are shown in appendix 4.3). While they did this activity I recorded some of the students telling me about their work. The first three students described the bars with reference to the specific shape they had drawn. They all used the word 'bisect' mirroring the way we had talked about the positions of the bars in the whole class discussions. The Pupil D called me over, very excited to tell me what she had just discovered:

Susan: *How could you say that the bars are positioned in that triangle?*
Pupil A: *This bar bisects the one on the bottom*

Susan: *How can you describe the bars in a kite?*
Pupil B: *This bar bisects the bigger bar*

Susan: *What have you made there?*
Pupil C: *An isosceles triangle*
Susan: *How are its bars positioned?*
Pupil C: *The longer one is bisecting the shorter one*

Pupil D: *A kite and an arrowhead have the same properties*
Susan: *A kite and an arrowhead have the same properties?*
Pupil D: *yes*
Susan: *They do don't they cause you've got those. Well what are the properties?*
Pupil D: *Two sets of adjacent equal lines, two pairs of equal angles and one line of symmetry*
Susan: *That's very good isn't it. So what's the difference between the kite and the arrowhead?*

Pupil D: *That BD is more, it's more concave*

Susan: *What about the bars for a kite and the bars for an arrowhead?*

Pupil D: *If you continue the line then they're like the same.*

Susan: *What is the same?*

Pupil D: *They both, the line BD bisects AC.*

Susan: *OK so that's what's the same about them*

Pupil D: *BD cutting, bisecting AC*

Susan: *OK and what's different about the bars for a kite and an arrowhead*

Pupil D: *That AC is further away from BD*

This last pupil had used the perceptual nature of the shape properties of kites and arrowheads to ascertain that they were the same shape. In this she was starting to develop reasoning at Van Hiele level three.

The shapes, which the students sketched, make it clear that they understood what each shape looked like and how the bars were positioned inside. At the least they were all operating at Van Hiele level two in that they were able to list the properties of shapes and describe the positions of the bars. Despite the fact that the instructions at the top of the worksheet asked the students to draw shapes made from two bars which were perpendicular and where one bar bisects the other, not all students described the bars as being perpendicular (although perhaps they thought it was a given and so they did not need to mention it). Some students described bars as bisecting and others wrote that the bars crossed each other. All students drew a kite and their descriptions of the kite are the most illuminating. Some described the bars for the specific kite they had drawn, e.g. the students whose worksheets can be found in appendices 4.3b, 4.3c and 4.3k described one bar as being above the middle or one third of the way down. Other students described the kite in more general terms, in particular the authors of the worksheets found in appendix 4.3l wrote that one bar crosses the other at any point and of 4.3m wrote that you can make an infinite number of these (kites) by moving the shorter bar up or down the longer bar. These last two descriptions are indicative of movement towards level three reasoning. However other students could show flashes of level three reasoning, for example the author of the worksheet found in appendix 4.3h made a connection between the bars of the kite and the bars of the arrowhead (the bars are like the kites but don't touch).

Next I showed the students the animation. This has already been illustrated in chapter 8 but here it is again.

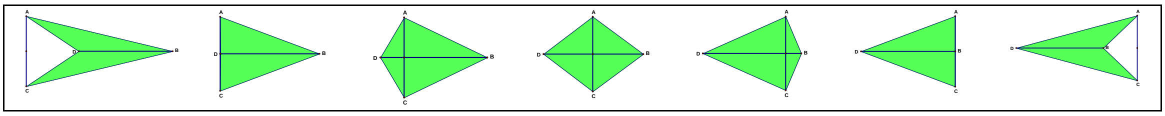


Figure 9.5 Seven positions of the animated figure

In the following dialogue I have noted pupil 1 who was the student who used level three reasoning most of the time and who seemed to be able to use more sophisticated reasoning than his peers. Otherwise I have used pupil to indicate any of the other pupils in the class.

Susan:	*So what is AC doing to DB all the time? (I said it the wrong way round!)*
Pupil:	*It's acting as the bisector*
Susan:	*It's always bisecting it isn't it. What angle between AC and BD?*
Pupil:	*Ninety*
Susan:	*Ninety degrees? So can we put those two words together. We've got AC bisects BD and it's also at right angles to it. What's at right angles? What's that word?*
Pupil 1:	*Perpendicular*
Susan:	*Perpendicular and a line which cuts another one in half is a*
Pupil 1:	*Bisector*
Susan:	*OK AC is the perpendicular bisector of BD because it's at right angles and it cuts it in half. So we've been making shapes that have the property that one diagonal is the perpendicular bisector of the other. In that perpendicular quadrilateral what is it most of the time? We've had kites and arrowheads and isosceles triangles and rhombuses. What is it most of the time*
Pupil 1:	*A kite*
Susan:	*OK and sometimes, go on.*
Pupil 1:	*An arrowhead*
Susan:	*An arrowhead. Are we going to agree that kites and arrowheads are the same thing. Some people are not quite sure about that. Some people are. We'll call it kite stroke arrowhead for now. So most of the time it's a kite or an arrowhead. And how many times do we have it being a rhombus?*
Pupil:	*One*
Susan:	*And an isosceles triangle*
Pupil:	*Twice*
Susan:	*And how many kites stroke arrowheads do we have?*
Pupil 1:	*Infinity*

It can be seen that the students had started to use the word 'bisector' rather than 'cuts in half' or 'crosses in the middle'. I had been careful in introducing terms such as 'bisector' and the students mimicked my use. It can also be seen that I introduced them to 'perpendicular bisector' by putting together the two words which they already understood. In this way I modelled use of mathematical vocabulary inducting the students into the mathematical discourse.

9.4.3 Lesson 3

I did not record the discussion at the beginning of lesson three and so this paragraph details the notes I wrote shortly after the lesson. First we did a recap of what we did during the previous lesson looking at the animation. Some of the students offered that there were an infinite number of kites. In the discussion the students appeared happy to accept the arrowheads as being concave versions of kites. We talked about the properties of kites being two pairs of adjacent equal sides and identified these sides on the animated figure. The arrowheads had the same properties and the students seemed to accept that this meant the kites and arrowheads were the same. They identified one rhombus and two isosceles triangles that occurred during the animation. We talked about the rhombus as being a special member of the kite family and even the two triangles belonging to the kite family in the context of the computer animation. Finally we identified the horizontal bar as being a line of symmetry.

In preparation for making a poster the students and I made a list of the vocabulary and the students were keen to suggest words (see Appendix 4.4 for a photograph of this list on the board). They were becoming familiar with the mathematical vocabulary and many of them wanted to use the correct terms.

Several conversations took place whilst pupils worked on their posters. Some students discussed whether there were an infinite number of kites or just a lot (but a finite number). The students who said there was a finite number appeared to be thinking about physically moving the bar a little bit along each time. The students who said there was an infinite number appeared to be thinking theoretically if the bar could be moved an infinitesimal amount (or as one boy said 0.0000000000000000 (recurring) 1).
Some students still found it difficult to accept a rhombus being a special kite even if the properties for a rhombus were also true for a kite.

It might be argued that I had set up the situation for the students to appear as if they accepted inclusive relations between the shapes in the animation and that they would mirror this in their posters. However my view was and is that the students would be honest in using the posters to tell me what they had learnt. They would not be able to imitate Van Hiele level three reasoning if they had not understood it (Vygotsky, 1978). Indeed an analysis of the posters show clearly the students who reasoned at level two and those who had begun to accept inclusive relations and were moving towards level three.

9.5 The posters

Thirty-one students made posters of what they had learnt during the lessons. I have analysed these according to level of reasoning indicated and identified common themes in the comments and drawings.

9.5.1 Assigning levels to the posters

I assessed the level (according to Van Hiele) of reasoning indicated on each poster, using the comments and diagrams to ascertain how each student reasoned about the shapes which were discussed in the lessons. Three examples of the comments on the posters which provided this evidence are given below:

<u>Comments on Poster A</u>
"These four shapes, namely a kite, isosceles triangle, arrowhead and rhombus all have one thing in common, they all have a perpendicular bisector. They all belong to the same family because all of them have a certain property when the bars AC bisect bars BD.
(Describing the animation) When slided along a certain line segment of a certain size, then 4 shapes are formed: an arrowhead, an isosceles triangle and a rhombus. The shapes always have two equal sides".

I assessed poster A as indicating Van Hiele level three reasoning because the student has clearly seen that keeping the property of one bar being the perpendicular bisector of the other has generated the four different kinds of shapes. They have also connected the sliding of the bar to this property. However, there is no suggestion that the other shapes are special cases of the kites but there appears to be an understanding that all four classes of shapes do belong to a larger class or family..

<u>Comments on Poster E</u>

"From a kite we managed to make a family of shapes which all had a line that cut at an angle, called a perpendicular bisector. They ranged from a kite to an arrowhead"

I assessed poster E as indicating Van Hiele level two / three reasoning. There is some indication of inclusivity in that the student has seen that the shapes originate in a kite. However two drawings on the poster of the rhombus and kite show measurements which indicate a partitional classification. So it appears that the student who designed this poster moves between level two and level three reasoning, indicative of starting to think at level three.

<u>Comments on Poster M</u>

"All these shapes are made from 2 bars. In every shape the lines AB/AD and BC/CD are the same."

I assessed poster M as indicating Van Hiele level two reasoning. Alongside the above comment the student had drawn a sequence of shapes under the animation. However the comments did not suggest to me that any connections were made between the shapes even though pairs of equal sides were listed.

9.5.2 Identifying themes in the posters

Appendix 4.5 shows a table of comments made on the posters. Five themes were identified:

- Description of the animated figure by drawing different positions of the figure and / or written commentary.
- Listing common properties of the shapes drawn and described on the poster
- Use of the term 'perpendicular bisector'
- Reference to a family of shapes either explicitly or by implication

- An indication of an infinite number of kites and arrowheads

Figure 9.6 shows a Venn diagram indicating whether each poster (labelled as in appendix 4.5) mentioned the animation, the family of shapes and the infinite number of kites. The Van Hiele levels have been written next to each label where A3, for example indicates that the student who made poster A demonstrated reasoning at level three, E2/3 indicates that the student demonstrated reasoning which was moving towards level three and M2 demonstrates reasoning at level two.

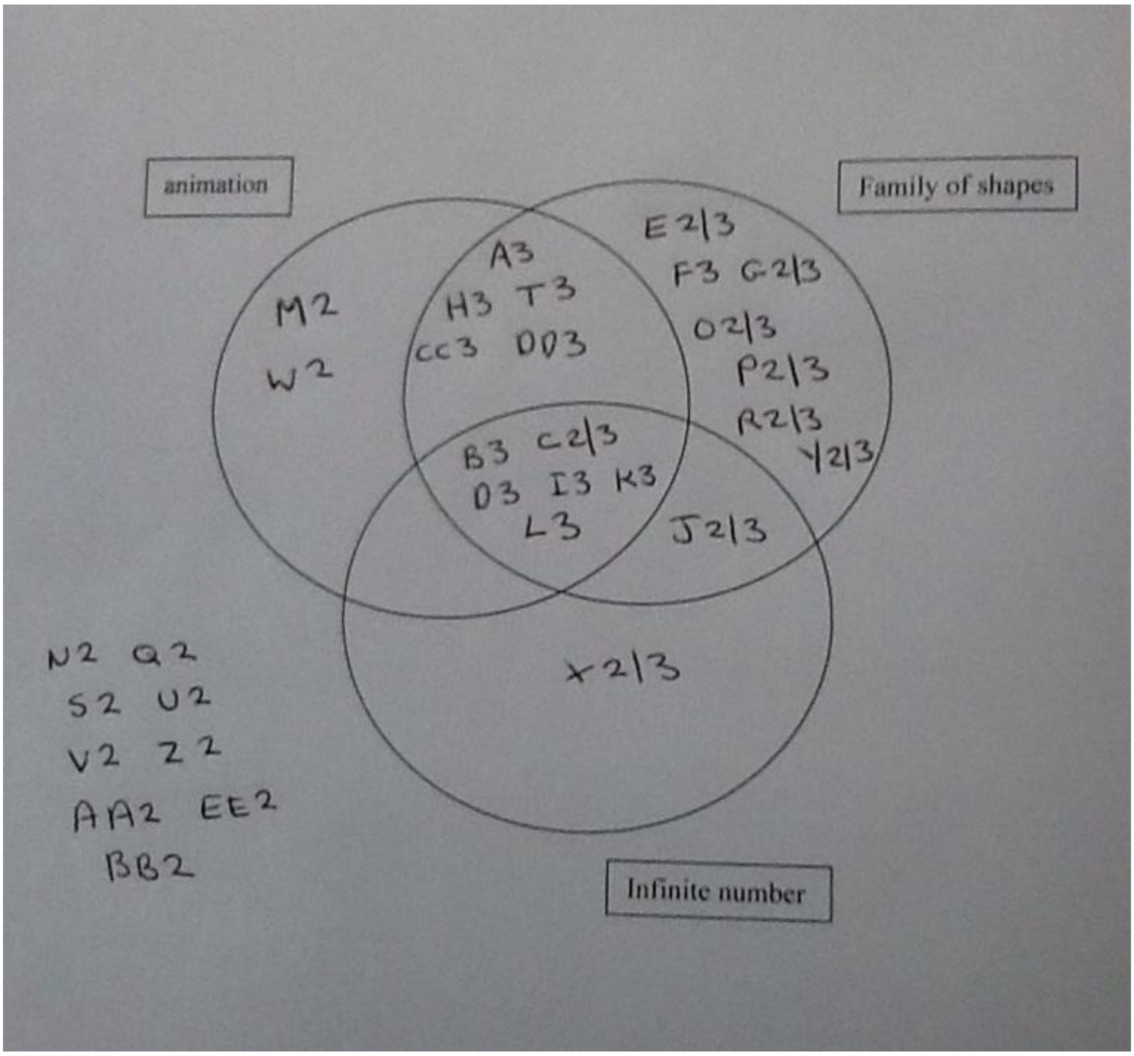


Figure 9.6 Posters, content and Van Hiele levels

9.5.3 Posters which mentioned animation

Thirteen of the thirty one posters illustrated and/or described the animation. Five of these solely illustrated the animation with diagrams without commentary in the form of text. From these only two (M and W) indicated that their authors probably held a partitional view of the classification of shapes. Poster M did not describe the animation in words but the student had drawn the figure in seven positions along the movement of the animation. Poster W also did not describe the animation in words but the student had drawn the figure in nine positions. This student wrote about the properties of the shapes according to a partitional classification.

Eleven posters which described or illustrated the animation included information which indicated that students were beginning to use the idea of a family of shapes, where some shapes (rhombus, isosceles triangles) were special versions of the kites and that the arrowheads were concave kites. Comments included ‘a family of shapes made by moving one bar along the other’, ‘part of the kite family’, ‘these shapes are mostly kites’, and ‘special kinds of kites’. Six of the posters also mentioned an infinite number of kites and arrowheads. One student wrote:

“You can move even one millimetre and it will be another shape”.

This comment does suggest that the student continued to think in terms of (a large number of) discrete positions. However this is a step towards the understanding of continuous change.

Another student wrote:

“The smaller line (i.e. bar) must always bisect the other. So is the isosceles triangle really acceptable? Or if it bisects it just before the end is it a kite technically?”

This comment suggests that the student had been perturbed by the new ideas with which they had been confronted. The student has wondered if the figure continues to be a kite right up until the point when it becomes an isosceles triangle.

Overall the students who referred to the animation either in words or by drawing the animation on their posters were more likely (11 out of 13) to write comments that indicated an acceptance of inclusive relations and all of the students who described the animation in words were able to describe inclusive relations.

Of the eighteen students whose posters did not mention the animated figure, the comments of nine of them indicated that they were beginning to move towards level three reasoning, the comments of seven students could either be considered to be partitional in nature or to provide insufficient evidence of level three reasoning, one poster (F) did indicate level three reasoning and a final poster simply included a glossary of terms which had been used during the lessons.

9.5.4 Posters which mentioned a family of shapes either explicitly or by implication.

Nineteen of the posters mentioned the family of shapes, either explicitly or implicitly. In the lessons, particularly in the second lesson, I had introduced the idea of a family of shapes to the students. It could therefore be the case that students were simply mimicking my suggestion that we had a family of shapes when they made their posters. I therefore needed to look at what the students wrote to ascertain whether they understood the concept of inclusive classifications and that we had a family of kites with the rhombus and isosceles triangles as special cases. The students had also been asked to consider the arrowhead as versions of a kite. Thus when analysing the comments of the eighteen posters which referred to a family of shapes I looked at the other comments on the poster to ascertain the level of understanding shown.

Eleven of the nineteen posters indicated reasoning at level three. Students who used level three reasoning gave descriptions of the family of shapes using reasons such as common properties (e.g. bar AC bisects bar BD for all the shapes), indicated that there were infinite kites or arrowheads which could be made, that arrowheads were special kites, the rhombus was a special kite, isosceles triangles were special kites, all shapes were members of the kite family and combinations of these.

Eight posters indicated that the students were moving towards level three reasoning or there was insufficient evidence to be confident that they had used level three reasoning. Some comments on these posters described how the shapes were in a family because they had all been generated in the same way, but without extra supporting statements, e.g. all shapes were made by moving one bar along the other. Or the comments might have expressed doubt, such as "the kite is possibly a special version of a rhombus".

The mention of a family of kites appears to imply at least a movement towards level three reasoning. The comments on the posters indicated that the students had formed a concept of a family of shapes in that they could identify common properties of the sides and angles or the movement of the bars. Students who were more secure in level three reasoning either mentioned the infinite number of kites or arrowheads which can be generated or implied the infinite number ("these shapes are mainly kites") otherwise they referred to common properties being true for shapes and their subsets ("kites and arrowheads have two pairs of adjacent equal lines").

9.5.5 Perceiving an infinite number of kites.

Eight posters explicitly mentioned the word 'infinite' in relation to the number of kites and arrowheads. Five students whose posters explicitly referred to an infinite number of kites demonstrated level three reasoning and three students demonstrated reasoning which was developing towards level three. I had originally thought that if the students could perceive that the dynamic figure generated an infinite number of kites this would help them to develop a concept of inclusive classification of rhombuses in kites since the rhombus appears as only one position along the dragging journey between kites. However, only five of the eleven students who demonstrated level three reasoning referred to an infinite number of kites, which surprised me. During the second lesson when the students watched the animation I had asked how many kites could be made and pupil 1 (who demonstrated the most sophisticated reasoning) had replied "infinity". We had also had a discussion in the third lesson, while the students were working on their posters, on how many kites could be made; was it just a large number or was it an infinite number. Nevertheless the majority of the students did not refer to an infinite number of kites.

9.6 Discussion

The whole class lessons included more activities than the sessions with pairs of students. This enabled the students to perceive the problem, of which shapes can be generated using two perpendicular bars, from a kinaesthetic perspective using geo-strips, and a visual perspective through the dynamic figure on the computer screen. There was also discussion with a greater number of people with the potential for many more ideas than can be generated with two students and one researcher. Despite this, the students did react to the dynamic figure in similar ways to students in previous iterations. They used GD and DMS to generate shapes and RD to refine shapes with reference to displayed measurements. When the class watched the animation they identified the shapes in the same way as the students in iteration three had done.

9.6.1 Changing discourse

Over the three lessons the discourse of geometrical reasoning had changed. In the first session I recorded student comments which I classed as level one and level two reasoning as reported in section 9.6.1 when one student had said:

"A square is regular because all the sides are the same length and the sides are going either sideways or up and down".

This comment suggests that the student was influenced by the orientation of the figure and had not separated orientation form the properties of the shape, which is suggestive of reasoning at level one. (Even though the students were in a class of high attainers as assessed by national tests and departmental tests they did not demonstrate more sophisticated reasoning than other students who participated in the study). However as the lessons progressed I was able to identify reasoning at level two and level three. In contrast to lesson one where students had found it difficult to articulate the positions of the bars inside the figure, by lesson three students were able to describe the positions, for example poster D states

"The bars are perpendicular and create four right angles as they bisect each other".

Pupil 1, in particular, demonstrated that he already reasoned at level three before participating in the study. His contributions to the class did provide an opportunity for the class to discuss the figure and to be confronted with arguments based on level three reasoning as when in the first session he suggested that the arrowhead is just a concave kite. This may have encouraged his classmates to consider inclusive relations that they may not have thought about previously. This effect, from working with a more able peer in the classroom had also been observed by Sinclair and Moss (2012).

9.6.2 The catalyst for change in the students' reasoning

The main focus of iteration four was to test whether the animation, shown to the students after they had already worked with the dynamic figure and discussed the properties of the shapes which were generated, could be the catalyst for their development of level three reasoning particularly the concept of inclusive relations. The evidence appears to show that this had happened. Studying the contents of the posters shows that ten of the eleven students who demonstrated level three reasoning mentioned both the animation and the family of shapes on their posters (see figure 9.6, the evidence for level three was taken from all comments made on the posters). If any factors are responsible for the development of level three reasoning amongst the students it would appear to be the animation which has acted as the catalyst. Observing the perpendicular bisector of BD move along the line of BD generating versions of kites, rhombus and isosceles triangles appears to have helped some students to perceive the rhombus as a special kite and even helped some students to see that the isosceles triangle could be a kite in the dynamic context. It appears that the continuous morphing of the figure rather than the demonstration of an infinite number of kites was the catalyst for the change in the level of reasoning.

9.6.3 Measuring the change in the class

After the three lessons with this class of students I realised that I had failed to collect an important set of data. I had not known before the sessions which students already held a hierarchical classification of shapes. It was clear that pupil 1 came to the lessons with a hierarchical view of shapes but I did not know about the other students. I was given the opportunity to repeat the lessons with the other top set in the year group who were taught by another teacher. In the event the lessons took place in the last week of term when there was a heat wave and the temperature in the classroom was not conducive to working. The regular class teacher was only prepared to let me work with the class for two lessons and so they did not make the posters. It was not the best circumstances for undertaking research but I worked with this class because I had been asked to do so as the head of department wanted the students to be given the same experience as the first class. However, I took the opportunity to find out whether the students' views on the classification of rhombuses as kites would change as a result of watching the animation. At the beginning of the second lesson I gave each student in this class a voting slip as shown in figure 9.6

A rhombus is a special case of a kite

I agree	
I disagree	

Figure 9.7 voting slip

I asked the students to decide whether they agreed or disagreed with the statement; a rhombus is a special case of a kite. I then asked the students to put the voting slip at the top of their desk. At the end of the session I asked the students to write on their voting slips whether they still thought the same or whether they had changed their mind. I gave them the option of including their reasons.

At the beginning of the lesson 12 students agreed that the rhombus is a special case of a kite and did not change their minds at the end of the lesson.

17 students disagreed that the rhombus is a special case of a kite. By the end of the lesson 6 of these students had changed their minds and decided the statement is true. Two gave reasons which appear to show that they were influenced by the animation:

"I have changed my mind because if the sides of a kite are stretched they are still bisected and simply converted into a different shape"

“I changed my mind because I saw the animation on the board”.

The students who did not change their minds gave arguments based on a partitional classification of kites and rhombuses.
“A rhombus has all of its sides equal and a kite has two pairs of adjacent sides”.

The voting slips were a simplistic tool to measure change but they do, at least provide extra evidence to suggest that the three lessons with the first class would have had an impact on students’ reasoning from classifying shapes into discrete classes to an inclusive classification of shapes which can be generated using the DMS strategy on the dynamic perpendicular quadrilateral.

9.7 Conclusion

Iteration four concludes the study. In this iteration I have demonstrated that the dynamic figure can be used as part of a sequence of lessons designed to develop the concept of inclusive relations between shapes. However the development of mathematical concepts by students is a complex process which necessitates that they are given time to work on tasks which support this development. This process will take different amounts of time and experience for each student.

Of thirty one students, by the end of the sessions, eleven demonstrated level three reasoning, ten demonstrated that they were beginning to use level three reasoning but were not secure in it and ten students continued to use level two reasoning. The discourse developed throughout the sessions providing evidence that there was a genuine change in the way the students engaged with the classifications of the shapes by properties and by inclusion in other shape families.

The animation appeared to be the catalyst for the acceptance in inclusive relations in some students. It helped them to observe the figure as continuously changing, rather than as discrete shapes produced at the end of dragging journeys. Watching the figure change between kites also seems to have helped the students to accept that kites are defined by their properties and not by the typical image of a kite. However the infinite number of kites which the figure theoretically generates (although computer screen pixels mean it displays a discrete number of kites) did not appear to be an influencing factor as I had thought originally. The dynamic change itself is likely to be the main factor and this will be explored in the next chapter.

10 Retrospective Analysis

Thus far this report has addressed each of the iterations in turn, describing the research process, the collection and analysis of data and the developing theory. Now that this has been completed the retrospective analysis will provide an overview of the data collected across all of the design cycles. First I set out the main themes and sub-themes into one table (see table 10.1).

Next the data will be analysed in the context of Duval's theory of cognitive apprehensions (Duval, 1995) taking each in turn to provide the focus and framework for analysing the data:

- Perceptual apprehension
- Operative apprehension
- Sequential apprehension
- Discursive apprehension

Finally this chapter addresses the dynamic element of Dynamic Geometry Software.

10.1 Themes common to all iterations

Table 10.1 illustrates the themes which have emerged from the data collected over iterations zero to four. Within each theme there are sub-themes some of which describe student activity (mostly on screen activity) and some of which describe student discourse (indicated by the verb *to describe* or *to comment*). The themes have been grouped into four areas.

- Geometrical reasoning (concerning the shapes generated from the dynamic perpendicular quadrilateral) is divided into holistic perception of shapes, knowledge and understanding of shape properties and the development of analytical reasoning which recognises that there are connections to be made between the properties of shapes. These aspects are descriptive of the development through Van Hiele levels one to three (Van Hiele, 1986).

- Dragging strategies as a theme relates to the instrumentation of the software tools by the students when generating shapes from the dynamic figure. The dragging strategies can act as tools of semiotic mediation when students are able to connect meanings attached to the use of dragging to geometrical theory of shapes and their properties and also of inclusive relations.

- Symmetry and orientation are two significant influences on the way students have perceived the shapes generated from the dynamic figure. Chapter two, particularly section 2.1 shows how research has identified the importance of symmetry and orientation in the way human beings perceive shapes.

10.1.1 The Van Hiele levels in the table of themes

The themes have been organised into table 10.1 and I have used Van Hiele levels one, two and three throughout the analysis of the work with the dynamic figure and each of these levels occur under the headings Geometrical Reasoning and Dragging Strategies.

Table 10.1 Overview of the themes

Geometrical reasoning	Holistic perception of a shape (VH1) • Generating the ‘three quarters’ kite • Comments e.g. “It looks like a rhombus” • Describing a rhombus as a parallelogram due to its orientation
	Describing a shape as made of triangles
	Describing properties of sides and angles (VH 2)
	Describing properties of the bars (VH2)
	Describing a family of shapes (towards VH 3) • Active DMS between rhombus, kite, isosceles triangle and arrowhead • Describing common properties of a rhombus and square • Describing, in multiple shapes, the common property that one bar always crosses the other at its midpoint
Dragging strategies	Wandering dragging: exploring what happens when bars are dragged randomly
	Guided dragging: dragging the bars intentionally to generate a specific shape (VH 1)
	Refinement dragging: small dragging movements to make sides and angles which should be equal to be as close as possible (VH 2)
	Dragging Maintaining Symmetry: dragging one bar along the perpendicular bisector of the other bar (towards VH 3)
	Describing an action in DMS “pull it up or down”
Symmetry	Active: Guided dragging of the bars into a symmetrical position.
	Active: Dragging Maintaining Symmetry
	Describing axes of symmetry as being in the middle of the shape or naming the bars as the axes of symmetry
	Describing the symmetry of the shape in a holistic way “it will look equal”
	Describing a process; visualisation of the shape being folded in half
Orientation	Describing the shape using its orientation: • “an angled kite” • “A rhombus is slanted and a square is straight”
	Commenting or acting on the non vertical frame of reference • “if you tilt your head”
	Describing the dragging as up or down when the figure is being dragged at an angle to the vertical

I have linked the holistic perception of the shape (which includes recognising shapes by their visual configuration and dragging the bars straight into position) to generate a shape with Van Hiele level one reasoning. When students used holistic reasoning it appeared that their concept image of the shape had a significant influence on the way they saw the shape, which could include its orientation as well as its proportions. When students dragged the bars into position to make a kite, they often preferred the three-quarters kite (with the horizontal bar roughly three quarters along the vertical bar) which is a typical presentation of the kite. Using another example, if students viewed a rhombus with two of its sides close to the horizontal, then they sometimes viewed it as a parallelogram because rhombuses are usually presented with all sides oblique and parallelograms are usually presented with a base sitting horizontally as shown in figure 10.1. Referring back to section 7.4.10 describes how Tara and Ruth perceived the oblique rhombus to be a parallelogram.

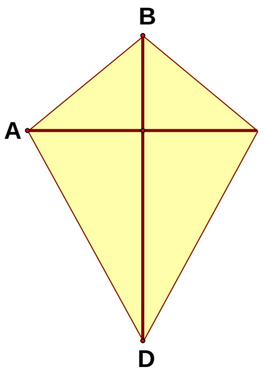
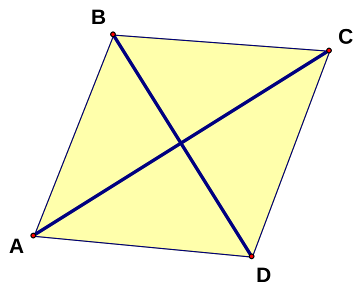
three quarters kite	a rhombus presented obliquely
	

Figure 10.1 Proportion and orientation affect perception of shapes so that the rhombus presented obliquely can sometimes be perceived as a parallelogram

This preference for shapes in specific orientations and proportions, which may arise from students' prior experience with shapes in school, is a naive kind of reasoning. As students develop a perceptual understanding of shape properties, we might expect that they would be able to separate the shape from its orientation and proportion, in other words to label a shape based on its properties rather than the way it looks. However the students in the study seemed to hold onto their concept images of shapes they generated and this overlapped with their understanding of the properties of the shapes, for an example see section 8.4.3 where Hemma and Seema discussed kites in different incarnations. In terms of their understanding of the figural concept, the students thought of the shape very much as a material object (in the spatio-graphical sense) whilst maintaining an understanding that the shape obeyed theoretical rules related to their concept definition of the shape. Usually the concept definition incorporated the 'official' mathematical definition of the shape but may also have included other implicit properties such as orientation and specific proportions.

Displaying an understanding that the shapes they generated had properties which they could check (using the measurement facility of GSP) demonstrates reasoning at Van Hiele level two. If students were secure in level two reasoning they would be expected to recognise shapes on the basis of their properties and thus recognise short fat kites, long thin arrowheads etc as being versions of the kites or arrowheads. Whilst all of the students were able to use the properties of the shapes to identify them they still typically showed a preference for 'three-quarters' kites (see section 7.7.5) which indicates that an ability to think at level two overlapped with thinking at level one.

In order to progress to Van Hiele level three, in the context of this study, it was necessary for students to recognise as kites, shapes which have the properties of a kite but which look like a rhombus or isosceles triangle. In the context of the dynamic figure a kite shape can be dragged so that it is very close to the rhombus or isosceles triangle position and look very much like the rhombus or isosceles triangle even though its properties identify it as a kite. Students who accepted the near isosceles triangle as a kite, say, demonstrated secure level two and it can be argued that they demonstrated early level three reasoning because they were being analytical, not merely descriptive in relating this position of the kite as being next to the position for the isosceles triangle as Hemma did in iteration three (see section 8.6.3).

When students dragged using the DMS strategy they were undertaking an activity which I have connected to level three reasoning. However, although all the students did use DMS the dialogue from the recordings does not indicate that they had all progressed to level three reasoning. My claim about DMS is that it has the potential to develop level 3 reasoning but it is not given that this will happen. Although it is straightforward to connect guided dragging to holistic level one reasoning and refinement dragging to the descriptive perceptual understanding of shapes at level two it is not so easy to connect DMS to level three reasoning. That is to say it is not easy to find evidence of level three reasoning if that has been interpreted as the accepting of inclusive relations leading to a hierarchical classification of shapes, as I have done.

10.1.2 Symmetry

Symmetry appeared as a common theme in the data, sometimes described as a property of the shapes by students and sometimes in use by guided dragging and DMS. However if symmetry was being used intuitively during dragging activity the students rarely articulated it. When the dynamic figure was oriented at an angle to the vertical the students still showed the facility for symmetrical dragging demonstrating that they were using symmetry rather than simply dragging up or down the computer screen.

10.2 The use of Duval's framework to analyse the overall findings

In chapter 2, section 2.5.2, Duval's framework of cognitive apprehensions and cognitive processes (Duval, 1995, 1998) was introduced. This framework will be used to analyse the overall findings of this study because it is a comprehensive framework which provides insight into how students use diagrams in a heuristic way to reason geometrically (Duval referred to these diagrams as figures, using this term in a different way than I have done). By way of a reminder table 10.2 indicates how the cognitive apprehensions and cognitive processes are connected

Table 10.2 Cognitive Apprehensions and how they link to cognitive processes

Cognitive apprehension	**What it involves and its links to cognitive processes**
Perceptual	Perceptual apprehension involves the recognition of the shape of the figure including its orientation and symmetry, whether it represents an object in two dimensions or in three dimensions and is a process of *visualisation.* It may also involve recognition of sub figures within the figure.
Discursive	Discursive apprehension is important for mental organisation of understandings of the geometric object and is important in the *reasoning* process. The definition of a geometrical object and a description of its construction are part of discursive apprehension. Visualisation can be embedded in a natural discursive process when a student describes what they are seeing and uses this as part of their *reasoning* process.
Sequential	Sequential apprehension is an understanding of how the figure is constructed. This is dependent on the use of tools in the process of constructing the diagram (representation) of the figure which works like a model of the object. The student learns about the figure through the *construction* of its representation.
Operative	Operative apprehension relates to physically or mentally operating on the figure by splitting the figure into subfigures, changing the position of sub-figures or transforming the figure. *Visualisation* also plays a basic heuristic role, through operative apprehension, and can provide a basis for *reasoning*.

10.3 Perceptual apprehension

Perceptual apprehension involves holistic recognition of the shape of the figure and its orientation, and whether it has symmetry. It also involves visualisation of the figure as divided into sub figures. When students from the study looked at the dynamic perpendicular quadrilateral in any of its various incarnations on the computer screen they were able to recognise and name the shape, identify line symmetry and frequently made a comment about its orientation.

10.3.1 Perceiving the figure as split into sub-triangles

Since the dynamic perpendicular quadrilateral was based around the two bars which split the figure from top to bottom and side to side it would be perfectly natural to perceive it as split into smaller figures and many students did so. Students often described the shape on the screen as being made of two triangles or four triangles and sometimes these were referred to as being the same, i.e. meaning congruent. For example Adam and Jack (iteration one) talked about the necessity of the sub-triangles being the same size and having the same angles if the shape were to have symmetry. Figure 10.2 shows the triangle (non-symmetrical) which they had generated.

<table>
<tr><th>Dialogue</th><th>On-screen</th></tr>
<tr><td>Susan: What might be true about that shape. What sort of shape is it?

Jack: It's a triangle, it has three sides.

Susan: Mmhmm, anything else

Adam: Well it could be two triangles cause it looks like it's split, split in the middle, one big triangle one smaller.

Susan: Mmhmm, right, anything else? What if you tried to make one that's got symmetry? Do you think that's got symmetry?

Jack: No

Adam: No

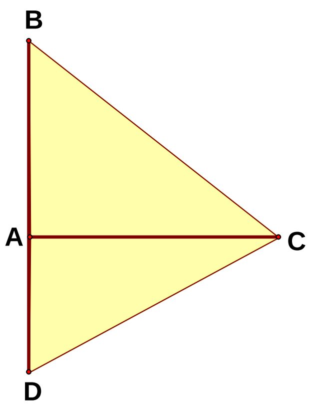
Jack: Cos the top one's bigger than the bottom</td><td>
</td></tr>
<tr><td colspan="2">Overview: Jack dragged the bar AC to generate a triangle which is not symmetrical.</td></tr>
</table>

Figure 10.2 Splitting the figure into sub-triangles is an example of operative apprehension

Jack referred to the property of the triangle (it has three sides) to justify labelling it a triangle and Adam pointed out that the figure was split into two different sized triangles. Jack said that the triangle was not symmetrical because the two triangles inside were not the same size. He recognised that to have a symmetrical figure the triangles inside the shape (the sub-triangles) needed to be the same shape and size.

Shortly after this episode Adam and Jack generated a symmetrical triangle and then referred to having equal sub-triangles inside the larger triangle as shown in figure 10.3.

<table>
<tr><th>Dialogue</th><th>On-screen</th></tr>
<tr><td>Susan: Think about the symmetry because you talked about folding it didn't you. And we'd also talked about two triangles inside the big triangle. So what's true about the two triangles that make up the big triangle?

Adam: They're both right angled, right angled triangles.

Susan: Yeah, anything else?

Adam: They're, I think both of the small triangles are the same, they've got all the same angles as each other

Jack: the opposite angles of each other are the same

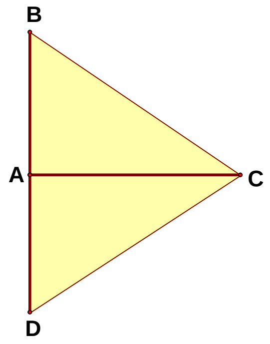
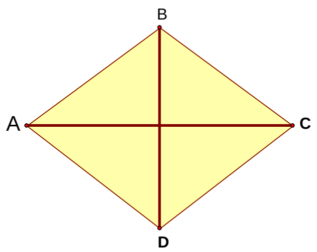
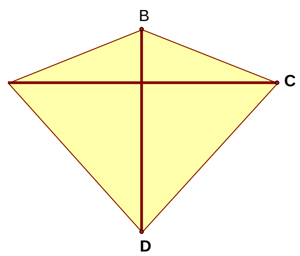
Adam: yeah so D and B are the same</td><td>
</td></tr>
<tr><td colspan="2">Overview: Here the boys talked about having two right angled triangles the same size with equal size angles. When Jack said the opposite angles were the same this refers to angles opposite bar AC which is the line of symmetry. These are the angles at B and D.</td></tr>
</table>

Figure 10.3 Adam and Jack refer to congruent sub-triangles in the figure.

These two examples show how the students' perception of symmetry was related to their observation of sub-triangles in the figure. In order to have symmetry Adam and Jack recognised that the sub-triangles BAC and DAC needed to be congruent. In part this perception naturally followed from their process view of symmetry where one half of the shape has to be folded over the line of symmetry onto the other half of the shape, which was used by all students who described symmetry during the study. When students imagined folding the shape in half to check symmetry perceptual apprehension was linked with operative apprehension (since operative apprehension requires the figure to be transformed in some way often by moving sub-figures around). Describing this process links it with discursive apprehension hence these three apprehensions were often used together.

10.3.2 The effect of the dynamic nature of the figure on perceptual apprehension

In both of the cases demonstrated by figures 10.2 and 10.3 Adam and Jack had been discussing the figure while it was a static image on the computer screen having completed an episode of dragging activity. However, it is also important to consider the dynamic aspect of the figure and how this dynamic nature affects students' development of geometrical reasoning. Leung (2011) stated that perceptual apprehension comes into play when students begin to explore a dynamic figure by dragging. Students observe the shape of the figure changing under dragging in a holistic manner. When the figure is subject to a dragging action these sides and angles usually change, in size or position. This often causes the figure to become configured into a different shape, for example the dynamic perpendicular quadrilateral may take the shape of a triangle, and after dragging it may take the shape of a kite. Dragging to change the figure from one shape to another is clearly an active process and the students in the study often articulated the movement in the bars needed to do this. In Figure 10.4 which shows an excerpt from the recording with Aftab and Rupen (iteration two) I had asked the boys what they would need to do to make a rhombus into a kite.

Dialogue	On-screen
Susan: OK then. What would you have to do to make that rhombus into a kite? *Aftab: Erm, pull it up or down a bit.* *Susan: Pull what up or down a bit?* *Aftab: The bar AC* *Susan: OK then, do you want to have a go? Can you describe to me how you decided to drag that? What have you been able to do?* *Aftab: I was watching the measurements on the side ....* *Susan: What did you do just there?* *Aftab: I just put the mid-point, the middle on there* *Susan: Could you make a kite in another position?*	At the beginning of this episode the boys had a rhombus on the screen  bar AC was dragged up to make a kite 

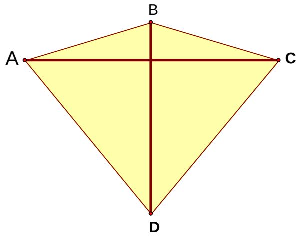
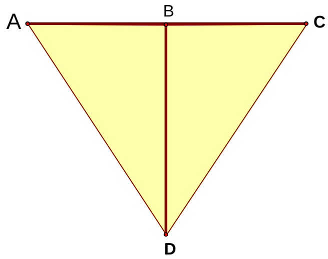
Rupen: You have to do it down, move a bit down. *Susan: You could, OK. How many kites do you think you could make?* *Aftab: Four* *Susan: You think you can make four kites? What if you move AC a little further up?* *Rupen: You'll make an arrowhead if you put it up* . *Susan: OK. Is that a kite, how it is at the moment?* *Aftab: Yeah* *S: Right, if he moves it a little bit more?* *Aftab: You can make loads of kites.* *Susan: You can can't you so what happens if you keep going up, if you go through* *Aftab: If you go past B it will make an arrowhead* *Susan: OK, what happens at B?* *Aftab: It will be a triangle.*	When Aftab talked about putting "the middle on there" he used the cursor to point at a constructed mid-point of AC and then to point along the bar AC either side of the mid-point. bar AC was dragged a little further up  bar AC was dragged to sit on point B 
Overview: Aftab dragged bar AC into different positions which began with the rhombus and then generated two different kites and the isosceles triangle. During this dragging bar AC continued to be the perpendicular bisector of the bar BD, i.e. using DMS. Aftab claimed to be attending to the displayed measurements whilst doing this. The dialogue indicates that the students thought about dragging as an activity which created different shapes from the figure. However, as I indicated in the introduction to chapter eight, it seems that in common with other students, Aftab and Rupen viewed dragging as a journey to an end point. They did not appear to see the figure as continuously changing.	

Figure 10.4 Aftab and Rupen describe dragging between shapes as moving one bar up or down and watching the displayed measurements

The dialogue in figure 10.4 indicates typical thinking by the students in the study. Firstly they talked about dragging the bars up or down but not necessarily noting that they were keeping one bar through the mid-point of the other even if they were clearly trying to do this. When I asked them if they were being careful how they dragged, like Aftab, they would tell me they were trying to keep one bar so that it crossed the other at its middle. The perpendicular aspect of the bars was an invariant property of the figure so it was natural that students would not mention this.

Similarly the students did not give any indication that they considered the figure to be continuously changing (rather than moving between discrete shapes). The many instances of students declaring that they could make a discrete number of kites testifies to this although some students, such as Kate and Jane (iteration two) did say that they thought it was possible to make an infinite number of kites. It was for this reason that I had decided to create an animation of the DMS strategy to ascertain whether sitting back and watching would encourage students to perceive the figure as continuously changing.

10.3.3 The effect of the animation of the DMS strategy on perceptual apprehension

I hoped that watching the animation would encourage students to see the figure continuously changing as a kite shape which at three points became a rhombus or two isosceles triangles for split seconds. In iteration three Stan and Eric and Hema and Sima watched the animation in a second session after they had worked with the dynamic figure oriented at an angle in the first session. Watching the animated figure under DMS for the first time elicited the following comments:

<u>Stan and Eric (iteration three)</u>

Eric: It's showing you the shapes you can make in between them

Stan: Apart from it's a bit slower so you can see what's happening.

Stan: It kinds of shows you the line of symmetry as well cause you've got like D and B and D so it shows all the shapes you can make with one line of symmetry.

Eric: The arrowheads are changing.

Stan: There's lots of different shapes. Well it goes into a rhombus anyway.

Susan: When does it go into a rhombus?

Stan: When they're both crossing in the middle. Then it's like

Eric: It's kind of a rhombus. Those two are the same. one o six, one o six

....

Susan: So erm, how many kites do you think you actually see when you animated the bar?

Eric: Two cause you see the ones

Stan: You actually see loads because each angle changes

Eric: There, where you see one down there. You see lots of different ones cause that's still a kite, kite, kite, kite, a rhombus, er kite. Then it goes into the kites.

From the dialogue it can be seen that Stan and Eric saw many more shapes than they had seen when they had dragged the figure. They talked about the changing of the figure with comments such as "the arrowheads are changing". They said there were loads of kites and noted the angle changing. They talked of the figure as "it goes into a rhombus". There appears to be an understanding here of a continuously changing figure with changing angles which, at different positions, takes on the configuration of different shapes. It is interesting to note that Eric originally said he could see only two kites and changed his mind when Stan pointed out that that he could see loads of kites.

<u>Hemma and Seema (iteration three)</u>

Seema: Oooh! Oh wow! Does it carry on? It stretches......It stretches from both sides then they both form the same shapes on each side

Susan: What shapes does it form?

Seema: The arrowhead

Hemma: arrowhead, that was it, and it was also a rhombus at one point and that was right in the centre

Seema: A rhombus

Susan: What shape's that.

Seema: A kite

Hemma: Look, it's still making kites. Is that a triangle?

Seema: It won't stop there. And the erm angles change

....

Susan: How would you describe what's going on in this file?

Seema: Basically A and C are moving but B and D are staying the same. And it's just stretching from both sides from different shapes.

Hemma: Do you know it actually looks like, if you have this in real life and not on the computer, it looks like a piece of material and someone's stretching it out and them two are just staying the same.

Susan: Mmmm. And what's staying the same?

Hemma: B and D

Seema: B and D aren't moving at all but they're still making the same symmetrical line because like I think the angles are changing only when it's ...

Hemma: Look it's

Seema: The distance measurements aren't as big but C and D and D and A are the same.

Hemma: Look these change by one millimetre each time.

Hemma and Seema noticed slightly different aspects of the changing figure than the boys (see figure 10.6). For example Seema described the figure as stretching and that the shapes on both sides were the same by which, I think, she was referring to the shapes generated either side of the rhombus position. When I asked the girls to describe what was happening Seema commented that BD stayed the same and AC was moving. Hemma, one of the students in the study who made the most insightful comments, described the figure as a piece of material which was being stretched. The girls went on to say that BD stayed where it was and continued to be the line of symmetry in the figure and the side lengths CD and DA were the same as each other. In this the girls had observed variance and invariance of the figure under the animated DMS. Hemma noted that the measurements of sides changed by one millimetre which, of course, was a result of the way that I had set up the degree of accuracy for the line measurements.

In conclusion it appeared that the animation of the figure under DMS did allow the students to perceive the figure as changing continuously between shapes as I had hoped it would. Furthermore the students had also perceived that there would be an infinite number of kites even though the constraints of the software probably mean that a discrete number of kites were presented.

10.4 Operative apprehension in relation to dragging strategies

Duval (1998) referred to operative apprehension as figurative change, an action which transforms the visual organisation of the figure. In his work, Duval gave examples in plane geometry where figures or diagrams are split into subfigures, or where subfigures are rearranged, in order to help with the solution of a problem. However, if operative apprehension is figurative change then that must surely mean that the change of a dynamic figure under dragging encourages students to engage in operative apprehension. Leung (2011) has certainly claimed that when students use specific dragging strategies in order to operate on the figure to discover its geometrical properties, operative apprehension is brought to the fore.

In the case of the dynamic perpendicular quadrilateral, the four dragging strategies appear to do different jobs and have different meanings for the students. In table 10.1, I have purposely listed the dragging strategies in a hierarchical order and this is because I perceive them to be attached to a hierarchy of reasoning which becomes more sophisticated further along the list.

How is it that using a dragging strategy (through the physical manipulation of a mouse connected through hardware to an object on a computer screen) can be imbued with meaning? To understand this, it is necessary to refer to the work of Vygotsky (1978) who explained that human cognition has developed through use of tools and by humans' ability to form a mental image and concept of the tool which he referred to as signs. Dragging in a DGS environment can therefore become a tool, and together with dragging strategies undertaken with intelligent purpose in the mind of the student it becomes a utilisation scheme (Verillon and Rabardel, 1995). Thus dragging strategies can become tools for semiotic mediation (Bartilini Bussi and Mariotti, 2008) which is to say that a dragging strategy itself can carry meaning about the geometrical figure, through the task using the dynamic figure.

In the next paragraphs I describe the meanings attached to the dragging strategies which the participating students were observed to use when working with the dynamic figure.

10.4.1.1 Wandering dragging

The strategy of wandering dragging was different from the other strategies in that it appeared to only have the purpose for the students of exploring what could happen to the dynamic figure, i.e. drag and see what happens. It was used infrequently during the sessions usually when students were unfamiliar with the figure and so I have not studied its use in any detail.

10.4.1.2 Guided dragging

Guided dragging was used by students to intentionally place the bars. At the beginning of the session, when the students were presented with two perpendicular bars separate on the screen, and were asked to put one bar so that it crossed the other, they all made a (close to) symmetrical arrangement. The students did this in response to my request, for example, in iteration two:

To Aftab and Rupen: "*OK then, so now, Aftab, if you get the mouse, you're going to drag one bar over the other bar*"

To Kate and Jane: "*Right, so drag one bar over the other*"

As can be seen from these two examples I did not give the students any clues as to how they should place one bar over the other. Nevertheless every single pair of students made an arrangement which would result in a kite or rhombus when the ends of the bars were connected with line segments. This led me to make the claim that the students were showing a preference for symmetrical arrangements.

Guided dragging was also used at other times in the sessions when the students dragged the figure from one configuration to another in a purposeful manner (but not symmetrically otherwise I would have coded the activity as DMS). The purposeful manner and short dragging time (usually between 2 to 8 seconds, for example see appendices 1.1a, 2.2a, 3.1a) led me to believe that the students understood precisely where the bars needed to go to make a desired shape which must have arisen from a strong holistic mental picture of the shape with the bars inside. They must have had these mental pictures before the sessions or else quickly developed them based on prior understanding. Assessing this kind of reasoning as Van Hiele level one due to the holistic view of the shape the students were using, as I have done up to now, hardly explains this complexity. At this point it would be useful to use Duval's theory of cognitive apprehensions which may provide more insight.

Duval (1998) described visualisation as a cognitive process involving the representation of a figure, identification of its constituent parts such as lines, angles and the shape of its boundary and the ability to split the figure into subfigures and even to rearrange subfigures. Duval also said that students use visualisation to explore the figure in a heuristic way. How this description of visualisation explains guided dragging may lie with the way in which the bars split the dynamic figure into four triangles, something which was mentioned by several students (eg Colin and Terry, Adam and Jack in iteration one). Students have also commented on triangles being the same, meaning congruent, for example when Adam and Jack described the two congruent sub-triangles inside the isosceles triangle as shown in Figure 10.3.

If the students did notice and react to the relative sizes of the sub-triangles in the figure while dragging this may also explain how they could make symmetrical shapes since if two triangles opposite a bar are congruent then the way the figure is constructed would mean these triangles would be mirror images. I have already noted research (e.g. Palmer, 1985, Shepard, 1994) which indicates that symmetry is a human intuition (section 2.1.2). This may explain how the innate notion of symmetry together with the making of sub-triangles to be congruent inside the figure supports guided dragging. In the case of the initial dragging of the bars, the sub-figures would have been equal sections of one or both bars.

Finally, students used guided dragging to generate right angled triangles by placing the bars end to end (e.g. Mike and Luke in iteration zero, see section 5.4.3). Once they had worked out one right angled triangle they easily made the others (there are four). However most of the shapes they generated, which have labels and known properties, and that the students would have learned about in mathematics lessons, have symmetry. If guided dragging mediates geometric knowledge about the shapes generated from the dynamic figure then that must concern the symmetry of the shapes and the resultant splitting into two equal halves.

10.4.1.3 Refinement dragging

Usually, whenever the students dragged the bars in order to make a specific shape the positions would be close to but not exactly perfect. Refinement dragging was then used by students to arrange the bars in exactly the right place to ensure side and angle properties of the desired shape (see section 6.5.1, Tilly and Alice, iteration one). In other words objects within the figure (bars) were being moved to make other objects (sides and angles) to have required measurements.

I have described in iteration one (section 6.7.3) how refinement dragging appeared to support students, such as Adam and Jack, in reviewing and revising their knowledge of side and angle properties and how it was used to support students in moving between spatio-graphical and theoretical geometry (Olivero and Robutti, 2007). Refinement dragging was different from the other dragging strategies because it was undertaken with respect to displayed side and angle measurements. It also entailed small movements with the computer mouse which must have required careful fine motor co-ordination of the hand and wrist. Refinement dragging took up more time than the other dragging strategies (see section 6.6) and occasionally took up to 100 seconds per episode of dragging. Often it was only when I played back the recording that I realised that the student with control of the mouse had continued to refine the figure even while we were talking (see section 6.4.1 and Table 6.2, Tilly and Alice, iteration one). It almost seems that it was a point of pride for some students to persevere to get the figure to be accurate!

Assessing refinement dragging activity as Van Hiele level two does appear to be justified since students did have to use knowledge of the properties of shapes. However, Duval may help to shed light on this strategy. It would seem that refinement dragging, with its reference to the students' perceptual understanding of shapes, is connected to discursive apprehension. This will be discussed further in section 10.5. However guided dragging must also take into account the bisecting of one bar by the other and hence, in the same way as guided dragging, probably uses operative apprehension but in a way which is focused on one small area in the figure, where the bars cross each other.

10.4.1.4 Dragging maintaining symmetry

The geometrical meanings mediated by DMS are complicated. First of all there were potential meanings understood by me as the expert mathematician but which were not necessarily realised by the students. I could see that DMS generated a figure where one diagonal was the perpendicular bisector of the other, with the default shape being the kite. Other shapes which were generated at specific points were the rhombus (in paper geometry known by expert mathematicians as being a special case of a kite) and isosceles triangles (not considered special kites in paper geometry by most mathematicians), as well as the concave kites. In my mind using DMS generated a specific kite family or ‘dragging family’. It was clear from the students’ comments that most of them did not consider the rhombus, for example as being a special kite, an understanding that I had hoped they would reach and on which I had pinned my expectations for the task using the dynamic figure. I concluded at the end of iteration two (section 7.6) that some students perceived there to be a discrete number of possible kites and suggested that this explained why they did not perceive the rhombus as a special member of the kite family. I postulated that if they could see that an infinite number of kites could be generated by dragging using the DMS strategy then the position which corresponds to the rhombus ought to be seen as a special kite.

It would be useful at this point to look at DMS through the lens of Duval’s framework. DMS was often quite a short dragging episode. Sometimes a DMS episode lasted only two seconds (see appendix 3.1a) even when the bars were oriented at an angle. This suggests that the students used a holistic view and probably tried to keep the two halves of the shape to be congruent. However, when I asked students how they dragged between, say, rhombus and a kite, they told me that they looked at the side measurements and tried to keep them equal. No doubt some students did keep an eye on the displayed measurements but I suspect that most dragged intuitively. Figure 10.5 is an excerpt from the data which records how Tilly and Alice (iteration one) dragged between a rhombus and kite.

Dialogue	On-screen activity
Susan: You did very well there. OK so what have we made so far? We've made an arrowhead, an isosceles triangle and a right angled triangle. So see what else you can make.	She drags the horizontal bar with big circular movements and settles it to make a kite. Wandering dragging then guided dragging
Tilly: we've got arrow, well kite. oh no that's a diamond or what do you? . Wait, that's a	She drags the horizontal bar down to the

<table>
<tr>
<td>diamond and that's a kite

Alice: Oh no that would be the

Tilly: no that's a kite I think

Alice: Isn't that,

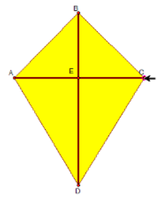
Tilly: diamonds are all four, just like a square...

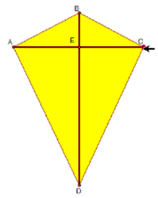
Alice: yeah, yeah, yeah

Susan: What's the difference between a diamond and a kite?

Tilly: erm, a diamond is basically like a square turned diagonally

Alice: It's more evened out so that, like, you've got more lines of symmetry so if you went down that way you'd have one line. And then if you went across you'd have another line because it's more equal</td>
<td>rhombus position then back up to the kite position. Dragging to maintain symmetry





She drags the horizontal bar down to the rhombus position then back up to the kite position</td>
</tr>
<tr>
<td colspan="2">The students said “that’s a diamond and that’s a kite” as the vertical bar was moved up and down. This appears to be an instance of ‘dragging to maintain symmetry’, moving between the rhombus and the kite. The students only allowed kites drawn in the three quarters position</td>
</tr>
</table>

Figure 10.5 An excerpt from the data showing DMS in moving between the rhombus and the kite

This episode shows that Tilly purposefully dragged the figure between the rhombus and kite positions (preferring the kite in its typical orientation, which I started to call the ‘three quarters kite’). This was the episode which first alerted me to the possibility that students might drag keeping symmetry a constant. The dialogue does not appear to show that students were aware that symmetry was maintained however, despite Alice’s description of the symmetry of the rhombus. It may be that keeping the congruency of two halves of the shape was the strategy the students used but the dialogue gives no evidence either way. This led me to believe that keeping symmetry constant is an intuitive action since it has not been verbalised. Hence DMS probably uses an intuitive notion of symmetry measured by the congruence of the two halves of the shape. This comes under the heading of operative apprehension (for the continuous figural change) and perceptual apprehension (for perceiving the shape as two symmetrical halves).

As far as the meanings which DMS may carry, the students recognised that they were generating shapes which had a common property (line of symmetry with the necessary equal sides and angles which can be deduced). However, the dragging family concept which I hoped the task would engender did not appear through the students' discourse. Development along these lines would require the services of discursive apprehension guided by myself in the role of teacher which was undertaken in iteration four.

10.5 Sequential apprehension

Sequential apprehension involves the understanding of how the figure is constructed and so is connected to discursive apprehension. In this study the students were not asked to construct the figure from scratch. However, because I had wanted them to appreciate that the figure was based around two unchangeable bars I had presented the pairs of students with the two bars and guided them in completing the figure around them. I had not done this when working with a whole class because I thought that it might be organisationally difficult to support a large number of students to complete the shape.

For the pairs of students who used the line segment tool to join the ends of the bars and the construction menu to colour in the interior of the shape there must have been some understanding of how the shape was made up. This created a shape with perpendicular diagonals (perpendicular quadrilateral and triangles) which could be dragged into specific shapes such as the kite.

However sequential apprehension may be deemed to have been used if students attempt to construct either a robust figure (whose properties are invariant under dragging) or soft figure (which keeps its properties only if objects on the figure are dragged along a specific locus) using the tools in the software (Leung 2011). Hence the DMS strategy can be connected to sequential apprehension because it is arguably a strategy which provides a soft construction of the kites (with occasional rhombus and isosceles triangles as special cases / positions along the dragging journey).

10.6 Discursive apprehension

Discursive apprehension involves verbal or mental reasoning about the figure. In the context of this study it involves describing relations between the geometric properties of the shapes generated by the figure in terms of a situated discourse about dragging activity. However, this has not proved to be an easy step for students in the study to make, particularly in terms of inclusive relations between shapes generated by DMS. It appears that more work needs to be done with the students to help them make sense of how they observe the figure under DMS. As Duval (1998) said, it is important that teachers guide students to describe changes in figures otherwise they may notice the visual changes but not develop the ability to reason about them.

10.6.1 Talking and reasoning about dynamic figures

When working with dynamic geometry figures, it is interesting to contrast how students and teachers speak of them as geometrical objects as opposed to how they talk about static figures. For example, in a study of a teacher and student teacher working with a class of fifteen to sixteen year old students Sinclair and Yurita (2008) observed how the teacher's geometrical discourse was affected by whether the class were working with static or dynamic images. Static images were spoken of as if they had always existed, for example "this is a square because it has four equal sides". On the other hand dynamic images were spoken of in terms of the human agency which had created them and thus there was an implicit reference to a point in time when these figures had not yet been brought into being (ibid). Saying "I made a rhombus by dragging the bar AC" would be an example of this sort of reasoning which is much more of a narrative than a description of a static figure because there is a sense of change happening over time.

Sinclair et al (2009) claim that students engage in narrative thinking whilst interacting with dynamic figures under dragging and that this narrative is important in developing their understanding of geometry. Here the word 'narrative' is used in the sense of recounting a series of events and interpreting these events. Narratives have a sequential process, being accounts of events occurring over time. Listening to the narration provides information about what the narrator has noticed as being important. In the following excerpt shown in Figure 10.6 Hemma used the dynamic nature of the figure to demonstrate to Seema that she could find more versions of a kite than Seema had at first imagined.

<table>
<tr><th>Dialogue</th><th>On screen activity</th></tr>
<tr><td>Susan: How many kites do you think you could make?

Seema: About two.

Susan: You think about two?

Hemma: I think a bit more because if I do that that's still a kite. And if I do that it's still a kite. Cause look they're the same and they're closer. Then if I also do that then that's still a kite and that's still a kite.

Susan: How many kites do you think you could make then.

Hemma: I think about six or seven.

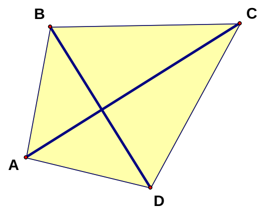
Susan: Could you not keep on moving it a little bit all the time and it still be a kite?

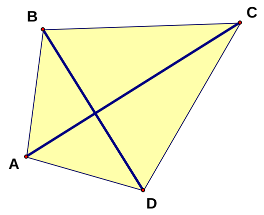
Hemma: Like a millimetre

Seema: It could go on for ages couldn't it.

Susan: So how many do you think it could be then if you can go on for ages?

Seema: About twenty.

Hemma: Twenty five or twenty six
.</td><td>

Hemma moved the bar BD a small amount and then another small amount


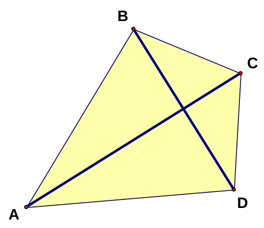
Hemma moved the bar BD so that she had a kite the other way up

</td></tr>
<tr><td colspan="2">Hemma used DMS for two short bursts to demonstrate to Seema that there were many positions for a kite. Seema then revised the number of kites to twenty.</td></tr>
</table>

Figure 10.6 Hemma demonstrated several different kites by moving the bar a small amount each time

During the discussion in Figure 10.6 Hemma used the mathematical meaning embodied in the DMS strategy to demonstrate a big idea to Seema, i.e. that many kites can be generated from the dynamic figure. Her explanation and demonstration was accomplished through the use of her own agency in carrying out dragging actions and her narrative which described what she was doing ("if I do that, that's still a kite"). Even though the girls did not make the leap of recognising an infinite number of kites they had made the important change in their understanding that there were many more kites in different positions. The other big idea which emerged from the on-screen activity in this excerpt is that using DMS generates this large number of kites, which all have one of the bars as a line of symmetry. Although Hemma did not articulate this verbally, her actions indicated it.

10.6.2 The role of the teacher in helping the students make links between their personal meanings and the mathematical content.

When students work with a dynamic figure or watch an animated figure, the descriptions of these events over time provide a narrative that the students can use to reason about mathematical relationships amongst the objects in the screen figure (Sinclair et al, 2009). However these personal meanings made by students interacting with dynamic figures need to be linked to mathematical knowledge and this is where the teacher has an important role in helping this to happen (Mariotti, 2009).

One example of where I, acting as the teacher, helped students to make personal meanings was during the second session with the year eight class in iteration four (section 9.4.2). From the beginning of the first session I had introduced the students to the words 'perpendicular' and 'bisector' and had modelled the use of these words with the students. In the second session I showed the class the animation and while they were watching it I got the students to put the two terms 'perpendicular' and bisector' together so that we had the concept of perpendicular bisector which is important in the meaning of the DMS strategy. I have not differentiated between the various pupils who contributed but pupil 1 is a specific boy who demonstrated more advanced geometrical reasoning than most of his peers. His contributions may also have helped other students in the class to make connections between what they observed and the mathematical meanings of shapes and their properties.

Susan: AC is just going back and forth. So what is AC doing to DB all the time?

Pupil: It's acting as the bisector

Susan: It's always bisecting it isn't it. What angle between AC and BD?

Pupil: Ninety

Susan: Ninety degrees? So can we put those two words together. We've got AC bisects BD and it's also at right angles to it. What's at right angles? What's that word?

Pupil1: Perpendicular

Susan: Perpendicular and a line which cuts another one in half is a

Pupil1: bisector

....

Susan: AC is the perpendicular bisector of BD because it's at right angles and it cuts it in half. So we've been making shapes that have the property that one diagonal is the perpendicular bisector of the other. In that perpendicular quadrilateral what is it most of the time? We've had kites and arrowheads and isosceles triangles and rhombuses. What is it most of the time?

Pupil1: A kite

Susan: OK and sometimes, go on.

Pupil1: An arrowhead

When I asked the students what the figure was most of the time, referring to the way that the figure morphed between different shapes, I was trying to encourage them to consider that the animation generated a kite family. I hoped that they would then perceive the rhombus and isosceles triangles as special positions in the animation and connect this to them being special cases of the dynamic kite. However the isosceles triangles do not have the requisite number of sides which is a difficulty if it is to be considered a special kite. It can only be considered a kite in the context of a dynamic figure.

In the third session the students made posters to demonstrate what they had learnt about geometry in the first two sessions. Some of the comments on the posters used the term perpendicular bisector, for example the students wrote that they had made shapes with perpendicular bisectors, describing what this meant and drawing the bars inside the shape. The following is an excerpt from poster A.

"The four shapes namely kite, isosceles triangle, arrowhead and rhombus all have one thing in common: they all have a perpendicular bisector. They all belong to the same family because each of them has a certain property when the bars AC bisect bars BD".

Modelling the use of this geometrical term and drawing the attention of students to the perpendicular bisector as having a meaning in the dynamic figure microworld idea did help many of the students in developing their own concept of the idea.

10.7 The particular value of the dynamic element of dynamic geometry figures.

When students work in a DGS environment the visual representation is clearly the main focus of the students' attention. The special feature of DGS is its dynamic nature which is brought into effect by the affordance of dragging; effectively the way in which students interact with the software (Leung and Lopez-Real, 2002, Jackiw and Sinclair, 2009). Under dragging the dynamic visual representation of a figure on the screen is akin to a small motion picture of the figure changing in real time (Lopez-Real and Leung, 2006). Dragging strategies used by the students act on the dynamic figure and as such become instruments of semiotic mediation which can help students make mathematical meanings (ibid). Lopez-Real and Leung (2006) viewed these meanings as bridging between dynamic geometry and Euclidean geometry, generally to help students develop understanding of proof. However it is likely that the semiotic potential of dragging is wider than this, particularly if we start to envision dynamic geometry as being a different kind of geometry in its own right. Dragging a figure on the screen is a visual actualisation of what humans often do when we mentally animate figures in order to perceive the variants and invariants (Leung, 2008, Sinclair et al 2009) and therein may lie the power of dynamic geometry as being more intuitive and closer to how our minds work than static geometry. Jackiw and Sinclair (2009, p.413) describe DGS environments as

"The powerful temporalised representation of continuity and continuous change and the sensory immediacy of direct interaction with mathematical representations".

In this sense a dynamic figure under dragging changes (morphs into different versions of itself) in a continuous fashion and under the direct control of the students. Since the figure embodies mathematical concepts this dragging has the potential for the students to engage with these concepts. Jackiw and Sinclair (2009) stated that a dynamic figure is a manifestation of a mathematical idea which can take all of the possible theoretical versions of that idea. In the case of the dynamic figure in this study a perpendicular quadrilateral can become various shapes under dragging and the particular DMS strategy generates the shapes which have one of the bars as a line of symmetry. The big idea in this is that the default shape produced by DMS is a kite and the other shapes are special cases of the kite. To understand this idea it is necessary to have perceptual understanding of the properties of the figure under DMS in that the symmetry of the shape results in its having two pairs of adjacent congruent sides (the necessary and sufficient conditions for a kite). However, as has already been noted students in the study did not necessarily develop the concept of the dragging family of kites. As Mariotti (2009) stated, the teacher has a role to play in helping students develop conceptual understanding from their own personal meanings, in this case constructed from working with the task using the dynamic figure.

Taking this further Jackiw and Sinclair (2009), described how dragging has first and second order effects. Students drag a specific figure and observe what happens to it, noting its example space (the many incarnations it can take). In this study students noted all the many triangles and quadrilaterals they made when they dragged the dynamic perpendicular quadrilateral, particularly using the DMS strategy. These were the first order effects. The second order effects of dragging are the developments in the students' geometrical reasoning which are mediated by thinking about the figure they could see morphing on the screen and the theoretical figure it represented. In this study when students dragged maintaining symmetry they observed that the shapes generated had a common line of symmetry and that there were always two pairs of congruent sides. However, I found that more needed to be done to support students to develop the concept of the dragging family of kites. One of the modifications I made to the task was to introduce the animated figure under DMS after having given the students time to work with the figure themselves. The other modification was to harness the power of the combined intelligences of a whole class of students using discussion to explore the big ideas behind the demonstration of the animation. So I could ask the class what the figure was most of the time, and pupil1 was able to answer that most of the time it was kites and arrowheads. Other pupils were able to answer that there was one rhombus and two isosceles triangles and this meant the class could think collectively about what this might mean. However I was mindful of the need to allow each student to make their own personal meanings and not to plant an undeveloped idea in their heads.

10.8 Reflection on Duval's theory of cognitive apprehensions as a tool of data analysis.

Using Duval's cognitive apprehension, when looking again at the data, has helped me to consider the particular value of using a dynamic figure in the task which I gave to the students. It appears that the changing figure under dragging provides the environment for students to talk about what they perceive as changing and unchanging in the form of a narrative, giving them more personal agency over the generating of shapes from the figure. This also supports the developing of students' reasoning as a cognitive process because describing changes in a dynamic figure may act as a bridge to describing properties in the theoretical figure.

The innate sense of symmetry which students appeared to employ when positioning the bars inside the shapes they generated suggests that students used visualisation as a cognitive process (Duval 1998) which can take a basic heuristic role when students begin to explore the figure by attending to its constituent parts and its sub figures (as when students pay attention to four right angled triangles inside the figure).

Generally, most activities undertaken by students whilst working on the figure could be said to employ perceptual, discursive and operative apprehension (and also sequential apprehension if dragging to generate shapes is included). However Duval (1995) was clear that the apprehensions should be taught separately, indeed need to be taught as he said that students do not necessarily develop these skills for themselves. So it appears from what Duval said and also from my findings from this study that the contribution of the teacher expert / more knowledgeable other has an important role to play if students are to develop more sophisticated geometrical reasoning.

In the next chapter I will consider how the dynamic perpendicular quadrilateral worked as a microworld for the students in the study. I make the claim that the dragging family is a situated abstraction on which to develop an understanding of inclusive relations between shapes. I will also consider the progress students did make towards level three reasoning even if they did not go as far as to use a hierarchical classification of the quadrilaterals which can be generated from the dynamic figure.

11 Discussion

In this chapter I explore the efficacy of the dynamic perpendicular quadrilateral as a microworld and consider how it has provided an environment for students to develop a greater understanding of kites (including those shapes which are subsets of the kites) and their properties. I describe the web of ideas which have supported the students in developing meanings within this microworld including the four dragging strategies that have emerged in the context of working with the dynamic perpendicular quadrilateral. I argue that Dragging Maintaining Symmetry (DMS) is a dragging utilisation tool which acts as a tool of semiotic mediation for the concept of the 'family of shapes', which is itself a situated abstraction. However, as I argued in chapter ten, to appreciate the meanings carried by DMS it is necessary to perceive its action on the dynamic figure as continuous rather than as a journey to a discrete position.

Finally I review the process of development in students' geometrical reasoning between Van Hiele levels two and three to ascertain how the students' geometrical reasoning has developed through the task.

11.1 The dynamic figure as a microworld

In chapter three (section 3.3) I described the concept of a computer microworld as an environment where students work intuitively, building on their current knowledge and understanding to make connections between mathematical objects and relationships (Noss and Hoyles, 1996) . Working in this way with the computer changes the way that students construct knowledge which is then situated within the context of the microworld (ibid). The dynamic perpendicular quadrilateral designed in the Geometers Sketchpad is itself a microworld within a microworld (of the DGS environment). It embodies a subset of Euclidean Geometry principles in that it can only be used to generate shapes which have diagonals at fixed lengths and at a fixed angle of ninety degrees. Working in the dynamic perpendicular quadrilateral microworld constrains the students' activity to generating shapes and constructing meanings about their conceptual properties (using the measuring facility). There is the potential in the microworld for students to develop more sophisticated meanings if they make connections between shapes generated by the DMS strategy.

11.1.1 How students construct knowledge through narrative description when working in Dynamic Geometry

As was discussed in chapter ten, section 10.6.1 students were observed to talk about dynamic figures differently from the way they talk about static figures. Dynamic figures were very often described in narrative terms in that they changed as a result of the students' own, sometimes exploratory sometimes intentional, activity through dragging objects in the figure. So the student was able to interact with the dynamic figure on the computer screen and to observe what happens during this interaction. When they verbalised this process they did so in a narrative fashion as a chain of events happening over a period of time and in which they are a main player (by causing these events to happen). It thus appears that the knowledge constructed in this microworld has a very personal meaning for the student. As was seen in chapter ten static figures were often perceived to exist in an atemporal sense (Sinclair et al, 2009) and consequently students may not feel any personal connection since even if they are asked to construct the static figure themselves using ruler and compasses they probably use a given algorithm to do this rather than devising their own methods.

However Sinclair et al (2009) view the making of mathematical meanings as requiring both the logical deductive reasoning used in static Euclidean geometry and the narrative experiential reasoning used in Dynamic Geometry. Students may find the second kind of reasoning to be easier and more intuitive and hence much research into learning in a DGS environment has addressed the efficacy of working with Dynamic Geometry to support students in developing skills of reasoning in the domain of Euclidean Geometry (Leung and Lopez-Real, 2002, Lopez-Real and Leung, 2006). This suggests that Dynamic Geometry continues to be viewed as the facilitator of Euclidean Geometry rather than being a new kind of geometry in its own right.

11.2 The web of ideas accessible through the dynamic perpendicular quadrilateral

Noss and Hoyles (1996) and Pratt and Noss (2002) describe a web as a network of links which includes external resources such as the environment, and tools which the students use to work on the task, and internal tools such as intuition and understanding of concepts. When students worked with the dynamic figure in this study, they had access to a web of supporting concepts which includes the figure itself, the Geometers Sketchpad program (and Euclidean Geometry principles, the measuring facility, the drag mode) and also the students' prior understanding of shapes and their properties, the discussion between students and between the students and me as the researcher. This webbing has allowed the students to construct new mathematical meanings by making connections between their prior concepts of shapes and what they have observed and reflected upon when dragging the figure and articulating what they understand about it. In particular many of the students who watched the animation of the figure under DMS have started to develop the concept of inclusive relations between the kites and the rhombuses as a result of making meanings about the 'family of shapes' (see section 9.6.2).

11.2.1 The perpendicular quadrilateral in the web of ideas

The figure with which the students worked is a dynamic perpendicular quadrilateral whose diagonals are 8cm and 6cm long. In reflecting on the data, I perceive the first layer of the webbing based on the figure itself as the concept that triangles and quadrilaterals can be generated by dragging the figure. The second layer of webbing based on the figure is that special triangles and quadrilaterals can be generated which can be tested using displayed measurements of sides and angles in the figure. When students first worked with the figure some of them (e.g. Adam and Jack, Stan and Eric) tried to make an equilateral triangle. They had seen that a large number of triangles could be made and so thought they should be able to make an equilateral triangle. However, with reference to the displayed measurements of sides and angles the students found that it was impossible to make an equilateral triangle although it was difficult for them to explain why. In chapter 6, section 6.7.4 I recounted an episode where Adam and Jack, in iteration one, used dragging and the displayed measurements to form a convincing situated proof that it is impossible to generate an equilateral triangle from perpendicular 8 cm and 6 cm bars.

Students in the study, having generated the rhombus, often tried to generate a square from the dynamic figure (e.g. Adam and Jack, Kate and Jane, Aftab and Rupen). Students in school A, especially, held the concept of the rhombus as a squashed square which, they said, they had been told in primary school (ages 5-11). Given this analogy the students thought they ought to have been able to make a square but found that they could not make one with the 8cm and 6cm bars. However, if I asked the students whether they would like to order a different set of bars in order to make a square they straightaway replied that the bars need to be the same length. In another example when I asked Hemma and Seema what would have to be true about the bars to make the rhombus which they had on the screen into a square Seema replied (transcript line 406):

> "*They would have to be the same size. Cause this one's 8 cm and 6 cm they have to be just 6 cm and 6 cm or 8 cm and 8 cm*".

Hence at this level in the webbing the students had begun to realise that there were constraints on which triangles and quadrilaterals they could generate from the figure, given the lengths of the bars. The fixed ninety degree angle also caused further constraints on which shapes could be generated. As a further challenge to some students I had prepared a file where the bars could be rotated away from the perpendicular and, for example, Kate and Jane in iteration two worked with these bars. However I decided that this line of enquiry detracted from the work with the perpendicular bars. Since I was interested in whether students could form a concept of a family of shapes, based on the kites, it seemed more fruitful to keep working with perpendicular bars.

In the next level of webbing, based on the dynamic perpendicular quadrilateral is the concept that, if one bar is intentionally dragged so that it continues to intersect the other bar at its mid-point, then shapes are generated which have properties in common. It is this activity which has the potential to mediate the concept of the 'family of shapes' and the inclusive relations of the rhombus in the family of kites.

11.2.2 Students' prior understanding of shapes and their properties in the web of ideas

It was clear from the sessions with the students that they brought prior knowledge about shapes to their work in the sessions. I have already noted the intuitive use of symmetry when students placed the bars and when they used the DMS strategy. I have also noted the concept image of shapes that student appeared to hold, which included notions of the kite having to be proportioned so that the shorter bar was positioned approximately three quarters of the way up the longer bar. This concept image, together with concept definitions which are definitely partitional (excluding, say a rhombus from the kites) may have acted against the conceptualising of a dragging family of shapes. The following is an excerpt from the recording with Tara and Ruth, iteration two.

Susan: *OK, so what makes a kite a kite? In other words what are the properties of a kite?*

Tara: *Erm, no parallel lines, no pairs of parallel sides. You've got two lines that are longer than the other two sides.*

Tara and Ruth definitely held the three quarters kite concept image as shown by the kites they presented when I asked how many different kites they could make (see figure 11.1).

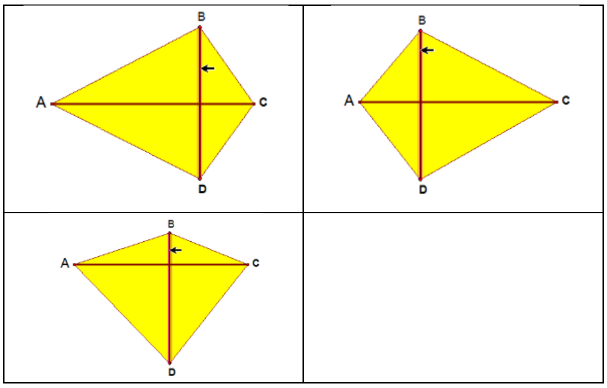


Figure 11.1 Tara and Ruth presented these kites which are close to the three quarters kite. This appears to be the typical and preferred proportion for a kite

Other concepts which the students brought to the sessions include line symmetry as a folding process, parallel lines as train tracks and (with Hemma and Seema) that triangles could not be described as both isosceles and obtuse angled at the same time (see section 8.4.1).

The rhombus as squashed square analogy was common and this often led to students thinking that the angles ought to be equal as well as the side lengths. Refinement dragging to try and make the angles equal served to support students in reviewing their knowledge of the angle properties of a rhombus.

11.2.3 Dragging strategies in the web of ideas

Dragging strategies have the potential to become utilisation schemes acting as tools of semiotic mediation for the concept of the family of shapes (Mariotti, 2009) which was discussed in chapter ten (section 10.4.1).

11.2.4 Discourse in the web of ideas

Discourse between students and teacher in an educational setting is a huge area of research and I do not have the space to go into the amount of detail the subject properly deserves. However it is an important subject because communication is the way that human beings share ideas and through which individuals' understanding is challenged (Donald, 2001).

In the session with Hemma and Seema which was described in chapter ten, (section 10.6.1) Hemma changed Seema's mind over how many kites it is possible to make (from an original two to twenty) by moving one bar in small increments and saying

> *"And if I do that it's still a kite. Cause look they're the same and they're closer. Then if I also do that then that's still a kite and that's still a kite".*

She demonstrated and gave good reasons for the claim that many more kites could be made than two and Seema revised her own thinking as a result.

When I recorded pairs of students working together clearly some of the pairs interacted more effectively than others. Adam and Jack, for example were two students who bounced ideas from one to the other and consequently the data from their recording is much richer than in cases where students tended to respond to my questions rather than discussing together.

In the whole class lesson in iteration four there were more opportunities for students to hear other viewpoints simply because more of them were in the same session. In a larger group it seems more likely that there will be individuals who have a more sophisticated level of reasoning which they might share with others. As an example there was a discussion of how many kites can be made (which I described in chapter nine, section 9.6.3), where a student said that one bar could be moved by 0.0000000000000000 (recurring) 1 cm and so there must be an infinite number of kites. Throughout the three whole class sessions the level of reasoning used when discussing shapes in their dialogue and their writing became progressively more sophisticated.

11.3 Purpose and utility of the task and associated mathematical concepts

The purpose of the task and utility of the mathematical concepts which arise through the task are important considerations when designing a learning task (Ainley et al, 2006, Pratt and Noss, 2010). In this section I consider to what extent purpose and utility became apparent as the students worked through the task.

The students certainly appeared to find purpose in the task generating shapes from the dynamic figure. It was a straightforward instruction to ‘drag the bars and see what shapes you can make’. I have already mentioned the pride some students took in using refinement dragging to get the position of the bars just right so that the properties of sides and angles were mirrored in the displayed measurements on the screen. The mathematics involved, for which the students appeared to develop an appreciation (of its utility) was in revising and reviewing the properties of shapes and also in developing an appreciation that shapes like the kite can be presented in different proportions. In this the students’ reasoning about shapes and their properties at Van Hiele level two were focused and consolidated. I believe that this is necessary before students can move on to reasoning at Van Hiele level three.

The use of the DMS strategy also appeared to have purpose for the students in that it was the action used to move between the positions for the shapes which have symmetry, namely kites, arrowheads, rhombus and isosceles triangles. Since humans appear to like symmetrical figures there may have been an aesthetic reason to generate shapes which have symmetry. Another reason may be that most of the shapes which are given labels and which students learn about in school have special properties such as symmetry. DMS generates shapes which are kites or special cases of kites and I hoped that the students would perceive them as members of a family of shapes all of which have properties in common based on line symmetry, and that this might lead to them including the rhombus as a special kite in the context of working with the dynamic figure. However the students did not easily appreciate the utility of this mathematical concept. It could be argued that there was no practical reason why this should be so. I had not given them a problem where using inclusive relations would make solving the problem more effective. I was simply asking them to consider a difficult idea just for the interest of it. Perhaps it is surprising that any of the students made connections between the shapes made under DMS!

11.4 The family of shapes as a Situated Abstraction: students did not necessarily develop this concept on their own

Situated abstraction refers to the making of meanings in mathematics within a specific context such as a microworld which means that the mathematical construct is situated within the context or situation within which the student has been working (Pratt and Noss, 2002). The tools they use and the way the students think and talk about the mathematics tends to be relevant to the specific context (ibid). In particular the concept of the ‘family of shapes’ (generated using DMS) is a situated abstraction not least because it is an idea which does not lend itself to the static geometry environment. Figure 11.2 shows how Dave and Evan (iteration two) were beginning to develop the dragging family concept. The figure was in the position for a kite and Evan used DMS to drag it into the isosceles triangle position. Evan explained how he dragged the bar BD to turn the kite into an isosceles triangle.

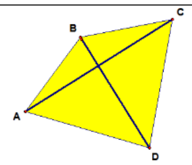
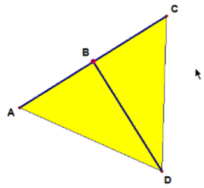
Dialogue	On-screen
Susan: OK. So how did you move that? You moved B didn't you. *Evan: Yeah* *Susan: From the kite position to the isosceles triangle position. What did you have to make sure you'd do to get it from a kite to an isosceles?* *Evan: Well we had to move it down, so near the solid line and I made sure it was in the middle so that it made the equal side lengths.*	 
Overview: The cursor traced the journey point B made to turn the shape from the kite to the isosceles triangle. In this case, when Evan talked about moving it down he actually meant towards the bottom right hand corner. It may be that he had mentally rotated what was on the screen so that in his mind he was dragging downwards.	

Figure 11.2 Description of moving the figure from a kite to isosceles triangle

Here Evan used the DMS strategy to move between two of the shapes on the DMS dragging journey. He talked about keeping one bar (BD) in the middle so that it made equal side lengths. It is not clear whether he meant that the bar should be kept in the middle of the shape or in the middle of bar AC. In either case this would maintain the symmetry of the figure. The mathematical meaning making in this episode concerns the use of a dragging strategy to move between two shapes which have in common the property of equal side lengths.

Other students identified a number of kites which could be generated using the DMS strategy. For example, Kate and Jane (iteration two) talked about being able to make millions of kites. Figure 11.3 shows how Kate perceived there to be millions of kites generated by moving one bar by very small amounts.

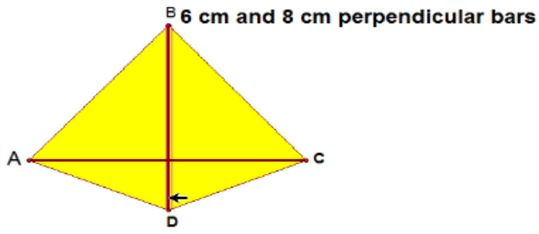
Dialogue	On-screen
Susan: Can you make another kite whose line of symmetry is BD? *Kate: Yeah, you can like pull it up* *Susan: So what are you pulling upwards?* *Kate: The BD line* *Susan: OK then.* *Kate: That's an upside down kite.* *Susan: How many kites do you think you could actually make?* *Kate: Quite a lot cause you can take it down from ...* *Susan: You could couldn't you. So what do you call that sort of number?* *Kate: millions. It's like 4.2, 4.1*	
Although Kate hah generated a three quarters kite, she could see the possible millions of kites which could be made by moving the bar by a very small amount (0.1 cm in this case).	

Figure 11.3 Kate said she can make millions of kites by moving the bar by a small amount

Kate and Jane realised that dragging the bar (BD) up or down could generate a large number of kites. Although neither of them mentioned the necessity of bar BD passing through the mid-point of bar AC, their dragging action indicated that this was their intention.

From these examples it can be seen that the students knew that dragging using the DMS strategy produced shapes that had the same line of symmetry and that the same pairs of sides were equal. However these appeared to be understood by the students at the level of observation rather than making any connections between the shapes generated using DMS. At the beginning of chapter 8 I wondered whether the reason was that students perceived the figure as changing between discrete positions at the end of the dragging journey rather than as a continuously morphing figure. There seems to be a dichotomy in this argument in the sense that I have claimed that students attended to the symmetry of the figure while using DMS, and they often reported that they looked at the displayed measurements while they dragged. It may be that the students did not consider the figure under dragging to be a proper shape until they had finished dragging, at which point they gave it status as a finished shape. In other words they perceived the dynamic figure during the task as being incarnated into a number of static shapes and dragging was used as a tool to move the figure between these static shapes.

11.4.1 Teacher / researcher intervention in the concept of the family of shapes

When the dynamic figure is dragged so that one bar is the perpendicular bisector of the other bar, in other words using DMS, a family of shapes is generated. These are a family of kites which include the rhombus (when both bars bisect each other), the isosceles triangles and arrowheads. Although the students observed that they generated these shapes when using DMS they did not themselves develop the concept of the family of shapes. In iteration three I introduced the idea of the family of shapes to the students: Stan and Eric and Hemma and Seema.

During the first session with Stan and Eric I suggested that all the shapes which could be made with one of the bars as a line of symmetry constituted a family of shapes. With Hemma and Seema I suggested that we could make a family of shapes where the mid-point of BD was always on bar AC. In neither of the cases did the students take up the idea of a family of shapes. They simply ignored that part of what I was trying to say because it meant nothing to them. For example, I suggested to Stan and Eric that they see what shapes they could make by keeping AC as a line of symmetry. When I asked the boys which shapes they had gone through, Eric replied:

"Started off with an isosceles, went to a rhombus then went to a kite. Then back to an isosceles".

I suggested to the boys that they had made a family of shapes by keeping AC as the line of symmetry. However, they did not themselves explore this idea and continued to talk about the shapes they made as simply being examples of what could be made by dragging.

When, in the second session, I showed the students the animation, this made an impact and they seemed ready to take up the idea of a family of shapes made through the animation. In chapter eight, (section 8.6.1) I described how, after having watched the animation, Stan and Eric began to talk about a family of shapes which included the arrowheads and kites as brothers (perhaps because they have the same properties but look very different shapes holistically) and also the rhombus as a member of the family. I thought that the boys were beginning to see the inclusivity of the relations between the shapes and probed further into why they thought the rhombus might be a member of this family of shapes. However, at this point their view of shapes through partitional classification became more dominant and as can be seen in the following dialogue the boys began to question whether the rhombus should be a member of the family.

Susan: OK then so, do you think the rhombus might be a member of that family as well then.

Stan: Well I suppose it kind of is, yeah. But it has two sets of equal angles. No it won't be cause it has two sets of equal angles.

Although the Stan and Eric had reverted to a partitional classification which excluded the rhombus from being a member of the kite family they did appreciate that there could be a connection between the rhombus and the kite because they suggested the rhombus could be the Dad in the family (and isosceles triangles could be uncles).

Hemma and Seema, after watching the animation, were also beginning to consider the possibility that there might be a family of shapes on the screen, although, again a partitional classification view and a holistic view prevailed over the perceptual understanding of the shape properties. When I had suggested that kites and arrowheads have the same properties and so might actually be in the same family Seema said:

"They might be but normally you wouldn't see it because they look so different".

This comment is akin to the kites and arrowheads 'as brothers' analogy made by Stan and Eric. Kites and arrowheads may have the same properties but they do not look alike. Overall I would claim that watching the animation helped the students to view the figure as morphing continuously between different shapes and they were more prepared to consider that these shapes were in a common family but their partitional view of shapes was not completely over-ridden. Partitional classification is a strongly held view, which had served the students well so far in their mathematics education (De Villiers et al 2009, Okazaki, 2009).

11.4.2 The whole class lessons: harnessing the power of shared meaning making

The web of support changed when I worked with the whole class in iteration four. Through the activity using the geo-strips I was able to introduce the concept of a constructive classification where properties are added incrementally. However, the most important part of the web was the class discussions and the comments from students who already demonstrated a more sophisticated level of reasoning about shapes than their peers. In this case the Zone of Proximal Development (Vygotsky, 1978) was active in giving many students who demonstrated Van Hiele level two reasoning the space to begin to perceive shapes as being connected to other shapes through inclusive relations. In listening to the comments made by their peers, students at level two often began to change the way they thought, as was noted by Sinclair and Moss (2012) when they worked with four and five year old children learning about different shape triangles.

The animation of the figure under DMS together with the power of class discussion among peers does appear to have influenced how the students reasoned about the dynamic figure and acted as the catalyst for many of them to develop the concept of the family of shapes. The data recorded from the work in iteration four indicates that students' reasoning developed over the sessions and that many students in the class were beginning to perceive the rhombus as being in the same family as the kites.

11.5 Development from Van Hiele level two to Van Hiele level three; the second period.

If Van Hiele level two reasoning is interpreted as recognition of the properties of shapes and recognising shapes from their properties then the students in this study were all able to reason at this level. Hardly any of them were secure in reasoning at level three by the end of their sessions, but that does not mean they made no progress at all.

The evidence from the data in this study indicates that working with the dynamic figure and watching the animation did act as a catalyst for the students to progress within the second period of development (Van Hiele, 1986). As in any building of mathematical concepts, that of inclusive relations needs to be carefully developed and will not necessarily be completed over one or two lessons. In this section I will consider the activities which may indicate progress within the second period.

11.5.1 From rhombus to square

Mike and Luke (iteration zero), Adam and Jack (iteration one), Aftab and Rupen (iteration two), Hemma and Seema, and Stan and Eric (iteration three) wanted to make a square with the 8 cm and 6 cm perpendicular bars and the nearest they could get to a square was a rhombus. At the point in time where they had tried to make a square but found that the figure could only be a rhombus I had asked them if they would like a different set of bars in order to make a square. All commented that to make a square they would need equal length bars. As Eric put it:

"It's to do with the eight and the six centimetres. If they're both eight or both six then it would be all equal so then it would work".

In deducing that equal length bars were needed to generate a square these students demonstrated reasoning at Van Hiele level three.

11.5.2 Perceiving similarities between the kites and the arrowheads

In iteration four one of the girls had noticed that she had written the same properties for the kite as for the arrowhead on her worksheet. In chapter nine (section 9.6.2) I described how she had called me over while the class wrote on their worksheets to tell me:

> *"A kite and an arrowhead have the same properties"*

When I asked her what those were she replied:

"Two sets of adjacent equal lines, two pairs of equal angles and one line of symmetry"

Stan and Eric, in iteration three, had suggested that kites and arrowheads could be brothers because they had the same properties. In both these cases the students had noticed the same properties but, possibly due to kites and arrowheads looking different, they did not make the step to perceiving kites and arrowheads to be the same shape.

11.5.3 Perceiving the figure as being able to generate a very large number or infinite number of kites / arrowheads.

- Aftab said "*You can make loads of kites*,"
- Kate said "*millions*" when I asked how many kites she thought she could make,
- Eric said they could make "*quite a few*" arrowheads,
- Hemma said they could make *twenty-six kites and a lot of arrowheads*.

In the whole class lesson pupil 1, who demonstrated inclusive reasoning about shapes, said there were infinity kites. (Students typically think of infinity as being a number rather than a tendency of numbers to keep increasing).

Being able to appreciate that there are many positions for the kites and the arrowheads is connected to the tendency to accept those positions where the proportions are not typical, such as where the cross-bar is close to the end of the other bar or near the middle. This requires that students use their perceptual understanding of shape properties to recognise shapes as being kites or arrowheads and that these are given preference over the figural nature of the shape. Thus understanding needs to be secure at Van Hiele level two, before the student can progress to level three reasoning.

11.5.4 Perceiving the figure to be a kite when the crossbar is close to the mid-point or the ends of the other bar

In chapter eight, section 8.8.3 Hemma had looked at the screen shown in Figure 8.12 which was very close to an isosceles triangle and suggested that the figure was a kite, "just a very odd kite with a very small top bit" She also linked this with their being "loads and loads and loads of kites" which connects the ability to accept atypical representations of kites with their being an infinite number of kites. I view Hemma's reasoning as being analytical because she showed that she privileged the properties of the kite over its figural nature, which would prompt most people to say it was a triangle including, most students in the study. Hemma demonstrated, by her appreciation that a figure which was a near triangle was still a kite, that her reasoning was more sophisticated than level two reasoning.

Some would look at Figure 8.12 and decide it was close enough to be considered a triangle whilst its displayed side and angle measurements indicate it to be a kite. A dichotomy that may arise from seeing a near triangle as a kite is that sometimes I have encouraged students to be happy with 'close enough' as when the measurements of expected equal sides and angles could not be made exact. This happened more in iteration zero and iteration one when the measurements were displayed to two decimal places and it was difficult to make expected equal measurements to be spot on.

Overall there was evidence that working on the task with the dynamic figure had a positive effect on the students in that their reasoning moved from level two towards level three.

11.6 Conclusion

In concluding this chapter, the dynamic perpendicular quadrilateral as a microworld has accomplished its role as an environment where students can develop their understanding of shapes and their properties and make connections between them. There is the potential for students to develop the concept of inclusive relations indicative of a move to reasoning at Van Hiele level three. There is evidence from iteration four that students were developing this concept but we cannot expect concepts to be built in a short space of time. Carefully designed activities together with discussion between teacher and students are necessary to build concepts such as inclusivity, particularly because students have to rebuild their original concepts about shapes which they have held since early childhood.

Chapter 12 Conclusions

In this concluding chapter I address the research questions. I also consider what using the Design Based research method brought to the study in terms of how undertaking the research in a series of iterations allowed me to reflect on the findings and to make improvements to the overall task which the students worked on.

12.1. Addressing the research questions

In chapter 3 (section 3.7), I posed two research questions, which arose from the review of the literature. The first question was to ascertain which aspect of the dynamic nature of DGS impacts on students' reasoning about 2 dimensional figures. The second question addressed whether the Dragging Maintaining Symmetry strategy acts as a dragging utilisation scheme to mediate the concept of inclusive relations, particularly between the rhombus and kites. Since the two questions are connected in the context of this study I have synthesised my answers to both.

Dragging Maintaining Symmetry appears to be a special case of Maintaining Dragging (itself a dragging utilisation scheme) which was described by Baccaglini-Frank and Mariotti (2010) as intentional dragging of a figure in order to keep a desired property constant. In the case of DMS the desired property is the symmetry of the shape. The ability to use DMS to drag the figure so that it keeps close symmetry would suggest that humans have an innate notion of symmetry which they use in spatial problem solving.

The symmetrical aspect of DMS gives the dragging strategy its meaning as a generator of a family of shapes with the common property that one diagonal is the perpendicular bisector of the other. The family of shapes is, in fact, a family of kites with the rhombus as a special case of the kites (also isosceles triangles and arrowheads). However, it was when I animated the figure under DMS that students were able to perceive the figure as continuously morphing through kites with the rhombus and isosceles triangles as special positions along the dragging journey. Thus DMS has generated a situated abstraction (the dragging family of shapes) which has potential for being the catalyst for students' progression in geometrical reasoning. Clearly DMS is a speciality situated within the specific microworld of the dynamic perpendicular quadrilateral.

An important factor in the development of the concept of the family of shapes was the dynamic nature of the figure, first under manual dragging and later when it was animated. When the students watched the animation they articulated a narrative to describe their observations which enabled them to form new concepts about the relations between the shapes in the kite family. The advantages of creating a narrative came to the fore when students discussed their observations with their peers. In this, the whole class context provided an extra dimension when many more ideas and observations could be shared, including from the students who were more advanced in their geometrical reasoning.

12.2 The value of higher level thinking in geometry.

In chapter two (section 2.5.2.2) I wrote of how Duval (1998) questioned the teaching of deductive reasoning and proof to all students, particularly as doing so often results in students losing their natural ability to use reasoning and justification. In agreement with Duval I would argue that we need to develop tasks for students which encourage the use of reasoning and justification, and which will give those who wish to study geometry further a firm foundation on which to develop deductive reasoning and proof skills. Concept building is important as a basis for higher mathematics and geometry is no exception to this.

> "If mathematics instruction were to concentrate on meaning and concepts first, that initial learning would be processed deeply and remembered well. A stable cognitive structure could be formed on which later skill development could build".
> (Heid, 1988. P.4)

I believe it is the dynamic nature of DGS which, through the narrative that students construct of the changing figure, has potential for developing such conceptual understanding.

12.3 Using the Design Based Research method

The advantage of using a Design Based Research method has been that it allowed me to treat the study as a journey and process of discovery. The emerging findings, from each iteration, have been taken into account and tested or used to justify the modification of the dynamic figure itself or the way that the task was presented to the students. There were three main modifications to the task before I decided it was ready to trial with a whole class.

12.3.1 First modification

The first modification to the direction of the study occurred after iteration zero when I decided not to continue with the construction of shapes task (although Stan and Eric did get to construct a kite in their second session in iteration three). When analysing the data collected during iteration zero I decided that the way the students interacted with the dynamic figure was interesting and that I would focus on looking at the dragging strategies the students used when generating shapes from the figure. Four distinct dragging strategies emerged and iteration one was designed to ascertain whether other students would use the same strategies. By the end of iteration one I had observed that pupils used a sense of symmetry when generating the shapes and that one dragging strategy, DMS, brought into play students' intuitive use of symmetry (with a preference for dragging keeping vertical symmetry).

12.3.2 Second modification

The modification of the figure in iteration two was designed to test whether students would continue to use a sense of symmetry, when dragging the bars inside a figure which was not oriented within a vertical – horizontal framework. The evidence from the recordings showed that students dragged the figure using the same strategies whether the figure was 'upright' or presented obliquely. However the students working with the oblique figure appeared to find it slightly uncomfortable and some of them asked me if they could not turn the figure so that it was the right way up!

12.3.3 Third modification

Although the students had observed that the use of DMS generated kites, rhombus, isosceles triangles and arrowheads, they did not develop the concept of the family of shapes which could lead to an understanding of inclusive relations. I wondered whether the problem was that the students perceived dragging as an action leading to discrete shapes rather than resulting in a continuously changing figure. The modification in iterations three and four to address this was the animation of the figure under DMS, shown to the students after they had themselves worked with the figure. The animation had an impact on the students who began to talk of the figure as 'showing the shapes you can make in between' or being 'like a piece of elastic material'. The students began to talk of the shapes as being members of the same family. The animation acted to mediate the concept of the family of shapes.

12.4 Limitations of the study and what might be done to improve it.

If I had to carry through this study again there are a number of things which I would do differently.

12.4.1Pre and post assessment of students' reasoning about geometrical shapes

In collecting data I took the view that I would assess the students' prior understanding of shapes through their work with the dynamic figure and that I would be able to observe if their reasoning about shapes developed during the session. To a large extent this worked with pairs because the students would name the shapes they generated and when I asked what would need to be true to make a specific shape a kite / isosceles triangle or whatever they would tell me what they thought to be the properties of the shape. I was able to asses these comments and assign a level of reasoning according to Van Hiele.

Certainly for the class lesson in iteration four, I regretted not finding out if any of the students had already developed the concept of inclusive relations before the lesson as pupil 1 clearly did. Collecting such data from the second class had to act as a proxy for the information I did not collect from the first class. However the data collected from the three lessons did indicate a development in the discourse between the students in the class, both written and verbal.

12.4.2 Introducing constructive classification to the students

In the sessions with pairs of students I had allowed them to explore freely which shapes could be generated. However, in iteration four with the whole class working with the geo-strips I followed this by introducing constraints. First I asked that the geo-strips should be kept perpendicular (as the bars are constructed to be) and then added a further constraint that one bar had to bisect the other (making sure the word 'bisect' was understood by the students). I had introduced this constraint to students in iterations two and three when I asked them to drag keeping one bar as a line of symmetry but because it was not given to the students in the logical manner of iteration four these students may not have appreciated the implications of generating subsets within sets of shapes which emerges when constraints are added.

12.4.3 Involving class teachers

It became clear to me that the ownership of the research methods and findings rested solely with me even though I had attempted to involve the contact teachers in School A and School B I through co- teaching whole class lessons. The task with the dynamic perpendicular quadrilateral was not used again after my involvement with the schools had finished. Sutherland (2007) describes the importance of collaboration between researchers and teachers in sustaining innovations which develop through classroom based research. I realise now that, for classroom teachers to take ownership of the research outcomes, they need to be involved in the research in terms of designing and undertaking activities, and discussing the analysis.

12.5 Did the task achieve the goals set down for Design Based research?

In chapter four (section 4.1) I described two important goals for Design Research; to advance new theory of how students learn and to use this theory to develop educational interventions, in the form of new activities and resources which can be used in the classroom to facilitate that learning (Design Based Collective, 2003, Lamberg and Middleton, 2009). In this section I will address whether this study has achieved these goals and aims.

Symmetry and orientation have emerged as important factors in how students perceived the shapes generated from the dynamic figure and this has been observed by previous research into how students perceive shapes in a static environment (e.g. Palmer, 1985, Shepard, 1994). The students' concept image of shapes (Tall and Vinner, 1981) clearly affected how they expected shapes to be presented, particularly the orientation and the proportions of the shapes. It is not surprising that students would bring preference for symmetry and their concept images of shapes in a static environment into their experiences working in a dynamic environment. However, it appears that pupils used their intuitive notion of symmetry when positioning the bars inside the dynamic figure, and in particular when using DMS to drag between symmetrical shapes.

What I have shown in this thesis is that, when working within the microworld of the dynamic perpendicular quadrilateral, students used dragging strategies which were situated within the specific context. These dragging strategies can be aligned to cognitive activity and of most interest during this study has been the Dragging Maintaining Symmetry strategy since it has the potential to mediate the concept of the 'dragging family of shapes'. However the students did not access this concept until they were able to perceive the dynamic figure as continuously morphing through the animation. I suggest that it was the animation of the figure under DMS which allowed the students to perceive the figure as being a kite which at certain points changed into a rhombus or isosceles triangle. The concept of the rhombus as special case of a kite follows from this observation although a strongly held partitional classification view can interfere with this development.

Finally the task using the dynamic perpendicular quadrilateral has shed some light onto how students perceive two dimensional shapes and has been shown to be an effective intervention to use in the classroom to facilitate the development of geometrical reasoning. Working with the dynamic figure appears to encourage the use of perceptual, operative and discursive apprehensions (Duval, 1995), in a heuristic way, allowing students to think about the shapes they generated in a new way.

12.6 Final Conclusion

Past research into Dynamic geometry Environments has focussed on the dragging tool as a way to test whether a construction is robust, or to generate a soft construction which keeps specified properties. However the potential of the dynamic nature of DGS to encourage a narrative through which students construct meanings in geometry is less researched. Further analysis of how students' reasoning develops as they work with dynamic representations in mathematics could prove to be an immensely fruitful area for research.

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Appendices 1 Iteration 1

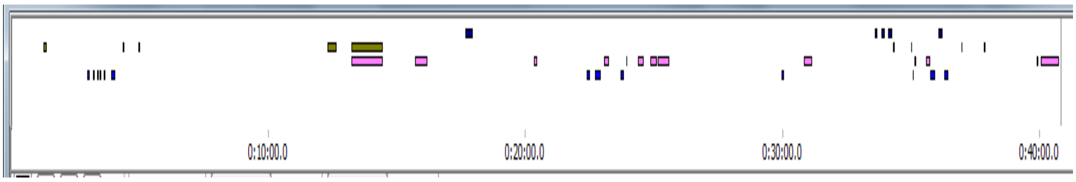
Appendix 1.1a Adam and Jack June 2010 Table of dragging episodes

	1.10-1.15	5 seconds	GD	One bar is dragged over the other.
2.40-4.21 Investigating what shapes can be made	2.49-2.53	4 seconds	DMS	The bar AC was dragged left and right to make a rhombus, then a triangle, then back to a rhombus. The bar BD was dragged to make an isosceles triangle then back to a rhombus, then to an arrowhead. All the while symmetry was maintained.
	3.05-3.10	3 seconds	DMS	
	3.16-3.19	3 seconds	DMS	
	3.21-3.26	5 seconds	DMS	
	3.31-3.35	4 seconds	DMS	
	3.49-3.58	9 seconds	DMS	
	4.16-4.20	4 seconds	GD	
4.21-4.57 Isosceles triangle	4.52-4.57	5 seconds	GD	The bar AC was dragged to make an isosceles triangle with BD as the base.
4.57-14.23Trying to make an equilateral triangle	12.15-12.35	20 seconds	GD	The boys first measured the angles. There was a discussion on how they positioned the bars by eye. AC was dragged up and down keeping point A touching bar BD.
	13.11-14.23	72 seconds	RD GD	
14.23-17.35 Isosceles triangle	15.38-16.09	31 seconds	RD	The bars were positioned to generate an isosceles triangle. When the boys were asked which angles should be equal Jack pointed to the pairs of smaller angles either side of bar AC. Refinement dragging was used to make the angles at B and D equal. The boys referred to two triangles being the same.
17.35-23.40 kite	17.36-17.53	17 seconds	WD	The bar AC was moved left, down, then up. The sides AD and CD were measured to check if they were equal and similarly the sides AB and BC. The bar AC was dragged to generate an upside down kite.
	20.18-20.25	7 seconds	RD	
	22.21-22.29	8 seconds	DMS	
	22.40-22.54	14 seconds	DMS	
	23.02-23.13	11 seconds	RD	
23.40-29.57 rhombus	23.40-23.47	7 seconds	DMS	The bar AC was moved down to the middle of bar BD. Refinement dragging was used to make the sides equal. There was a discussion on the sizes of the angles.
	23.53-23.55	2 seconds	RD	
	24.20-24.35	15 seconds	RD	

	24.50-25.04	14 seconds	RD	
	25.07-25.34	27 seconds	RD	
29.57-33.34 arrowhead	29.57-30.03	6 seconds	DMS	The bar AC was dragged below point D and it was symmetrical by eye. There followed a discussion on symmetry by folding. There was refinement dragging of bar AC whilst measures of sides were attended to. There was a discussion on which sides and angles should be equal.
	30.49-31.08	19 seconds	RD	
33.34-36.49 Trying to make a square with the 8 cm and 6 cm perpendicular bars	33.35-33.41	6 seconds	WD	The bar BD was dragged right and left and back to an arrowhead. Then BD was dragged up, then down, to the side (wandering dragging). The boys made a right angled triangle and called it half a square. The bars were dragged back to the rhombus position.
	33.51-33.58	7 seconds	WD	
	34.05-34.16	11 seconds	WD	
	34.17-34.22	5 seconds	GD	
	35.00-35.02	2 seconds	GD	
	35.03-35.06	3 seconds	DMS	
	35.07-35.12	5 seconds	RD	
	35.34-35.43	9 seconds	RD	
	35.44-35.56	12 seconds	DMS	
	36.03-36.13	10 seconds	WD	
	36.16-36.26	10 seconds	DMS	
36.49-42.59 Square using two 8 cm perpendicular bars	36.57-37.00	3 seconds	GD	The bars were dragged to a vaguely symmetrical shape (about both bars). The angles of the shape were measured. Refinement dragging was used to get the angles close to 90 degrees.
	37.50-37.52	2 seconds	GD	
	39.52-39.57	5 seconds	RD	
	40.03-40.46	43 seconds	RD	

Appendix 1.1b Adam and Jack June 2010 Visualisation window

The screen shot below shows the visualisation window in the Transana file for the 2 boys June 2010. As can be seen the intervals of dragging are grouped together and correspond to the generation of the trial shapes, isosceles triangle, right angled triangle, kite, rhombus and square. This particular pair of boys was quite interesting in that they used DMS early in the session, being able to visualise where they should put the bars for each shape. It was only later in the session, when I asked them to make a square in the first file (with the 8 cm and 6 cm perpendicular bars) that they used wandering dragging over a period of time trying to do the impossible!
Two long bars coincide; RD and GD. This corresponds to an interesting episode where Jack and Adam tried to make an equilateral triangle and appeared to be using guided measuring.



Appendix 1.2a Colin and Terry Jan 2011 Table of dragging episodes

0.00-5.54 Exploring by using WD	2.26-2.35	9 seconds	WD	The bars were dragged randomly and the boys generated a number of triangles and quadrilaterals including the kite, rhombus and isosceles triangle.
	4.11-4.18	9 seconds	WD	
	4.28-4.35	7 seconds	GD	
	4.45-5.44	59seconds	WD	
5.54-14.40 kite	5.51-5.56	5 seconds	DMS	The boys generated a kite and identified two pairs of equal sides and one pair of equal angles.
	6.15-6.17	2 seconds	DMS	
	9.35-9.38	3 seconds	RD	
	9.54-10.17	23 seconds	RD	
	11.03-11.12	9 seconds	RD	
14.40-35.06 rhombus	14.40-15.06	26 seconds	GD	The boys generated something which was close to a rhombus and could get the four sides to be within 0.4 cm of each other and they could not identify angles which ought to be equal. At my suggestion they put infinite lines over sides AB and CD and getting these to be parallel helped the boys to make the sides of the quadrilateral much closer and they were able to identify pairs of equal angles. Later they added the mid-points to bars AC and BD and lined them up to make a fairly accurate rhombus.
	15.17-15.27	10 seconds	RD	
	15.42-16.07	25 seconds	RD	
	16.17-16.21	4 seconds	RD	
	16.27-16.29	2 seconds	RD	
	16.53-17.37	44 seconds	RD	
	20.41-20.48	7 seconds	RD	
	21.28-22.15	47 seconds	RD	
	22.57-23.14	17 seconds	RD	
	24.25-24.27	2 seconds	GD	
	25.43-25.45	2 seconds	GD	
	26.53-27.01	8 seconds	RD	
35.06-42.42 Isosceles triangle	35.06-35.52	46 seconds	DMS	They generated an isosceles triangle and identified its side and angle properties
	37.14-37.50	36 seconds	RD	
	38.17-38.21	4 seconds	RD	

	38.49-39.28	6 seconds	RD	
42.42-43.23 Between kite and rhombus	42.42-42.53	7 seconds	DMS	The boys used DMS to move from the isosceles triangle. Through the kite to the rhombus.

Colin and Terry made very little use of the DMS strategy during the session but they did make a great deal of use of the RD strategy. They were very careful when dragging the bars and may have been carefully watching the measurements change while they dragged.
Since Colin and Terry mentioned parallel sides earlier when we were talking about the rhombus I decided to explore this idea. I told them how to put infinite sides over the top of the sides AB and CD so that they could see whether the sides were parallel. Making the sides parallel did help Colin and Terry to generate a fairly accurate rhombus but, in the end I decided that parallel lines were not a helpful line of enquiry, and I did not explore this any further.

Appendix 1.2b Colin and Terry Jan 2011 Visualisation window
The visualisation window below indicates that most dragging activity is RD with some WD, GD and a little DMS.

Appendix 1.3 The strategy of Dragging maintaining Symmetry

This episode shows Adam and Jack beginning to use DMS as a strategy. In this episode the boys began with an isosceles triangle with horizontal symmetry. This was unusual because most students made shapes with vertical symmetry. Jack had control of the mouse and he first used Dragging Maintaining Symmetry horizontally followed by dragging (without maintaining symmetry) vertically.

However as the vertical dragging continued the shape became closer to symmetrical until, at the end, the ‘equal’ angles were only two degrees apart.

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This series of screen shots depicts the figure at stages during the dragging process.

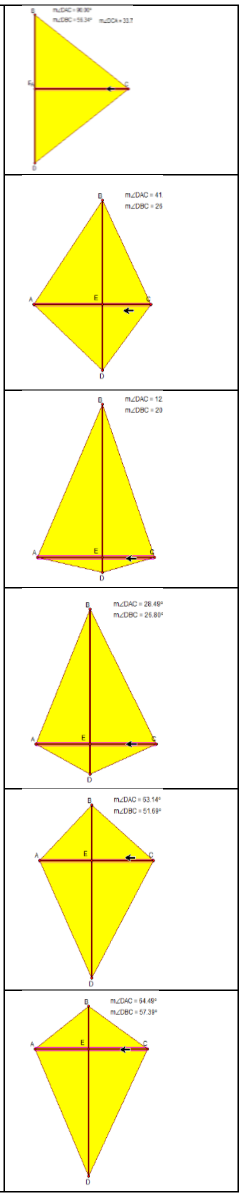
First Jack dragged bar AC to the right and AC stayed close to the perpendicular bisector of BD while he was doing this.

Then Jack dragged bar AC straight down the computer screen but not maintaining symmetry.

At the lower limit of the dragging he started to move bar AC slightly over to the left so that the figure became closer to having symmetry.

The figure became closer to symmetrical as he dragged bar AC up the screen.

By the time he finished dragging bar AC upwards Jack had got the shape to be close to symmetrical. Angles DAC and DCA were less than 2 degrees apart. These angles lie between the bar AC and the lower sides of the kite.



Adam and Jack: an example of dragging maintaining symmetry

Appendix 1.4 Two episodes per pair of students where displayed measurements indicate DMS.

	Time interval	Differences between expected equal measurements	comments
Adam and Jack	22.21-22.29 35.1-35.56	Angles < 3 degrees Sides < 0.2 cm Angles < 2 degrees Sides < 0.2cm	
Tilly and Alice	13.45-13.50 41.15-41.23	No angles displayed Sides < 0.3 cm Angles < 11 degrees Sides < 0.7 cm	Alice had the mouse and dragged carefully. Tilly had the mouse and tended to sway the vertical bar hovering around the middle position.
Colin and Terry	35.06-35.40 42.42-42.53	Angles < 3 degrees Sides < 0.1 cm Angles < 4 degrees Sides < 0.4 cm	
Gill and Sara	18.28-18.30 42.11-42.16	Angles < 9 degrees Sides < 0.4 cm Angles < 6 degrees Sides < 0.2 cm	 Illustrated example in chapter six

Clearly some students have a steadier hand than others when moving the mouse to drag the bars. The differences given are the greatest differences observed during the DMS episode which often indicated more accurate dragging over most of the time interval.

Appendix 1.5 Table of episodes for the recording with Adam and Jack

Line numbers from narrative	Time interval in recording	Description
3	1.10-1.15	Bars put straight into symmetrical position
7-17	2.49-3.58	DMS while investigating shapes
24-31	4.20-4.50	Triangle split into two right angled triangles. There cannot be symmetry as "the top triangle is bigger than the bottom triangle". *Discursive and operative apprehension.*
158-172	13.11-14.23	Boys prove they cannot make an equilateral triangle. *Discursive and operative apprehension.*
173-192	14.23-16.09	Symmetry leads to certain angles being equal. Symmetry as a process (by folding one half of the shape over the other). Refinement dragging with instructions. *VH level 2* *Discursive and operative apprehension.*
193-207	16.09-17.26	Boys talk of the isosceles triangle being made of two right angled triangles which are the same and opposite angles being the same. *VH level 2* *Discursive and operative apprehension.*
249-252	20.17-20.25	RD to within a difference of 0.01. Boys identify equal sides. *VH level 2*
283-287	23.40-23.47	DMS "move AC bar into the middle" *VH level 2*
298	24.20-24.35	RD with instructions and reasoning *Discursive and operative apprehension*
300-306	24.50-26.05	RD Jack works out the difference between expected equal lengths as being 0.01. *VH level 2*
307-334	26.05-28.15	Properties of a rhombus Position of bars for a rhombus Symmetry of a rhombus with a functional explanation *VH level 2*
336-366	28.18-29.41	Boys review properties of the angles in a rhombus. VH level 2
372-378	30.08-30.34	Symmetry of arrowhead by folding *Discursive and operative apprehension* *VH level 2*
385-395	31.28-32.12	Properties of an arrowhead *VH level 2*

432-451	35.26-36.38	Boys try to make a square using unequal length bars and don not succeed. They decide they need equal length perpendicular bars to make a square. *Towards VH level 3?*
463-465	37.55-38.28	Position of bars to make a square. Comparison of bar positions in a rhombus. Mention of orientation ("looking at it at an angle to see whether it's perfect"). *Towards VH level 3?*
481-490	39.54-40.46	RD with instructions *Discursive and operative apprehension* *VH level 2*
4910497	40.46-41.01	Close enough measurements, but Jack says that it is not definitely a square since the angle measurements are 0.07 either of 90 degrees. (need for the perfect square but getting close is an indication that it could be perfect). *Figural concept*

Appendix 1.6 Table of episodes for the recording with Tilly and Alice

Line numbers from narrative	Time interval in recording	Description
5-7	0.43-0.48	Bars put straight into symmetrical position
53-57	4.25-4.38	Triangles in the shape
65-68	4.54-4.59	Line of symmetry identified
73-95	5.21-6.58	Girls used a functional meaning of symmetry derived from how they had been taught to check for symmetry *Situated abstraction*
96-103	6.58-7.37	Use of symmetry to identify equal side measurements *VH level 2*
144-152	10.04-10.48	RD with dialogue, girls cannot get measurements to be exact
153-168	10.48-12.41	Alice attends to the measurements whilst dragging and Tilly instructs her to keep the bar BD in the middle of the shape “we’re trying to get BD in the middle of the shape”
169-177	12.41-13.54	Tilly takes the mouse and makes an isosceles triangle, uses RD to get it accurate then moves from that position to make an arrowhead and a kite using DMS
189-194	15.14-15.44	DMS. Connecting position of bar in the middle of the shape to getting equal length sides *Towards VH level 3?*
198-219	16.01-17.25	Properties of isosceles triangle *VH level 2*
286-293	22.04-22.37	RD with dialogue
298-300	23.19-23.25	“you sort of know” a right angle *VH level 1*
305-345	23.50-26.08	Discussion on whether the right angled triangle is half a square or half a rectangle, visualising a transformation
350-358	26.38-27.09	DMS “that’s a diamond and that’s a kite” “ a diamond is basically like a square turned diagonally”
392-455	29.09-33.51	Girls compare properties of rhombus and square, they review their knowledge about angles of a rhombus *VH level 2*
463-469	34.48-35.27	Girls check and review angle properties *VH level 2*
529	39.41	Typical kite with obtuse angle at the top
540-541	40.37-41.08	Girls start with a rhombus and drag bar AC higher

Appendix 1.7 Table of episodes for the recording with Colin and Terry

Line numbers from narrative	Time interval in recording	Description
21-23	2.38-2.42	Description of the isosceles triangle as two triangles *operative apprehension*
24-28	2.48-3.11	It cannot be an equilateral triangle because the sides are not all the same size *VH level 2*
67-70	5.40-5.52	Rhombus is a pushed over square
71-72	5.52-6.04	"if you look at it this way it looks a bit like a rhombus or a parallelogram" *Holistic view, possibly orientation,* *VH level 1*
73-81	6.04-7.17	Description of a kite in typical orientation (2 sides at the bottom are usually bigger). *VH level 1* Split into 4 triangles, *operative apprehension*
102-110	8.54-9.20	Fairly accurate kite straight off
281-287	25.30-25.56	Rhombus split into triangles
309	28.13-28.38	Properties of bars, *VH level 2*
439-443	40.01-40.22	Satisfied with near accuracy
459-461	41.27-41.35	Satisfied with near accuracy

Appendices 2 Iteration 2

Appendix 2.1a Lesson plan Geometers Sketchpad session

<table>
<tr><td colspan="2">Topic: properties of 2D shapes</td></tr>
<tr><td colspan="2">Lesson objectives To use dynamic geometry software to investigate 2 D shapes that can be generated with diagonals of fixed length and orientation.</td></tr>
<tr><td colspan="2">Outcomes Pupils learn about shape properties from a fresh perspective</td></tr>
<tr><td>Vocabulary
Dynamic, dragging, properties, symmetry</td><td>Resources
Computers loaded with the Geometers Sketchpad software (GSP)
A file in the GSP which contains 2 perpendicular bars (rigid lines) of 8 cm and 6 cm.</td></tr>
<tr><td colspan="2">Starter
From a bag of cardboard shapes, pupils pick them out at random. They name the shape and give the remembered properties of the shape.</td></tr>
<tr><td colspan="2">Main activity: Pupils work in pairs with a computer.

Task 1. Pupils open the file then spend 5 minutes using the arrow tool to drag the shape, investigating what specific shapes they can make.

Questions: What changes and what stays the same when you drag in the shape?
What shapes is it possible to make?

Mini plenary – pupils share their answers to the questions.

Task 2. Pupils click on the buttons (on the screen): Show lengths and Show Angles

Pupils drag to generate a shape which has the vertical bar BD as a line of symmetry.
They need to check the lengths and angles of the shape to check how symmetrical their shape is.

Mini plenary – The first pair of pupils come to the IWB (or computer linked to projector) to demonstrate their symmetrical shape. Discuss what lengths and angles need to be equal and the position of the bars to generate the shape.

The next pair to come to the front, are asked what shape they made. They are asked how they need to drag the bars in order to generate their shape. They then drag the bars. Discuss the properties of their shape and the position of the bars.

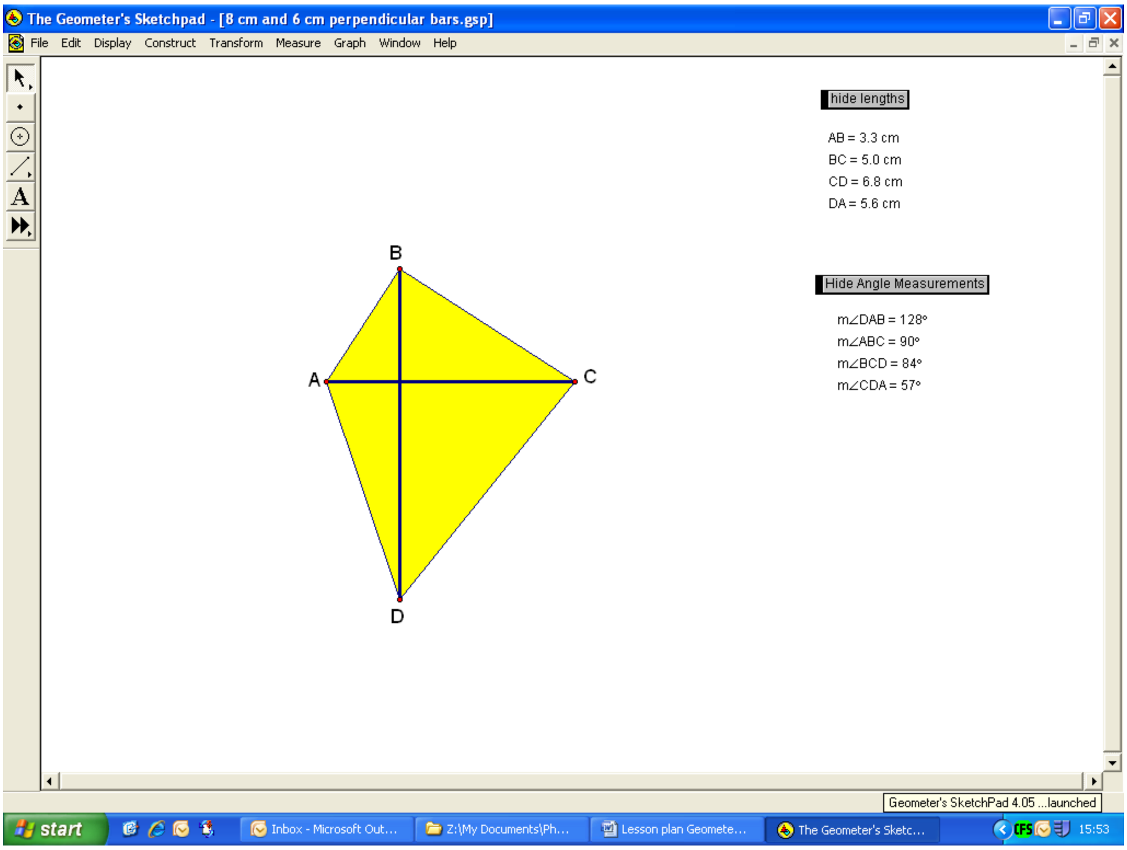
Repeat with further pairs, hoping to get the rhombus, kite, concave kite and isosceles triangle.

Task 3. Pupils asked to make the horizontal bar AC the line of symmetry and to see what they can make.

Task 4. Pupils generate a shape of their choice from the file. They use the text tool to write the name of the shape, a description of how the bars are positioned and the properties of the shape. They could save this or print it off.</td></tr>
<tr><td colspan="2">Plenary
Question: If you had to describe how to position the bars to make a kite, what would you say?

If you had to change that kite into a rhombus what would you do?</td></tr>
</table>

Appendix 2.1b The computer file given to the classes



Appendix 2.1c Whole class lessons

I had another objective for iteration two which was to ascertain whether the task could be used with whole classes. I took the opportunity to work with two classes from the School A and two classes from School B. This was the completion of my work at the School A and the beginning of my work at School B. I wanted to thank the mathematics department at School A for allowing me access to their students and the contact teacher and I agreed we would work together with two top set year seven classes. Although I had previously been working with year eight students the school were keen to offer new experiences (i.e. working with computers in mathematics lessons) to their ‘more able’ students. An account of these lessons with one of the year 7 classes has been published (Forsythe and Cook, 2012).

At School A the Head of Mathematics and I took two year eight ‘middle ability’ classes. In both schools the whole class lessons used the 8 cm vertical and 6 cm horizontal bars. When I worked with pairs of students from each of the classes they worked with the bars in different orientations.

The files used with the classes were modified versions of the files used with the pairs that contained the 8 cm vertical bar and 6 cm horizontal bar. The reason for this was that I would not be able to easily show 30 children how to construct the shape. So the shape was constructed in the file and also the side lengths and angles were measured and hidden with a hide/show button. A copy of the file is shown in appendix 7.1b.

The first lesson

At the beginning of the lesson I showed the class a number of different paper triangles and quadrilaterals to test whether the pupils knew their names and properties. The pupils were knowledgeable about both names and properties and only needed to be reminded of the name for the rhombus (which they remembered as a diamond).

The task had been designed so that the students, who had not used the software before, did not need to learn many of its features. First they were asked to drag the bars and find out what changes and what stays the same. Next they were asked to investigate what shapes they could make. When the students made a particular shape, they were asked to give their properties and then to check these using measurements of sides and angles.

The pupils easily dragged the bars to form different shapes. At the first plenary, when asked what stayed the same, the pupils observed that the lines inside the shape (the bars) stayed the same. When asked what changed the pupils observed that the shape changed.

During the lesson the pupils were asked to make specific shapes and to note the position of the bars within the shape. At first they found it hard to articulate this. After hearing some examples given from pupils who were prepared to offer suggestions for the positions, all the pupils were able to describe the bars. In the final plenary I asked the class to tell me what bars they could need to make a square. The suggestion came back that they would need two equal length bars. At the end of the lesson the pupils were given a worksheet to complete where they named shapes they had made, drew the positions of the bars and described the positions in words.

The second lesson

For the next lesson, the pupils worked in pairs at the computers. When devising the follow on task for the next lesson, I decided to give the pupils two choices. One choice required them to construct a square from scratch **which would remain a square when dragged** and they had the use of a construction booklet to give them instructions on how to make mid pints, perpendicular lines, rotations, etc. Many pupils successfully constructed a square, usually by drawing a line, constructing a mid point and then rotating the line 90 degrees about the mid-point. A different method that was observed was to draw a line, construct its mid-point, then draw a circle with the mid-point as centre and the circumference passing through the ends of the line. Then the pupil constructed a perpendicular to the line through the mid-point. The particular boy who devised this method had to be given some help in using the software to be successful but the idea was his. On reflection I wish that I had spent more time questioning the children as to how they thought of the method which they used to construct the square. Inevitably some pupils made squares which did not remain squares when dragged.

The other choice of task was more structured. I showed the class the four files containing, 2 unequal perpendicular bars, 2 equal perpendicular bars, 2 equal bars at an adjustable angle and 2 unequal bars at an adjustable angle. The pupils who chose this task had to try to make a rectangle, a parallelogram and a right angled isosceles triangle using the appropriate file. Then they completed the worksheet drawing the bars within the shape and describing the bars in words. To help them in their descriptions I had 3 questions on the worksheet as hints: are the bars equal lengths or unequal lengths, are the bars perpendicular or at another angle, where do the bars cross each other. When some of the pupils appeared to be stuck on this question, I found that by asking them to trace and then describe the positions of the bars inside the diagram of the shape on the worksheet, helped them to identify which file they needed to use.

Overall the pupils were able to access the task and to observe the type and positions of bars which generated certain shapes. Since, in a quadrilateral, these bars are the diagonals this activity might help pupils to observe the properties of the diagonals and to reflect on the connection between the diagonal properties and the side and angle properties.

Appendix 2.2a Table of dragging strategies used by Dave and Evan

	Time interval	Length of time interval	Dragging strategy	Description of student activity
Original placement of bars	0.35-0.39	4 seconds	GD	Bars placed fairly symmetrically
Attempt to make a square	2.32-2.34	2 seconds	RD	Point C is dragged down and up
	4.32-4.40	8 seconds	DMS	Bar BD dragged down to make an isosceles triangle which is adjusted to get angles A and C closer
The boys tried to make an equilateral triangle but instead made an isosceles triangle.	5.01-5.03	2 seconds	RD	
	7.07-7.14	7 seconds	GD	An attempt to make an equilateral triangle. Instead an isosceles triangle is made.
	7.18-7.20	2 seconds	GD	
	7.55-8.01	6 seconds	RD	
	11.20-11.21	1 second	DMS	The boys constructed the mid-point of the horizontal bar and moved bar BD on to it.
	11.42-11.44	2 seconds	GD	
	11.46-11.50	4 seconds	DMS	
Vertical and horizontal kites. Discussion of how many kites can be made by dragging the bars.	13.56-14.05	9 seconds	DMS	Bar BD dragged to generate a kite with vertical symmetry
	15.40-15.44	4 seconds	DMS	Another vertical kite is generated
	16.18-16.23	5 seconds	GD	A kite with horizontal symmetry is generated and adjusted for measures.
	16.24-16.36	12 seconds	RD	
	16.55-17.02	7 seconds	RD	
	18.49-18.54	5 seconds	DMS	Two more horizontal kites are made.
	19.00-19.04	4 seconds	RD	
	19.38-19.42	4 seconds	DMS	
	19.57-19.59	2 seconds	DMS	
rhombus	20.29-20.33	4 seconds	RD	the mid-points of both bars are put together to make a rhombus
new file with bars oriented at an angle	21.22-21.24	2 seconds	GD	one bar dragged over the other, result is reasonably symmetrical
kite	27.42-27.48	6 seconds	RD	no measurements shown
	29.44-29.46	2 seconds	RD	adjustment to get sides equal
isosceles triangle	32.43-32.45	2 seconds	DMS	isosceles triangle is generated and refined
	32.45-32.49	4 seconds	RD	
	32.55-33.00	5 seconds	RD	
rhombus	34.15-34.17	2 seconds	DMS	rhombus is generated and refined
	34.18-34.22	4 seconds	RD	
	34.58-35.04	6 seconds	RD	
arrowhead	35.54-35.58	4 seconds	GD	the bars were dragged up and

	36.01-36.06	5 seconds	GD	down, though not with accurate symmetry although the shapes generated during dragging were symmetrical
	36.07-36.11	4 seconds	GD	
	36.11-36.15	4 seconds	RD	
	36.15-36.22	7 seconds	GD then RD	seamless GD and RD
	36.40-36.59	19 seconds	WD	point B is dragged with big movements
	37.00-37.09	9 seconds	RD	an accurate arrowhead is achieved
	40.21-40.26	5 seconds	DMS	one arrowhead is changed into a different arrowhead
	40.26-40.35	9 seconds	RD	the new arrowhead is made accurate
How many isosceles triangles can you make?	40.43-40.46	3 seconds	DMS	an isosceles triangle is generated
	40.46-40.49	3 seconds	RD	
	41.22-41.43	21 seconds	WD then RD	seamless WD into RD
	41.45-41.47	2 seconds	DMS	another isosceles triangle is generated
	41.51-41.52	1 second	RD	

Appendix 2.2b Table of episodes from the recording with Dave and Evan

line numbers from narrative	time interval in recording	description
54-86	7.31-11.35	Isosceles triangle with vertical symmetry, boys identify side and angle properties and position of bars *shape: side and angle properties*
94-96	13.34-13.53	From isosceles triangle to kite “you’ve got to keep the dot (i.e.mid-point) central as you’re going up” *focus: attending to bars*
106-120 and 144-168	15.22-16.14 18.48-20.19	When asked how many kites it is possible to make the boys identify a discrete number of kites “about eight” *discrete number of kites*
169-200	20.19-22.03	Comparison of rhombus and square. Boys discover the rhombus has 4 equal sides and 2 pairs of equal angles *shape: squashed square, side and angle properties*
212-223	22.59-25.15	Symmetry as a folding action and cutting a shape in half. Dave deduces which side-length of the rhombus another would sit on another under this operation *symmetry: folding, equal sides*
249-266	30.11-31.34	File with bars at an angle boys identify the properties of the bars needed to make a kite including the perpendicular property. The boys were unable to deduce why having one bar in the middle of the other would result in the shape having equal side-lengths *shape: properties of bars*
285-286	33.28-33.48	When moving from kite to isosceles triangle Evan describes how one bar is dragged ‘down’ to make sure it was in the middle and that this leads to equal side lengths. Yet the dragging had to be at an angle. *Orientation perception: ‘down’ when at an angle*
289-294	34.08-34.41	Boys accept close enough measures in identifying two sets of equal angles “B and D are two degrees out and A and C are just one”. *measures: close enough*
301-311	35.10-35.43	Boys identify that both bars cross at their mid-points to make a rhombus and that only one rhombus can be made. *shape: properties of bars*
338-346	39.42-40.36	I ask how many arrowheads can be made. At first Evan suggests there are two. When I suggest that they try to make others Dave says they could just

		keep pulling it (i.e. the bar) down. Evan adjusts his idea of the number of possible arrowheads to "quite a few". *orientation perception: 'down' when at an angle* *many arrowheads*

Appendix 2.3a Table of dragging strategies used by Tara and Ruth

	Time interval	Length of time interval	Dragging strategy	Description of student activity
dragging of separate bars one over the other to the rhombus position	0.53-0.56	3 seconds	GD	Original dragging of one bar over the other to the rhombus position then adjusted angle measurements (displayed) until perfect
	0.56-1.26	30 seconds	RD	
The kite. Tara and Ruth position the bars where they think they ought to go and then refine the shape using RD by attending to the measures. They report that they can make four kites.	13.18-13.25	7 seconds	DMS	The girls make a kite by dragging one bar through the middle of the other bar. Then they adjust the bars to make the kite close to perfect.
	13.25-14.13	48 seconds	RD	
	15.08-15.18	10 seconds	RD	The girls try to get the kite to be perfect.
	17.36-17.48	12 seconds	GD	The girls demonstrate the four different positions of the ‘three quarter’ kite
	19.34-19.41	7 seconds	GD	The four kites in the ¾ position.
	19.47-20.05	18 seconds	DMS	She generates kites with the cross bar in different positions
	20.16-20.26	10 seconds	RD	She makes adjustments to get the kite perfect.
isosceles triangle. Tara and Ruth say they put the end of one bar on the middle of the other bar. They report that they can make four isosceles triangles.	21.38-21.44	6 seconds	GD	Girls exploring what happens if bar BD is dragged randomly
	21.52-21.57	5 seconds	GD	Girls generate an isosceles triangle
	22.16-22.19	3 seconds	GD	
	22.22-22.29	7 seconds	RD	Isosceles triangle made close to perfect.
	22.47-22.51	4 seconds	RD	
	23.24-23.26	2 seconds	DMS	The bar BD is dragged to demonstrate the four possible isosceles triangles.
	23.26-23.30	4 seconds	GD	
	23.30-23.34	4 seconds	DMS	
how many kites?	23.42-23.46	4 seconds	DMS	bar BD is moved along in small increments to show kites in different proportions
	23.46-23.52	6 seconds	RD	
	24.10-24.16	6 seconds	DMS	
new file with angled bars	24.43-24.48	5 seconds	GD	Girls again place the bars so that they are in the rhombus position
	24.48-24.59	11 seconds	RD	
First a rhombus and then kites.	28.21-28.25	4 seconds	DMS	A kite is generated and is adjusted so it is close to perfect.
	28.25-28.50	25 seconds	RD	

Squashed square analogy, keeping the bar in the middle, 4 discrete kites can be made.	30.30-31.05	35 seconds	GD	The girls demonstrate the other three ¾ kites using these bars
	31.17-31.28	11 seconds	WD	The girls try to make arrowhead kites. They seem unsure of how to make an arrowhead but when they have made one they refine it.
	31.28-31.50	22 seconds	RD	
	31.50-31.55	5 seconds	DMS	BD is dragged to make sides equal
arrowhead kites an interest in infinite lines, an isosceles triangle with another isosceles triangle bit missing, reflex angle.	31.59-32.03	4 seconds	DMS	Use of DMS followed by RD is used to make a good arrowhead.
	32.03-32.26	23 seconds	RD	
	34.33-34.36	3 seconds	WD	BD is dragged seemingly randomly then brought back onto the perpendicular bisector of |AC
	34.36-44.44	8 seconds	DMS	
	40.41-40.46	5 seconds	DMS	Bar BD is dragged along the perpendicular bisector of bar AC to make an arrowhead.
	42.20-42.26	6 seconds	GD	A kite is generated.
	43.24-43.30	6 seconds	GD	having constructed the mid-point of BD the bar AC is dragged over to it.
	43.35-43.39	8 seconds	GD	
	44.07-44.16	9 seconds	DMS	careful dragging through kites to the arrowhead.
	44.52-45.00	8 seconds	DMS	
	45.00-45.06	6 seconds	RD	
	45.15-45.21	6 seconds	GD	the two mid-points are moved together and a rhombus results
	45.21-45.27	6 seconds	RD	

Appendix 2.3b Table of episodes from the recording with Tara and Ruth

line numbers from narrative	**time interval in recording**	**description**
2-10	0.55-1.35	Tara places one bar over the other and tries to get the bars to cross at the mid-points. Tara says she was trying to get the two pairs of angle measurements to be equal *shape: properties of bars* *focus: attending to measures*
24-30	3.05-3.38	Tara and Ruth prefer the rhombus in this orientation! *orientation: looking at the shape this way*
39-41	5.14-5.32	The girls list the properties of a rhombus *shape: side and angle properties*
55-63	7.54-8.34	The girls refer to the rhombus as a squashed square. They compare the angle properties of the rhombus and square. They say a square has straight sides. *shape: angle properties* *shape: squashed square* *orientation descriptive: straight sides*
74-101	9.51-11.32	I question the girls about the properties of the rhombus which arise because of its symmetry by asking them to imagine folding in half. The girls identify equal sides and angles. *symmetry: equal sides and angles* *symmetry process*
106	12.18-12.30	In a rhombus the folding lines are the bars *symmetry: naming axes*
109	12.55-13.05	You can fold the rhombus into halves and quarters *symmetry process* *shape: split into triangles*
130-132	15.39-15.58	Tara gives a description of purposeful dragging of bars to make a kite followed by refinement to get measures accurate. *focus: attending to bars* *focus: attending to measures*
136-141	16.20-16.46	Ruth describes making the shape look equal and it having one line of symmetry via folding. *shape: holistic* *symmetry process*
151-158	17.26-17.51	Tara identifies that four kites would be made from the figure. Ruth questions whether a short squat kite is really a kite and Tara replies that it is a different shaped kite. *discrete number of kites* *shape: typical proportion*
159-168	17.58-19.08	The girls and I have a discussion of how we know a shape is a kite. Tara says that normal kites are quite thin. Ruth says that you have to have a line you can fold it over and equal sides.

		shape: typical proportion *shape: side and angle properties* *symmetry process*
170-174	19.31-19.41	The girls identify four discrete kites but I suggest they slide the cross bar over a bit more and a bit more. *discrete number of kites*
176-185	19.52-21.32	I ask the girls how they are being careful to move the bar BD. They reply that they are being careful to keep the measurements of sides BC and DC and sides AB and AD the same. Ruth comments that they are always in a cross shape. *focus: attending to measures* *focus: attending to bars*
189-193	22.16-22.35	The girls describe how the bars need to be dragged to make an isosceles triangle. *focus: attending to bars*
202-207	23.13-23.40	The girls identify four isosceles triangles which they demonstrate on the screen and one rhombus.
208-212	23.40-24.19	The girls identify four kites which can be made from the figure. Tara says there could be more than four kites if the line (ie cross bar) is moved over a little more. *discrete number of kites*
new file with bars at an angle 213-215	24.38-25.03	Ruth positions the bars into a symmetrical arrangement and refines to get the displayed angles being equal *focus: attending to bars* *focus: attending to measures*
224	26.20-26.22	Ruth says that the figure looks like a parallelogram. *orientation: shape looks like....*
225-226	26.25-26.35	Ruth wants to measure the sides before committing to the shape being a rhombus *measures*
244	28.20-28.25	Ruth says of the figure “it’s a bit of an angled kite” *orientation: descriptive*
248-255	29.09-30.05	I asked the girls how they had moved the bars to make a kite. They talked of bringing the bar up and moving it up, even though the dragging was in a direction at an angle to the vertical. *orientation perception: up when at an angle*
279-280	32.34-32.53	I ask the girls how the bars are positioned in an arrowhead and Ruth replies “not on AC but still in the centre”. Although the bars do not cross each other when an arrowhead is generated the girls still knew that they had to position one bar so that it would line up with the mid-point of the other one. *shape: properties of bars*
289-292	34.10-34.43	I ask how the bar would be moved to generate another arrowhead. Ruth replies that it would be moved upwards. Tara says that they would make sure the

		guideline went through the point (they had drawn an infinite line over bar BD and constructed the mid-point of AC). *orientation perception: up when at an angle* *shape: properties of bars*
388-389	43.46-43.56	The girls identify four or eight potential kites *discrete number of kites*
398-399	44.53-45.08	The girls identify thousands of potential arrowhead kites. *many arrowheads*

Appendix 2.4a Table of dragging strategies used by Kate and Jane

Kate and Jane worked through the task quickly, seeming to have remembered all they had learned from the class lesson. In consequence we spent some time working with the files containing the bars at adjustable angles.

	Time interval	Length of time interval	Dragging strategy	Description of student activity
Kites, kites in different orientations, millions of kites.	0.10-0.12	2 seconds	GD	The bars are moved to a symmetrical kite position
	2.32-2.34	2 seconds	RD	RD to get equal angles
	3.12-3.14	2 seconds	DMS	A kite in the opposite orientation is generated.
	3.16-3.20	4 seconds	RD	
Rhombus	3.57-3.59	2 seconds	GD	A rhombus is generated.
	4.08-4.10	2 seconds	WD	
	4.13-4.17	4 seconds	GD	
	4.20-4.23	3 seconds	RD	
Isosceles triangles, in different orientations, millions	4.50-4.52	2 seconds	DMS	Bar BD is dragged up to make an isosceles triangle and then refined in one seamless movement.
	4.52-4.55	3 seconds	RD	
	5.40-5.44	4 seconds	DMS	Bar BD is dragged to generate an isosceles triangle in the opposite orientation followed by an arrowhead.
	5.48-5.52	4 seconds	DMS	
	6.04-6.07	3 seconds	RD	
A description of DMS by pulling bars through the middle of the other and making different shapes.	7.39-7.45	6 seconds	DMS	Bar BD is dragged down to generate a rhombus
	7.58-8.04	6 seconds	DMS	BD is the line of symmetry
	8.04-8.07	3 seconds	DMS	AC is the line of symmetry
	8.23-8.26	3 seconds	DMS	kite to isosceles triangle to arrowhead.
	8.27-8.31	4 seconds DMS		
	8.49-8.54	5 seconds	RD	
	8.54-8.57	3 seconds	DMS	
Girls give reasons why they cannot make a square	9.19-9.20	1 second	WD	WD to see if they can make a square. They conclude with a rhombus which they adjust to make accurate.
	9.23-9.35	2 seconds	WD	
	9.29-9.32	3 seconds	GD	
	9.32-9.35	3 seconds	RD	
new file with bars at an angle	11.19-11.21	2 seconds	GD	One bar is dragged over the other and refined in a rhombus position
The girls discover they cannot make a square with the perpendicular bars at an	11.28-11.30	2 seconds	RD	
	13.13-13.22	9 seconds	RD	The girls try to make a square. There are four series of RD but overall the RD went on from 11.28-13.53. i.e. the girls used RD to try and make a square for 2
	13.25-13.30	5 seconds	RD	
	13.38-	8 seconds	RD	

angle. They successfully drag the figure into kite, rhombus, isosceles triangle and arrowhead.	13.46			minutes and 25 seconds.
	13.50-13.53	3 seconds	RD	
	15.54-15.59	5 seconds	DMS	An arrowhead is generated and refined.
	16.03-16.17	14 seconds	RD	
	16.24-16.27	3 seconds	DMS	A perfect isosceles triangle is generated straightaway.
	16.57-17.05	8 seconds	DMS	Bar BD is dragged to generate kites and a rhombus and an isosceles triangle.
	17.09-17.12	3 seconds	DMS	
	17.24-17.28	4 seconds	GD	
	17.30-17.38	8 seconds	DMS	
	17.39-17.42	3 seconds	GD	
new file with unequal bars at adjustable angles	18.29-18.39	10 seconds	WD	Jane takes over the mouse from Kate. She drags the bar BD around trying to make a parallelogram.
Parallelogram	18.46-18.48	2 seconds	GD	Jane continues to try and make a parallelogram.
	18.51-18.55	4 seconds	RD	
	19.40-19.45	5 seconds	WD	
	19.47-19.55	8 seconds	RD	
	19.59-20.00	1 second	RD	
	20.44-20.53	9 seconds	RD	Jane tries to get the shape to be as regular as possible.
	20.55-21.03	8 seconds	RD	
The girls adjust the bars so that they are at an angle of 90 degrees and generate a kite. Then they adjust the angle and generate a shape which	28.17-28.20	3 seconds	GD	the angle between the bars has been changed. The girls make a right angled triangle.
	28.40-28.48	8 seconds	WD	The bars are adjusted to be perpendicular then the angle is changed, the figure looks like a kite seen from the side. Then the bars are changed back to perpendicular and the figure changes to a kite.
	28.52-28.56	4 seconds	GD	
	29.05-29.08	3 seconds	GD	
	29.46-29.49	3 seconds	WD	The girls experiment with changing the angle between the

looks like a kite if viewing the computer from the side.	29.54-29.58	4 seconds	GD	bars when the figure looks like a kite seen from a different perspective.
	29.58-30.07	9 seconds	RD	
parallelogram	30.51-30.58	7 seconds	GD	The bars are dragged so that they both cross at their mid-points and this is refined to make an accurate parallelogram.
	31.04-31.15	11 seconds	RD	
	31.50-31.58	8 seconds	GD	Jane changes the angle between the bars again.
new file with equal length bars at an adjustable angle	32.35-32.37	2 seconds	GD	The bars are dragged so that they cross each other, the shape is constructed around the bars. The mid-points of the bars are constructed.
square	33.26-33.30	4 seconds	GD	Jane places the mid-points of the bars together
	36.28-36.36	8 seconds	RD	She tries to get the measurements exact but cannot do so though they are within 0.1 cm and 1 degree
	36.54-36.55	1 second	RD	
rectangle	37.44-37.48	4 seconds	GD	Having adjusted the angle between the bars another rectangle is generated which is accurate.

Appendix 2.4b Table of episodes from the recording with Kate and Jane

line numbers from narrative	**time interval in recording**	**description**
1	0.10-0.12	One bar is dragged over the other so the result is a nearly symmetrical kite position
4-7	1.11-1.29	Kate identifies the shape as a kite because it has two shorter sides at the top and two longer sides at the bottom. *Shape: side properties*
9-11	2.13-2.21	Kate identifies two equal angles in the kite *shape: angle properties*
14-17	2.47-2.58	Kate identifies the line of symmetry in the kite *symmetry: naming axis*
18-31	2.58-3.47	Kate says that to make another kite with BD as the line of symmetry you need to pull BD up. She demonstrates this and labels the result an upside down kite. She identifies that it is possible to make millions of kites. *focus: attending to bars* *orientation descriptive: upside down* *infinite number of kites*
39-44	4.24-4.35	Kate identifies the two lines of symmetry in the rhombus *symmetry: naming axes*
47-51	4.49-5.12	The girls identify the properties of the isosceles triangle. *shape: side and angle properties*
55	5.33-5.35	Kate refers to an ‘upside down triangle’ *orientation descriptive: upside down*
58-60	5.46-6.05	The girls identify properties of an arrowhead *shape: side and angle properties*
62-64	6.19-6.35	After they had made the kite, rhombus, isosceles triangle and arrowhead I asked the girls how they moved the bars to get all those shapes. Kate replied that you just have to move one of the lines (bars) either up or down. *focus: attending to bars, describing DMS*
67-79	6.50-7.31	When I asked them where the bars crossed each other for all the shapes they had made the girls replied ‘in the middle’. For a rhombus the girls said that the bars crossed ‘in the middle of both of them’. shape: properties of bars
85	7.52-7.54	Kate says the bars are both lines of symmetry in a rhombus *symmetry: naming axes*
93	8.10-8.12	Kate refers to the figure as a ‘kite on its side’ *orientation descriptive: on its side*
94-95	8.13-8.17	Kate said they could make millions of kites on their

		side. *infinite number of kites*
97	8.22-8.24	Kate refers to a sideways triangle. *orientation descriptive: on its side*
99-103	8.34-8.47	The girls describe the properties of the bars for an isosceles triangle and an arrowhead *shape: properties of bars*
107	9.16-9.18	I suggest that the girls make a shape without symmetry and Kate refers to the shape as 'just random' *random shape? Lack of special properties?*
113-117	9.31-9.50	The girls say that they cannot make a square with these bars because the bars are not the same length. *deduction: square needs equal length bars*
new file with perpendicular bars at an angle 128	11.19-11.21	The girls move one bar over the other to the rhombus position *symmetry: intuitive*
128-131	11.19-11.37	The girls identify that if the bars cross in the middle the shape should be a rhombus *shape: properties of bars*
133	11.43-11.45	I ask Jane how she is looking at the shape and she says you have to turn your head. *orientation perception: turning the head*
138-143	11.51-12.14	I ask which bar they are trying to make the right way up if they turn their head to the left (bar DB) and to the right (bar AC) *orientation perception: turning the head*
149	13.32-13.36	When I ask how the girls position the bars at first Jane says they try to make things look equal first. *shape: holistic perception*
150-153	13.39-14.00	The girls appear to find it more difficult to position the bars when the rhombus is at a different orientation.
166-169	16.29-16.48	Kate says she keeps the bar BD in the middle of the bar AC when moving from the arrowhead to the triangle *description of DMS*
new file with unequal length bars at an adjustable angle 185	20.03-20.06	The girls have made a parallelogram. Jane says it is a parallelogram on its side. *orientation descriptive: on its side*
193-199	21.16-21.22	In the parallelogram the girls cannot get the pairs of angles to be equal. Kate says they are not that far off. Jane says two degrees.

		I ask the girls where the bars should cross and they tell me they should cross in the middle. I suggest they construct the mid-points of the bars. The girls are then able to get the measurements exact. *measures: close enough* *shape: properties of the bars*
209-252	23.00-27.20	The girls identify that the parallelogram does not have line symmetry but does have rotational symmetry. They perceive rotation as turning the shape round on itself (through 180 degrees). However they find it difficult to visualise congruent triangles within the parallelogram which would turn onto each other. *symmetry process: rotational by turning*
258-263	28.55-29.39	The girls rotate the bars so that they are at 90 degrees and make a kite which is accurate. Jane says there are 2 shorter sides and 2 longer sides. I ask what the angle is between the bars and she says it looks like 90 degrees. *shape: side properties* *shape: properties of bars*
263-268	29.39-30.33	I ask the girls if they can make a kite with bars which are not at 90 degrees. They make a figure which they label as looking like a 'flat kite' i.e. if you look at it in a perspective. *shape: holistic perception albeit at a perception from the side*
269-273	30.35-31.41	I ask the girls if they could make a rhombus. Kate says they need to get the bars to cross in the middle. They do this and generate a parallelogram. *shape: properties of bars*
273-274	31.41-32.03	I ask if they can make a rectangle and Jane asks if we can make the bars the same length. *deduction: rectangle needs equal length bars*
new file with equal bars at an adjustable angle 277-300	33.37-35.41	The girls identify that if they have equal bars at 90 degrees which cross at their middles they will have a square *deduction: bars needed to generate a square*
317-324	38.13-38.50	The girls have adjusted the angle of the bars and generated a rectangle. They are able to identify congruent triangles within the rectangle. *symmetry process: rotational by turning*
329-335	39.16-39.54	The girls identify the properties of the bars necessary to generate a rectangle and also that the bars are the diagonals of the rectangle. *deduction: bars needed to generate a rectangle*

Appendix 2.5a Table of dragging strategies used by Aftab and Rupen

	Time interval	Length of time interval	Dragging strategy	Description of student activity
kite	0.07-0.10	3 seconds	GD	one bar is dragged over to the other in a kite position
The boys try to make a square but instead manage to make a rhombus	3.49-4.04	15 seconds	RD	The boys try to make the figure into a square and keep checking the angle measurements on the screen.
	4.11-4.18	7 seconds	RD	
	4,22-4.39	17 seconds	RD	
	4.40-4.46	6 seconds	RD	
	4.47-5.01	14 seconds	DMS	
	5.04-5.11	7 seconds	DMS	
	5.14-5.18	4 seconds	RD	
	5.23-5.25	2 seconds	RD	
	5.29-5.30	1 second	RD	
kite	9.49-9.56	7 seconds	DMS	Aftab drags bar AC to generate a kite which he then refines. After constructing the mid-point of AC he drags it onto BD.
	9.59-10.07	8 seconds	RD	
	10.15-10.17	2 seconds	GD	
	10.41-10.49	8 seconds	DMS	A kite in two different positions is generated by dragging bar AC up
	10.53-10.54	1 second	DMS	
isosceles triangle and arrowhead	11.02-11.04	2 seconds	DMS	an isosceles triangle is generated
	12.20-12.21	1 second	DMS	an arrowhead is generated
shapes with horizontal symmetry	13.00-13.12	12 seconds	DMS	the figure is dragged into a kite with vertical symmetry.
	13.37-14.03	26 seconds	DMS	then the figure is dragged into a rhombus
	15.42-16.08	26 seconds	DMS	various shapes with horizontal symmetry are generated. The boys appear to find it harder to drag horizontally.
	16.08-16.23	15 seconds	RD	
	16.30-17.10	40 seconds	DMS	
	17.13-17.17	4 seconds	RD	
	17.20-17.25	5 seconds	RD	
	17.41-17.50	9 seconds	RD	
	17.53-17.58	5 seconds		
	18.14-19.08	54 seconds	DMS	
	19.11-19.22	11 seconds	DMS	
new file with perpendicular bars at an angle	19.55-19.59	4 seconds	GD	one bar is dragged over the other to the rhombus position
rhombus	20.18-20.21	3 seconds	RD	the mid points are constructed and then moved together to generate the rhombus
	21.24-21.28	4 seconds	RD	
kites, then isosceles triangles	24.28-24.30	2 seconds	DMS	the bar is dragged with reasonable accuracy to generate a kite

	25.18-25.22	4 seconds	GD	a different kite is generated
	25.47-25.49	2 seconds	GD	a kite with BD as line of symmetry is generated
	26.32-26.34	2 seconds	DMS	BD is dragged further along to generate another kite
	27.35-27.41	6 seconds	GD	an isosceles triangle is generated
	27.45-27.54	27 seconds	WD	different shapes are demonstrated.

Appendix 2.5b Table of episodes from the recording with Aftab and Rupen

line numbers from narrative	**time interval in recording**	**description**
10-12	1.33-1.47	Aftab identifies two equal sides in the kite *shape: side properties*
16	2.54-3.00	Aftab identifies two equal angles in the kite *shape: angle properties*
29-72	4.14-7.18	The boys try to drag the figure into a square. They construct the mid-points of the bars because they have identified that the bars need to cross at the mid-points. They decide in the end that they have got a rhombus *shape: properties of bars*
74	7.21-7.23	Aftab says it is a rhombus because it looks like a rhombus *shape: holistic*
76-77	7.34-7.42	The boys identify the side and angle properties of a rhombus *shape: side and angle properties*
93-100	9.00-9.23	The boys identify the lines of symmetry in a rhombus *symmetry: naming axes*
104	9.30-9.34	Aftab says the bars in a rhombus cross at the mid point *shape: properties of bars*
106-115	9.40-10.33	When I ask how the rhombus can be changed into a kite Aftab says bar AC has to be pulled up or down. Aftab said that he put the mid-point of one bar on the other and moved it down to make another kite *description of DMS*
116-123	10.33-10.56	Aftab reports that it is possible to make four kites. When I suggest that he could move the bar up a little bit more he changes this to "loads of kites". *discrete kites* *many kites*
159	13.20-13.22	when I ask Aftab if he looked at the bars or the measurements when he was dragging he said that he looked at the bars *focus: attending to bars*
181-184	14.50-15.02	The boys said that they made a kite, an isosceles triangle and an arrowhead when the middle of bar AC was kept on bar BD *shape: common properties*
188-189	15.18-15.28	I suggest they make shapes with AC as the line of symmetry and the boys say they could make the same shapes but on the side *orientation description: on the side*
197-204	15.38-16.20	The boys dragged horizontally and it appeared to be harder for them to keep symmetry than when they

		dragged vertically. *cognitive load?*
new file with perpendicular bars at an angle 235	19.55-19.59	one bar is dragged over the other bar to the rhombus position *shape: holistic*
236-247	19.59-20.33	The boys construct the mid-points of the bars and place them together. They decide they should be able to make a rectangle. *deduction though incorrect*
265-271	23.28-24.02	The boys turn their head to look at the figure. Aftab says of the shape “you can tell it’s like a kite this way”. *orientation perception: this way*
274-276	24.06-24.20	Rupen says he thinks the shape is a rhombus because the sides are the same and two angles are the same. *shape: side and angle properties*
279-280	24.24-24.31	I ask how the rhombus could be made into a kite and Rupen demonstrates by dragging bar AC down and left. *DMS to go between shapes*
283-284	24.48-25.00	When I ask the boys how they dragged the bars to change form the rhombus to the kite Rupen replied that he made sure AC stayed on the mid-point of BD *shapes: common properties of bars*
308	27.05-27.18	Aftab talked about pulling the bar down when he meant at an angle. *orientation perception: down when it is really at an angle*

In the remainder of the session Aftab and Rupen used the bars with adjustable angle to generate a parallelogram. With some help they recognised 2 congruent triangles within the parallelogram by imagining turning the parallelogram through 180 degrees.

Appendix 2.6 Accurate DMS episodes

I have included one DMS episode from the file containing the 6 cm vertical and 8 cm horizontal bars and two episodes from the file containing perpendicular bars oriented at an angle.

	orientation of bars	Time interval	Differences between expected equal measurements
Dave and Evan	horizontal and vertical	4.32-4.40	angles < 4 degrees
	oriented at an angle	40.21-40.26 40.43-40.46	angles < 1 degree, sides < 0.6 cm angles < 1 degree, sides < 0.3 cm
Tara and Ruth	horizontal and vertical	13.18-13.25	angles < 4 degrees, sides < 0.2 cm
	oriented at an angle	31.59-32.03 40.42-40.46	angles < 5 degrees, sides < 0.6 cm angles < 5 degrees, sides < 0.6 cm
Kate and Jane	horizontal and vertical	5.40-5.44	angles < 2 degrees, sides < 0.4 cm
	oriented at an angle	15.54-15.59 16.24-16.27	angles < 5 degrees, sides < 0.6 cm angles < 0 degrees, sides < 0.1 cm
Aftab and Rupen	horizontal and vertical	13.00-13.12	angles > 2 degrees, sides < 0.1 cm
	oriented at an angle	24.28-24.30 26.32-26.34	angles < 4 degrees, sides < 0.1 cm angles < 4 degrees, sides < 0.3 cm

Appendices 3 Iteration 3

Appendix 3.1a Table of dragging strategies used by Hemma and Seema

	Time interval	**Length of time interval**	**Dragging strategy**	**Description of student activity**
Initial exploration of shapes.	0.45-0.47	2 seconds	GD	Bar BD is dragged over bar AC reasonably symmetrically
	3.06-3.08	2 seconds	GD	A triangle is generated
	3.19-3.25	6 seconds	GD	The figure moves through kites
	3.36-3.39	3 seconds	WD	The figure is dragged into concave shapes
	4.03-4.11	8 seconds	WD	The girls explore the figure
	4.14-4.15	1 second	GD	an isosceles triangle is made
	4.19-4.30	11 seconds	WD	Crossed quadrilaterals are formed and the shape ends as an isosceles triangle.
	4.38-4.43	5 seconds	WD	Bars are dragged around and a right angled triangle is formed at the end.
	4.53-4.59	6 seconds	GD	Arrowheads and a kite are formed.
Kites	6.46-6.50	4 seconds	GD	The kite is turned into an isosceles triangle but accurate symmetry is not apparent.
	7.36-7.38	2 seconds	RD	This dragging was an adjustment to make the triangle symmetrical.
	7.54-7.58	4 seconds	GD	Isosceles triangle on base AC is moved to isosceles triangle on base BD
	8.03-8.11	8 seconds	GD	From isosceles triangle to kite.
	8.20-8.23	3 seconds	GD	From one kite to another by moving both bars in turn.
	8.25-8.30	5 seconds	GD	
	8.31-8.35	4 seconds	GD	
	8.47-8.50	3 seconds	RD	Making measures more accurate
	8.55-8.59	4 seconds	WD	BD is dragged to generate different kites.
	10.12-10.15	3 seconds	RD	The girls try to get one of the exterior angles to look like a right angle.
rhombus	11.07-11.47	40 seconds	RD	The girls try to get an accurate rhombus

	12.35-12.41	6 seconds	RD	This makes a rhombus with four equal sides.
	12.48-12.57	9 seconds	RD	
	13.25-13.35	10 seconds	RD	The girls try to get two pairs of equal angles.
	14.19-14.20	1 second	RD	
	14.24-14.29	5 seconds	RD	
	15.22-15.23	1 second	RD	Having constructed the mid-points of the bars the girls get the angles close to equal.
	15.33-15.37	4 seconds	RD	
kites	19.06-19.10	4 seconds	GD	A kite is generated and refined.
	19.12-19.15	3 seconds	RD	
	20.12-20.14	2 seconds	RD	Two sides are made to be exactly equal.
	20.59-21.01	2 seconds	DMS	2 different kites are generated with a small movement of bar BD.
	21.03-21.08	5 seconds	DMS	
	22.42-22.44	2 seconds	DMS	The girls make another small movement of BD to make another kite.
	22.48-22.53	5 seconds	DMS and RD	
	26.38-26.55	17 seconds	GD	A kite is moved to another with AC as the line of symmetry.
	27.00-27.14	14 seconds	RD	A little adjustment generates a rhombus.
	27.17-27.28	11 seconds	DMS	Another kite is generated.
	27.45-27.47	2 seconds	DMS	Little movements of the bar demonstrate different kites which can be generated.
	27.48-27.49	1 second	DMS	
	27.55-27.57	2 seconds	DMS	
	27.58-28.00	2 seconds	DMS	
	28.10-28.11	1 second	DMS	two more different kites.
	28.14-28.15	1 second	DMS	
Isosceles triangle	28.37-28.39	2 seconds	GD	The bar is moved purposefully but not keeping symmetry to make an isosceles triangle.
	28.43-28.47	4 seconds	GD	
Arrowheads	28.57-29.35	38 seconds	DMS	Slow, careful dragging to generate many different arrowheads.
	29.39-29.44	5 seconds	DMS	Dragging back and forth to demonstrate more arrowheads.
	29.47-29.49	2 seconds	DMS	
	29.52-30.02	10 seconds	DMS	The arrowhead with the right angle (exterior to the reflex angle) is demonstrated.
	31.47-31.56	9 seconds	WD	BD is dragged around and the end result is an isosceles triangle.
	32.00-32.05	5 seconds	DMS	To make an arrowhead
Rhombus	32.45-33.00	15 seconds	GD	To make a rhombus
	34.45-34.47	2 seconds	RD	Hemma tries to get side AD

				on the parallel line
	34.53-34.55	2 seconds	GD	She moves AD onto the parallel line
Equal length bars and the square	36.34-36.38	4 seconds	GD	BD dragged over AC into a symmetrical arrangement.
	38.16-38.22	6 seconds	RD	To make the square more accurate looking.
	40.07-40.17	10 seconds	RD	More accuracy on square
Kites	41.22-41.24	2 seconds	DMS	2 kites are generated and refined.
	41.25-41.27	2 seconds	RD	
	41.31-41.35	4 seconds	DMS	
	43.12-43.20	8 seconds	DMS	Many kites are demonstrated using DMS.
	43.30-43.38	8 seconds	RD	The kite is made accurate.
	44.12-44.22	12 seconds	DMS	Short bursts of dragging demonstrate different kite positions.
	44.24-44.32	8 seconds	DMS	
	44.36-44.39	3 seconds	RD	Refinement of the kite shape.
Arrowheads	45.06-45.17	11 seconds	GD	AC is dragged away from BD to generate arrowheads.
	45.18-45.35	17 seconds	DMS	Many different arrowheads are demonstrated.

Appendix 3.1b Table of episodes from the recording with Hemma and Seema

line numbers from narrative	time interval in recording	description
6-11	3.05-3.28	Seema perceives the figure with the bars as a 2D representation of a 3D figure. *shape: holistic perception*
27-29	4.31-4.38	Seema recognises and names the shape as an isosceles triangle and Hemma identifies that it has a line of symmetry. *shape; holistic perception* *symmetry; naming axis*
33-38	5.02-5.22	The girls discuss the properties of a kite as having 2 longer sides at the bottom. *shape; side properties* *shape; typical representation*
48	6.26-6.28	Seema says that two sides which are 0.2 cm apart are quite similar *measures; close enough*
81	8.37-8.39	Seema says that two kites can be made from the figure. *discrete number of kites*
86-90	8.48-9.10	The girls describe kites as having longer sides at the bottom and shorter sides at the top in comparison to a rhombus which has all sides the same. *shape; side properties* *shape; typical representation* *shape; partitional classification*
121-129	11.07-11.56	Seema tries to make a rhombus. The girls identify that all four sides need to be the same. Seema refers to the measurements as centimetres. *shape: side properties*
140	13.06-13.11	Hemma describes angles in a rhombus which are the same (74 degrees) and angles which are similar (105 degrees and 108 degrees). *measures; close enough*
157	14.23-14.25	Hemma says that the bars cross in the middle to make a rhombus. *shape; properties of bars*
164-167	15.14-15.43	Seema appears to view the shapes holistically. She describes a rhombus as a “kind of diagonal square”. *shape; holistic perception*
171-175	15.48-16.17	Seema identifies the need to know the half way point of the bars in order to make them cross in the middle. “You know how that one’s eight centimetres, where four centimetres is”. *shape; properties of bars* *measures; used to check properties*
180-186	16.52-17.24	When Seema positions the bars to generate a kite she reports that she looks at the shape while moving the

		bars and tries to get the shape "kite looking". *shape; holistic perception*
201-205	18.37-19.02	Seema thinks that two rhombuses could be made if the shape could be turned round. *orientation; perception*
213-220	19.35-19.56	A discussion of the properties of the kite. The girls have a partitional view of kites and rhombuses because rhombuses have four equal sides and kites have two longer sides on the bottom. This demonstrates that they have a stereotypical view of a kite. *shape; side properties* *shape; partitional classification* *shape; typical representation*
224-226	20.02-20.20	Seema used RD to adjust the kite so that the bottom two sides are equal. *shape; side properties*
232-236	21.04-21.27	When I ask where the bars should cross in a kite, Seema says they should cross in the middle of BD and Hemma comments that is the same as the other one (ie the other kite they made previously). *shape; properties of bars*
260-267	23.31-24.06	The kite they have made has its 'top' pointing left and downwards. However the girls talk about the kite as if they had mentally rotated it so that the 'top' pointed upwards. The girls have split the kite into two triangles; an isosceles triangle and an obtuse triangle. *orientation perception; looking this way* *shape; split into triangles*
270-281	24.10-25.05	Following on from this the girls do not think that the obtuse angled triangle can be labelled isosceles. They seem to think it must be one thing or the other but not both.
282-291	25.05-26.18	The girls forget that they labelled the shape a kite (in line 260) and suggest other things the shape could be eg 3D pyramid, trapezium. It may be that the orientation of the figure affects their recognition of the shape as a kite. *orientation perception*
300-304	27.14-27.36	After dragging bar BD along AC the girls identify the shape as a kite and that two sides at the bottom are the same and two sides at the top are the same. Hemma also says that the measures are close enough (sides differ by 0.1 cm and angles differ by 1 degree). *shape; side properties* *shape; typical representation* *measures; close enough*
306-318	27.42-28.23	I ask the girls how many kites they think they could make. Seema says two. Hemma then demonstrates

		that many more kites can be made by dragging the bar BD to different positions on AC. Hemma thinks they can make six or seven kites. I suggest they move the bar a little bit. “Like a millimetre” Hemma asks. Seema suggests it could “go on for ages” and that they could make twenty kites. Hemma suggests 25 or 26 kites. *discrete number of kites* *many kites*
329-330	29.19-29.28	Hemma says that they can make quite a lot of arrowheads and demonstrates by moving the bar along. *many arrowheads*
331-337	29.28-29.58	The girls notice that as the bar BD is moved further away from point C the exterior angle at C becomes more acute. *effect of moving bars*
339-340	30.04-30.11	The girls consider that the line of symmetry in an arrowhead is just the part which is inside the shape. It appears that the analogy of a fold line as a line of symmetry may cause difficulties when they need to view a line of symmetry as being an infinite line. *symmetry process*
351-357	30.50-31.30	The girls have learnt that parallel lines are like train tracks. *parallelism process*
394	35.05-35.11	Hemma says that the shape is not a square because the angles are not ninety degrees. *shape; angle properties*
395-406	35.11-36.17	The girls see that they need the bars to be equal length in a square but they also want them to be ‘straight’. shape; properties of bars *deduction; square needs equal length bars* *orientation perception*
407	36.28-36.57	Hemma opens the new file and drags one bar over the other to make a symmetrical arrangement. *symmetry; intuitve*
410-416	37.57-38.24	The girls identify that the bars need to cross in the middle, but they would still prefer ‘straight’ bars and the reassurance of using measurements to check. *shape; properties of bars* *orientation perception* *measures; check on properties*
448-455	41.17-41.48	Using equal length bars Hemma has made a kite. It’s proportions are not pleasing to the girls but they acknowledge that it is a kite because it has the properties of a kite (two equal short sides, two equal long sides and made up of two isosceles triangles). *shape; typical representation* *shape; side properties*

461-472	42.04-43.22	The girls discuss how the unequal length bars made more accurate kites which are easier to make! *shape; typical representation*
484-485	44.10-44.41	Hemma demonstrates that a number of kites can be made using fairly accurate DMS. *many kites*
489-491	45.12-45.24	Hemma demonstrates that a number of arrowheads can be made. *many arrowheads*

Appendix 3.2a Table of dragging strategies used by Stan and Eric

	Time interval	**Length of time interval**	**Dragging strategy**	**Description of student activity**
Exploring shapes which can be generated.	1.55-1.57	2 seconds	DMS	A kite, triangle, rhombus and arrowhead (not symmetrical) are made.
	1.58-2.00	2 seconds	DMS	
	2.05-2.17	12 seconds	WD	
	2.36-2.55	19 seconds	WD	
Isosceles triangle	2.57-3.01	4 seconds	GD	The boys try to generate an equilateral triangle but find that they cannot do so.
	3.26-3.57	31 seconds	RD	
	4.00-4.10	10 seconds	GD	
	4.14-4.20	6 seconds	RD	
	4.21-4.24	3 seconds	RD	
	5.42-5.45	3 seconds	GD	The boys make a scalene triangle then go back to the isosceles triangle.
	5.54-5.55	1 second	GD	
	6.55-7.02	7 seconds	WD	The boys investigate making the two base angles equal
	7.07-7.21	14 seconds	WD	
	7.30-7.40	10 seconds	RD	
	7.44-7.48	4 seconds	RD	
	11.01-11.10	9 seconds	RD	The boys adjust the angles whilst discussing that the measured angles sum to 179 or 181.
	11.16-11.18	2 seconds	RD	
Rhombus	12.49-12.57	8 seconds	DMS	The boys make a rhombus and then refine it.
	13.22-13.33	11 seconds	RD	
	13.34-13.41	7 seconds	RD	
	13.50-14.20	30 seconds	RD	
Isosceles triangle	17.36-17.38	2 seconds	DMS	They demonstrate the four possible isosceles triangles which can be made with the figure.
	17.38-17.43	5 seconds	GD	
	17.50-17.58	8 seconds	RD	
	18.38-18.41	3 seconds	DMS	
	18.42-18.56	14 seconds	RD	
	18.56-19.04	8 seconds	GD	
Rhombus	19.39-19.42	3 seconds	DMS	Eric focuses on the measurements of the sides when dragging the figure into the rhombus position.
	19.42-19.48	6 seconds	RD	
	20.34-20.51	17 seconds	DMS	
Arrowhead	21.16-21.22	6 seconds	DMS	The boys generate arrowheads with BD as a line of symmetry. They say they can make 4 arrowheads by moving AC by small amounts.
	21.24-21.30	6 seconds	DMS	
	21.38-21.43	5 seconds	RD	
	21.46-21.50	4 seconds	RD	
	22.11-22.29	18 seconds	DMS	
	22.30-22.34	4 seconds	DMS	
Shapes with BD as line of symmetry	22.50-22.59	9 seconds	DMS	Arrowheads to rhombus through kites.
	22.59-23.01	2 seconds	DMS	
	23.07-23.09	2 seconds	DMS	
	23.22-23.27	5 seconds	RD	

Shapes with AC as line of symmetry	23.48-23.54	6 seconds	DMS	Isosceles triangles and arrowheads. The boys hardly seem to notice the kites. They make what they call a fat kite.
	23.56-24.12	16 seconds	RD	
	24.15-24.27	12 seconds	DMS	
	24.29-24.43	14 seconds	WD	
	24.54-25.02	8 seconds	GD	
	25.02-25.10	8 seconds	RD	
	25.20-25.25	5 seconds	DMS	with BD as line of symmetry
	25.25-25.32	7 seconds	DMS	with AC as line of symmetry
	25.32-25.40	8 seconds	RD	
A number of arrowheads	25.40-25.45	5 seconds	GD	They drag slowly to demonstrate a number of arrowheads.
	25.47-25.54	7 seconds	WD	
	25.55-26.02	7 seconds	DMS	
	26.02-26.06	4 seconds	GD	
	26.06-26.08	2 seconds	DMS	
	26.08-26.12	4 seconds	WD	
	26.14-26.20	6 seconds	DMS	BD as line of symmetry
	26.20-26.21	1 second	GD	AC as line of symmetry
Rhombus	26.47-26.51	4 seconds	DMS	They make a rhombus and compare it with a square.
	26.51-27.00	9 seconds	RD	
	27.28-27.30	2 seconds	RD	
Square	33.40-34.35	55 seconds	RD	In a new file with equal length bars they place the bars so they cross in the middle. RD does not get it perfect so the mid-points are constructed and a perfect square is made.
	35.45-35.51	6 seconds	RD	
	36.00-36.03	3 seconds	RD	
	37.09-37.15	6 seconds	DMS	
	39.37-39.44	7 seconds	DMS	
Isos RA triangle	39.45-39.51	6 seconds	WD	Bars end to end
Dragging to make different shapes	40.16-40.25	9 seconds	GD	Different scalene triangles are made and the boys make some arrowheads at the end.
	40.25-40.38	13 seconds	WD	
	41.10-41.13	3 seconds	GD	
	41.16-41.18	2 seconds	GD	
	41.18-41.34	16 seconds	WD	
	41.54-42.00	6 seconds	GD	
	42.00-42.10	10 seconds	WD	

Appendix 3.2b Table of episodes from the recording with Stan and Eric

line numbers from narrative	**time interval in recording**	**description**
33-48	4.28-5.11	The boys perceive the angle between the bars AC and BD as other than a right angle. Stan says he can't tell because it is at a weird angle. *orientation; perception*
85-92	8.10-8.39	I ask the boys what they could call a triangle which has two equal angles. Stan half remembers the label 'isosceles'. Naming the shape an isosceles triangle appears to jog the boys' memory about the side properties. *shape: side and angle properties*
100-108	9.45-10.13	The boys find it difficult to see which of the bars is the longer one. This may be due to the orientation of the bars. Stan says that if the triangle was tilted up the line would be up there. He may be mentally rotating the shape. *orientation; perception*
113-129	10.37-11.28	The boys notice that the sum of the displayed angle measurements is 179 or, after dragging, 181. This bothers them. *measures*
131	11.44-12.00	I explain about how the computer is rounding the angle sizes to the nearest whole number. Stan then suggests possible real measurements for the angles. *measure*s
135-140	12.23-12.39	Stan and Eric discuss the symmetry of the isosceles triangle and decide there is only one line of symmetry. *symmetry; naming axes*
153	13.50-13.52	Eric decides that the tilted rhombus is a parallelogram. It is a special parallelogram but Eric's comment is more likely to be the result of the orientation of the figure which is close to the typical orientation of a parallelogram *orientation; typical*
156-159	14.28-14.41	The boys identify the properties of the bars for a rhombus. *shape; properties of bars*
163	14.52-14.56	Eric identifies the side properties for a rhombus *shape; side properties*
167-172	15.24-15.48	The sides of the rhombus are measured. Eric conjectures they will add up to 20 and they do.
177-182	15.58-16.38	The boys discuss the lines of symmetry in a rhombus. Eric discounts Stan's suggestion that the lines which go through the middles of opposite sides are lines of symmetry because the angles either side are not equal. *symmetry; naming axes*

		symmetry; equal angles
191-196	17.48-18.29	They make an almost perfect isosceles triangle. Stan says the angles add up to 181 but then Eric suggests this is due to a rounding error. *measures*
197-209	18.29-19.27	The boys demonstrate the four isosceles triangles which can be generated from the figure.
211-220	19.35-20.14	I suggest that the boys turn the isosceles triangle into a rhombus and ask them how they do this. Eric had the mouse and explained that he dragged BD down (although it was left and down). He said that he watched the measurements of the sides while dragging. *orientation; perception ('down' when at an angle)* *focus; attending to measurements*
224-230	20.47-21.06	While they have the rhombus there is a suggestion that this is a square as all four sides are equal. Stan points out that the angles are not all ninety degrees. Eric notes the two pairs of equal angles. *shape; angle properties*
233-238	21.13-21.34	The boys drag quickly from the rhombus to the arrowheads and do not seem to notice the kites they pass through on the way. *discrete number of shapes*
241-247	21.53-22.34	The boys only consider the four arrowheads in each of four positions. *discrete number of arrowheads*
269-272	24.12-24.32	Eric recounts that they made an isosceles triangle, rhombus and kite. These shapes appear to be discrete positions at the end of a dragging movement. *discrete number of shapes*
291-295	25.44-26.06	I suggest to the boys that they try to make different arrowheads by moving the bar slowly. Sam says you can make a lot of arrowheads by moving slowly. This suggests a perception of continuous change as opposed to all other references to discrete positions. *many arrowheads*
302	26.58-27.06	Eric lists the properties of a square. *shape; side and angle properties*
309-315	29.15-29.35	The boys have difficulty seeing the right angles at the intersection of the bars. This may be due to the orientation. *orientation perception*
336	31.50-32.00	Eric says they need equal length bars. I am not sure whether he means equal bars to make a square or equal bars will make the angles at their intersection look more like ninety degrees.
344	32.33-32.37	The new bars are placed symmetrically in a square position. Stan says they should cross in the centre. *shape; properties of bars*

349-354	33.20-34.27	The square is made but the boys are not convinced due to its orientation. Eric gets the angles to be close to ninety but not spot on except for a split second. *orientation; perception*
356-357	34.34-34.49	I ask where the bars should cross and Stan says they should cross with BD and AC straight. Eric says they should cross in the centre. *orientation; perception* *shape; properties of bars*
361-364	35.01-35.21	The boys bemoan the orientation of the square and say it is slanted and does not look like a square. They say it would help if the square was straight. *orientation; typical*
366	36.03-36.08	They boys construct the mid-points of the bars which helps them to get four right angles. However Stan does not quite believe it. *orientation; perception*
368-372	36.14-36.35	Stan says the shape is a square because it has four right angles. I remind him that rectangles also have four right angles. Stan says a square has four sides that look the same and a rectangle is longer. *shape; partitional classification*
390	40.07-40.14	The bars have been put end to end and the result is an isosceles right angled triangle. Stan says isosceles triangles don't always have to be big. They can be small and fat. *shape; typical representation*
399-403	41.27-41.48	The boys generate an obtuse angled scalene triangle and check with me that it is a real shape. *shape; typical representation*

Appendix 3.3 Accurate DMS episodes

	Time interval	Differences between expected equal measurements
Hemma and	21.03-21.08	angles differ < 4 degrees, sides differ < 0.2 cm
Seema	44.12-44.22	angles differ < 3 degrees, sides differ < 0.2 cm
Stan and Eric	17.36-17.38	angles differ < 3 degrees, sides differ < 0.4 cm
	26.47-26.40	angles differ < 6 degrees, sides differ < 0.3 cm

Appendices 4 Iteration 4

Appendix 4.1 Lesson plans for the whole class lesson

Making shapes from a dynamic perpendicular quadrilateral

Resources: geo-strips, class set of laptop computers, my laptop comouter, radio mouse, ipad for photos of pupils work, digital voice recorder, worksheets, poster paper

Lesson 1

Pupils log in to laptops and leave them booting up.

On computer attached to projector display the 8 cm vertical and 6 cm horizontal bars.

Keeping the bars perpendicular

Pupils are each given two geo strips of different lengths as concrete representations of the bars.

Keeping the geo strips at right angles students put one over the other and imagine what shapes they could make if they joined the ends of the bars. Students to sketch some of these shapes on mini white boards.

Class discussion on which shapes the pupils think it is possible to make.

Adding constraints 1

Pupils now asked to keep bars perpendicular but also to keep one bar so that it always crosses the other bar at the mid point. Paired discussion then class discussion on what shapes can be made.

Adding constraints 2

What happens when both bars cross in the middle?

Working with the dynamic perpendicular quadrilateral

Display the dynamic perpendicular quadrilateral on the board. Tell the pupils

A perpendicular quadrilateral is a 4 sided shape whose diagonals cross at right angles. This shape is a perpendicular quadrilateral and its bars can be dragged on the screen in the same way as you moved the geo strips to make different shapes.

On pairs working with the dynamic figure in the GSP on laptops pupils encouraged to drag the bars inside the dynamic shape to make all the shapes they have previously made using the geo strips but also to drag one bar so that it is below or to the side of the other bar. Paired discussion then class discussion of what shapes can be made.

Plenary

Bars perpendicular and one bar crosses the other at its mid point:

Which shapes can you make?

What do these shapes have in common?

How many rhombuses can you make?

How many isosceles triangles can you make?

How many kites can you make?

How many arrowheads can you make?

Lesson 2

Discussion on the shapes we made yesterday, properties of bars and properties of shapes

Listing shapes and their properties.

Recap the perpendicular quadrilateral and its properties.

Project dynamic figure onto screen through my laptop. Use the radio mouse so that volunteer pupils can drag the figure into different shapes.

Discuss shapes which can be made, properties of bars, properties of shapes.

Pupils fill in a sheet with columns for bar positions and shape properties.

shape	How the bars are positioned	Properties of the shape

Remembering what we discussed yesterday how many kites do you think it is possible to make? What happens if you move one bar a little bit?
How many arrowheads is it possible to make?

Animation of the dynamic shape whilst the bars are kept so that one crosses the other at its mid point
Whole class watches the animation. Class discussion of the changing shape.

Questions for class discussion:
Develop use of mathematical language: When you drag one bar so that it always crosses the other bar in the middle we say that one bar *bisects* the other. The bars are at right angles so one bar is the perpendicular bisector of the other bar.

Think about the kites and the arrowheads. What is different about them? What is the same about them?

How are the bars positioned for a kite?

How are the bars positioned for an arrowhead?

What happens when both bars are the perpendicular bisector of each other?

Is a rhombus a special kite?

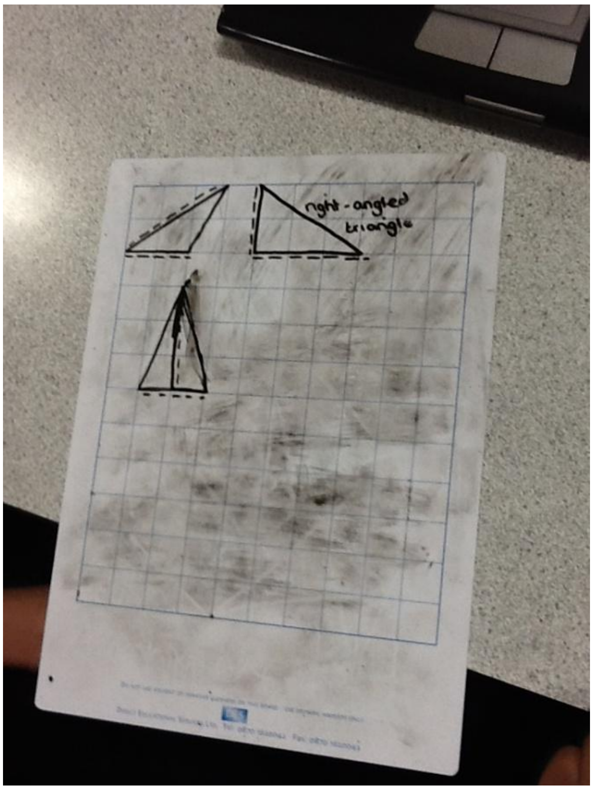
Plenary
Large circle on board. This is the perpendicular quadrilaterals. Circle inside this. What shapes could be in this (that are the shapes made in the animation). What shapes are inside the second circle?

Lesson 3

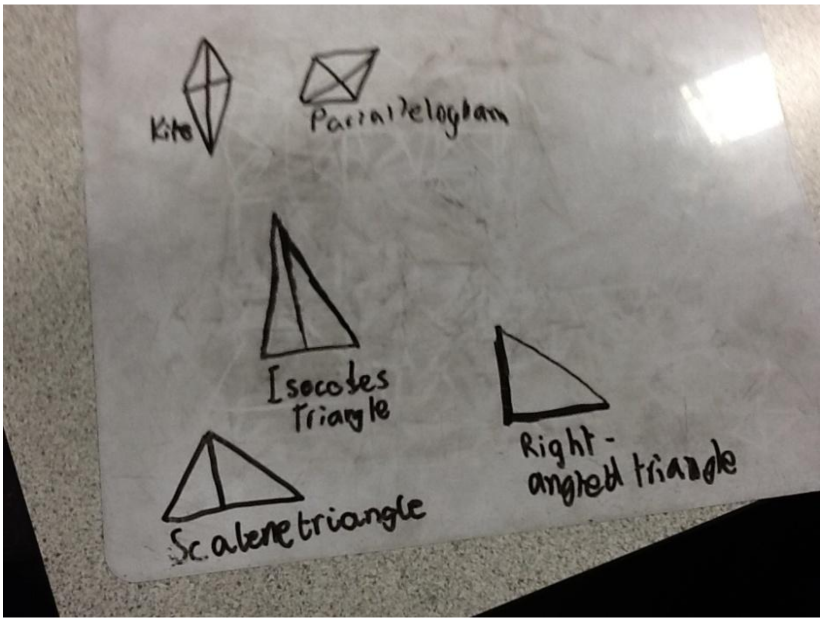
Pupils make posters of what they have learned about the shapes, the properties of the bars and the properties of sides and angles.

Appendix 4.2 Examples of the shapes students sketched on their mini whiteboards.

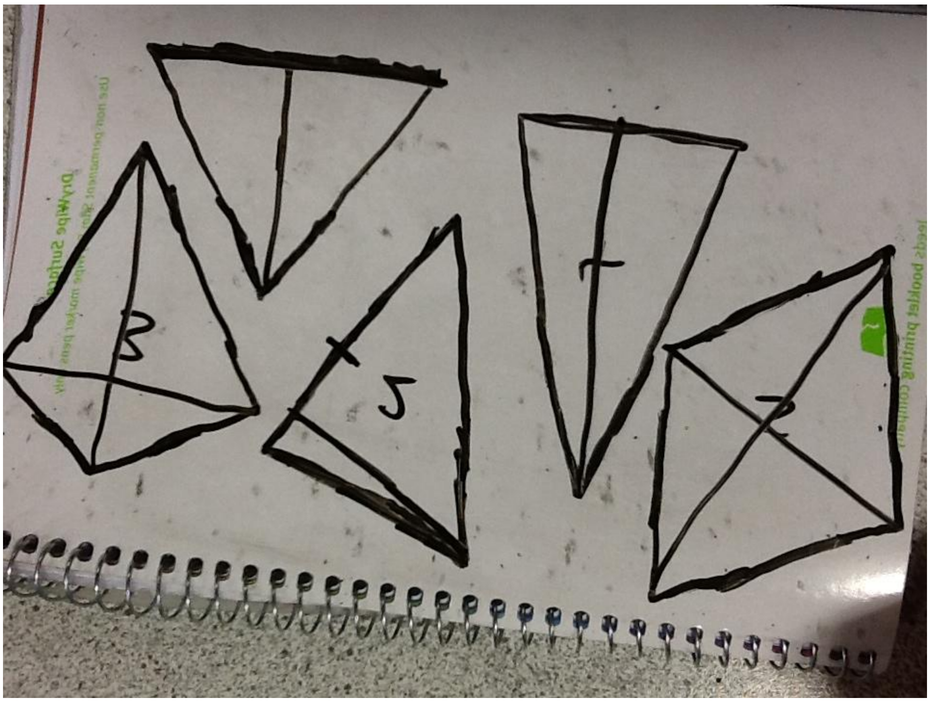
4.2a



4.2b

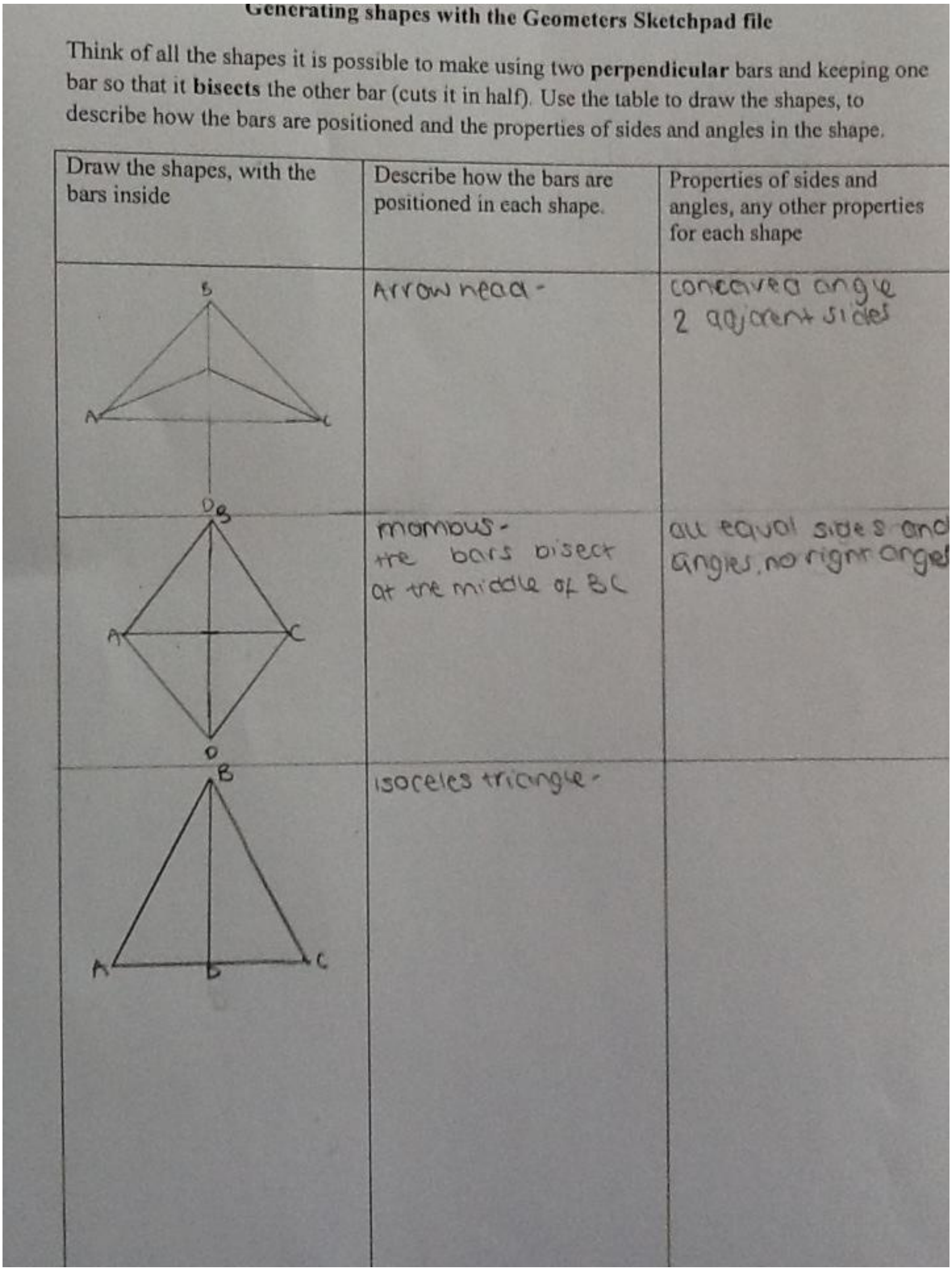

Kite
Parrallelogram
Isocoles
Triangle
Right-
angled triangle
Sc.alene triangle

4.2c



Appendix 4.3 Examples of completed worksheets

4.3a

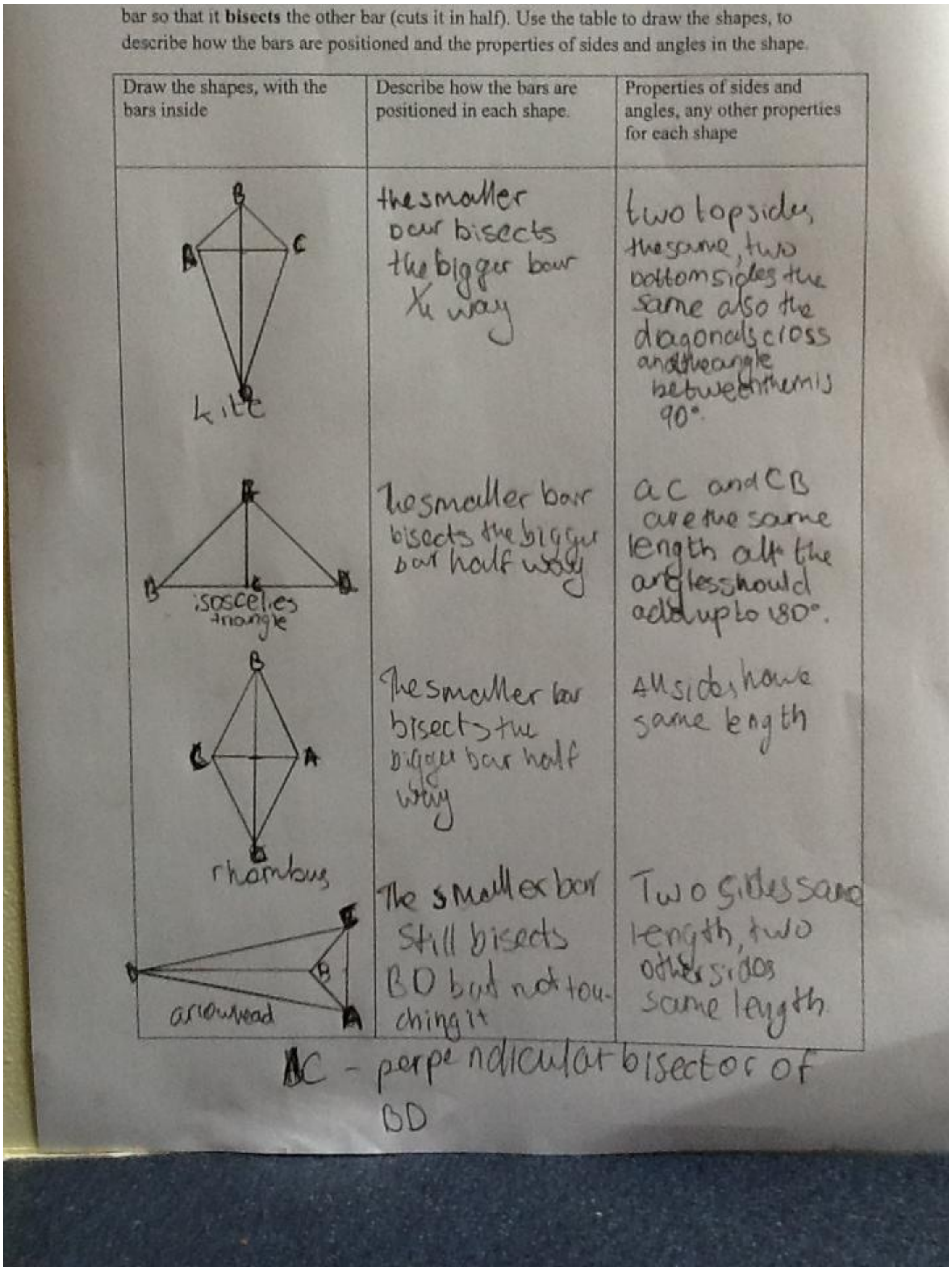


Generating shapes with the Geometers Sketchpad file

Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
	Arrow head -	concaved angle 2 adjacent sides
	rhombus - the bars bisect at the middle of BC	all equal sides and angles, no right angles
	isoceles triangle -	

4.3b

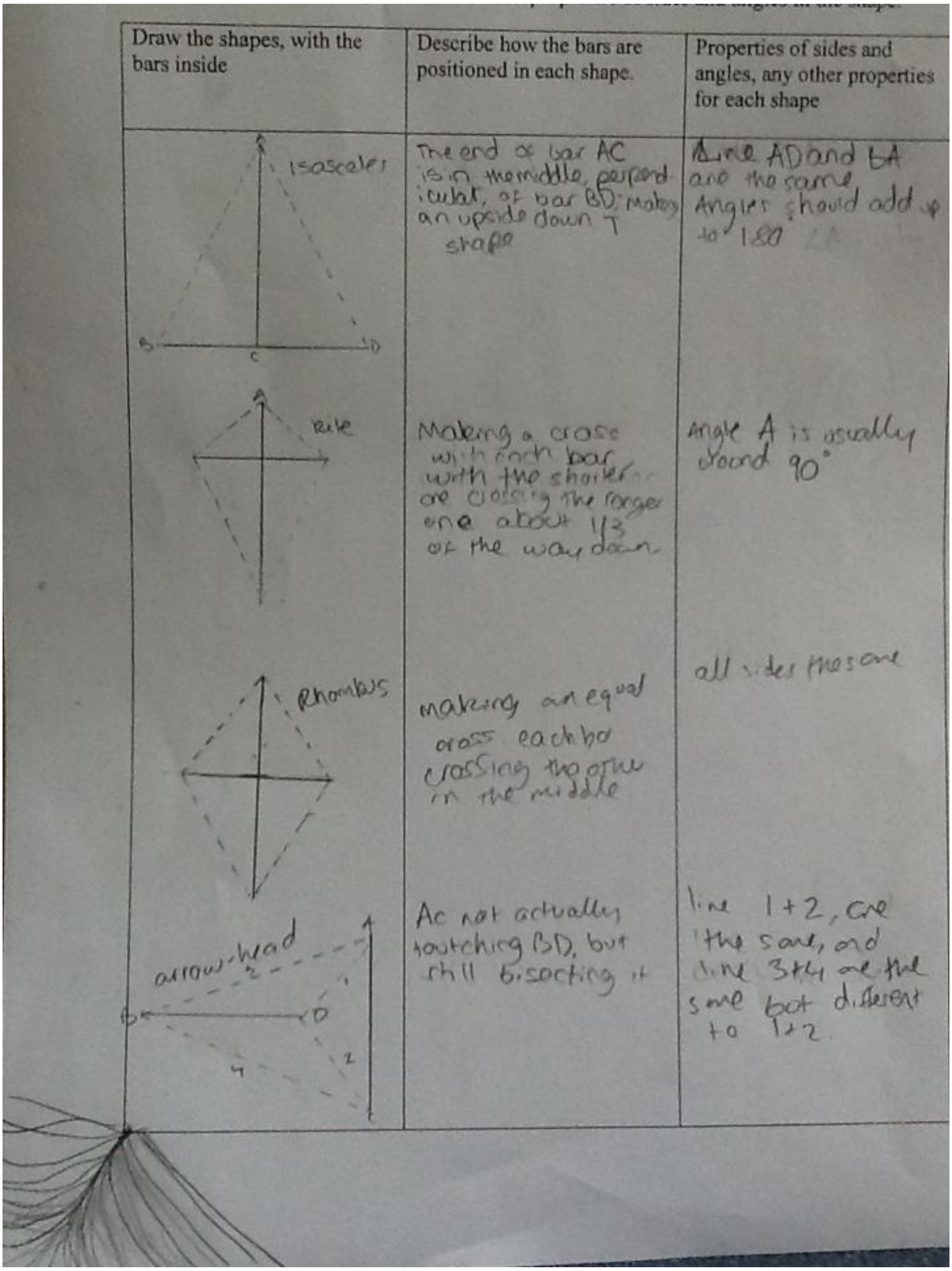


bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
B, A, C, D kite	the smaller bar bisects the bigger bar 1/2 way	two top sides the same, two bottom sides the same also the diagonals cross and the angle between them is 90°.
A, B, C, D isoscelies triangle	The smaller bar bisects the bigger bar half way	AC and CB are the same length all the angles should add up to 180°.
B, C, A, D rhombus	The smaller bar bisects the bigger bar half way	All sides have same length
C, D, B, A arrowhead	The smaller bar still bisects BD but not touching it	Two sides same length, two other sides same length.

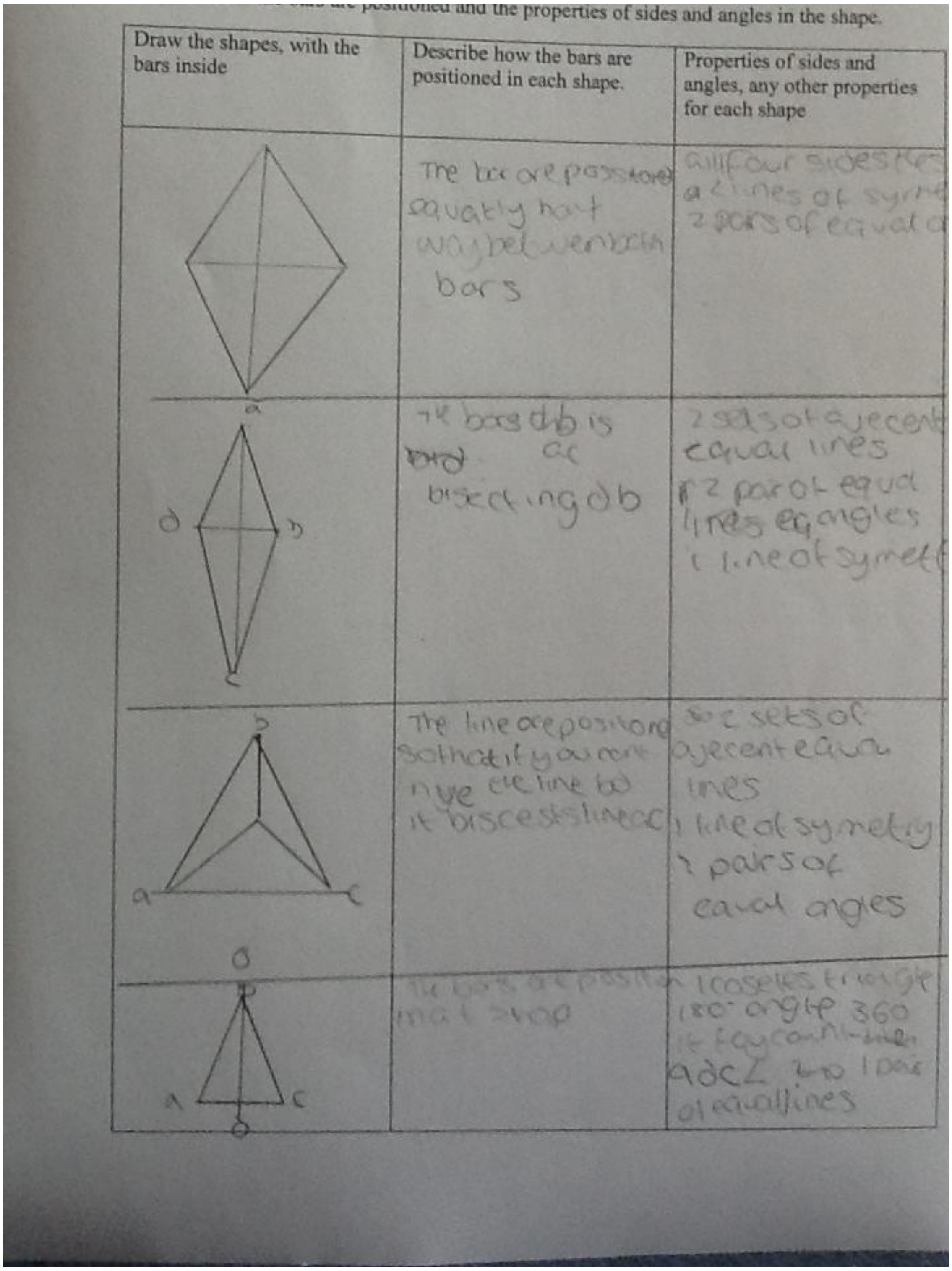
AC - perpendicular bisector of BD

4.3c



Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
Isoscales	The end of bar AC is in the middle, perpend-icular, of bar BD; makes an upside down T shape	Line AD and BA are the same, Angles should add up to 180
kite	Making a cross with each bar with the shorter one crossing the longer one about 1/3 of the way down.	Angle A is usually around 90°
Rhombus	making an equal cross each bar crossing the other in the middle	all sides the same
arrow-head	AC not actually touching BD, but still bisecting it	line 1+2, are the same, and line 3+4 are the same but different to 1+2.

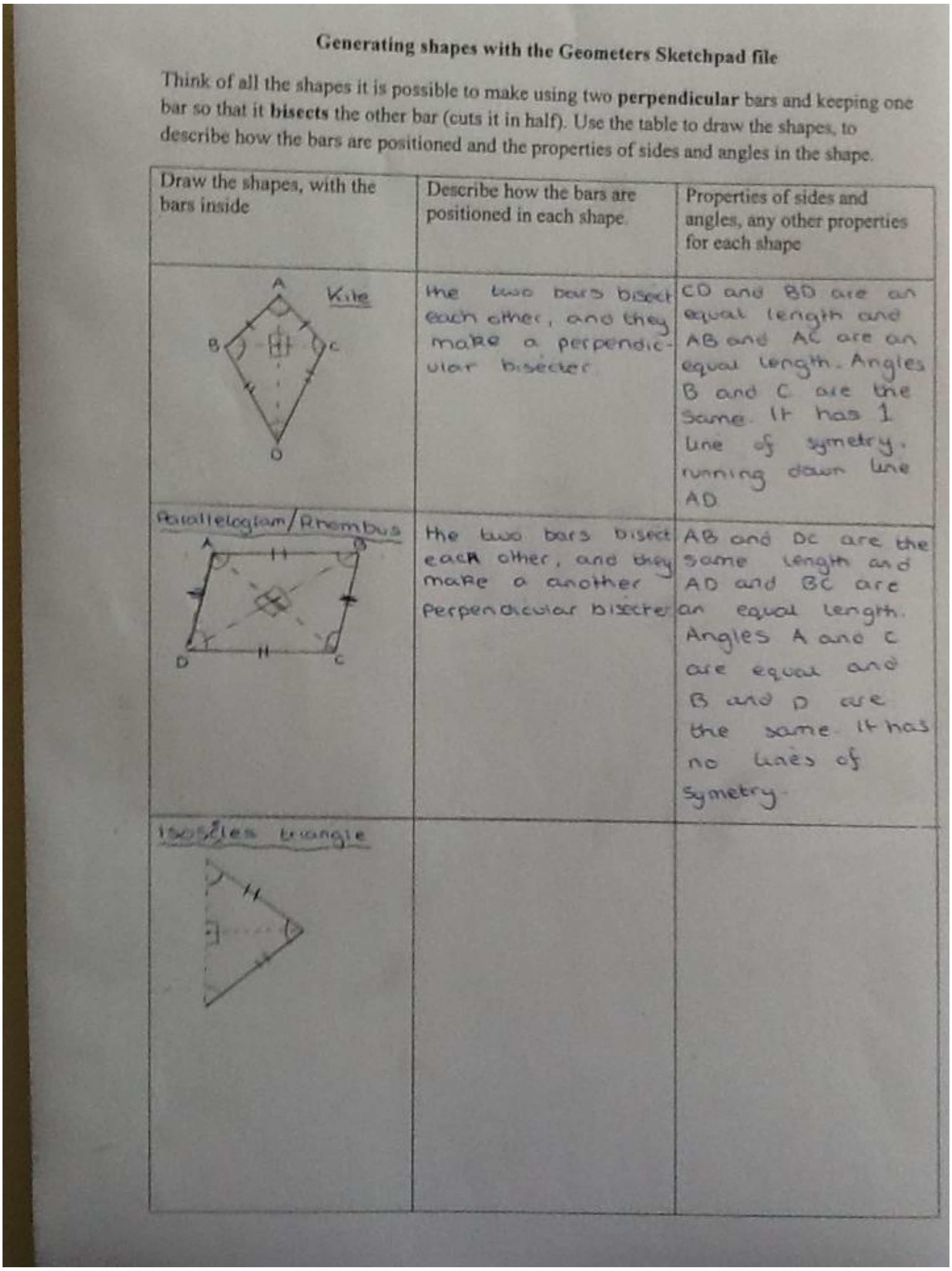
4.3d



... positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
	The bar are passitoned equatly half way between both bars	all four sides the[illegible] 2 lines of sym[illegible] 2 pairs of equal a[illegible]
d b c	The bars db is ~~bid~~ ac bisecting db	2 sets of ajecent equal lines 2 pair of equal lines eq angles 1 line of symet[illegible]
a b c	The line are positiond so that if you cont nue the line to it bisects line ac	2 sets of ajecent equal lines 1 line of symetry 2 pairs of equal angles
d b a c d	The bars are position mat stop	Icoseles triangle 180° angle 360 if you count [illegible] adc∠ two 1 pair of equal lines

4.3e

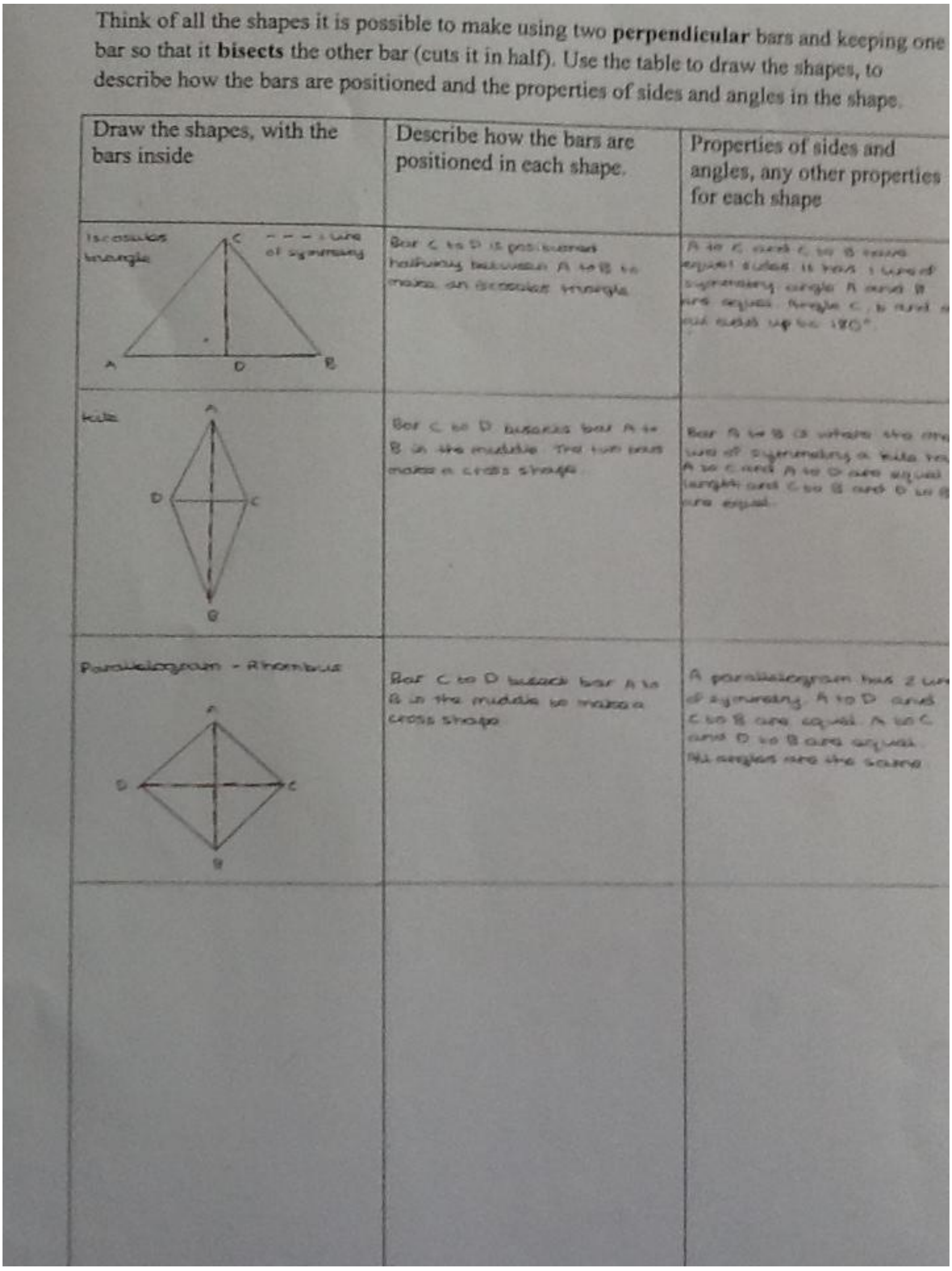


Generating shapes with the Geometers Sketchpad file

Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
Kite A B C D	the two bars bisect each other, and they make a perpendicular bisecter	CD and BD are an equal length and AB and AC are an equal length. Angles B and C are the same. It has 1 line of symetry, running down line AD
Parallelogram/Rhombus A B C D	the two bars bisect each other, and they make a another perpendicular bisecter	AB and DC are the same length and AD and BC are an equal length. Angles A and C are equal and B and D are the same. It has no lines of symetry.
Isosceles triangle		

4.3f



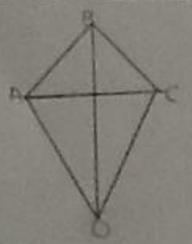
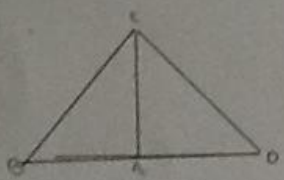
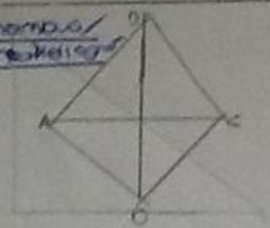
Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
Iscosules triangle C --- : Line of symmetry A D B	Bar C to D is positioned halfway between A to B to make an iscosules triangle	A to C and C to B have equal sides. It has 1 line of symmetry. angle A and B are equal. Angle C, B and A add up to 180°.
Kite A D C B	Bar C to D bisects bar A to B in the middle. The two bars make a cross shape.	Bar A to B is where the line of symmetry a kite has. A to C and A to D are equal length and C to B and D to B are equal.
Parallelogram - Rhombus A D C B	Bar C to D bisects bar A to B in the middle to make a cross shape.	A parallelogram has 2 lines of symmetry. A to D and C to B are equal. A to C and D to B are equal. All angles are the same

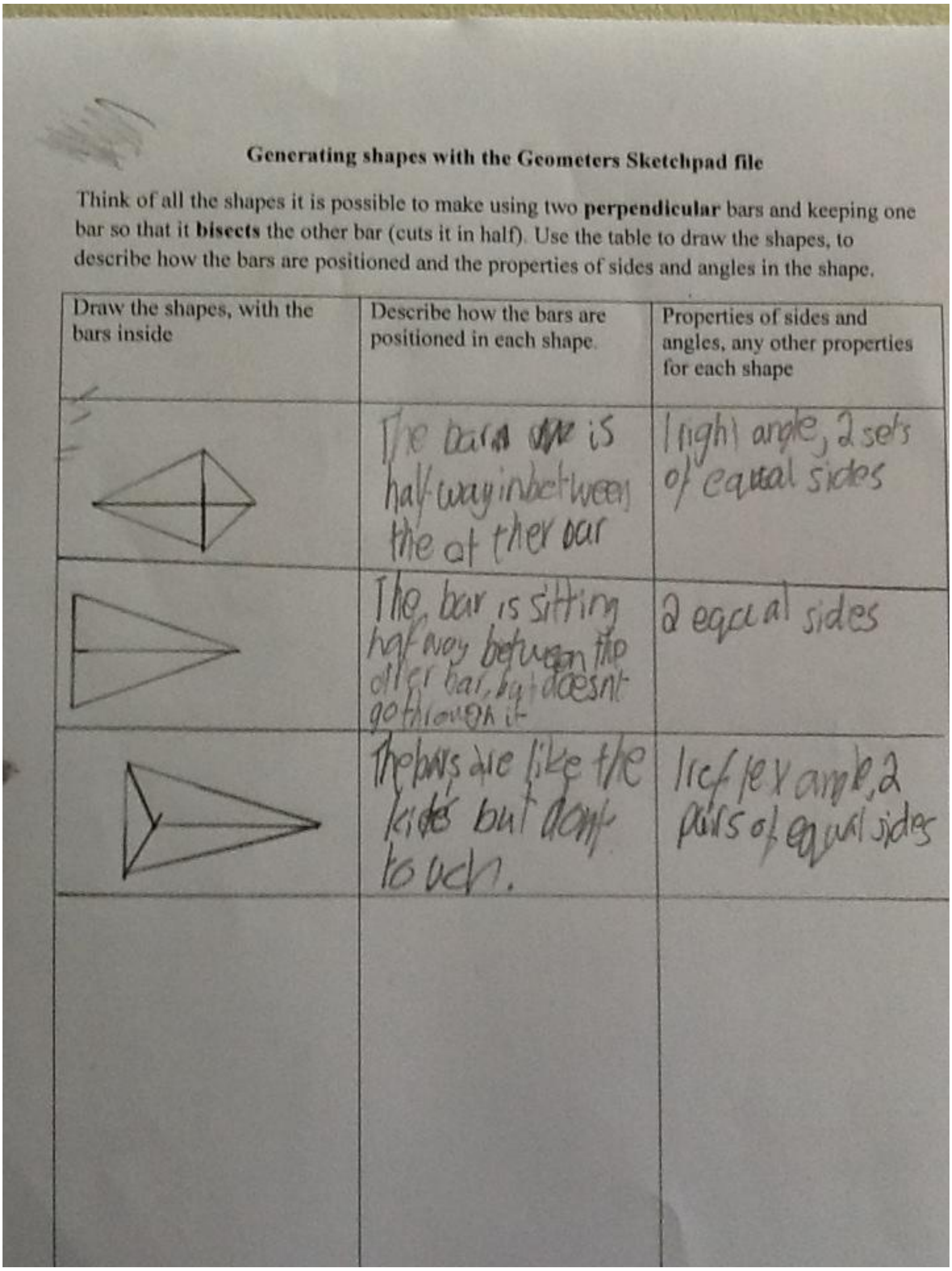
4.3g

Generating shapes with the Geometers Sketchpad file

Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
Kite 	Bars a ~~and~~ to c are bisecting bar B ~~and~~ to D, making a kite.	A ~~and~~ to C and B to D are different lengths but A to B to C are equal and a to d to C are too
isosceles triangle. 	Bar a to c are in the middle of bar D to B, making an isosceles triangle.	A to c and B to d have equal sides B to da is longer but B to c to d are equal
Rhombus/ Parallelogram 	Bar A to C are bisecting B and D, slanting the lines making a a parrallelogram Rhombus.	A ~~parrallelogram~~ Parallelogram has equal opposites so a to b and D to C are equal. B to C and A to D are aswell.

4.3h



Generating shapes with the Geometers Sketchpad file

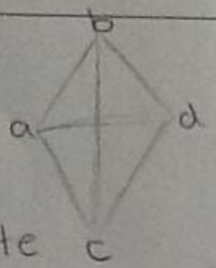
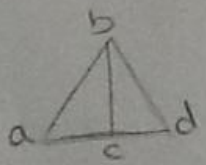
Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
	The bars ~~are~~ is half-way inbetween the other bar	1 right angle, 2 sets of equal sides
	The bar is sitting half way between the other bar, but doesnt go through it	2 equal sides
	The bars are like the kites but dont touch.	1 reflex angle, 2 pairs of equal sides

4.3i

Generating shapes with the Geometers Sketchpad file

Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

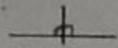
Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
 Kite	Line bc is positioned right in the middle of line ad	2 lines of symmetry 2short equal sides 2longequal sides
	Position line ad at the bottom of bc.	1 line of symmetry 2 equalsides not base

4.3j

| bars

Generating shapes with the Geometers Sketchpad file

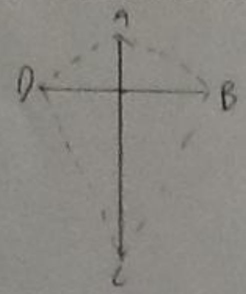
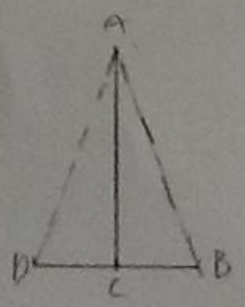
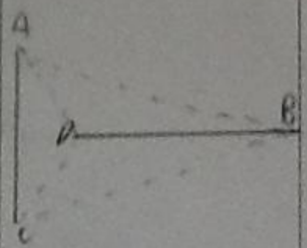
Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
 Kite	The bars are positioned in each shape by being perpendicular In the kite you can see the lines cross each other to create this 	If the kite was accurate, the properties of the side and angles are that the angles could be equal and there are two small lines and two larger lines
 iccosim triangle	The bars are positioned in each shape by being perpendicular In the iccosies triangle preform this shape 	If the icossoles was accurate, the properties of the side and angles would have to equal 180 because that is how many angles are in a triangle

4.3k

Generating shapes with the Geometers Sketchpad file

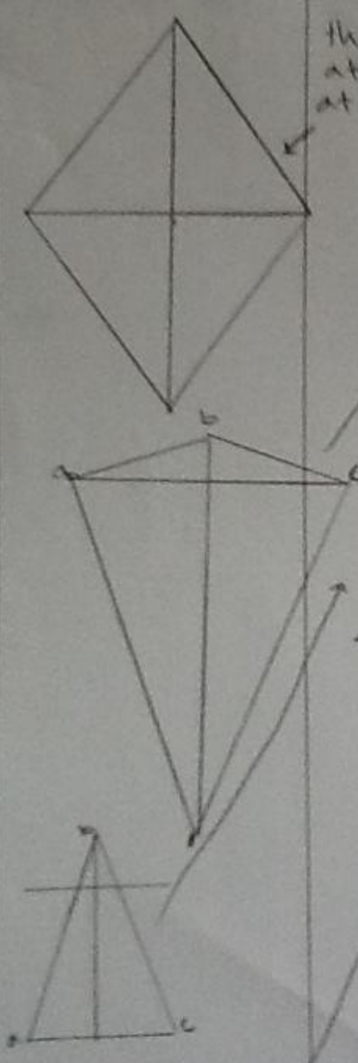
Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape.	Properties of sides and angles, any other properties for each shape
	DB is crossing AC above the middle	DA, AB are the same length DC and BC are the same length
	C is sitting on the middle of DB.	
	DB is not in contact with AC, B is in the middle.	

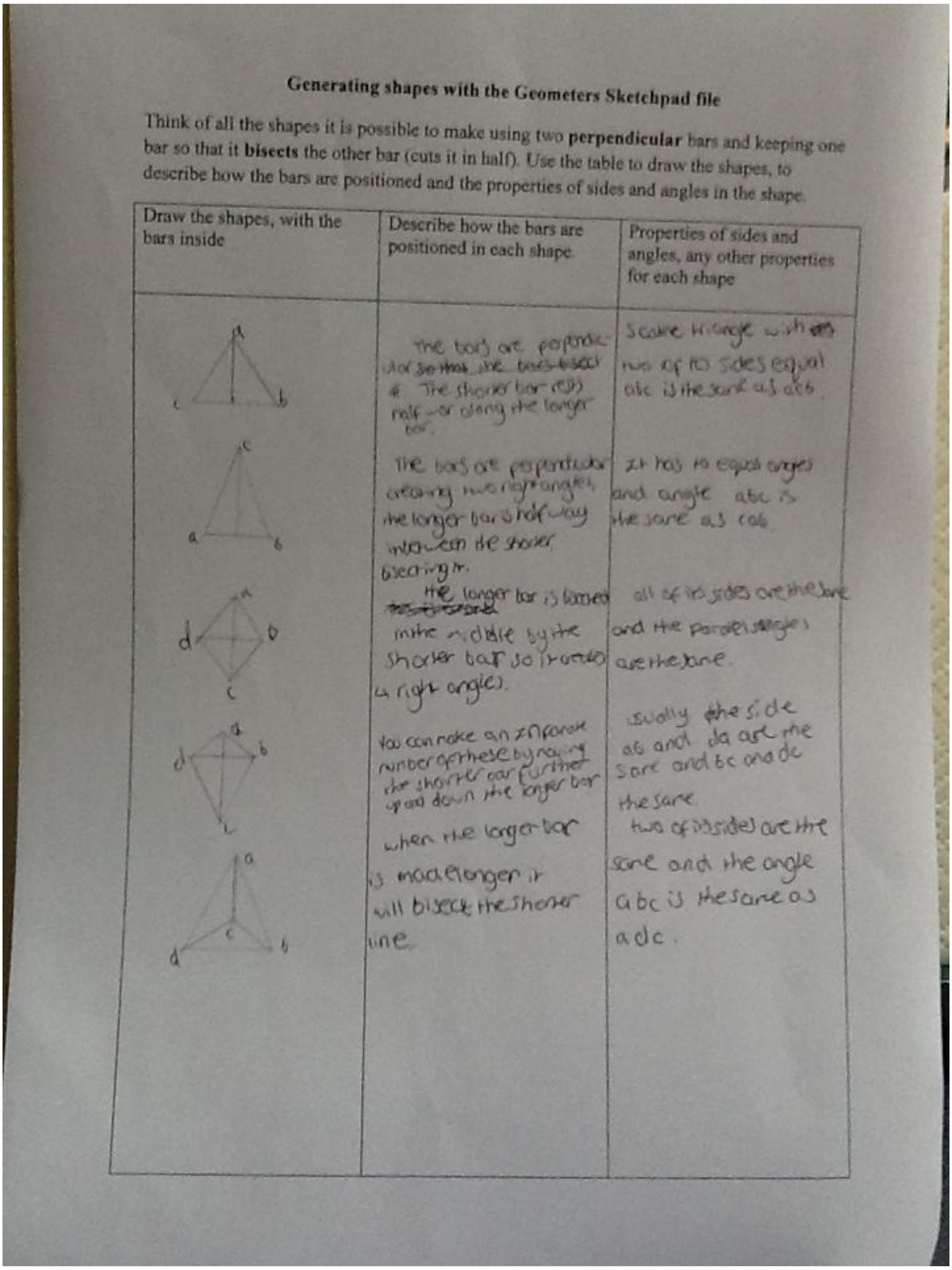
4.3l

Generating shapes with the Geometers Sketchpad file

Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape	Properties of sides and angles, any other properties for each shape
	they meat at the centre at right angle's 1 crosses the other at any point thay make T They form a right angle	all sides the same lenth a-b and c-b are equal, a-d and c-d are equal a-b and b-c are equal

4.3m

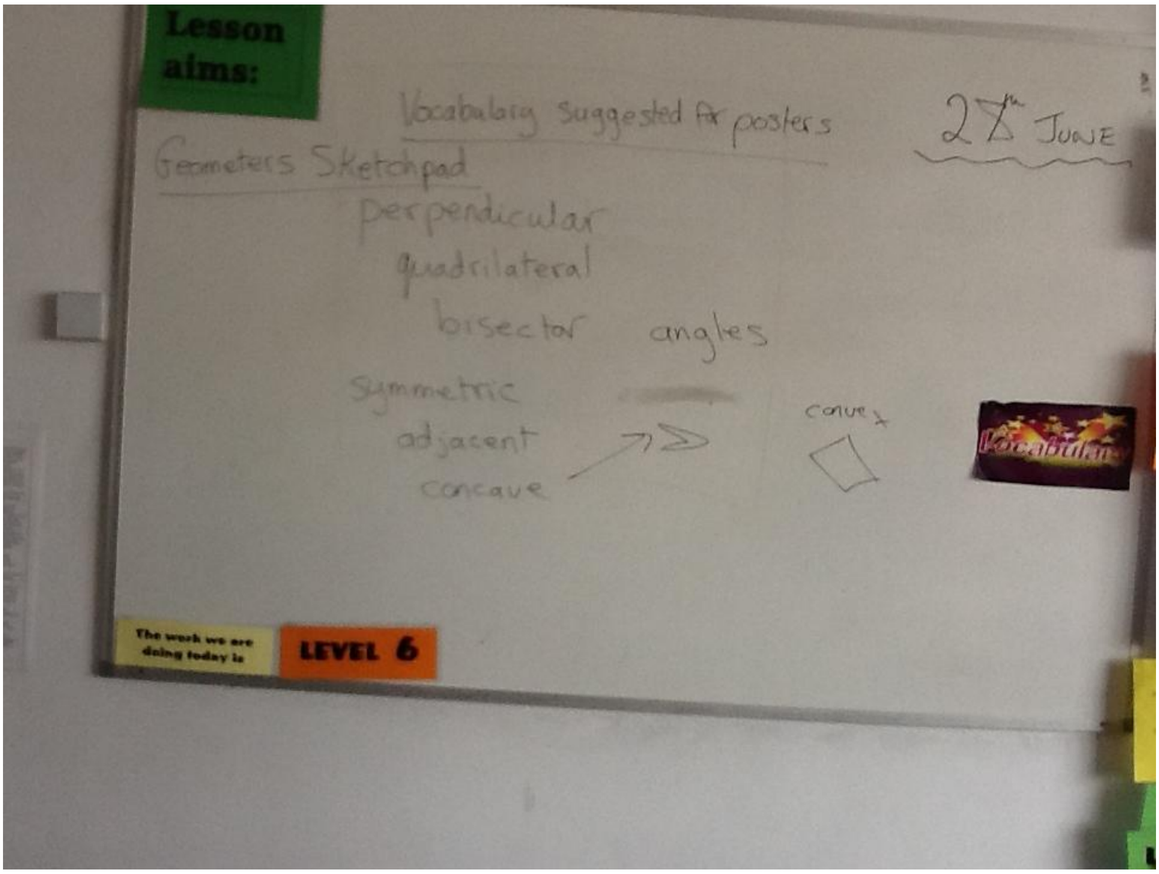


Generating shapes with the Geometers Sketchpad file

Think of all the shapes it is possible to make using two **perpendicular** bars and keeping one bar so that it **bisects** the other bar (cuts it in half). Use the table to draw the shapes, to describe how the bars are positioned and the properties of sides and angles in the shape.

Draw the shapes, with the bars inside	Describe how the bars are positioned in each shape	Properties of sides and angles, any other properties for each shape
	The bar's are perpendicular ~~so that the bars bisect~~ The shorter bar rests half-war along the longer bar.	Scalene triangle with two of its sides equal abc is the same as acb
	The bars are perpendicular creating two right angles, the longer bar is half way inbetween the shorter bisecting it.	It has two equal angles and angle abc is the same as cab
	The longer bar is bisected in the middle by the shorter bar so it creates 4 right angles.	all of its sides are the same and the parallel angles are the same.
	You can make an infinite number of these by moving the shorter bar further up and down the longer bar	usually the side ab and da are the same and bc and dc the same.
	when the longer bar is made longer it will bisect the shorter line.	two of its sides are the same and the angle abc is the same as adc.

Appendix 4.4 Screen shot of the board showing the vocabulary the class discussed that they should use for their posters.



Appendix 4.4 The contents of 31 posters

30 of the posters displayed the shapes that could be generated from the dynamic figure indicating the bars inside. 10 mentioned using geo-strips. 10 mentioned the Geometers Sketchpad programme.

	animation	properties	perp. bisector	family of shapes	infinite kites / arrowheads?
A	7 positions drawn of the animated figure	all shapes have 2 equal sides	perpendicular bisector	same family due to the property of bar AC bisecting bar BD	
B	description of AC moving along BD. AC is always perpendicular. infinite number of kites if you move AC along	named 2 sets of equal sides and 2 equal angles	perpendicular	mentions these properties as the defining properties of a kite. A kite is similar to a triangle	∞
C	They (shapes) are all created by moving one of the diagonals along the other		when diagonals are perpendicular	in terms of dynamic geometry they (shapes) are very similar	∞
D	family of shapes made by moving one bar along the other continuously	2 pairs of equal sides, but if all 4 are the same it is classed as a rhombus	perpendicular bisector	in this family there are 1 rhombus, 2 isos triangles and an infinite variation of kites and an infinite number of arrowheads	∞
E		drawing of a specific kite and a specific rhombus with equal sides shown by measurements.	perpendicular bisector	from a kite we managed to make a family of shapes which all had a perpendicular bisector	
F		arrowheads are concave kites because the top half of a kite has been folded in on itself		arrowheads are concave kites. Rhombus is a special parallelogram with equal sides	
G		perpendicular bars	perpendicular	family of shapes made from 2 perpendicular bars	
H	the rhombus uses the same bars as the kite (implies a movement between rhombus and kite)	the bars were perpendicular and bisected the other. you need equal bars for a square	perpendicular bisector	these shapes are mainly kites apart from the isosceles triangle. the rhombus is a special kind of kite as it uses the same bars as the kite. arrowhead kite	
I	7 positions drawn	all shapes	perpendicular	arrowheads/concave	∞

	of the animated figure	have a perpendicular bisector	bisector	kites all the shapes are from the same family	
J		all shapes have a perpendicular bisector	perpendicular bisector	all the shapes are from the same family	∞
K	you can move even 1 mm and it will be another shape.	2 pairs of equal sides, all shapes have a perpendicular bisector	perpendicular bisector	arrowheads are like concave kites. |In a way they are part of the kite family. 2 pairs of equal sides like a kite. Isosceles triangles belong to the family too. A rhombus also belongs to the kite family	∞
L	it may be considered infinite but you have to stop at one point (describing kites)	2 pairs of equal sides	perpendicular bisector	Family of shapes where bars have perpendicular bisectors. arrowheads are a special kind of kite. Using the GSP an isosceles triangle could be considered a special kind of kite but the only property it does not share is 4 sides. A rhombus can be considered a special kind of kite but instead of 2 pairs of equal sides all of its sides are the same.	∞
M	no words, but 7 positions drawn of the animated figure	lines AB and AD, BC and DC are the same			
N					
O		AB = BC CD = DA		a concave kite has the same properties as a convex kite but in the shape of an arrowhead	
P		the rhombus has 2 sets of adjacent sides and unlike a kite the pairs of adjacent sides are equal to each other		the kite is possibly a special version of a rhombus the isosceles triangle is in the rhombus family	
Q		the line bisectors are perpendicular	bisector, perpendicular		
R		annotated diagrams to indicate equal sides and right angles.	bisects	kite family, arrowhead is a special kind of kite	

		a kite has 2 sets of equal sides, BD bisects AC, <DCB = <DAB, one line of symmetry			
S		annotated diagrams to show equal sides and right angles			
T	(starts with isosceles triangle). If you move this line (i.e. base of triangle) up the middle line and bisect it like this it turns into a kite. If you carry on and bisect in the middle like this it's a rhombus. If you carry on for a long while and over the vertical line (points to an arrowhead).	kites and arrowheads have 2 pairs of adjacent equal lines.	bisects	an arrowhead is a concave kite. It is an irregular kite. a rhombus is a special parallelogram. It is not a square because all the angles are different.	
U			perpendicular bisector		
V		partitional classification of shapes by properties	perpendicular bisector		
W	no words, but 9 positions drawn of the animated figure	partitional classification of shapes by properties			
X					∞
Y		kites and rhombuses described as 2 pairs of adjacent angles and sides. In a rhombus all sides are equal.		a kite could possibly be a special version of a rhombus	
Z		they all have perpendicular bisectors	perpendicular bisector		
AA		2 sides equal length	perpendicular bisector		
BB		partitional classification of shapes by properties, specific			

		instance of a kite			
CC	if you push the line along to the left a bit (in an arrowhead) you get a kite. If you align the bars so that they cross exactly in the middle you get a rhombus, if the bars are the same length it becomes a square.	DB is the line of symmetry. kites and arrowheads have. 2 long sides and 2 short sides. the smaller line must always bisect the other. So is the isosceles triangle really acceptable? Or if it bisects it just before the end is it a kite technically?		arrowhead is a special kind of kite.	
DD	diagram of the animation showing the bars and dashed lines indicate different shapes which can be made superimposed on each other.	arrowheads are concave kites because the different sides are equal like a kite. Kites have similar if not the same properties as arrowheads		arrowheads are concave kites, they are part of the kite family. rhombus are equilateral kites, just look like a tipped over square. isosceles triangles are actually in the kite family even though they have 3 sides. The base is just like the point of a kite flattened.	
EE		in every shape there was a perpendicular bisector	perpendicular bisector		

Appendix 5

Letter to Head teacher May 30th 2012

Dear,
As part of my studies for a PhD I am undertaking a research project into children's geometrical reasoning in a Computer Geometry environment. I have devised a task in a program called the Geometers Sketchpad and I have already worked with a number of pairs of children, observing them while they work together on the task. Last year I came into School B to work with two of (HOD)'s classes and afterwards worked with two pairs of pupils chosen from those classes.

I am writing to ask your permission to come into School B again this year to work with pairs of year 8 pupils on these tasks (between two and four pairs depending on availability of times). The pairs of year 8 pupils will be asked to work in pairs at the computer and will be encouraged to discuss the task together. I will bring a computer for them to use which is loaded with an image capture software program. This records the screen activity and the dialogue so that they can be analysed.

If you are happy for this to go ahead then I have given (HOD) an electronic copy of a letter to parents of the pairs of children to ask for their permission. As the letter to the parents states, I have ethical approval from the ethics committee at the university to undertake this research. I would be very grateful to be allowed to work with School B pupils and hope that the maths department would also benefit from being involved.

Yours sincerely

Sue Forsythe

Letter to head teacher May 20th 2013

Dear,
As part of my studies for a PhD I am undertaking a research project into children's geometrical reasoning in a Computer Geometry environment. I have devised a task in a program called the Geometers Sketchpad and have worked with a number of pairs of children (including at School B), observing them while they work together on the task. I have also developed the task so that it can be used in a whole class setting and used this in School B two years ago. I am at the point of writing up the PhD and have studied the task enough to have a better idea of how it works conceptually. As a way of concluding the research I would like to trial a new lesson plan with a whole class.

I am writing to ask your permission to come into School B to work with year 8 pupils. I have approached (HOD), Head of Mathematics at School B, and she is happy for me to come into school so that we can co teach the whole class lesson to her year 8 class for two consecutive lessons with the option of a third lesson if needed. I will bring a computer which is loaded with an image capture software program. This records the on-screen activity and the dialogue so that they can be analysed. I would also like to bring a digital recorder to record interesting comments the pupils make and my i Pad to take photos of pupils' work. I will not make video recordings or take photos of pupils and any contributions will be anonymised.

If you are happy for this to go ahead then I have given (HOD) an electronic copy of a letter to parents of children in the class to ask for their permission. As the letter to the parents states, I have ethical approval from the ethics committee at the university to undertake this research. I would be very grateful to be allowed to work with School B pupils and hope that the maths department would also benefit from being involved.

Yours sincerely

Sue Forsythe

Letter to parents 2012

Dear Parent / Guardian,

There have been a number of developments in mathematics teaching over the last few years. Children now have opportunities to learn in a computer environment and this can be very valuable given the right kind of task. There is more emphasis placed on learning through task-based activities and the profile of geometry is being raised. Our colleague Mrs Sue Forsythe from the School of Education, University of Leicester, is researching into the kinds of activities which promote effective learning of geometry, using tasks in a computer geometry environment.

The children taking part in this project will work in pairs at the computer with Mrs Forsythe. The computer being used will have been loaded with image capture software, which records what happens on the computer screen. The pupils' conversations will also be recorded but there will be no video recording. The information collected will remain anonymous, and any pupil asked to participate will be able to withdraw at any time.

The information collected will be held in a secure place and will only be used by Mrs Forsythe and (HOD)to aid understanding of how successful the tasks have been. It is hoped that the work completed on the computer activities would be used for eventual academic publication in education journals to share any good practice with others in the education sector.

If you have any further questions concerning this matter, please feel free to get in contact with Miss Green. If you are happy for your son/daughter to be involved in the collection of information, please complete the slip below and return to (HOD)

Yours faithfully

✂____

We/I am happy for our son/daughter to take part in the computer geometry project. We understand that some audio recordings may be made in class and work may be photographed.

Name of student:__________________________

Signature:_____________________________ Date:__________

Student signature:_____________________________ Date:_________

Letter to parents 2013

Dear Parent / Guardian,

There have been a number of developments in mathematics teaching over the last few years. Children now have opportunities to learn in a computer environment and this can be very valuable given the right kind of task. There is more emphasis placed on learning through task-based activities and the profile of geometry is being raised. Our colleague Mrs Sue Forsythe from the School of Education, University of Leicester, is researching into the kinds of activities which promote effective learning of geometry, using tasks in a computer geometry environment.

The children taking part in this project will work in their usual mathematics lessons with their regular class teacher, (HOD) and Mrs Forsythe. On screen activity of the class computer will be recorded and a digital recorder will be used to record some of the pupil comments during the lesson. Photographs will be taken of pupils' work. There will be no video recording and all pupils' contributions will be anonymised. Any pupil asked to participate will be able to withdraw at any time.

The information collected will be held in a secure place and will only be used by Mrs Forsythe and (HOD) to aid understanding of how successful the tasks have been. It is hoped that the work completed on the computer activities would be used for eventual academic publication in education journals to share any good practice with others in the education sector.

If you have any further questions concerning this matter, please feel free to get in contact with Miss Green. If you are happy for your son/daughter to be involved in the collection of information, please complete the slip below and return to (HOD).

Yours faithfully

✂____

We/I am happy for our son/daughter to take part in the computer geometry project. We understand that some audio recordings may be made in class and work may be photographed.

Name of student:__________________________

Signature:_____________________________ Date:__________

Student signature:_____________________________ Date:_________