Zames-Falb multipliers for absolute stability: from O'Shea's contribution to convex searches

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Abstract

Absolute stability attracted much attention in the sixties. Several stability conditions for loops with slope- restricted nonlinearities were developed. Results such as the Circle Criterion and the Popov Criterion form part of the core curriculum for students of control. Moreover, the equivalence of results obtained by different techniques, specifically Lyapunov and Popov's stability theories, led to one of the most important results in control engineering: the KYP Lemma.

For Lurye[1] systems this work culminated in the class of multipliers proposed by O'Shea in 1966 and formalised by Zames and Falb in 1968. The superiority of this class was quickly and widely accepted. Nevertheless the result was ahead of its time as graphical techniques were preferred in the absence of readily available computer optimization. Its first systematic use as a stability criterion came twenty years after the initial proposal of the class. A further twenty years have been required to develop a proper understanding of the different techniques that can be used. In this long gestation some significant knowledge has been overlooked or forgotten. Most significantly, O'Shea's contribution and insight is no longer acknowledged; his papers are barely cited despite his original parameterization of the class.

This tutorial paper aims to provide a clear and comprehensive introduction to the topic from a user's viewpoint. We review the main results: the stability theory, the properties of the multipliers (including their phase properties, phase-equivalence results and the issues associated with causality), and convex searches. For clarity of exposition we restrict our attention to continuous time multipliers for single-input single-output results. Nevertheless we include several recent significant developments by the authors and others. We illustrate all these topics using an example proposed by O'Shea himself.

Keywords: Absolute stability, Slope-restricted nonlinearities, Multiplier theory

1. Introduction

A feedback interconnection of a linear system and a static nonlinearity is said to be *absolutely stable* if the interconnection is stable (in some sense) for every nonlinearity in a given class. The theory of absolute stability has occupied an important portion of the control theory literature due to its relevance to a variety of practical control/systems engineering problems. The absolute stability problem can be studied, broadly, from either the perspective of internal stability, or from that of input-output stability. The former, and perhaps more common, approach typically involves the search for the parameters of a proposed Lyapunov function which can be used to guarantee asymptotic stability of the origin for as large a class of nonlinearities as possible. The latter approach involves the use of transfer functions called *multipliers*. In their classical interpretation they are used to translate one nonlinear passivity-type problem into another linear, easier to solve, passivity-type problem. The aim, again, is to choose a multiplier within a predefined

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Figure 1: R. P. O'Shea, reproduced with kind permission of The Tribune-Democrat (2009).

class of multipliers which allows input-output stability to be guaranteed for as large a class of nonlinearities as possible. In this paper, attention is focused on input-output stability from the perspective of passivity and in particular on the properties of the so-called Zames-Falb multipliers.

The multiplier approach attracted much attention from the control community in the 1960's. One reason for this was, without the computing power of today, researchers were able to glean a great deal about the absolute stability of a system purely from the properties of the linear part. In an early paper the concept of multiplier was used by Brockett and Willems (1965) and the idea developed rapidly from this (O'Shea, 1966, 1967), etc. Despite this early promise and flurry of activitity, probably the most widely known absolute stability tools today are the Circle and Popov Criteria (see Vidyasagar, 1993; Khalil, 2002) which have become standard, in part due to their simplicity and in part due to their graphical interpretations. However, when a tighter description of the nonlinearity is available, these criteria are well-known to be conservative. In such cases, the use of more general multiplier methods can be useful and, in particular, the so-called Zames-Falb multipliers can often be used to improve predictions made about stability and performance of the interconnection.

Despite their moniker, Zames-Falb mulitipliers were actually discovered by O'Shea (1966, 1967). While the treatment of O'Shea (1966) was restricted to causal multipliers, the aim of O'Shea (1967) was to extend this definition to noncausal multipliers: "this modification allows greater freedom in the phase variation of $G(j\omega) + 1/k$ outside of the $\pm 90^o$ band". There were several correspondence items discussing these (Zames and Falb, 1967; Willems et al., 1967; Falb and Zames, 1967). A rigorous and correct treatment was first given in the much-cited paper by Zames and Falb (1968). The contribution of O'Shea was fully acknowledged by all concerned at the time. As an example, Desoer and Vidyasagar (1975) state that the "idea of using noncausal multipliers is due to O'Shea."

However, the class of multipliers aroused little further interest for twenty years, until the proposal of Safonov and Wyetzner (1987) for computer-aided search and the illustration by Megretski and Rantzer (1997) of multiplier analysis embedded within the framework of IQCs. In these and subsequent papers the pioneering work of O'Shea was largely overlooked. The terminology "Zames-Falb multiplier" appears to have been coined by Chen and Wen (1995, 1996) in their proposal for a convex search. This development, while rightly acknowledging the important work of Zames and Falb, has had an unfortunate consequence. Zames and Falb (1968) focus on the relation of the nonlinearity to the monotone and bounded static nonlinearity; O'Shea's insights into the phase properties of the multipliers have been largely forgotten (with one notable exception: the discussion of Megretski (1995) on phase limitation).

In this tutorial paper we re-examine Zames-Falb multipliers and, in particular, use an example of O'Shea (1967) to discuss the phase properties of the Zames-Falb multipliers and how they can be used advantageously in the study of the absolute stability problem.

The remainder of the paper is structured as follows. In Section 2 we provide a brief motivating example explaining the significance of Zames-Falb multipliers, and in Section 3 we review the basics of the absolute stability problem and some approaches to its solution. In Section 4 we address at length an example previously discussed by O'Shea (1967). In particular we discuss how a number of input-output stability methods can be used for analysis. This

section includes a comprehensive treatment of the application of a multiplier originally proposed by O'Shea. In Section 5 further properties of Zames-Falb multipliers are discussed and in Section 6 a brief review of start-of-the-art computational searches is given. Finally in Section 9 we conclude and point to some other recent developments in the use of Zames-Falb multipliers. While we emphasise the tutorial aspect of this overview, some mathematical formalism and machinery is inevitable; this is given in the appendix.

2. Motivating example

Remark 1 *Several concepts in this section are formally defined in Section III and/or the appendix.*

Since saturation is a memoryless and slope restricted nonlinearity, the Zames-Falb multipliers can be used to study the stability/robust stability of systems involving saturation (Jönsson and Rantzer, 2000). We shall illustrate such analysis with an anti-windup example (Kothare and Morari, 1999) where robust stability is to be established (Turner et al., 2007; Morales et al., 2014). $U(s)$ and $Y(s)$ are the Laplace transform of the plant's input and output, respectively.

Consider a plant with additive uncertainty

$$Y(s) = \left(G(s) + \frac{1}{\gamma}\Delta\right)U(s), \tag{1}$$

where $G(s)$ is the nominal SISO transfer function and Δ represents additive uncertainty with, for any bounded signal u,

$$\|\Delta u\|_2 \leq \|u\|_2. \tag{2}$$

In the case where Δ is restricted to be a linear time invariant (LTI) system we may write this as the familiar $\mathbf{H}_\infty$ norm condition

$$\|\Delta\|_\infty \leq 1. \tag{3}$$

Suppose the controller has the internal model control structure given by

$$U(s) = -Q(s)\left(Y(s) - G(s)U(s)\right). \tag{4}$$

and illustrated in Fig. 2.

The robustness of such controllers are discussed at length by Morari and Zafiriou (1989). Briefly, if both G and Q are stable, then it follows from a small gain argument that the loop is stable provided

$$\|Q\|_\infty < \gamma. \tag{5}$$

Suppose now there is saturation in the loop, as in Fig. 3. Since the saturation operator is in series with Δ, a similar small gain argument (Turner et al., 2007) says that the loop remains stable provided (5) is satisfied. In other words, the antiwindup scheme preserves the robustness (to additive uncertainty) of the loop without saturation.
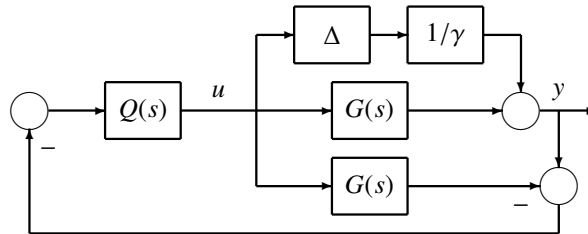

Figure 2: Internal model control where the plant dynamics are assumed known save for an additive uncertainty.
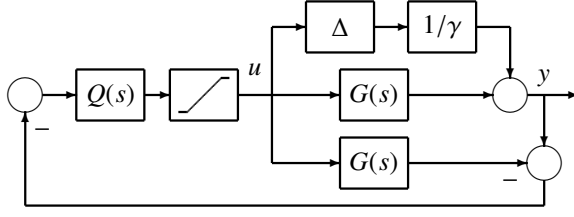
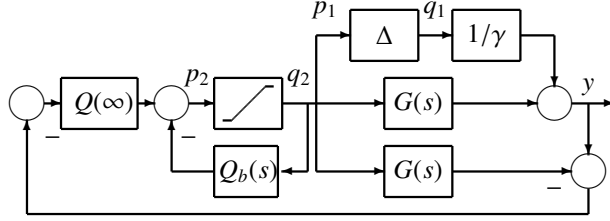Figure 3: Internal model control with saturation.

Figure 4: The antiwindup scheme of Zheng et al. (1994).

But the antiwindup scheme of Fig. 3 can be notoriously sluggish. To improve matters, one suggestion in the literature (Zheng et al., 1994) is the scheme of Fig. 4 with

$$Q_b(s) = \frac{Q(\infty)}{Q(s)} - 1, \tag{6}$$

so that

$$Q(s) = (1 + Q_b(s))^{-1} Q(\infty). \tag{7}$$

This often has much better performance, but there is no longer an *a priori* guarantee of stability. In our example we consider a case where the Zames-Falb multipliers can be used to establish such stability.

Suppose G is first order with a delay (a standard model in the process industries):

$$G(s) = e^{-sd} \frac{b}{s+a}. \tag{8}$$

A natural choice for Q is then:

$$Q(s) = \frac{c}{b} \frac{s+a}{s+c}. \tag{9}$$

In the unconstrained (saturation $\equiv$ identity) case, robust stability is established via (5) which in this case reduces to

$$\max\left(\frac{c}{b}, \frac{a}{b}\right) < \gamma. \tag{10}$$

In the constrained (saturation $\not\equiv$ identity) case, establishing robust stability is much more difficult. However the constrained loop is stable (Morales et al., 2014) provided there exists a multiplier (of some form) M such that

$$1 - Q_b^* M^* - MQ_b - M - M^* + M^* MQ(\infty)^2/\gamma^2 < 0, \tag{11}$$

at all frequencies. It may not be possible to satisfy this inequality with a constant M (and it cannot be satisfied with constant M if $\|Q\|_\infty \to \gamma$), but for our example, it is straightforward to check that the inequality is satisfied if we choose

$$M(s) = \frac{s+a}{s+c}. \tag{12}$$

It transpires that this belongs to the class of first order Zames-Falb multipliers provided $0 < a < 2c$. In this case the robust stability of the constrained loop is established using a Zames-Falb multiplier.
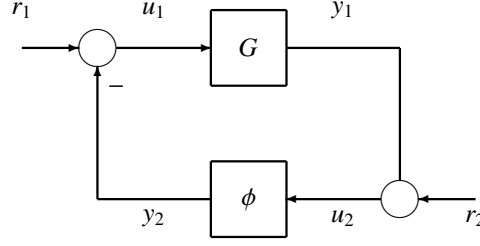
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Figure 5: Lurye system. Both G and ϕ are assumed to be causal.

3. Preliminaries

In an early paper Brockett and Willems (1965) use the concept of a multiplier. The aim of its use is to reduce the conservatism of the open loop approach which is used to analyse the stability of the problem. The advantage of this approach is that the condition to be tested will only depend on the linear system G and the maximum slope of the nonlinearity k.

3.1. The Lurye problem

The Lurye problem consists of finding conditions on the linear system G such that the feedback interconnection between G and any nonlinearity ϕ that belong to some class of nonlinearities is stable. As stability must be ensured for the whole class, the adjective *absolute* is added, and this problem is also known as the absolute stability problem (see Liberzon, 2006, for an overview).

The feedback interconnection is defined by

$$\begin{cases} u_1 = r_1 - \phi u_2, \\ u_2 = r_2 + G u_1. \end{cases} \tag{13}$$

It is usual, although not necessary, to assume that G is strictly proper. This is enough to ensure the feedback between G and any slope-restricted nonlinearity is well-posed (i.e. that u_1 and u_2 are uniquely defined given r_1 and r_2, all on the extended spaces defined in the appendix). The system is said to be input/output stable if for any $r_1 \in \mathcal{L}_2$, $r_2 \in \mathcal{L}_2$ we also have $u_1 \in \mathcal{L}_2$ and $u_2 \in \mathcal{L}_2$.

In this paper we consider the class of static nonlinearities with slope less than or equal to k. With an abuse of notation we use ϕ to denote both the memoryless operator ($\phi : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$) and its associated nonlinear function $\phi : \mathbb{R} \to \mathbb{R}$.

Definition 1 *A static nonlinearity ϕ is said to be slope restricted $\phi \in S[0, k]$ if for any real number x and y we have*

$$0 \le \frac{\phi(x) - \phi(y)}{x - y} \le k. \tag{14}$$

The LTI system G is given by

$$\dot{x} = Ax + Bu, \tag{15}$$
$$y = Cx + Du, \tag{16}$$

and its transfer function is $G(s) = C(sI - A)^{-1}B + D$. Henceforth, we will no longer distinguish between LTI operators and their transfer functions. The Rosenbrock system matrix

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{17}$$

will be used as shorthand. G^* denotes the adjoint of G and it is given by $G^*(s) = G'(-s)$.

We assume G is stable (i.e. A is Hurwitz), hence we may assume $r_1 = 0$ without loss of generality. If G is unstable, the loop would be unstable with ϕ the zero operator.

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3.2. The Nyquist value and the Kalman conjecture

The class of slope restricted nonlinearities $\phi \in S[0, k]$ includes the linear gains τk with $\tau \in [0, 1]$. This provides some insight to the absolute stability problem. In particular it is *necessary* for absolute stability that the Lurye system be stable with any such linear gain. The Nyquist value k_N is the maximum value of k for which this holds:

Definition 2 (Nyquist value) *Let $G \in \mathbf{RH}_\infty$ with Rosenbrock system matrix (17). The Nyquist value is given by*

$$k_N = \sup \left\{ k > 0 : \quad A - \tau BCk(1 + \tau kD)^{-1} \text{ is Hurwitz for all } \tau \in [0, 1] \right\}. \tag{18}$$

It is tautologous to say that for absolute stability we require $k < k_N$. As a result, the inverse of the linear system $(1 + \tau kG)$ needs to be bounded for all $\tau \in [0, 1]$. This fact implies that for absolute stability to hold the phase of the system $(1 + kG)$ must be within the interval $(-180^o, 180^o)$.

Kalman (1957) made the conjecture that consideration of feedback with linear gain was also *sufficient* for absolute stability:

Kalman Conjecture (Kalman, 1957) *Let ϕ be a memoryless nonlinearity slope-restricted on $S \in [0, k]$. Then, the Lurye system in Fig. 5 is asymptotically stable if $A - \tau BCk(1 + \tau kD)^{-1}$ is Hurwitz for all $\tau \in [0, 1]$.*

The conjecture has played an important part in the development of the absolute stability of feedback systems containing slope-restricted nonlinearites. It is true for first, second and third-order continuous-time systems (Barabanov, 1988). Thus we know *a priori* that a third order system is absolutely stable provided $\phi \in S[0, k_N)$ and we can benchmark a test for stability by seeking the maximum slope value and comparing with this upper bound (e.g Safonov and Wyetzner, 1987; Carrasco et al., 2014b). But the conjecture is false in general and the fourth-order counterexamples proposed more than 40 years ago (Fitts, 1966; O'Shea, 1967; Willems, 1971; Leonov and Kuznetsov, 2013) can also be used as benchmarks as they can be very challenging for stability tests. We illustrate such a benchmark in this paper.

3.3. Passivity, loop transformations and multipliers

Passivity theory provides an important stability test for closed-loop systems. Conditions for stability are simplified since one element of the Lurye system (13) is LTI stable. We can assume $r_1 = 0$ without loss of generality. It is sufficient for closed-loop stability that ϕ be passive and G be strictly input passive (SIP). A stable operator $\phi : \mathcal{L}_2 \to \mathcal{L}_2$ is said to be passive[2] if there exists some $\beta \leq 0$ such that

$$\langle \phi u, u \rangle \geq \beta \text{ for all } u \in \mathcal{L}_2.$$

A stable LTI system G is SIP (Brogliato et al., 2006) if and only if there is a $\delta > 0$ such that

$$\text{Re}\{G(j\omega)\} \geq \delta \text{ for all } \omega.$$

If the nonlinearity ϕ is sector bounded on the interval $[0, k]$ then the map from $\tilde{u}_2 = u_2 - y_2/k$ to y_2 is passive. But the system shown in Fig. 6 is stable if and only if our original Lurye system is stable. Hence, via a loop transformation argument, it is sufficient for stability for $G + 1/k$ to be SIP. This is the Circle Criterion.

Similarly, suppose M (a "multiplier") is biproper transfer function whose zeros and poles are all in the left half plane. Then the system shown in Fig. 7 is stable if and only if our original Lurye system is stable. If ϕM^{-1} is passive, then it suffices for stability that MG be SIP.

3.4. Zames-Falb theorem

O'Shea (1966, 1967) proposed a set of multipliers appropriate for slope-restricted nonlinearities. This included an extension to *noncausal* multipliers. The machinery was formalised by Zames and Falb (1968) in their seminal paper.

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[2]General definitions of passivity are given by Brogliato et al. (2006) and Khalil (2002).

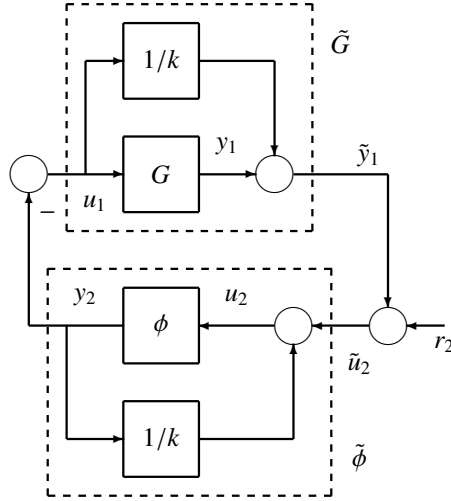Figure 6: Loop transformation. If the map $\tilde{\phi}$ from $\tilde{u}_2 = u_2 - y_2/k$ to y_2 is passive, then it suffices for stability to test whether $\tilde{G} = G + 1/k$ is strictly input passive, where $\tilde{y}_1 = y_1 + u_1/k$
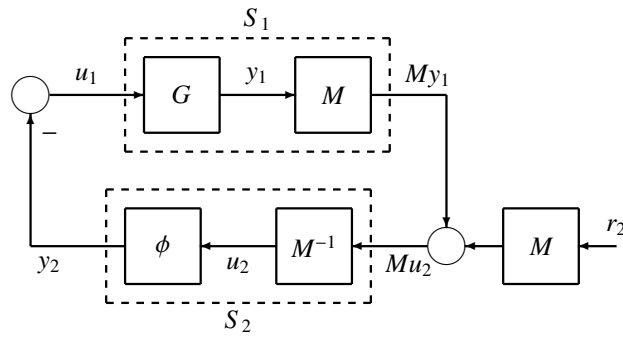

Figure 7: Multiplier theory. If the map $S_1 = \phi M^{-1}$ is passive then it suffices for stability to test whether $S_2 = MG$ is strictly input passive.

Theorem 1 (Zames and Falb, 1968) *Consider the feedback system in Fig. 5 with $G \in \mathbf{RH}_\infty$, and a slope-restricted nonlinearity $\phi \in S[0, k]$. Assume that the feedback interconnection is well-posed. Then suppose that there exists a convolution operator $M : \mathcal{L}_2(-\infty, \infty) \to \mathcal{L}_2(-\infty, \infty)$ whose impulse response is of the form*

$$m(t) = \delta(t) - \sum_{i=1}^{\infty} h_i \delta(t - t_i) - h(t), \tag{19}$$

where δ is the Dirac delta function and

$$\sum_{i=1}^{\infty} |h_i| < \infty, \quad h \in \mathcal{L}_1, \quad and \quad t_i \in \mathbb{R} \; \forall i \in \mathbb{N}. \tag{20}$$

Assume that:

(i)

$$\|h\|_1 + \sum_{i=1}^{\infty} |h_i| < 1, \tag{21}$$

(ii) either $h(t) \geq 0$ for all $t \in \mathbb{R}$ and $h_i \geq 0$ for all $i \in \mathbb{N}$, or ϕ is odd; and
(iii) there exists $\delta > 0$ such that

$$Re\{M(j\omega)(1 + kG(j\omega))\} \geq \delta \quad \forall \omega \in \mathbb{R}. \tag{22}$$

Then the feedback interconnection (13) is $\mathcal{L}_2$-stable. ∎

The corresponding class of multipliers is known as the class of Zames-Falb multipliers.

Definition 3 *The class of Zames-Falb multipliers $\mathcal{M}$ is given by all transfer functions $M \in \mathbf{L}_\infty$ whose inverse Laplace transform[3] is given by*

$$m(t) = \delta(t) - \sum_{i=1}^{\infty} h_i \delta(t - t_i) - h(t), \tag{23}$$

where

$$\|h\|_1 + \sum_{i=1}^{\infty} |h_i| < 1. \tag{24}$$

The above class will be used for slope-restricted and odd nonlinearities. If the nonlinearity is non-odd, only a subclass of multiplier can be used.

Definition 4 *The class of positive Zames-Falb multipliers $\mathcal{M}_+$ is given by all transfer function $M \in \mathcal{M}$ such that the inverse Laplace transform (23) satisfies that $\mathcal{L}^{-1}(1 - M) = h(t) \geq 0$ for all $t \in \mathbb{R}$ and $h_i \geq 0$ for all $i \in \mathbb{N}$.*

Although such definitions may appear formidable at first sight, it is usual to consider only subclasses. Most searches are restricted to rational Zames-Falb multipliers (the class $\mathcal{RM}$), where $h_i = 0$ for all i and $M \in \mathbf{RL}_\infty$. An exception is the search of Safonov and Wyetzner (1987) where instead $h(t) = 0$ for all t, so the multiplier is a sum of delayed impulses.

In addition, if M is a Zames-Falb multiplier we can always find a factorization $M = M_a M_c$ where $M_c, M_c^{-1} \in \mathbf{H}_\infty$ and $M_a, M_a^{-1} \in \mathbf{H}_\infty^-$, i.e. $M_a^*, (M_a^*)^{-1} \in \mathbf{H}_\infty$. This is the cornerstone of Zames and Falb (1968) to formalise the use of the class of multiplier proposed by O'Shea (1967). In the jargon, this factorization is referred to as a canonical factorization (see Carrasco et al., 2012b, and references therein).

It is emphasized that the causality assumption of the real systems G and ϕ is not required on the multiplier, since it is just a mathematical "device". Hence M is not required to be causal. It is required to be bounded in the sense that its impulse response has finite $\mathcal{L}_1$-norm (24). In particular this ensures M can be factorized into a causal and bounded operator M_c and an anticausal and bounded operator M_a. For LTI systems, the use of the bilateral Laplace transform leads to duality properties[4]. Loosely speaking, if a system is assumed to be causal, $M \in \mathbf{H}_\infty^-$ means that the impulse
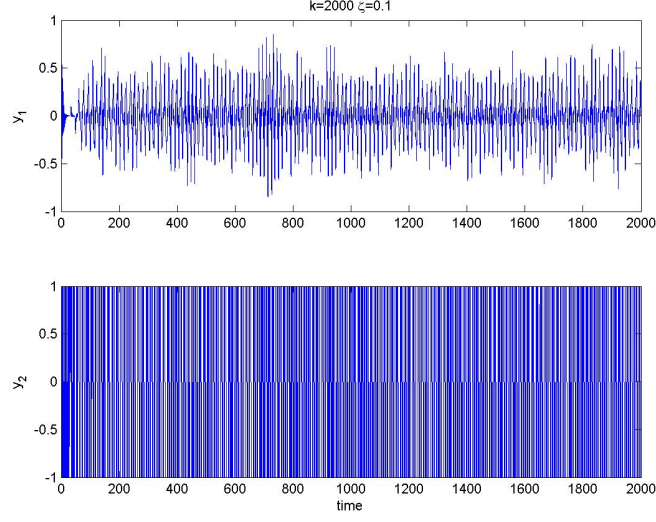
Figure 8: Combination of the loop transformation in Fig. 6, multiplier approach in Fig. 7, and factorization $M = M_a M_c$. All LTI blocks are in $\mathbf{H}_\infty$. The structure is used to justify noncausal multipliers.

response of the system is unbounded; if a system is assumed to be bounded, $M \in \mathbf{H}_\infty^-$ means that the impulse response $m(t)$ is zero for all $t > 0$.

It can be shown that the $\mathcal{L}_1$ norm condition (21) on M and the slope-restriction on ϕ ensures

$$\langle y_2, M\tilde{u}_2 \rangle \geq 0, \tag{25}$$

This guarantees the block S_2 in Fig. 7 to be positive. Similarly, the phase property on $M(G + 1/k)$ (22) ensures

$$\langle M\tilde{y}_1, u_1 \rangle \geq 0. \tag{26}$$

Nevertheless, stability cannot be ensured since M is not causal. Zames and Falb used the canonical factorization $M = M_a M_c$ to show stability. The properties of the inner product in (25) and (26) mean we can write

$$\langle M_a^* y_2, M_c \tilde{u} \rangle \geq 0, \text{ and } \langle M_c \tilde{y}_1, M_a^* u_1 \rangle \geq 0. \tag{27}$$

As a result, both the blocks S_1 and S_2 in Fig. 8 are stable and positive, and hence passive. Therefore, the feedback interconnection between S_1 and S_2 is stable (by passivity) and equivalent to our original Lurye problem.

4. O'Shea's example

Our standard problem is the Lurye problem (13) depicted in Fig. 5. The nonlinearity ϕ from u_2 to y_2 is static and slope-restricted to the interval $S[0, k]$.

Brockett and Willems (1965) suggested plants with the structure

$$G(s) = \frac{s^2}{s^4 + as^3 + bs^2 + c + d}, \tag{28}$$

would be challenging to analyse. O'Shea (1967) chose a subclass of the form

$$G(s) = \frac{s^2}{(s^2 + 2\zeta s + 1)^2}, \tag{29}$$

where the symmetry aids both intuitive understanding and the ability to find solutions by hand. The symmetry is given by $G(j\omega) = G^*(j\omega^{-1})$. This turns out to be a challenging feature for several classes of multipliers.

If the nonlinearity ϕ is replaced with a linear gain k, then the loop is stable for all k and for all $0 < \zeta \leq 1$; i.e. the Nyquist value is infinite when ζ is in this range (the phase never drops below -180°, see Fig. 12). But if ϕ is a saturation

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[3] Since $m : \mathbb{R} \to \mathbb{R}$, the bilateral Laplace transform is required.
[4] It also leads to intrinsic difficulties (Georgiou and Smith, 1995).

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Figure 9: Simulated values for signals y_1 and y_2 when $k = 2000$ and $\zeta = 0.1$. The loop is apparently unstable.
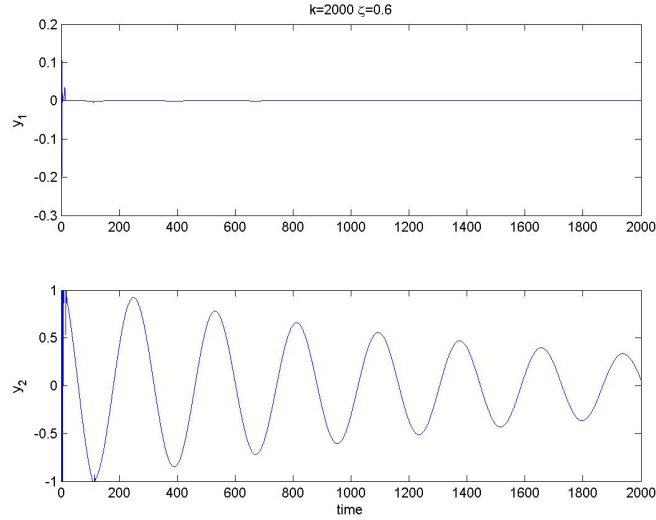

Figure 10: Simulated values for signals y_1 and y_2 when $k = 2000$ and $\zeta = 0.6$. The loop is apparently stable, and O'Shea's analysis proves it to be.

block in series with a gain k, then it is possible to find values of k and ζ that are apparently unstable. For example, Fig. 9 shows the result generated in Simulink when $\zeta = 0.1$ and $k = 2000$. **Such phenomena were first observed by Fitts (1966), and have attracted much attention as counterexamples to the Kalman conjecture. Barabanov (1988) questioned the validity of Fitts' original counterexample; this has led to considerable discussion (Leonov and Kuznetsov, 2011; Kuznetsov et al., 2011; Leonov and Kuznetsov, 2013)**. O'Shea (1967) showed that such loops could be guaranteed stable for all $k > 0$ provided $1/2 < \zeta \leq 1$. For most of our discussion we will fix $\zeta = 0.6$. Fig. 10 shows such stable behaviour generated in Simulink when $\zeta = 0.6$ and $k = 2000$.

In the following subsections, we will consider how various standard criteria can be used to judge stability. In particular we will be able to associate a particular range of k for each criterion where stability can be guaranteed.

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Figure 11: The $\mathbf{H}_\infty$ norm of G is 0.6944.

4.1. Passivity

In our problem the nonlinearity ϕ from u_2 to y_2 is passive. It would therefore suffice for the phase of G to lie on the interval $(-90^o, +90^o)$. However G is *not* passive; its phase approaches $+180^o$ at low frequency and -180° at high frequency (Fig. 12). Hence the passivity theorem cannot be used directly to establish stability for any $k > 0$.

4.2. Small gain theorem

The $\mathcal{L}_2$ gain of the nonlinearity from u_2 to y_2 is k. That is to say, for any $u_2 \in \mathcal{L}_2$, we must have $\|y_2\|_2 \leq k\|u_2\|_2$. It follows by the small gain theorem that the loop is guaranteed stable provided

$$k\|G\|_\infty < 1.$$

The $\mathbf{H}_\infty$ norm of G is $\|G\|_\infty = 0.6944$. Hence we may conclude the loop is stable for

$$k < \frac{1}{0.6944} = 1.44.$$

4.3. Circle Criterion

Although the passivity theorem cannot be invoked directly, it can be used indirectly. The nonlinearity ϕ is sector bounded; it follows we can use a loop transformation and apply the Circle Criterion (Fig. 6). It is thus sufficient for $G + 1/k$ to be SIP for stability. For our example we find (Fig. 12)

$$\text{Re}\,\{G(j\omega)\} > -0.0868 \text{ for all } \omega.$$

It follows that the loop is stable provided
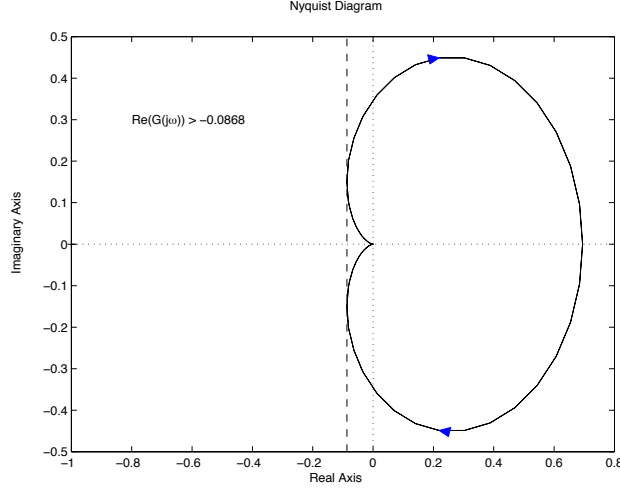
$$k < \frac{1}{0.0868} = 11.52.$$

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Figure 12: Nyquist plot of G when $\zeta = 0.6$. The plot never encircles the -1 point, so the Nyquist value is infinite. The phase drops from $+180^o$ to -180°, so G is not passive. The Nyquist plot of G is always to the right of -0.0868, so the Circle Criterion confirms stability for $k < 1/0.0868$.
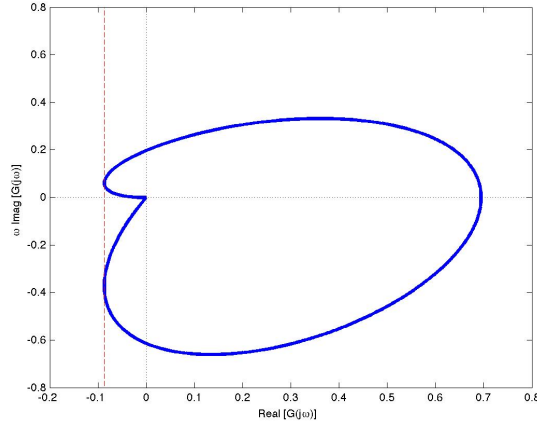

Figure 13: Popov plot for G when $\zeta = 0.6$

4.4. Popov Criterion

For the Popov Criterion we test whether $M(G + 1/k)$ is SIP where M is a Popov multiplier of the form

$$M(s) = 1 + \eta s \text{ with } \eta \in \mathbb{R}. \tag{30}$$

This is a standard and well-known result, although the case with $\eta < 0$ is often ignored (Khalil, 2002). In fact it can be derived as a corollary of the Zames-Falb theorem (Brockett and Willems, 1965; Carrasco et al., 2013).

One might naïvely expect the Popov Criterion to offer an improvement over the Circle Criterion. However for this example the symmetry of G ensures this is not the case. The Popov plot in Fig. 13 provides a result no better than the Circle Criterion and shows that the implicit Popov multiplier is $1 + 0s$, since the dashed line is vertical. The reason is simple: suppose $k > 11.52$ (the maximum k for which the Circle Criterion guarantees stability). There is a frequency interval where the phase of $1 + kG(j\omega)$ is greater than $+90^o$ and a frequency interval where it is less than -90°. See Fig. 14 for the case $k = 15$. Any Popov multiplier that raises the phase at high frequency (i.e. with positive coefficient)

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cannot reduce the phase at low frequency. Conversely, any Popov multiplier that reduces the phase at low frequency cannot raise the phase at high frequency. In brief, if $k > 11.52$ and given some $\eta \in \mathbb{R}$, there must exist some frequency ω where

$$|\angle \{(1 + \eta j\omega)(1 + kG(j\omega)\}| > 90^o.$$

A typical result with η positive is shown in Fig. 15; a negative η would result in a similar but opposite effect.

In the sixties and seventies, several other frequency domain conditions based on stability multipliers were tested using graphical interpretations; see Narendra and Taylor (1973) as a classical textbook on this topic and Section 3.4.1 in Altshuller (2013) for a recent overview. In particular, Table 3.1 in Altshuller (2013) can be used to show that the symmetry of the phase prevents other classes of multipliers, such as the Yakubovich multipliers (Yakubovich, 1965), the RL multipliers (Brockett and Willems, 1965) or the RC multipliers (Brockett and Willems, 1965), from improving on the Circle Criterion for this example. Similarly the Off-axis Circle Criterion Cho and Narendra (1968) uses either RL or RC multipliers. Note that Park (2002) provides a convex search for the Yakubovich multipliers.
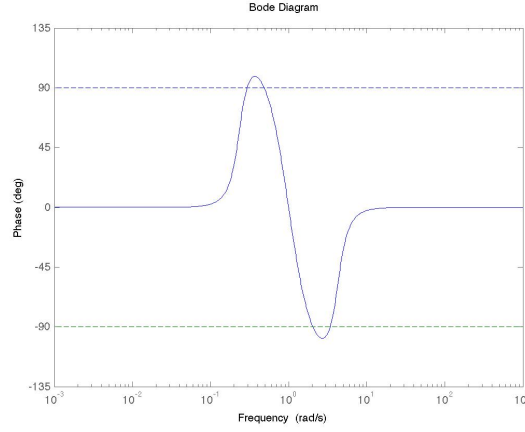

Figure 14: Phase of $1 + 15G(j\omega)$. There are value of $\omega < 1$ where the phase is greater than $+90^o$ and values of $\omega > 1$ where the phase is less than -90°.
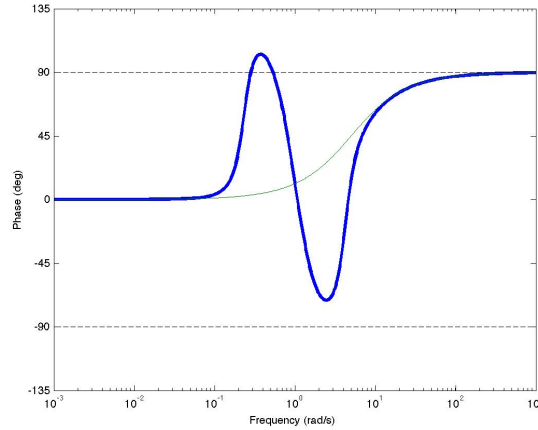

Figure 15: In blue, phase of $(1 + 0.2j\omega)(1 + 15G(j\omega))$. In green, the phase of $(1 + 0.2j\omega)$. The multiplier raises the phase at all frequencies.
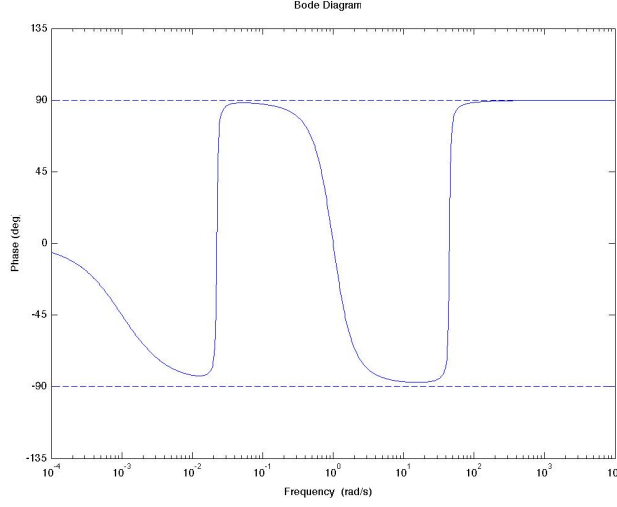
13

Figure 16: Phase of $M(j\omega)(G(j\omega) + 1/k)$ with $k = 2000$ and $p = 0.001$. The phase is always between -90° and $+90^o$.

4.5. O'Shea's multiplier

O'Shea (1967) proposed the multiplier

$$M(s) = \frac{(s + 1)(-s + p)}{-s + 1}, \tag{31}$$

with $p > 0$ sufficiently small. This is sufficient to ensure the phase of $M(G + 1/k)$ lies above -90° and below $+90^o$ as in see Fig. 16. This in turn is sufficient for stability *even though the multiplier has a pole in the right half plane, i.e. the multiplier is noncausal*.

In particular, the existence of O'Shea's multiplier M with the property that $M(G + 1/k)$ has phase on the interval $(-90^o, 90^o)$ guarantees the existence of a Zames-Falb multiplier (Definition 3) satisfying the conditions of the Zames-Falb theorem (Theorem 1). Hence the existence of O'Shea's multiplier is sufficient for stability.

The multiplier M suggested by O'Shea (31) itself is not within the class of Zames-Falb multipliers $\mathcal{M}$. We can write

$$\frac{1}{2 - p} M(s) = M_{ZF}(s) + \eta s, \tag{32}$$

with

$$M_{ZF}(s) = 1 - \frac{2 - 2p}{2 - p} \frac{1}{-s + 1} \text{ and } \eta = \frac{1}{2 - p}.$$

Since $M_{ZF} \in \mathcal{M}_a$ we can write M as the sum of a Zames-Falb mutliplier and a Popov term ηs. We require a phase-equivalence result (Carrasco et al., 2013): if $M(G + 1/k)$ has phase in $(-90^o, +90^o)$ and M can be written $M(s) = M_{ZF}(s) + \eta s$ with $M_{ZF} \in \mathcal{M}$, then there exists a phase-equivalent $M_{PE} \in \mathcal{M}$ such that $M_{PE}(G + 1/k)$ has phase in $(-90^o, +90^o)$. In this case, such a M_{PE} can be constructed as follows. Put $q = (2 - 2p)/(2 - p)$ and choose $\rho > 0$ such that $q < 1 - \rho$, for example $\rho = p/(4 - 2p)$. We can then write

$$\frac{1}{2 - p} M(s) = 1 - \frac{q}{-s + 1} + \eta s = \left(1 - \rho - \frac{q}{-s + 1} \right) + \rho \left(1 + \frac{\eta}{\rho} s \right),$$

Then we can write

$$M_{PE}(s) = \left(1 - \rho - \frac{q}{-s + 1} \right) + \rho \left(\frac{1 + \eta s/\rho}{1 + \varepsilon s} \right),$$
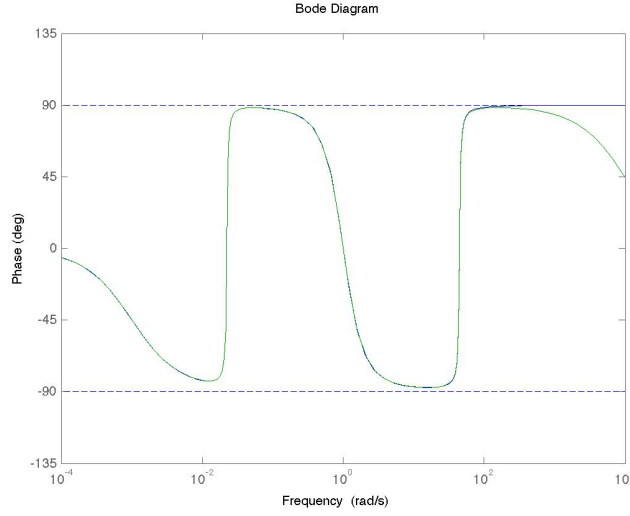
14

Figure 17: Phases of $M(j\omega)(G(j\omega) + 1/k)$ and $M_{PE}(j\omega)(G(j\omega) + 1/k)$ where M is O'Shea's multiplier and M_{PE} is a phase-equivalent Zames-Falb multiplier. The parameter ε is chosen as 10^{-4}. As O'Shea's multiplier includes a Popov term (with positive parameter η) the corresponding phase tends to $+90^o$ at high frequency, while that for M_{PE} tends to 0°.

for $\varepsilon > 0$ sufficiently small. The phases of $M(G + 1/k)$ and $M_{PE}(G + 1/k)$ are compared in Fig. 17.

In short, the noncausal multiplier (31) can be used to guarantee the absolute stability of our example for *any* positive k provided $p > 0$ is chosen sufficiently small. A similar analysis guarantees stability for any positive k provided the damping ratio $\zeta > 0.5$. A formal proof requires the concept of *phase-equivalence* (Carrasco et al., 2013, 2014a) which we discuss further in the next section.

5. Further properties

The class of Zames-Falb multiplier is one of the possible classes that have been proposed for analysing the stability of the Lurye system (13). The aims of this section are to discuss the phase properties of the Zames-Falb multipliers and the equivalence between different classes of multipliers, where it will be shown that phase is a key factor.

5.1. Positivity

One trivial property of the multiplier is that it must be a positive system. By definition, a multiplier is required to preserve the positivity of the class of nonlinearities. As mentioned by Carrasco et al. (2012b), when a scaled identity is within the class of nonlinearities, then the multiplier itself needs to be positive. Referring to Fig. 8, this can easily be seen because

$$\langle M_a^* \tilde{\phi} \tilde{u}_2, M_c \tilde{u}_2 \rangle = \langle \tilde{\phi} \tilde{u}_2, \underbrace{M_a M_c}_{=M} \tilde{u}_2 \rangle \geq 0,$$

where $\tilde{u}_2 = u_2 - y_2/k$. Thus if we consider the particular case $\tilde{\phi} \tilde{u}_2 = k\tilde{u}_2$ with $k > 0$, then

$$\langle \tilde{\phi} \tilde{u}_2, M\tilde{u}_2 \rangle = \langle k\tilde{u}_2, M\tilde{u}_2 \rangle \geq 0,$$

and hence

$$\langle \tilde{u}_2, M\tilde{u}_2 \rangle \geq 0,$$

for all $\tilde{u}_2 \in \mathcal{L}_2$.

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The phase of the multiplier is required to be within the interval $[-90^o, 90^o]$. However, we cannot consider this as a limitation of the multiplier as the phase of $(1 + kG)$ must belong to the interval $(-180^o, 180^o)$ to satisfy the necessity of the Kalman Conjecture.

In the Nyquist diagram, we can find further restrictions on the Nyquist plot of the multiplier $M \in \mathcal{M}$. In particular, we can use $\mathcal{L}_1$ norm properties to ensure that the Nyquist plot belongs to a circle with centre $(1, 0)$ and radius 1 (Safonov and Kulkarni, 2000); see Fig. 18. Loosely speaking, it is due to the fact that the $\mathcal{L}_1$ norm of a system is always greater than its $\mathbf{H}_\infty$ norm. See Vidyasagar (1993) for further details.
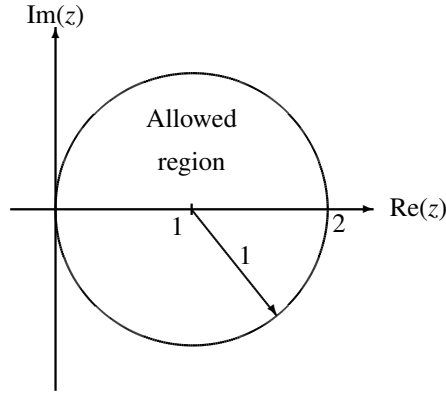

Figure 18: Allowed region for the Nyquist plot of Zames-Falb multipliers (see further details in Carrasco et al., 2012a)

5.2. Noncausal multipliers

Undoubtedly, O'Shea's main contribution to multiplier theory was the introduction of noncausal multipliers (see Desoer and Vidyasagar, 1975, page 227). The motivation was to increase the flexibility of the phase of the multiplier. In this section, we demonstrate this concept. An analytic result can be obtained for rational first order Zames-Falb multipliers.

Lemma 1 *Given $\epsilon > 0$, there exists a first-order causal Zames-Falb multiplier such that its phase is $90° - \epsilon$ at some frequency. However, if M_c is a causal rational first-order Zames-Falb multiplier, then $\angle M_c(j\omega) > -\arcsin(1/3)$ for all $\omega \in \mathbb{R}$.*
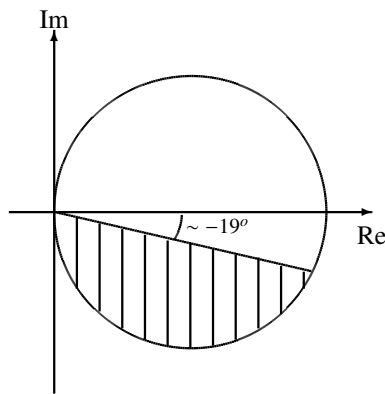

Figure 19: A fist order multiplier cannot reach a phase below $-19°$. A symmetric figure can be drawn for anticausal multipliers.

Lemma 2 *Given $\epsilon > 0$, there exists an anticausal first-order Zames-Falb multiplier such that its phase is $-90° + \epsilon$ at some frequency. However, if M_{ac} is an anticausal rational first-order Zames-Falb multiplier, then $\angle M_{ac}(j\omega) < \arcsin(1/3)$ for all $\omega \in \mathbb{R}$.*

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The proofs of these results are straightforward, but they show the significant reduction on the selection of the phase of the multiplier if we limit ourselves solely to causal or anticausal multipliers for a fixed order. However, it can be easily shown that a causal multiplier can reach any phase by considering an infinite dimensional multiplier.

Lemma 3 *Given* $\theta \in (-90°, 90°)$, *there exist causal or anticausal Zames-Falb multipliers with phase* θ.

Once again the result is trivial by using the multiplier $M(s) = 1 + z_1 e^{\pm s}$. Hence one could be tempted to think that there is no phase limitation if the order is infinite.

5.3. Phase limitations of Zames-Falb multipliers

It has been established that the conditions on the Zames-Falb multiplier require some limitation in the selection of the phase[5]. The lack of such limitation would imply that we can make any biproper plant appear passive if its phase is within the interval $(-180°, 180°)$; hence the Kalman conjecture would be true. As the Kalman conjecture is known to be false, the $\mathcal{L}_1$ condition on the multiplier must have an interpretation as a phase limitation. One such characterization of a limitation is given by Megretski (1995).

Lemma 4 (Megretski, 1995) *Let* $b > a > 0$, $r > b/a$ *be real numbers. Let*

$$\rho = \rho(a, b, r) = \sup_{t>0} \frac{|\psi(t)|}{\phi(t)},$$

where

$$\psi(t) = -\frac{\cos(rbt)}{rt} + \frac{r\cos(rbt)}{t} - \frac{r\cos(rat)}{t} + \frac{\cos(rat)}{rt},$$

and

$$\phi(t) = (r + 1)(b - a) + \frac{r\sin(at)}{t} - \frac{r\sin(bt)}{t} + \frac{\sin(rat)}{rt} + \frac{\sin(rat)}{rt},$$

Then $\rho < \infty$ *and there exist no multiplier* $M(s) \in \mathcal{M}_+$ *such that*[6]

$$\angle(M(j\omega)) > \tan^{-1}\rho, \omega \in [a, b]$$

and

$$\angle(M(j\omega)) < -\tan^{-1}\rho, \omega \in [a, b]$$

Remark 2 *A symmetric result, i.e. negative phase at low frequency and positive phase at high frequency, can be straightforwardly developed.*

If it has been shown that there is no limitation in the phase of a causal or anticausal multiplier, Megretski's result shows that there is a limitation based on the rate of change of the phase. It can be easily shown with O'Shea's example. Let us consider $k = \infty$, then the required phase properties of the multiplier are presented in Table 1.

As O'Shea (1967) mentions, the proposed multiplier ensures stability for any gain for $\zeta > 0.5$. For $\zeta < 0.5$, the problem remains unsolved, and searches can be tested with this example. However, Fig. 9 shows that no multiplier can ensure stability for large k when $\zeta = 0.1$. So the limiting factor for the phase of multiplier is not the phase itself, but *how fast it changes*. So, for $\zeta = 0.5$ there exists a multiplier able to change its phase from -79° up to 79° in two decades; but for $\zeta = 0.1$, there is no Zames-Falb multiplier which can change its phase form -88° up to 88° in two decades, preserving the properties for the rest of frequencies.

The analysis of Megretski (1995) is not definitive. It is clear that restrictions to subclasses of Zames-Falb multipliers, such as the causal multipliers, impose further limitations in phase. However no analytic result has yet been provided. Fig. 21 shows the phase of two causal but irrational multipliers. Their phase spans the interval $(-90^o, +90^o)$ with very fast transitions from -90° up to $+90^o$ but slow transitions from $+90^o$ down to -90°.

---

[5]For a further discussion on the phase of Zames-Falb multipliers see Freedman (1972).
[6]The original results is given for all the Zames-Falb multipliers. However, it holds only if $h(t) \geq 0$ for all t. This is discussed in a conference paper under submission.
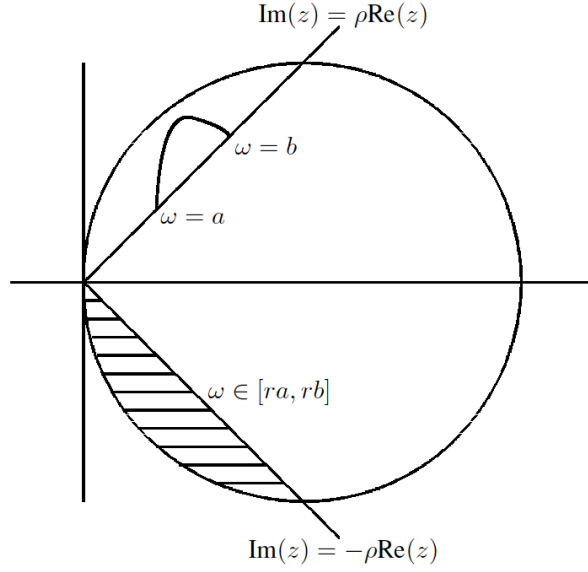
Figure 20: Megretski's phase limitation for Zames-Falb multipliers. If the phase of a multiplier is above $\tan^{-1}\rho$ for a range of frequencies (a,b), then the multiplier cannot reach a phase below $-\tan^{-1}\rho$ over the range of frequencies (ra,rb).

ζ	$-\infty < \omega < 10^{-1}$	$10^{-1} < \omega < 10^{1}$	$10^{1} < \omega < \infty$
0.9	$(-90^o, -70^o)$	$(-70^o, 70^o)$	$(70^o, 90^o)$
0.7	$(-90^o, -74^o)$	$(-74^o, 74^o)$	$(74^o, 90^o)$
0.5	$(-90^o, -79^o)$	$(-79^o, 79^o)$	$(79^o, 90^o)$
0.3	$(-90^o, -83^o)$	$(-83^o, 83^o)$	$(83^o, 90^o)$
0.1	$(-90^o, -88^o)$	$(-88^o, 88^o)$	$(88^o, 90^o)$

Table 1: Phase properties of a multiplier in order to show that O'Shea example is stable for any slope

5.4. Equivalences

The first equivalence result between classes of multiplier was given by Falb and Zames (1968). In this paper, they show that given any RL and RC multiplier (for definitions, see Falb and Zames, 1968; Brockett and Willems, 1965), a Zames-Falb multiplier with the same phase can be found. Then it is no longer important that the class of Zames-Falb multipliers does not include some RL and RC multipliers, since we can always find an "substitute" in the class.

A formal definition is required. However, we need to limit our set of interest. It is a key step in order to be able to establish formal substitution and equivalence results. As we have mentioned, the necessity of the Kalman conjecture is required in order to ensure absolute stability; hence we will restrict our attention to plants where this property is required.

Definition 5 *The set $\mathcal{SR}$ is given by the plants $\tilde{G} = 1 + kG$ with the following properties*

- *$1 + kG$ is stable.*

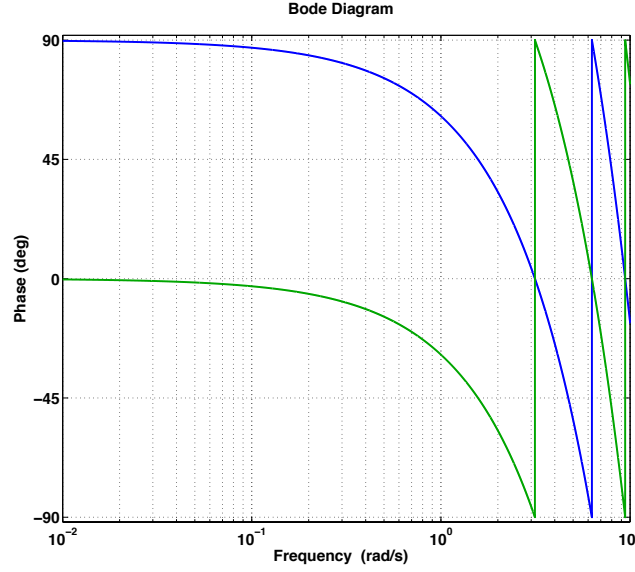- *$(1 + \tau kG)^{-1}$ is stable for any $\tau \in [0, 1]$.*

Figure 21: Phase of the multiplier $1 \pm e^{-s}$.

Loosely speaking, given some plant G and maximum slope k, then $(1 + kG) \notin \mathcal{SR}$, the existence of a Zames-Falb multiplier such that $(1 + kG)M$ is positive can be dismissed. Once we have introduced this class of LTI system, then formal definitions can be given.

Definition 6 (Phase-substitute) *Let M_a and M_b be two multipliers and $\widehat{G} \in \mathcal{SR}$. The multiplier M_b is a phase-substitute of the multiplier M_a when*

$$Re\left\{M_a(j\omega)\widehat{G}(j\omega)\right\} \geq \delta_1 \quad \forall \omega \in \mathbb{R},$$

for some $\delta_1 > 0$ implies

$$Re\left\{M_b(j\omega)\widehat{G}(j\omega)\right\} \geq \delta_2 \quad \forall \omega \in \mathbb{R},$$

for some $\delta_2 > 0$.

This property has already been used in Section 4.5, and the phase-substitution of multipliers with the addition of the Popov term has been explained. Another simple but insightful equivalence is described as follows:

Definition 7 (Park's multipliers, Park, 2002) *The class of Park's multipliers is defined as follows:*

$$M_P(s) = 1 + \frac{bs}{-s^2 + a^2}, \tag{33}$$

for any two scalars a and b.

It is straightforward to show that Park's multipliers can be linked with the class developed by Yakubovich (1965). However, a more detailed analysis will show that we can find a Zames-Falb multiplier with the same phase properties.

Firstly, let us find the zeros of a Park multiplier. If $b > 0$, then the zeros will be labelled as follows

$$
\begin{aligned}
z_1 &= \frac{b - \sqrt{b^2 + 4a^2}}{2}, \\
z_2 &= \frac{b + \sqrt{b^2 + 4a^2}}{2};
\end{aligned}
$$

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whereas if $b < 0$ we will use

$$z_1 = \frac{b + \sqrt{b^2 + 4a^2}}{2},$$

$$z_2 = \frac{b - \sqrt{b^2 + 4a^2}}{2}.$$

This ensures that $|z_1| < |z_2|$. Note that $z_1 z_2 < 0$.

Secondly, the phase of $M_P(j\omega)$ is given by the phase of its zeros, since the phase of its poles cancels out, i.e.

$$\angle(M_P(j\omega)) = \angle(j\omega - z_1) + \angle(j\omega - z_2). \tag{34}$$

Finally, the zero with larger absolute value, z_2 can be transformed into a pole reflected in the imaginary axis since $\angle(j\omega - z_2) = -\angle(j\omega + z_2)$. As a result, the phase of any Park multiplier is given by

$$\angle(M_P(j\omega)) = \angle(j\omega - z_1) - \angle(j\omega + z_2) = \angle\left(\frac{j\omega - z_1}{j\omega + z_2}\right). \tag{35}$$

So we have found a phase-substitute multiplier that can be rewritten as follows:

$$M(s) = \frac{s - z_1}{s + z_2} = 1 - \frac{z_2 + z_1}{s + z_2}. \tag{36}$$

Then, we can check that $M \in \mathcal{M}$ since

$$\|m\|_1 = \frac{z_2 + z_1}{z_2} < 1, \tag{37}$$

where m is the inverse Laplace transform of

$$\frac{z_2 + z_1}{s + z_2}$$

and we have used that $z_1 z_2 < 0$ and $|z_1| < |z_2|$.

This example and the previous example illustrate the concept of phase-substitution. Using this concept, it can be shown that a search over the whole class of Zames-Falb multipliers would be enough to obtain the best possible result compared with any other class of multipliers in the literature (Carrasco et al., 2013). Nevertheless the significant difficulties in obtaining a convex search over the whole class of Zames-Falb multipliers mean that the parameterizations of other classes of multipliers may still be useful. We discuss searches in the following section.

6. Convex Searches

Section 4 has illustrated how various stability tests can be used to find the maximum slope of the nonlinearity ϕ for which stability is guaranteed. This, and other sections, have also illustrated how the selection of an appropriate Zames-Falb multiplier can enable more accurate statements regarding the absolute stability of a Lurye system to be made. Using an example from O'Shea (1967) it has been shown how the selection of such a multiplier may be achieved for a relatively simple system. For more complex systems, it is somewhat more difficult to choose the most "appropriate" Zames-Falb multiplier because the set of such multipliers is extremely large and, in fact, infinite dimensional. Typically, we would like to choose a multiplier which allows us to make the least conservative statements about, for example:

1. the size of slope for which stability is guaranteed;
2. the $\mathcal{L}_2$ gain from a given input to a given output (Turner and Kerr, 2011; Megretski et al., 2004).

Choosing such a multiplier which either maximises the slope size or minimises the $\mathcal{L}_2$ gain is not trivial. In this section, several automated searches for rational ($M \in \mathcal{RM}$) multipliers are introduced. The searches described are based on Chen and Wen (1995, 1996), which is similar to that implemented in the IQC toolbox (Megretski et al., 2004; Kao et al., 2004; D'Amato et al., 2001), and also more recent approaches of the authors (e.g. Turner et al., 2009b; Carrasco et al., 2014b). The technique of Safonov and Wyetzner (1987) and Gapski and Geromel (1994), which has recently been updated (Chang et al., 2012) is briefly discussed in Section 6.7.

6.1. Linear search in k

Modern searches for multipliers are somewhat different to classical graphical criteria (see Narendra and Taylor, 1973). In graphical criteria with simple multipliers (e.g. Circle, Popov), k_{max} is found directly via a plot of $G(j\omega)$, even though an auxiliary multiplier is implicit (See Fig. 13).

The search for more sophisticated multipliers requires a different approach. In this case, a linear search over k is carried out. Given a value k_f, then a search over $\mathcal{M}$ is carried out to find a suitable multiplier for $\tilde{G} = G + 1/k_f$, i.e. we search for a multiplier $M \in \mathcal{M}$ such that

$$\text{Re}\{M(j\omega)(1/k + G(j\omega))\} \geq 0, \tag{38}$$

for all $\omega > 0$. If the search is successful, the multiplier found is suitable for any $k < k_f$, i.e.

$$\text{Re}\{M(j\omega)(1/k + G(j\omega))\} = \text{Re}\{M(j\omega)/k\} + \text{Re}\{M(j\omega)G(j\omega)\} \geq$$
$$\text{Re}\{M(j\omega)/k_f\} + \text{Re}\{M(j\omega)G(j\omega)\} > 0 \quad \forall \omega \in \mathbb{R}, \tag{39}$$

where we have used that $\text{Re}\{M(j\omega)\} > 0$; hence we can increase k_f. If the search is unsuccessful, we reduce k_f until a successful search is obtained.

Before computational methods were available attempts were made to interpret the Zames-Falb multipliers graphically (e.g. Freedman, 1972; Lipatov, 1981). It must be highlighted that these geometrical methods also require a linear search. In this tutorial, we focus on the development of convex searches, but Section 3.4.1 of Altshuller (2011) provides an overview of such graphical methods (see Altshuller, 2010, for an example).

6.2. Time and Frequency domain conditions

Recall that the system in Fig. 5 is absolutely stable if there exists a multiplier $M = 1 - H \in \mathcal{RM}$ which satisfies the following conditions:

1. a <u>frequency domain</u> condition

$$\text{Re}\{M(j\omega)(1 + kG(j\omega))\} \geq 0 \quad \forall \omega \in \mathbb{R}; \tag{40}$$

2. a <u>time domain</u> condition

$$\|h\|_1 \leq 1. \tag{41}$$

A central issue, which will recur throughout this section, is that of combining, in an efficient and tractable manner, these two conditions. In particular, it is generally difficult to provide a frequency domain characterisation of the $\mathcal{L}_1$ norm, although, as discussed in Section 5, the $\mathcal{L}_1$ norm requirement appears to place a limit on the rate-of-change of phase of the multiplier (Megretski, 1995).

A result which we shall invoke several times in this section is the so-called *Positive Real Lemma* given below. This can be interpreted as a special case of the KYP lemma (Rantzer, 1996).

Lemma 5 (Rantzer, 1996) *Let $G(s)$ be a transfer function with Rosenbrock matrix (17) such that $\det(j\omega I - A) \neq 0$ for all $\omega \in \mathbb{R}$.*

1.

$$G(j\omega)^\star + G(j\omega) \geq 0 \qquad \forall \omega \in \mathbb{R},$$

if and only if (A, B) is controllable and there exists a $P = P'$ such that

$$\begin{bmatrix} A'P + PA & PB - C' \\ \star & -D - D' \end{bmatrix} \leq 0.$$

2.

$$G(j\omega)^\star + G(j\omega) > 0 \qquad \forall \omega \in \mathbb{R},$$

if and only if there exists a $P = P'$ such that

$$\begin{bmatrix} A'P + PA & PB - C' \\ \star & -D - D' \end{bmatrix} < 0.$$

The Positive Real Lemma provides a connection between positive realness in the frequency domain and a matrix inequality. It is of central importance in casting conditions involving Zames-Falb multipliers as LMI's.

$$\left[\begin{array}{cc} \left[\begin{array}{cc} A_p & 0 \\ -\mathbf{B_H}kC_p & \mathbf{A_H} \end{array}\right]' \mathbf{P} + \mathbf{P}\left[\begin{array}{cc} A_p & 0 \\ -\mathbf{B_H}kC_p & \mathbf{A_H} \end{array}\right] & \mathbf{P}\left[\begin{array}{c} B_p \\ \mathbf{B_H}(I-kD_p) \end{array}\right] - \left[\begin{array}{c} kC_p'(I-\mathbf{D_H}) \\ \mathbf{C_H}' \end{array}\right] \\ \star & -(I-\mathbf{D_H})(I-kD_p)-(I-kD_p)(I-\mathbf{D_H}) \end{array}\right] < 0 \qquad (47)$$

6.3. Preliminary Manipulations

The main goal of this section is to translate the positive real condition (40) and the $\mathcal{L}_1$ condition (41) into tractable, automated searches. Two approaches to this will be described, but both ways share some initial manipulation which will be covered here. Assume first that $H(s)$ has state-space realisation

$$H(s) \sim \left[\begin{array}{c|c} \mathbf{A_H} & \mathbf{B_H} \\ \hline \mathbf{C_H} & \mathbf{D_H} \end{array}\right], \qquad (42)$$

where the matrices $\mathbf{A_H, B_H, C_H, D_H}$ are to be determined[7]. Here and elsewhere G has the following state-space realisation

$$G(s) \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right]. \qquad (43)$$

Given these two state-space realisations, it then follows that

$$M(j\omega)(kG(j\omega)+1) \sim \left[\begin{array}{c|c} A_I & B_I \\ \hline C_I & D_I \end{array}\right], \qquad (44)$$

where

$$\left[\begin{array}{c|c} A_I & B_I \\ \hline C_I & D_I \end{array}\right] = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ -\mathbf{B_H}kC_p & \mathbf{A_H} & \mathbf{B_H}(I-kD_p) \\ \hline k(I-\mathbf{D_H})C_p & \mathbf{C_H} & (I-\mathbf{D_H})(I-kD_p) \end{array}\right]. \qquad (45)$$

Our objective then becomes a search, over the multiplier state-space matrices $(\mathbf{A_H, B_H, C_H, D_H})$, in order to maximise the scalar k which represents the slope restriction of our nonlinearity. Invoking Lemma 5 now shows that the positive real condition (40) can, equivalently, be expressed as a search over real symmetric matrices $\mathbf{P}$ such that the following matrix inequality is satisfied.

$$\left[\begin{array}{cc} A_I'\mathbf{P} + \mathbf{P}A_I & \mathbf{P}B_I - C_I' \\ \star & -D_I - D_I' \end{array}\right] < 0. \qquad (46)$$

Using the realisation (45), this inequality can be written as inequality (47) (at the top of next page). Note that, because the matrices (A_I, B_I, C_I, D_I) are affine functions of multiplier matrices $(\mathbf{A_H, B_H, C_H, D_H})$ this matrix inequality is nonlinear, due to products of $\mathbf{A_H, B_H}$ and $\mathbf{P}$, and therefore not amenable to efficient solution. While this section does not seem to have eased the difficulty of the search for Zames Falb multipliers, it transpires that the inequality (47) is a useful stepping stone towards a convex search. The following two sections will show two different approaches for simplifying this inequality and imposing the $\mathcal{L}_1$ bounds (41).

6.4. Structured Multipliers

The problems with searches for multipliers as they stand are two-fold: the troublesome $\mathcal{L}_1$ condition (41) and the nonlinear matrix inequality (47) that arises as a result of the positive real condition (40). The approach advocated by Chen and Wen (1995, 1996) (see also Jönsson, 1996; Megretski et al., 2004), is to structure the multipliers in such a way that (i) $\mathcal{L}_1$ norm bounds may easily be obtained and (ii) the nonlinear matrix inequality (47) becomes a *linear* matrix inequality. The approach described here follows Chen and Wen (1995). The first observation to make is that if a transfer function, $H(s)$, is given a first order structure, it is easy to calculate its $\mathcal{L}_1$ norm as illustrated below.

---

[7]Colour highlights when these matrices are considered as variables, other variables within LMI's are typed in bold.

Example 1 (A first order multiplier) *Let*

$$H(s) = \frac{\kappa}{s + a}; \quad then \; \|h\|_1 = \frac{\kappa}{a}.$$

Thus the $\mathcal{L}_1$ bound is simply $\kappa/a < 1$. For fixed a this is simply a linear inequality in κ and the associated state-space matrices are

$$A_H = -a \quad B_H = 1 \quad \mathbf{C_H} = \kappa \quad D_H = 0.$$

Notice that, under the assumption that a is constant, three of the four state-space matrices are constant, which means that inequality (47) is actually linear. Therefore, with this structure of multiplier we have obtained a linear inequality for the $\mathcal{L}_1$ norm and a linear matrix inequality for the positive real condition: a tractable search.

The basic approach by Chen and Wen (1995) is to extrapolate from the above example. By restricting attention to a sub-class of Zames-Falb multipliers, the $\mathcal{L}_1$ norm conditions become simple linear inequalities. Also, because this ensures that matrices A_H and B_H are constant, the nonlinear matrix inequality (47) becomes linear.

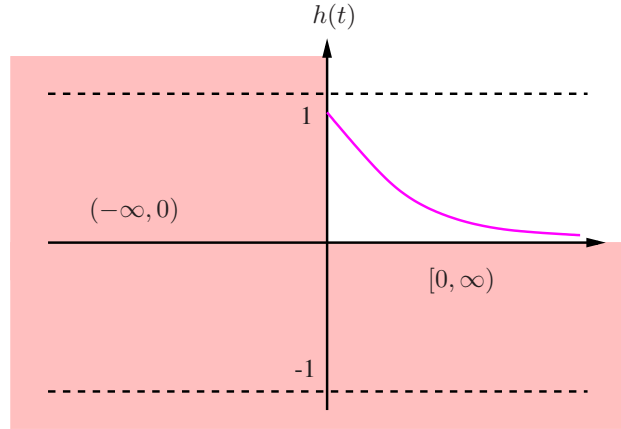*6.4.1. A class of causal positive multipliers*

Figure 22: A causal multiplier with positive impulse response

In order to describe the work of Chen and Wen (1995) concisely, we first consider the following sub-class of Zames-Falb multipliers:

$$\mathcal{RM}_{+c} = \{M(s) = 1 - H_{+c}(s)\} \tag{48}$$

where $h_{+c}(t) = \mathcal{L}^{-1}\{H_{+c}(s)\}$ is such that $h_{+c}(t) = 0 \quad t < 0$ and $h_{+c}(t) \geq 0 \quad \forall t$. In other words, the multiplier is assumed causal and the impulse response $h_{+c}(t)$ is positive. An example of such a function is shown in Fig. 22. These assumptions will be relaxed in subsequent sections.

This representation of the multiplier has several advantages: firstly it significantly reduces the complexity of the $\mathcal{L}_1$ inequality, viz:

$$
\begin{aligned}
\|h\|_1 &= \int_{-\infty}^{\infty} |h_{+c}(t)| dt \\
&= \int_0^{\infty} |\mathbf{h_{+c}(t)}| dt && \text{(causality)} \\
&= \int_0^{\infty} \mathbf{h_{+c}(t)} dt && \text{(positivity)} \\
&< 1.
\end{aligned}
$$

In this case, the following Theorem can be used for approximating such functions:

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$$\left[\begin{array}{cc} \begin{bmatrix} A_p & 0 \\ -B_H k C_p & A_H \end{bmatrix}' \mathbf{P} + \mathbf{P} \begin{bmatrix} A_p & 0 \\ B_H k C_p & A_H \end{bmatrix} & \mathbf{P} \begin{bmatrix} B_p \\ B_H(I - kD_p) \end{bmatrix} - \begin{bmatrix} kC'_p \\ \mathbf{C_H}' \end{bmatrix} \\ \star & -(I{-}kD_p){-}(I{-}kD_p) \end{array}\right] < 0 \tag{51}$$

Theorem 2 (On approximation in $\mathcal{L}_1[0, \infty)$, Szegö, 1975) *For any $h(t) \geq 0 \in \mathcal{L}_1[0, \infty)$ and any $\epsilon > 0$ there exists κ_i^{+c} and N such that*

$$\int_0^\infty \left| h(t) - \sum_{i=0}^N \kappa_i^{+c} e^{-t} t^i \right| dt < \epsilon.$$

The implications of this are the following: by choosing N large enough, $h_{+c}(t)$ can be arbitrarily well approximated by a sum of first order functions. This means that, for $h_{+c}(t)$, we can always find a phase equivalent multiplier represented by a sum of first order functions. In addition, with this approximation, the $\mathcal{L}_1$ norm can be calculated explicitly and the $\mathcal{L}_1$ constraints are linear inequalities in κ_i^{+c}:

$$\|h_{+c}\|_1 = \sum_{i=0}^N (\kappa_i^{+c})i! \qquad \Rightarrow \qquad \sum_{i=0}^N (\kappa_i^{+c})i! < 1. \tag{49}$$

The above follows from considering the Laplace Transform of $e^{-t} t^i$ for $i \in \{0, 1, \dots, N\}$, i.e.

$$H_{+c}(s) = \sum_{i=0}^N \frac{\kappa_i^{+c}}{(s+1)^{i+1}} \sim \left[\begin{array}{c|c} A_H & B_H \\ \hline \mathbf{C_H} & 0 \end{array} \right] \tag{50}$$

The main issue in the application of this result is the requirement that $h_{+c}(t) \geq 0$ for all $t \geq 0$. To guarantee this, note that $h_{+c}(t)$ is given by

$$h_{+c}(t) = \sum_{i=0}^N \kappa_i^{+c} e^{-t} t^i \tag{52}$$

for some N and for some κ_i^{+c} to be chosen. Then $h_{+c}(t) \geq 0$ for all $t \geq 0$ if

$$\sum_{i=0}^N \kappa_i^{+c} t^i \geq 0 \quad \forall t \geq 0. \tag{53}$$

Chen and Wen (1996) have shown that this time-domain condition is equivalent to

$$\underbrace{\frac{\sum_{i=0}^N \kappa_i^{+c} (-1)^i s^{2i}}{(1+s)^N (1-s)^N}}_{S_{+c}(s)} \geq 0 \quad \forall s = j\omega. \tag{54}$$

A state-space realisation of $S_{+c}(s)$ can then be given as

$$S_{+c}(s) \sim \left[\begin{array}{c|c} A_{+c} & B_{+c} \\ \hline \mathbf{C_{+c}} & \mathbf{D_{+c}} \end{array} \right], \tag{55}$$

where the structure of the above state-space matrices is given by Chen and Wen (1996). In particular $\mathbf{C_{+c}}$ and $\mathbf{D_{+c}}$ are structured *affine* functions of the κ_i^{+c}. Then applying Lemma 5 to inequality (54) we obtain the matrix inequality

$$\left[\begin{array}{cc} A'_{+c} \mathbf{X_{+c}} + \mathbf{X_{+c}} A_{+c} & \mathbf{X_{+c}} B_{+c} + \mathbf{C_{+c}} \\ \star & \mathbf{D_{+c}} + \mathbf{D_{+c}}' \end{array} \right] \leq 0, \tag{56}$$

for some $\mathbf{X_{+c}} = \mathbf{X_{+c}}'$. Noting that A_{+c} and B_{+c} are constant, and that $\mathbf{C_{+c}}(\kappa_i^{+c})$ and $\mathbf{D_{+c}}(\kappa_i^{+c})$ are affine functions of κ_i^{+c}, we have an LMI in $\mathbf{X_{+c}}$, and structured $\mathbf{C_{+c}}(\kappa_i^{+c})$ and $\mathbf{D_{+c}}(\kappa_i^{+c})$. Thus letting $H(s) = H_{+c}(s)$, inequality (47) can be expressed as the inequality (51) at the top of the page. For a fixed k, inequalities (54), (56) and (51) form a system of linear matrix inequalities. Thus, in this case, the search for a multiplier becomes an LMI problem.

6.4.2. Noncausal multipliers

A restriction in the results derived so far is that we have assumed the multiplier, $M(s)$ is causal, that is $h(t) = 0$ $\forall t < 0$. As explained earlier in the paper, this may cause some conservatism. However, this assumption can be removed relatively easily by structuring $h(t)$ as the sum of a causal part and an anticausal part, viz,

$$h(t) = h_{+c}(t) + h_{+a}(t) \tag{57}$$

where $h_{+c} \in \mathcal{L}_1[0, \infty)$ and $h_{+a} \in \mathcal{L}_1(-\infty, 0]$. An example of such an $h(t)$ is shown in Fig. 23. A similar approximation to that given in Theorem 2 can then be used to approximate the anti-causal part of $h(t)$ as

$$h_{+a}(t) \approx \sum_{i=0}^{N} \kappa_i^{+a} e^t t^i, \tag{58}$$

which is again a sum of first order terms. Recalling the approximation (52) for the causal part of $h_+(t)$, the $\mathcal{L}_1$ constraint (41) can again be replaced by the linear inequality

$$\|h\|_1 = \sum_{i=0}^{N} \left(\kappa_i^{+c} + (-1)^i \kappa_i^{+a} \right) i! < 1, \tag{59}$$

which is a linear inequality in κ_i^{+c} and κ_i^{+a}. In the same way as before if we assign

$$S_{+a}(s) \sim \left[\begin{array}{c|c} A_{+a} & B_{+a} \\ \hline \mathbf{C_{+a}} & \mathbf{D_{+a}} \end{array} \right], \tag{60}$$

where A_{+a} and B_{+a} are constant, and $\mathbf{C_{+a}}(\kappa_i^{+a})$ and $\mathbf{D_{+a}}(\kappa_i^{+a})$ are affine functions of κ_i^{+a}. Then following a similar line of reasoning to that in equations (53)-(54), it follows that $h_{+a}(t) \geq 0$ providing there exists a symmetric matrix $\mathbf{X_{+a}}$ satisfying the LMI:

$$\left[\begin{array}{cc} A_{+a}' \mathbf{X_{+a}} + \mathbf{X_{+a}} A_{+a} & \mathbf{X_{+a}} B_{+a} + \mathbf{C_{+a}} \\ \star & \mathbf{D_{+a}} + \mathbf{D_{+a}}' \end{array} \right] \leq 0,$$

In this case, a state-space realisation for $H(s)$ is given by

$$H(s) = \sum_{i=0}^{N} \left(\frac{\kappa_i^{+c}}{(s+1)^{i+1}} - \frac{\kappa_i^{+a}}{(s-1)^{i+1}} \right) \sim \left[\begin{array}{c|c} A_H & B_H \\ \hline \mathbf{C_H} & 0 \end{array} \right].$$

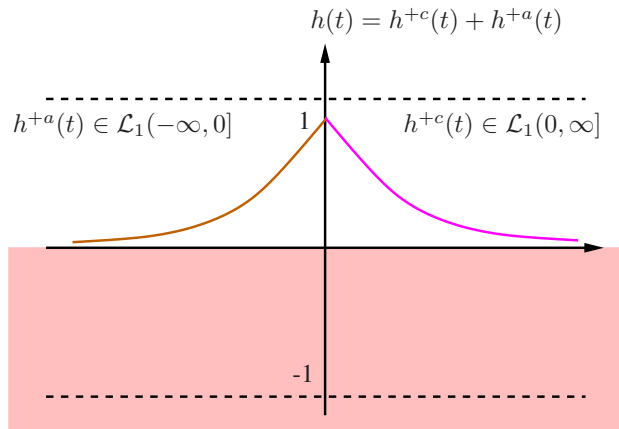This realisation can then be used to obtain an LMI from (47).

Figure 23: A noncausal multiplier with positive impulse response

$$\begin{bmatrix} \mathbf{S_{11}}A_p+A'_p\,\mathbf{S_{11}} & \mathbf{S_{11}}A_p+A'_p\,\mathbf{P_{11}} + kC'_p\,\mathbf{B_H}'+\mathbf{A_H}' & \mathbf{S_{11}}B_p+\mathbf{C_H}' - kC'_p(1-\mathbf{D_H}) \\[2mm] \star & \mathbf{P_{11}}A_p+A'_p\,\mathbf{P_{11}}+k(\,\mathbf{B_H}C_p+C'_p\,\mathbf{B_H}) & \mathbf{P_{11}}B_p+\mathbf{B_H}(kD_p+1)- kC'_p(1-\mathbf{D_H})' \\[2mm] \star & \star & -(kD_p+1)'(1-\mathbf{D_H})' - (1-\mathbf{D_H})(kD_p+1) \end{bmatrix} < 0 \qquad (65)$$

6.4.3. Non-positive multipliers

Previously it was assumed that $h(t) = h_+(t) \geq 0 \quad \forall t$. This assumption can be relaxed by assuming $h(t)$ is the difference between two positive impulse responses

$$h(t) = h_+(t) - h_-(t) \qquad h_+(t), h_-(t) \geq 0 \quad \forall t \qquad (61)$$

From this it follows that a bound on the $\mathcal{L}_1$ norm is given by

$$\|h\|_1 \leq \|h_+\|_1 + \|h_-\|_1. \qquad (62)$$

Because both $h_+(t)$ and $h_-(t)$ are assumed positive for all t, arguments mirroring those in the two above subsections can be used to derive LMI conditions for both the strict positive real condition (47) and the guarantees of positivity. The $\mathcal{L}_1$ constraint can again be simplified to the inequality

$$\sum_{i=0}^{N} (\kappa_i^{+c} + (-1)^i\kappa_i^{+a} + \kappa_i^{-c} + (-1)^i\kappa_i^{-a})i! < 1, \qquad (63)$$

which is linear in κ_i^{+c}, κ_i^{+a}, κ_i^{-c} and κ_i^{-a}: we have a system of LMI's as before.

6.4.4. Remarks on the structured approach

There are two main criticisms which could be levelled at the fixed structure approach.

Complexity/Conservatism of approximation. In common with all Zames-Falb multiplier searches, the approach of Chen and Wen (1995) searches over only a subset of these multipliers, namely over the set

$$\mathcal{RM}_N = \left\{ M(s) = 1 - H(s) : h(t) = \sum_{i=0}^{N} \kappa_i^{+c}e^{-t}t^i + \kappa_i^{+a}e^{t}t^i + \kappa_i^{-c}e^{-t}t^i + \kappa_i^{-a}e^{t}t^i \right\} \subset \mathcal{RM}. \qquad (64)$$

The order of the multiplier is proportional to $4N$ in this case, where N is a free parameter indicating the accuracy of the approximation: large N will imply that $\mathcal{RM}_N$ is in some sense a denser approximation of $\mathcal{RM}$, but large N implies a large computational burden as the larger number of states imply numerous LMI variables. In short: there is a clear trade-off between conservatism and computational efficiency, and the choice of the integer N may be problem-dependent. In particular, as N increases, the problem becomes ill-conditioned.

The decomposition $h(t) = h^+(t) - h^-(t)$. Although this decomposition is not conservative when N approaches infinity (see Veenman and Scherer, 2014), for finite values of N it may introduce some conservatism, since a triangular inequality is used to bound the $\mathcal{L}_1$-norm. This is confirmed by numerical results (Carrasco et al., 2014a). In particular, if $h(t)$ changes sign a nonsmooth function must be approximated with smooth functions, which is only possible if N approaches infinity. As previously mentioned, the problem becomes ill-conditioned as N increases; hence this decomposition may be conservative in practice.

6.5. Plant-order multipliers

An issue with the structured multipliers introduced in the foregoing section is that the conservatism is heavily dependent on the choice of N which determines the order of the multiplier. Indeed, for certain choices of multiplier a very large N would need to be chosen in order to reduce the conservatism to acceptable levels. In addition, although
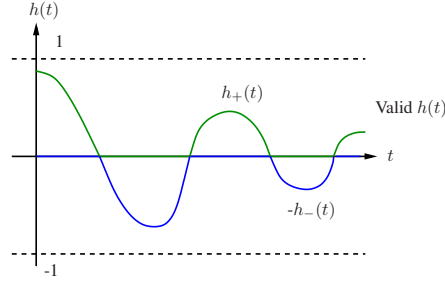
Figure 24: Smooth $h(t)$ difficult to decompose as smooth $h_+(t)$ and $h_-(t)$

there is a clear trade-off between computational requirements and conservatism, the choice of N for a particular problem is by no means obvious.

For this reason, in a series of papers (Turner et al., 2009b, 2010; Turner and Kerr, 2011; Carrasco et al., 2012a; Turner et al., 2012; Carrasco et al., 2014b) the authors have developed an alternative method for searching for multipliers based on a change of variables similar to that used in $\mathbf{H}_\infty$ controller design (Scherer et al., 1997). The main idea is that, if the multiplier is unstructured, but its order is the same as the plant, then a change of variables may be used to "linearise" some of the resulting matrix inequalities. The approach also uses uses a pseudo-LMI approach in order to bound the $\mathcal{L}_1$ norm of the transfer function $H(s)$; the approach is conservative but it can be easily accommodated in the plant-order approach. The restriction of attention to plant order multipliers, $\mathcal{RM}_{PO} \subset \mathcal{RM}$ obviously introduces some conservatism *a priori* but the resulting search procedure is, with a slight caveat, entirely systematic.

6.5.1. Causal Plant order multipliers

In this section we restrict ourselves to causal plant order multipliers

$$\mathcal{RM}_{PO} = \{M(s) = 1 - H(s) : \quad H(s) \in \mathbf{RH}_\infty\}, \tag{66}$$

and where $\deg(H(s)) = \deg(G(s)) = n_p$. The assumption that $H \in \mathbf{RH}_\infty$ implies that the matrix A_H is Hurwitz, but no structure is imposed. This absence of structure in the multiplier is required for the change of variables proposed later, but it also means that calculating $\|h\|_1$ accurately is generally difficult. Instead, a convenient upper bound from the literature will be used.

Theorem 3 (Scherer et al., 1997; Abedor et al., 1996) *Let $H \in \mathbf{RH}_\infty$. Then*

$$\|h\|_1 \le \xi \quad \text{if} \quad \exists \mathbf{Y} = \mathbf{Y}' > 0, \mu > 0, \lambda > 0 \quad \text{s.t.}$$

$$\begin{bmatrix} \mathbf{A_H}'\mathbf{Y} + \mathbf{Y}\mathbf{A_H} + \lambda\mathbf{Y} & \mathbf{Y}\mathbf{B_H} \\ \star & -\mu I \end{bmatrix} \quad < \quad 0, \tag{67}$$

$$\begin{bmatrix} \lambda\mathbf{Y} & 0 & \mathbf{C_H}' \\ \star & (\xi - \mu)I & \mathbf{D_H}' \\ \star & \star & \xi I \end{bmatrix} \quad \ge \quad 0. \tag{68}$$

It is emphasized that Theorem 3 gives only an upper *bound* on the $\mathcal{L}_1$ norm, $\|h\|_1 \le \xi$; it may be extremely conservative. Another issue with Theorem 3 is that the two matrix inequalities are "not quite" LMIs due to the presence of the free scalar $\lambda > 0$. The consequence of this is that the plant order searches proposed here will take the form of LMI's plus a line search, which is computationally cumbersome compared to an LMI, but relatively easy - and entirely systematic - to implement.

The first step in obtaining convenient plant-order multiplier searches is the partitioning of the matrix $\mathbf{P} = \mathbf{P}'$ in the positive real condition (47). It is assumed that $\mathbf{P} > 0$ and therefore that its inverse $\mathbf{Q}^{-1} = \mathbf{P}$ exists. This allows one to write

$$\begin{bmatrix} \mathbf{Q_{11}} & \mathbf{Q_{12}} \\ \mathbf{Q_{12}}' & \mathbf{Q_{22}} \end{bmatrix}\begin{bmatrix} \mathbf{P_{11}} & \mathbf{P_{12}} \\ \mathbf{P_{12}}' & \mathbf{P_{22}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{69}$$

where each of the sub-matrices, $\mathbf{P_{ij}}, \mathbf{Q_{ij}} \in \mathbb{R}^{n \times n}$. Based on this partitioning, the following matrices are defined:

$$\mathbf{\Pi_1} := \begin{bmatrix} \mathbf{Q_{11}} & I \\ \mathbf{Q_{12}}' & 0 \end{bmatrix} \quad \mathbf{\Pi_2} := \begin{bmatrix} I & 0 \\ \mathbf{P_{11}} & \mathbf{P_{12}} \end{bmatrix}. \tag{70}$$

Using the congruence transformation diag($\mathbf{\Pi_1}, I$) and noting that $\mathbf{\Pi_1 P} = \mathbf{\Pi_2}$, the positive real condition (47) is equivalent to

$$\begin{bmatrix} \mathbf{\Pi_1}' A_I \mathbf{\Pi_2}' + \mathbf{\Pi_2} A_I \mathbf{\Pi_1} & \mathbf{\Pi_2} B_I - \mathbf{\Pi_1}' C_I \\ \star & -D_I - D_I' \end{bmatrix} < 0. \tag{71}$$

After some algebra, it can be deduced that this inequality is equivalent to inequality (65), which is an LMI in $\mathbf{P_{11}}, \mathbf{S_{11}} > 0$, $\mathbf{A_H}, \mathbf{B_H}, \mathbf{C_H}, \mathbf{D_H}$ for fixed k, where $\mathbf{S_{11}} = \mathbf{Q_{11}}^{-1}$ and

$$\begin{aligned}
\mathbf{A_H} &:= \mathbf{P_{12}} \mathbf{A_H} \mathbf{Q_{12}'} \mathbf{S_{11}}, & (72) \\
\mathbf{B_H} &:= \mathbf{P_{12}} \mathbf{B_H}, & (73) \\
\mathbf{C_H} &:= \mathbf{C_H} \mathbf{Q_{12}'} \mathbf{S_{11}}, & (74) \\
\mathbf{D_H} &:= \mathbf{D_H}. & (75)
\end{aligned}$$

A similar congruence transformation can be applied to the $\mathcal{L}_1$ inequalities (67) and (68) in order to arrive at expressions in the new coordinates ($\mathbf{A_H}, \mathbf{B_H}, \mathbf{C_H}, \mathbf{D_H}$). In order for this to work, the choice $\mathbf{Y} = \mathbf{P_{22}}$ in inequalities (67) and (68) is made. With this choice, the congruence transformation diag(Q_{12}, I) is applied to inequality (67) and the congruence transformation diag($\mathbf{Q_{11}}^{-1} \mathbf{Q_{12}}, I, I$) is made to inequality (68). Under these congruence transformations, inequalities (67) and (68) then become equivalent to

$$\begin{bmatrix} -\mathbf{A_H} - \mathbf{A_H}' + \lambda(\mathbf{P_{11}} - \mathbf{S_{11}}) & \mathbf{B_H} \\ \star & -\mu \end{bmatrix} < 0, \tag{76}$$

$$\begin{bmatrix} \lambda(\mathbf{P_{11}} - \mathbf{S_{11}}) & 0 & \mathbf{C_H}' \\ \star & 1-\mu & \mathbf{D_H}' \\ \star & \star & 1 \end{bmatrix} > 0. \tag{77}$$

Together, for fixed k, inequalities (65), (76) and (77) form a system of linear matrix inequalities plus a line search over $\lambda > 0$. This problem can be solved relatively easily using modern software and the multiplier can be recovered by using equations (72)-(75).

6.5.2. Anticausal multipliers

A key restriction so far is that $M(s)$ ($H(s)$) is assumed causal. Similar results but with $M(s)$ assumed anticausal can be obtained with the aid of the following result

Theorem 4 (Carrasco et al., 2014b) *Let $H \in \mathbf{RH}_\infty^-$. Then*

$$\|h\|_1 \leq \xi \quad \text{if} \quad \exists \, \mathbf{Y} = \mathbf{Y}' < 0, \; \mu > 0, \; \lambda > 0 \quad \text{s.t.}$$

$$\begin{bmatrix} \mathbf{A_H}' \mathbf{Y} - \mathbf{Y} \mathbf{A_H} + \lambda \mathbf{Y} & \mathbf{Y} \mathbf{B_H} \\ \star & -\mu I \end{bmatrix} \quad < \quad 0, \tag{78}$$

$$\begin{bmatrix} -\lambda \mathbf{Y} & 0 & \mathbf{C_H}' \\ \star & (\xi - \mu)I & \mathbf{D_H}' \\ \star & \star & \xi I \end{bmatrix} \quad \geq \quad 0. \tag{79}$$

The consequence of $\mathbf{Y} < 0$ is that when applying the KYP Lemma (5), for nonsingularity of $\mathbf{P}$ to be guaranteed (as we need to use $\mathbf{Q} = \mathbf{P}^{-1}$), instead of stipulating $\mathbf{P} > 0$, instead it is stipulated that $\mathbf{P} < 0$. Using a similar reasoning to before, it then follows that the positive real condition is satisfied if the matrix inequality (65) is satisfied. Similarly, invoking Theorem 4 and applying similar reasoning to the causal case, the $\mathcal{L}_1$ inequalities become the following:

$$\begin{bmatrix} -\mathbf{A_H} - \mathbf{A_H}' - \lambda(\mathbf{P_{11}} - \mathbf{S_{11}}) & \mathbf{B_H} \\ \star & -\mu \end{bmatrix} < 0, \tag{80}$$

$$\begin{bmatrix} -\lambda(\mathbf{P_{11}} - \mathbf{S_{11}}) & 0 & \mathbf{C_H}' \\ \star & 1 - \mu & \mathbf{D_H}' \\ \star & \star & 1 \end{bmatrix} \geq 0. \tag{81}$$

Together inequalities (65), (80) and (81) form a system of LMI's plus a line search over $\lambda > 0$. This set of LMI's is similar to the causal result given earlier but, since it results in the return of anticausal multipliers can sometimes yield much less conservative results.

6.5.3. Including Popov Multipliers

Popov multipliers are not bounded on the imaginary axis and so, strictly speaking, do not belong to the class of Zames-Falb multipliers. However, following arguments given in Section 4 (see Carrasco et al., 2013), they can be considered as anticausal relaxations in the case of causal Zames-Falb multipliers; or, as causal relaxations in the case of anticausal multipliers. Space prohibits a full discussion, but it suffices to say that they are useful in the plant-order searches proposed earlier (see Turner and Kerr, 2011; Turner et al., 2012).

6.5.4. Remarks on the plant-order approach

The plant order approach is a systematic search for Zames-Falb multipliers, but the nature of the approach is inherently restrictive: the order is fixed *a priori*, the manner of including the $\mathcal{L}_1$ constraint has several sources of conservatism and, excluding the Popov terms, the multiplier returned is either causal or anticausal. A common criticism of multiplier techniques is the poor scaling of dynamic multiplier searches with problem complexity. The plant order approach described here also suffers from that due to the inclusion of the line search over λ, and, to a lesser extent, due to the full-block nature of the matrix variables in the LMIs. Another subtle issue with the plant order approach is that, as described here, it is only applicable to the case of ϕ being *odd*. This is not the case with the structured approach of Chen and Wen (1996) or the approach using irrational multipliers (Chang et al., 2012).

6.6. Application of search techniques to O'Shea's example

This section illustrates the application of the search techniques described to O'Shea's example. The Zames-Falb searches are compared to the well-known Circle Criterion, Park's Criterion (Park, 2002) and also the non-rational Zames-Falb searches of Chang et al. (2012), which have not been described in detail in the paper. The Zames-Falb searches used are the causal (Turner et al., 2009b) and anticausal plant-order searches (Carrasco et al., 2014b) described in Section 6.5, these same searches with the addition of Popov multipliers (Turner and Kerr, 2011; Turner et al., 2012; Carrasco et al., 2014b), and the structured searches of Chen and Wen (1995) from Section 6.4. The plant-order searches were performed by solving the LMI's given earlier together with a 100 element line search over logarithmically spaced λ. The Chen and Wen search is performed with a 18th order multiplier, comprising an 9th order causal and an 9th order anticausal part. Results are tested using the IQC toolbox (Kao et al., 2004).

The results of the various searches are shown in Table 2 for O'Shea's example using a variety of damping ratios, ζ. For all ζ, the Nyquist Value is infinite. As mentioned earlier, due to the symmetry in the example, Park's Criterion cannot out-perform the Circle Criterion, leading to identical maximum slope predictions from both criteria. Safonov and Chang's method gives the greatest slope value for $\zeta = 0.6$, but for the remaining ζ gives values similar to Park/Circle. The remaining Zames-Falb searches all do better than Park for all values of ζ and, perhaps not surprisingly given the phase symmetry of the problem, the causal and anticausal plant searches provide exactly the same slope values in all cases.

The structured search of Chen and Wen deserves some explanation: the performance of this technique is highly dependent on multiplier order. For $\zeta \in [0.2, 0.6]$ with a 18th order multiplier, Chen and Wen's method took a similar

Max slope for guaranteed stability

| ζ | Circle | Park (2002) | Plant order Zames-Falb Searches | | | | Safonov | Structured ZF 18th order |
| | | | Causal | Causal+Pop. | Anticausal | Anticausal+Pop. | | |
			Turner et al. (2009b)	Turner and Kerr (2011)	Carrasco et al. (2014b)	Carrasco et al. (2014b)	Chang et al. (2012)	Chen and Wen (1995)
0.6	11.52	11.52	19.9351	23.6229	19.9351	23.6229	399.6129	45.0599
0.55	9.6799	9.6799	17.1323	19.9064	17.1323	19.9064	9.6800	37.5353
0.5	8	8	14.5146	16.5478	14.5146	16.5478	8.0000	32.1810
0.45	6.48	6.48	12.0855	13.5025	12.0855	13.5025	6.4800	27.1045
0.4	5.12	5.12	9.849	10.7913	9.849	10.7913	5.1200	21.9573
0.35	3.92	3.92	7.8131	8.3913	7.8131	8.3913	3.9200	15.9843
0.3	2.88	2.88	5.9839	6.3238	5.9839	6.3238	2.8800	11.0796
0.25	2	2	4.352	4.5276	4.352	4.5276	2.0000	6.7336
0.05	0.08	0.08	0.2156	0.2165	0.2156	0.2165	0.0086	0.1916
0.01	0.0032	0.0032	0.0088	0.0088	0.0088	0.0088	0.0032	0.0034
0.005	0.0008	0.0008	0.0022	0.0022	0.0022	0.0022	0.0008	0.0007

Table 2: Maximum slope sizes predicted by search methods for various damping ratios in O'Shea's example. When $\zeta > 0.5$ the maximum possible gain is infinite. The Nyquist value is infinity for all values of ζ.

computation time compared with other methods but provided significantly greater slope values. For higher order multipliers, the results deteriorate. This is likely to be due to numerical issues associated with the factorials in the basis representation. For $\zeta < 0.05$, Chen and Wen's method provided more conservative estimates of slope size, irrespective of multiplier order, than the plant order search. For low order multipliers this is due to the restricted bases $(1/(s \pm 1)^N)$ used, while for high order multipliers numerical issues again become significant.

This suggests that results can be improved by a better selection of poles and this is indeed the case. Note that the IQC-toolbox can be used as a manual search tool by choosing an adequate location of the poles, hence the multiplier is given by

$$M(s) = 1 - \left(\sum_{i=0}^{N} \frac{\kappa_i^c}{(s + p_c)^{i+1}} + \sum_{i=0}^{N} \frac{\kappa_i^a}{(s - p_a)^{i+1}} \right) \tag{82}$$

for some selection $p_c > 0$ and $p_a > 0$. For a discussion in the selection of these values, (see Jönsson, 1996). An inexperience user may perform the search using a swap over these two values or a Monte Carlo approach. However performance then depends on the user's ability. For example, when $\zeta > 0.5$, experienced users will be understand that the phase of this plant requires a selection of the poles that concurs with the solution proposed by O'Shea: a very fast causal pole and an anticausal pole at -1. It does not seem surprising as O'Shea proposed such solution by hand 50 years ago.

O'Shea's example illustrates clearly the complexities in finding the "best" multiplier, and the current lack of a complete and tractable method.

6.7. Safonov's search

It is the first tractable search proposed in the literature, Safonov and Wyetzner (1987) proposed a search where the parametrisation of the multiplier contained irrational terms, i.e.

$$m(t) = \delta(t) - \sum_{i=1}^{N} z_i \delta(t - t_i).$$

The main advantage of this search is the simplicity to test the time domain condition. However, it is not possible to check the frequency domain condition in a convex manner. The lack of an LMI implementation reduces the usefulness of this search since it cannot straightforwardly be combined with other classes of multiplier.

Originally, the impulses where equally distributed over a range of times resulting in a large optimization problem. To reduce the computational burden, Gapski and Geromel (1994) reduced the size of the optimization by proposing an iterative method where the position of a new impulse $\delta(t - t_{N+1})$ is obtained if the search with N impulses fails. Recently, a new sub-algorithm has been proposed to improve this selection of the new impulse (Chang et al., 2012). In table 2, we have used the code developed by Chang and Safonov[8]. The results for $\zeta > 0.55$ are very good, but this search is not able to improve Circle criterion results for $\zeta < 0.5$. Once again, anti-symmetry of the phase of $(1 + kG)$ is a possible explanation. The search for the new t_{N+1} is designed to correct only one region of frequencies where there is a lack of positivity. So it is possible this the selected exponential for the multiplier is only able to "fix" one of both regions where there is lack of positivity. Note that as ζ approaches zero, non-positive regions are closer each other.

7. Further Developments

7.1. The IQC framework

We have framed our discussion in terms of passivity rather than IQCs (integral quadratic constraints) (Megretski and Rantzer, 1997) which provided a new framework for multiplier theory. Whereas passivity, dissipativity, and Lyapunov theories can be used in any nonlinear interconnection, the IQC framework restricts its attention to the Lurye system. It therefore provides a natural framework in which to work with Lurye problems in general, and Zames-Falb multipliers in particular. **Although Megretski and Rantzer (1997) provide a self-contained frequency domain**

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[8]Code is available on http://www.michaelwchang.com/zf/

result, the IQC framework was developed in a combination of frequency and time domains by Yakubovich (1965). A natural extension of the original IQC framework to obtain stability conditions including Zames-Falb multipliers the using convolution results by Falb and Zames (1967) has been developed in (Altshuller et al., 2004; Altshuller, 2013).

Furthermore IQC theory provides not only self-contained stability results, but also computational tools to test stability conditions. For example, the search of Chen and Wen (1995, 1996) is encapsulated within the IQC-β toolbox (Kao et al., 2004; Megretski et al., 2004). The IQC framework is especially useful if there is more than one nonlinearity or uncertainty in a feedback loop (e.g. Jönsson and Rantzer, 2000); the motivating example of Section 2 is most naturally expressed in the IQC framework.

Zames-Falb multipliers were used as an illustrative example by Megretski and Rantzer (1997). This sparked renewed interest in their properties and applications. It is therefore worth asking what advantages the IQC framework offers over and above the user-friendliness noted above. IQC theory dispenses with the requirement that multipliers can be factorized, and this is often claimed as an advantage over classical methods (Megretski and Rantzer, 1997). But the requirement that the Zames-Falb multipliers must be factorizable is no restriction on their generality (Zames and Falb, 1968). It turns out that this is also the case whenever multipliers are used for a class of nonlinearities that includes a finite gain (Carrasco et al., 2012b). On the other hand the IQC framework allows additional properties of the nonlinearities to be included in the analysis (for example Turner and Kerr, 2011 include a tighter sector condition); to the best of the authors' knowledge the results of such analysis cannot be obtained via classical techniques.

The relation between passivity theory and the IQC theorem is explored by Carrasco et al. (2012b). The relation between dissipativity (and hence Lyapunov methods) and the IQC theorem is beginning to be understood (Willems and Takaba, 2007; Veenman and Scherer, 2013; Seiler, 2015).

7.2. MIMO nonlinearities

Similarly, our treatment has been restricted to SISO systems with slope restricted nonlinearities. The generalisation of the Zames-Falb theorem to MIMO nonlinearities is discussed by Safonov and Kulkarni (2000). In particular, it is necessary that the nonlinearity can be expressed as the derivative of a convex potential function. This condition is natural for SISO nonlinearities, but may be restrictive in the MIMO case. In fact the condition was recognised in the classical literature (Willems, 1972). It can be shown that the quadratic program used in input-constrained model predictive control satisfies the conditions for Zames-Falb multipliers (Heath and Wills, 2007). Often attention is limited to diagonal nonlinearities; if the nonlinearities are repeated then the symmetry may be usefully exploited (D'Amato et al., 2001; Kulkarni and Safonov, 2002; Turner et al., 2009a). More generally, it is possible to construct specific multipliers appropriate for nonlinearities with repeated blocks (Mancera and Safonov, 2005).

7.3. Discrete-time multipliers

O'Shea also pioneered the equivalent multipliers for use with discrete-time systems (O'Shea and Younis, 1967). Formal treatments can be found by Willems and Brockett (1968) and Willems (1971); their MIMO extension and necessity properties are given by Kulkarni and Safonov (2002) and Mancera and Safonov (2005). The discrete-time counterpart to the search for causal multipliers of Turner et al. (2009b) (and the anti-causal search of Carrasco et al., 2014b) is developed by Ahmad et al. (2013). A search for a first-order FIR Zames-Falb multiplier is presented by Ahmad et al. (2015). A generalisation to FIR Zames-Falb multipliers is presented by Wang et al. (2014). This approach appears highly promising, as it seems to combine the best aspects of the searches of Chen and Wen (1995) and Safonov and Wyetzner (1987).

7.4. Different nonlinearities

Both Rantzer (2001) and Materassi and Salapaka (2011) allow relaxations on the condition of the nonlinearity. The nonlinearity considered by Rantzer (2001) is a perturbation of a nominal odd saturation function, while that considered by Materassi and Salapaka (2011) is some perturbation of a more general nominal odd nonlinearity. In both cases positivity is preserved by further limiting the $\mathcal{L}_1$ norm condition. The multipliers of Rantzer (2001) can be shown to be applicable to a wider class of nonlinearity (Kulkarni et al., 2011b,a).

8. Open questions

Although there have been many recent advances in Zames–Falb mutliplier theory, some open question are still open even for SISO systems in continuous-time. Here we provide some of these open questions.

Complete search. Different searches have been proposed in the literature. Results in this paper and comparisons by Carrasco et al. (2014a) show that no complete tractable search can be found in the literature. Advantages and drawbacks of each technique have been mentioned in Section 6. The development of a complete and tractable search remains an open challenge.

Instability criteria. Currently, searches are tested using only the Nyquist value as an upper bound. However, it is well know that this is not a tight bound for fourth-order systems or higher order. For O'Shea's example small values of ζ produce unstable behaviour. The maximum slope for searches decreases significantly as ζ approaches zero, though the Nyquist value remains infinity. Tractable instability criteria must be developed to be able to understand the real conservativeness of the classes of multipliers and searches over these classes. Existence of limit cycle for this particular problem is discussed by Leonov and Kuznetsov (2013), whereas general results are also available in the literature, (e.g. Manchester and Slotine, 2014).

Dual problem. The dual problem in robustness analysis has been proposed by Jönsson (1996) and Jönsson and Rantzer (1997). The limitations given by Megretski (1995) provide a method to find when a Zames-Falb multiplier cannot be found. However, it is not understood how to use these limitations to discard the existence of a Zames-Falb multiplier suitable for a plant G and slope k.

Stability conjecture. Related with the two previous questions, we can state the following conjecture:

Conjecture 1 *Suppose we have a stable plant G and a constant $k < k_N$ such that there is no Zames-Falb multiplier M satisfying*

$$Re\{M(j\omega)(1 + kG(j\omega))\} \geq \delta$$

for any $\delta > 0$. Then there exists a slope-restricted nonlinearity $\phi \in S[0, k]$ such that the feedback interconnection between G and ϕ is not $\mathcal{L}_2$-stable.

Completeness. In discrete-time, the Zames–Falb multipliers are the only multipliers that can preserve the positivity of monotone and bounded nonlinearities (Willems, 1971; Kulkarni and Safonov, 2002; Mancera and Safonov, 2005). In continuous time other multipliers, such as Popov multipliers, can also preserve the positivity of the nonlinearity yet are not themselves Zames–Falb multipliers. Nevertheless, all other classes of multipliers in the literature have been shown to be phase-equivalent to Zames-Falb multipliers (Carrasco et al., 2013, 2014a). It is an open question whether any possible class of multipliers preserving the positivity of monotone and bounded nonlinearities is phase-contained in the class of Zames–Falb multipliers.

Synthesis. The use of multipliers for synthesis has already been proposed by Veenman and Scherer (2014). Moreover, Zames–Falb multipliers have been already used in synthesis techniques (Moreno et al., 2010; Kerr et al., 2011). The development of a convex synthesis technique using the Zames-Falb multipliers for anti-windup design is still open.

Local stability. The use of input-output stability criteria provides many advantages. However, open-loop unstable systems only can be locally stable in closed-loop. The use of IQCs for local stability has been considered by Fang et al. (2008). However, it is unknown whether multiplier theory has any role in the analysis of local stability.

9. Conclusion

This tutorial has attempted to provided a coherent introduction to the topic of Zames-Falb multipliers. We have shown a motivating example for using Zames-Falb multipliers in the robustness analysis of antiwindup. Their definitions, phase properties, and searches have been presented.

We have devoted a significant part of the paper to describing O'Shea's contribution, most notably O'Shea's observation that noncausal multipliers provide significant advantages over causal multipliers, in particular with respect to their phase properties. We have also shown, using O'Shea's original set of examples, the complexity in multiplier searches: for some ($\zeta > 0.5$), a manual search remains best; while for others ($\zeta < 0.5$) the best achievable slope remains unknown and for different values of ζ different automated searches appear better.

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11. References

Abedor, J., Nagpal, K., Poolla, K., 1996. A linear matrix inequality approach to peak-to-peak gain minimization. International Journal of Robust and Nonlinear Control 6 (9-10), 899–927.

Ahmad, N., Carrasco, J., Heath, W., 2013. LMI searches for discrete-time Zames-Falb multipliers. In: IEEE 52nd Annual Conference on Decision and Control, 2013. pp. 5258–5263.

Ahmad, N., Carrasco, J., Heath, W., 2015. A less conservative LMI condition for stability of discrete-time systems with slope-restricted nonlinearities. IEEE Transactions on Automatic Control, in press, DOI: 10.1109/TAC.2014.2361398.

Altshuller, D., Proskurnikov, A., Yakubovich, V., 2004 Frequency-domain criteria for dichotomy and absolute stability for integral equations with quadratic constraints involving delays. In: Doklady Mathematics, 2004, 70, 998-1002.

Altshuller, D., 2010. Frequency-domain criteria for robust stability for a class of linear time-periodic systems. In: The 2010 American Control Conference, 4247-4252.

Altshuller, D.A., 2011. Delay-Integral-Quadratic Constraints and Stability Multipliers for Systems With MIMO Nonlinearities, IEEE Transactions on Automatic Control, 56 (4), 738 – 747.

Altshuller, D.A., 2013. Frequency Domain Criteria for Absolute Stability, Springer

Barabanov, N. E., 1988. On the Kalman problem. Siberian Mathematical Journal 29, 333–341.

Brockett, R., Willems, J. L., 1965. Frequency domain stability criteria–part i. IEEE Transactions on Automatic Control 10 (3), 255 – 261.

Brogliato, B., Maschke, B., Lozano, R., Egeland, O., 2006. Dissipative Systems Analysis and Control: Theory and Applications. Springer.

Carrasco, J., Heath, W., Li, G., Lanzon, A., 2012a. Comments on "On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities";. IEEE Transactions on Automatic Control 57 (9), 2422–2428.

Carrasco, J., Heath, W. P., Lanzon, A., 2012b. Factorization of multipliers in passivity and IQC analysis. Automatica 48 (5), 609–616.

Carrasco, J., Heath, W. P., Lanzon, A., 2013. Equivalence between classes of multipliers for slope-restricted nonlinearities. Automatica 49 (6), 1732 – 1740.

Carrasco, J., Heath, W. P., Lanzon, A., 2014a. On multipliers for bounded and monotone nonlinearities. Systems & Control Letters 66, 65 – 71.

Carrasco, J., Maya-Gonzalez, M., Lanzon, A., Heath, W. P., 2014b. LMI searches for anticausal and noncausal rational Zames-Falb multipliers. Systems & Control Letters 70, 17 – 22.

Chang, M., Mancera, R., Safonov, M., 2012. Computation of Zames-Falb multipliers revisited. IEEE Transactions on Automatic Control 57 (4), 1024–1029.

Chen, X., Wen, J., 1995. Robustness analysis of LTI systems with structured incrementally sector bounded nonlinearities. In: Proceedings of the American Control Conference, 1995. Vol. 5. pp. 3883 –3887.

Chen, X., Wen, J., 1996. Robustness analysis for linear time-invariant systems with structured incrementally sector bounded feedback nonlinearities. Applied Mathematics and Computer Science 6, 623–648.

Cho, Y.-S., Narendra, K., 1968. An off-axis circle criterion for stability of feedback systems with a monotonic nonlinearity. IEEE Transactions on Automatic Control. 13 (4), 413 – 416.

D'Amato, F. J., Rotea, M. A., Megretski, A. V., Jönsson, U. T., 2001. New results for analysis of systems with repeated nonlinearities. Automatica 37 (5), 739 – 747.

Desoer, C. A., Vidyasagar, M., 1975. Feedback Systems: Input-Output Properties. Academic Press, Inc., Orlando, FL, USA.

Falb, P., Zames, G., 1967. On cross-correlation bounds and the positivity of certain nonlinear operators. IEEE Transactions on Automatic Control 12 (2), 219 – 221.

Falb, P., Zames, G., 1968. Multipliers with real poles and zeros: An application of a theorem on stability conditions. IEEE Transactions on Automatic Control 13 (1), 125 – 126.

Fang, H., Lin, Z., Rotea, M., 2008. On IQC approach to the analysis and design of linear systems subject to actuator saturation. Systems & Control Letters 57 (8), 611 – 619.

Fitts, R., 1966. Two counterexamples to Aizerman's conjecture. IEEE Transactions on Automatic Control 11 (3), 553 – 556.

Freedman, M. I., 1972. Phase function norm estimates for stability of systems with monotone nonlinearities. SIAM Journal on Control 10 (1), 99 – 111.

Gapski, P., Geromel, J., 1994. A convex approach to the absolute stability problem. IEEE Transactions on Automatic Control 39 (9), 1929 – 1932.

Georgiou, T., Smith, M., 1995. Intrinsic difficulties in using the doubly-infinite time axis for input-output control theory. IEEE Transactions on Automatic Control 40 (3), 516 – 518.

Heath, W. P., Wills, A. G., 2007. Zames-Falb multipliers for quadratic programming. IEEE Transactions on Automatic Control 52 (10), 1948 – 1951.

Jönsson, U. T. Robustness Analysis of Uncertain and Nonlinear Systems. PhD dissertation. Department of Automatic Control, Lund Institute of Technology, 1996

Jönsson, U., Rantzer, A., 1997. Duality bounds in robustness analysis. Automatica 33 (10), 1835 – 1844.

Jönsson, U. T., Rantzer, A., 2000. Optimization of integral quadratic constraints. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, Ch. 6, pp. 109 – 127.

Kalman, R. E., 1957. Physical and mathematical mechanisms of instability in nonlinear automatic control systems. Trans. ASME 79 (3), 553 – 566.

Kao, C.-Y., Megretski, A., Jonsson, U. T., Rantzer, A., 2004. A MATLAB toolbox for robustness analysis. In: IEEE Int. Symposium on Computer Aided Control Systems Design. pp. 297 – 302.

Kerr, M., Turner, M. C., Villota, E., Jayasuriya, S., Postlethwaite, I., 2011. A robust anti-windup design procedure for SISO systems. International Journal of Control 84 (2), 351 – 369.

Khalil, H. K., 2002. Nonlinear Systems. Prentice Hall.

Kothare, M. V., Morari, M., 1999. Multiplier theory for stability analysis of anti-windup control systems. Automatica 35 (5), 917 – 928.

Kulkarni, V., Safonov, M., 2002. All multipliers for repeated monotone nonlinearities. IEEE Transactions on Automatic Control 47 (7), 1209 – 1212.

Kulkarni, V. V., Pao, L. Y., Safonov, M. G., 2011a. On stability analysis of systems featuring a multiplicative combination of nonlinear and linear time-invariant feedback. International Journal of Robust and Nonlinear Control 21 (18), 2101 – 2108.

Kulkarni, V. V., Pao, L. Y., Safonov, M. G., 2011b. Positivity preservation properties of the Rantzer multipliers. IEEE Transactions on Automatic Control 56 (1), 190 – 194.

Kuznetsov, N.V., Leonov, G.A., Seledzhi, S.M., 2011. Hidden oscillations in nonlinear control systems. IFAC Proceedings Volumes , 18(1), 2506 – 2510.

Leonov, G.A., Kuznetsov, N.V., 2011. Analytical-numerical methods for investigation of hidden oscillations in nonlinear control systems, In: IFAC Proceedings Volumes 18(1), 2494–2505.

Leonov, G., Kuznetsov, N., 2013. Hidden attractors in dynamical systems. from hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits. International Journal of Bifurcation and Chaos 23 (10), 1330002.

Liberzon, M., 2006. Essays on the absolute stability theory. Automation and remote control 67 (10), 1610 – 1644.

Lipatov, A. V., 1981. Stability of continuous systems with one nonlinearity. Doklady Akademii nauk SSSR, 267, 1069-1072.

Mancera, R., Safonov, M. G., 2005. All stability multipliers for repeated MIMO nonlinearities. Systems & Control Letters 54 (4), 389 – 397.

Manchester, I. R., Slotine, J.-J. E. (2014). Transverse contraction criteria for existence, stability, and robustness of a limit cycle. Systems & Control Letters 63, 32 – 38.

Materassi, D., Salapaka, M., 2011. A generalized Zames-Falb multiplier. IEEE Transactions on Automatic Control 56 (6), 1432 – 1436.

Megretski, A., 1995. Combining $\mathcal{L}_1$ and $\mathcal{L}_2$ methods in the robust stability and performance analysis of nonlinear systems. In: Proceedings of the 34th IEEE Conference on Decision and Control, 1995. pp. 3176 – 3181.

Megretski, A., Kao, C., Jonsson, U., Rantzer, A., 2004. A guide to IQC-β: A MATLAB toolbox for robust stability and performance analysis. Tech. rep., MIT.

Megretski, A., Rantzer, A., 1995. System analysis via integral quadratic constraints: Part I. Tech. rep., Departament of Automatic Control, Lund Institute of Technology.

Megretski, A., Rantzer, A., 1997. System analysis via integral quadratic constraints. IEEE Transactions on Automatic Control 42 (6), 819 –830.

Morales, R., Li, G., Heath, W., 2014. Anti-windup and the preservation of robustness against structured norm-bounded uncertainty. International Journal of Robust and Nonlinear Control 24 (17), 2640 – 2652.

Morari, M., Zafiriou., E., 1989. Robust Process Control. Prentice Hall International.

Moreno, J. C., Baños, A. B., Berenguel, M., 2010. A QFT framework for antiwindup control systems design. Journal of Dynamic Systems, Measurement, and Control 132 (2), 021012.

Narendra, K., Taylor, J., 1973. Frequency domain criteria for absolute stability. Academic Press, New York and London.

O'Shea, R., 1966. A combined frequency-time domain stability criterion for autonomous continuous systems. IEEE Transactions on Automatic Control 11 (3), 477 – 484.

O'Shea, R., 1967. An improved frequency time domain stability criterion for autonomous continuous systems. IEEE Transactions on Automatic Control 12 (6), 725 – 731.

O'Shea, R., Younis, M., 1967. A frequency-time domain stability criterion for sampled-data systems. IEEE Transactions on Automatic Control 12 (6), 719 – 724.

Park, P., 2002. Stability criteria of sector- and slope-restricted Lur'e systems. IEEE Transactions on Automatic Control 47 (2), 308 – 313.

Rantzer, A., 1996. On the Kalman–Yakubovich–Popov lemma. Systems & Control Letters 28 (1), 7 – 10.

Rantzer, A., 2001. Friction analysis based on integral quadratic constraints. International Journal of Robust and Nonlinear Control 11 (7), 645–652.

Safonov, M., Wyetzner, G., 1987. Computer-aided stability analysis renders Popov criterion obsolete. IEEE Transactions on Automatic Control 32 (12), 1128 – 1131.

Safonov, M. G., Kulkarni, V. V., 2000. Zames-Falb multipliers for MIMO nonlinearities. International Journal of Robust and Nonlinear Control 10 (11-12), 1025 – 1038.

Scherer, C., Gahinet, P., Chilali, M., 1997. Multiobjective output-feedback control via LMI optimization. IEEE Transactions on Automatic Control

42 (7), 896 – 911.

Seiler, P., 2015. Stability analysis with dissipation inequalities and integral quadratic constraints. IEEE Transactions on Automatic Control, in press, DOI: 10.1109/TAC.2014.2361004.

Szegö, G., 1975. Orthogonal Polynomials. American Mathematical Society, Providence, RI.

The Tribune-Democrat, 2009. Obituary of Richard P. O'Shea:
http://www.tribdem.com/archives/richard-paul-o-shea-ph-d/article%208ab76949-87f8-5652-b9f6-64d6d3f2b951.html
Retrieved: 31/10/14.

Turner, M., Herrmann, G., Postlethwaite, I., 2007. Incorporating robustness requirements into antiwindup design. IEEE Transactions on Automatic Control 52 (10), 1842 – 1855.

Turner, M., Kerr, M., Postlethwaite, I., 2009a. On the existence of multipliers for MIMO systems with repeated slope-restricted nonlinearities. In: ICCAS-SICE, 2009. pp. 1052 – 1057.

Turner, M., Kerr, M., Postlethwaite, I., Sofrony, J., 2010. $\mathcal{L}_2$ gain bounds for systems with slope-restricted nonlinearities. In: American Control Conference (ACC), 2010. pp. 2671 – 2676.

Turner, M., Kerr, M. L., Postlethwaite, I., 2012. Authors reply to "Comments on 'On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities' ". IEEE Transactions on Automatic Control 57 (9), 2428 – 2430.

Turner, M. C., Kerr, M. L., 2011. $\mathcal{L}_2$ gain bounds for systems with sector bounded and slope-restricted nonlinearities. International Journal of Robust and Nonlinear Control 22 (13), 1505 – 1521.

Turner, M. C., Kerr, M. L., Postlethwaite, I., 2009b. On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities. IEEE Transactions on Automatic Control 54 (11), 2697 – 2702.

Veenman, J., Scherer, C. W., 2013. Stability analysis with integral quadratic constraints: A dissipativity based proof. In: 52nd IEEE Conference on Decision and Control, 2013. pp. 3770 – 3775.

Veenman, J., Scherer, C. W., 2014. IQC-synthesis with general dynamic multipliers. International Journal of Robust and Nonlinear Control 24 (17), 3027 – 3056.

Vidyasagar, M., 1993. Nonlinear systems analysis. Prentice-Hall International.

Wang, S., Heath, W. P., Carrasco, J., 2014. A complete and convex search for discrete-time noncausal FIR Zames-Falb multipliers. In: 53rd IEEE Conference on Decision and Control, 2014. pp. 3918 – 3923.

Willems, J. C., 1971. The analysis of Feedback Systems. The MIT Press.

Willems, J. C., 1972. Dissipative dynamical systems part I: General theory. Archive for Rational Mechanics and Analysis 45, 321 – 351.

Willems, J. C., Brockett, R., 1968. Some new rearrangement inequalities having application in stability analysis. IEEE Transactions on Automatic Control 13 (5), 539 – 549.

Willems, J. C., Gruber, M., O'Shea, R., 1967. Comments on "A combined frequency-time domain stability criterion for autonomous continuous systems". IEEE Transactions on Automatic Control 12 (2), 217 – 219.

Willems, J. C., Takaba, K., 2007. Dissipativity and stability of interconnections. International Journal of Robust and Nonlinear Control 17 (5-6), 563 – 586.

Yakubovich, V., 1965. Frequency conditions for the absolute stability and dissipativity of control systems with a single differentiable nonlinearity. Soviet Mathematics Doklady 6, 98 – 101.

Zames, G., Falb, P., 1967. On the stability of systems with monotone and odd monotone nonlinearities. IEEE Transactions on Automatic Control 12 (2), 221 – 223.

Zames, G., Falb, P. L., 1968. Stability conditions for systems with monotone and slope-restricted nonlinearities. SIAM Journal on Control 6 (1), 89 – 108.

Zheng, A., Kothare, M. V., Morari, M., 1994. Anti-windup design for internal model control. International Journal of Control 60 (5), 1015–1024.

Appendix A. Further notation

This appendix provides some technicalities about common notation in multiplier theory that has been used tacitly in the main text.

Appendix A.1. Signal Spaces

Let $\mathcal{L}_2(-\infty, \infty)$ be the Hilbert space of all square integrable and Lebesgue measurable functions (usually signals) $f : (-\infty, \infty) \to \mathbb{R}$ with inner product defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt, \tag{A.1}$$

and norm defined as $\|f\|_2 = \langle f, f \rangle^{1/2}$, for $f, g \in \mathcal{L}_2(-\infty, \infty)$. The function $f \in \mathcal{L}_2(-\infty, \infty)$ belongs to the subspace $\mathcal{L}_2[0, \infty)$ if $f(t) = 0$ for all $t < 0$ and the subspace $\mathcal{L}_2(-\infty, 0]$ if $f(t) = 0$ for all $t > 0$. For brevity we often use $f \in \mathcal{L}_2$ as shorthand for $f \in \mathcal{L}_2[0, \infty)$. A truncation of the function f at T is given by $f_T(t) = f(t)$, $\forall t \leq T$ and $f_T(t) = 0$, $\forall t > T$. The function f belongs to the extended space $\mathcal{L}_{2e}[0, \infty)$ if $f_T \in \mathcal{L}_2[0, \infty)$ for all $T > 0$.

Let $\mathcal{L}_1(-\infty, \infty)$ be the space of all absolutely integrable and Lebesgue measurable functions (usually impulse responses of LTI systems) $f : (-\infty, \infty) \to \mathbb{R}$ with norm

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(t)|dt. \tag{A.2}$$

The function $f \in \mathcal{L}_1(-\infty, \infty)$ belongs to the subspace $\mathcal{L}_1[0, \infty)$ if $f(t) = 0$ for all $t < 0$ and the subspace $\mathcal{L}_1(-\infty, 0]$ if $f(t) = 0$ for all $t > 0$. For brevity we often use $f \in \mathcal{L}_1$ as shorthand for $f \in \mathcal{L}_1[-\infty, \infty)$.

Appendix A.2. System Spaces

The space $\mathbf{L}_\infty$ ($\mathbf{RL}_\infty$) is the space of (real rational) transfer functions, $G(s)$, bounded and analytic on the imaginary axis. The $\mathbf{L}_\infty$-norm is defined as

$$\|G\|_\infty = \sup_{\omega \in (-\infty, \infty)} |G(j\omega)|$$

The space $\mathbf{H}_\infty$ ($\mathbf{RH}_\infty$) is the space of (real rational) transfer functions analytic in the closed right half plane. The space $\mathbf{H}_\infty^-$ ($\mathbf{RH}_\infty^-$) is the space of (real rational) transfer functions analytic in the closed left half plane. It can be shown, for $G \in \mathbf{RH}_\infty$, that

$$\|G\|_\infty = \sup_{w \in \mathcal{L}_2[0,\infty), w \neq 0} \frac{\|Gw\|_2}{\|w\|_2} \tag{A.3}$$

A system G is said to be causal if $(Gu)_T = (Gu_T)_T$ for any $T > 0$. A formal definition of anticausal system would require a different truncation, but an LTI operator is anticausal if its adjoint is causal. If an LTI operator has a bounded impulse response $h(t)$, i.e. $h \in \mathcal{L}_1$, its transfer function belongs to $\mathbf{H}_\infty$ ($\mathbf{H}_\infty^-$) if and only if $h(t) = 0$ for all $t < 0$ ($t > 0$), i.e. the LTI operator is causal (anticausal).

Appendix A.3. Nonlinearities

A nonlinearity $\phi : \mathcal{L}_{2e}[0, \infty) \to \mathcal{L}_{2e}[0, \infty)$ is said to be memoryless if there exists $N : \mathbb{R} \to \mathbb{R}$ such $(\phi v)(t) = N(v(t))$ for all $t \in \mathbb{R}$. We assume that $N(0) = 0$. Moreover, ϕ is slope-restricted in the interval $S[0, k]$, if

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq k \tag{A.4}$$

for all $x_1 \neq x_2$. The nonlinearity ϕ is said to be odd if $N(x) = -N(-x)$ for all $x \in \mathbb{R}$. Let $\Phi(k)$ be the class of slope-restricted nonlinearities with slope within the interval $S[0, k]$. Our prime example is a saturation function, which is slope-resricted to the interval $S[0, 1]$. A saturation function in series with a linear gain k is a memoryless nonlinearity slope-restricted to the interval $S[0, k]$. It is odd if the absolute value of the upper bound is equal to the absolute value of the lower bound.