Dragging Maintaining Symmetry: Can it generate the concept of inclusivity as well as a family of shapes?

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Key words: dynamic figure, symmetry, inclusive, dragging maintaining symmetry

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Abstract

This paper describes a project to ascertain whether a pedagogical task based on a dynamic figure designed in a Dynamic Geometry Software (DGS) program could be instrumental in developing students' geometrical reasoning. A dragging strategy which I have named 'Dragging Maintaining Symmetry' (DMS) was shown to be important for the making of mathematical meanings in the context of Dynamic Geometry. In particular, it encouraged students' development of the concept of inclusive relations between shapes generated from the dynamic figure, especially the rhombus as a special case of the kites. This development was not automatic and in addition to their work with the dynamic figure the students were shown an animation of the figure under DMS. Watching the animation allowed the students to attend to the continuous nature of the changing figure and proved to be the catalyst for moving their reasoning towards perceiving inclusive relations between the rhombus and kite.

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Introduction

This paper describes a research study focused on a pedagogic task where students generated shapes from a dynamic figure designed in a Dynamic Geometry Software (DGS) program. The aim was to explore whether the task could develop the geometrical reasoning of 13 year old students beyond viewing shapes as being in discrete classes towards an appreciation of some categories of shapes as special cases of others.

The dynamic figure, which I call a dynamic perpendicular quadrilateral, constructed using the Geometers Sketchpad version 4.05 (Jackiw, 2001) is based around two unequal fixed length (8 cm and 6 cm) perpendicular diagonals, referred to as *bars*. Dragging (moving) the bars inside the dynamic perpendicular quadrilateral generates a number of different shapes such as those in figure 1.

Figure 1 here

The shapes generated from the dynamic figure can be said to be special cases or subsets *included* in the set of the perpendicular quadrilaterals. If the figure is dragged so that symmetry is maintained (a strategy I called Dragging Maintaining Symmetry or DMS) it effectively adds another constraint: that one bar must be the perpendicular bisector of the other. This generates a set of kites whose diagonals are of length 8 cm and 6 cm. If, during DMS, the bars are perpendicular bisectors of each other a rhombus is generated and hence it would be reasonable mathematically to suggest that this rhombus is a special case included in the set of kites. However to appreciate this logic it is necessary to hold the concept of inclusivity, and the development of this concept is the focus of this study. Thus the research question which the study aimed to address was: Can Dragging Maintaining Symmetry mediate the meaning of inclusive relations between shapes generated from the dynamic perpendicular quadrilateral, in particular between the rhombus and kites.

Theoretical background.

The Van Hiele model of development in geometrical reasoning.

Van Hiele (1986) developed a set of levels to describe the development of mathematical thinking. One of the examples given was the structure of reasoning about two dimensional shapes in geometry. Van Hiele (1986) described how learners progress through levels of thinking and how at each level they build a mental structure of the concept (for example of two dimensional shapes). He viewed the levels as being hierarchical with each level built on the foundations of the previous one. The first three levels, which are pertinent to this study, are described below.

Level one: the visual level at which learners recognise shapes in an intuitive fashion. They may not use any language to explain why they recognise a shape. They may just say “this is a square because it looks like it”.

Level two: the descriptive analytic level when learners are able to describe a shape as having a collection of properties. For example they understand that one diagonal of a kite is the perpendicular bisector of the other and that two pairs of adjacent sides are equal in length.

Level three: the abstract / relational level where learners understand how one property follows from another. For example, they know that a rhombus is a quadrilateral whose diagonals bisect each other at right angles, and can be included in the kites because the properties of a kite are also true for a rhombus.

The Van Hiele model has been the basis of much research, either to test the theory as an accurate description of learners’ development in geometrical reasoning or to use the Van Hiele levels to assess geometrical reasoning which has been observed in research studies (Battista, 2007). However it has been noted that, rather than a straightforward progressing from one level to the next, learners appear to use different Van Hiele levels in different situations, oscillating from one level to another during the same task and even regressing to earlier levels (Burger and Shaughnessy, 1986).

Papademetri-Kachrani (2012) suggests that we consider the Van Hiele model as modes of thinking rather than as a hierarchy of levels. I have found this approach to be most helpful when analysing the data from my study and will later show that the students participating in my research have used all of the first three levels of thinking while they worked on the task and that each level of thinking has been valuable to the students when working with the dynamic figure. At the same time, through the iterations in the study, I have modified the task to ascertain whether it could encourage development of analytic thinking at the third level. I hold the view that a learner operates up to and including the highest level of thinking that they have achieved.. So a learner who would be assessed as being at the analytical stage will also use the visual and descriptive modes of thinking depending on what suits them best in any particular context.

The Concept image and concept definition

Tall and Vinner (1981) noted that students have experience of shapes before formally learning about them in mathematics lessons and so have already developed a mental structure of shape concepts. These can include prototypical representations and common orientations of shapes which together form the *concept image* (ibid). This is demonstrated when students view a square, presented at a 45 degree angle, as a diamond rather than a square. They may view the square as having sides which are horizontal and vertical, as well as equal in length, whereas a mathematical definition (four equal sides and two pairs of perpendicular sides) is not dependent on its orientation.

On the other hand, the *concept definition* (Tall and Vinner, 1981) is a description of the concept in words. In the case of shapes and their definitions, Fujita and Jones (2007) referred to the ‘personal figural concept’ which is the student’s own personal definition and the ‘formal figural concept’ which is the official definition used in mathematics. If, for example, we hold a personal figural concept of a rhombus as ‘a squashed square which consequently has four equal sides’, and a kite as ‘a shape which must have two smaller sides at the top and two longer sides at the bottom’ then we are likely to have difficulty accepting that a rhombus could be a special case of a kite.

Classifying quadrilaterals: exclusive versus inclusive, partitional versus hierarchical (from Van Hiele level 2 to level 3 reasoning)

Typically students in early school years are taught about 2 dimensional shapes as being distinct from each other. For example, they are introduced to squares and rectangles as being two discrete classes of shapes as opposed to the more inclusive view which would be that squares are special cases of rectangles since the properties of a rectangle also hold true for a square (Dabell, 2014). Classifying 2D shapes discretely is known as partitional classification (De Villiers, 1994). A partitional view can be held very strongly since it has been developed from an early age, so when in secondary school students are presented with the concept of the inclusion of some classes of shapes within others (eg squares as a special case of rectangles, rhombuses as special kites), they often find this difficult to accept (Okazaki, 2009).

I*nclusivity* is the idea that particular mathematical concepts are included within, i.e. form subsets of general mathematical concepts (De Villiers, 1994). A hierarchical (or inclusive) classification views particular shapes as being subsets of other shapes, so that squares are seen as special cases of rectangles and rhombi are included in the set of kites. When considering degrees of sophistication of geometrical reasoning the partitional classification view of 2 dimensional shapes can be viewed at level two, the descriptive analytic level. The hierarchical classification view, of which inclusive classification of shapes is an example, is reasoning at level three, the abstract, relational level (ibid). Use of hierarchical definitions is thought of as indicating higher level geometrical reasoning that can lead to more efficient proving of theorems since a proof for a class of shapes immediately implies the proof for the subsets of that class (De Villiers, 1994).

The progression from Van Hiele level two to three is difficult (Fujita and Jones, 2007), and it is thought that students' personal figural concepts, particularly when they are based on definitions which use a partitional classification, can be a strong confounding factor. In order to move from classifying shapes in a partitional manner to a hierarchical one, students need to re-construct how they categorise shapes (Tall et al, 2001). This requires students to work on the conceptual nature of the shapes and to allow the figural aspect less importance, but students' personal figural concepts are so influential that they dominate the way the student defines the properties of shapes (Fujita and Jones, 2007, Okazaki, 2009).

The affordance of dragging as a cognitive tool

The drag mode is an important affordance in DGS as it allows students to move objects in a figure on the computer screen whilst any properties embedded into its construction remain constant. The experience for students of working in a DGS environment is qualitatively different from that of working in the static pencil and paper environment. When students engage with a DGS figure by dragging it they observe it morphing through many versions of itself and this experience has an effect on how they perceive the figure and thus how they reason about it (Jackiw and Sinclair, 2009, Sinclair et al, 2009).

Arzarello et al, (2002) showed how dragging can be linked to cognitive activity and allows students to move back and forth between their perception of the figure on the screen and the theoretical attributes of the geometrical figure it represents. They observed different dragging strategies being used by students, which they classified according to whether these strategies allow students to move from the practical geometry of the figure on the computer screen to theoretical geometry (which they called an ascending process) or from the theoretical to the practical domain of geometry (which they called a descending process).

Some of the dragging strategies pertinent to this study which Arzarello et al described are listed below:

- *Wandering dragging* is used by students to explore a figure in order to discover its properties and so allows students to move towards theoretical geometry.
- *Guided dragging* is used by students to drag the figure into a particular configuration and is an example of moving from theoretical to practical geometry.
- *Dragging test,* which is used to drag objects and check that the figure maintains its constructed properties.
- *Dummy locus dragging* (moving an object on a figure so that the figure keeps a specific property, the movement of the object traces an invisible path although the student may not consciously intend to move the object in this way, hence 'dummy locus').

Further work on dragging strategies has produced an extrapolation of the dummy locus dragging mode. When the student **intentionally** drags an object to keep a certain property constant, Baccaglini-Frank and Mariotti (2010) refer to it as *maintaining dragging* which they describe as dragging (a base point) so that the (DGS) figure maintains a certain property.

"Maintaining dragging involves the recognition of a particular configuration as interesting, and the user's attempt to induce the particular property to become an invariant under dragging." (ibid, p. 230).

The Dragging Maintaining Symmetry strategy can be seen to be a specific form of Maintaining dragging as will be discussed later.

Methods

The Design Based Research methodology (Cobb et al, 2003, Barab and Squire, 2004) was used to explore whether the task using the dynamic figure could be the catalyst for change in the reasoning of 13 year old students. This paper deals specifically with three iterations which involved myself as the researcher working with pairs of students on the task using the dynamic perpendicular quadrilateral. (A fourth and final iteration involved working over several lessons with one whole class on a modified version of the task.) After each iteration had been completed and analysed the observations and conclusions were used to modify the task to improve it and to test the mini theory which emerged (Cobb et al, 2003, Design Based Research Collective, 2003).

The research took place in two different schools. Student subjects were chosen by their mathematics teachers on the basis of whether they would be happy to work at a task using the computer, with an adult (myself as the researcher) whom they had not met before. In each of iterations one and two, data was collected from 50 minute interviews with four pairs of students. In iteration three, two pairs of students participated in two 50 minute interviews.

The students, who had no prior experience of working with DGS, worked in pairs with one computer, which was loaded with the Geometers Sketchpad™ version 4 (Jackiw, 2001). The pairs worked with the researcher (myself) in a quiet room in their school. On screen activity and dialogue were recorded and these were analysed at three levels: by the dialogue, by onscreen activity (dragging strategies and pointing to objects with the cursor) and by the overarching narrative to connect dialogue and onscreen activity. The sessions took the form of task-based interviews, between pairs of students (two girls or two boys) and the researcher. Task-based interviews entail two or more students working together on a mathematical task with the possibility that they may discuss the problem together and make further progress than each might have made on their own (Evens and Houssart, 2007). However this method explores the knowledge and understanding of the pair or group rather than that of individuals.

My aim was to encourage the students to explore the shapes which could be generated by dragging the bars inside the dynamic perpendicular quadrilateral and to engage in geometrical reasoning to explain how placing the bars in particular positions should result in particular shapes. Since the task-based interview relies on student talk to shed light on their conceptualisations it does rely on students being able to describe their thought processes which some may find difficult. Hence as the researcher I attempted to choose questions carefully to be able to probe students' understanding in a way that helped them to articulate shape properties and how these were generated by dragging activity. The task based interview is akin to a semi-structured clinical interview whose technique is defined by flexibility and the asking of questions contingent on student subject responses (Ginsburg, 1997).

Figure 2 here

The task

The students were presented with a Sketchpad file which contained perpendicular fixed length bars AC and BD. The students were asked to place one bar over the other (as shown in figure 2), to use the line tool to join the ends of the bars and then to use the Construct menu to fill the resulting shape (ABCD) with colour (as shown in Figure 2). When they claimed to have made a specific shape, such as a kite, students were asked to identify the properties of the shape and to use the displayed measurements to check if they could make the shape accurate as indicated by the necessary equal sides and angles within the figure. When the students made a particular shape, they were asked to give its properties and then to use the Measure menu to check side and angle properties. They were also asked **"how are the bars positioned with each other?"** to encourage them to observe the properties of the diagonals (in a quadrilateral) and the base and height (for a triangle). However the main focus of this study was to ascertain whether the task could encourage the development of students' conceptual understanding of shapes and their properties and in particular if the students could develop the concept of inclusive relations between shapes.

Analysing the data

The on-screen activity and dialogue were recorded in each session, and the constant comparison method was used to identify themes in the data (Thomas, 2009). This process entailed listening to and watching the session recordings several times in order to identify emerging themes. I decided to code episodes where the on-screen activity and dialogue appeared to address some aspect or aspects of students' understanding about the geometric properties of the figure, known as a *natural unit of meaning* (Cohen et al, 2003). It became clear from repeatedly re-playing the recordings that the on-screen activity was as important as the dialogue for shedding light on students' conceptions of geometrical figures, particularly the dragging strategies which the students were observed to use. Four main themes emerged which are described below. These themes were present in all iterations, including iteration two where the students were presented with the same bars but tilted so that the dynamic figure was at an angle.

Themes which emerged from the data

The data took the form of audio recordings of dialogue and videos of the on-screen activity. Four main themes emerged from the data. Themes which were gleaned from dialogue included students naming the shapes, describing their properties, describing symmetry, and reference to the orientation of the figure. Themes which were developed from the on-screen activity included the four identified dragging strategies and the use of shape properties and symmetry. In the descriptions of the themes below I have indicated the Van Hiele levels which describe the students' thinking according to the activity.

Geometrical reasoning about the shapes generated from the dynamic figure

Figure 3 shows the dynamic perpendicular quadrilateral in the kite position. Whilst looking at this shape students typically made observations which could be characterised as below:

- A holistic perception of a shape e.g. commenting "it looks like a kite" (Van Hiele 1)
- Describing the shape as split into sub-triangles, for example the smaller isosceles triangle ABC and the larger isosceles triangle ACD) (Van Hiele 1/2)
- Describing properties of sides and angles, or properties of the bars e.g. "AB is equal to BC and AD is equal to CD" (Van Hiele 2)
- Describing common properties of shapes e.g. line of symmetry or congruent sides / angles generated by dragging according to the DMS strategy. (Van Hiele 2)

Figure 3 here

Dragging strategies used by the students in the study:

In the study all pairs of students were observed to drag the bars inside the dynamic figure using four distinct dragging strategies described in an earlier paper (Forsythe 2011) and which are situated within the task using the dynamic perpendicular quadrilateral. The first two dragging strategies align with and therefore have been given the same label as two of the strategies described by Arzarello et al (2002).

- *Wandering dragging* to explore what shapes can be generated from the dynamic figure
- *Guided dragging* to move the bars straight into position to generate a desired shape such as a kite. (Van Hiele 1)

- *Refinement dragging* of the bars using small movements to refine their positions so that the measurements indicate that the properties of the shape are upheld, for example to make the angles at A and C to be the same number of degrees (or so close that we might agree to ignore the difference). (Van Hiele 2)
- *Dragging maintaining symmetry* (DMS) when students drag one of the bars through the middle of the shape with an intention to maintain symmetry. The DMS strategy is thus a special case of Maintaining dragging described by Baccaglini-Frank and Mariotti (2010) because there is an intention to keep the property of shape symmetry. However DMS also has the potential to create mathematical meaning, specifically the concept of inclusivity as will be discussed later. *This strategy emerged in iteration one of the study and the students were not specifically encouraged to use it until iteration three.* (Van Hiele 3)

Symmetry

A sense of symmetry emerged as important in the way students viewed the dynamic figure:

- Initial Guided dragging of the bars into a symmetrical position. (Van Hiele 1)
- Dragging Maintaining Symmetry strategy (potential to move from Van Hiele 2 to 3)
- Describing axes of symmetry as being in the middle of the shape or naming the bars as the axes of symmetry (Van Hiele 2)
- Describing the symmetry of the shape in a holistic way "it will look equal" (Van Hiele 1)

Student preferences of orientation and proportion in the shapes generated from the dynamic figure

The preference for the way the figure was presented is connected to the concept image (Tall and Vinner, 1981). Students appeared to prefer the ‘upright’ version of the figure. Those students in iteration two, who worked with the dynamic perpendicular quadrilateral tilted at an angle, typically commented about the orientation e.g. referring to an ‘angled kite’. Students often positioned the bars inside the kite so that the cross bar was approximately three quarters of the way along the other bar. Many students reported that four kites could be made using the bars (one in each of the relative positions). When questioned as to whether they could move the bar a little to make another kite the number was often revised to a multiple of four, e.g. eight or twelve kites. It appeared that they were reluctant to consider kites with unusual proportions.

An account of how the pedagogical design developed

In the narrative which follows, extracts from the data are presented in chronological order to give an account of the way in which the Dragging Maintaining Symmetry strategy emerged as an important activity and how I modified the task to use DMS as a catalyst for development of the concept inclusivity, in particular of the rhombus as a special case of the kites.

Iteration one

At first glance DMS appeared to be simply an efficient form of guided dragging where one bar is dragged so that it always passes through the mid-point of the other bar. Sometimes this dragging strategy was accompanied by the students explaining what they were trying to do, eg “we're trying to keep BD in the middle of the shape”. In the following excerpt (Figure 4) from the recording of the session with Tilly and Alice, Tilly used DMS to move between the rhombus (which she called a diamond) and the kite.

Figure 4 here

I make the claim that DMS is a special strategy which employs a sense of symmetry and is not just a case of students using Guided Dragging to make the shape look like a kite (for example) using their judgement (placing the bars 'by eye' as it were). In order to test this claim it is necessary to take a close look at the measurements which were displayed on the screen during episodes of DMS. These measurements revealed further evidence to suggest that the students were purposefully maintaining symmetry while dragging. Stopping the on-screen recording at intervals and checking measurements of sides and angles which could be expected to be equal (if the dynamic figure maintained its symmetry) revealed a high degree of accuracy which indicated that the dragging action was keeping (near) symmetry constant. An example is given below (Figure 5). Gill and Sara had a kite on the screen. They changed the figure from a kite through a rhombus and an inverted kite to an arrowhead. The congruent sides were within 0.2 cm of each other and the congruent angles (A and C) were almost always within 2 degrees of each other. These measurements showed a strong indication that these students were demonstrating DMS.

Figure 5 here

Iteration two

It could be argued that the DMS strategy was simply the result of students dragging the bars up and down the screen. Therefore to test whether students would use DMS if the bars were in a different orientation on the computer screen I constructed two new files: the first file contained the dynamic figure constructed around a longer horizontal bar and shorter vertical bar (Figure 6a), and in the second the dynamic figure was constructed around perpendicular bars oriented at an angle to the vertical (Figure 6b).

Figures 6a and 6b here

In iteration two the students were observed to use the same dragging strategies as had been observed in iteration one. When working with the longer horizontal bar the shapes generated still indicated a preference for vertical symmetry over horizontal symmetry. When working with the perpendicular bars oriented an angle the students talked about tilting their head, or they physically tilted their head,

"you can tell it's like a kite this way".

Ruth bemoaned the orientation of the kite which caused her to feel less capable of doing the task.

"Erm, it's a bit of an angled kite. I'm rubbish at this."

Some students asked if the figure couldn't be turned so it was the right way up, which indicates a preferred orientation!

The data indicated that participating students were confident in their knowledge of shapes and their properties, being able to recognise shapes and identify their side and angle properties, parallel lines and axes of symmetry. In this they displayed reasoning at Van Hiele level two.

I had hoped the students would notice that using the DMS strategy generated kites (and arrowheads) most of the time, with the rhombus and isosceles triangles as special cases at particular points on the dragging journey. This might lead to an appreciation that DMS generated the kite as the default shape and that the rhombus and isosceles triangles (and even the arrowheads) could be special cases of the kite. However the participating students in iterations one and two had tended to visualise a discrete number of kites rather than a possibly infinite number of kites. Some students in particular had a preference for kites in the ‘three quarter’ position and, when asked how many kites it was possible to make by dragging the bars inside the figure, a common answer was ‘four’.

If I suggested that they move the cross bar a little bit to make different kites they revised their estimation of the possible number of kites to eight or twelve. They seemed to not perceive that the cross bar could be moved in very small amounts which would produce an infinite number of kites and arrowheads. In effect I was trying to get the students to attend to the figure as a shape which changes continuously, rather than as a figure which generates a number of discrete shapes. Mason (1998) has said that teaching is concerned with directing students’ attention to new mathematical structures. My role, therefore, was to find a way that I could direct the participating students to focus their attention on continuous change as a way to help them perceive the concept of inclusivity. In this I found it helpful to read the work of Mamon Erez and Yerushalmy.

Mamon Erez and Yerushalmy (2007) observed that students who conceptualised 2D shapes according to a hierarchical classification tended to visualise a figure under dragging as moving continuously through different cases of the same figure and continuing to embody its invariant properties, indicating Van Hiele level three reasoning. Hence when the figure is at a stage where it displays a particular shape these students viewed it as being a special instance of the general figure. Students who conceptualised 2D shapes according to a partitional classification tended to visualise the figure under dragging as changing from one discrete shape to another, indicating level two reasoning. These students had difficulty appreciating the geometric logic underpinning the dragging mode (ibid).

With reference to the work of Mamon-Erez and Yerushalmy I postulated that the students participating in the study viewed the dragging action as resulting in a discrete shape rather than as a possible continuous action leading to a continually morphing figure. I wondered if this was the crux of the matter. If I could find a way to allow the students to sit back from the figure and observe it continuously changing it might help them to see the infinity of kites with the rhombus as one specific position along the dragging journey and thus help to move their reasoning from level two to level three. Therefore I decided to develop an animation of the DMS activity and show this to the students in iteration three.

Iteration three

In this section I describe the next iteration in greater detail because the important findings regarding the catalyst for developing the concept of inclusive relations between the rhombus and the kite emerged in this part of the research. Stan and Eric, and Hemma and Seema participated in iteration three over two sessions to ascertain whether the animation of the DMS strategy could be the catalyst for the development of the concept of the dragging family. In the first session the students worked with the bars (oriented at an angle) and in the second session they were shown the animated figure to see whether it would affect their perception of the shapes generated by the dynamic figure.

In session one the students were asked to drag through shapes which had a common line of symmetry thus encouraging them to use the DMS strategy. Hemma and Seema had the following discussion (figure 7) where Hemma demonstrated that she could make a number of kites or arrowheads. This was a strong indication of a student using DMS and linking it to a possibly infinite set of kites. She even managed to convince Seema that there are a large number of kites which can be made (they settled on twenty six.)

Figure 7 here

During the discussion Hemma used the mathematical meaning embodied in the DMS strategy to demonstrate a big idea to Seema, i.e. that many kites can be generated from the dynamic figure. Her explanation and demonstration was accomplished through the use of her own agency in carrying out dragging actions and her narrative which described what she was doing ("if I do that, that's still a kite"). Even though the girls did not make the leap of recognising an infinite number of kites they had made the important change in their understanding that there were many more kites in different positions.

Whilst Stan and Eric dragged the figure through the kites they were more interested in isosceles triangles and arrowheads. They decided that there were four possible arrowheads with the bars in each of the four relative positions (figure 8).

Figure 8 here

When I suggested that they move the bar just a little to generate another arrowhead Sam said

"You can make loads if you do it slowly. One, two, three".

(Meanwhile he dragged the bar AC using small movements to demonstrate more arrowheads). This suggests that Stan had started to view the figure as changing continuously although he only identified a discrete number of arrowheads.

In the second session I showed Stan and Eric the new file and asked them what they could see on the screen. The figure on the screen was in the shape of a kite (figure 9).

Figure 9 here

Stan said

"I see a kite".

When asked why he thought it was a kite Eric replied

“Because it’s shaped like one”.

This suggests that Eric first looked at the shape in a holistic way recognising it as a kite by its general shape.

Next I asked the boys to tell me the properties of the kite. Eric said:

“Two pairs of equal sides and four sides”.

Whilst this statement is not sufficient to provide the definition of the kite, since Eric did not say which pairs of sides were equal, he moved the cursor over the sides AD and DC and then moved the cursor to trace over the sides AB and BC. This activity indicates he did know which sides made equal pairs.

When asked about the properties of the bars AC and BD:

Eric: AC and BD, they’re not the same length

Susan: What about the angle between them?

Eric: It’s a right angle

Susan: And where do they cross each other?

Eric: Mmm, here (the cursor pointed to the intersection of the bars).

Susan: How would you describe that?

Eric: Not in the middle but near the top.

After we had discussed the kite I pointed out the Animate Bar button on the screen. I asked the boys to click on the button and see what happened. The button animated the bar AC so that it moved along the line (which was slightly longer than BD on both sides). As bar AC moved along this line the figure changed shape in a continuous fashion through kites, the rhombus, isosceles triangles and the arrowheads. Eric and Stan had previously generated these shapes in session 1 when they used the Dragging Maintaining Symmetry strategy. The animation was designed to mimic what they were doing, allowing them to observe the figure continuously changing. The intention was that they would notice many more positions for the kite and arrowhead and that this might lead to a concept of a dragging family. Figure 10 shows examples of the shapes which were generated.

Figure 10 here

As soon as the animation started the boys began to relate which shapes they were seeing.

Eric: Triangle, arrowhead

Stan: Arrowhead. Now it’s gone back to a kite.

Eric: It’s showing you all the shapes it can be. Oh rhombus, back to a kite.

Stan: Isosceles triangle. It’s going to another arrowhead.

Susan: What do you think is actually happening?

Eric: Well you know when we moved the line last week, it’s showing the shapes you can make in between them.

Stan: Apart from it’s a bit slower, so you can see what’s happening. It kind of shows you the line of symmetry.

Susan: Which one’s the line of symmetry?

Stan: You’ve got BD so it shows all the shapes you can make with one line of symmetry.

Eric: The arrowheads are changing.

In this excerpt from the dialogue it appears that Eric and Stan noticed two different aspects of the animated figure. Eric noticed that there were more shapes in between those that he had dragged the figure to in the previous session. The other shapes had been the many kites and arrowheads between the discrete kites and arrowheads he had made by dragging. I had hoped that the animated figure might enable the boys to visualise the large number of kites and arrowheads which can be generated when the bar which is being bisected at right angles is moved along the other bar. In theory there should be an infinite number of kites and arrowheads as the bar would be moved by myriad infinitesimal amounts. In practice, the constraints of the computer software probably mean that the bar makes small discrete movements leading to a finite number of possible shapes.

Stan, on the other hand, had noticed that all the shapes being generated had the bar BD as a line of symmetry. The common property of the symmetry of the shapes made by the animated figure might lead to the concept of a dragging family. Next, I asked some questions to ascertain whether the boys would be able to appreciate the significance of this.

Susan: How many kites do you think you actually see when you animate the bar?

Eric: Two cause you see the ones

Stan: You actually see loads cause each angle changes

Eric: You see lots of different one's cause that's still a kite. Kite, kite, kite, rhombus, er kite

At this point Eric began to see that many different kites can be generated from the figure. I got the boys to stop the animation when the shape was an arrowhead.

Susan: What are the properties of an arrowhead?

Stan: It has two pairs the same size lengths. It has two angles that are the same.

Susan: Can you give me a pair of equal sides

Stan: AD and CD and then AB and BC

Susan: Right and where are the two equal angles?

Stan: DAB and BCD

Susan: That's right. You know when you described a kite, weren't you saying the same things?

Eric: Yes I suppose they're like brothers.

Susan: So the arrowhead and the kite might be in the same family, brothers together?

Eric: Same size, two same size angles, and one line of symmetry, two pairs of equal sides

Stan: That's why, same as a rhombus as well.

Eric: So you can see that, all those are in the same family.

Stan: And the triangle and the rhombus.

At this stage in the session it did look as if the boys were beginning to include the shapes made under DMS in the same family. The boys had noted that the kites and the arrowheads shared the same properties and therefore could be related (although they were not able to see that they might actually be versions of the same shape). Next they suggested that these properties held true for the rhombus and isosceles triangle too. However as the boys thought about this again they reverted to a partitional classification by considering that a rhombus has two pairs of equal angles whereas the kite has one pair of equal angles.

Susan: Why do you say that the rhombus is in the family of kites?

Stan: Because it's an 8 by 6 bar so you've got two different line sizes obviously. So you have two different lengths.

Susan: Do you think the rhombus might be a member of that family as well?

Stan: Well I suppose it kind of is. But it has two sets of equal angles. No it won't because it has two sets of equal angles.

Next I suggested that the properties which hold for a kite are also true for a rhombus:

Susan: If you say that BD is a line of symmetry and AD has to be equal to DC and AB has to be equal to BC, that's what you are saying for the kite. Is that also true for a rhombus?

Eric: I think so

Susan: So maybe you could say that a rhombus is a special member of the family.

Eric: Yeah, maybe it's the dad.

After a discussion about the properties of the isosceles triangle the boys decided that the isosceles triangles were the uncles in the family. In summary they decided that the rhombus is the father, isosceles triangles the uncles and the arrowheads and kites the children! This seems to be the opposite of inclusivity in that the most defined shape is seen as being the important one and relaxing its properties produces other members of its family.

Hemma and Seema were looking at the figure in the position below in figure 11 where the animation had been stopped. This is very close to the isosceles triangle but Hemma realised that it was not exactly in the correct place for a triangle.

Figure 11 here

Hemma made the following observations:

"I actually think it might be a kite. But it's just a very odd kite. Cause do you know how we were saying that you can make loads and loads and loads of kites, I suppose it is still a kite. It's just got a very small top bit".

It appears that Hemma was able to use the conceptual properties of a kite to identify it rather than relying on a previously held concept image of a kite.

Another interesting comment by Seema was to say that bar BD split the figure (seen in figure 12) in half so anything which happened to that side (here the cursor pointed to the top triangle ABD) also happened to that side (then the cursor pointed to the bottom triangle DBC). This is a nice observation of the effects of symmetry in a dynamic environment and suggests that Seema was checking the congruence of the two sub-triangles.

Later in the session I asked the girls if kites and arrowheads could be in the same family since they had the same properties.

"They might be" came the reply "but normally you wouldn't see it because they look so different".

When I suggested that the arrowheads and kites might just be different versions of the same shape I got a very unconvinced "mmmm"! When we looked at the rhombus and discussed if this could be a member of the family the girls started to look at the similarities between the kites and rhombus, particularly that AB = BC and AD=DC. When they turned their attention to the isosceles triangle they noticed that there were still two pairs of equal sides (just that one pair were on the same line). The dialogue suggested that they were beginning to think in a more inclusive way about the shapes which were generated by the animation of the dynamic figure, beginning to notice the similarities between the shapes whilst still being aware of the differences.

Discussion

The advantage of using a Design Based Research method has been that it allowed me to treat the study as a journey and process of discovery. The mini-theory emerging from each of the iterations has been taken into account and tested or used to justify the modification of the dynamic figure itself or the way that the task was presented to the students. In iteration one, a sense of symmetry appeared to be used by the students when they dragged the figure between the kites and rhombus or isosceles triangles. This was tested in iteration two when I gave the students a differently oriented figure and observed that the DMS strategy was used even when this resulted in the students dragging at an angle to the vertical.

The shapes formed from the dynamic perpendicular quadrilateral under DMS could be considered to be a 'dragging family' of shapes whose common property is that one bar is the perpendicular bisector of the other bar. Most of the time during DMS the figure takes the shape of a kite, including the concave kites which the participating students knew as arrowheads and which they did not in general include as kites. At special points along the dragging journey the figure passes through the rhombus (when both bars are the perpendicular bisector of each other) and the isosceles triangles (when the end of one bar sits on the mid-point of the other bar). The rhombus being a special case of a kite is mirrored in static geometry but the isosceles triangle might only be considered to be a special kite in dynamic geometry.

My claim is that DMS is a form of Maintaining Dragging which appears to be important for the making of mathematical meanings in the context of the dynamic perpendicular quadrilateral. In particular it has the potential to be a catalyst for students' development of the concept of inclusivity. The use of DMS on the dynamic perpendicular quadrilateral, generates a kite family with the rhombus, isosceles triangles and arrowhead kites at positions along the dragging journey.

The animation used in iteration three was designed to allow the students to focus on what happens when one bar is dragged through the figure using a DMS strategy and it certainly did give the students the opportunity to notice the figure as continuously changing through many more shapes than they had identified in session one. When they viewed the animation Stan and Eric noticed that there were many more kites generated between the rhombus and isosceles triangle for example. Hemma and Seema noticed that many shapes were generated which had something in common (BD as a line of symmetry) and that the changing position of AC changed the lengths of sides and the angles in the figure. That the dynamic figure morphed into kites, arrowheads, isosceles triangle and a rhombus seemed to suggest to the students that they had more in common than they realised at first.

The students in iterations one and two found it difficult to accept the idea that kites and arrowheads were the same shape, or that a rhombus was a special instance of a kite (when the bar AC was at the mid-point of bar BD). However, both Stan and Eric and Hemma and Seema were beginning to accept the idea that shapes can be related if they have a common property which would indicate a (small) movement towards inclusive reasoning.

My premise in creating the animation was that, if the students saw the rhombus as a discrete position among an infinite number of kites, then they might come to see that the rhombus is a special case of a kite. In order to accept the concept of an infinite number of kites it is necessary to accept unusual looking kites into the set of kites. Hemma showed that she was able to do this by accepting the shape in figure 11 as being a kite, if an odd one. She was able to recognise kites in all manner of positions and used the properties of the kite in order to do this using analytic reasoning.

Conclusion

When students work in a DGS environment the visual representation is clearly the main focus of their attention. Under dragging the dynamic visual representation of a figure on the screen is akin to a small motion picture of the figure changing in real time (Lopez-Real and Leung, 2006). Dragging a figure on the screen is a visual actualisation of what mathematicians often do when mentally animating figures in order to perceive the variants and invariants (Leung, 2008, Sinclair et al 2009) Therein may lie the power of dynamic geometry as being more intuitive and closer to how our minds work than static geometry and it may also help students who find such mental visualisations difficult.

In the case of the dynamic figure in this study a perpendicular quadrilateral morphed into various shapes under dragging and the particular DMS strategy generates the shapes which have one of the bars as a line of symmetry. The big idea in this is that the default shape produced by DMS is a kite and the other shapes are special cases of the kite. However, students in the study did not automatically develop the concept of the dragging family of kites. The opportunity afforded by watching the animation of DGS was important in challenging their partitional classification view of shapes and nudging it towards a more inclusive one. It enabled the students to observe the figure as it changed between many more examples of kites in many more different proportions (than they had noticed before) with a rhombus in the middle position. In this way the animation has moved the students' thinking closer towards the concept of inclusive classification. Whilst we should not expect perceptions of shape to change in only one or two 50 minute sessions, the work with the dynamic figure and the animation of the figure under DMS appears to have moved the participating students' reasoning further along the spectrum from Van Hiele level two reasoning towards level three reasoning. The potential of the dynamic nature of DGS in facilitating students to construct meanings in geometry could prove to be a fruitful area for research.

References

Arzarello, F., F. Olivero, D. Paola and O. Robutti (2002). A cognitive analysis of dragging practices in Cabri environments, *ZDM*, Vol. 34(3), pp. 66-72.

Baccaglini-Frank, A and Mariotti, M. A. (2010). Generating conjectures in Dynamic geometry: The maintaining dragging model. *International Journal of Computing in Mathematics Learning* 15: 225-253

Barab, S., and Squire, K.(2004). Design-based research: Putting a stake in the ground. *Journal of the Learning Sciences,* 13(1) : 1-14

Battista, M.T. (2007). The development of geometric and spatial thinking. In Lester, F. (Ed). *Second handbook of research in mathematics thinking and learning*. pp. 843-908. NCTM. Reston, VA: National Council of Teachers of Mathematics.

Burger, W.F. and Shaugnessy, J.M. (1986) Characterising the Van Hiele levels of development in geometry. Journal for Research in Mathematics Education 17(1): 31-48

Cobb, P., Confrey, J., diSessa, A., Lehrer. R., Schauble. L. (2003). Design Experiments in Educational Research. *Educational Researcher,* 32 (1) : 9-13.

Cohen, L., Manion, L. and Morrison, K. (2003) Research methods in education. Routledge-Falmer. London

Dabell, J, (2014) Faulty Rectangles, Mathematics Teaching, Association of Teachers of Mathematics, Derby: 239, pp. 38-39

De Villiers, M. (1994) The role and function of a hierarchical classification of quadrilaterals. *For the learning of mathematics* 14 (1): 11-18

Design-based research collective, (2003). Design based research: An emerging paradigm for educational enquiry. *Educational researcher,* 32 (1): 5-8

Evens, H. And Houssart, J. (2007) Paired interviews in mathematics education. In D. Kuchemann (Ed.) Proceedings of the British Society for Research into learning Mathematics, 27, 2 http://m.bsrlm.org.uk/IPs/ip27-2/BSRLM-IP-27-2-04.pdf

Fujita, T. and Jones, K. (2007). Learners‟ understanding of the definitions and hierarchical classification of quadrilaterals: towards a theoretical framing. Research in Mathematics Education 9(1) :3-20

Forsythe, S. K. (2011) Current report: Developing perceptions of symmetry in a Dynamic Geometry Environment. Research in Mathematics Education 13(2): 225-226

Ginsburg, Herbert. *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press, 1997.

Jackiw, N. (2001). *The Geometers Sketchpad version 4.0* [Computer software]. Emeryville CA: Key Curriculum Press.

Jackiw, N., & Sinclair, N. (2009). Sounds and pictures: dynamism and dualism in Dynamic Geometry. *ZDM*, *41*(4), 413-426..

Leung, A. (2008). Dragging in a Dynamic Geometry Environment Through the Lens of Variation. *International Journal of Computers for Mathematical Learning* 13:135-157

Lopez-Real, F. and A. Leung (2006) Dragging as a conceptual tool in dynamic geometry environments. *International journal of mathematical education in science and technology* 37 (6), p665-679

Mamon Erez M and Yerushalmy M, 2007. “If you can turn a rectangle into a square, you can turn a square into a rectangle...” Young students experience the dragging tool. International journal of computers for mathematical learning 11: 271-299

Mason, J. (1998) Enabling teachers to be real teachers: necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education* 1:243-267

Okazaki, M. (2009) Process and means of reinterpreting tacit properties in understanding the inclusion relations between quadrilaterals. In Tzekaki, M., Kaldrimidou, M and Sakonidis, H. (Eds). Proceedings of the 33rd Conference of the International Group for the

Psychology of Mathematics education, Vol 4:249-256, Greece.IGPME http://igpme.org/index.php/publications

Papademetri-Kachrimani, C. (2012) Revisiting Van Hiele *For the learning of mathematics* 32(3): 2-7

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM*, *41*(4), 441-452.

Tall, D. and Vinner, S. (1981) Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics* 12: 151-169

Tall, D., Gray, E., Ali, M. B., Crowley, L., DeMarois, P., McGowen, M., ... & Yusof, Y. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Math, Science & Technology Education*, *1*(1), 81-104.

Thomas, G. (2009) How to do your Research project. *Sage* London

Van Hiele, P. M. (1986) Structure and Insight. Academic press Orlando Florida

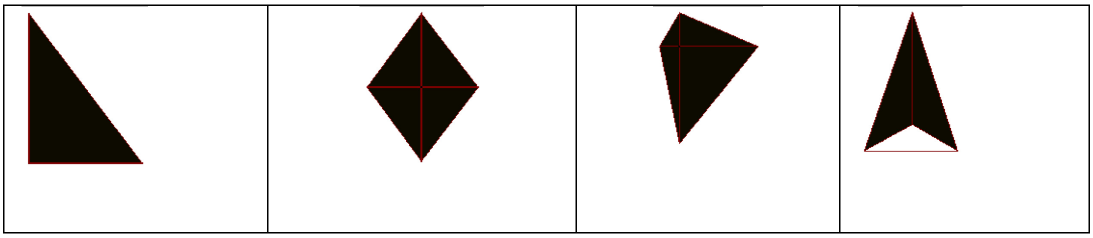


Figure 1 Examples of shapes which can be generated by moving the bars inside the dynamic perpendicular quadrilateral.

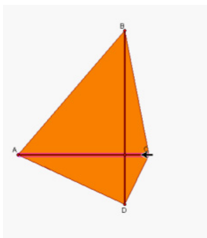


Figure 2 The Dynamic Perpendicular Quadrilateral

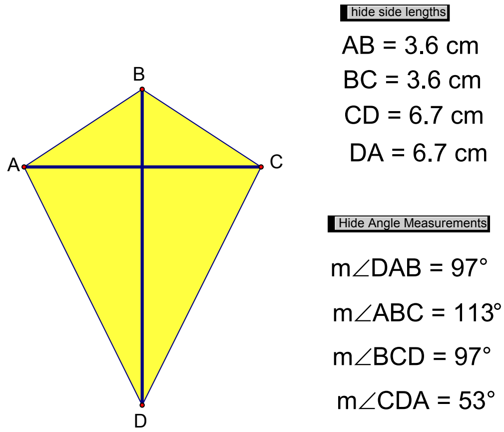
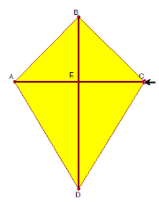
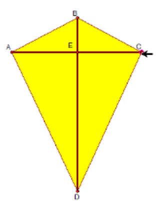


Figure 3 The dynamic perpendicular quadrilateral in the kite position

Dialogue		On-screen activity
Tilly:	*We've got arrow, well kite. oh no that's a diamond or what do you? . Wait, that's a diamond and that's a kite.*	*Tilly drags the horizontal bar with big circular movements and settles it to make a kite. Wandering dragging then guided dragging*
Alice:	*Oh no that would be the*	*She drags the horizontal bar down to the rhombus position then back up to the kite position. Dragging to maintain symmetry*
Tilly:	*No that's a kite I think*	
Alice:	*Isn't that,*	
Tilly:	*Diamonds are all four, just like a square...*	
Alice:	*yeah, yeah, yeah*	
Susan:	*What's the difference between a diamond and a kite?*	
Tilly:	*Erm, a diamond is basically like a square turned diagonally*	
Alice:		*She drags the horizontal bar down to*

It's more evened out so that, like, you've got more lines of symmetry so if you went down that way you'd have one line. And then if you went across you'd have another line because it's more equal	*the rhombus position then back up to the kite position*
Tilly said "that's a diamond and that's a kite" as the vertical bar was moved up and down. This appears to be an instance of 'dragging to maintain symmetry', moving between the rhombus and the kite.	

Figure 4 An excerpt from the data showing DMS

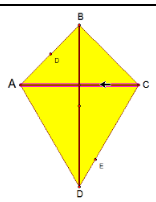
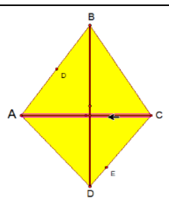
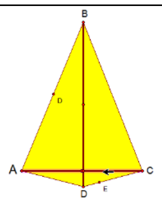
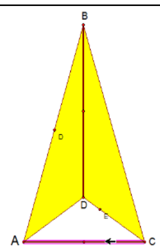
			
m∠ABC = 91° m∠BCD = 104° m∠CDA = 62° m∠DAB = 104°	m∠ABC = 68° m∠BCD = 109° m∠CDA = 81° m∠DAB = 103°	m∠ABC = 45° m∠BCD = 83° m∠CDA = 151° m∠DAB = 81°	m∠ABC = 33° m∠BCD = 39° m∠CDA = 111° m∠DAB = 39°
CB = 4.2 cm AB = 4 DC = 5.9 cm DA = 5 DB = 8.0 cm	CB = 5.3 cm AB = 5.5 cm DC = 4.5 cm DA = 4.7 cm DB = 8.0 cm	CB = 7.8 cm AB = 7.8 cm DC = 3.0 cm DA = 3.2 cm DB = 8.0 cm	CB = 10.5 cm AB = 10.5 cm DC = 3.7 cm DA = 3.6 cm DB = 8.0 cm

Figure 5: Gill and Sara use DMS with high levels of accuracy

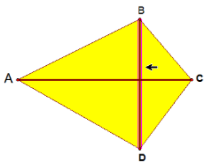


figure 6a

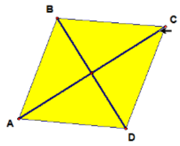
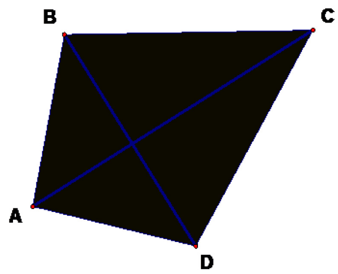


figure 6b

A kite and rhombus generated from bars in different positions

dialogue		on screen activity
Susan:	*They are close enough aren't they. OK so how many kites do you think you could make?*	
Seema:	*About two.*	
Susan:	*You think about two?*	
Hemma:	*I think a bit more because if I do that that's still a kite*	
Susan:	*That's true.*	
Hemma:	*And if I do that it's still a kite. Cause look they're the same and they're closer. Then if I also do that then that's still a kite and that's still a kite.*	Hema moved the bar BD a small amount and then another small amount
Susan:	*How many kites do you think you could make then.*	
Hemma:	*I think about six or seven.*	

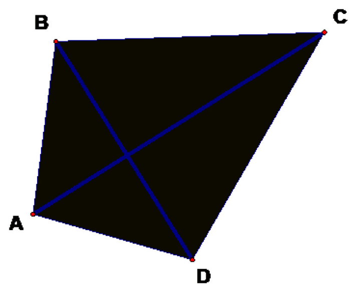
<table>
<tr>
<td>
Susan: Could you not keep on moving it a little bit all the time and it still be a kite?

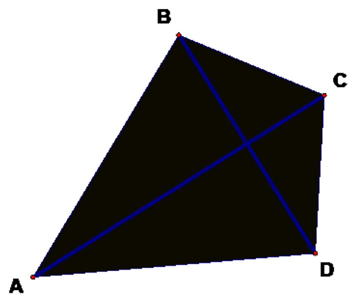
Hemma: Like a millimetre

Seema: It could go on for ages couldn't it.

Susan: So how many do you think it could be then if you can go on for ages?

Seema: About twenty.

Hemma: Twenty five or twenty six
</td>
<td>


Hema moved the bar BD so that she had a kite the other way up


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<tr>
<td colspan="2">Hemma intended to show Seema that there were many more kites than two. In this she used DMS for two short bursts to demonstrate that there were many positions for a kite that Seema had not thought of. Seema then revised the number of kites to twenty and Hemma thought there might be twenty-six.</td>
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</table>

Figure 7 A discussion about the number of kites which can be generated

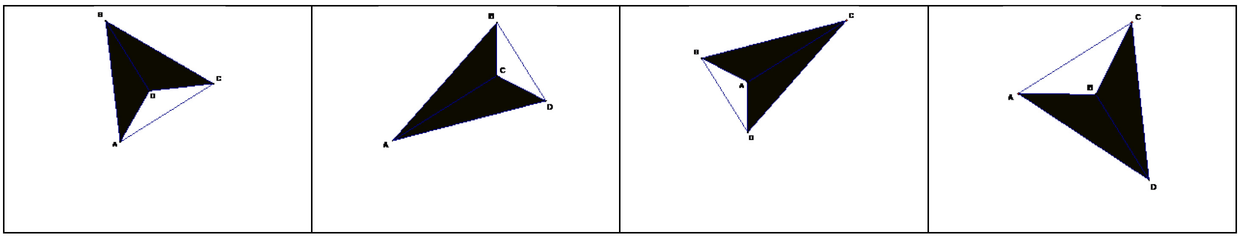

Figure 8 The four relative positions of arrowheads.

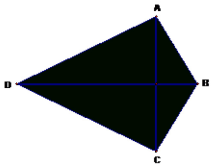

Figure 9 The figure which was animated

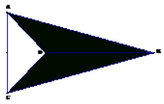	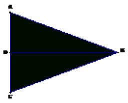	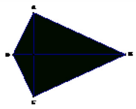	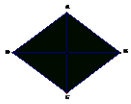	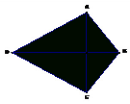	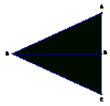	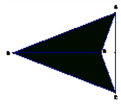
arrowhead	Isosceles triangle	kite	rhombus	kite	Isosceles triangle	arrowhea d

Figure 10 shapes generated during the animation

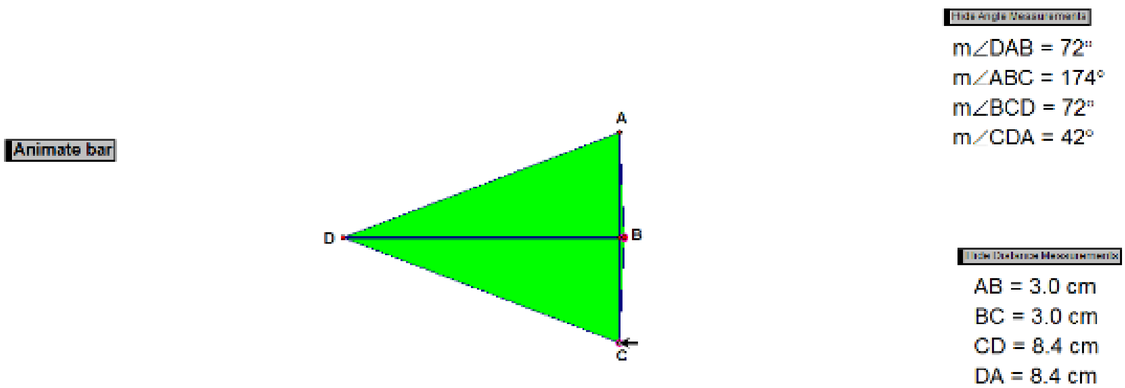

Figure11 A kite near to the isosceles triangle position

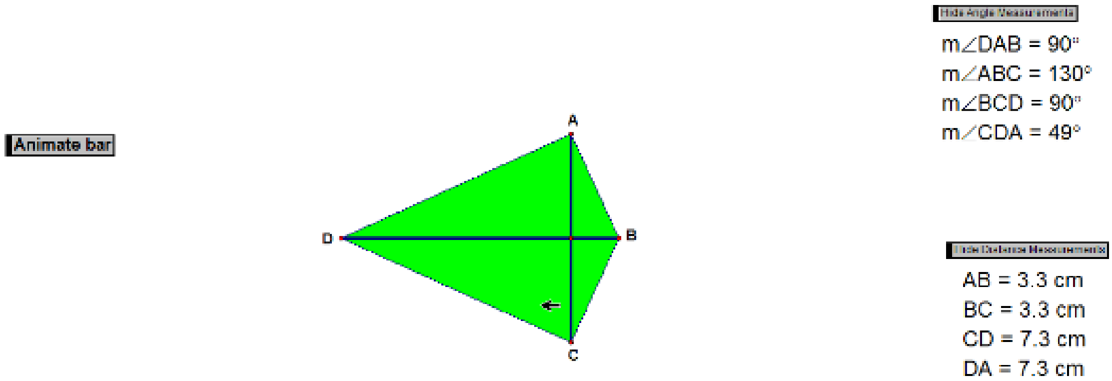

Figure 12 the kite