THE WATER WAVE - ICE FLOE INTERACTION AND ASSOCIATED INTEGRAL EQUATION PROBLEMS

Submitted for the degree of Doctor of Philosophy of the University of Leicester

by

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SUMMARY

The water wave - ice floe interaction is introduced by reviewing the work done to date on the problem. Several mathematical models, incorporating hitherto unexplored and possibly significant mechanisms of the interaction, are then constructed and investigated.

In the first place, the effect of a plane wave incident at any angle upon a semi-infinite elastic sheet of constant thickness is considered, using linearised shallow water theory. The solution for the velocity potential under the ice is discussed for various values of the physical parameters, and in the most interesting case, numerical calculations are made to determine the relevance of such factors as ice thickness and angle of incidence.

Secondly, a semi-infinite sheet of variable thickness is examined and the particular case treated when this thickness has a sinusoidal form. Ranges of incident wavelengths corresponding to a progressive wave solution under the ice are calculated. Also, an ice thickness having a rectangular wave form is considered with similar results.

Attention is then turned to the problem of the existence of a progressive wave in an infinite array of rigidly held, equally spaced floes. Two different approaches are employed to reduce the resulting potential problem to weakly singular integral equations, which in turn are solved by a perturbation method, and, in the general

case, by a numerical technique. It is found that complex wave groups can be constructed satisfying the problem, but that simple progressive waves do not exist.

In an attempt to make analytic inroads on the above mentioned integral equations, some aspects of singular integro-differential equations are investigated, and methods developed by which these may be solved. The closely associated generalised Riemann-Hilbert problem is also discussed and two integro-differential equations arising in aerodynamic theory are solved as examples of the techniques proposed.

CHAPTER I

INTRODUCTION AND REVIEW OF THE WATER WAVE - ICE FLOE INTERACTION

It is intuitive that floating pack ice must impede ocean waves incident upon it to a certain extent, that the energy of such waves will be partly reflected from the ice, and that the energy penetrating the ice may manifest itself in a different form. Travellers in the polar regions have been aware for many centuries of the attenuation of water waves by icefields and the consequent comparative safety which these guarantee from the fury of a storm.

In spite of this, very little is known about the interactions between seas and ice floes, and present day factors press for an increase in this knowledge. Facts which have always been relevant to the animal habitat in the arctic zones are becoming more and more important to man. The size of floes, in terms of both surface dimension and thickness, their cover of the sea and the predominance of intervening open water, the effect of wave motion on floes, the breaking and dividing of pack ice, are all questions which need to be posed and answered.

An understanding of these phenomena will help to assure the safety of the submarines which now move under the ice, and will extend comprehension of the factors governing the motion of surface vessels in pack ice belts. It will also be of use in determining locations of

the many geophysical stations which are being set up on ice islands, as regards their stability and hence usefulness as bases for precise instrumentational techniques. Finally, this knowledge, which must in part be instinctive in the wild life of such regions, will enlighten our understanding of the biological environment.

We shall hereafter be concerned with the specific problem of the interactions between ocean waves and ice floes. As the work which has been done to date on this topic is not extensive, it seems worthwhile to assimilate and summarise the current knowledge. The following sections of the chapter are designed to do this, to indicate the trends which researches have followed, and perhaps to establish their shortcomings and point the way to future developments.

This discourse can conveniently be divided into three categories, concerned respectively with observational, theoretical and experimental research, though of course these are linked in the sense that the last two are primarily aimed at reflecting the results of the first, and of each other.

1. Observational Research.

Considering the extent to which many natural phenomena have been studied and the wealth of information that has been gathered about various aspects of the physical environment that we are subject to, observations of the interactions of ocean waves with fields of pack ice

are curiously rare. The difficulty of collecting controlled data in the polar regions is perhaps the reason for this state of affairs, an exact knowledge of the many parameters involved being virtually impossible to obtain. A further deterrent is perhaps one of economy as many ship hours are required to negotiate the pack ice and perform regular stops for the purpose of taking readings. Ideally too, a second vessel is needed to record the ocean characteristics outside the pack ice simultaneously with measurements inside the icefield, while a completely accurate record of the prevailing ice conditions can only be obtained from an aerial vantage point.

Tidal records in the polar regions for the last few decades have indicated that sea level oscillations with periods of around two minutes occur in areas of extensive ice cover. Hunkins [1]* observed, on four drifting ice stations in the Arctic Ocean, continuous vertical oscillations of the ice. Periods in the range 15 - 60 seconds were detected, the amplitude being roughly proportional to the square of the period, and equal to $\frac{1}{2}$ mm. at 30 seconds. These motions were due in part to wind generation, but in one case they were identified as propagating waves. Longer period disturbances were also observed which, Hunkins suggests, may be due to conditions peculiar to ice regions,

* Figures denoted thus refer to the bibliography following the corresponding chapter. The notation for tables and figures is exactly similar in meaning.

perhaps meteorological in origin.

The wave energy acceptance of Antarctic pack ice should be more marked than this, the region being exposed to a larger surrounding ocean area and generally heavier seas and swells. In fact one of the earliest recorded contributions to this topic was made in the Antarctic by the master of the Endurance in 1916, who witnessed the periodic rise and fall of large floes.

A major contribution to this field of study was made some ten years ago by Robin [2], [3] who undertook to investigate wave penetration at large distances into the Antarctic pack ice. His study highlighted the problem of accurately recording ice conditions as he found that visual estimates of floe sizes made by experienced polar travellers were some 30 per cent too low, while ice cover appraisals were subject to even greater error.

However Robin made rigorous measurements of wave characteristics by means of a ship-borne recorder, and was able to deduce from these and corrected visual observations several conclusive results.

He found that periods less than 8 seconds disappeared from the wave spectrum in the pack ice, while waves with periods exceeding 8 seconds suffered a decrease in energy if the floes were no larger then 40 metres in diameter. For waves with a 16 second period there was little energy loss in floes 40 metres or less across, and energy was still detectable in floes up to several kilometres in diameter,

provided that they were no thicker than two metres. His results indicate that the horizontal dimensions of the floes are a major factor controlling wave penetration at shorter wavelengths, while for longer wavelengths traversing fields of larger floes, the thickness of these floes may be significant.

More specifically, Robin deduced that when the floe diameters are small compared to wavelength, a cut-off in transmission occurs for wavelengths three times the floe size. The "cut-off" wavelength is defined somewhat arbitrarily: Robin's data was measured in terms of spectral energy density, and the cut-off wavelength is one at which this energy density in the pack ice falls well below that in the open sea for all periods. The wavelength at which attenuation becomes appreciable is approximately twice that of cut-off, namely six times the floe diameter.

In the case when floe diameters are large, Robin concludes that major energy changes occur at the 400 metre wavelength at the seaward boundary of large floe fields, when the floe diameters change from a fraction of a wavelength to several wavelengths across.

Further, Robin was able to deduce that for large floes, the fraction of wave energy transmitted is proportional to the fourth power of the wavelength, elastic bending being a significant mechanism in such cases, and inversely proportional to the cube of the ice thickness. As thickness is a difficult parameter to measure this latter deduction

is perhaps less relevant than the λ^4 hypothesis.

The most complete set of observational results to date have been gathered by Dean [4], who was concerned with wave energy behaviour near the pack ice - sea boundary. Having thoroughly reviewed and compared the possible methods of wave recording in icefields [5], Dean used more refined instrumentation than Robin, and, no doubt benefitting from Robin's experience, employed a more rigorous technique of observing the ice parameters. Before each reading he ensured that (i) the ice cover was uniform over the area, (ii) the ocean/pack ice boundary was sharply defined, and (iii) this boundary was straight and parallel to the wave crests over a large distance. At each reading, Dean took photographs of the prevailing ice conditions, and these were later analysed methodically, while estimates of floe thickness were carefully made using upturned floes. Measurements of the open ocean energy spectrum were made before and after each reading.

He found that waves of periods less than 6 seconds lost 70-75% of their energy in the first 100 metres of the pack ice; waves with an 8 second period suffered a 5 - 10% energy loss in half a kilometre, and for waves with a 16 second period there was no detectable loss in two kilometres. The quantitative results of the rate of decay with distance presented by Dean are unique and indicate a rapid energy loss near the ocean - ice boundary at shorter wavelengths. This decay, which lasts only a few wavelengths, was found to be of an exponential nature, and

further into the ice Dean was able to confirm the λ^4 hypothesis of Robin. Dean therefore suggests that the law governing transmission of energy needs to be modified so that it relates to the level of energy being measured, the λ^4 proportionality being valid at the energies present well into the pack ice.

It would seem that the initial rapid decay at the shorter wavelengths can be related to the proximity of the floes. Near the ocean - ice boundary the floes will be closer due to wind and drift, so that energy loss could be partly accounted for by the bumping of floes and some degree of turbulence in the narrow stretches of intervening water.

2. Theoretical Research.

We now turn to the more numerous theoretical inroads which have been made upon the problem. The construction of a mathematical model, incorporating dependence on all the parameters involved in the physical situation, is of course impossible, and one must isolate specific mechanisms and value their significance by comparing the resulting theoretical predictions with the observed phenomena. In particular, two dimensional analyses, to which the mathematician is often restricted for reasons of tractability, do not necessarily reflect the complexities of the three dimensional problem.

The simplest way of simulating an icefield in a mathematical

model is to regard it as a continuous rigid sheet of uniform thickness, having a straight boundary line where it meets the ocean. This so-called "Dock Problem" was tackled by Heins [6] on a two dimensional basis, considering a semi-infinite sheet defined in the above way. By assuming that the velocity potential decayed exponentially under the ice, Heins was able to derive a Wiener - Hopf type of integral equation. This yielded a somewhat complex solution for the velocity potential which Heins did not discuss with respect to the physical problem.

Following this, the main group of theoretical studies was that of New York University, started in 1950 by Peters [7]. He formulated a second method of describing ice structures mathematically, which has since been widely used, and consists of regarding the floes as made up of individual non-interacting point particles. Peters resolved to determine how such a semi-infinite "mass-loading" ice floe reacted to a progressing wave incident on its edge. Assuming the water to be infinitely deep, he solved the problem by complex variable techniques and established that the nature of the disturbance in the ice depends upon the sign of

$$c = \rho/\{\rho_1\omega^2/g - \rho\},$$

where ρ_1 is the density of the ice, ρ the water density, and ω the time frequency.

If $c < 0$, the wave is transmitted unattenuated into the ice with a changed wavelength and amplitude. If $c > 0$, the wave decays in

the ice, its amplitude diminishing like a positive power of $1/d$ when the distance d into the ice is sufficiently large.

Weitz and Keller [8] modified this problem, by retaining an arbitrary depth of water and allowing the incident wave to impinge on the ice at any angle. Their method of solution was very similar to that which Heins used for his dock problem, and they obtained various criteria for a progressive wave to pass undamped into the ice. These can be conveniently stated in terms of the parameter c defined above: if c is positive and sufficiently large such a propagating wave exists, while if c is positive, but not large, or negative, there is no propagation beneath the ice. Thus the effect of keeping the water depth finite is a shift in the critical value of c from zero to some positive non-explicit value.

Using the results of this work, Weitz and Keller [9] published a second paper in which they derived explicit expressions for the reflection and transmission coefficients at the ice edge. This latter publication was probably inspired by the work of Shapiro and Simpson [10], who undertook to utilise the theory developed by Weitz and Keller and, to a lesser extent, Peters, to produce some practical results regarding the effect of a broken icefield on water waves.

They were able to deduce several significant results from this numerical study, the principal ones being as follows:

(i) the damping of waves in the ice results both from the pressure

effect of the ice and multiple reflections,

(ii) for a fixed period wave, less energy is transmitted as the ice thickness increases,

(iii) the longest waves penetrate the ice with the least energy loss,

(iv) for normal incidence, and ice thickness up to ten feet, waves of period greater than 5 seconds have at least 90% of their energy transmitted,

(v) wavelengths are decreased on entering the ice, the reduction being proportional to the ice thickness.

Shapiro and Simpson also found that the incident energy was not compatible with the sum of the reflected and transmitted energies in the model. To explain this, they introduced a new concept which they called a "pressure energy", and postulated that there is, in addition to the energy of the penetrating wave, a second component of transmitted energy which is in evidence as an upward pressure directed against the overlying icefield.

It is at once evident from (iv) above that these results are not quantitatively in accord with observations. Robin [3] examined the energy penetration predicted by Shapiro and Simpson for all wavelengths and, comparing them graphically with his data, found a considerable discrepancy. He suggested that this might be explained by the fact that the theoretical work is based on a continuous broken icefield, while his observations relate to waves which have traversed discrete patches

of floes.

Keller and Goldstein [11] then formulated the same problem on the basis of shallow water theory, obtaining the same qualitative results. Although they were restricted to the cases when water depth is small compared to wavelength, this was compensated somewhat by the fact that the ultimate results were readily translated into practical terms. They further considered the situation when surface tension forces are present in the floating ice, and were able to deduce that this is a possible factor controlling wave penetration.

Curiously, one of the most important advances was made as a result of a totally different motivation, when Stoker [12] examined the effectiveness of a floating elastic beam as a breakwater. That the elastic properties of large floes is an inherent device governing wave transmission has already been indicated by observational conclusions. Stoker dealt with the case of normally incident waves and as his primary objective differed from our present interest, the only relevant conclusion we can draw from his treatment is that the rigidity of the beam increases the reflection of the incident wave.

Stoker's approach paved the way for further study of the ice-wave interaction, notably by Hendrickson and his colleagues. The case of a floating elastic sheet of finite length in water of arbitrary depth has been tackled by Hendrickson [13] using a finite difference technique to resolve this difficult problem numerically. The results

reflect that flexural rigidity is an important parameter in determining the wave energy acceptance of floes, confirming Stoker's conclusion.

Before this, a more tractable model had been proposed by Hendrickson and Webb [14], in which the elastic sheet was semi-infinite and the water infinitely deep. The primary concern of this project was to examine the internal stresses in the sheet due to wave action. The problem was found to succumb analytically to a certain extent, and results indicate that the stress in the sheet increases gradually to a peak situated well into the ice, before decreasing again. This peak was found to be of greater magnitude for thinner floes.

A far more ambitious model was then formulated in which the finite submergence of the sheet was taken into account, this modification resulting in the necessary use of a totally numerical approach. An incident wavelength of 10 feet and an ice thickness of 5 feet were recognised by Hendrickson and Webb as rather unrealistic figures, and were imposed as a result of computer economics. The transient condition due to the submerged edge appeared to die away after several wavelengths, indicating that reflection and wavelength change take place near the leading edge. It was the complete investigation of this phenomenon without increasing the finite difference grid beyond certain limits which led to the above choice of parameters. The stress peak was found to be much nearer to the leading edge than in the case of zero submergence, and this prompted the authors to suggest that the breaking up of

a floe near its edge may be a progressive procedure. That is, the edge fractures due to stress, transient conditions occur near the new edge, which eventually breaks, and so on.

Before answering this on the basis of theoretical results, it is necessary to inspect the correlation between such results and those obtained experimentally from a wave tank model of the situation.

3. Experimental Research.

As a result, such an experiment was commissioned, and performed by NESCO [15]. The wave periods were restricted so that comparison with the deep water problem was valid, and all the parameters were subjected to a rigorous scaling, so that future correlation with observations could also be made.

The ice floe was simulated by a single polythene sheet and was acted upon by waves of varying characteristics. The deformation of the sheet, resultant bending stresses and the pressure at the interface between the sheet and the underlying water were measured. Qualitatively the results were in good agreement with those predicted by Hendrickson and Webb, but in the cases of deformation and stress were found to be in excess of the theoretical values by a factor as high as 4 in some instances. The pressure proved difficult to measure, so no comparison with other estimates could reasonably be made.

Even though the stress attained much larger values at positions

along the floe than the mathematical model indicated, its full scale equivalent has been calculated to be insufficient to break the floe, apart from minor fractures at its edge, and so one must look elsewhere for the cause of this observed phenomenon. It has been suggested that thermal stresses and sudden barometric fluctuations may weaken floes to a point at which the stresses caused by wave motion are sufficient to cause fissures, but this is unproven.

There remain two notable experiments which have recently carried out in this field. The first of these was performed by Ofuya and Reynolds [16] in an attempt to determine the relevance of experiments in understanding the ice floe problems. The laboratory layout remained as in the previous model, a polythene sheet of suitable density and rigidity representing a single floe, but measurement of the wave heights at various points in the channel was now the prime interest.

The readings taken and translated into full scale parameters were found to bear little relationship to the results of Shapiro and Simpson, but there was some similarity to the results of Hendrickson and Webb, particularly at the longer wavelengths. In an attempt to increase the wave energy acceptance of the sheet, indentations were made in the leading and trailing edges, giving a saw-tooth configuration to these edges. It was found that optimum wave transmission occurred when these cuts were $\frac{1}{3}$ of the length of the incident wave,

but this artificial device has little bearing on the current topic. The qualitative results that long waves pass under the sheet with little energy loss, while short waves experience a high degree of attenuation was confirmed. Ofuya found that if a similar experiment were performed using a fluid of a particular viscosity, the value of which he calculates, the results of this would compare favourably with Robin's observed data. Hence he suggests that this may be the way to link experiments with the true physical situation, but this empirically based hypothesis has no theoretical meaning.

Ofuya and Reynolds also deduced from their results that boundary layer effects under the ice could be a major factor, and this postulate has yet to be tackled from a theoretical standpoint.

Finally, we mention the report by Henry [17] on the experiment carried out at the Stevens Institute in New Jersey. This incorporated by far the most ambitious and realistic simulation of an icefield of any venture to date, in that many loosely constrained square blocks of polythene were arranged to cover a variable percentage of the "icefield" area. An additional parameter was the floe size, three different values of this being used.

Unfortunately, the results obtained were not significant; no consistent change was observed, in the relationship between lengths, speeds and frequencies of the waves, from the free surface state, when the floes were introduced and their configuration and surface

cover varied. The results also inferred that the floes did not affect the bottom pressures in the fluid.

It was deduced that the predominant dissipating mechanism in the laboratory model was a linear viscous decay both with and without the floes, the bottom and side boundary layer effects being the major factors, rather than internal stresses in the fluid induced by a velocity gradient.

It is of interest to compare this model with the previous one of Ofuya and Reynolds, who were able to obtain concrete results. The difference in floe simulation is clearly not relevant, and the other basic variation is in the water depth, which was about four times greater in Ofuya's model - 0.61 metres compared with 5.88 inches which was Henry's value. This would seem to substantiate Henry's conclusion concerning viscous effects, the boundary layer at the bottom being apparently the more important, in that the water depth of the last discussed model was insufficient to override such an effect.

4. Conclusion.

It is clear from the preceding account of the present knowledge of the water wave - ice floe interaction that much has still to be learned about the processes involved, and that the conclusions reached by Robin and Dean have not been adequately explained either on a theoretical basis or in a laboratory environment. The onus is there-

fore on the mathematician and the experimentalist to illuminate the known degrees of attenuation of progressing waves in an icefield.

It has been shown that the elastic nature of the floes plays a fundamental role, and this leads us to examine in Chapter II the effect of a propagating wave incident on a semi-infinite ice sheet with a built-in flexural rigidity factor. We allow the wave to impinge on the sheet at an arbitrary angle, and use linearised shallow water theory to formulate the problem. This latter restriction of course limits the relevance of the results, but is offset to a certain extent by the fact that we are able to develop and discuss all aspects of the problem comparatively easily.

All the mathematical models to date have taken the ice floe to be of constant thickness, and it is therefore of interest to investigate cases in which this dimension varies in a prescribed manner. We devote Chapter III to the examination of such a problem, with special reference to the cases when the ice thickness is described by sinusoidal and rectangular waves, so that limiting cases exist when the ice cover can be thought of as made up of an infinity of discrete floes. Again we restrict the discussion to shallow water theory, and even with this simplification the problem is found to be somewhat complex. However we are able to reach several qualitative conclusions which indicate that further investigations of this type of problem would be perhaps

fruitful.

In all researches of the wave - ice phenomena, with the exception of the "Dock Problem", and in the two chapters of the present project outlined above, the Peters representation of surface matter has been used. We recall that this device regards the ice as the aggregate of non-interacting floating point masses. It is therefore an artificial approach in that it does not reflect the inherent rigidity of floes. One way of introducing this latter concept would be to take account of friction between the particles, so that their relative motion is subject to constraint. On the other hand, it is thought worthwhile to supercede the mass loading type of model altogether by a formulation in which the floes are in fact rigid objects.

In Chapter V we derive what can be called the "many dock" problem as a preliminary step into this area of investigation. An attempt is made to find periodic solutions of this problem and the formulation illustrates a useful but little known theory for solving potential problems with periodic boundary conditions, namely that of automorphic functions.

At a point in this last mentioned problem it appears that a knowledge of the methods of solving singular integro-differential equations would be of use. This was found to be a not particularly well documented field, and so the contents of Chapter IV were assembled in an attempt to remedy this. In it, known methods of dealing with

singular integro-differential equations are reviewed, further techniques are developed for the most general case of such an equation, and two examples arising in aerodynamic theory are solved as examples.

At this stage we mention that the computing system used in the numerical side of the subsequent work comprised an Elliot 4130, with standard peripheral assemblies, the appropriate variant of ALGOL being the computer language employed.

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CHAPTER II

WAVE TRANSMISSION IN A FLEXIBLE SEMI-INFINITE ICE FIELD OF UNIFORM THICKNESS, ON THE BASIS OF LINEARISED SHALLOW WATER THEORY.

1. Formulation of the equations.

The ice field in $o<y<\infty$, $-\infty<x<\infty$ is made up of elements of constant mass floating on water of a constant depth h (see Fig. 1 a). A plane wave from $y = -\infty$ is incident obliquely on the edge $y = o$ of the field. Equations for the velocity potential $\Phi(x, y, z, t)$ of the liquid motion are derived using linearised shallow water theory; Φ satisfies

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \qquad o<z<h, \tag{1.1}$$

and

$$\Phi_z = 0, \qquad z = o. \tag{1.2}$$

Suppose that the equation of the free surface $y<o$, $-\infty<x<\infty$ is given by

$$z = h + \eta(x, y, t),$$

then since the normal velocity of a point on the surface is equal to the normal velocity of a particle at the surface,

$$\eta_t - \Phi_x \eta_x - \Phi_y \eta_y = -\Phi_z ,$$

or, linearising,

$$\eta_t = -\Phi_z . \tag{1.3}$$

From Bernoulli's equation,

$$p - p_o = \rho(\Phi_t - gz + gh - \tfrac{1}{2}q^2),$$

and neglecting the second order term, on the free surface we require

$$g\eta = \Phi_t, \qquad z = h. \tag{1.4}$$

Equations (1.3) and (1.4) combine to give the free surface condition

$$\Phi_{tt} + g\Phi_z = 0, \qquad -\infty<y<o, \; z = h. \tag{1.5}$$

Similarly, if $\epsilon(x, y, t)$ represents the elevation of the ice sheet above the undisturbed level in $y>o$, we require

$$\epsilon_t = -\Phi_z, \qquad z = h. \tag{1.6}$$

The pressure difference across the ice sheet

$$p - p_o = \rho(\Phi_t - g\epsilon), \tag{1.7}$$

where p_o is atmospheric pressure, displaces the sheet and subjects each element to bending stresses. If the ice has a uniform thickness h_o and density ρ_i, then the motion of an element is governed by the equation

$$p - p_o = D\nabla^4\epsilon + \rho_i h_o \epsilon_{tt}, \tag{1.8}$$

in which the flexural rigidity constant D is given by

$$D = Eh_o^3/12(1 - \sigma^2),$$

where E is Young's modulus, σ Poisson's ratio, and

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

The first term on the right hand side of (1.8) is derived, for example, in [1].

At this stage we invoke the shallow water theory approximation [2], and introduce $\phi(x, y, t)$ by

$$\Phi(x,y,z,t) = \phi(x,y,t) - \tfrac{1}{2}z^2\{\phi_{xx} + \phi_{yy}\} + O(z^3), \qquad (1.10)$$

which satisfies (1.1) to the first order in h and which enables (1.6) to be written in the form

$$\epsilon_t = h\nabla^2\phi + O(h^2). \qquad (1.11)$$

Equating (1.7) and (1.8) and differentiating with respect to t

$$\Phi_{tt} - g\epsilon_t = (D/\rho)\nabla^4\epsilon_t + sh_o\epsilon_{ttt}.$$

Using (1.11) this can be written

$$\phi_{tt} - gh\nabla^2\phi - (hD/\rho)\nabla^6\phi - shh_o\nabla^2\phi_{tt} = 0, \quad y>o, \qquad (1.12)$$

where $s = \rho_i/\rho$.

Similarly the free surface condition (1.5) becomes

$$\phi_{tt} - gh\nabla^2\phi = 0, \qquad y<o. \qquad (1.13)$$

We assume that $\phi(x,y,t)$ contains the time harmonically,

$$\phi(x,y,t) = \exp(-i\omega t)F(x,y), \qquad (1.14)$$

so that $F(x,y)$ is subject to

$$(hD/\rho)\nabla^6F + h\{g - \omega^2 sh_o\}\nabla^2F + \omega^2F = 0, \quad y> o,$$

and

$$\nabla^2F + (\omega^2/gh)F = 0, \qquad y<o,$$

and since the ice sheet extends infinitely in both positive and negative x directions we can remove the x dependence from F by

$$F(x,y) = \exp(ikx)\ G(y), \qquad (1.15)$$

where k is real, giving

$$(hD/\rho)\{d^2/dy^2 - k^2\}^3G + h\{g - \omega^2 sh_o\}\{d^2/dy^2 - k^2\}G + \\ + \omega^2G = 0, \quad y>o \qquad (1.16)$$

and

$$\{d^2/dy^2 - k^2\}G + (\omega^2/gh)G = 0, \qquad y<o. \tag{1.17}$$

<u>The boundary conditions at y = o.</u>

We require continuity of the velocity components in the liquid at y = o, and this is equivalent to imposing the continuity of

$$G(y) \text{ and } dG/dy \text{ at } y = o. \tag{1.18}$$

Also, at the free edge y = o of the ice sheet, we require the vanishing of the bending and twisting moments along the edge and of the vertical shearing force. These three components are respectively [3],

$$M_y = -D\{\epsilon_{yy} + \sigma\epsilon_{xx}\},$$

$$M_{xy} = D(1 - \sigma)\epsilon_{xy},$$

$$Q_y = -D\frac{\partial}{\partial y}\{\epsilon_{xx} + \epsilon_{yy}\},$$

and the requirement that they vanish simultaneously has been shown to be equivalent to the two conditions

$$\left.\begin{aligned} M_y &= 0, \\ Q_y - \frac{\partial}{\partial x}M_{xy} &= \epsilon_{yyy} + (2 - \sigma)\epsilon_{xxy} = 0. \end{aligned}\right\} \tag{1.19}$$

This necessary combining of two of the equations was first indicated by Kirchoff. Using (1.11) and (1.15), equations (1.19) easily become

$$\left.\begin{aligned} \{d^2/dy^2 - \sigma k^2\}\{d^2/dy^2 - k^2\}G &= 0, \qquad y = 0, \\ \frac{d}{dy}\{d^2/dy^2 - (2-\sigma)k^2\}\{d^2/dy^2 - k^2\}G &= 0, \qquad y = 0, \end{aligned}\right\} \quad (1.20)$$

or, more exactly, we require these equations to be satisfied by the solution for $G(y)$ in the limit $y \to 0+$.

We shall see subsequently that equations (1.18), (1.20) are sufficient to determine all the arbitrary constants arising in the solution.

2. The solution of the equations.

The solution in $y<o$ of (1.17), corresponding to a real progressive wave moving from $y = -\infty$ to $y = o$, is

$$G(y) = e^{im_o y} + Re^{-im_o y}, \qquad (2.1)$$

in which

$$m_o^2 = k_o^2 - k^2, \qquad k_o^2 = \frac{\omega^2}{gh} > k^2. \qquad (2.2)$$

The coefficient of the term corresponding to the incident wave is taken to be unity, and the second term in the solution represents ultimately a wave reflected at the ice sheet. The coefficient R is a complex quantity and the velocity potential is subsequently described by the real part of (1.14).

We seek the solution of (1.16), for $y>o$, in the form

$$G(y) = C_j\ e^{i(k_j^2 - k^2)^{1/2} y}, \qquad (2.3)$$

for some constants C_j, and it is easily seen that k_j must be a root of

$$k_j^6 + ak_j^2 - b = 0, \tag{2.4}$$

where

$$\left.\begin{aligned} a &= \frac{\rho}{D}\{g - \omega^2 sh_0\}, \\ \text{and} \quad b &= \frac{\omega^2\rho}{Dh}. \end{aligned}\right\} \tag{2.5}$$

The solutions of (2.4) may conveniently be expressed in the form

$$\left.\begin{aligned} k_1^2 &= \alpha + \beta, \\ k_2^2 &= \nu\alpha + \nu^2\beta, \\ k_3^2 &= \nu^2\alpha + \nu\beta, \end{aligned}\right\} \tag{2.6}$$

in which

$$\left.\begin{aligned} \alpha &= [\tfrac{1}{2}b\{1 + \Delta\}]^{\frac{1}{3}}, \\ \beta &= [\tfrac{1}{2}b\{1 - \Delta\}]^{\frac{1}{3}}, \end{aligned}\right\} \tag{2.7}$$

and

$$\Delta^2 = 1 + 4a^3/27b^2. \tag{2.8}$$

ν and ν^2 are the imaginary cube roots of unity, namely,

$$\nu = \tfrac{1}{2}\{-1 + i\sqrt{3}\},$$

$$\nu^2 = \tfrac{1}{2}\{-1 - i\sqrt{3}\},$$

so that

$$k_2^2 = \overline{k_3^2}. \tag{2.9}$$

Writing the trial solution (2.3) in the form

$$G(y) = C\, e^{imy},$$

then the allowable values of m are given by

$$m = \pm(k_j^2 - k^2)^{1/2}, \qquad j = 1, 2, 3,$$

from which we must discard those values of m which have a negative imaginary part, to avoid unbounded terms as $y \to \infty$.

The complete solution $\phi(x,y,t)$ for the shallow water potential will contain a term representing an undamped progressive wave in the y direction only if at least one of the roots m is completely real, that is, if one of the k_j^2 is real and greater than k^2.

Now $b > o$, and α, β are real provided that Δ is real, i.e. provided

$$1 + 4a^3/27b^2 > 0, \tag{2.10}$$

and if this inequality is satisfied, k_1^2 is real and positive while k_2^2 and k_3^2 are complex.

If

$$1 + 4a^3/27b^2 = 0, \quad (a < 0), \tag{2.11}$$

then the roots are easily seen to be

$$k_1^2 = (4b)^{\frac{1}{3}}, \; k_2^2 = k_3^2 = -\tfrac{1}{2}(4b)^{\frac{1}{3}},$$

and if

$$1 + 4a^3/27b^2 < 0, \quad (a < 0), \tag{2.12}$$

it can be shown that the roots are all real and

$$k_1^2 = \sqrt{(-4a/3)}\, \cos\tfrac{1}{3}\theta,$$

$$k_2^2 = \sqrt{(-4a/3)}\, \{-\tfrac{1}{2} \cos\tfrac{1}{3}\theta - \sqrt{3}\, \sin\tfrac{1}{3}\theta/2\},$$

$$k_3^2 = \sqrt{(-4a/3)}\, \{-\tfrac{1}{2} \cos\tfrac{1}{3}\theta + \sqrt{3}\, \sin\tfrac{1}{3}\theta/2\},$$

where

$$\cos\theta = -\frac{3b}{a}\sqrt{(-3/4a)} > 0,$$

so that we may choose $0 < \theta < \pi/2$, and it follows that $k_1^2 > 0$, while $k_2^2 < 0$ and $k_3^2 < 0$.

Hence k_1^2 is real and positive in all cases, while the two further roots are complex if (2.10) is satisfied, and real but negative otherwise. The criterion that a progressive wave can proceed unattenuated in the y direction is thus

$$\alpha + \beta = k_1^2 > k^2. \tag{2.13}$$

We note that (2.11) and (2.12) are satisfied only if $a < 0$, that is, if

$$\omega^2 > g/\mathrm{sh}\,h_0,$$

or, in terms of the incident wavelength λ,

$$\lambda^2 < (2\pi)^2\,\mathrm{sh}\,h h_0.$$

In particular, (2.11) is found to be satisfied in the typical case in which $h = 10$ metres and $h_0 = 2$ metres, (for other parameters see 4) by $\lambda = 0.3$ metres approximately.

Hence the cases for which

$$1 + 4a^3/27b^2 \leqslant 0$$

are of little significance physically, and we investigate in detail only values for which (2.10) holds.

3. The solution of the problem in a particular case.

The solution is investigated in detail in the case $a > 0$, which is physically the most interesting; that is, we restrict the discussion to the wavelengths

$$\lambda > \lambda_{min} = 2\pi\sqrt{shh_0}, \tag{3.1}$$

and, as we have seen, k_1^2 is real and positive while k_2^2 and k_3^2 are complex conjugate.

We take

$$m_1 = +\{k_1^2 - k^2\}^{\frac{1}{2}}, \tag{3.2}$$

and choose m_2 and m_3 so that they contain a positive imaginary part,

$$\left.\begin{aligned} m_2 &= \gamma + i\delta, \\ m_3 &= -\gamma + i\delta = -\overline{m_2}, \end{aligned}\right\} \tag{3.3}$$

where

$$\gamma^2 - \delta^2 = -\tfrac{1}{2}(\alpha + \beta) - k^2,$$

$$2\gamma\delta = \tfrac{1}{2}\sqrt{3}(\alpha - \beta),$$

so that

$$\left.\begin{aligned} 2\gamma^2 &= -\tfrac{1}{2}(\alpha + \beta) - k^2 + \sqrt{\{[\tfrac{1}{2}(\alpha + \beta) + k^2]^2 + \tfrac{3}{4}(\alpha - \beta)^2\}}, \\ \delta &= \sqrt{3}(\alpha - \beta)/4\gamma. \end{aligned}\right\} \tag{3.4}$$

The solution in $y > 0$ for $G(y)$ is now

$$G(y) = T_1 e^{im_1 y} + T_2 e^{im_2 y} + T_3 e^{im_3 y}, \tag{3.5}$$

and we recall that in $y < 0$,

$$G(y) = e^{im_0 y} + Re^{-im_0 y}. \tag{2.1}$$

Introducing an incident angle θ, and a transmitted angle ψ by (see Fig. 1 b),

$$k = k_0 \sin\theta = k_1 \sin\psi, \tag{3.6}$$

so that

$$m_0 = k_0 \cos\theta, \quad m_1 = k_1 \cos\psi,$$

and replacing the x dependence, the incident wave has the form

$$e^{ik_0 y \cos\theta + ik_0 x \sin\theta},$$

and the reflected wave is

$$e^{-ik_0 y \cos\theta + ik_0 x \sin\theta},$$

the incident and reflected angles being equal. If (2.13) holds, the undamped part of the transmitted wave is

$$e^{i\{k_1^2 - k_0^2 \sin^2\theta\}^{\frac{1}{2}} y + ik_0 x \sin\theta}$$

$$= e^{ik_1 y \cos\psi + ik_1 x \sin\psi}. \tag{3.7}$$

Writing (2.13) as

$$k_1 > k_0 \sin\theta,$$

and (2.4) in the form

$$k_1^6 + \frac{\rho g}{D} \{ 1 - k_0^2 \operatorname{sh} h_0 \} k_1^2 - \frac{\rho g}{D} k_0^2 = 0,$$

the criterion for real propagation can be stated more conveniently. A critical wavelength λ_c is defined by $k_1 = k_0$, and is easily seen to be

$$\lambda_c = 2\pi \sqrt{\left\{ \frac{D}{g\rho \operatorname{th} h_0} \right\}}. \tag{3.8}$$

Using (1.9) we see that, for a constant water depth, λ_c is directly proportional to the ice thickness h_0.

We consider four cases for the length of the incident wave λ:

(i) $\lambda = \lambda_c$

Incident waves pass undamped into the ice without deflection from their original path, by (3.6), and with the same wavelength $\lambda = 2\pi/k_0 = 2\pi/k_1$.

(ii) $\lambda < \lambda_c$

In this case $k_1 < k_0$ and there exists a critical angle of incidence θ_c, defined by

$$\theta_c = \sin^{-1}(k_1/k_0), \tag{3.9}$$

such that for waves incident on the ice edge at an angle $\theta < \theta_c$, there is real propagation into the ice. By (3.6), we see that such waves are deflected away from the normal on entering the ice. Further, the length of the wave increases as it passes into the icefield. For an incident angle $\theta > \theta_c$ the expression $(k_1^2 - k_0^2 \sin^2\theta)^{\frac{1}{2}}$ becomes imaginary, and no real propagation occurs.

(iii) $\lambda > \lambda_c$

In this case $k_1 > k_0$ and there is real transmission into the ice for any incident angle. Equation (3.6) indicates that the propagated waves are deflected towards the normal, while their lengths are diminished on entering the icefield.

(iv) λ large

Writing in (2.4)

$$a = \frac{\rho g}{D}\{1 - k_0^2 \operatorname{sh} h_0\}, \qquad b = \frac{\rho g}{D} k_0^2,$$

and letting $k_0^2 = (2\pi/\lambda)^2$ be sufficiently small, so that its square may be neglected, we can expand the expressions (2.7) for α and β in powers of k_0^2 to give

$$\alpha = \sqrt{\left\{\frac{\rho g}{3D}\right\}} + \tfrac{1}{2}k_0^2\left[1 - \operatorname{sh} h_0\sqrt{\left\{\frac{\rho g}{3D}\right\}}\right] + O(k_0^4),$$

$$\beta = -\sqrt{\left\{\frac{\rho g}{3D}\right\}} + \tfrac{1}{2}k_0^2\left[1 + \operatorname{sh} h_0\sqrt{\left\{\frac{\rho g}{3D}\right\}}\right] + O(k_0^4).$$

Hence,

$$k_1^2 = \alpha + \beta = k_0^2 + O(k_0^4), \tag{3.10}$$

so that as λ becomes large, $k_1 \to k_0$, and such incident waves, impinging at any angle, pass undeflected into the icefield, there being no change in their wavelengths.

We note that for normal incidence ($\theta = 0$), waves of any length may propagate into the ice.

To determine the four complex constants in (2.1) and (3.5) we apply in turn the boundary conditions (1.18) and (1.20), which yield

$$\left.\begin{aligned} 1 + R &= T_1 + T_2 + T_3, \\ m_0(1 - R) &= m_1T_1 + m_2T_2 + m_3T_3, \end{aligned}\right\} \tag{3.11}$$

$$\left.\begin{aligned} 0 &= a_1T_1 + a_2T_2 + a_3T_3, \\ 0 &= m_1b_1T_1 + m_2b_2T_2 + m_3b_3T_3, \end{aligned}\right\} \tag{3.12}$$

in which

$$\left.\begin{aligned} a_i &= k_i^2 \{k_i^2 - (1 - \sigma)k^2\}, \\ b_i &= k_i^2 \{k_i^2 + (1 - \sigma)k^2\}. \end{aligned}\right\} \quad i = 1,2,3. \qquad (3.13)$$

From (3.12)

$$A_1 T_1 + A_0 T_2 = 0,$$

$$A_2 T_1 - A_0 T_3 = 0,$$

where

$$A_0 = a_2 m_3 b_3 - a_3 m_2 b_2 ,$$

$$A_1 = a_1 m_3 b_3 - a_3 m_1 b_1 ,$$

$$A_2 = a_1 m_2 b_2 - a_2 m_1 b_1 ,$$

and using these expressions in (3.11) we find that

$$R = \frac{P - Q}{P + Q}, \qquad T_1 = \frac{2}{P + Q}, \qquad (3.14)$$

where

$$P = 1 - \frac{A_1}{A_0} + \frac{A_2}{A_0} , \quad m_0 Q = m_1 - m_2 \frac{A_1}{A_0} + m_3 \frac{A_2}{A_0} . \qquad (3.15)$$

If we write

$$P = p_1 + ip_2 , \qquad Q = q_1 + iq_2 ,$$

then

$$\left.\begin{aligned} |R|^2 &= \frac{(p_1 - q_1)^2 + (p_2 - q_2)^2}{(p_1 + q_1)^2 + (p_2 + q_2)^2} , \\ |T_1|^2 &= \frac{4}{(p_1 + q_1)^2 + (p_2 + q_2)^2} , \end{aligned}\right\} \qquad (3.16)$$

which can be combined to give

$$E^2 |T_1|^2 + |R|^2 = 1, \qquad (3.17)$$

where

$$E^2 = p_1 q_1 + p_2 q_2 .$$

Using the fact that m_0, m_1, and k_1^2 are real and $k_2^2 = \overline{k_3^2}$, $m_2 = -\overline{m_3}$, it is easily found that

$$\left.\begin{aligned} p_1 &= 1 - a_1(m_3 b_3 - m_2 b_2)/A_0 \ , \\ p_2 &= -i m_1 b_1 (a_3 - a_2)/A_0 \ , \\ m_0 q_1 &= m_1 + m_1 b_1 (a_3 m_2 - a_2 m_3)/A_0 \ , \\ m_0 q_2 &= i a_1 m_2 m_3 (b_3 - b_2)/A_0 \ . \end{aligned}\right\} \qquad (3.18)$$

From (2.4) we know that

$$k_1^2 + k_2^2 + k_3^2 = 0,$$

$$k_1^2 k_2^2 + k_3^2 k_2^2 + k_3^2 k_1^2 = a,$$

$$k_1^2 k_2^2 k_3^2 = b,$$

and using these expressions, we find after some manipulation that

$$\begin{aligned} E^2 &= \frac{m_1}{m_0}\left\{1 + \frac{2k_1^6}{b}\right\} , \\ &= \frac{m_1}{m_0}\left\{3 - \frac{2ak_1^2}{b}\right\} . \end{aligned} \qquad (3.19)$$

As λ becomes large and $k_1 \to k_0$, it is easily seen that

$$E^2 \to 1 + O(k_0^2).$$

Reflection, transmission and pressure coefficients.

In defining transmission and pressure coefficients in $y > 0$, we consider only the undamped part of the solution, and for sufficiently large y,

$$G(y) \approx T_1 \, e^{m_1 i y} \ .$$

To determine the elevation above the undisturbed level, we have

$$\epsilon_t = h\nabla^2\phi \ ,$$

ignoring second order terms, and so

$$\epsilon_t = h \left\{\partial^2/\partial y^2 - k^2\right\} G(y)\ e^{ikx - i\omega t} \ .$$

For the transmitted wave therefore

$$\epsilon = -\frac{ih}{\omega}\left\{m_1{}^2 + k^2\right\} T_1\ e^{im_1 y + ikx - i\omega t} \ ,$$

having an amplitude

$$\frac{hk_1{}^2}{\omega}\left|T_1\right| \ .$$

Similarly for the incident wave

$$\epsilon = -\frac{ihk_0{}^2}{\omega}\ e^{im_0 y + ikx - i\omega t} \ ,$$

with an amplitude

$$\frac{hk_0{}^2}{\omega} \ ,$$

and the reflected wave has amplitude

$$\frac{hk_0{}^2}{\omega}\left|R\right| \ .$$

We define the transmission coefficient τ to be the ratio of the amplitude of the transmitted wave to the amplitude of the incident wave,

$$\tau = \frac{k_1{}^2}{k_0{}^2}\left|T_1\right| \ , \qquad (3.20)$$

and the reflection coefficient κ to be the ratio of the amplitude of the reflected wave to the amplitude of the incident wave,

$$\kappa = |R|.$$

From (3.14) and (3.17) we have an immediate relation between τ and κ,

$$\frac{k_0{}^4 m_1}{k_1{}^4 m_0}\left\{1 + \frac{2k_1{}^6}{b}\right\}\tau^2 + \kappa^2 = 1, \tag{3.21}$$

which serves to check the computed values of τ and κ.

Bernoulli's equation gives the pressure on the bottom

$$p = p_0 + \rho[\Phi_t]_{z=0} + \rho gh,$$

so that the pressure on $z = 0$ in excess of atmospheric pressure and the normal hydrostatic pressure is

$$p(x,y,0,t) = \rho[\Phi_t]_{z=0} ,$$

where

$$\Phi_t = -i\omega G(y)\, e^{ikx - i\omega t} .$$

Hence the amplitudes of the pressure fluctuation on $z = 0$ are

$$\rho\omega|T_1| \qquad \text{(transmitted wave, } y > 0),$$

$$\rho\omega \qquad \text{(incident wave, } y < 0).$$

Defining the pressure coefficient Θ to be the ratio of these two amplitudes, we have simply

$$\Theta = |T_1| .$$

4. Numerical Results

From Robin [4] we have the following values for ice

Young's modulus, $E = 5 \times 10^{10}$ dynes/cm^2 ,

Poisson's ratio, $\sigma = 0.3$.

Also

density of sea water, $\rho = 1.025$ gm/cm^3 ,

density of ice, $\rho_i = 0.92$ gm/cm^3 ,

$g = 980.7$ cm/sec^2.

Hence we find that

$$D = 4.58 \times 10^9 \times h_o^3,$$

$$a = \frac{1005}{D}\left\{1 - \frac{\lambda_{min}^2}{\lambda^2}\right\},$$

$$b = \frac{39670}{D \times \lambda^2},$$

$$\lambda_{min}^2 = 35.42 \times h \times h_o,$$

$$\lambda_c = 0.209 \sqrt{\left\{\frac{D}{hh_o}\right\}}.$$

Water depths of 10 and 20 metres are considered together with ice thicknesses of 0.5, 1, 2, and 4 metres. For each value of h and h_o, λ_{min} and λ_c are evaluated. Using the expressions derived in section 3, the values of τ, κ, Θ and ψ are computed for a range of incident wavelengths, and the incident angles 0°, 15°, 30°, 45° and 60°. In addition θ_c is calculated for $\lambda < \lambda_c$ in each case.

As a check on the values of τ and κ, the expression

$$E^2 \frac{k_0^4}{k_1^4} \tau^2 + \kappa^2 - 1,$$

where E^2 is given by (3.19), is computed and by (3.21) should be identically zero. It is found to be negligible in all cases.

The Critical Angle.

In Figs. 2 and 3 the critical angle θ_c is plotted against the incident wavelength λ, θ_c being equal to $\pi/2$ for $\lambda = \lambda_c$.

As was shown in section 3, if $\lambda < \lambda_c$ then there is real propagation into the ice provided $\theta < \theta_c$, while there is total reflection if $\theta > \theta_c$. For each value of the ice thickness h_0, the incident angle θ, and the depth h, we can determine from the corresponding curve the value λ_0 of λ for which $\theta = \theta_c$. It follows that incident waves for which $\lambda < \lambda_0$ are totally reflected, while those for which $\lambda > \lambda_0$ undergo some real propagation. It is found that, for a fixed h_0 and θ, the value λ_0 is almost the same for the two selected depths.

The values of λ_0 are not displayed explicitly, but are used in the graphs of incident wavelength against τ, κ and Θ.

The Transmitted Angle and Wavelength.

The values obtained for the transmitted angle ψ are not displayed. It is found that ψ decreases from $\pi/2$ at λ_0 so that $\psi = \theta$, the incident angle, at $\lambda = \lambda_c$. For $\lambda > \lambda_c$,ψ decreases further but finally increases again with increasing wavelength, so that for sufficiently long waves it is always equal to the incident angle.

This confirms the result that $k_1 \to k_0$ for large λ.

Closely associated with ψ is the transmitted wavelength, which is simply $2\pi/k_1$, and it is found to follow the trends indicated above, tending to the incident wavelength for sufficiently large values of the latter.

The Transmission and Reflection Coefficients.

τ and κ are plotted against the incident wavelength for the various incidence angles and values of ice thickness. For $\theta = 0^0$, the curves representing τ have been extended to pass through the origin, though in fact we see from (3.20) that τ strictly becomes infinite when the incident wavelength is zero.

It is seen from the curves that the water depth is hardly significant in the transmission of wave energy into the icefield, except at the smaller wavelengths. In contrast the thickness of the ice is seen to be a crucial factor; for example, in the case of normal incidence and a wavelength of 300 metres, the transmitted amplitude undergoes a diminution of about 25% when the ice thickness is increased from 2 metres to 4 metres. In other words, since the energy of a wave is proportional to the square of its amplitude, there is approximately a 40% reduction in the transmitted energy when the ice thickness is doubled.

Further, it is seen from Figs. 4 - 9 that the incident angle plays a dominant part in wave energy acceptance by the ice structure.

At the various values of λ_0 determined from the critical angle versus wavelength curves, the graphs of τ drop to zero, corresponding to total reflection, for wavelengths less than λ_0. Provided that transmission occurs, the transmitted amplitude is seen to increase with the incident angle for a fixed value of λ, although for the larger ice thicknesses and greater incident angles , the curves representing τ suffer some distortion and decrease to a minimum value at some wavelength, before steadily approaching unity. This phenomenon is first seen for h_0 = 4.0 metres when the angle of incidence is only 30^0, and becomes more marked as θ increases, until at an incident angle of 60^0 occurs for all values of the ice thickness but the smallest, h_0 = 0.5 metres.

As one would intuitively expect, transmission is almost total at the longer wavelengths, and for ice thicknesses 0.5, 1.0 and 2.0 metres, and all incidence angles, the transmitted amplitude is almost 100% of the incident amplitude for $\lambda > 600$ metres, while in the case h_0 = 4.0 metres this proportion is approximately 95% for the same range of λ.

Reflection of the incident wave is small except at the lower wavelengths, for which it is slightly less at the greater depth , 20 metres.

The Pressure Coefficient.

Figs. 10 - 14 display the pressure coefficient as a function of the incident wavelength, for the various values of ice thickness and incident angle. As we would expect, Θ tends to unity as the wavelength increases.

For normal incidence, the pressure on the bottom under the ice is found to be less than the corresponding pressure in the absence of ice, this phenomenon being more marked for the larger values of ice thickness. However, as the incident angle is increased to 15°, these curves change radically so that in general, Θ is at a maximum at the first value of λ for which transmission occurs, decreases to a minimum value, and finally tends to unity. This occurs for all subsequent incidence angles, the band of wavelengths for which Θ is actually greater than unity being more marked as θ increases. For $\theta = 60^{\circ}$ and an ice thickness of 4.0 metres, the pressure on the bottom for an incident wavelength of about 290 metres is found to be almost double the pressure in the absence of ice.

Once again the water depth is found to be of little significance except that the peaks of pressure are slightly greater for the deeper water and larger incident angles.

5. Conclusion.

Robin [4] deduced from his observational data that when floes are over 100 metres in diameter, there is little transmission of energy for wavelengths less than 200 metres. Measurements leading to this conclusion were made in an icefield of loosely floating floes in which the wavelengths were large compared to the open leads of water between the floes, and to the thickness of the floes. It is therefore reasonable to assume that the wave energy is not confined to these open leads, and that the bending of the ice must be an inherent part of the propagation process. Robin also found that for very large floes of the order of 1000 metres or more across, there is some energy loss for wavelengths of less than 400 metres, and also that the thickness of the ice contributes to major energy changes.

It is perhaps unrealistic to attempt to correlate results based on shallow water theory with observed phenomena. However, qualitatively our above results certainly confirm the fact that ice thickness is a decisive factor in the wave energy acceptance of large floes. Quantitatively, there is some discrepancy between the two sets of results; for example, in the case of normal incidence, when $\lambda = 85$ metres and $h_0 = 2.0$ metres, then $\kappa \approx 0.3$, so that more than 90% of the energy is transmitted. The numerical dissimilarity between the present results and the conclusions of Robin is almost certainly due to the use of the shallow water approximation, and suggests that the retention of an

arbitrary ocean depth would yield more meaningful results. This problem has been attempted but a suitable method of resolving it has yet to be found.

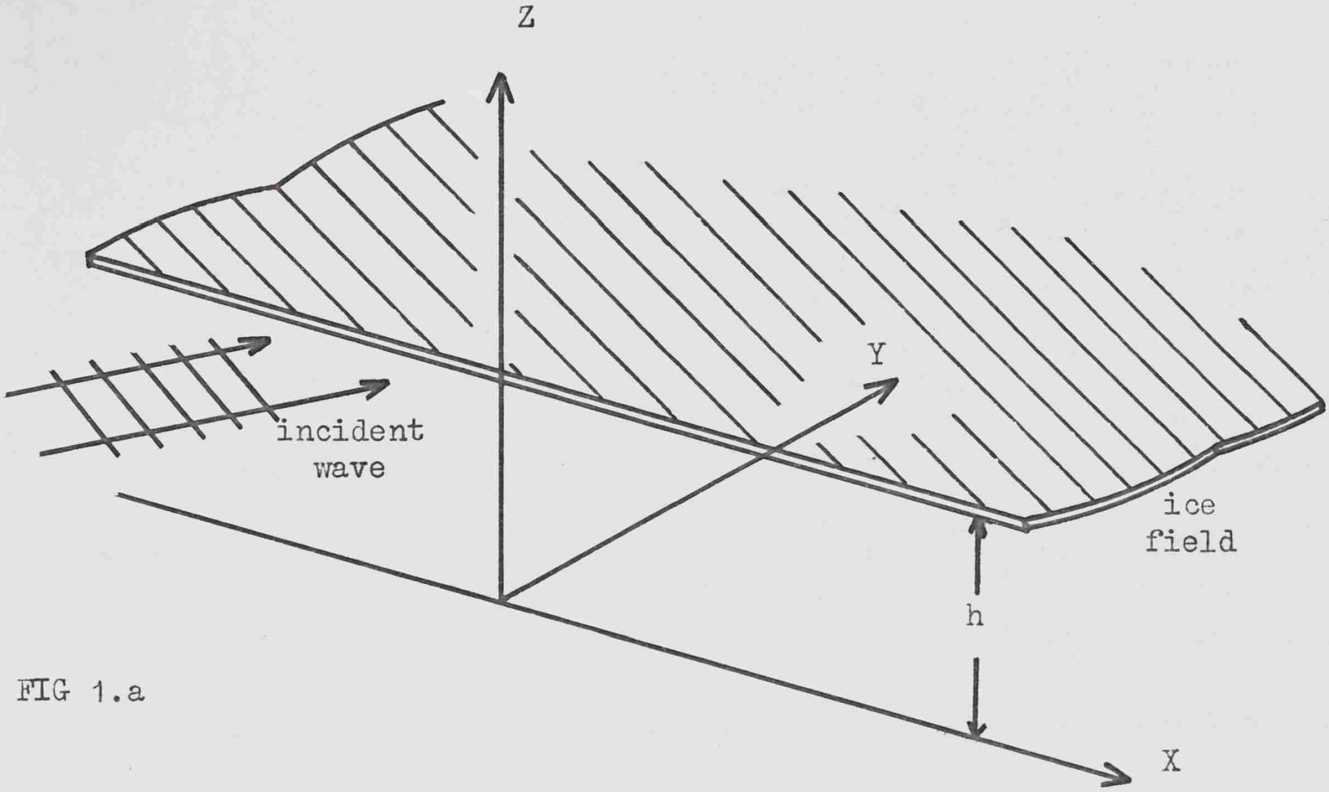

Z
Y
incident
wave
ice
field
h
X

FIG 1.a

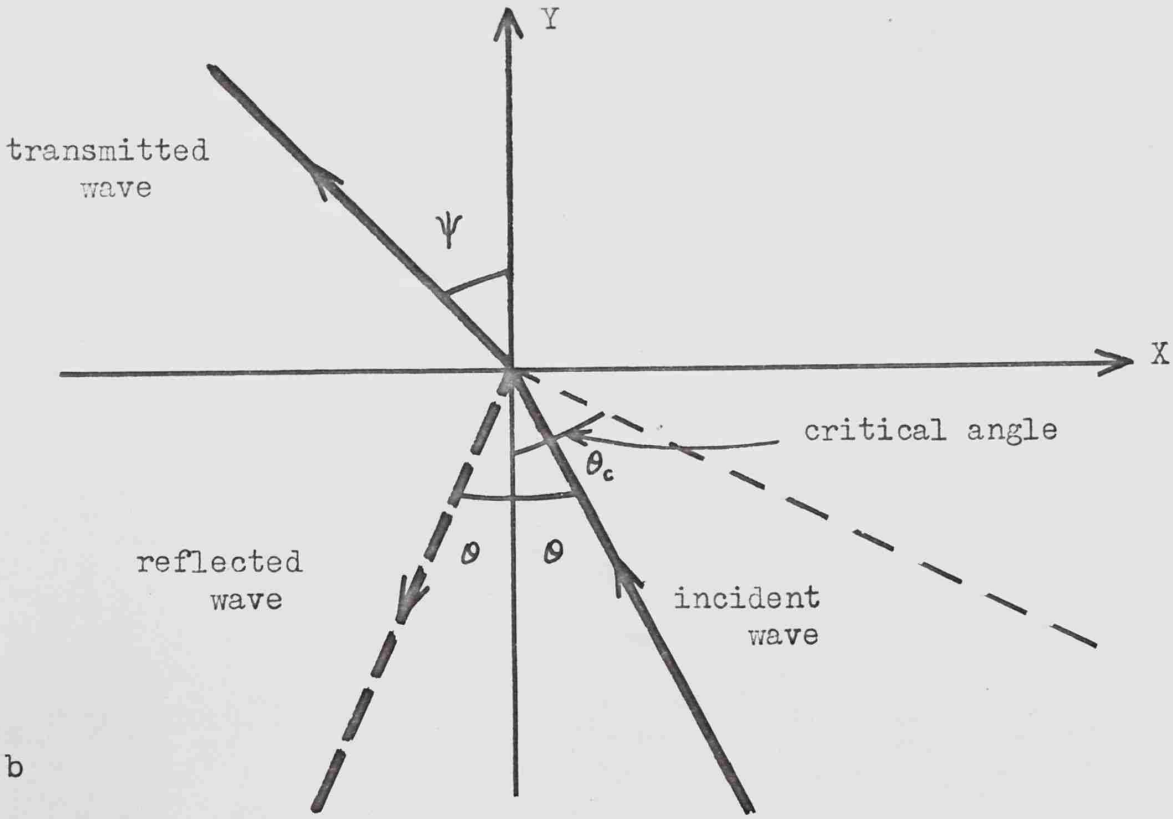

Y
transmitted
wave
ψ
X
critical angle
θ_c
reflected
wave
θ
θ
incident
wave

FIG 1.b

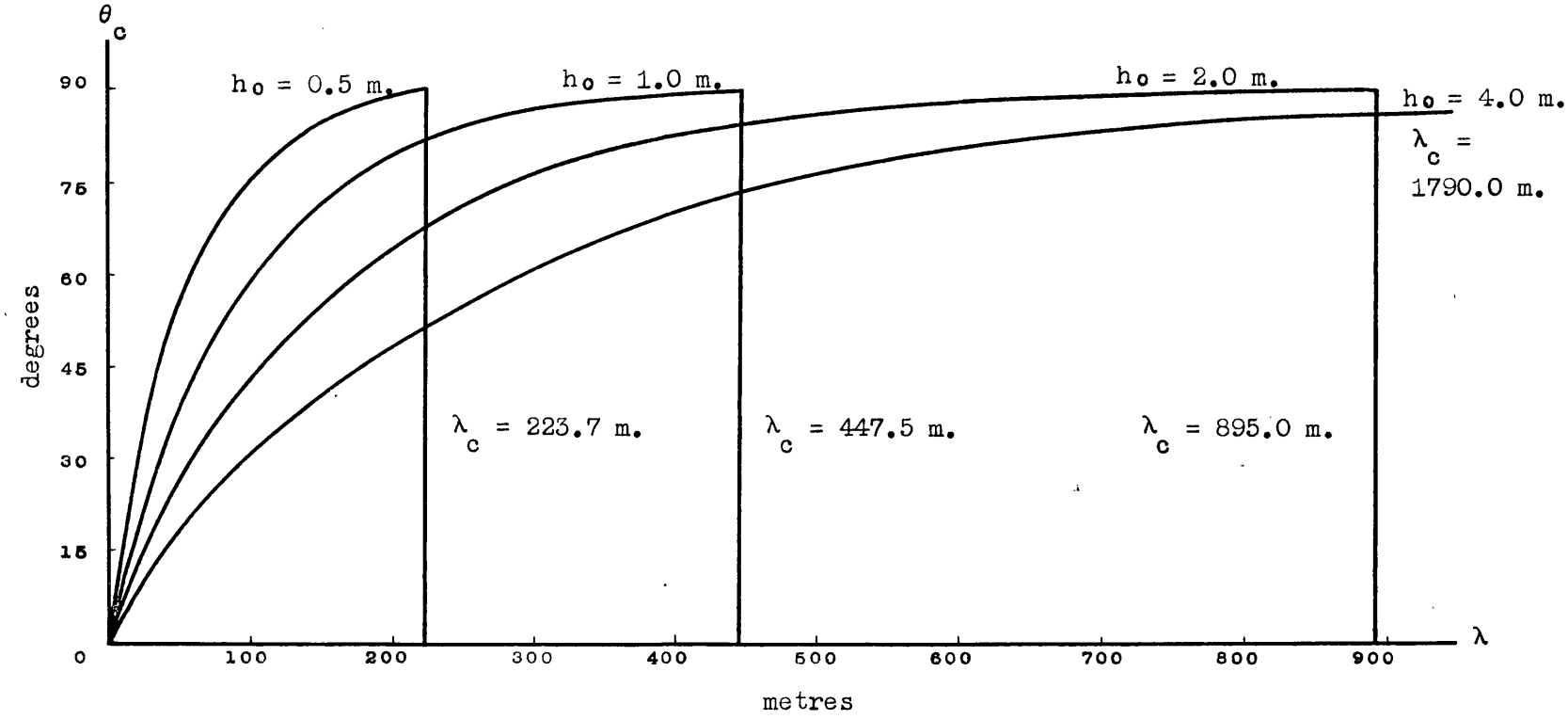


FIG. 2 : incident wavelength vs. θ_c , h = 10 metres.

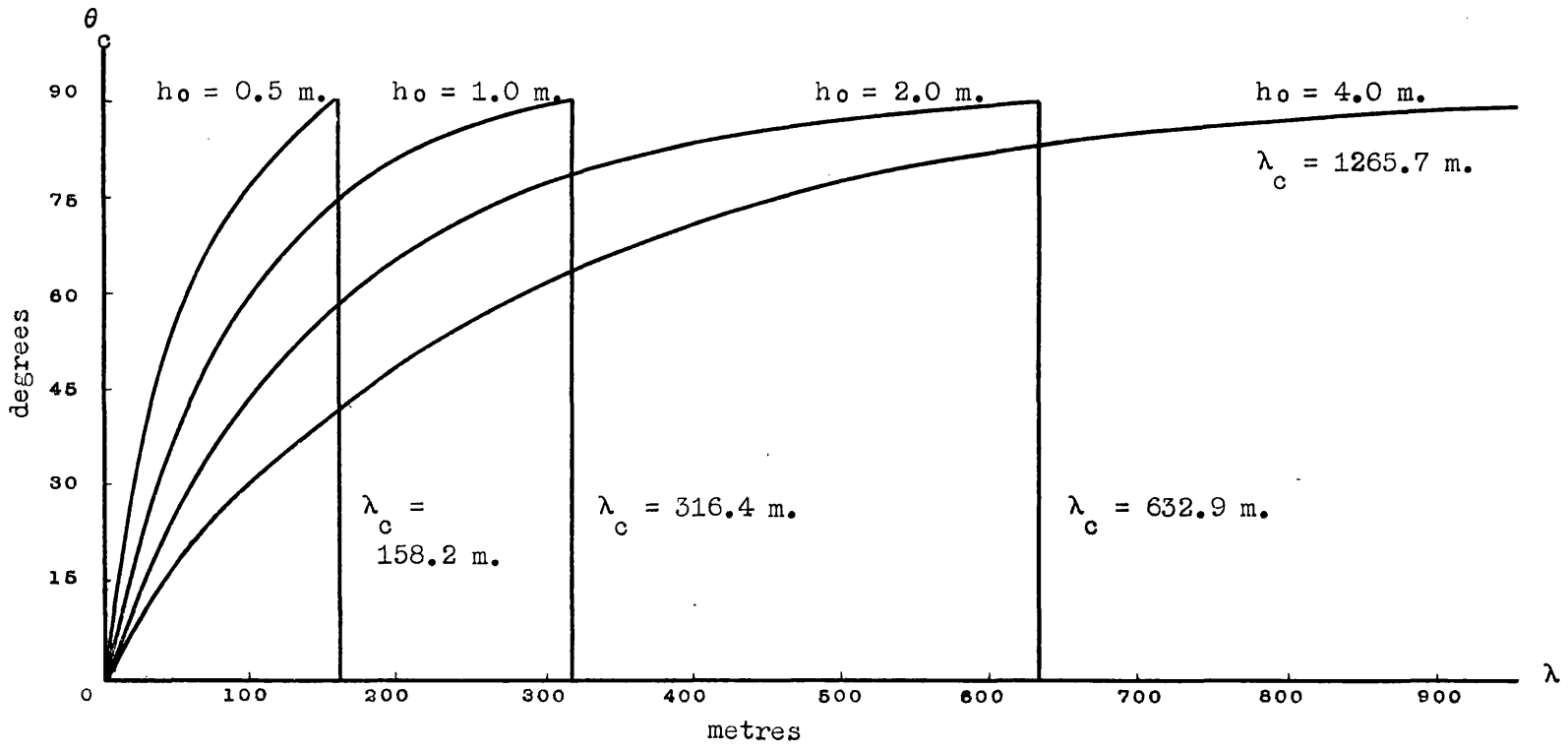


FIG. 3 : incident wavelength vs. θ_c , h = 20 metres.

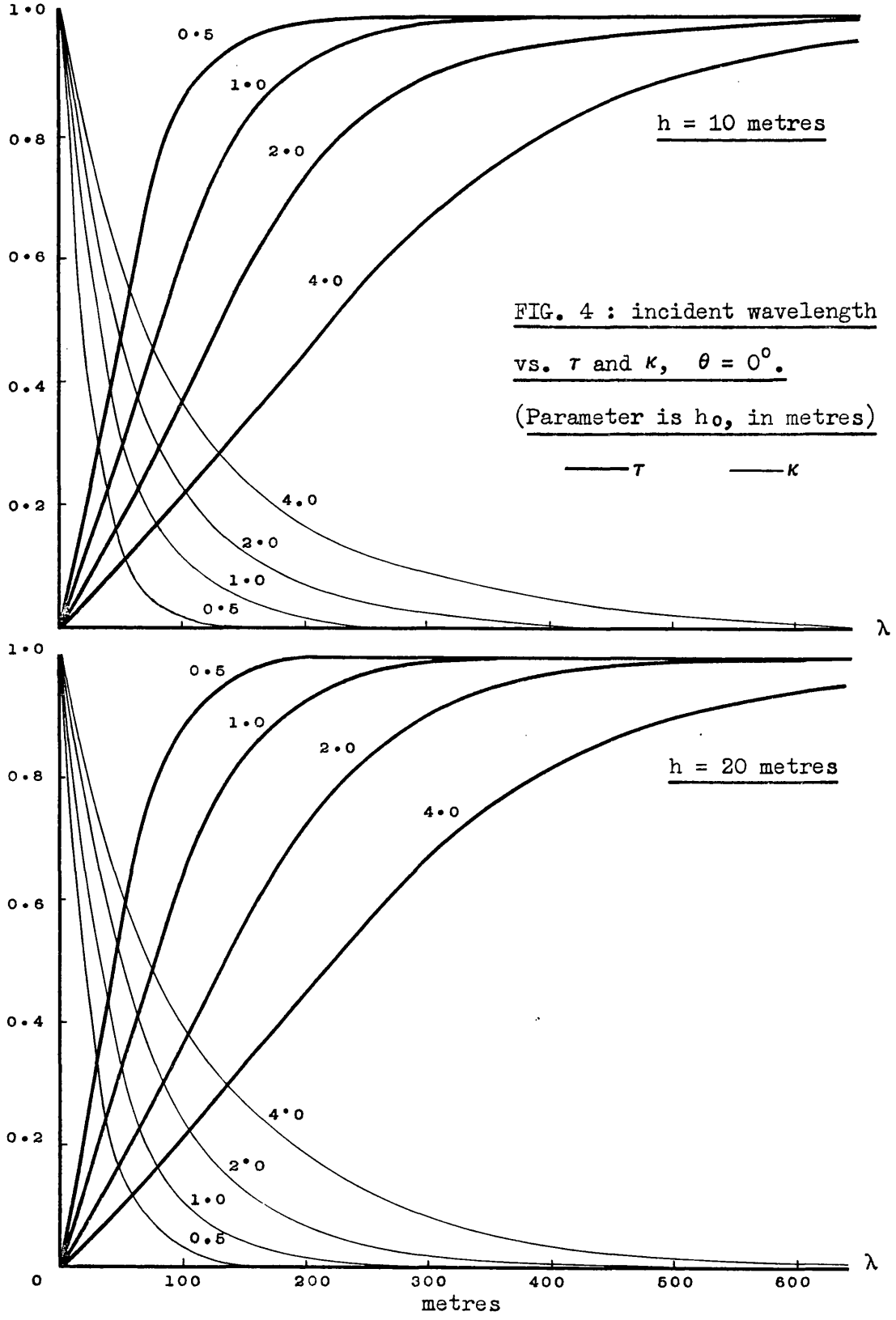


FIG. 4 : incident wavelength vs. τ and κ, $\theta = 0^{\circ}$.

(Parameter is h_0, in metres)

——— τ ——— κ

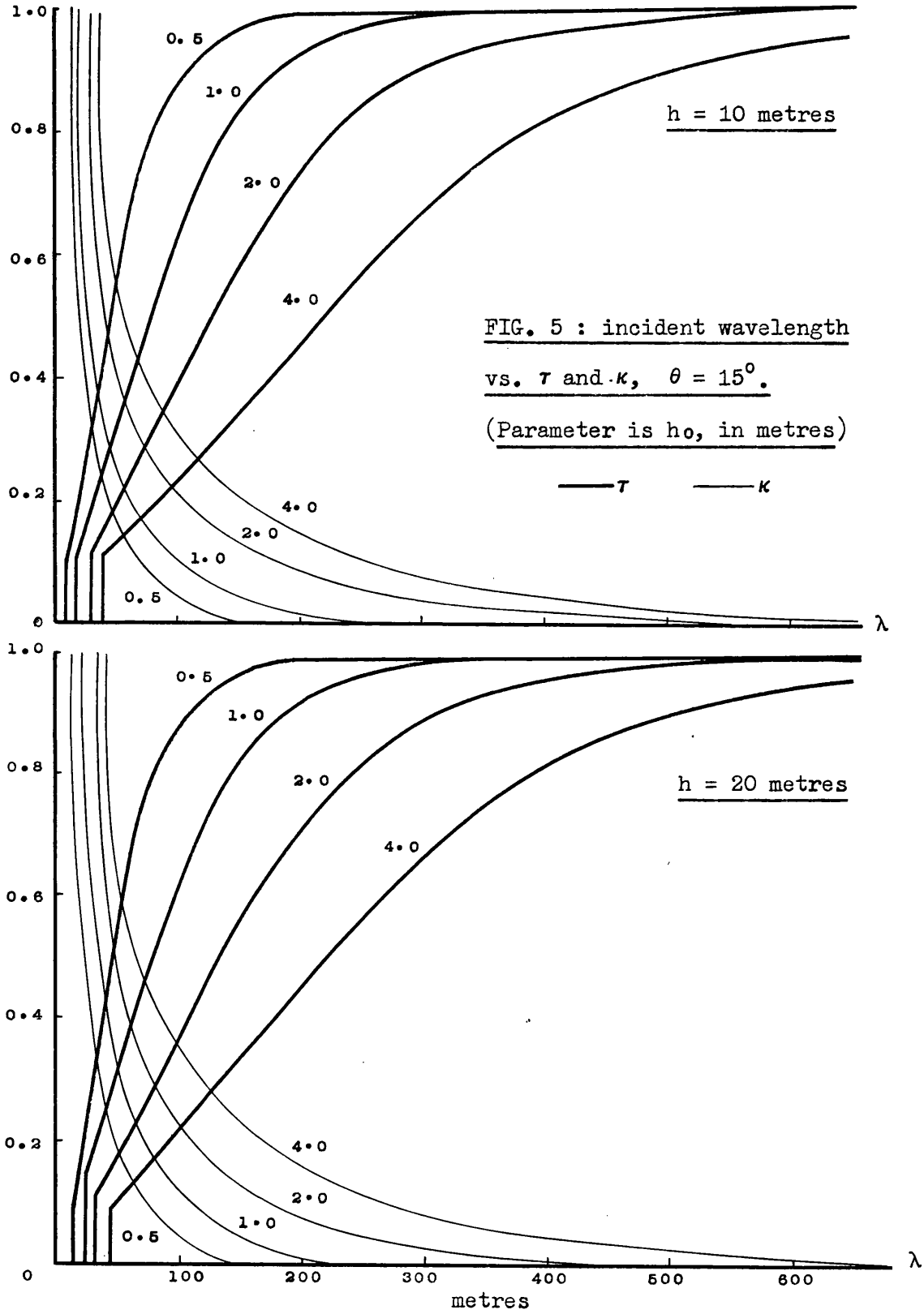


FIG. 5 : incident wavelength vs. τ and κ, $\theta = 15^{\circ}$.

(Parameter is h_0, in metres)

—— τ —— κ

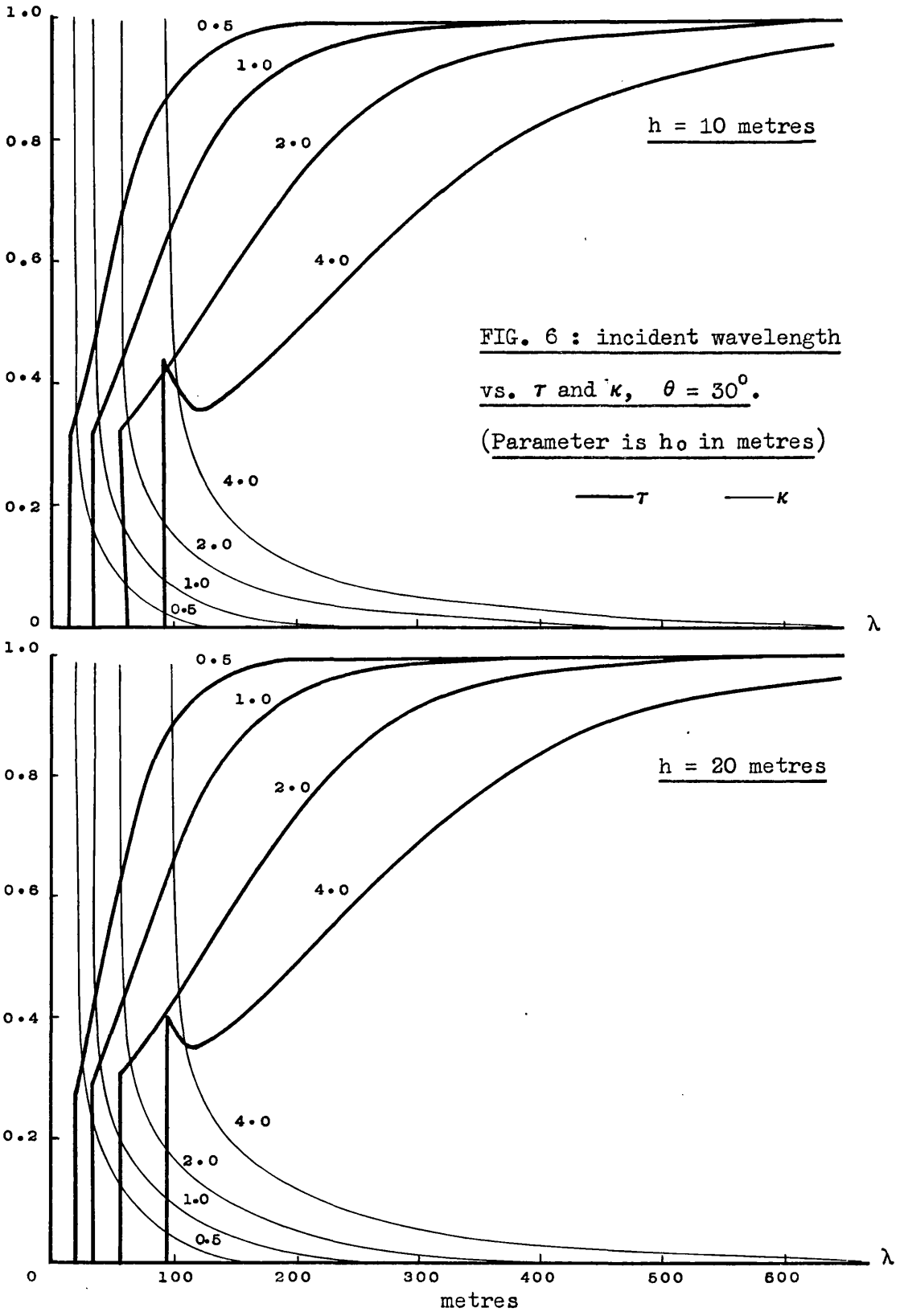
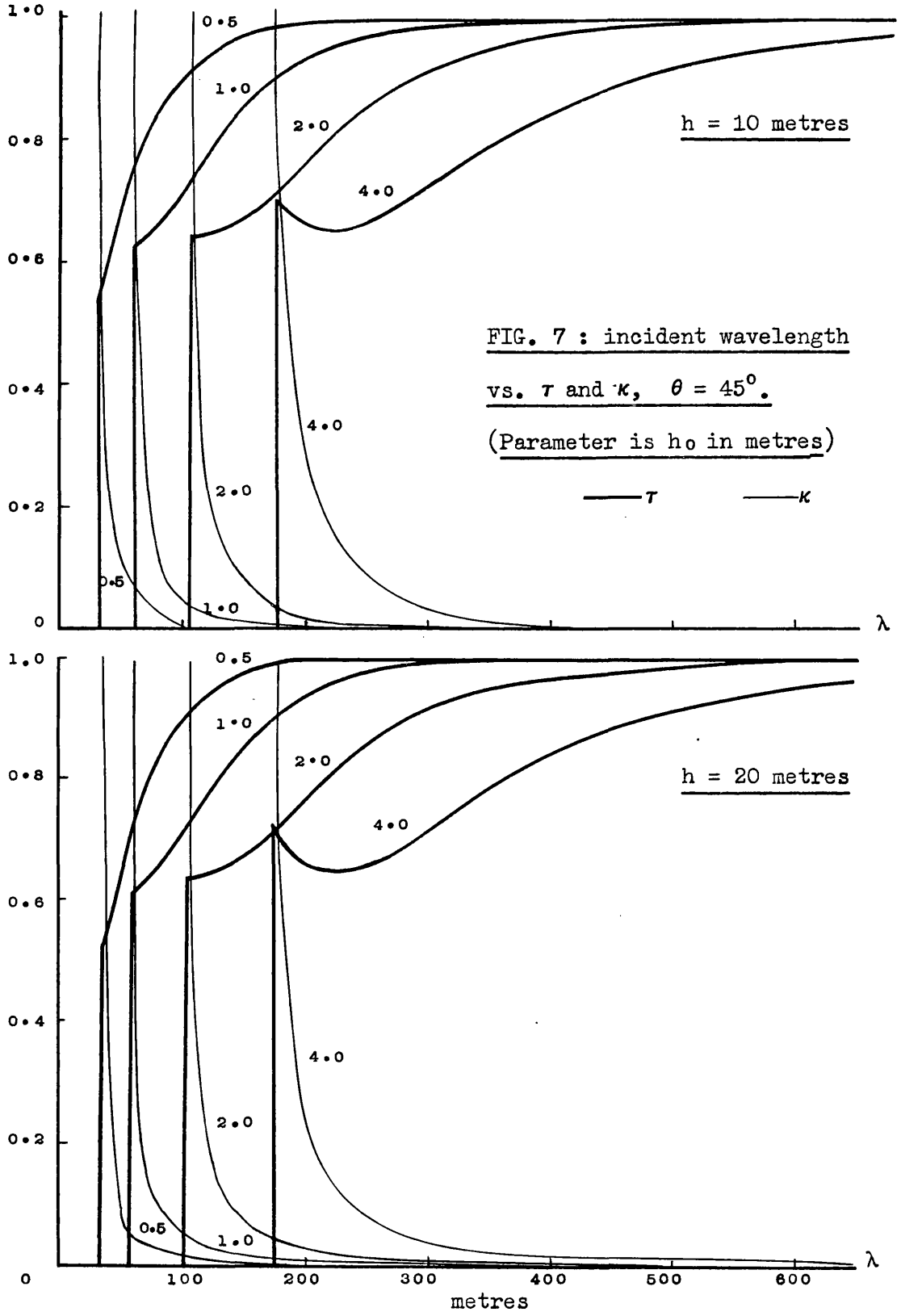


FIG. 6 : incident wavelength vs. τ and κ, $\theta = 30^{\circ}$.

(Parameter is h_0 in metres)

—— τ —— κ


1.0
0.8
0.6
0.4
0.2
0
0.5
1.0
2.0
4.0
h = 10 metres
FIG. 7 : incident wavelength
vs. τ and κ, θ = 45°.
(Parameter is h₀ in metres)
τ
κ
λ
h = 20 metres
100
200
300
400
500
600
metres

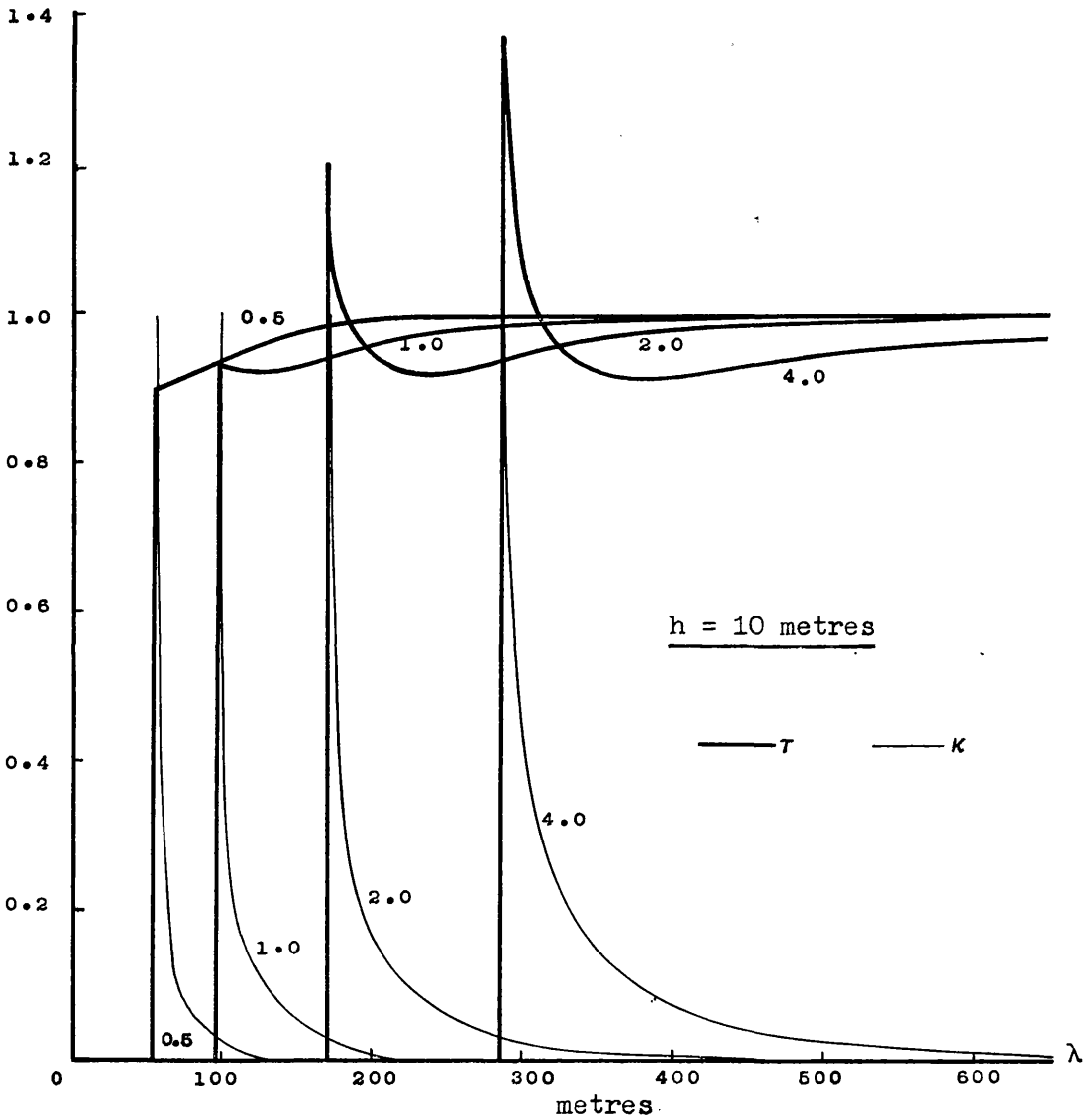


FIG. 8 : incident wavelength vs. τ and κ, $\theta = 60^{\circ}$.

(Parameter is h_0 in metres)

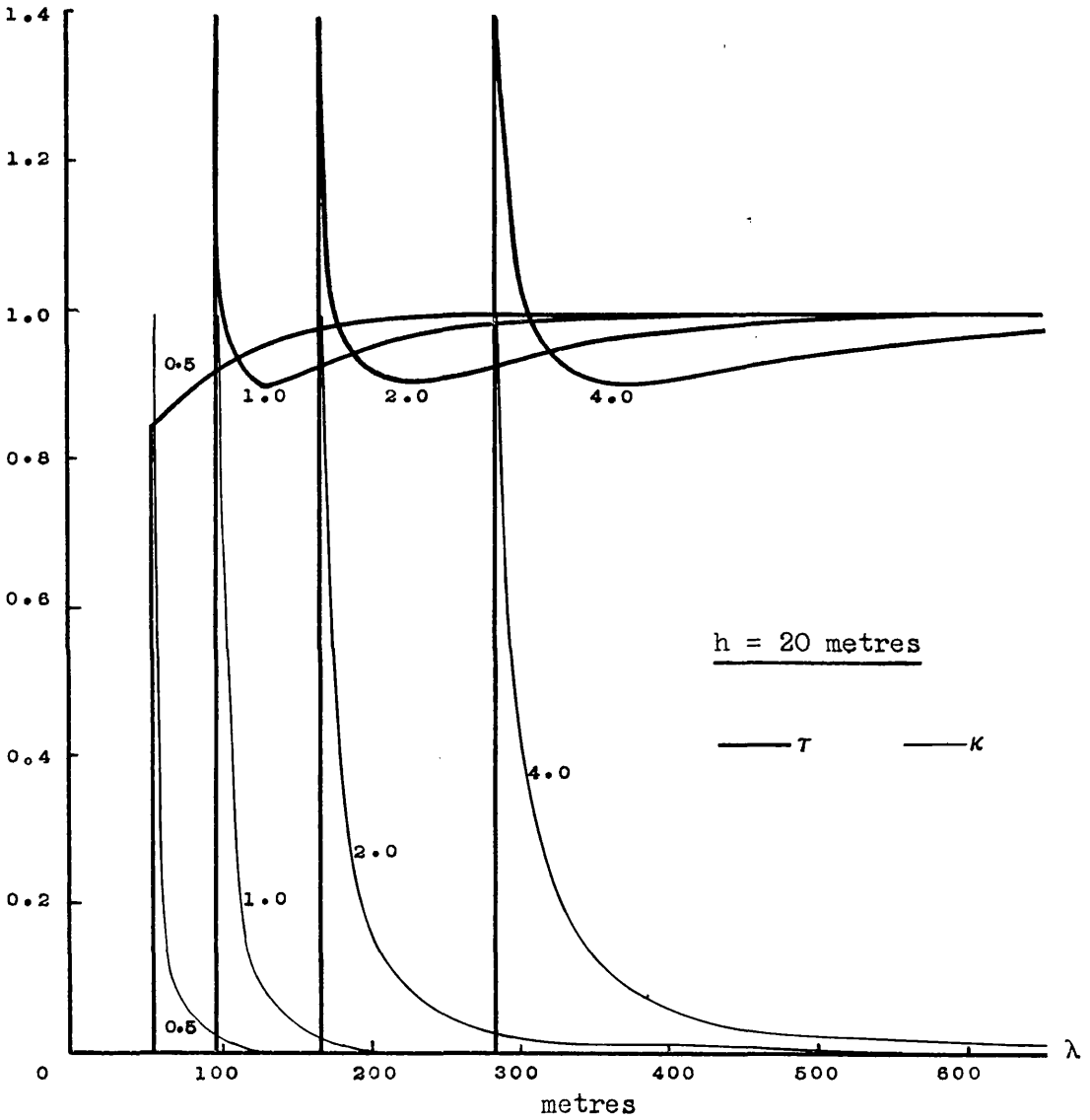
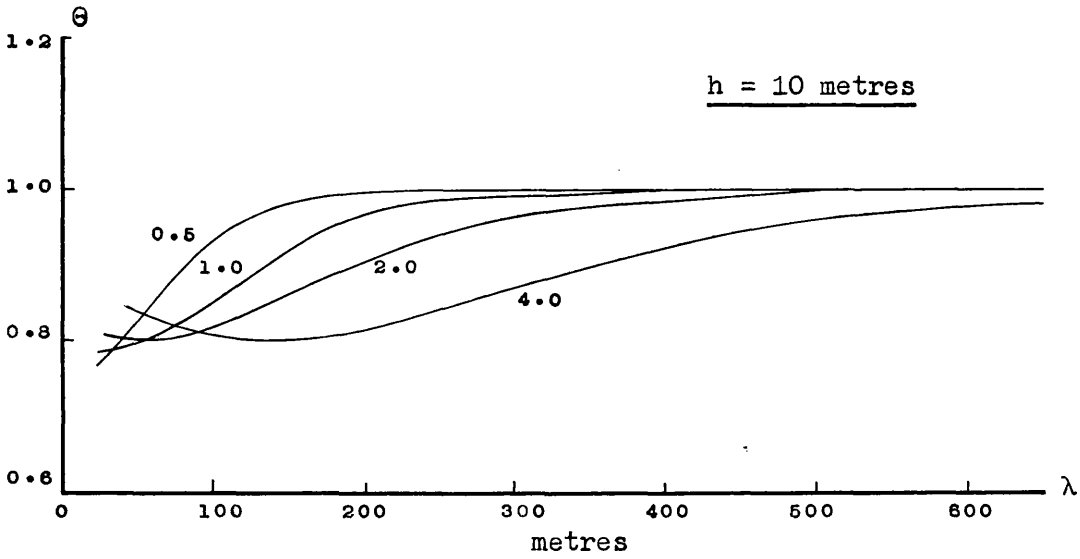


FIG. 9 : incident wavelength vs. τ and κ, $\theta = 60^{\circ}$.

(Parameter is h_0 in metres)



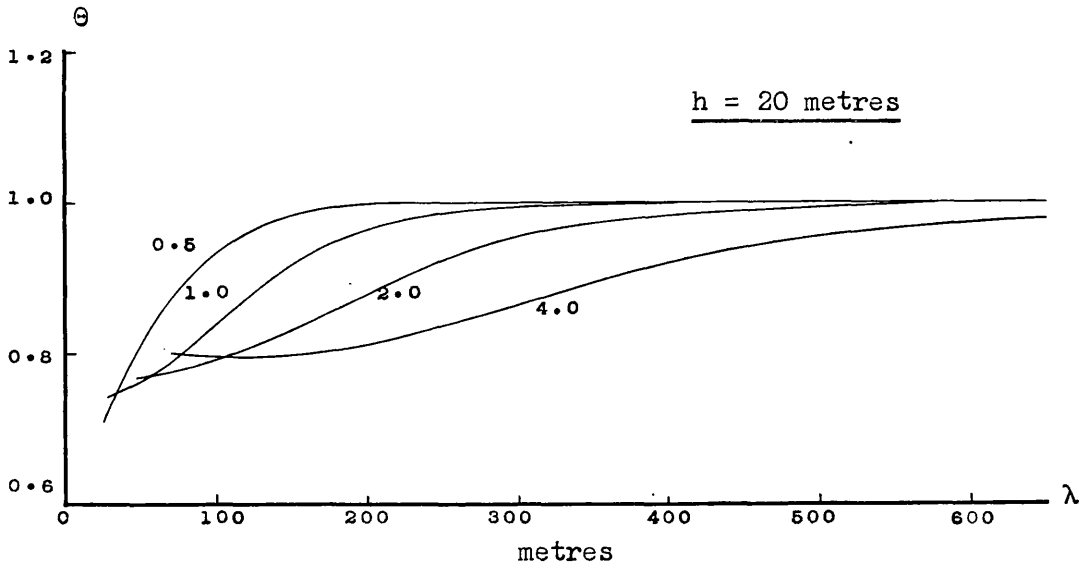
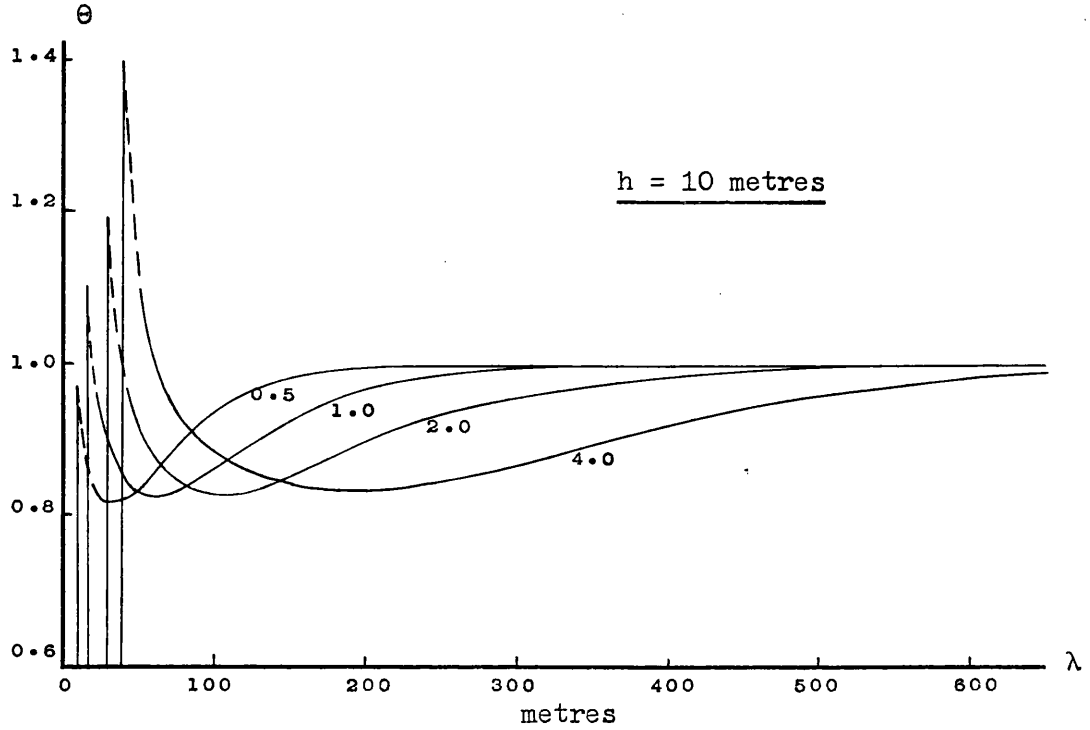


FIG. 10 : incident wavelength vs. Θ, $\theta = 0^{\circ}$.

(Parameter is h_0 in metres)



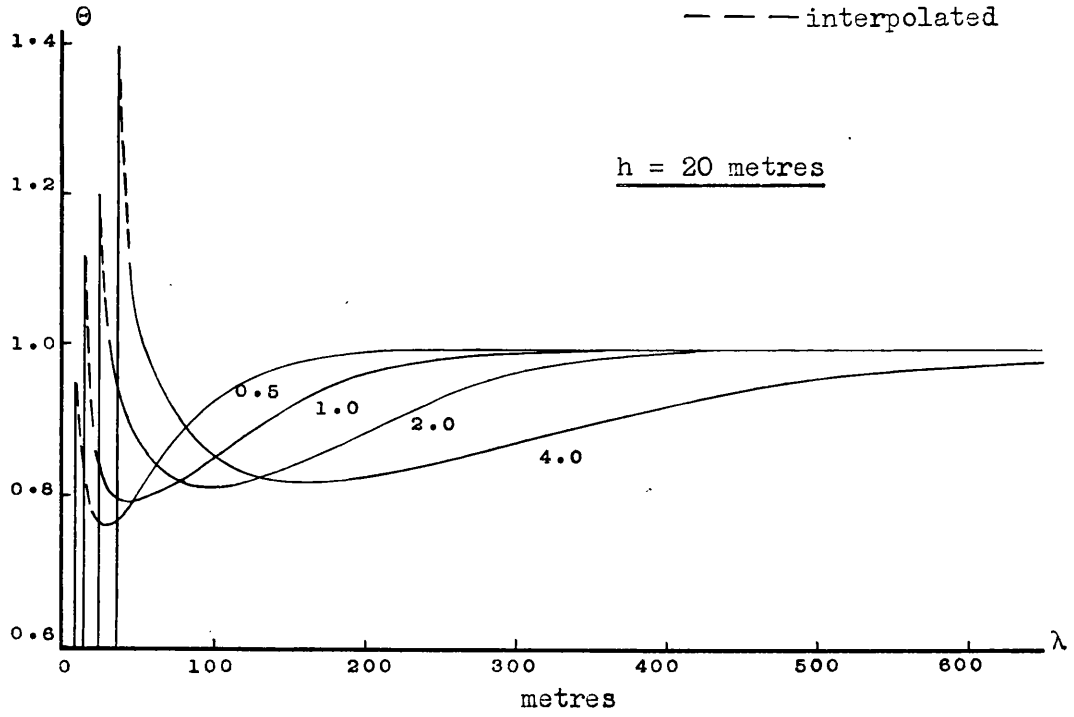
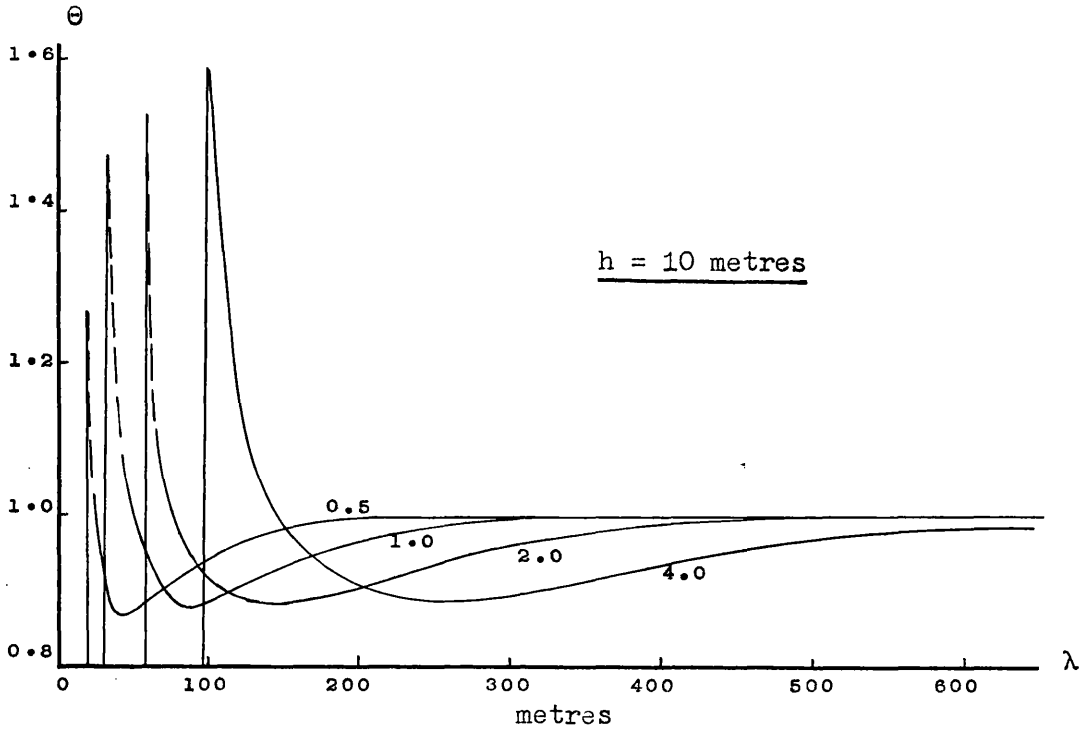


FIG. 11 : incident wavelength vs. Θ, $\theta = 15^{\circ}$.

(Parameter is h_0 in metres)



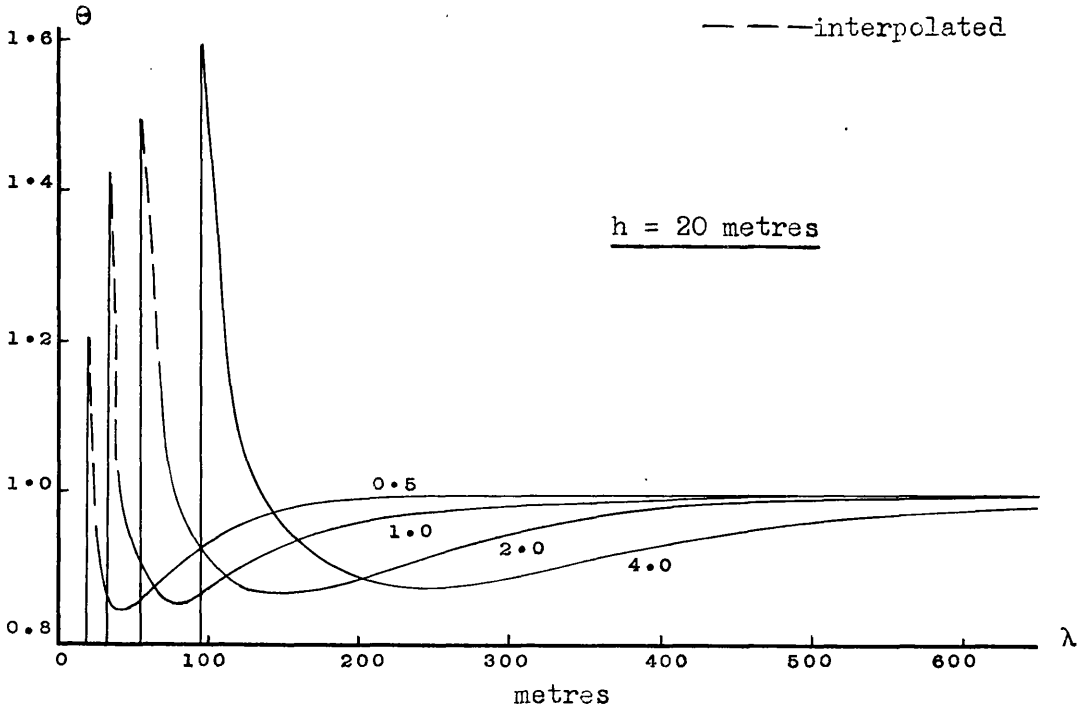
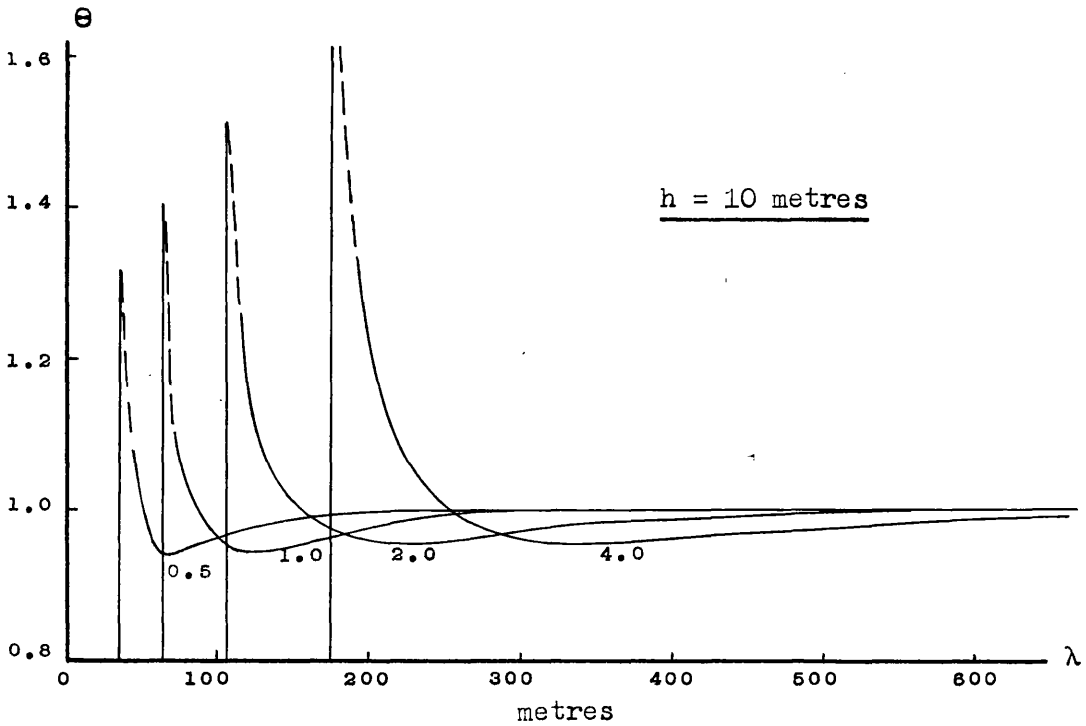


FIG. 12 : incident wavelength vs. Θ, $\theta = 30^{\circ}$.

(Parameter is h_0 in metres)



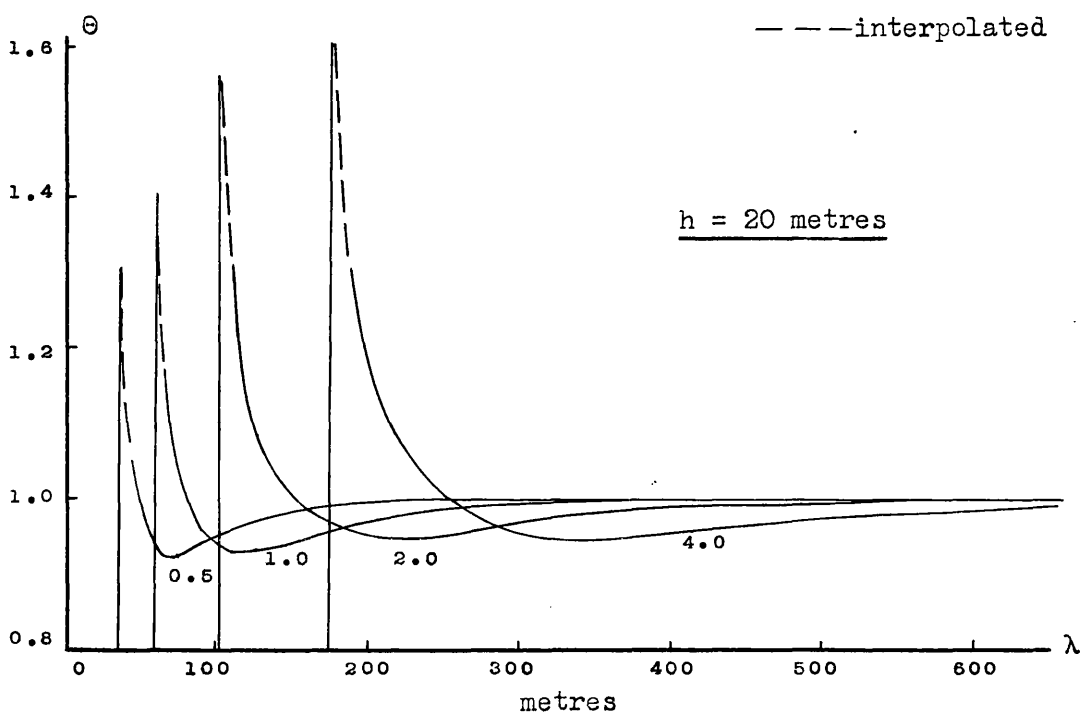


FIG. 13 : incident wavelength vs. Θ, $\theta = 45^{\circ}$.

(Parameter is h_0 in metres)

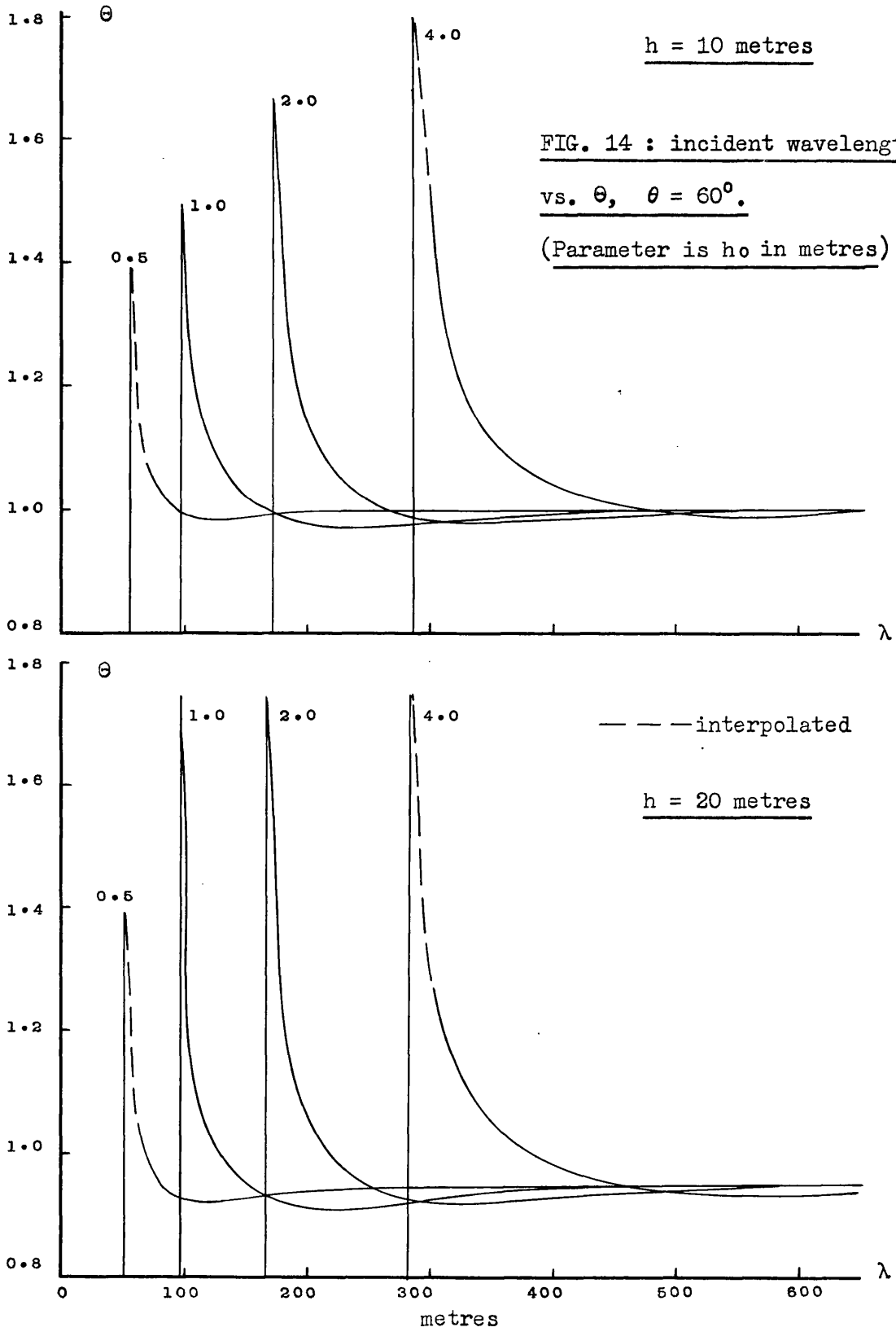


FIG. 14 : incident wavelength vs. Θ, $\theta = 60^{\circ}$.

(Parameter is h_0 in metres)

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[1] Chi-Teh Wang, "Applied Elasticity",Chapter 11, McGraw-Hill.

[2] Keller, J.B. and Goldstein, E., "Water wave reflection due to surface tension and floating ice", Transactions, American Geophysical Union, vol. 34, no. 1, (1953).

[3] Timoshenko, S. and Woinowsky-Krieger, S., "Theory of Plates and Shells", Chapter 4, McGraw-Hill, 1959.

[4] Robin, G. de Q., "Wave propagation through fields of pack ice", Philosophical Transactions of the Royal Society of London, A255, 313 - 339, (1963).

CHAPTER III

WAVE TRANSMISSION IN A SEMI-INFINITE ICE FIELD OF VARIABLE THICKNESS, ON THE BASIS OF LINEARISED SHALLOW WATER THEORY.

1. Formulation of the Equations.

As in Chapter II this problem is approached on the basis of linearised shallow water theory; that is, we assume that the wavelength of the incident ocean wave is long compared with the depth of the water. The icefield is ~~again~~ supposed made up of non-interacting elements, this time of thickness $H(x,y)$, floating on water of uniform depth h in $o < y < \infty$, $-\infty < x < \infty$. In the first place the ice thickness is assumed to vary sinusoidally about a mean value, and secondly, using a different approach, a rectangular wave type of ice thickness is considered.

The velocity potential for the fluid motion satisfies

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \qquad o < z < h, \tag{1.1}$$

$$\Phi_z = 0, \qquad z = o. \tag{1.2}$$

As in Chapter II the free surface condition reduces to

$$\Phi_{tt} + g\Phi_z = 0, \qquad -\infty < y < o, \; z = h. \tag{1.3}$$

Assuming that the ice sheet is represented by

$$z = h + \eta(x,y,t),$$

then the pressure difference across it is given by Bernoulli's equation,

$$p - p_o = \rho\{\Phi_t - g\eta\},$$

where p_o is atmospheric pressure. In the absence of surface tension forces and flexural rigidity forces, each element of the ice sheet is governed by the equation of motion

$$p - p_o = \rho_i \, H(x,y) \, \eta_{tt} \, ,$$

so that

$$\Phi_t - g\eta = s \, H(x,y) \, \eta_{tt} \tag{1.4}$$

where $s = \rho_i/\rho$.

Introducing the shallow water potential by

$$\Phi(x,y,z,t) = \phi(x,y,t) - \tfrac{1}{2}z^2 \, \{\phi_{xx} + \phi_{yy}\} + O(z^3),$$

which satisfies (1.1) to the first order in h, and using the linearised kinematic boundary condition

$$\eta_t = -\Phi_z \, ,$$

then the boundary condition (1.4) in terms of $\phi(x,y,t)$ is

$$\phi_{tt} - gh\nabla^2\phi - shH(x,y) \, \nabla^2\phi_{tt} = 0, \quad o < y < \infty, \tag{1.5}$$

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/dy^2.$$

Similarly equation (1.3) reduces to

$$\phi_{tt} - gh\nabla^2\phi = 0, \qquad -\infty < y < o. \tag{1.6}$$

We assume that

$$\phi(x,y,t) = \exp(-i\omega t) \, F(x,y), \tag{1.7}$$

so that (1.5) and (1.6) become

$$\left\{1 - \frac{\omega^2 sH(x,y)}{g}\right\} \nabla^2 F + \frac{\omega^2}{gh} F = 0, \quad o < y < \infty, \tag{1.8}$$

$$\nabla^2 F + \frac{\omega^2}{gh} F = 0, \quad -\infty < y < o. \tag{1.9}$$

We require that the respective solutions of these equations be continuous at $y = 0$, together with their first derivatives F_y, to ensure continuity of the velocity components in the fluid.

2. Sinusoidal Ice Thickness Variation in the y Direction with Normally Incident Waves.

We consider here waves which are incident normally on the water - ice interface $y = 0$, and restrict the ice thickness variation to the y direction. The shallow water potential $\phi(y,t)$ is thus determined by

$$\left\{1 - \frac{\omega^2 sH(y)}{g}\right\} \frac{d^2F}{dy^2} + \frac{\omega^2}{gh} F = 0, \quad o < y < \infty, \tag{2.1}$$

$$\frac{d^2F}{dy^2} + \frac{\omega^2}{gh} F = 0, \quad -\infty < y < o \tag{2.2}$$

$$\phi(y,t) = \exp(-i\omega t)\, F(y).$$

To simulate the effect of ice floes we take

$$H(y) = h_0 - h_1 \cos ky, \tag{2.3}$$

in which

$$h_0 \geqslant h_1 > o,$$

and the differential equation for $F(y)$ in $o < y < \infty$ becomes,

$$\left[\left\{1 - \frac{\omega^2 sh_0}{g}\right\} + \frac{\omega^2 sh_1}{g} \cos ky\right] \frac{d^2F}{dy^2} + \frac{\omega^2}{gh} F = 0. \qquad (2.4)$$

It is not necessary to work with this equation as it stands, for if we insert typical values of the various parameters, say $h_0 = 1$ metre, $h_1 = 0.5$ metre, then we find that

$$\frac{\omega^2 sh_0}{g} << 1, \qquad \frac{\omega^2 sh_1}{g} << 1,$$

provided that the periodic times of the incident waves are sufficiently large, that is, greater than about four seconds.

Restricting the investigation to this class of waves, we see that

$$\frac{\omega^2 sh_1}{g}\left[1 - \frac{\omega^2 sh_0}{g}\right]^{-1} << 1,$$

so that we may expand the coefficient of d^2F/dy^2 in (2.4) in a convergent series to yield the equation

$$\frac{d^2F}{dy^2} + \frac{\omega^2/gh}{(1 - \omega^2 sh_0/g)}\left\{1 - \frac{sh_1\omega^2/g}{(1 - \omega^2 sh_0/g)} \cos ky\right\} F = 0. \qquad (2.5)$$

Making the simple change of independent variable

$$ky = 2\eta,$$

then (2.5) may be written,

$$\frac{d^2F}{d\eta^2} + \{a - 2q \cos 2\eta\} F = 0, \qquad (2.6)$$

where

$$a = \frac{4\omega^2}{ghk^2} \frac{1}{(1 - \omega^2 sh_0/g)} \qquad (2.7)$$

$$q = \frac{2\omega^2}{ghk^2} \quad \frac{\omega^2 sh_1 / g}{(1 - \omega^2 sh_0/g)^2} \tag{2.7}$$

Equation (2.6) is the standard form of Mathieu's equation [1], and we need to discuss its solutions and their dependence on the parameters a and q. We observe that

$$\frac{2q}{a} = \frac{\omega^2 sh_1}{(g - \omega^2 sh_0)} \quad << 1, \tag{2.8}$$

and that a, q > o.

(a) Mathieu's Equation and the Stability Diagram.

In the particular case q = 0, solutions of (2.6) are obtained by taking $a = n^2$, for integral n, and are

$$1, \cos \eta, \cos 2\eta, \ldots.$$
$$\sin \eta, \sin 2\eta, \ldots. .$$

The Mathieu functions of integral order, purely periodic solutions of the full equation (2.6), which reduce to these as $q \to 0$ are denoted respectively by *

$$ce_0(\eta,q), \; ce_1(\eta,q), \; ce_2(\eta,q), \ldots.$$
$$se_1(\eta,q), \; se_2(\eta,q), \ldots.$$

* That these harmonic functions are the only solutions of Mathieu's equation appropriate to the present problem is guaranteed by a theorem which states that Mathieu's equation never possesses two basically periodic solutions for the same values of a and q, apart from the trivial case q = 0. This was first proved by E. L. Ince, Proc. Phil. Soc. Camb., 21, 117 - 120, (1922).

and the corresponding characteristic values of the equation are

$$a_0,\ a_1,\ a_2,\ \ldots.$$

$$b_1,\ b_2,\ \ldots.\ .$$

It is convenient to discuss the solution with reference to the stability diagram, in which the parameter a is ordinate and q is abscissa. The curves a_0, b_1, a_1, ... divide the a - q plane into regions and may be plotted using [1]

$$\begin{aligned}
a_0 &= -\tfrac{1}{2}\, q^2 + \frac{7}{128}\, q^4 + O(q^6),\\
b_1 &= 1 - q - \frac{1}{8}\, q^2 + \frac{1}{64}\, q^3 - \frac{1}{1536}\, q^4 + O(q^5),\\
a_1 &= 1 + q - \frac{1}{8}\, q^2 - \frac{1}{64}\, q^3 - \frac{1}{1536}\, q^4 + O(q^5),\\
b_2 &= 4 - \frac{1}{12}\, q^2 + \frac{5}{13824}\, q^4 + O(q^6), \qquad (2.9)\\
a_2 &= 4 + \frac{5}{12}\, q^2 - \frac{763}{13824}\, q^4 + O(q^6),\\
b_3 &= 9 + \frac{1}{16}\, q^2 - \frac{1}{64}\, q^3 + \frac{13}{20480}\, q^4 + O(q^5),\\
a_3 &= 9 + \frac{1}{16}\, q^2 + \frac{1}{64}\, q^3 + \frac{13}{20480}\, q^4 + O(q^5),
\end{aligned}$$

which hold for sufficiently small q.

Thus, corresponding to a point (q,a) lying on one of these curves, the solution of (2.6) will be purely periodic and is denoted by

$$ce_{2n}(\eta,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2r\eta \ , \qquad (a_{2n}),$$

$$ce_{2n+1}(\eta,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)\eta \ , \qquad (a_{2n+1}),$$

$$se_{2n+1}(\eta,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)\eta, \qquad (b_{2n+1}),$$

$$se_{2n+2}(\eta,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)\eta \ , \qquad (b_{2n+2}).$$

Whittaker[2] introduced a method of finding the solution of (2.6), for points (q,a) not on these curves, which is of interest here. Invoking the Floquet theory (see for example [3]), we know that the general solution may be written in the form

$$F(\eta) = e^{\mu\eta}\, \Psi_1(\eta) + e^{-\mu\eta}\, \Psi_2(\eta), \tag{2.10}$$

where $\Psi_1(\eta)$, $\Psi_2(\eta)$ are periodic functions, and μ is the characteristic exponent of the theory. Whittaker proposed the introduction of a new parameter, σ, and taking the first term of (2.10) sought the solution in the form

$$F(\eta,\sigma) = e^{\mu\eta}\, \Psi_1(\eta,\sigma), \tag{2.11}$$

in which μ, together with a, is expressed as a power series in q, with coefficients dependent on σ.

In particular, he considered the region in the (q,a) plane

between the characteristics a_0 and b_2, and took $\Psi_1(\eta,\sigma)$ to be of the form,

$$\Psi_1(\eta,\sigma) = \sin(\eta - \sigma) + s_3\sin(3\eta - \sigma) + s_5\sin(5\eta - \sigma) + \ldots.$$
$$\ldots. + c_3\cos(3\eta - \sigma) + c_5\cos(5\eta - \sigma) + \ldots. \qquad (2.12)$$

By substitution into (2.6) it was found that

$$a(\sigma,q) = 1 - q\cos2\sigma + \tfrac{1}{4}q^2\ \{-1 + \tfrac{1}{2}\cos4\sigma\} + \frac{1}{64}q^3\ \cos2\sigma + \ldots. \qquad (2.13)$$

$$\mu(\sigma,q) = -\tfrac{1}{2}q\ \sin2\sigma + \frac{3}{128}q^3\ \sin2\sigma - \ldots. \qquad (2.14)$$

$$s_3 = -\frac{1}{8}q + \frac{1}{64}q^2\ \cos2\sigma - \frac{1}{512}q^3\ \{-\frac{14}{3} + 5\cos4\sigma\} + \ldots.$$

$$c_3 = \frac{3}{64}q^2\ \sin2\sigma - \frac{3}{512}q^3\ \sin4\sigma + \ldots.$$

$$s_5 = \frac{1}{192}q^2 - \frac{1}{1152}q^3\ \cos2\sigma + \ldots. \qquad (2.15)$$

$$c_5 = -\frac{7}{2304}q^3\ \sin2\sigma + \ldots.$$

$$s_7 = -\frac{1}{9216}q^3 + \ldots.$$

$$s_{m+2},\ c_m = O(q^4), \qquad m = 7, 9, 11, \ldots.$$

We note that

$$a(-\pi/2,q) = a_1\ , \qquad a(0,q) = b_1\ ,$$

and deduce that for a fixed value of q, $-\pi/2 < \sigma < 0$ yields the values of a intermediate between a_1 and b_1. Whittaker found that the value

$$\sigma = -\tfrac{1}{2}\pi + i\theta$$

describes the region of the (q,a) plane between a_1 and b_2, as θ moves from zero on a_1 through real positive or negative values until b_2 is attained.

Similarly the value

$$\sigma = i\theta$$

describes the region between a_0 and b_1 as θ moves from zero on b_1 through real positive or negative values until a_0 is reached.

We note that since μ is an odd function of θ in these cases, then taking $\theta > 0$ to yield the first term of (2.10), the negative values of θ will provide the second part of the Floquet solution. This resolves the apparent arbitrariness in the definition of θ.

It is seen that when σ is real between a_1 and b_1, then the characteristic exponent $\mu(\sigma,q)$ is real, and the region is designated an unstable area. When the region between a_0 and b_1 is described, σ is complex, so that $\mu(\sigma,q)$ is purely imaginary and so the region is designated a stable area. Similarly the region between a_1 and b_2 is a stable area. The remainder of the (q,a) plane can be investigated and described in a similar manner and the results depicted diagrammatically as in Fig. 1. The variation of the parameter σ is shown in Fig. 2.

The second solution is, using the Floquet result,

$$F(\eta) = e^{-\mu\eta}\,\Psi_1(\eta,-\sigma),$$

so that the general solution may be written

$$F(\eta) = A\, e^{\mu\eta}\, \Psi_1(\eta,\sigma) + B\, e^{-\mu\eta}\, \Psi_1(\eta,-\sigma). \tag{2.16}$$

The solution for the shallow water potential is

$$\phi(y,t) = \exp(-i\omega t)\, F(\tfrac{1}{2}\, ky), \tag{2.17}$$

indicating that the stable regions of the (q,a) plane correspond to a solution respresenting a progressive wave in $y > 0$, while the unstable regions correspond to solutions representing attenuated waves. The degree of attenuation will depend on the magnitude of $\mu(\sigma,q)$. If (q,a) lies on one of the characteristic curves $a_0, b_1, a_1, \ldots$ then the character of the corresponding transmitted wave will be periodic; on the other hand if (q,a) lies in a stable region the wave will not be periodic, but almost periodic, in the sense of H. Bohr; that is, it will be oscillatory and bounded but non-periodic.

(b) The Solution for (q,a) Lying Between a_1 and b_3.

We find here the solution in Whittaker's form which reduces to b_2 and a_2 when $\sigma = 0$ and $\sigma = -\frac{1}{2}\pi$ respectively. Let

$$F(\eta) = e^{\mu\eta}\, \Psi(\eta,\sigma),$$

where

$$\Psi(\eta,\sigma) = \sin(2\eta - \sigma) + q h_1(\eta,\sigma) + q^2 h_2(\eta,\sigma) + \ldots \tag{2.18}$$

and

$$\mu = q\, g_1(\sigma) + q^2 g_2(\sigma) + \ldots$$

$$a = 4 + qf_1(\sigma) + q^2 f_2(\sigma) + \dots .$$

We exclude the term $\cos(2\eta - \sigma)$ from $\Psi(\eta,\sigma)$ as it would give rise to a non-periodic term of the form $\eta\sin(2\eta - \sigma)$ in the solution.

Substituting into (2.6) we obtain

$$-4\sin(2\eta - \sigma) + qh_1'' + q^2 h_2'' + \dots + 2\{qg_1 + q^2 g_2 + \dots\} \times$$
$$\times\{2\cos(2\eta - \sigma) + qh_1' + q^2 h_2' + \dots\} + \{(qg_1 + q^2 g_2 + \dots)^2 +$$
$$+(4 + qf_1 + q^2 f_2 + \dots) - 2q\cos 2\eta\}\{\sin(2\eta - \sigma) + qh_1 + q^2 h_2 + \dots\}$$
$$= 0.$$

Equating powers of q;

q^0: satisfied identically

$$q: \quad h_1'' + 4h_1 + 4g_1\cos(2\eta - \sigma) + f_1\sin(2\eta - \sigma)$$
$$-2\cos 2\eta\sin(2\eta - \sigma) = 0.$$

We use the identity

$$-2\cos 2\eta\sin(2\eta - \sigma) = -\sin(4\eta - \sigma) + \sin\sigma,$$

and require the coefficient of $\sin(2\eta - \sigma)$ to be unity in the solution, conforming with the usual normalisation, and the coefficient of $\cos(2\eta - \sigma)$ to vanish, preventing non-periodic terms. Hence we must choose

$$f_1 = 0, \qquad g_1 = 0,$$

leaving

$$h_1'' + 4h_1 = \sin(4\eta - \sigma) - \sin\sigma,$$

so that

$$h_1 = -\frac{1}{12}\sin(4\eta - \sigma) - \frac{1}{4}\sin\sigma. \tag{2.19}$$

$$q^2: \quad h_2'' + 4h_2 + 4g_2\cos(2\eta - \sigma) + f_2\sin(2\eta - \sigma)$$
$$-2h_1\cos 2\eta = 0.$$

Again equating to zero the terms in $\sin(2\eta - \sigma)$ and $\cos(2\eta - \sigma)$ we find that

$$g_2 = -\frac{1}{16}\sin 2\sigma, \quad f_2 = -\tfrac{1}{4}\cos 2\sigma + \frac{1}{6},$$

and

$$h_2'' + 4h_2 = -\frac{1}{12}\sin(6\eta - \sigma),$$

or

$$h_2 = \frac{1}{384}\sin(6\eta - \sigma).$$

Proceeding in this way we find that

$$f_3 = g_3 = 0,$$

$$h_3 = \frac{1}{288}\sin 2\sigma\cos(4\eta - \sigma) - \frac{1}{144}\{\tfrac{1}{2}\sin^2\sigma - \frac{5}{96}\}\sin(4\eta - \sigma) -$$
$$- \frac{1}{23040}\sin(8\eta - \sigma) + \frac{1}{32}\{\tfrac{1}{3} - \tfrac{1}{2}\cos 2\sigma\}\sin 2\sigma ; \qquad (2.21)$$

$$g_4 = \frac{1}{8}\{\frac{1}{9} - \tfrac{1}{4}\sin 2\sigma\}\sin 2\sigma,$$

$$f_4 = \frac{7}{144}\sin^2\sigma - \frac{3}{256}\cos^2 2\sigma + \frac{265}{1384},$$

$$h_4 = -\frac{1}{1024}\{\frac{1}{16} - \frac{1}{9}\sin 2\sigma\}\cos(6\eta - \sigma) + \frac{1}{18432}\{5\sin^2\sigma - \frac{167}{60}\} \times$$
$$\times \sin(6\eta - \sigma) + \frac{1}{2211840}\sin(10\eta - \sigma). \qquad (2.22)$$

Hence

$$a = 4 + q^2\{\frac{1}{6} - \tfrac{1}{4}\cos 2\sigma\} + q^4\{\frac{7}{144}\sin^2\sigma - \frac{3}{256}\cos^2 2\sigma + \frac{256}{13824}\} + O(q^5)$$
$$(2.23)$$

$$\mu = -q^2 \frac{1}{16} \sin 2\sigma + q^4 \frac{1}{8} \{\frac{1}{9} - \frac{1}{4} \sin 2\sigma\} \sin 2\sigma + O(q^5) \qquad (2.24)$$

and we observe that the expression for a reduces to the appropriate characteristic values on choosing σ as indicated in Fig. 2.

A similar solution was obtained by Young [4] who, however took μ to be of the form

$$\mu = - Nq^2 \sin 2\sigma$$

only. Young also determined the solution reducing to b_3 and a_3 for the appropriate values of σ, taking

$$\mu = - Nq^3 \sin 2\sigma.$$

(c) The Solution of the Problem with Reference to the Stability Diagram.

The features of the stability diagram outlined above can be interpreted in the most useful way by returning to the expressions (2.7) for a and q, and eliminating between these the incident wave frequency ω, we obtain

$$q = \left\{\frac{\pi^2 h_1 hs}{2L^2}\right\} a^2, \qquad (2.25)$$

in which $L = 2\pi/k$ is the wavelength of the ice thickness fluctuations. For specified values of L and the geometrical quantities h and h_1, the equation (2.25) will represent a parabola Γ in the (q,a) plane, which will intersect the curves b_1, a_1, b_2, ... at points B_1, A_1,

B_2, ... as in Fig. 3.

If we also fix h_0 then at each point of intersection we can determine from either a or q, a corresponding value of ω, all other parameters being specified. We denote the frequencies obtained in this manner by $\omega(B_1)$, $\omega(A_1)$, $\omega(B_2)$, ... respectively. It follows that the bands of ω defined by

$$0 < \omega \leqslant \omega(B_1),$$

$$\omega(A_1) \leqslant \omega \leqslant \omega(B_2),$$

$$\omega(A_2) \leqslant \omega \leqslant \omega(B_3), \ \ldots.$$

correspond to waves which are transmitted undamped, while the bands

$$\omega(B_1) < \omega < \omega(A_1),$$

$$\omega(B_2) < \omega < \omega(A_2), \ \ldots.$$

correspond to waves which are attenuated.

Since all of the characteristic curves intersect the axis $a = 0$ for sufficiently large q (McLachlan [1], section 3.24) then there exists an infinity of frequency bands which are associated alternately with damped and undamped waves. However for sufficiently large positive q, the curves a_m and b_{m+1} are mutually asymptotic and so the bands of frequencies corresponding to real propagation are indistinguishable from points. Hence all wavelengths below a certain one undergo damping apart from an infinite number of discrete wavelengths. The most important frequency is $\omega(B_1)$, and the corresponding wavelength

$$\lambda(B_1) = \frac{2\pi\sqrt{gh}}{\omega(B_1)}$$

is such that all $\lambda > \lambda(B_1)$ give rise to undamped transmitted waves.

The crucial parameter which determines the relative breadth of the frequency bands is $h_1/2A$ where

$$A = \frac{L^2}{\pi^2 sh}, \tag{2.26}$$

in terms of which, Γ may be written

$$q = \frac{h_1}{2A} a^2. \tag{2.27}$$

As $h_1 \to 0$, Γ degenerates to the axis $q = 0$, and no bands of ω which correspond to a damped solution exist. This can of course also be seen by considering equation (2.6), which for a constant ice thickness h_0 reduces to

$$\frac{d^2F}{d\eta^2} + aF = 0.$$

As h_1 increases from zero, the width of the cut-off bands increases. A similar argument can be made with regard to L, the breadth of the cut-off bands increasing as L decreases.

(d) Determination of the Frequency Bands.

We consider first the intersection of Γ with the characteristics

$$\left.\begin{aligned} b_1 &= a = 1 - q - \frac{1}{8} q^2 + O(q^3), \\ a_1 &= a = 1 + q - \frac{1}{8} q^2 + O(q^3). \end{aligned}\right\} \tag{2.28}$$

At the points of intersection, using (2.27) we obtain the quartics in a :

$$\left.\begin{aligned} b_1 &: \ \frac{1}{8}\{h_1/2A\}^2\ a^4 + \{h_1/2A\}\ a^2 + a - 1 = 0, \\ a_1 &: \ \frac{1}{8}\{h_1/2A\}^2\ a^4 - \{h_1/2A\}\ a^2 + a - 1 = 0. \end{aligned}\right\} \quad (2.29)$$

Inserting realistic values of the parameters into

$$h_1/2A = \pi^2 shh_1/L^2$$

we find that the term is moderately small. In view also of the factor of 1/8 in the leading terms of (2.29), we may therefore obtain approximate solutions of these equations by solving the quadratics

$$b_1 : \{h_1/2A\}\ a^2 + a - 1 = 0,$$

$$a_1 : \{h_1/2A\}\ a^2 - a + 1 = 0,$$

to give

$$a(B_1) = \frac{A}{h_1}\left\{-1 + \sqrt{\left[1 + \frac{2h_1}{A}\right]}\right\},$$

$$a(A_1) = \frac{A}{h_1}\left\{1 - \sqrt{\left[1 - \frac{2h_1}{A}\right]}\right\},$$

the signs of the square roots being chosen so that

$$a(B_1) \to 1, \quad a(A_1) \to 1,$$

as $A/h_1 \to \infty$.

To determine closer approximations to the solutions of (2.29) let us consider a quartic of the form

$$\epsilon a^4 + \alpha a^2 + a - 1 = 0, \qquad (2.30)$$

having solution

$$a = a_{00} + \epsilon a_{11} + \epsilon^2 a_{22} + \dots$$

where a_{00} is the appropriate root of

$$\alpha a^2 + a - 1 = 0,$$

and $\epsilon \ll \alpha$, $\epsilon \ll 1$.

Substituting the trial solution into (2.30) we easily find a_{11} and a_{22} by equating powers of ϵ, and that

$$a(B_1) = a_{00} - \frac{1}{8}\{h_1/2A\}^2 \frac{a_{00}^4}{\{h_1/A\}a_{00} + 1} +$$

$$+ \frac{1}{64}\{h_1/2A\}^4 \frac{a_{00}^7}{\{h_1/A\}a_{00} + 1}\left\{4 - \frac{\{h_1/A\}a_{00}}{\{h_1/A\}a_{00} + 1}\right\} + \dots \quad (2.31)$$

where

$$a_{00} = \frac{A}{h_1}\left\{-1 + \sqrt{\left[1 + \frac{2h_1}{A}\right]}\right\},$$

and

$$a(A_1) = a_{00} - \frac{1}{8}\{h_1/2A\}^2 \frac{a_{00}^4}{1 - \{h_1/A\}a_{00}} +$$

$$+ \frac{1}{64}\{h_1/2A\}^4 \frac{a_{00}^7}{1 - \{h_1/A\}a_{00}}\left\{4 - \frac{\{h_1/A\}a_{00}}{1 - \{h_1/A\}a_{00}}\right\} + \dots \quad (2.32)$$

where now

$$a_{00} = \frac{A}{h_1}\left\{1 - \sqrt{\left[1 - \frac{2h_1}{A}\right]}\right\}.$$

An exactly similar procedure can be used to determine the intersections of Γ with the characteristics b_2 and a_2, which are

$$b_2 = a = 4 - \frac{1}{12} q^2 + O(q^4),$$

$$a_2 = a = 4 + \frac{5}{12} q^2 + O(q^4),$$

and we find that they are given by

$$a = a_{00} - \epsilon a_{00}^4 + 4\epsilon^2 a_{00}^7 - 22\epsilon^3 a_{00}^{10} + \ldots. \qquad (2.33)$$

where $a_{00} = 4$. Choosing

$$\epsilon = \frac{1}{12} \{h_1/2A\}^2 , \qquad \epsilon = - \frac{5}{12} \{h_1/2A\}^2 ,$$

we determine respectively the quantities $a(B_2)$ and $a(A_2)$ for values of ϵ for which (2.33) converges.

(e) The Damped Solutions.

The damping term in the solutions corresponding to the unstable regions of the (q,a) plane is

$$\exp(- \tfrac{1}{2}k\mu y),$$

and we briefly discuss it in the first two such regions, in terms of

$$\tau = 2/k\mu = L/\pi\mu , \qquad (2.34)$$

the distance in which the elevation in $y > 0$ is reduced by a factor $1/e$.

In the first unstable region between b_1 and a_1,

$$\mu = - \tfrac{1}{2}q \sin 2\sigma + \frac{3}{128} q^3 \sin 2\sigma + O(q^4), \qquad (2.14)$$

and in the second unstable region between b_2 and a_2, we have from (b)

$$\mu = -\frac{1}{16} q^2 \sin 2\sigma + \frac{1}{8} q^4 \left\{\frac{11}{9} - \frac{1}{4} \sin 2\sigma\right\} \sin 2\sigma + O(q^5) \tag{2.24}$$

for values of q sufficiently small to ensure the convergence of these series. In each case $-\pi/2 < \sigma < 0$.

We see immediately from (2.34) that the damping is less marked as the wavelength of the ice thickness fluctuations, L, increases. Further, the expressions for μ have maxima at $\sigma = -\pi/4$, and hence the attentuation of an incident wave is predominant at approximately the mid point of each unstable region. In addition, for smaller values of q, we see that in the first unstable region

$$\tau_{min} \simeq 2L/\pi q ,$$

while in the second unstable region

$$\tau_{min} \simeq 16L/\pi q^2 ,$$

indicating that the more pronounced damping occurs for (q,a) in the first unstable region.

(f) Obliquely Incident Waves.

To consider the more general case of waves which are incident obliquely on the icefield, we return to equations (1.7), (1.8) and (1.9). We assume that the solution of (1.9) for the incident wave has the form

$$F(x,y) = e^{im_0 x + imy}, \quad -\infty < y < 0, \ -\infty < x < \infty,$$

where m_0 and m are real and satisfy

$$m_0^2 + m^2 = k_0^2 = \omega^2/gh.$$

That is,

$$F(x,y) = e^{i(k_0^2 - m_0^2)^{\frac{1}{2}}y + im_0 x},$$

in $-\infty < y < 0$, $-\infty < x < \infty$, and introducing the incident angle θ by

$$m_0 = k_0 \sin\theta,$$

then

$$F(x,y) = e^{ik_0 y \cos\theta + ik_0 x \sin\theta}. \tag{2.35}$$

For continuity of the velocity components in the fluid, the solutions in $y < 0$ and $y > 0$ must be continuous at $y = 0$, and it is therefore necessary that the solution of (1.8) contains exactly the same x dependence as (2.35). We may write the solution of (1.8) in the form,

$$F(x,y) = e^{ik_0 x \sin\theta} F_1(y), \qquad 0 < y < \infty, \ -\infty < x < \infty \tag{2.36}$$

so that $F_1(y)$ must satisfy

$$\left\{1 - \frac{\omega^2 sH(x,y)}{g}\right\}\left\{\frac{d^2}{dy^2} - k_0^2 \sin^2\theta\right\} F_1(y) + \frac{\omega^2}{gh} F_1(y) = 0.$$

Assuming as before that the ice thickness varies in the y direction only, and that is has the form

$$H(x,y) = h_0 - h_1 \cos ky,$$

then, under the same restrictions as previously,

$$\omega^2 sh_0/g << 1, \qquad \omega^2 sh_1/g << 1,$$

we may write

$$\left\{\frac{d^2}{dy^2} - k_0^2 \sin^2\theta\right\} F_1 +$$

$$+ \frac{\omega^2/gh}{1 - \omega^2 sh_0/g}\left\{1 - \frac{\omega^2 sh_1/g}{1 - \omega^2 sh_0/g} \cos ky\right\} F_1 = 0.$$

Performing the variable change $2\eta = ky$, then this becomes

$$\frac{d^2 F_1}{d\eta^2} + \{a_1 - 2q_1 \cos 2\eta\} F_1 = 0, \tag{2.37}$$

where

$$a_1 = \frac{4\omega^2/gh}{k^2(1 - \omega^2 sh_0/g)} - \frac{4k_0^2}{k^2}\sin^2\theta = a - \frac{4k_0^2}{k^2}\sin^2\theta, \tag{2.38}$$

and

$$q_1 = q = \frac{2\omega^2/gh}{k^2} \frac{\omega^2 sh_1/g}{(1 - \omega^2 sh_0/g)^2} . \tag{2.39}$$

Again we may eliminate the frequency between (2.38) and (2.39), but the resulting curve will not be as simple as the parabola obtained for $\theta = 0$. However we can deduce several qualitative results from the expressions for a_1 and q_1, since the latter is simply equal to q and is independent of θ. For a fixed value of q, we see that the effect of increasing the incident angle is to decrease the value of the parameter a, and since a_1 can be written in the form

$$a_1 = \frac{4k_0^2}{k^2}\left\{\frac{1}{1 - \omega^2 sh_0/g} - \sin^2\theta\right\},$$

then for θ approaching $\pi/2$, the value of a_1 will be very small.

The effect of a non-zero incident angle is thus to move some

points in the stability diagram from stable to unstable areas and vice versa. In particular, certain points in the first unstable region will move into the stable region enclosed by the axes and the characteristic b_1, as θ increases; that is, the wavelength $\lambda(B_1)$ will decrease as the incident angle increases. We recall that $\lambda(B_1)$ is the largest wavelength which is an upper bound of a band of wavelengths corresponding to a damped solution. Hence we can conclude that the reflection of an incident wave decreases as the angle of incidence of that wave increases.

(g) Modification for Displaced Ice Structure.

In writing

$$H(y) = h_0 - h_1 \cos ky, \qquad [(2.3)]$$

we are assuming that the ice thickness has a minimum at the edge $y = 0$ of the sheet. It is therefore of interest to investigate how the subsequent solution alters if the ice thickness at $y = 0$ remains arbitrary, and the simplest way to do this is to introduce a phase angle κ and write

$$H(y) = h_0 - h_1 \cos(ky + \kappa). \qquad (2.40)$$

By following the initial formulation of this section through, it is not difficult to see that we again arrive at the Mathieu equation (2.6) by making the change of independent variable

$$ky + \kappa = 2\eta,$$

and if the solution of this equation is denoted by $F(\eta)$ as before, the resulting velocity potential will now be

$$\phi(y,t) = \exp(-i\omega t)\ F(\tfrac{1}{2}ky + \tfrac{1}{2}\kappa). \tag{2.41}$$

That is, the velocity potential undergoes a phase change equal to $\frac{1}{2}\kappa$. The significance of this can be made clear by taking a particular solution of the Mathieu equation. Let us assume that (q,a) lies in the stable region between a_1 and b_2 so that the velocity potential represents a progressive wave in $y > 0$. We can make use of the results of (b), which are, on writing $\sigma = i\theta$ as required by Whittaker's theory,

$$F(\eta) = e^{\mu\eta}\ \{\sin(2\eta - i\theta) + O(q)\}, \tag{2.42}$$

$$\mu = -\frac{1}{16}\, iq^2\ \sinh 2\theta + O(q^4)$$

$$= -i\mu_1 + O(q^4),$$

$$a = 4 + q^2\ \{\tfrac{1}{6} - \tfrac{1}{4}\cosh 2\theta\} + O(q^4). \tag{2.43}$$

Imposing $\theta > 0$, we must take the second part of the Floquet solution (2.3) to obtain a wave moving in the appropriate direction, and by virtue of (2.41)

$$\phi(y,t) = B\ e^{\frac{1}{2}i\kappa\mu_1}\ e^{i(\frac{1}{2}k\mu_1 y - \omega t)}\ \{\sin(ky + \kappa + i\theta) + O(q)\} \tag{2.44}$$

In general, the coefficient $T = B\ \exp(\frac{1}{2}i\kappa\mu_1)$ is complex, and by the velocity potential we strictly mean the real part of the function (2.44). We remark that on imposing the continuity of (2.44)

and the solution of (1.9) in $y < 0$, and their first derivatives, we shall determine T as a function of θ, and this coefficient will necessarily remove the apparent unboundedness in (2.44) as θ becomes large.

For simplicity let us take only one term of the real part of (2.44),

$$\phi_1(y,t) = T_1 \sin(\tfrac{1}{2}k\mu_1 y - \omega t) \sin(ky + \kappa), \tag{2.45}$$

where the θ dependence has been absorbed into the real coefficient T_1. It is at once clear that if the ice structure suffers a displacement κ/k then so does the progressive wave in $y > 0$. Defining the wavelengths

$$L_1 = 4\pi/k\mu_1 \ , \qquad L_2 = 2\pi/k \ ,$$

of the two periodic terms in (2.45), then we require a knowledge of the ratio

$$L_1/L_2 = 2/\mu_1 \ . \tag{2.46}$$

Using (2.9) we see that the expression (2.43) for a is coincident with the characteristic b_2 when $\theta = 0$, and for a fixed value of q , represents a point on the characteristic a_1 for a value θ_0 of θ given approximately by

$$1 + q - \frac{1}{8} q^2 = 4 + q^2 \{\tfrac{1}{6} - \tfrac{1}{4} \cosh 2\theta_0\},$$

from which we deduce that

$$\frac{q^2 \sinh 2\theta_0}{16} = \tfrac{1}{4}\sqrt{\left\{(3 - q)^2 + \frac{7}{8} q^2 + O(q^3)\right\}} < 2,$$

for sufficiently small q.

Hence since $\mu_1 = q^2 \sinh 2\theta/16$, and by virtue of (2.46), we conclude that $L_1 > L_2$ for all points in the (q,a) plane lying between a_1 and b_2.

Therefore the progressive wave part of (2.45) will provide bounding curves for the motion while the final factor gives rise to an oscillatory motion within these bounds. The net effect is that of a group moving with velocity $2\omega/k\mu_1$, and is shown in Fig. 4 for both κ zero and non-zero, together with the corresponding ice forms.

As $\phi(y,t)$ consists of terms like $\phi_1(y,t)$ superposed, it will behave in the same manner. Retension of further terms in $F(\eta)$ of order q and smaller will complicate the structure of the progressive wave, but will not vary its general appearance from that described above.

3. Rectangular Wave Ice Thickness Variation in the y direction, with Normally Incident Waves.

As in section 2 the shallow water potential $\phi(y,t)$ is determined from

$$\left\{1 - \frac{\omega^2 sH(y)}{g}\right\} \frac{d^2F}{dy^2} + \frac{\omega^2}{gh} F = 0, \qquad 0 < y < \infty, \tag{3.1}$$

$$\frac{d^2F}{dy^2} + \frac{\omega^2}{gh} F = 0, \qquad -\infty < y < 0, \tag{3.2}$$

and

$$\phi(y,t) = \exp(-i\omega t)\, F(y).$$

Let the ice thickness $H(y)$ be a general rectangular periodic function defined by (see Fig. 5)

$$\left.\begin{aligned} H(y) &= h_0 + h_1, \quad 0 < y < \ell_1, \\ H(y) &= h_0 - h_2, \quad \ell_1 < y < \ell_1 + \ell_2, \end{aligned}\right\} \tag{3.3}$$

having period $d = \ell_1 + \ell_2$. The quantity h_0 is the mean ice thickness, so that $h_1 \ell_1 = h_2 \ell_2$. Equation (3.1) now becomes

$$\left\{1 - \frac{\omega^2 s h_0}{g} - \frac{\omega^2 s h_1}{g}\right\} \frac{d^2 F}{dy^2} + \frac{\omega^2}{gh} F = 0, \quad 0 < y < \ell_1,$$

$$\left\{1 - \frac{\omega^2 s h_0}{g} + \frac{\omega^2 s h_2}{g}\right\} \frac{d^2 F}{dy^2} + \frac{\omega^2}{gh} F = 0, \quad \ell_1 < y < \ell_1 + \ell_2,$$

and provided that the coefficients of F_{yy} do not vanish, we may write these in the form

$$\left.\begin{aligned} \frac{d^2 F}{dy^2} - x_1{}^2 F &= 0, \quad 0 < y < \ell_1, \\ \frac{d^2 F}{dy^2} - x_2{}^2 F &= 0, \quad \ell_1 < y < \ell_1 + \ell_2, \end{aligned}\right\} \tag{3.4}$$

where

$$\left.\begin{aligned} x_1{}^2 &= -\frac{\omega^2}{gh}\left[1 - \frac{\omega^2 s h_0}{g} - \frac{\omega^2 s h_1}{g}\right]^{-1}, \\ x_2{}^2 &= -\frac{\omega^2}{gh}\left[1 - \frac{\omega^2 s h_0}{g} + \frac{\omega^2 s h_2}{g}\right]^{-1}. \end{aligned}\right\} \tag{3.5}$$

The solutions of (3.4) can be written down at once,

$$\left.\begin{aligned} F(y) &= Ae^{x_1 y} + Be^{-x_1 y}, \quad 0 < y < \ell_1, \\ F(y) &= Ce^{x_2 y} + De^{-x_2 y}, \quad \ell_1 < y < \ell_1 + \ell_2, \end{aligned}\right\} \tag{3.6}$$

where A, B, C, D are arbitrary constants.

The whole solution must fit Floquet's theorem (see, for example, [3]) with period d, that is, choosing the positive exponent of the theorem,

$$F(y) = e^{\mu y}\, G(y), \tag{3.7}$$

where G(y) has period d, so that

$$F(y) = e^{\mu d}\, F(y - d).$$

The general Floquet solution can be expressed in terms of G(y) by

$$F(y) = C_1 e^{\mu y}\, G(y) + C_2 e^{-\mu y}\, G(-y),$$

where C_1 and C_2 are constants.

Applying the periodicity relation for F(y) to the interval $\ell_1 + \ell_2 < y < \ell_2 + 2\ell_1$, then

$$F(y) = Ae^{\mu d}\, e^{x_1 (y - d)} + Be^{\mu d}\, e^{-x_1 (y - d)}, \quad d < y < d + \ell_1, \tag{3.8}$$

which together with (3.6) defines the whole solution. We must have continuity of F(y) and its first derivative at $y = \ell_1$ and $y = d$, yielding four equations between the constants,

$$Ae^{x_1 \ell_1} + Be^{-x_1 \ell_1} = Ce^{x_2 \ell_1} + De^{-x_2 \ell_1},$$

$$x_1 Ae^{x_1 \ell_1} - x_1 Be^{-x_1 \ell_1} = x_2 Ce^{x_2 \ell_1} - x_2 De^{-x_2 \ell_1},$$

$$Ce^{x_2 d} + De^{-x_2 d} = Ae^{\mu d} + Be^{\mu d},$$

$$x_2 Ce^{x_2 d} - x_2 De^{-x_2 d} = x_1 Ae^{\mu d} - x_1 Be^{\mu d},$$

which are consistent provided that the determinant of coefficients vanishes,

$$\begin{vmatrix} e^{x_1 \ell_1} & e^{-x_1 \ell_1} & e^{x_2 \ell_1} & e^{-x_2 \ell_1} \\ x_1 e^{x_1 \ell_1} & -x_1 e^{-x_1 \ell_1} & x_2 e^{x_2 \ell_1} & -x_2 e^{-x_2 \ell_1} \\ e^{\mu d} & e^{\mu d} & e^{x_2 d} & e^{-x_2 d} \\ x_1 e^{\mu d} & -x_1 e^{\mu d} & x_2 e^{x_2 d} & -x_2 e^{-x_2 d} \end{vmatrix} = 0,$$

and writing $\zeta = e^{\mu d}$, this can be expressed as a quadratic in ζ^2 on expanding,

$$\zeta^2 - 2\zeta \left\{ \cosh x_1 \ell_1 \ \cosh x_2 \ell_2 + \tfrac{1}{2}\left(\frac{x_1}{x_2} + \frac{x_2}{x_1} \right) \sinh x_1 \ell_1 \ \sinh x_2 \ell_2 \right\} + 1 = 0.$$

The roots ζ_1, ζ_2 of this equation are such that $\zeta_1 \zeta_2 = 1$, hence

$$\zeta_1 = e^{\mu d}, \qquad \zeta_2 = e^{-\mu d},$$

and

$$\zeta_1 + \zeta_2 = 2\cosh \mu d = 2\alpha, \tag{3.9}$$

where we have written

$$\alpha = \cosh x_1 \ell_1 \ \cosh x_2 \ell_2 + \tfrac{1}{2}\left(\frac{x_1}{x_2} + \frac{x_2}{x_1} \right) \sinh x_1 \ell_1 \ \sinh x_2 \ell_2 \ . \tag{3.10}$$

We note that:

(i) if $\alpha > 1$, μ is real,

(ii) if $-1 \leqslant \alpha \leqslant 1$, μ is purely imaginary,

(iii) if $\alpha < -1$, μ is complex.

In view of the solution (3.7), this implies that case (ii) corresponds to propagated unattenuated waves passing through the icefield, while cases (i) and (iii) correspond to waves which are attenuated. According to the sign of μ we may need to choose the second part of the Floquet solution with the negative exponent, to produce a wave travelling in the appropriate direction, or to avoid unboundedness, whichever is applicable to the form of solution.

The bounds of case (ii) are given by

$$\cosh\mu d = \alpha = \pm 1,$$

and these can be depicted graphically as curves in the x_1, x_2 plane, producing a stability diagram of the type resulting from the Mathieu equation. This analysis applies to a perfectly general rectangular wave and was first carried out by Brillouin [5]. We now consider a particular case in which

$$\ell_1 = \ell_2, \qquad h_1 = h_2,$$

so that $d = 2\ell_1$. Changing the independent variable in the above analysis by

$$y = d\eta/\pi,$$

then (3.4) becomes

$$\left.\begin{aligned} &\frac{d^2F}{d\eta^2} - \frac{d^2}{\pi^2}\, x_1{}^2\, F = 0, \qquad 0 < \eta < \pi/2, \\ &\frac{d^2F}{d\eta^2} - \frac{d^2}{\pi^2}\, x_2{}^2\, F = 0, \qquad \pi/2 < \eta < \pi. \end{aligned}\right\}$$

We now have a problem which has the same period $d' = \pi$ as the Mathieu problem, and in which

$$x_1'^{\,2} = \frac{d^2}{\pi^2}\, x_1{}^2 = \frac{4}{k^2}\, x_1{}^2,$$

$$x_2'^{\,2} = \frac{d^2}{\pi^2}\, x_2{}^2 = \frac{4}{k^2}\, x_2{}^2,$$

and $\ell_1' = \ell_2' = \pi/2$, where we have written $d = 2\pi/k$ to preserve the parameters of section 2.

Referring to (3.5) and imposing the restrictions of section 2. , namely,

$$\omega^2 sh_0/g \ll 1, \qquad \omega^2 sh_1/g \ll 1,$$

then we have

$$\left.\begin{aligned} -x_1'^{\,2} &= \frac{4\omega^2}{k^2 gh}\, \frac{1}{(1 - \omega^2 sh_0/g)} \left\{1 + \frac{\omega^2 sh_1/g}{(1 - \omega^2 sh_0/g)}\right\} = a + 2q, \\ -x_2'^{\,2} &= \frac{4\omega^2}{k^2 gh}\, \frac{1}{(1 - \omega^2 sh_0/g)} \left\{1 - \frac{\omega^2 sh_1/g}{(1 - \omega^2 sh_0/g)}\right\} = a - 2q, \end{aligned}\right\} \qquad (3.11)$$

where

$$a = \frac{4\omega^2}{k^2 gh}\, \frac{1}{(1 - \omega^2 sh_0/g)}, \qquad q = \frac{2\omega^2}{k^2 gh}\, \frac{\omega^2 sh_1/g}{(1 - \omega^2 sh_0/g)^2},$$

are the parameters occuring in section 2.

When $a > 2q > 0$, (3.10) becomes

$$\alpha = \cos\gamma_1 \cos\gamma_2 - \frac{1}{2}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right)\sin\gamma_1 \sin\gamma_2, \tag{3.12}$$

and when $a < 2q$, $q > 0$,

$$\alpha = \cos\gamma_1 \cosh\gamma_3 - \frac{1}{2}\left(\frac{\gamma_1}{\gamma_3} - \frac{\gamma_3}{\gamma_1}\right)\sin\gamma_1 \sinh\gamma_3, \tag{3.13}$$

where we have replaced the quantities in (3.10) by their primed counterparts, so that

$$\gamma_1^2 = (\pi/2)^2 \{a + 2q\},$$
$$\gamma_2^2 = (\pi/2)^2 \{a - 2q\},$$
$$\gamma_3^2 = (\pi/2)^2 \{2q - a\}.$$

Using these equations, a stability diagram can be plotted with q and a as variables, the curves bounding the regions being given by $\alpha = \pm 1$. Such a diagram was first produced by Van der Pol and Strutt [6], who took the coordinates a and 2q. It will be seen (Fig. 6) that this stability diagram is similar to that of the Mathieu equation, except for the intersection of some of the bounding curves, and that

(i) for $a > 2q > 0$, the (q,a) plane consists mainly of stable areas,

(ii) for $-2q < a < 2q$, the (q,a) plane consists mainly of unstable areas, apart from the bounding curves which correspond to a purely periodic solution,

(iii) for $a < -2q$, the (q,a) plane consists wholly of an unstable area.

From (2.8), $2q \ll a$, so that in the present problem the solutions mainly represent real propagating waves, apart from narrow bands of frequencies for which the solution is an attenuated wave. Rearranging (3.13),

$$\alpha = \cos(\gamma_1 + \gamma_2) - \frac{(\gamma_1 - \gamma_2)^2}{2\gamma_1\gamma_2} \sin\gamma_1 \sin\gamma_2 , \qquad (3.14)$$

and writing $\nu = 2q/a \ll 1$ for brevity, we can expand

$$\gamma_1 = \frac{\pi}{2}\sqrt{a}\,\{1 + \tfrac{1}{2}\nu - \frac{1}{8}\nu^2 + \ldots\},$$

$$\gamma_2 = \frac{\pi}{2}\sqrt{a}\,\{1 - \tfrac{1}{2}\nu - \frac{1}{8}\nu^2 - \ldots\},$$

so that

$$\gamma_1 + \gamma_2 = \pi\sqrt{a}\,\{1 - \frac{1}{8}\nu^2 + O(\nu^4)\},$$

$$\gamma_1 - \gamma_2 = \frac{\pi}{2}\sqrt{a}\,\nu + O(\nu^3),$$

and

$$\gamma_1\gamma_2 = (\pi\sqrt{a}/2)^2\,\{1 - \tfrac{1}{2}\nu^2 + O(\nu^4)\}.$$

Hence

$$\cos(\gamma_1 + \gamma_2) = \cos\pi\sqrt{a}\cos\tfrac{1}{8}\pi\sqrt{a}\nu^2 + \sin\pi\sqrt{a}\sin\tfrac{1}{8}\pi\sqrt{a}\nu^2 ,$$

$$= \cos\pi\sqrt{a} + \frac{1}{8}\pi\sqrt{a}\nu^2 \sin\pi\sqrt{a} + O(\pi\sqrt{a}\nu^2/8)^2,$$

while

$$\cos(\gamma_1 - \gamma_2) = 1 - \pi^2 a\nu^2/8 + O(\pi\sqrt{a}\nu/2)^4 .$$

Using these expressions in (3.14) we find that

$$\alpha = \cos\pi\sqrt{a} - \tfrac{1}{2}\nu^2\{\sin^2\tfrac{1}{2}\pi\sqrt{a} - \tfrac{1}{4}\pi\sqrt{a}\sin\pi\sqrt{a}\} + O(\nu^4). \qquad (3.15)$$

When $q = \nu = 0$, then the points at which the bounding curves intersect the a axis are easily seen to be given by $a = n^2$, $n = 0, 1, 2, \ldots$.

Equation (3.15) holds provided that a is sufficiently small for the approximate expansion of the circular functions to be valid. However we can see that we are justified in deducing from (3.15) the curves which pass through $q = 0$, $a = 1$ as the expansions certainly hold when a is near unity. These curves correspond to $\alpha = -1$, and we make use of the fact that $2q \ll 1$ in assuming a solution for them of the form

$$\sqrt{a} = 1 + c_1 q + c_2 q^2 + \ldots . \qquad (3.16)$$

Hence

$$\begin{aligned}
\cos\pi\sqrt{a} &= -\cos\pi(c_1 q + c_2 q^2 + \ldots) \\
&= -1 + \tfrac{1}{2}\pi^2 q^2 c_1{}^2 + \pi^2 q^3 c_1 c_2 + O(q^4), \\
\sin^2\tfrac{1}{2}\pi\sqrt{a} &= 1 - \tfrac{1}{4}\pi^2 q^2 c_1{}^2 + O(q^3), \\
\sin\pi\sqrt{a} &= -\pi q c_1 - \pi q^2 c_2 + O(q^3),
\end{aligned}$$

and

$$(2q/a)^2 = 4q^2\{1 - 4qc_1\} + O(q^3).$$

Substituting into (3.15) and putting $\alpha = -1$, we obtain

$$0 = q^2\ \{\pi^2 c_1{}^2/2 - 2\} + q^3\ \{\pi^2 c_1 c_2 + 8c_1 - \pi^2 c_1/2\} + O(q^4),$$

from which

$$c_1 = \pm 2/\pi, \qquad c_2 = \tfrac{1}{2} - 8/\pi^2,$$

so that in (3.16),

$$\sqrt{a} = 1 \pm \frac{2}{\pi} q - \{8/\pi^2 - \tfrac{1}{2}\} q^2 + O(q^3),$$

or

$$a = 1 \pm \frac{4}{\pi} q - \{12/\pi^2 - 1\} q^2 + O(q^3). \tag{3.17}$$

By analogy with section 2. we therefore write

$$\left.\begin{aligned} b_1 &= 1 - \frac{4}{\pi} q - \{12/\pi^2 - 1\} q^2 + O(q^3), \\ a_1 &= 1 + \frac{4}{\pi} q - \{12/\pi^2 - 1\} q^2 + O(q^3), \end{aligned}\right\} \tag{3.18}$$

which are similar to the characteristics of the Mathieu equation. We recall that in that case the coefficients of q and q^2 are unity and 1/8 respectively. Thus for a fixed small q, the breadth of the first unstable region for a rectangular wave is greater than that for a sinusoidal wave by a factor $4/\pi$. The first band of cut-off wavelengths can be determined from (3.18) by a similar method to that used in section 2, but due to the close similarity with that investigation, this is not pursued here.

Finally we examine the case when we ignore the terms of order ν^2 in the expansions of γ_1 and γ_2, taking

$$\gamma_1 = \tfrac{1}{2}\pi\sqrt{a}[1 + \tfrac{1}{2}\nu],$$

$$\gamma_2 = \tfrac{1}{2}\pi\sqrt{a}[1 - \tfrac{1}{2}\nu],$$

in which case it is easily shown that (3.14) reduces to

$$\alpha = \cos \pi\sqrt{a} - \tfrac{1}{2}\nu^2 \sin^2 \tfrac{1}{2}\pi\sqrt{a} + O(\nu^4). \tag{3.19}$$

Since we have taken the period d' to be equal to π, we also have

$$\alpha = \cosh \mu\pi = \cos i\mu\pi,$$

so that (3.19) can be written

$$\sin^2 \tfrac{1}{2} i\mu\pi = \{1 + \tfrac{1}{4}\nu^2\} \sin^2 \tfrac{1}{2}\pi\sqrt{a} + O(\nu^4), \tag{3.20}$$

which, on ignoring the term $O(\nu^4)$, is exactly similar to the equation obtained by Hill [7] for the determination of the characteristic exponent μ in the solution of the equation

$$F_{yy} + \{a + \psi(y)\}F = 0,$$

where $\psi(y)$ has period π and, in the particular case treated by Hill, is continuous. However, as we have seen, equation (3.20) holds for the determination of μ only when approximations are made and is not a general property of the solution as in Hill's case.

4. Numerical Results and Conclusion.

For the sinusoidal ice thickness case we determine the wavelengths $\lambda(B_1)$, $\lambda(A_1)$, ... , $\lambda(B_6)$, and consider the particular case $h_0 = h_1$ when the thickness distribution can be thought of as representing a semi-infinite array if discrete ice floes in the y direction. We take the values $h_0 = h_1 = 0.5, 1.0, 2.0, 4.0$ metres and the floe lengths $L = 10, 20, 50, 100, 200$, metres. The water

depth is assigned the two values 10 and 20 metres.

Solving (2.7) for $\lambda^2 = (2\pi)^2 gh/\omega^2$, we have

$$\left.\begin{aligned} \lambda^2 &= \frac{4L^2}{a^2} + (2\pi)^2 \operatorname{sh} h_0, \\ \lambda^2 &= \sqrt{\left\{\frac{2L^2(2\pi)^2 \operatorname{sh} h_0}{q}\right\}} + (2\pi)^2 \operatorname{sh} h_0. \end{aligned}\right\} \qquad (4.1)$$

Using the expressions derived in section 2 (d) for the values of a at the intersection of Γ with the characteristics b_1, a_1, b_2, a_2, we can at once compute, using the first of (4.1), the values of λ corresponding to these points, omitting those values for which $A < 2h_0 = 2h_1$.

The curves b_1, a_1, ... , b_6 are plotted using the tables of Appendix II in MacLachlan [1] for $q > 1$, and the series expansions (2.9) for $q < 1$. The parabola Γ is then plotted for the various values of the parameters, and the points of intersection with the characteristic curves tabulated. This procedure is not so tedious as may first appear, since the parabola Γ, we recall, is given by

$$a^2 = \left\{\frac{2L^2}{\pi^2 \operatorname{sh} h_0}\right\} q,$$

in the case $h_0 = h_1$, so that the term L^2/hh_0, and hence Γ, is the same for several combinations of the parameters. For graphical reasons it is convenient to plot the curves in such a way that we obtain the value of q more accurately than that of a at the inter-

sections, and so we use the second of (4.1) to determine the corresponding wavelengths. We also insert the line $a = 2q$ to distinguish those values for which $2q < a$ from those which violate the inequality. These results supplement those obtained by direct computation as indicated above, and the aggregate of the results determined by the two methods is shown in Table 1.

We include those wavelengths for which $2q > a$, and though these are not strictly valid conclusions of the theory quantitatively they serve to complete the picture qualitatively. Such values are denoted by an asterisk.

To understand the table it is sufficient to recall that those wavelengths lying between $\lambda(B_n)$ and $\lambda(A_n)$ are damped on entering the ice sheet. If a value $\lambda(B_n)$ is such that all shorter wavelengths are damped then this is indicated. All pairs of intersections $q(B_n)$, $q(A_n)$ which are virtually points are omitted from the table. All lengths are in metres.

The results indicate that cut-off bands of λ exist for increasing wavelengths as the ice thickness increases, and that for a particular thickness, although there are more cut-off bands for the longer floes, the particular wavelengths below which no propagation occurs do not vary significantly.

Robin [8] draws the following conclusions from the case of floe diameters small compared to wavelength: an approximately linear

relationship exists between the floe size and the wavelength at which damping first occurs; the cut off in wave transmission takes place for wavelengths less than three times the floe size.

From section 2. (d) we have the approximate value of the parameter a at the first cut-off,

$$a(B_1) = \frac{A}{h_1}\left\{-1 + \sqrt{\left(1 + \frac{2h_1}{A}\right)}\right\}$$

$$\simeq 1 - \frac{\pi^2 shh_1}{L^2} + O(h_1/A)^2,$$

and in the first of (4.1) this gives, when $h_1 = h_0$,

$$\lambda^2(B_1) = 4L^2 + 8\pi^2 shh_0 + O(h_0/A)^2,$$

or

$$\lambda(B_1) = 2L\{1 + \pi^2 shh_0/L^2 + O(h_0/A)^2\}.$$

This indicates that, while there is a dominant linear relation between the first cut-off wavelength and L in the present formulation, $\lambda(B_1)$ is further influenced by a term of order hh_0/L which in the case of shorter floes will be appreciable.

To compare the above observational conclusions of Robin with our numerical results, we consider the cases L = 10, 20 metres, and the thicknesses h_0 = 0.5, 1.0, 2.0 metres which are, according to Robin's data, typical for floes of this length. We at once see that the first cut-off of transmission occurs for a wavelength between 2 and 4 times the length of the floe. The factor of three postulated

by Robin is in evidence in the case of the shorter floe particularly. In general our first cut-off wavelength falls surprisingly close to that observed by Robin.

In conclusion, this problem has indicated qualitatively the existence of a wavelength, dependent on the exact nature of the ice form, at which the first major cut-off in transmission occurs, and a further wavelength below which no waves are allowed to propagate into the ice field. It is difficult to relate the results derived on the basis of shallow water theory to observational data, but the general comparison which we have been able to make is particularly favourable, and suggests that a similar approach for water of arbitrary depth would prove fruitful, especially for smaller floes whose bending is not a significant factor.

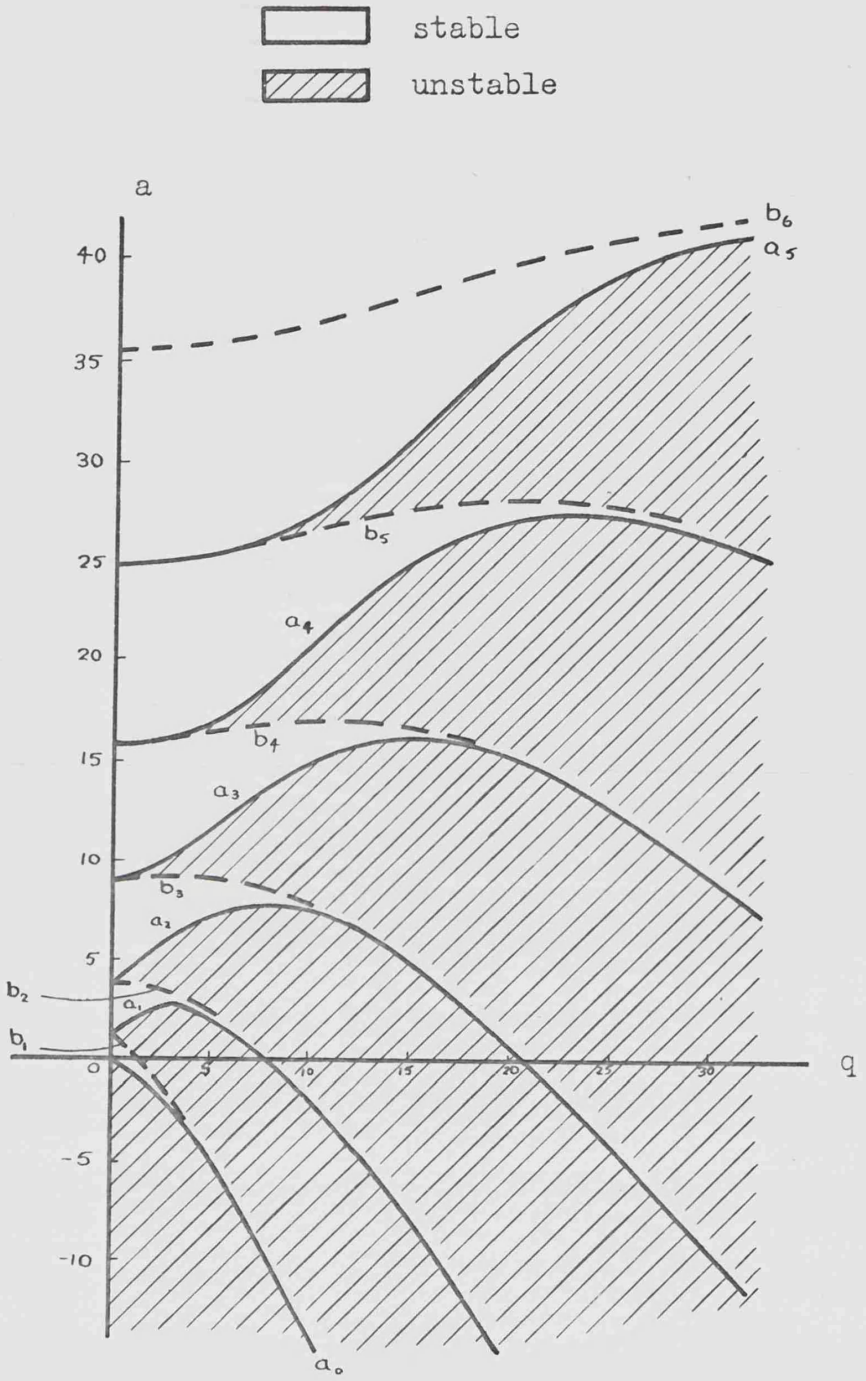

FIG. 1: The stability diagram for the Mathieu equation (taken from MacLachlan [1], page 40)

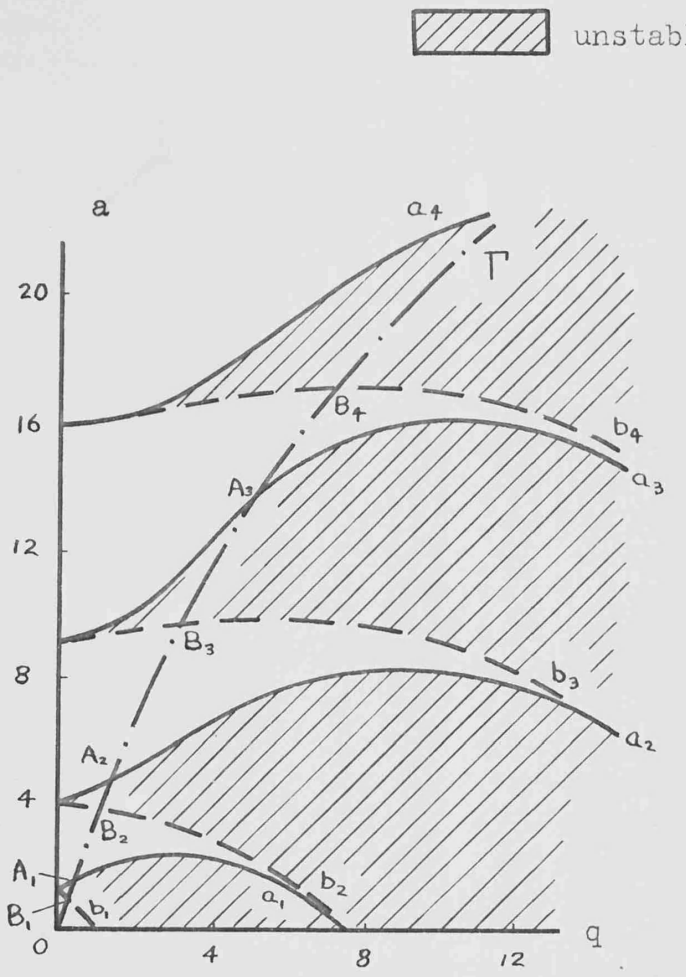

FIG. 3.

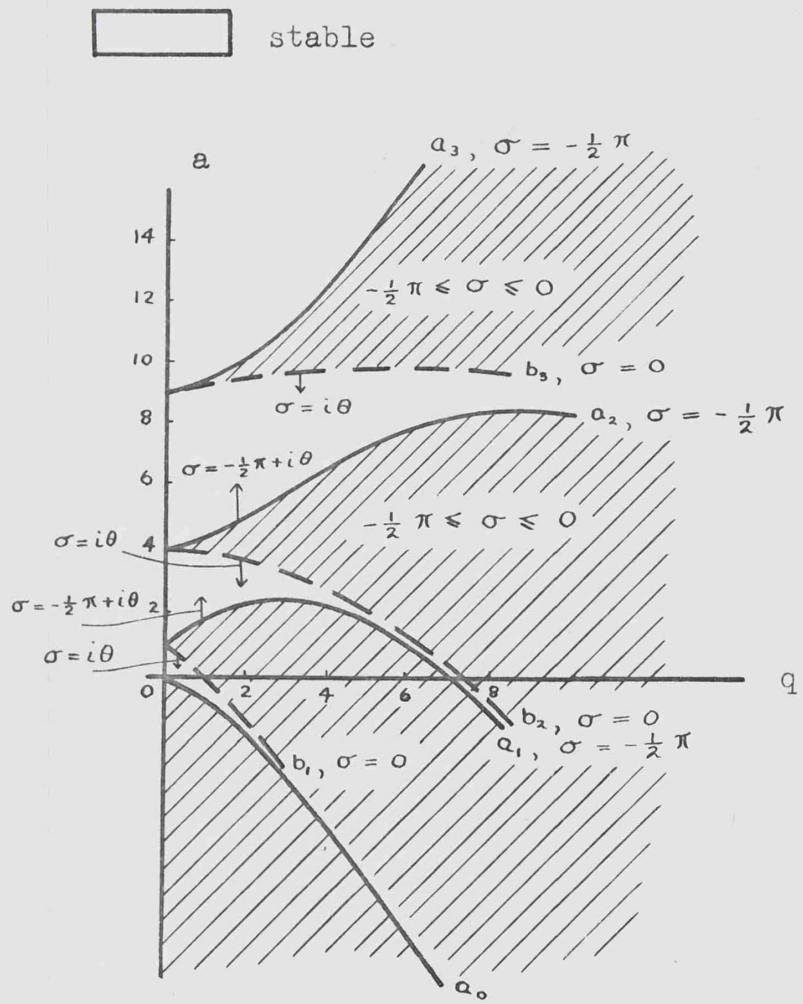

FIG. 2 : The distribution of σ.

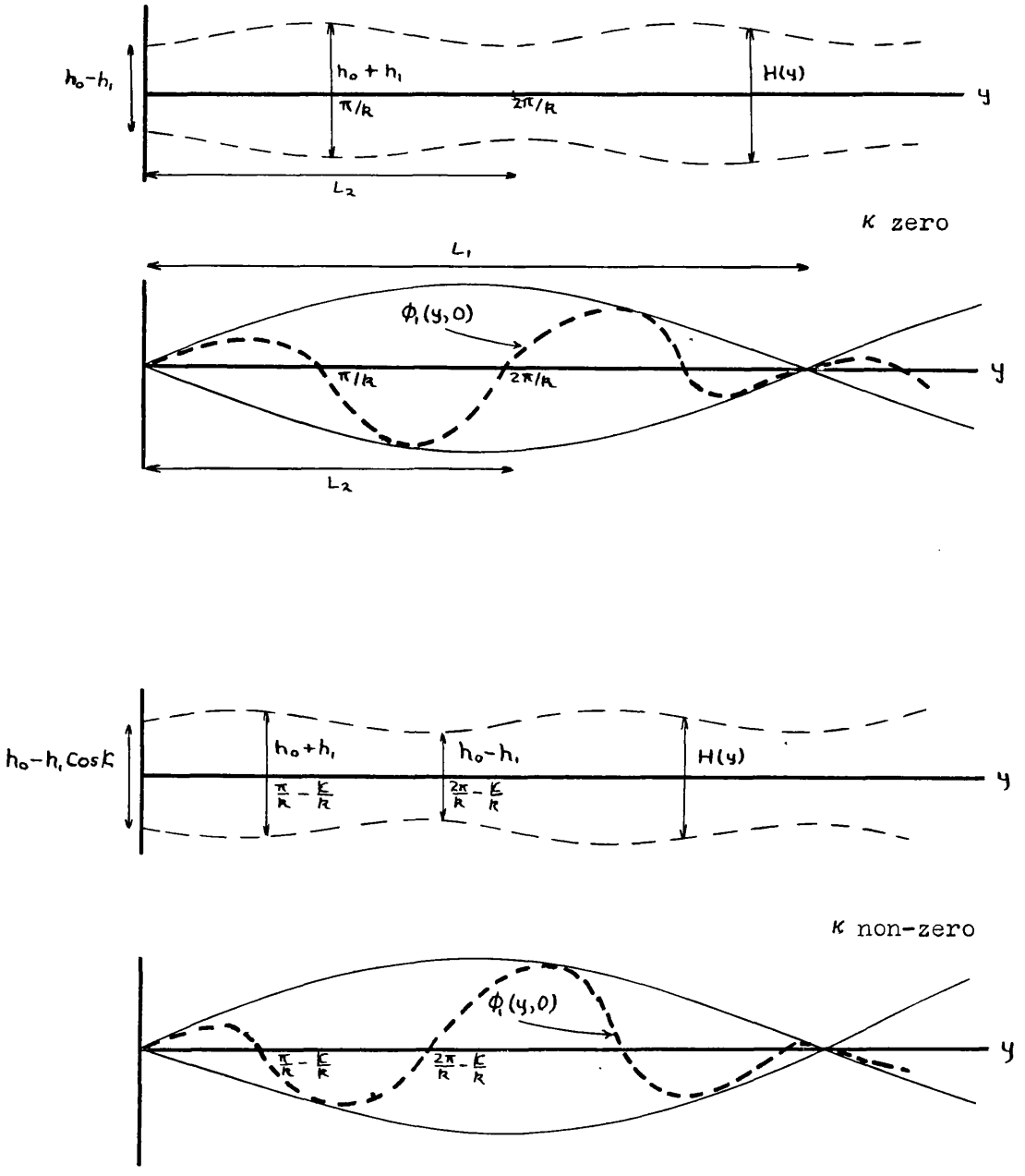
$h_0 - h_1$
$h_0 + h_1$
$H(y)$
π/k
$2\pi/k$
y
L_2
κ zero
L_1
$\phi_1(y,0)$
π/k
$2\pi/k$
y
L_2
$h_0 - h_1 \cos\kappa$
$h_0 + h_1$
$h_0 - h_1$
$H(y)$
$\frac{\pi}{k} - \frac{\kappa}{k}$
$\frac{2\pi}{k} - \frac{\kappa}{k}$
y
κ non-zero
$\phi_1(y,0)$
$\frac{\pi}{k} - \frac{\kappa}{k}$
$\frac{2\pi}{k} - \frac{\kappa}{k}$
y

FIG. 4.

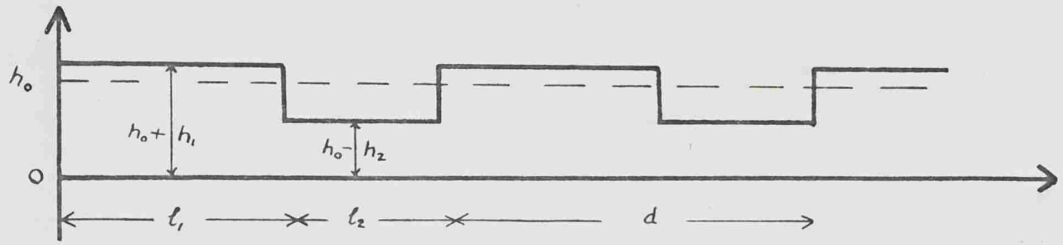

FIG. 5 : Section of the rectangular ice form

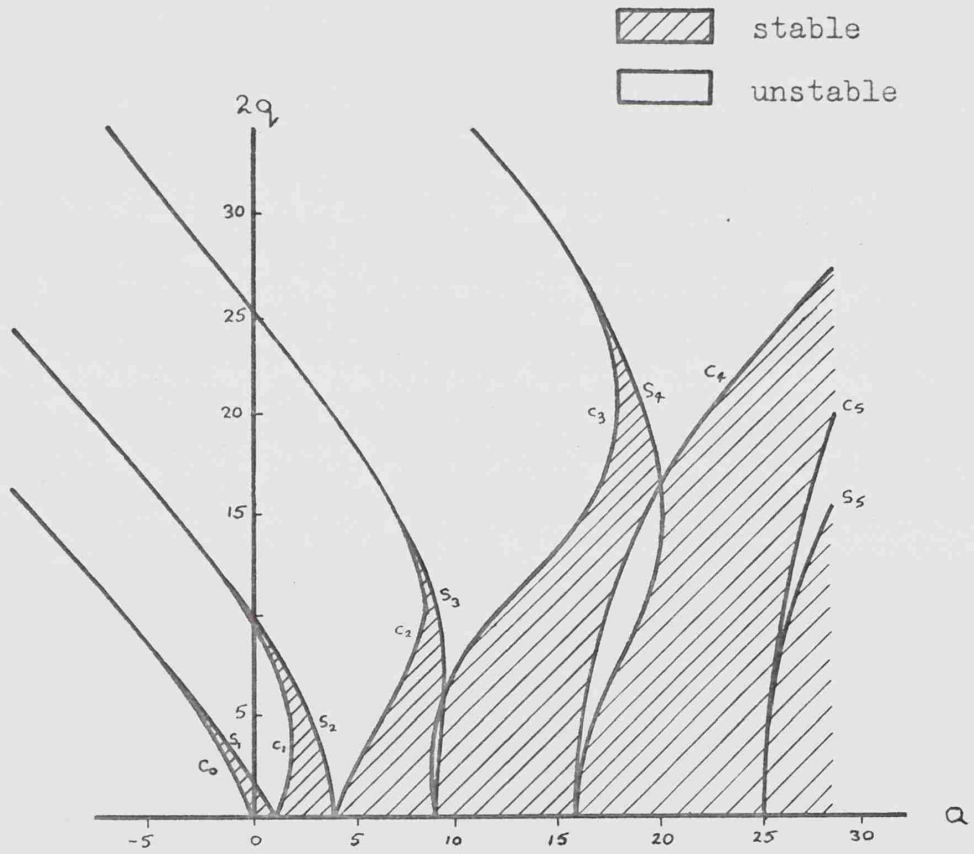

FIG. 6 : The stability diagram for the rectangular ice form (taken from [6])

h	h_0	L	$\lambda(B_1)$	$\lambda(A_1)$	$\lambda(B_2)$	$\lambda(A_2)$	$\lambda(B_3)$	$\lambda(A_3)$	$\lambda(B_4)$	$\lambda(A_4)$	$\lambda(B_5)$	$\lambda(A_5)$	$\lambda(B_6)$
10	0.5	10	25.6	21.9	16.9*	→	damped						
		20	43.2	41.1	25.6	21.3	18.7*	→	damped				
		50								undamped		→	31.6
		100							undamped		→	52.0	
		200							undamped		→	80.9	
	1.0	10	27.9	22.6*	22.2*	→	damped						
		20	46.1	42.1	29.2	20.9*	20.3*	→	damped				
		50			53.6	50.6	38.2	37.0	31.9	28.9	26.8	→	damped
		100							undamped		→	43.0	
		200							undamped		→	48.7	
	2.0	10	30.2*	27.7*	27.6*	→	damped						
		20	51.3	44.0	36.3*	26.7*	→	damped					
		50			57.0	54.1	42.3	39.6	35.8	→	damped		
		100								undamped		→	41.9
		200							undamped		→	52.0	

TABLE I (i).

h	h_0	L	$\lambda(B_1)$	$\lambda(A_1)$	$\lambda(B_2)$	$\lambda(A_2)$	$\lambda(B_3)$	$\lambda(A_3)$	$\lambda(B_4)$	$\lambda(A_4)$	$\lambda(B_5)$	$\lambda(A_5)$	$\lambda(B_6)$
10	4.0	10	40.0*	38.6*	38.5*	→	damped						
		20	59.9	45.3*	44.5*	→	damped						
		50	110.0	103.4	64.7	52.5	50.1	→	damped				
		100			107.2	102.3	78.3	76.0	63.8	61.2	42.9	→	damped
		200							undamped		→	113.1	
20	0.5	10	27.9	22.6*	22.2*	→	damped						
		20	46.1	42.1	29.2	20.9	20.3*	→	damped				
		50			53.6	50.6	38.2	37.0	31.9	28.9	26.8	→	damped
		100							undamped		→	43.0	
		200							undamped		→	48.7	
	1.0	10	30.2*	27.7*	27.6*	→	damped						
		20	51.3	44.0	36.3*	26.7*	→	damped					
		50			57.0	54.1	42.3	39.6	35.8	→	damped		
		100								undamped		→	41.9
		200								undamped		→	84.1

TABLE I (ii)

h	h_o	L	$\lambda(B_1)$	$\lambda(A_1)$	$\lambda(B_2)$	$\lambda(A_2)$	$\lambda(B_3)$	$\lambda(A_3)$	$\lambda(B_4)$	$\lambda(A_4)$	$\lambda(B_5)$	$\lambda(A_5)$	$\lambda(B_6)$
20	2.0	10	40.0*	38.6*	38.5*	→	damped						
		20	59.9	45.3*	44.5*	→	damped						
		50	110.0	103.4	64.7	52.5	50.1	→	damped				
		100			107.2	102.3	78.3	76.0	63.8	61.2	42.9	→	damped
		200							undamped		→	113.1	
	4.0	10	77.8*	61.8*	61.7*	→	damped						
		20	73.4	59.7*	59.4*	→	damped						
		50	80.2	74.6	60.8	56.9*	56.8*	→	damped				
		100			114.0	107.3	84.7	81.2	71.7	→	damped		
		200								undamped		→	83.7

TABLE I (iii) - showing the wavelengths which are damped. All lengths are in metres.

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CHAPTER IV

LINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS, AND THE GENERALISED RIEMANN-HILBERT PROBLEM.

1. Introduction.

The investigation carried out in this chapter was motivated by the derivation of two integral equations with weakly singular kernels, arising in connection with the water wave - ice floe model subsequently discussed in Chapter V. It can be shown that the classical Fredholm techniques, which reduce to a numerical investigation at an early stage, are applicable to these integral equations. However, on differentiation, a weakly singular integral equation gives rise to a simple linear integro-differential equation having a kernel which contains a Cauchy type singularity. This suggests a further method of obtaining information about the solutions of the original integral equations analytically. Actual ways of dealing with singular integro-differential equations were found to be not particularly well documented, and consequently we include in this chapter a survey of the methods of handling such equations, with a view to making use of the techniques developed on the integral equations of Chapter V cited above.

Integro-differential equations arise from physical problems in many fields, and as examples of the procedures described we solve two such equations occurring in aerodynamic theory. As the foregoing

theory is stated in general terms on the basis of several assumptions, it must be adapted to tackle each specific problem individually,and these examples serve to indicate difficulties which may evolve and devices by which they can be resolved.

A linear singular integro-differential equation may be written in the following general form,

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \sum_{r=0}^{n} \frac{1}{\pi} \int_L \frac{K_r(x,t)}{t-x}\, \phi^{(r)}(t)\, dt = f(x), \qquad (1.1)$$

where $x \in L$, $\phi^{(r)}(x)$ denotes the rth derivative of the unknown function $\phi(x)$, and $\alpha_r(x)$ and $K_r(x,t)$ are prescribed regular functions on L.

A means of solving (1.1) in the case when L is a simple closed curve has been proposed by Magnaradze, and consists of reducing the integro-differential equation, by means of direct manipulation, to an integral equation of the form

$$A(x)\, \phi(x) + \oint_L K(x,t)\, \phi(t)\, dt = F(x), \qquad (1.2)$$

for $\phi(x)$, where $K(x,t)$ is either a completely regular kernel, or is quasi-regular. Magnaradze's paper [1] on this topic comprises only a brief statement of the results he obtained and a number of suggestions as to possible applications of the method. A second publication [2] by Magnaradze treats the particular case of (1.1) when $m = n = 1$,

and $\alpha_0(x)$, $\alpha_1(x)$, $K_0(x,t)$, $K_1(x,t)$ and $f(x)$, together with the unknown $\phi(x)$, are matrices of a given order. Again the regular and quasi-regular integral equations to which this system can be reduced are given in the form of a summary of results.

It is the purpose of the initial sections of this chapter to develop similar results to those stated by Magnaradze, for the most general case when L is the union of smooth, non-intersecting arcs.

Equation (1.1) is related to the generalised Riemann-Hilbert problem in the theory of functions, which may be stated: to determine the function $\Phi(z)$, sectionally holomorphic in the plane apart from on the line L, where it satisfies the boundary condition,

$$\sum_{r=0}^{m} a_r(x) \frac{d^r \Phi_+(x)}{dx^r} - \sum_{r=0}^{n} b_r(x) \frac{d^r \Phi_-(x)}{dx^r} = f_1(x), \qquad (1.3)$$

where $x \in L$. The functions $a_r(x)$, $b_r(x)$ and $f_1(x)$ satisfy the Holder condition, and $\Phi_+(x)$ and $\Phi_-(x)$ are the limiting values of $\Phi(z)$ as z approaches L from each side. Expressing $\Phi(z)$ in the form of a Cauchy integral with an unknown density $\phi(t)$, which is Holder continuous

$$\Phi(z) = \int_L \frac{\phi(t)\, dt}{t - z}, \quad z \notin L, \qquad (1.4)$$

then as z tends to L from each side, we have by the Plemelj formulae

$$\Phi_{\pm}(x) = \pm i\pi\phi(x) + \int_L \frac{\phi(t)\,dt}{t - x} \quad . \tag{1.5}$$

$\Phi(z)$ as defined by (1.4) is the simplest example of a Carleman function. Using the fact that

$$\frac{d}{dx}\left\{\int_L \frac{\phi(t)\,dt}{t - x}\right\} = \frac{d}{dx}\left\{[\phi(t)\log|t - x|]_L - \int_L \phi^{(1)}(t)\log|t - x|\,dt\right\}$$

$$= \frac{d}{dx}\left\{[\phi(t)\log|t - x|]_L\right\} + \int_L \frac{\phi^{(1)}(t)\,dt}{t - x} ,$$

then r such operations used in (1.5) will give

$$\frac{d^r\Phi_{\pm}(x)}{dx^r} = \pm i\pi\phi^{(r)}(x) + \int_L \frac{\phi^{(r)}(t)\,dt}{t - x} + fr(x), \tag{1.6}$$

where $fr(x)$ is a function of x containing, in the case when L is an arc, or a union of arcs, the values of $\phi(x)$ and its first r derivatives at the end points of L. In the case when L is a simple closed contour, $fr(x)$ will of course vanish. Using (1.6) in (1.3) it is easily shown that one arrives at an equation of the type (1.1) for the density function $\phi(x)$.

Conversely, it may be possible to deduce from an equation of the form (1.1), a Riemann-Hilbert problem, by the introduction of a Carleman function similar to (1.4), but this depends on the exact form of the $K_r(x,t)$. In particular, if all $K_r(x,t)$ are independent of t, it is always possible to determine the boundary condition (1.3)

corresponding to the integro-differential equation, and this generalised Riemann-Hilbert problem,if soluble, provides a straightforward method of dealing with (1.1).

When L is a closed curve, or an aggregate of closed curves, methods have been developed to solve Riemann-Hilbert boundary equations. Integral representations for the two functions $\Phi_+(z)$ and $\Phi_-(z)$ which are defined inside and outside the contour respectively, are derived in terms of an unknown density $\mu(\tau)$, in such a way that the highest order derivatives of each of these functions present in the boundary condition are given by,

$$\frac{d^n\Phi_+(z)}{dz^n} = \frac{1}{2\pi i}\int_L \frac{\mu(\tau)\,d\tau}{\tau - z} \quad , \qquad \frac{d^m\Phi_-(z)}{dz^m} = \frac{z^{-m}}{2\pi i}\int_L \frac{\mu(\tau)\,d\tau}{\tau - z} \; .$$

The first of these is analytic inside L, the second is analytic outside L and vanishes at infinity. The derivatives of $\Phi_+(z)$ up to and including the $(n - 1)$ th, and those of $\Phi_-(z)$ up to and including the $(m - 1)$ th can be deduced from these two expressions in the form of integrals of $\mu(\tau)$ with Fredholm kernels. Substituting for the limiting values of $\Phi_+(z)$ and $\Phi_-(z)$ and their derivatives on L, into the generalised Riemann-Hilbert equation, one is led to the singular integral equation

$$a(x)\,\mu(x) + \frac{b(x)}{\pi i}\int_L \frac{\mu(t)\,dt}{t - x} + \int_L M(x,t)\,\mu(t)\,dt = f_2(x), \qquad (1.7)$$

for $\mu(x)$, in which $M(x,t)$ is a determined Fredholm kernel. This method was first proposed by I.N.Vekua [3]. A more recent development is the construction of a new integral representation by Rogozhin [4], which reduces the Riemann-Hilbert boundary condition to a completely regular integral equation

$$a_1(x)\ \mu(x) + \int_L N(x,t)\ \mu(t)\ dt = f_3(x).$$

Thus the outcome of this method is essentially the same as the direct approach of Magnaradze outlined above.

In section 6 we discuss the generalised Riemann-Hilbert problem, when L is the union of several arcs, from the point of view of a possible direct solution.

2. The Inversion of a Cauchy Integral.

Let us denote by $c_1, c_2, \ldots, c_{2n_1}$ the ends of the arcs $L_1, L_2, \ldots, L_{n_1}$, and define a function of the class

$$h(c_1, c_2, \ldots, c_{n_2})$$

to be that function which is bounded at the ends $c_1, c_2, \ldots, c_{n_2}$, and in fact zero there, and integrable at the remaining ends $c_{n_2+1}, \ldots, c_{2n_1}$. The solution of

$$\frac{1}{\pi i}\int_L \frac{\psi(t)\,dt}{t-x} = f_0(x), \tag{2.1}$$

where $L = L_1 + L_2 + \ldots + L_{n_1}$, in the form given by Muskhelishvili [5] (Chapter 11), is as follows: for $n_1 - n_2 \geq 0$, the solution of class $h(c_1, c_2, \ldots, c_{n_2})$ always exists and is given by

$$\psi(x) = \frac{1}{\pi i}\sqrt{\frac{R_1(x)}{R_2(x)}}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{f_0(t)\,dt}{t-x} + P_{n_1-n_2-1}\sqrt{\frac{R_1(x)}{R_2(x)}}, \tag{2.2}$$

where

$$R_1(x) = \prod_{k=1}^{n_2}(x - c_k), \qquad R_2(x) = \prod_{k=n_2+1}^{2n_1}(x - c_k),$$

and $P_{n_1-n_2-1}(x)$ is an arbitrary polynomial of degree not greater than n_1-n_2-1, and is identically zero if $n_1 = n_2$. If $n_1 - n_2 < 0$, the unique solution of class $h(c_1, c_2, \ldots, c_{n_2})$ exists if and only if

$f_0(x)$ satisfies the conditions

$$\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\; t^k\, f_0(t)\, dt = 0, \quad k = 0, 1, \ldots, n_2-n_1-1, \tag{2.3}$$

and is given by (2.2) with $P_{n_1-n_2-1}$ equal to zero.

In the particular case when L is a single arc (a,b), the solutions of (2.1) are as follows: the solution unbounded at both ends is

$$\psi(x) = \frac{1}{\pi i}\,\frac{1}{\sqrt{\{(x-a)(b-x)\}}}\int_a^b \frac{\sqrt{\{(t-a)(b-t)\}}f_0(t)\,dt}{t-x} + \\ + \frac{C}{\sqrt{\{(x-a)(b-x)\}}}, \tag{2.4}$$

where C is an arbitrary constant; the solution of class h(a), bounded at a, is

$$\psi(x) = \frac{1}{\pi i}\sqrt{\frac{x-a}{b-x}}\int_a^b \sqrt{\frac{b-t}{t-a}}\;\frac{f_0(t)\,dt}{t-x}, \tag{2.5}$$

and the solution of class h(a,b), bounded at both ends, is

$$\psi(x) = \frac{\sqrt{\{(x-a)(b-x)\}}}{\pi i}\int_a^b \frac{f_0(t)}{\sqrt{\{(t-a)(b-t)\}}}\;\frac{dt}{t-x}, \tag{2.6}$$

provided that

$$\int_a^b \frac{f_0(t)\,dt}{\sqrt{\{(b-t)(t-a)\}}} = 0. \tag{2.7}$$

We shall refer to these formulae in the following sections.

3. Reduction of a Singular Integro-differential Equation to an Integral Equation with a Regular Kernel.

We consider the case in which the line of integration has the form $L = L_1 + L_2 + \dots + L_{n1}$, and we deduce from (1.1) a regular integral equation for $\phi(x)$. The form of this equation is complicated in the general case, and we indicate some simplifications which occur when the line of integration is a closed curve. We assume that the $\alpha_r(x)$, $K_r(x,t)$ and $f(x)$ are sufficiently differentiable for the following steps to be valid, and that the various integrations by parts are permissible.

We arrange (1.1) in the form

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \sum_{r=0}^{n} \beta_r(x)\, \frac{1}{\pi}\int_L \frac{\phi^{(r)}(t)\, dt}{t - x} = f(x) + \sum_{r=0}^{n} \frac{1}{\pi}\int_L Q_r(x,t)\, \phi^{(r)}(t)\, dt, \qquad (3.1)$$

where

$$\beta_r(x) = K_r(x,x),$$

and

$$Q_r(x,t) = \frac{K_r(x,t) - K_r(x,x)}{t - x}$$

is regular. We may integrate the components of the final term of (3.1) by parts,

$$\int_L Q_r(x,t)\, \phi^{(r)}(t)\, dt = [Q_r(x,t)\phi^{(r-1)}(t)]_L - \int_L \frac{\partial Q_r(x,t)}{\partial t}\, \phi^{(r-1)}(t)dt$$

$$= \sum_{k=1}^{r} (-1)^{k-1} \left[\frac{\partial^{k-1} Q_r(x,t)}{\partial t^{k-1}} \phi^{(r-k)}(t) \right]_L + (-1)^r \int_L \frac{\partial^r Q_r(x,t)}{\partial t^r} \phi(t)\, dt,$$

after r such operations. Hence (3.1) becomes

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \sum_{r=0}^{n} \beta_r(x) \frac{1}{\pi} \int_L \frac{\phi^{(r)}(t)\, dt}{t - x} =$$

$$= F_1(x) + \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^r Q_r(x,t)}{\partial t^r} \phi(t)\, dt \tag{3.2}$$

where

$$F_1(x) = f(x) + \frac{1}{\pi} \sum_{r=0}^{n} \sum_{k=1}^{r} (-1)^{k-1} \left[\frac{\partial^{k-1} Q_r(x,t)}{\partial t^{k-1}} \phi^{(r-k)}(t) \right]_L \tag{3.3}$$

involves the values of $\phi(x)$ and its derivatives at the end points of the arc.

We now express (1.1) in the alternative form

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \sum_{r=0}^{n} \frac{1}{\pi} \int_L \frac{\beta_r(t)\, \phi^{(r)}(t)\, dt}{t - x} = f(x) +$$

$$+ \sum_{r=0}^{n} \frac{1}{\pi} \int_L P_r(x,t)\, \phi^{(r)}(t)\, dt$$

$$= F_2(x) + \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^r P_r(x,t)}{\partial t^r} \phi(t)\, dt, \tag{3.4}$$

where

$$P_r(x,t) = \frac{K_r(x,t) - K_r(t,t)}{t - x} ,$$

and

$$F_2(x) = f(x) + \frac{1}{\pi}\sum_{r=0}^{n}\sum_{k=1}^{r}(-1)^{k-1}\left[\frac{\partial^{k-1}P_r(x,t)}{\partial t^{k-1}}\,\phi^{(r-k)}(t)\right]_L. \qquad (3.5)$$

It is convenient to write (3.4) as

$$\sum_{r=0}^{n}\frac{1}{\pi}\int_L\frac{\beta_r(t)\,\phi^{(r)}(t)\,dt}{t-x} = \sum_{r=0}^{m}\alpha_r(x)\,\phi^{(r)}(x) - G(x),$$

where $G(x)$ represents the right hand side of (3.4). Inverting this equation by means of the formulae of section 2., the solution for

$$\sum_{r=0}^{n}\beta_r(x)\,\phi^{(r)}(x)$$

in the class $h(c_1, c_2, \ldots, c_{n_2})$ is given by

$$\sum_{r=0}^{n}\beta_r(x)\,\phi^{(r)}(x) = -\frac{1}{\pi}\sqrt{\frac{R_1(x)}{R_2(x)}}\int_L\sqrt{\frac{R_2(t)}{R_1(t)}}\;\frac{\sum_0^m\alpha_r(t)\,\phi^{(r)}(t) - G(t)}{t-x}\,dt +$$

$$+ P_{n_1-n_2-1}\sqrt{\frac{R_1(x)}{R_2(x)}}\,, \qquad (3.6)$$

where the arbitrary polynomial is chosen in accordance with the results stated in section 2. In particular, if the solution (3.6) is required to be bounded at more than half of the end points of L, then the function

$$\sum_{r=0}^{m}\alpha_r(x)\,\phi^{(r)}(x) - G(x)$$

must satisfy the corresponding solubility conditions (2.3); that is,

the final solution of the problem is only valid providing that it is found to satisfy these conditions. If the function on the right hand side of (1.1) contains a number of arbitrary constants, then it may be possible to select these in such a way that the solubility conditions hold.

Writing

$$R(x) = \sqrt{\{R_1(x)/R_2(x)\}},$$

and dropping the subscript from the arbitrary polynomial, (3.6) is

$$\sum_{r=0}^{n} \beta_r(x)\, \phi^{(r)}(x) = -\, R(x) \sum_{r=0}^{m} \frac{1}{\pi} \int_L \frac{\alpha_r(t)\, \phi^{(r)}(t)\, dt}{R(t)\{t - x\}} +$$

$$+ \frac{R(x)}{\pi} \int_L \frac{G(t)\, dt}{R(t)\{t - x\}} + R(x)\, P(x),$$

$$= -\frac{1}{\pi} \sum_{r=0}^{m} \alpha_r(x) \int_L \frac{\phi^{(r)}(t)\, dt}{t - x} + R(x) \sum_{r=0}^{m} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^r \sigma_r(x,t)}{\partial t^r}\, \phi(t)\, dt +$$

$$+ \frac{R(x)}{\pi} \sum_{r=0}^{m} \sum_{k=1}^{r} (-1)^{k-1} \left[\frac{\partial^{k-1} \sigma_r(x,t)}{\partial t^{k-1}}\, \phi^{(r-k)}(t) \right]_L +$$

$$+ \frac{R(x)}{\pi} \int_L \frac{G(t)\, dt}{R(t)\{t - x\}} + R(x)\, P(x) \qquad (3.7)$$

in which

$$\sigma_r(x,t) = \left\{ \frac{\alpha_r(x)}{R(x)} - \frac{\alpha_r(t)}{R(t)} \right\} \frac{1}{t - x},$$

is regular.

Now

$$\int_L \frac{G(t)\,dt}{R(t)\{t-x\}} =$$

$$= \int_L \frac{F_2(t)\,dt}{R(t)\{t-x\}} + \int_L \frac{dt}{R(t)\{t-x\}} \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^r P_r(t,\tau)}{\partial \tau^r} \phi(\tau)\,d\tau$$

$$= \int_L \frac{F_2(t)\,dt}{R(t)\{t-x\}} + \frac{1}{\pi} \sum_{r=0}^{n} (-1)^r \int_L \phi(t)\,dt \int_L \frac{\partial^r P_r(\tau,t)}{\partial t^r} \frac{d\tau}{R(\tau)\{\tau-x\}},$$

since changing the order of integration is permissible in the last term, and using this, we find that (3.7) can be written

$$\sum_{r=0}^{n} \beta_r(x)\, \phi^{(r)}(x) + \sum_{r=0}^{m} \alpha_r(x) \frac{1}{\pi} \int_L \frac{\phi^{(r)}(t)}{t-x}\,dt = F_3(x) + \frac{1}{\pi} \int_L Z(x,t)\phi(t)\,dt, \tag{3.8}$$

where

$$Z(x,t) = R(x) \sum_{r=0}^{m} (-1)^r \frac{\partial^r \sigma_r(x,t)}{\partial t^r} + R(x) \sum_{r=0}^{n} (-1)^r \times$$

$$\times \frac{1}{\pi} \int_L \frac{\partial^r P_r(\tau,t)}{\partial t^r} \frac{d\tau}{R(\tau)\{\tau-x\}}, \tag{3.9}$$

and

$$F_3(x) = \frac{R(x)}{\pi} \left\{ \sum_{r=0}^{m} \sum_{k=1}^{r} (-1)^{k-1} \left[\frac{\partial^{k-1} \sigma_r(x,t)}{\partial t^{k-1}} \phi^{(r-k)}(t) \right]_L + \int_L \frac{F_2(t)\,dt}{R(t)\{t-x\}} \right\}$$

$$+ R(x)\, P(x). \tag{3.10}$$

We now have the two equations (3.2) and (3.8),

$$\left.\begin{aligned} \sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \sum_{r=0}^{n} \beta_r(x)\, \frac{1}{\pi}\int_L \frac{\phi^{(r)}(t)\, dt}{t - x} &= X(x), \\ \sum_{r=0}^{n} \beta_r(x)\, \phi^{(r)}(x) + \sum_{r=0}^{m} \alpha_r(x)\, \frac{1}{\pi}\int_L \frac{\phi^{(r)}(t)\, dt}{t - x} &= Y(x), \end{aligned}\right\} \qquad (3.11)$$

where $X(x)$ and $Y(x)$ represent the right hand sides of (3.2) and (3.8) respectively, and contain $\phi(x)$ only in integrals with regular kernels. We must now eliminate from the equations (3.11) the singular integrals containing the $\phi^{(r)}(t)$. To do this we consider the function

$$\psi(x) = \frac{1}{\pi}\int_L \frac{\phi\,(t)\, dt}{t - x}\ ,$$

which, on integrating by parts and differentiating with respect to x gives

$$\psi'(x) = \frac{1}{\pi}\frac{d}{dx}\left[\phi(t)\, \log|t - x|\ \right]_L + \frac{1}{\pi}\int_L \frac{\phi(t)\, dt}{t - x}\ ,$$

and after a further (r-1) such procedures yields

$$\psi^{(r)}(x) = \frac{1}{\pi}\int_L \frac{\phi^{(r)}(t)\, dt}{t - x} + f_r(x), \qquad (3.12)$$

where

$$f_r(x) = \frac{1}{\pi}\sum_{k=1}^{r} \frac{d^k}{dx^k}\left[\ \phi^{(r-k)}(t)\, \log|t - x|\ \right]_L .$$

In terms of $\psi(x)$, (3.11) becomes

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \sum_{r=0}^{n} \beta_r(x)\, \psi^{(r)}(x) = X(x) - \sum_{r=0}^{n} \beta_r\,(x)\, f_r(x)$$

$$= X_1(x), \tag{3.13}$$

$$\sum_{r=0}^{n} \beta_r(x)\, \phi^{(r)}(x) + \sum_{r=0}^{m} \alpha_r(x)\, \psi^{(r)}(x) = Y(x) + \sum_{r=0}^{m} \alpha_r(x)\, f_r\,(x)$$

$$= Y_1(x), \tag{3.14}$$

and from these two equations we must eliminate the $\psi^{(r)}(x)$, $r = 0, 1, \ldots. \max(m,n)$. Hereafter we assume that $m \geqslant n$, a similar procedure applying in the case $m < n$. Differentiating (3.13) 0, 1, ..., m times, multiplying the resulting equations by $\mu_0(x)$, $\mu_1\,(x)$, ... $\mu_m(x)$ respectively and adding them together, we obtain

$$\sum_{p=0}^{m} \mu_p(x)\, \frac{d^p}{dx^p} \left\{ \sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) \right\} - \sum_{p=0}^{m} \mu_p(x)\, \frac{d^p}{dx^p} \left\{ \sum_{r=0}^{n} \beta_r(x)\, \psi^{(r)}(x) \right\} =$$

$$= \sum_{p=0}^{m} \mu_p(x)\, \frac{d^p X_1}{dx^p}\ . \tag{3.15}$$

Repeating this for equation (3.14), using the multipliers $\lambda_0(x)$, $\lambda_1(x)$, ... , $\lambda_m(x)$, and subtracting the resulting expression from (3.15), we obtain,

$$\sum_{p=0}^{m} \left\{ \mu_p(x)\, \frac{d^p}{dx^p} \sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) - \lambda_p(x)\, \frac{d^p}{dx^p} \sum_{r=0}^{n} \beta_r(x)\, \phi^{(r)}(x) \right\} -$$

$$- \sum_{p=o}^{m} \left\{ \mu_p(x) \frac{d^p}{dx^p} \sum_{r=o}^{n} \beta_r(x) \psi^{(r)}(x) + \lambda_p(x) \frac{d^p}{dx^p} \sum_{r=o}^{m} \alpha_r(x) \psi^{(r)}(x) \right\} = T(x)$$

or,

$$\sum_{\ell=o}^{2m} a_\ell(x) \phi^{(\ell)}(x) - \sum_{\ell=o}^{m+n} b_\ell(x) \phi^{(\ell)}(x) - \sum_{\ell=o}^{m+n} c_\ell(x) \psi^{(\ell)}(x) -$$

$$- \sum_{\ell=o}^{2m} d_\ell(x) \psi^{(\ell)}(x) = T(x), \qquad (3.16)$$

where $a_\ell(x)$, $b_\ell(x)$, ... are functions of $\mu_p(x)$, $\lambda_p(x)$, $\alpha_r(x)$ and $\beta_r(x)$ which we can determine explicitly, and

$$T(x) = \sum_{p=o}^{m} \left\{ \mu_p(x) \frac{d^p X_1}{dx^p} - \lambda_p(x) \frac{d^p Y_1}{dx^p} \right\},$$

$$= \frac{1}{\pi} \int_L R(x,t) \phi(t) \, dt + g(x), \qquad (3.17)$$

on simplifying, where

$$R(x,t) = \sum_{p=o}^{m} \left\{ \mu_p(x) \sum_{r=o}^{n} (-1)^r \frac{\partial^{r+p} Q_r(x,t)}{\partial t^r \partial x^p} - \lambda_p(x) \frac{\partial^p Z(x,t)}{\partial x^p} \right\}, \qquad (3.18)$$

and

$$g(x) = \sum_{p=o}^{m} \left\{ \mu_p(x) F_1^{(p)}(x) - \mu_p(x) \frac{d^p}{dx^p} \sum_{r=o}^{n} \beta_r(x) f_r(x) - \lambda_p(x) F_3^{(p)}(x) - \right.$$

$$\left. - \lambda_p(x) \frac{d^p}{dx^p} \sum_{r=o}^{m} \alpha_r(x) f_r(x) \right\} . \qquad (3.19)$$

For all the $\psi^{(\ell)}(x)$ in (3.16) to vanish, we must have

$$\left.\begin{aligned} c_\ell(x) + d_\ell(x) &= 0, \quad \ell = 0, 1, \ldots, m+n, \\ d_\ell(x) &= 0, \quad \ell = m+n+1, \ldots, 2m, \end{aligned}\right\} \qquad (3.20)$$

giving 2m+1 equations between the 2m+2 unknown functions $\mu_0(x), \ldots, \mu_m(x), \lambda_0(x), \ldots, \lambda_m(x)$. Suppose that these functions have been chosen in accordance with (3.20), together with some additional condition which will be discussed later, then we have

$$\sum_{\ell=0}^{2m} a_\ell(x)\, \phi^{(\ell)}(x) - \sum_{\ell=0}^{m+n} b_\ell(x)\, \phi^{(\ell)}(x) = T(x), \qquad (3.21)$$

in which all the coefficients are known. Regarding the right hand side as a known function, this represents a differential equation of order 2m for $\phi(x)$. The solution of this equation is

$$\sum_{r=1}^{2m} C_r\, \phi_r(x),$$

the solution of the homogeneous equation

$$\sum_{\ell=0}^{2m} a_\ell(x)\, \phi^{(\ell)}(x) - \sum_{\ell=0}^{m+n} b_\ell(x)\, \phi^{(\ell)}(x) = 0,$$

plus a particular integral of (3.21). Appealing to the theory of differential equations (see for example [6]), such a particular integral is

$$\phi(x) = \sum_{r=1}^{2m} K_r(x)\, \phi_r(x),$$

where the $K_r(x)$ are subject to

$$\sum_{r=1}^{2m} \frac{dK_r}{dx}\, \phi_r^{(j-1)}(x) = 0, \qquad j = 1, 2, \ldots, 2m-1,$$

$$\sum_{r=1}^{2m} \frac{dK_r}{dx}\, \phi_r^{(2m-1)}(x) = \frac{T(x)}{a_{2m}(x)},$$

assuming that a_{2m} is not identically zero. Using Cramer's rule to solve this system of equations for the $K_i'(x)$, we have

$$\frac{dK_i}{dx} = M_i(x)\, T(x)/\{W(x) a_{2m}(x)\},$$

where $W(x)$ is the Wronskian of the system, and $M_i(x)$ is the cofactor of the i th element of the last row of the Wronskian. Hence

$$K_i(x) = \int^x \frac{M_i(\zeta)\, T(\zeta)}{W(\zeta)\, a_{2m}(\zeta)}\, d\zeta,$$

and the general solution of (3.21) is thus given by

$$\phi(x) = \sum_{i=1}^{2m} C_i\, \phi_i(x) + \sum_{i=1}^{2m} \phi_i(x) \int^x \frac{M_i(\zeta)\, T(\zeta)}{W(\zeta)\, a_{2m}(\zeta)}\, d\zeta$$

$$= \sum_{i=1}^{2m} C_i\, \phi_i(x) + \sum_{i=1}^{2m} \phi_i(x) \int^x \frac{M_i(\zeta)\, d\zeta}{W(\zeta)\, a_{2m}(\zeta)} \left\{\frac{1}{\pi}\int_L R(\zeta,t)\phi(t)\, dt + g(\zeta)\right\},$$

on substituting for $T(x)$, and this can be rearranged to give

$$\phi(x) - \frac{1}{\pi}\int_L K(x,t)\, \phi(t)\, dt = f_0(x), \qquad (3.22)$$

in which

$$K(x,t) = \sum_{i=1}^{2m} \phi_i(x) \int^{x} \frac{M_i(\zeta)\, R(\zeta,t)}{W(\zeta)\; a_{2m}(\zeta)}\, d\zeta, \tag{3.23}$$

and

$$f_o(x) = \sum_{i=1}^{2m} C_i\, \phi_i(x) + \sum_{i=1}^{2m} \phi_i(x) \int^{x} \frac{M_i(\zeta)\; g(\zeta)}{W(\zeta)\; a_{2m}(\zeta)}\, d\zeta. \tag{3.24}$$

Equation (3.22) represents an integral equation for $\phi(x)$ in which the kernel $K(x,t)$ is regular.

We now consider the determination of the multipliers $\mu_i(x)$ and $\lambda_i(x)$. The first term in (3.16) comes from

$$A(x) = \sum_{p=o}^{m} \mu_p(x)\, \frac{d^p}{dx^p} \left\{ \sum_{r=o}^{m} \alpha_r(x)\; \phi^{(r)}(x) \right\}$$

$$= \sum_{p=o}^{m} \mu_p(x) \sum_{r=o}^{m} \sum_{k=o}^{p} \binom{p}{k} \phi^{(r+k)}(x)\; \alpha_r^{(p-k)}(x),$$

on performing the differentiation, and by rearranging the terms and making a change of summation variable, we can show that this can be written

$$A(x) = \sum_{\ell=o}^{m} \sum_{p=o}^{m} \mu_p(x) \sum_{\substack{k=o \\ k\leq \ell}} \binom{p}{k} \phi^{(\ell)}(x)\; \alpha_{\ell-k}^{(p-k)}(x) +$$

$$+\sum_{\ell=m+1}^{2m}\ \sum_{p=\ell-m}^{m}\mu_p(x)\sum_{\substack{k=o\\ k\geq \ell-m}}^{p}\binom{p}{k}\phi^{(\ell)}(x)\ \alpha_{\ell-k}^{(p-k)}(x)=\sum_{\ell=o}^{2m}a_\ell(x)\ \phi^{(\ell)}(x),$$

where

$$\left.\begin{aligned} a_\ell(x) &= \sum_{p=o}^{m}\mu_p(x)\ \lambda_{p\ell}(x), \qquad 0\leq \ell \leq m,\\ &= \sum_{p=\ell-m}^{m}\mu_p(x)\ \lambda'_{p\ell}(x), \qquad m+1 \leq \ell \leq 2m,\end{aligned}\right\} \tag{3.25}$$

and

$$\lambda_{p\ell}(x)=\sum_{\substack{k=o\\ k\leq \ell}}^{p}\binom{p}{k}\alpha_{\ell-k}^{(p-k)}(x),\qquad \lambda'_{p\ell}(x)=\sum_{\substack{k=o\\ k\geq \ell-m}}^{p}\binom{p}{k}\alpha_{\ell-k}^{(p-k)}(x). \tag{3.26}$$

Similarly, it can be shown that

$$\left.\begin{aligned} b_\ell(x) &= \sum_{p=o}^{m+n}\lambda_p(x)\ \mu_{p\ell}(x), \qquad 0 \leq \ell \leq n,\\ &= \sum_{p=\ell-n}^{m}\lambda_p(x)\ \mu'_{p\ell}(x), \qquad n+1 \leq \ell \leq n+m,\end{aligned}\right\} \tag{3.27}$$

where

$$\mu_{p\ell}(x)=\sum_{\substack{k=o\\ k\leq \ell}}^{p}\binom{p}{k}\beta_{\ell-k}^{(p-k)}(x),\qquad \mu'_{p\ell}(x)=\sum_{\substack{k=o\\ k\geq \ell-n}}^{p}\binom{p}{k}\beta_{\ell-k}^{(p-k)}(x)\ ; \tag{3.28}$$

and that

$$\left.\begin{aligned} c_\ell(x) &= \sum_{p=0}^{m} \mu_p(x)\ \mu_{p\ell}(x), \qquad 0 \leqslant \ell \leqslant n, \\ &= \sum_{p=\ell-n}^{m} \mu_p(x)\ \mu'_{p\ell}\ (x), \qquad n+1 \leqslant \ell \leqslant n+m, \end{aligned}\right\} \tag{3.29}$$

$$\left.\begin{aligned} d_\ell(x) &= \sum_{p=0}^{m} \lambda_p(x)\ \lambda_{p\ell}(x), \qquad 0 \leqslant \ell \leqslant m, \\ &= \sum_{p=\ell-m}^{m} \lambda_p(x)\ \lambda'_{p\ell}(x), \qquad m+1 \leqslant \ell \leqslant 2m. \end{aligned}\right\} \tag{3.30}$$

Using these relations, the conditions (3.20), that the $\psi^{(\ell)}(x)$ all vanish, become

$$\sum_{p=0}^{m} \mu_p(x)\ \mu_{p\ell}(x) + \sum_{p=0}^{m} \lambda_p(x)\ \lambda_{p\ell}(x) = 0, \qquad 0 \leqslant \ell \leqslant n,$$

$$\sum_{p=\ell-n}^{m} \mu_p(x)\ \mu'_{p\ell}(x) + \sum_{p=0}^{m} \lambda_p(x)\ \lambda_{p\ell}(x) = 0, \qquad n+1 \leqslant \ell \leqslant m,$$

$$\sum_{p=\ell-n}^{m} \mu_p(x)\ \mu'_{p\ell}(x) + \sum_{p=\ell-m}^{m} \lambda_p(x)\ \lambda'_{p\ell}(x) = 0, \qquad m+1 \leqslant \ell \leqslant n+m,$$

$$\sum_{p=\ell-m}^{m} \lambda_p(x)\ \lambda'_{p\ell}(x) = 0, \qquad n+m+1 \leqslant \ell \leqslant 2m.$$

As we have seen, we are free to impose one further condition on the multipliers, to determine them uniquely. Let this condition be

$$\mu_0(x) + \dots + \mu_m(x) + \lambda_0(x) + \dots + \lambda_m(x) = h(x),$$

where $h(x)$ can be chosen later. The matrix of coefficients of this system is as follows:

$$\begin{bmatrix}
\mu_{00} & \mu_{10} & \mu_{20} & \cdot & \cdot & \cdot & \cdot & \mu_{m0} & \lambda_{00} & \lambda_{10} & \lambda_{20} & \cdot & \cdot & \cdot & \cdot & \lambda_{m0} \\
\mu_{01} & \mu_{11} & \mu_{21} & \cdot & \cdot & \cdot & \cdot & \mu_{m1} & \lambda_{01} & \lambda_{11} & \lambda_{21} & \cdot & \cdot & \cdot & \cdot & \lambda_{m1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\mu_{0n} & \mu_{1n} & \mu_{2n} & \cdot & \cdot & \cdot & \cdot & \mu_{mn} & \lambda_{0n} & \lambda_{1n} & \lambda_{2n} & \cdot & \cdot & \cdot & \cdot & \lambda_{mn} \\
0 & \mu'_{1,n+1} & \mu'_{1,n+2} & \cdot & \cdot & \cdot & \cdot & \mu'_{m,n+1} & \lambda_{0,n+1} & \lambda_{1,n+1} & & \cdot & \cdot & \cdot & \cdot & \lambda_{m,n+1} \\
0 & 0 & \mu'_{2,n+2} & \cdot & \cdot & \cdot & \cdot & \mu'_{m,n+2} & \lambda_{0,n+2} & \lambda_{1,n+2} & & \cdot & \cdot & \cdot & \cdot & \lambda_{m,n+2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & . & \mu'_{m-n,m} & \cdot & \cdot & \mu'_{m\,m} & \lambda_{0m} & \lambda_{1m} & \lambda_{2m} & \cdot & \cdot & \cdot & \cdot & \lambda_{mm} \\
0 & 0 & 0 & . & 0 & \mu'_{n-m+1,m+1} & . & \mu'_{m,m+1} & 0 & \lambda'_{1,m+1} & \lambda'_{2,m+1} & \cdot & \cdot & \cdot & \cdot & \lambda'_{m,m+1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & . & 0 & 0 & . & \mu'_{m,n+m} & 0 & 0 & 0 & . & \lambda'_{n,m+n} & . & . & \lambda'_{m,n+m} \\
0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 & 0 & . & 0 & \lambda'_{n+1,m+n+1} & . & \lambda'_{m,n+m+1} \\
\cdot & \cdot & \cdot & , & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 & 0 & . & 0 & 0 & . & \lambda'_{m,2m} \\
1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & . & 1
\end{bmatrix}$$

By Cramer's rule

$$\nu_r(x) = h(x)\, A_{2m+2,r}(x)/W_1(x), \quad r = 0, 1, \dots, 2m+2,$$

where

$$\nu_r(x) = \mu_r(x), \quad r = 0, 1, \dots, m,$$

and $\nu_r(x) = \lambda_{m+1-r}(x), \quad r = m+1, \ldots, 2m+2;$

$W_1(x)$ is the determinant of the coefficients, and $A_{2m+2,r}(x)$ is the cofactor of the r th element of the final row. Choosing $h(x) = W_1(x)$, then

$$\nu_r(x) = A_{2m+2,r}(x), \quad r = 0, 1, \ldots, 2m+2,$$

determines the multipliers.

It is seen that this method is effective in reducing (1.1) to a regular integral equation, provided that the various integrations by parts do not give rise to terms unbounded at one or more ends of the arcs L_i, and that the inversion step can be performed, the solubility conditions being satisfied ultimately if necessary. In the case in which L is a closed curve, or a union of non-intersecting closed curves, these difficulties do not occur. We can easily deduce the corresponding results in this case, bearing in mind that all the terms evaluated around L become zero, and, by reference to section 2., noting that putting $P(x) = 0$, and $R(x) = 1$, the inversion formula for arcs is identical to that for closed contours as given by Muskhelishvili [5].

Also

$$\frac{\partial^p Z(x,t)}{\partial x^p} = \sum_{r=0}^{m} (-1)^r \frac{\partial^{r-p}\sigma_r(x,t)}{\partial t^r \partial x^p} - \sum_{r=0}^{n} (-1)^r \frac{d^p}{dx^p} \frac{1}{\pi} \int_L \frac{\partial^r P_r(\tau,t)}{\partial t^r} \frac{d\tau}{\tau - x}$$

$$= \sum_{r=o}^{m} (-1)^r \frac{\partial^{r-p} \sigma_r(x,t)}{\partial t^r \partial x^p} - \sum_{r=o}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^{r-p} P_r(\tau,t)}{\partial t^r \partial \tau^p} \frac{d\tau}{\tau - x} ,$$

on integrating by parts p times. Hence, for a closed curve,

$$R(x,t) = \sum_{p=o}^{m} \left\{ \mu_p(x) \sum_{r=o}^{n} (-1)^r \frac{\partial^{r-p} Q_r(x,t)}{\partial t^r \partial x^p} + \lambda_p(x) \sum_{r=o}^{m} (-1)^r \frac{\partial^{r-p} \sigma_r(x,t)}{\partial t^r \partial x^p} \right.$$

$$\left. - \lambda_p(x) \sum_{r=o}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^{r-p} P_r(\tau,t)}{\partial t^r \partial \tau^p} \frac{d\tau}{\tau - x} \right\} ,$$

while it is not hard to see that $g(x)$ simplifies to

$$g(x) = \sum_{p=o}^{m} \left\{ \mu_p(x) \, f^{(p)}(x) - \lambda_p(x) \frac{1}{\pi} \int_L \frac{f^{(p)}(t)}{t - x} \, dt \right\}.$$

Apart from these simplifications, the formulae remain as above. These last results coincide with those stated by Magnaradze in his paper [3].

It is easy to see that if the $P_r(x,t)$, $Q_r(x,t)$ and $\sigma_r(x,t)$ are rational functions, then the kernel (3.22) is of the separable type, that is, it can be written in the form

$$K(x,t) = \sum_j K_j(x) \, L_j(t) ,$$

and in such a case, the solution of (3.22) is readily found, by constructing a system of linear equations for the unknown constants

$$E_j = \int_L L_j(t) \, \phi(t) \, dt.$$

4. Reduction of a Singular Integro-differential Equation to a Quasi-Regular Integral Equation.

This method of reduction is simpler in that it does not involve the determination of the unknown multipliers of the previous section. From (3.2) we have

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) = G(x), \tag{4.1}$$

where

$$G(x) = F_1(x) + \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\partial^r Q_r(x,t)}{\partial t^r} \phi(t)\, dt +$$

$$+ \sum_{r=0}^{n} \beta_r(x) \frac{1}{\pi} \int_L \frac{\phi^{(r)}(t)}{t - x}\, dt,$$

and, solving (4.1) by the method invoked in section 3., we have

$$\phi(x) = \sum_{\ell=1}^{m} C_\ell\, \phi_\ell(x) + \sum_{\ell=1}^{m} \phi_\ell(x) \int^{x} \frac{G(\zeta)\, M_\ell(\zeta)}{W(\zeta)\, \alpha_m(\zeta)}\, d\zeta, \tag{4.2}$$

where the first summation on the right hand side represents the general solution of the homogeneous equation

$$\sum_{r=0}^{m} \alpha_r(x)\, \phi^{(r)}(x) = 0,$$

$W(x)$ is the Wronskian of the system $\phi_1(x), \phi_2(x), \ldots, \phi_m(x)$, and $M_\ell(x)$ is the cofactor of the ℓ th element of the last row of the Wronskian. Replacing $G(x)$ in (4.2),

$$\phi(x) = \sum_{\ell=1}^{m} \phi_\ell(x) \int^{x} \frac{M_\ell(\zeta)\, d\zeta}{W(\zeta)\, \alpha_m(\zeta)} \left\{ \frac{1}{\pi} \int_L \sum_{r=0}^{n} (-1)^r \frac{\partial^r Q_r(\zeta,t)}{\partial t^r} \phi(t)\, dt \right\}$$

$$+ \sum_{\ell=1}^{m} \phi_\ell(x) \int^{x} \frac{M_\ell(\zeta)\, d\zeta}{W(\zeta)\, \alpha_m(\zeta)} \sum_{r=0}^{n} \beta_r(\zeta) \frac{1}{\pi} \int_L \frac{\phi^{(r)}(t)}{t - \zeta}\, dt + F_2(x), \qquad (4.3)$$

where

$$F_2(x) = \sum_{\ell=1}^{m} \phi_\ell(x) \left\{ C_\ell + \int^{x} \frac{M_\ell(\zeta)\, F_1(\zeta)}{W(\zeta)\, \alpha_m(\zeta)}\, d\zeta \right\}. \qquad (4.4)$$

We must rearrange the second term in (4.3). If we write for brevity

$$C_{r\ell}(\zeta) = \frac{M_\ell(\zeta)\, \beta_r(\zeta)}{W(\zeta)\, \alpha_m(\zeta)} ,$$

and introduce, as in section 3.,

$$\psi(x) = \frac{1}{\pi} \int_L \frac{\phi(t)}{t - x}\, dt ,$$

so that

$$\psi^{(r)}(x) - f_r(x) = \frac{1}{\pi} \int_L \frac{\phi^{(r)}(t)}{t - x}\, dt,$$

then the term in question is

$$I(x) = \sum_{r=0}^{n} \sum_{\ell=1}^{m} \phi_\ell(x) \int^{x} C_{r\ell}(\zeta)\, \{\psi^{(r)}(\zeta) - f_r(\zeta)\}\, d\zeta .$$

Since

$$\int^{x} C_{r\ell}(\zeta)\, \psi^{(r)}(\zeta)\, d\zeta = C_{r\ell}(x)\, \psi^{(r-1)}(x) - C^{(1)}_{r\ell}(x)\, \psi^{(r-2)}(x) +$$

$$+ C_{r\ell}^{(2)}(x)\ \psi^{(r-3)}(x) + \dots + (-1)^{r-1} C_{r\ell}^{(r-1)}(x)\ \psi(x) +$$

$$+ (-1)^r \int^x C_{r\ell}^{(r)}(\zeta)\ \psi(\zeta)\ d\zeta,$$

then

$$I(x) = \sum_{\ell=1}^{m} \phi_\ell(x) \left\{ \int^x \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \frac{\phi(t)\ dt}{t-\zeta}\ C_{r\ell}^{(r)}(\zeta)\ d\zeta + \right.$$

$$\left. + \sum_{k=0}^{n-1} \sum_{r=k+1}^{n} C_{r\ell}^{(r-k-1)} (-1)^{r-k-1} \frac{1}{\pi} \int_L \frac{\phi^{(k)}(t)dt}{t-x} - \sum_{r=0}^{n} \int^x C_{r\ell}(\zeta)\ f_r(\zeta)\ d\zeta \right\}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \phi(t)\ dt \int^x \frac{D_r(\zeta)}{t-\zeta}\ d\zeta -$$

$$- \sum_{\ell=1}^{m} \phi_\ell(x) \sum_{r=0}^{n} \int^x C_{r\ell}(\zeta)\ f_r(\zeta)\ d\zeta + \sum_{r=0}^{n-1} \eta_r(x) \frac{1}{\pi} \int_L \frac{\phi^{(r)}(t)}{t-x}\ dt,$$

in which we have omitted explicitly the x dependence from

$$D_r(\zeta) = \sum_{\ell=1}^{m} \phi_\ell(x)\ C_{r\ell}^{(r)}(\zeta),$$

and in which

$$\eta_r(x) = \sum_{k=r+1}^{n} \sum_{\ell=1}^{m} \phi_\ell(x)\ C_{k\ell}^{(k-r-1)}(x)\ (-1)^{k-r-1}\ .$$

The expression for I(x) may be further rearranged to give

$$I(x) = \sum_{r=0}^{n} (-1)^r \frac{1}{\pi} \int_L \phi(t)\ dt \int^x \frac{D_r(\zeta)\ d\zeta}{t-\zeta} + \sum_{r=0}^{n-1} \frac{1}{\pi} \int_L \frac{\eta_r(t)\phi^{(r)}(t)}{t-x} dt -$$

$$- \sum_{r=0}^{n-1} (-1)^r \frac{1}{\pi} \int_L \phi(t) \frac{\partial^r \mu_r(x,t)}{\partial t^r} dt - \sum_{\ell=1}^{m} \phi_\ell(x) \sum_{r=0}^{n} \int^x C_{r\ell}(\zeta) f_r(\zeta) d\zeta$$

$$- \frac{1}{\pi} \sum_{r=0}^{n-1} \sum_{k=1}^{r} (-1)^{k-1} \left[\frac{\partial^{k-1} \mu_r(x,t)}{\partial t^{k-1}} \phi^{(r-k)}(t) \right]_L ,$$

where

$$\mu_r(x,t) = \frac{\eta_r(t) - \eta_r(x)}{t - x} ,$$

and using this in (4.3), on changing the order of integration in the first term on the right hand side, we obtain

$$\phi(x) = \frac{1}{\pi} \int_L R(x,t) \phi(t) dt + \sum_{r=0}^{n-1} \frac{1}{\pi} \int_L \frac{\eta_r(t)\phi^{(r)}(t)dt}{t - x} + F_3(x). \qquad (4.5)$$

Here we have written

$$R(x,t) = \sum_{\ell=1}^{m} \phi_\ell(x) \sum_{r=0}^{n} (-1)^r \int^x \frac{M_\ell(\zeta)}{W(\zeta)\ \alpha_m(\zeta)} \frac{\partial^r Q_r(x,t)}{\partial t^r} d\zeta +$$

$$+ \sum_{r=0}^{n} (-1)^r \int^x \frac{D_r(\zeta)\ d\zeta}{t - \zeta} + \sum_{r=0}^{n-1} (-1)^{r+1} \frac{\partial^r \mu_r(x,t)}{\partial t^r} \qquad (4.6)$$

and

$$F_3(x) = F_2(x) - \frac{1}{\pi} \sum_{r=0}^{n-1} \sum_{k=1}^{r} (-1)^{k-1} \left[\frac{\partial^{k-1} \mu_r(x,t)}{\partial t^{k-1}} \phi^{(r-k)}(t) \right]_L -$$

$$- \sum_{\ell=1}^{m} \phi_\ell(x) \sum_{r=0}^{n} \int^x C_{r\ell}(\zeta) f_r(\zeta) d\zeta .$$

We now invert (4.5), choosing that solution for

$$\sum_{r=0}^{n-1} \eta_r(x)\ \phi^{(r)}(x)$$

which has the appropriate behaviour at the end points of L,

$$\sum_{r=0}^{n-1} \eta_r(x)\ \phi^{(r)}(x) = -\frac{R(x)}{\pi}\int_L \frac{d\tau}{R(\tau)\{\tau - x\}} \left\{\phi(\tau) - F_3(\tau) - \right.$$

$$\left. - \frac{1}{\pi}\int_L R(\tau,t)\ \phi(t)\ dt \right\} + R(x)\ P(x),$$

in the notation of section 2., omitting the subscript from the arbitrary polynomial. Hence

$$\sum_{r=0}^{n-1} \eta_r(x)\ \phi^{(r)}(x) = \frac{1}{\pi}\int_L R_1(x,t)\ \phi(t)\ dt + \frac{R(x)}{\pi}\int_L \frac{F_3(\tau)\ d\tau}{R(\tau)\{\tau - x\}} +$$

$$+ R(x)\ P(x), \qquad (4.7)$$

where

$$R_1(x,t) = R(x)\int_L \frac{R(\tau,t)\ d\tau}{R(\tau)\{\tau - x\}} - \frac{R(x)}{R(t)\{\ t - x\}},$$

has a singular part. Finally, we solve (4.7) in the usual way as a differential equation, giving

$$\phi(x) = \sum_{\ell=1}^{n-1} C_\ell\ \psi_\ell(x) + \sum_{\ell=1}^{n-1} \psi_\ell(x)\int^x \frac{N_\ell(\zeta)\ d\zeta}{\eta_{n-1}(\zeta)\ V(\zeta)} \left\{ \frac{1}{\pi}\int_L R_1(x,t)\phi(t)\ dt \right\}$$

$$+ F(x), \qquad (4.8)$$

where the first term on the right hand side is the general solution of the homogeneous equation

$$\sum_{\ell=0}^{n-1} \eta_\ell(x)\, \phi^{(\ell)}(x) = 0,$$

$V(x)$ is the Wronskian of the system $\psi_1(x), \psi_2(x), \ldots, \psi_{n-1}(x)$, and $N_\ell(x)$ is the cofactor of the ℓ th element of the last row of the Wronskian. Also we have placed

$$F(x) = \sum_{\ell=1}^{n-1} \psi_\ell(x) \int^{x} \frac{N_\ell(\zeta)\, d\zeta}{\eta_{n-1}(\zeta)\, V(\zeta)} \left\{ \frac{R(\zeta)}{\pi} \int_L \frac{F_3(\tau)\, d\tau}{R(\tau)\{\tau - \zeta\}} + R(\zeta)\, P(\zeta) \right\}.$$

On rearranging, (4.8) becomes

$$\phi(x) - \frac{1}{\pi} \int_L K(x,t)\, \phi(t)\, dt = \sum_{\ell=1}^{n-1} C'_\ell\, \psi_\ell(x) + F(x), \tag{4.9}$$

where

$$K(x,t) = \sum_{\ell=1}^{n-1} \psi_\ell(x) \int^{x} \frac{N_\ell(\zeta)\, R_1(\zeta,t)}{\eta_{n-1}(\zeta)\, V(\zeta)} d\zeta\ , \tag{4.10}$$

is weakly singular, due to the singular term in $R_1(\zeta,t)$.

Some simplification again occurs if L is closed, and it can be easily shown that in such a case,

$$R_1(x,t) = \int_L \frac{R(\tau,t)dt}{\tau - x} - \frac{1}{t - x}\ ,$$

and

$$F(x) = \sum_{\ell=1}^{n-1} \psi_\ell(x) \int^{x} \frac{N_\ell(\zeta)\, d\zeta}{\eta_{n-1}(\zeta)\, V(\zeta)} \frac{1}{\pi} \int_L \frac{f(\tau)d\tau}{\tau - \zeta} ,$$

otherwise, the various expressions are unchanged. These last formulae differ from those stated by Magnaradze [1] for a closed curve, suggesting that he adopted a different procedure to the one employed here.

5. The Reduction of a Particular Integro-differential Equation.

We turn from the integro-differential equation (1.1) in its most general form, to the particular case

$$a(x)\,\phi(x) + b(x)\,\phi'(x) - \frac{1}{\pi}\int_L \frac{p(x,t)\,\phi(t) + q(x,t)\,\phi'(t)}{t - x} = f(x), \quad (5.1)$$

where $x \in L$. This equation has important applications, in, for instance, the field of aerodynamics. For this reason it is thought of value to enumerate the results obtained above, and in this simpler case we can do this more concisely and explicitly. The method of approach is exactly parallel to that of section 3., to reduce (5.1) to a regular equation; namely, to eliminate the Cauchy type integrals, obtaining a differential equation for $\phi(x)$, the right hand side containing the unknown function inside an integral with a regular kernel.

Rearranging (5.1) in the usual way, we obtain

$$a(x)\,\phi(x) + b(x)\,\phi'(x) - \frac{p(x)}{\pi}\int_L \frac{\phi(t)}{t - x}\,dt - \frac{q(x)}{\pi}\int_L \frac{\phi'(t)}{t - x}\,dt =$$

$$= \frac{1}{\pi}\int_L p_1(x,t)\,\phi(t)\,dt + f_1(x), \qquad (5.2)$$

where

$$p_1(x,t) = \frac{p(x,t) - p(x,x)}{t - x} - \frac{\partial}{\partial t}\left\{\frac{q(x,t) - q(x,x)}{t - x}\right\},$$

is regular, and

$$f_1(x) = f(x) + \frac{1}{\pi}\left[\frac{\{q(x,t) - q(x,x)\}}{t - x}\phi(t)\right]_L .$$

Also

$$p(x) = p(x,x), \qquad q(x) = q(x,x).$$

We can alternatively arrange (5.1) in the form

$$a(x)\,\phi(x) + b(x)\,\phi'(x) - \frac{1}{\pi}\int_L \frac{p(t)\,\phi(t) + q(t)\,\phi'(t)}{t - x}\,dt =$$

$$= \frac{1}{\pi}\int_L q_1(x,t)\,\phi(t)\,dt + f_2(x), \qquad (5.3)$$

in which

$$q_1(x,t) = \frac{p(x,t) - p(t,t)}{t - x} - \frac{\partial}{\partial t}\left\{\frac{q(x,t) - q(t,t)}{t - x}\right\},$$

and

$$f_2(x) = f(x) + \frac{1}{\pi}\left[\frac{\{q(x,t) - q(t,t)\}}{t - x}\,\phi(t)\right]_L .$$

The inverse of (5.3) in the class $h(c_1, c_2, \ldots, c_{n_2})$ is

$$p(x)\,\phi(x) + q(x)\,\phi'(x) = -\frac{R(x)}{\pi}\int_L \frac{dt}{R(t)\{t - x\}}\left\{a(t)\,\phi(t) + b(t)\,\phi'(t)\right.$$

$$\left. - \frac{1}{\pi}\int_L q_1(t,\tau)\,\phi(\tau)\,d\tau - f_2(t)\right\} + R(x)\,P(x),$$

where $R(x) = \sqrt{\{R_1(x)/R_2(x)\}}$, in the notation of section 2., and this equation can be written

$$p(x)\,\phi(x) + q(x)\,\phi'(x) + \frac{a(x)}{\pi}\int_L \frac{\phi(t)}{t-x}\,dt + \frac{b(x)}{\pi}\int_L \frac{\phi'(t)}{t-x}\,dt =$$

$$= -\frac{1}{\pi}\int_L R_1(x,t)\,\phi(t)\,dt + f_3(x), \qquad (5.4)$$

where $R_1(x,t)$ is the regular kernel

$$R_1(x,t) = -\frac{R(x)}{\pi}\int_L \frac{q_1(\tau,t)d\tau}{R(\tau)\{\tau - x\}} + \frac{R(x)}{t-x}\left\{\frac{a(t)}{R(t)} - \frac{a(x)}{R(x)}\right\} -$$

$$- \frac{\partial}{\partial t}\left\{\frac{R(x)}{x-t}\left(\frac{b(t)}{R(t)} - \frac{b(x)}{R(x)}\right)\right\},$$

and

$$f_3(x) = \frac{R(x)}{\pi}\int_L \frac{f_2(t)\,dt}{R(t)\{t-x\}} - \frac{1}{\pi}\left[\frac{R(x)}{t-x}\left(\frac{b(t)}{R(t)} - \frac{b(x)}{R(x)}\right)\phi(t)\right]_L +$$

$$+ R(x)\,P(x).$$

Equations (5.2) and (5.4) correspond to (3.2) and (3.8) in the general case, and from them we must eliminate the singular integrals. This can be done without recourse to the multipliers explicitly, and after some manipulation we can show that the resulting equation is

$$\phi''(x) + A(x)\,\phi'(x) + B(x)\,\phi(x) = \frac{1}{\pi}\int_L R(x,t)\,\phi(t)\,dt + F(x), \qquad (5.5)$$

where

$$A(x) = \frac{b(a+b') + q(p+q') - bq(bc+qd)}{b^2 + q^2},$$

$$B(x) = \frac{ba' + qp' - bq(ac + pd)}{b^2 + q^2},$$

$$c(x) = -\left(\frac{pb - qa}{qb}\right)\left\{\frac{a}{pb - qa} + \frac{d}{dx}\left(\frac{b}{pb - qa}\right)\right\},$$

$$d(x) = -\left(\frac{pb - qa}{qb}\right)\left\{\frac{p}{pb - qa} + \frac{d}{dx}\left(\frac{q}{pb - qa}\right)\right\},$$

and

$$R(x,t) = \frac{1}{b^2 + q^2}\left\{b\,\frac{\partial p_1(x,t)}{\partial x} - q\,\frac{\partial R_1(x,t)}{\partial x} - bq\,\{cp_1(x,t) - dR_1(x,t)\}\right\}$$

$$F(x) = \frac{1}{b^2 + q^2}\left\{b\,\frac{df_1(x)}{dx} + q\,\frac{df_3(x)}{dx} - bq\,\{cf_1(x) + df_3(x)\}\right\},$$

where we have omitted the arguments in $a(x)$, $b(x)$, $c(x)$, ... on the right hand side for clarity, and where an accent denotes differentiation with respect to x. We have also assumed the existence on L of

$$b^2 + q^2\ , \quad pb - qa\ .$$

Equation (5.5) may be solved in the usual way, and we state the regular integral equation ultimately obtained for $\phi(x)$:

$$\phi(x) = C_1\,\phi_1(x) + C_2\,\phi_2(x) + \frac{1}{\pi}\int_L K(x,t)\,\phi(t)\,dt + \int^x W(x,t)F(t)dt, \tag{5.6}$$

where $\phi_1(x)$ and $\phi_2(x)$ are linearly independent solutions of

$$\phi''(x) + A(x)\,\phi'(x) + B(x)\,\phi(x) = 0,$$

$$W(x,t) = \frac{\phi_1(x)}{d\phi_2/dt}\left\{\frac{d\phi_1/dt}{d\phi_2/dt} - \frac{\phi_1(t)}{\phi_2(t)}\right\} + \frac{\phi_2(x)}{d\phi_1/dt}\left\{\frac{d\phi_2/dt}{d\phi_1/dt} - \frac{\phi_2(t)}{\phi_1(t)}\right\},$$

and

$$K(x,t) = \int^{x} W(x,\tau)\, R(\tau,t)\, d\tau.$$

We can also deduce a singular integral equation for $\phi(x)$ from (5.1), for from (5.2) we have, assuming that $b(x)$ does not vanish on L,

$$\phi'(x) + \frac{a(x)}{b(x)}\,\phi(x) = \frac{G(x)}{b(x)}\,, \tag{5.7}$$

where

$$G(x) = \frac{p(x)}{\pi}\int_L \frac{\phi(t)}{t-x}\,dt + \frac{q(x)}{\pi}\int_L \frac{\phi'(t)}{t-x}\,dt + \frac{1}{\pi}\int_L p_1(x,t)\phi(t)dt + f_1(x),$$

$$= \frac{p(x)}{\pi}\int_L \frac{\phi(t)}{t-x}\,dt + \frac{q(x)}{\pi}\,\frac{d}{dx}\left\{\int_L \frac{\phi(t)}{t-x}\,dt - [\phi(t)\log|t-x|]_L\right\}$$

$$+ \frac{1}{\pi}\int_L p_1(x,t)\,\phi(t)\,dt + f_1(x).$$

Denoting

$$\phi_0(x) = \exp\left\{\int^{x} \frac{a(\zeta)}{b(\zeta)}\,d\zeta\right\},$$

then (5.7) can be integrated at once to give

$$\phi_0(x)\,\phi(x) = \int^{x} \frac{G(\zeta)}{b(\zeta)}\,\phi_0(\zeta)\,d\zeta + C$$

where C is some constant, and replacing $G(x)$, integrating the second

term by parts, we obtain

$$\phi_0(x)\ \phi(x) = \frac{1}{\pi}\int_L K_1\ (x,t)\ \phi(t)\ dt + F_1(x)\ , \tag{5.8}$$

where

$$K_1(x,t) = \frac{\phi_0(x)q(x)}{b(x)\{t - x\}} + \int^x \left\{ \frac{\phi_0(\zeta)p(\zeta)}{b(\zeta)} - \frac{d}{d\zeta}\left(\frac{\phi_0(\zeta)q(\zeta)}{b(\zeta)}\right)\right\} \frac{d\zeta}{t - \zeta}$$

$$+ \int^x \frac{\phi_0(\zeta)}{b(\zeta)}\ p_1(\zeta,t)\ dt,$$

and

$$F_1(x) = \int^x \frac{\phi_0(\zeta)f_1(\zeta)}{b(\zeta)} d\zeta - \frac{\phi_0(x)}{b(x)}\ \frac{q(x)}{\pi}\ [\phi(t)\ \log|t - x|]_L +$$

$$+ \frac{1}{\pi}\int^x \frac{d}{d\zeta}\left(\frac{\phi_0(\zeta)q(\zeta)}{b(\zeta)}\right) [\phi(t)\ \log|t - \zeta|]_L\ d\zeta + C\ .$$

Equation (5.8) represents an integral equation for $\phi(x)$ with a kernel containing a singular part and a regular part.

If $b(x)$ is zero on L, so that (5.1) has the form

$$a(x)\ \phi(x) - \frac{1}{\pi}\int_L \frac{p(x,t)\ \phi(t) + q(x,t)\ \phi'(t)}{t - x} dt = f(x),$$

then we can invert the equation, and assuming that $q(x)$ does not vanish on L, use the method just described to obtain a singular equation for $\phi(x)$.

There remains one outstanding difficulty in this method of reducing a singular integro-differential equation to a regular, quasi-regular or singular integral equation, and that is the equivalence of the derived equation and the original one. By this we imply that, while each solution of the original equation will also be a solution of the derived equation, the converse is not in general true.

However, if all the solutions of the derived equation can be determined, it is always possible to construct from them the general solution of the original equation. If the integro-differential equation arises from a specific physical situation, then in general it is possible to appeal to this problem to ensure that the ultimate solution of the integral equation is appropriate.

6. The Generalised Riemann-Hilbert Problem.

We discuss the Riemann-Hilbert problem (1.2) as a possible method of solving an integro-differential equation for an open arc, or union of such arcs.

Considering only the case m = n, then equation (1.2) can be written

$$\sum_{r=0}^{m} \left[a_r(x) \frac{d^r\Phi_+(x)}{dx^r} - b_r(x) \frac{d^r\Phi_-(x)}{dx^r} \right] = f_1(x), \; x \in L, \quad (6.1)$$

and if each $a_r(x)$, $b_r(x)$ are such that

$$a_r(x) = F_r^+(x), \quad b_r(x) = F_r^-(x), \quad x \in L, \; r = 0, 1, \ldots, m, \quad (6.2)$$

for some $F_r(z)$, then (6.1) can be written as

$$\Psi_+(x) - \Psi_-(x) = f_1(x), \; x \in L, \quad (6.3)$$

where

$$\Psi(z) = \sum_{r=0}^{m} F_r(z) \frac{d^r\Phi(z)}{dz^r} .$$

The simple jump problem (6.3) can be solved for $\Psi(z)$ by standard procedures, and $\Phi_\pm(x)$ can be determined on solving the differential equations

$$\sum_{r=0}^{m} F_r^+(x) \frac{d^r\Phi_+(x)}{dx^r} = \Psi_+(x), \quad \sum_{r=0}^{m} F_r^-(x) \frac{d^r\Phi_-(x)}{dx^r} = \Psi_-(x).$$

If it is not possible to express the $a_r(x)$, $b_r(x)$ in this way, it would appear that the problem (6.1) is intractable in closed

form. This suggests that the solution of the corresponding integro-differential equation does not exist in closed form.

Functions which satisfy a relationship of the type (6.2) are usually self-evident. However, given two functions $a(x)$ and $b(x)$, we can derive an expression for a function $F(z)$ such that

$$a(x) = F_+(x), \qquad b(x) = F_-(x), \quad x \in L, \tag{6.4}$$

provided that $a(x)$ and $b(x)$ satisfy a necessary and sufficient condition. We restrict ourselves to the particular case in which $F(z)$ is required to be holomorphic in the plane, except on $L = L_1 + L_2 + \dots + L_{n_1}$, tending to zero at infinity, although as we shall see in the following section, the jump problem (6.3) can be solved for a $\Psi(z)$ which has known singularities off the line L.

Let us write

$$\left.\begin{aligned} F(z) &= \phi(x,y) + i\psi(x,y), \\ a(x) &= a_1(x) + ia_2(x), \\ b(x) &= b_1(x) + ib_2(x), \end{aligned}\right\} \tag{6.5}$$

so that on L

$$\left.\begin{aligned} \phi(x,0+) &= a_1(x), \qquad \phi(x,0-) = b_1(x), \\ \psi(x,0+) &= a_2(x), \qquad \psi(x,0-) = b_2(x). \end{aligned}\right\} \tag{6.6}$$

The determination of the two functions $\phi(x,y)$ and $\psi(x,y)$ subject to the limiting values (6.6) on L constitutes two Dirichlet problems. Following Muskhelishvili [5], (section 91), the solutions for $\phi(x,y)$ and $\psi(x,y)$ in the class $h(c_1, c_2, \dots , c_{n_2})$, vanishing at infinity,

are

$$\phi(x,y) = R\ell\left\{\frac{1}{2\pi i}\sqrt{\frac{R_1(z)}{R_2(z)}}\int_L\sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{\{a_1(t)+b_1(t)\}}{t-z}\,dt + \right.$$

$$\left. + \frac{1}{2\pi i}\int_L\frac{\{a_1(t)-b_1(t)\}}{t-z}\,dt + P(z)\sqrt{\frac{R_1(z)}{R_2(z)}}\right\},$$

$$\psi(x,y) = R\ell\left\{\frac{1}{2\pi i}\sqrt{\frac{R_1(z)}{R_2(z)}}\int_L\sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{\{a_2(t)+b_2(t)\}}{t-z}\,dt + \right.$$

$$\left. + \frac{1}{2\pi i}\int_L\frac{\{a_2(t)-b_2(t)\}}{t-z}\,dt + Q(z)\sqrt{\frac{R_1(z)}{R_2(z)}}\right\},$$

where $P(z)$ and $Q(z)$ are arbitrary polynomials with real coefficients, of degree not greater than $n_1 - n_2 - 1$, the remaining notation being consistent with section 2. Clearly the class of solution required is generally determined by the behaviour of $a(x)$ and $b(x)$ near the end points. If we write the above solutions in the form

$$\phi(x,y) = R\ell\{F_1(z)\}, \quad \psi(x,y) = R\ell\{F_2(z)\},$$

then it is clear from (6.5) that

$$F(z) = F_1(z) = iF_2(z), \tag{6.7}$$

so that the required function can be written

$$2F(z) = F_1(z) + iF_2(z)$$

$$= \frac{1}{2\pi i}\sqrt{\frac{R_1(z)}{R_2(z)}}\int_L\sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{\{a(t)+b(t)\}}{t-z}\,dt + \frac{1}{2\pi i}\int_L\frac{\{a(t)-b(t)\}}{t-z}\,dt$$

$$+ \tfrac{1}{2}P^*(z)\sqrt{\{R_1(z)/R_2(z)\}}, \tag{6.8}$$

where

$$P^*(z) = 2P(z) + 2iQ(z)$$

is a polynomial of degree not greater than $n_1 - n_2 - 1$. The solution (6.8) is valid provided that the second equality in (6.7) is satisfied, and this furnishes a condition on $a(x)$ and $b(x)$ which can be arranged in the form

$$\frac{1}{2\pi i}\sqrt{\frac{R_1(z)}{R_2(z)}}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{\{\overline{a(t)} + \overline{b(t)}\}}{t - z}\,dt + \frac{1}{2\pi i}\int_L \frac{\{\overline{a(t)} - \overline{b(t)}\}}{t - z}\,dt +$$

$$+ \tfrac{1}{2}\,\overline{P}^*(z)\sqrt{\{R_1(z)/R_2(z)\}} = 0, \quad (6.9)$$

where

$$\overline{P}^*(z) = \overline{P^*(\overline{z})} = 2\,P(z) - 2i\,Q(z).$$

If $F(z)$ is required to be bounded at more than half the end points of L, then we must impose solubility conditions on the two solutions of the Dirichlet problems to ensure the required behaviour at infinity. These conditions are parallel to those of section 2. for the inversion problem, and require that

$$\left.\begin{aligned} &\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\{a_1(t) + b_1(t)\}\,t^k\,dt = 0, \\ &\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\{a_2(t) + b_2(t)\}\,t^k\,dt = 0, \end{aligned}\right\} \quad k = 0, 1, \ldots, n_2-n_1-1,$$

or, what is the same thing,

$$\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\{a(t) \pm b(t)\}\,t^k\,dt = 0, \quad k = 0, 1, \ldots, n_2-n_1-1.$$

In this case we must also place $P(z)$ and $Q(z)$, and hence $P^*(z)$, equal to zero.

Taking the limiting values of (6.9) as z tends to a point x on L from above and below, remembering that $\sqrt{\{R_1(z)/R_2(z)\}}$ changes sign around the branch points c_i, and alternately adding and subtracting the two resulting equations, we obtain,

$$\{\overline{a(x)} + \overline{b(x)}\} + \frac{1}{\pi i}\int_L \frac{\{\overline{a(t)} - \overline{b(t)}\}}{t - x}\, dt = 0, \quad x \in L,$$

and

$$\frac{1}{\pi i}\sqrt{\frac{R_1(x)}{R_2(x)}}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{\{\overline{a(t)} + \overline{b(t)}\}}{t - x}\, dt + \{\overline{a(x)} - \overline{b(x)}\} +$$

$$+ P^*(x)\sqrt{\{R_1(x)/R_2(x)\}} = 0, \; x \in L.$$

The second of these is simply the inverse of the first in the class $h(c_1, c_2, \ldots, c_{n_2})$, consistent with the assumption that $F(z)$ belongs to this class. The two equations are thus equivalent, and we need only consider the first of them, or its conjugate, which is

$$a(x) + b(x) = \frac{1}{\pi i}\int_L \frac{\{a(t) - b(t)\}}{t - x}\, dt, \quad x \in L. \tag{6.10}$$

Thus, $F(z)$ as given by (6.8) is the required solution provided that (6.9) is satisfied by $a(x)$ and $b(x)$, and in particular that (6.10) is satisfied. Let us assume (6.10) to be true for $a(x)$ and $b(x)$, and substitute it into (6.9), giving

$$\frac{1}{2\pi i}\sqrt{\frac{R_1(z)}{R_2(z)}}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{dt}{t-z}\left\{-\frac{1}{\pi i}\int_L \frac{\{\overline{a(s)}-\overline{b(s)}\}}{s-t}\,ds\right\}+$$

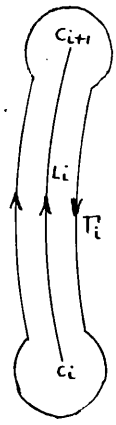
$$+\frac{1}{2\pi i}\int_L \frac{\{\overline{a(t)}-\overline{b(t)}\}}{t-z}\,dt+\tfrac{1}{2}\,\bar{P}^*(z)\sqrt{\frac{R_1(z)}{R_2(z)}}=0.$$

Since z is not on L, it is permissible to reverse the order of integration in the first term, giving

$$-\frac{1}{2\pi i}\sqrt{\frac{R_1(z)}{R_2(z)}}\int_L \frac{\{\overline{a(s)}-\overline{b(s)}\}}{s-z}\,ds\,\frac{1}{\pi i}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\left\{\frac{1}{t-z}-\frac{1}{t-s}\right\}dt$$

$$+\frac{1}{2\pi i}\int_L \frac{\{\overline{a(t)}-\overline{b(t)}\}}{t-z}\,dt+\tfrac{1}{2}\,\bar{P}^*(z)\sqrt{\frac{R_1(z)}{R_2(z)}}=0. \qquad (6.11)$$

Let us define



$$\Omega(z)=\frac{1}{\pi i}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{dt}{t-z},$$

$$=\frac{1}{2\pi i}\oint_\Gamma \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{dt}{t-z},$$

where Γ is the union of the Γ_i, each Γ_i surrounding L_i as shown, being sufficiently close to the arc that the point z lies outside each Γ_i. By a consequence of Cauchy's residue theorem we can now evaluate $\Omega(z)$,

$$\Omega(z)=\sqrt{\frac{R_2(z)}{R_1(z)}}-P^{**}(z),$$

where $P^{**}(z)$ is a polynomial defined by

$$\sqrt{\{R_2(z)/R_1(z)\}} = P^{**}(z) + O(1/z),$$

for large $|z|$. Hence it is of degree n_1-n_2 in z (and zero if $n_2 > n_1$). Further,

$$\frac{1}{\pi i}\int_L \sqrt{\frac{R_2(t)}{R_1(t)}}\,\frac{dt}{t-x} = \tfrac{1}{2}\{\Omega_+(x) + \Omega_-(x)\}, \qquad x \in L,$$

$$= -P^{**}(x),$$

and using these results in (6.11) we find that

$$\bar{P}^*(z) = \frac{1}{\pi i}\int_L \frac{\{\overline{a(t)} - \overline{b(t)}\}\{P^{**}(t) - P^{**}(z)\}}{t-z}\,dt,$$

or,

$$P^*(z) = -\frac{1}{\pi i}\int_L \frac{\{a(t) - b(t)\}\{\bar{P}^{**}(t) - \bar{P}^{**}(z)\}}{t-z}\,dt. \qquad (6.12)$$

Both sides of this equation are of degree at most n_1-n_2-1 in z, and so we can regard it as uniquely determining the polynomial $P^*(z)$.

We conclude that (6.8) furnishes the required function, provided that (6.10) is satisfied and that $P^*(z)$ is determined by means of (6.12).

As a simple example, we show that this theory holds for

$$a(x) = -b(x) = \sqrt{\frac{R_1(x)}{R_2(x)}},$$

and we consider the case $n_1 > n_2$.

Equation (6.10) becomes

$$0 = \frac{2}{\pi i}\int_L \sqrt{\frac{R_1(t)}{R_2(t)}}\,\frac{dt}{t - x},$$

which by a similar method to that used above is seen to be identically satisfied, while

$$2\,F(z) = \tfrac{1}{2}\,P^*(z)\sqrt{\frac{R_1(z)}{R_2(z)}} + \frac{1}{\pi i}\int_L \sqrt{\frac{R_1(t)}{R_2(t)}}\,\frac{dt}{t - z},$$

$$= \{\tfrac{1}{2}\,P^*(z) + 1\}\,\sqrt{\{R_1(z)/R_2(z)\}}.$$

From (6.12)

$$P^*(z) = -\frac{2}{\pi i}\int_L \sqrt{\frac{R_1(t)}{R_2(t)}}\,\frac{\bar{P}^{**}(t)}{t - z}\,dt + \frac{2\bar{P}^{**}(z)}{\pi i}\int_L \sqrt{\frac{R_1(t)}{R_2(t)}}\,\frac{dt}{t - z}$$

$$= -2\left\{\bar{P}^{**}(z)\sqrt{\frac{R_1(z)}{R_2(z)}} - Q^*(z)\right\} + 2\bar{P}^{**}(z)\sqrt{\frac{R_1(z)}{R_2(z)}},$$

$$= 2\,Q^*(z),$$

where $Q^*(z)$ is such that

$$Q^*(z) = \bar{P}^{**}(z)\sqrt{\frac{R_1(z)}{R_2(z)}} + O(1/z), \text{ large } |z|.$$

But

$$\bar{P}^{**}(z) = P^{**}(z) = \sqrt{\frac{R_2(z)}{R_1(z)}} + O(1/z), \text{ large } |z|,$$

and so $Q^*(z) = 1$, and $P^*(z) = 2$, which gives the required result

$$F(z) = \sqrt{\{R_1(z)/R_2(z)\}}.$$

In the classes of functions for which $n_2 > n_1$, it is easily seen that the solubility conditions are violated, and no solution for $F(z)$, tending to zero at infinity, exists, which is the conlusion one expects.

7. A new solution of the Prandtl equation for a wing of finite span.

As an example of an integro-differential equation for which the corresponding generalised Riemann-Hilbert problem is soluble, we consider a particular case of the Prandtl equation in aerodynamic theory,

$$\frac{\phi(x)}{B(x)} - \frac{1}{\pi}\int_{-a}^{a} \frac{\phi'(t)\,dt}{t - x} = f(x), \quad -a < x < a, \tag{7.1}$$

where, in standard notation,

$$B(x) = m\, b(x)/8, \qquad f(x) = 4V\alpha(x).$$

$b(x)$ is the chord of profile of a wing of span $2a$, $\alpha(x)$ is the angle of incidence, V the velocity of the airflow at infinity and m some determined constant. The unknown function $\phi(x)$ represents the circulation of the airflow around the profile at x. By symmetry we have

$$\phi(x) = \phi(-x), \quad B(x) = B(-x), \quad f(x) = f(-x),$$

and it is generally assumed that

$$\phi(a) = \phi(-a) = 0.$$

Equation (7.1) is thus of considerable practical interest, and is the subject of a large amount of work in the literature. The methods of solution parallel to this section have been given by I.N. Vekua and L.G.Magnaradze, and are discussed by Muskhelishvili [5]. Magnaradze [7] reduced (7.1) to a weakly singular Fredholm equation, while Vekua [8] derived an equivalent regular Fredholm equation, which has the advantage that if $B(x)$ is of the form

$$B(x) = \sqrt{\{a^2 - x^2\}}/P(x), \qquad (7.2)$$

where $P(x)$ is a rational function, then the kernel of the equation is separable, and the solution is readily found. The chord of profile given by (7.2) covers many cases of practical importance, and it is this choice of $B(x)$ which renders the Riemann-Hilbert problem tractable. Writing

$$R(x) = \sqrt{\{a^2 - x^2\}},$$

we choose the definite branches defined by

$$R_+(x) = - R_-(x) = R(x), \qquad -a < x < a,$$

where the subscripts $\pm$ denote, as usual, the limiting values on $(-a,a)$ from above and below. Clearly also

$$P_+(x) = P_-(x) = P(x).$$

We formulate the Riemann-Hilbert problem by introducing the Carleman function

$$\Phi(z) = \int_{-a}^{a} \frac{\phi(t)\, dt}{t - z},$$

from which the Plemelj formulae give

$$\left.\begin{aligned} \Phi_+(x) - \Phi_-(x) &= 2\pi i\phi(x), \\ \Phi_+(x) + \Phi_-(x) &= 2\int_{-a}^{a} \frac{\phi(t)\, dt}{t - x}, \end{aligned}\right\} -a < x < a,$$

and, on differentiating,

$$\Phi_+'(x) + \Phi_-'(x) = 2\int_{-a}^{a} \frac{\phi'(t)\, dt}{t - x}, \qquad -a < x < a,$$

since $\phi(\pm a) = 0$. Using these expressions in (7.1), we obtain

$$\left\{ -i\Phi_+'(x) + \frac{\Phi_+(x)}{B(x)} \right\} + \left\{ -i\Phi_-'(x) - \frac{\Phi_-(x)}{B(x)} \right\} = 2\pi i f(x),$$

or, substituting for $B(x)$ and multiplying through by $R(x)$,

$$R_+(x) \left\{ -i\Phi_+'(x) + \frac{\Phi_+(x)\, P_+(x)}{R_+(x)} \right\} - R_-(x) \left\{ -i\Phi_-'(x) + \frac{\Phi_-(x)\, P_-(x)}{R_-(x)} \right\}$$

$$= 2\pi i\, R(x)\, f(x).$$

Let us introduce the auxiliary Carleman function $F(z)$ by

$$F(z) = \int_{-a}^{a} \frac{f(t)\, R(t)}{t - z}\, dt,$$

so that

$$F_+(x) - F_-(x) = 2\pi i\, f(x)\, R(x), \quad -a < x < a,$$

then the Riemann-Hilbert equation can be written,

$$\left[R_+(x) \left\{ -i\Phi_+'(x) + \frac{\Phi_+(x)\, P_+(x)}{R_+(x)} \right\} - F_+(x) \right] -$$

$$- \left[R_-(x) \left\{ -i\Phi_-'(x) + \frac{\Phi_-(x)\, P_-(x)}{R_-(x)} \right\} - F_-(x) \right] = 0, \quad (7.3)$$

and on putting

$$D(z) = R(z) \left\{ -i\Phi'(z) + \frac{\Phi(z)\, P(z)}{R(z)} \right\} - F(z), \qquad (7.4)$$

this becomes

$$D_+(x) - D_-(x) = 0, \quad -a < x < a.$$

That is, $D(z)$ has no discontinuity on the arc $(-a,a)$; the behaviour of $D(z)$ in the remainder of the plane and at infinity depends upon the nature of $P(z)$, and from a knowledge of the possible poles of $P(z)$, and its behaviour at infinity, we can determine the function $D(z)$ by means of Liouville's generalised theorem. In particular,

$$D_+(x) = D_-(x) = D(x),$$

on $(-a,a)$, and returning to (7.3),

$$\left.\begin{aligned} -i\Phi_+'(x) + \frac{\Phi_+(x)\,P_+(x)}{R_+(x)} &= \frac{D(x) + F_+(x)}{R_+(x)}\,, \\ -i\Phi_-'(x) + \frac{\Phi_-(x)\,P_-(x)}{R_-(x)} &= \frac{D(x) + F_-(x)}{R_-(x)}\,. \end{aligned}\right\} \qquad (7.5)$$

We define

$$\lambda(x) = \int^x \frac{P(s)\,ds}{R(s)} = \int^x \frac{ds}{B(s)}\,,$$

in terms of which we can integrate (7.5) to give

$$\Phi_+(x) = C_1\,e^{-i\lambda(x)} + ie^{-i\lambda(x)}\int^x \frac{\{D(t) + F_+(t)\}\,e^{i\lambda(t)}\,dt}{R_+(t)}\,,$$

$$\Phi_-(x) = C_2\,e^{i\lambda(x)} + ie^{i\lambda(x)}\int^x \frac{\{D(t) + F_-(t)\}\,e^{-i\lambda(t)}\,dt}{R_-(t)}\,,$$

where C_1 and C_2 are arbitrary constants. That is,

$$\Phi_+(x) = C_1\,e^{-i\lambda(x)} + i\int^x \frac{\exp[-i\{\lambda(x) - \lambda(t)\}]}{R(t)}\left\{\int_{-a}^{a} \frac{f(s)\,R(s)\,ds}{s - t} + \right.$$

$$+ \pi i\, f(t)\, R(t) + D(t) \Big\},$$

$$\Phi_-(x) = C_2\, e^{i\lambda(x)} - i \int^{x} \frac{\exp[i\{\lambda(x) - \lambda(t)\}]}{R(t)} \Big\{ \int_{-a}^{a} \frac{f(s)\, R(s)\, ds}{s - t} -$$

$$- \pi i\, f(t)\, R(t) + D(t) \Big\}.$$

The solution for $\phi(x)$ is given by

$$2\pi i \phi(x) = \Phi_+(x) - \Phi_-(x),$$

and so

$$\phi(x) = A \cos\lambda(x) + B \sin\lambda(x) + \int^{x} \sin[\lambda(x) - \lambda(t)]\, f(t)\, dt +$$

$$+ \frac{1}{\pi} \int^{x} \frac{\cos[\lambda(x) - \lambda(t)]}{R(t)} \int_{-a}^{a} \frac{f(s)\, R(s)\, ds}{s - t}\, dt + \frac{1}{\pi} \int^{x} \frac{\cos[\lambda(x) - \lambda(t)]\, D(t) dt}{R(t)}.$$

Writing

$$\psi(x) = \int^{x} \sin[\lambda(x) - \lambda(t)]\, f(t)\, dt + \frac{1}{\pi} \int^{x} \frac{\cos[\lambda(x) - \lambda(t)]\, D(t)\, dt}{R(t)} +$$

$$+ \frac{1}{\pi} \int^{x} \frac{\cos[\lambda(x) - \lambda(t)]}{R(t)} dt \int_{-a}^{a} \frac{f(s)\, R(s)\, ds}{s - t}, \qquad (7.6)$$

then

$$\phi(x) = A \cos\lambda(x) + B \sin\lambda(x) + \psi(x).$$

Imposing the requirement that $\phi(x)$ be even, we see that we must place $B = 0$. Also if we choose the lower limit of the indefinite integrals in $\psi(x)$ to be $t = 0$, then clearly

$$\phi(x) = \phi(0) \cos \lambda(x) + \psi(x),$$

and finally, using $\phi(a) = 0$, we find that

$$\phi(x) = \psi(x) - \frac{\cos \lambda(x)}{\cos \lambda(a)} \psi(a), \tag{7.7}$$

provided that $\cos \lambda(a) \neq 0$.

The remaining difficulty in this solution is the determination of $D(z)$, and hence its value $D(t)$ on $(-a,a)$. We state Liouville's generalised theorem for reference:

Let the function $f_1(z)$ be analytic in the entire plane, except at the points $a_0 = \infty$, a_k $(k=1, 2, \ldots, n)$, where it has poles, and suppose that the principal parts of the expansion of the function $f_1(z)$ in the neighbourhoods of these poles have the form;

a_0, $\quad G_0(z) = C_{01} z + C_{02} z^2 + \ldots + C_{0n_0} z^{n_0}$,

a_k, $\quad G_k\left(\frac{1}{z - a_k}\right) = \frac{C_{k1}}{(z - a_k)} + \frac{C_{k2}}{(z - a_k)^2} + \ldots + \frac{C_{kn_k}}{(z - a_k)^{n_k}}$.

Then $f_1(z)$ is a rational function and is representable by

$$f_1(z) = C + G_0(z) + \sum_{k=1}^{n} G_k\left(\frac{1}{z - a_k}\right),$$

where C is some constant.

Returning to the problem under consideration, we note that for large values of $|z|$,

$$\Phi'(z) \approx O(|z|^{-2}), \quad \Phi(z) \approx O(|z|^{-1}), \quad F(z) \approx O(|z|^{-1}),$$

and so

$$D(z) \approx O(1/|z|)\ \{1 + P(z)\}.$$

We consider two cases for $P(z)$;

(i) $P(z) = p_0$,

where p_0 is a constant, i.e.

$$B(x) = \sqrt{\{a^2 - x^2\}}/p_0, \qquad \lambda(x) = p_0 \arcsin(x/a).$$

$P(z)$, and hence $D(z)$, has no poles in the plane, and

$$D(z) \approx O(1/|z|),$$

for large $|z|$. Therefore, by Liouville's theorem,

$$D(z) = 0,$$

and in particular

$$D(t) = 0, \qquad -a < t < a.$$

The solution therefore becomes

$$\phi(x) = \psi(x) - \frac{\cos \lambda(x)}{\cos \lambda(a)}\, \psi(a),$$

where now

$$\psi(x) = \int_0^x \sin[\lambda(x) - \lambda(t)]\ f(t)\ dt +$$

$$+ \frac{1}{\pi} \int_0^x \frac{\cos[\lambda(x) - \lambda(t)]\ dt}{R(t)} \int_{-a}^{a} \frac{f(s)\ R(s)\ ds}{s - t}\ .$$

In the particular case $f(x) = \text{constant} = f_0$, corresponding to a constant

incident angle, and assuming that $p_0 \neq \pm 1$, then this expression can be easily integrated to give

$$\psi(x) = \frac{f_0}{1 + p_0}\left\{\sqrt{\{a^2 - x^2\}} - a \cos \lambda(x)\right\},$$

yielding the solution

$$\phi(x) = f_0 \sqrt{\{a^2 - x^2\}}/(1 + p_0), \tag{7.8}$$

which is well known.

(ii)
$$P(z) = p_0 \left\{\frac{a^2 + \mu z^2}{a^2 + \nu z^2}\right\},$$

where $\mu, \nu > -1$, to ensure that $P(x)$ has no zeros or poles on $(-a,a)$. $P(z)$ has poles at

$$z = \pm \alpha = \pm ia/\nu^{1/2},$$

and for large $|z|$,

$$D(z) \simeq O(1/|z|).$$

We recall that

$$D(z) = p_0 \left\{\frac{a^2 + \mu z^2}{a^2 + \nu z^2}\right\} \int_{-a}^{a} \frac{\phi(s)\, ds}{s - z} - iR(z)\, \Phi'(z) - F(z),$$

and so the residue of $D(z)$ at $z = \alpha$ is

$$r_+ = \frac{p_0}{\nu} \frac{a^2 + \mu\alpha^2}{2\alpha} \int_{-a}^{a} \frac{\phi(s)\, ds}{s - \alpha},$$

and at $z = -\alpha$ is

$$r_- = -\frac{p_o}{\nu}\frac{a^2 + \mu\alpha^2}{2\alpha}\int_{-a}^{a}\frac{\phi(s)\,ds}{s+\alpha},$$

so that

$$r_+ + r_- = \frac{p_o}{\nu}(a^2 + \mu\alpha^2)\int_{-a}^{a}\frac{\phi(s)\,ds}{s^2 - \alpha^2},$$

and

$$r_+ - r_- = \frac{p_o}{\nu\alpha}(a^2 + \mu\alpha^2)\int_{-a}^{a}\frac{s\phi(s)\,ds}{s^2 - \alpha^2} = 0,$$

since $\phi(s)$ is even.

By Liouville's generalised theorem we have

$$D(z) = \frac{r_+}{z-\alpha} + \frac{r_-}{z+\alpha},$$

and on $(-a,a)$,

$$D(t) = \frac{r_+}{t-\alpha} + \frac{r_-}{t+\alpha}$$

$$= \frac{(r_+ + r_-)t + (r_+ - r_-)\alpha}{t^2 - \alpha^2}$$

$$= \frac{p_o}{\nu}\frac{(a^2 + \mu\alpha^2)t}{t^2 - \alpha^2}\int_{-a}^{a}\frac{\phi(s)\,ds}{s^2 - \alpha^2}$$

$$= \frac{p_o\, a^2(\nu - \mu)t}{\nu t^2 + a^2}\, C$$

where C is the constant determined by

$$C = \int_{-a}^{a} \frac{\phi(s)\, ds}{a^2 + \nu s^2} .$$

Thus, in the solution we obtain

$$\psi(x) = \int_0^x \sin[\lambda(x) - \lambda(t)]\, f(t)dt + \frac{1}{\pi}\int_0^x \frac{\cos[\lambda(x) - \lambda(t)]}{R(t)} dt \int_{-a}^{a} \frac{f(s)R(s)ds}{s - t}$$

$$+ \frac{p_o}{\nu}(\nu - \mu)\, C \int_0^x \frac{\cos[\lambda(x) - \lambda(t)]\, tdt}{1 + \nu t^2/a^2} ,$$

$$= \psi_1(x) + \frac{p_o}{\pi}(\nu - \mu)\, C\, \psi_2(x),$$

say, and so

$$\phi(x) = \psi_1(x) - \frac{\cos\lambda(x)}{\cos\lambda(a)}\psi_1(a) + \frac{p_o}{\pi}(\nu - \mu)\, C \left\{\psi_2(x) - \frac{\cos\lambda(x)}{\cos\lambda(a)}\psi_2(a)\right\}$$

$$= \Psi_1(x) + \frac{p_o}{\pi}(\nu - \mu)\, C\, \Psi_2(x),$$

and finally, determining C from the above formula, we find that

$$\phi(x) = \Psi_1(x) + \frac{p_o}{\pi}(\nu - \mu)\, \Psi_2(x) \int_{-a}^{a} \frac{\Psi_1(s)\, ds}{a^2 + \nu s^2} \Big/ \left\{ 1 - \right.$$

$$\left. - \frac{p_o}{\pi}(\nu - \mu) \int_{-a}^{a} \frac{\Psi_2(s)\, ds}{a^2 + \nu s^2} \right\} . \qquad (7.9)$$

The solution derived by Vekua [8] for this B(x) is in error; however, on rectifying it, we find that it coincides with the expression (7.9) resulting from the present method.

Finally we note that this method is applicable for any expression representing the chord of profile given by equation (7.2), provided that $P(x)$ is even and such that

$$P_+(x) = P_-(x) = P(x), \qquad -a < x < a,$$

this being a wider class of functions than that for which the method of Vekua offers a solution in closed form.

8. An integro-differential equation in aerodynamic theory, and its reduction to quasi-regular and regular integral equations.

As an example of the methods described in earlier sections, we examine the integro-differential equation arising from the two dimensional problem of the thin jet-flapped wing in aerodynamics. The jet-flap is a device designed to create jet-induced lift of an aerofoil by the emission of a high momentum sheet of air near the trailing edge. The problem was first formulated by Spence [9], who considered an inviscid, incompressible flow passing over the jet-flapped aerofoil, in the case of an infinitesimally thin jet of finite momentum, which he replaced by a vortex sheet. In a later paper, Hung-Ta Ho [10] considered the problem of a thin, jet-flapped hydrofoil with the added difficulty that at a sufficient speed, cavitation occurs on the upper surface of the foil. In the particular case when the cavity is taken to extend to infinity, Ho is able to deduce an integro-differential equation which differs only slightly from that obtained by Spence. We consider in detail Spence's equation, an exactly similar approach being applicable for the equation of Ho.

We briefly discuss the setting up of the aerodynamical problem. The incident angle α, and the angle of deflection of the jet as it exits from the trailing edge, τ, are both assumed to be small, so that the linearised theory is applicable (see Fig. 1). The assumptions of thin aerofoil theory are invoked, and the jet is replaced by a vortex

distribution

$$\gamma_j(x) = -\tfrac{1}{2} U_0 C_j y''(x),$$

along $(1,\infty)$ of the real axis. Here, $y(x)$ represents the jet profile, U_0 the undisturbed stream velocity, and C_j the momentum coefficient, a dimensionless quantity defined by

$$C_j = J/\tfrac{1}{2}\rho_0 U_0^2,$$

in which J is the momentum flux of the jet per unit span, and ρ_0 the undisturbed density. The aerofoil is replaced by the vortex distribution

$$\gamma_a(x) = U_0 f(x),$$

where $f(x)$ is unknown, on the segment $(0,1)$ of the real axis. Since the chord of the foil is taken to be unity, it does not occur explicitly in these expressions.

The downwash, or downward velocity, normal to both the span and direction of motion, due to these vortex distributions is (see for example [11], section 11.22),

$$w(x) = -(U_0/2\pi)\int_0^1 \frac{f(t)}{t-x}\,dt + (U_0 C_j/4\pi)\int_1^\infty \frac{y''(t)}{t-x}\,dt, \qquad (8.1)$$

at a point on the axis of x. For $x > 0$, one of these integrals is to be interpreted as a Cauchy principal value. The downwash on the axis can also be expressed as

$$\begin{aligned} w(x) &= U_0\epsilon(x), \qquad 0 < x < 1, \\ &= U_0 y'(x), \qquad 1 < x < \infty, \end{aligned}$$

where $\epsilon(x)$ is the inclination of the aerofoil to the axis, being the sum of the incident angle α and the camber of the foil. For x in $(1,\infty)$ it has been assumed that the downwash on the axis is identical to that on the jet.

From (8.1) we obtain the pair of simultaneous equations

$$\frac{1}{\pi}\int_0^1 \frac{f(t)dt}{t-x} - \tfrac{1}{2}C_j\,\frac{1}{\pi}\int_1^\infty \frac{y''(t)dt}{t-x} = \begin{cases} -2\epsilon(x), & 0 < x < 1, \\ -2y'(x), & 1 < x < \infty, \end{cases} \qquad (8.2)$$

for $f(x)$ and $y(x)$. The boundary conditions on the slope of the jet are

$$y'(1) = \alpha + \tau, \qquad y'(\infty) = 0,$$

while $f(x)$ is assumed to have integrable infinities at the leading and trailing edges, $x = 0$ and 1.

Spence solves (8.2) for $f(x)$ in terms of $y''(x)$ and $\epsilon(x)$ when $0 < x < 1$. This is a simple inversion in the class of functions unbounded at both ends, and the resulting expression is substituted into (8.2) for $1 < x < \infty$, giving an integro-differential equation in $y(x)$. Spence finds that, in the case of an uncambered aerofoil, when $\epsilon(x) = \alpha$, this equation is

$$\frac{1}{\pi}\left(\frac{x-1}{x}\right)^{1/2}\int_1^\infty \left(\frac{t}{t-1}\right)^{1/2}\frac{\phi'(t)dt}{t-x} - \lambda\phi(x) = 2\alpha\left\{1 - \left(\frac{x-1}{x}\right)^{1/2}\right\}, \qquad (8.3)$$

where $1 < x < \infty$, and

$$\phi(x) = -\tfrac{1}{2}C_j\,y'(x), \qquad \lambda = 4/C_j\,,$$

so that the boundary conditions on $\phi(x)$ are

$$\phi(1) = -2(\alpha + \tau)/\lambda, \qquad \phi(\infty) = 0. \tag{8.4}$$

The equation relating $\phi(x)$ to $f(x)$ is, for $0 < x < 1$,

$$f(x) = \frac{1}{\pi}\left(\frac{1-x}{x}\right)^{1/2} \int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{\phi'(t)dt}{t-x} + 2\alpha\left(\frac{1-x}{x}\right)^{1/2}. \tag{8.5}$$

We remark here that the essential difference between (8.3) and Ho's equation is that the latter contains an additional factor t^{-1} in the kernel.

Spence deals with equation (8.3) by mapping the range of integration onto $(0,\pi)$ and assuming a solution for $\phi'(x)$ as the sum of a term containing a logarithmic singularity at the trailing edge, and a general Fourier series. The existence of a singularity is determined by inspection of (8.3) in relation to the boundary condition at $x = 1$. The leading coefficients of the Fourier series are determined by Gaussian interpolation on the resulting equations.

In constructing the solution of (8.3) by reduction to a quasi-regular integral equation, we shall not use the formulae of section 5. explicitly. In fact the present problem contains one of the difficulties of the reduction method indicated previously; namely the determination of the inverse singular equation in the appropriate class of functions, and we shall indicate a method by which this difficulty is surmounted.

Construction of the solution of (8.3)

The existence of a singularity in $\phi'(x)$ at the trailing edge is apparent, since by (8.4) we require

$$\lim_{x\to 1} \frac{1}{\pi}\sqrt{\frac{x-1}{x}}\int_1^\infty \sqrt{\frac{t}{t-1}}\,\frac{\phi'(t)dt}{t-x} \to -2\tau.$$

The function defined by

$$F'(x) = x^{-5/2}\log(x-1), \tag{8.6}$$

is such that

$$\frac{1}{\pi}\sqrt{\frac{x-1}{x}}\int_1^\infty \sqrt{\frac{t}{t-1}}\,\frac{F'(t)dt}{t-x} = \pi x^{-5/2} + \frac{1}{x}\sqrt{\frac{x-1}{x}},$$

and so serves to represent this singularity if we introduce the coefficient $-2\tau/\pi$, and assume that

$$\phi(x) = -(2\tau/\pi)\,F(x) + 2\tau\,G(x) + 2\alpha\,H(x), \tag{8.7}$$

where, from (8.6),

$$F(x) = \tfrac{2}{3}(1 - x^{-3/2})\log(x^{1/2} - 1) - \tfrac{2}{3}(1 + x^{-3/2})\log(x^{1/2} + 1) + \\ + (4/3)\,x^{-1/2} \tag{8.8}$$

vanishes at infinity.

On substituting (8.7) into the integro-differential equation and equating the terms in α and τ, we obtain the two equations

$$\frac{1}{\pi}\sqrt{\frac{x-1}{x}}\int_1^\infty \sqrt{\frac{t}{t-1}}\,\frac{H'(t)dt}{t-x} - \lambda H(x) = 1 - \sqrt{\frac{x-1}{x}} \tag{8.9}$$

and

$$\frac{1}{\pi}\sqrt{\frac{x-1}{x}}\int_1^\infty \sqrt{\frac{t}{t-1}}\,\frac{G'(t)dt}{t-x} - \lambda G(x) = \frac{1}{\pi^2}\sqrt{\frac{x-1}{x}}\int_1^\infty \sqrt{\frac{t}{t-1}}\,\frac{F'(t)dt}{t-x}$$

$$- (\lambda/\pi)\, F(x),$$

$$= x^{-5/2} + \frac{1}{\pi}\, x^{-3/2}(x-1)^{1/2} - (\lambda/\pi)\, F(x), \quad (8.10)$$

and since

$$F(1) = (4/3)\,\{1 - \log 2\},$$

these are subject to the boundary conditions

$$\left.\begin{array}{l} H(1) = -1/\lambda, \quad H(\infty) = 0, \quad H'(1) = 0, \quad H'(\infty) \text{ finite}, \\ G(1) = -1/\lambda + (4/3\pi)\{1 - \log 2\}, \quad G(\infty) = 0, \quad G'(1) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G'(\infty) \text{ finite} \end{array}\right\} \quad (8.11)$$

Equation (8.9) is exactly similar to the equivalent equation in Spence's solution, while (8.10) differs from the latter due to the present representation of the logarithmic singularity. This departure from Spence's method will be justified at a later stage.

<u>The equation (8.9)</u>

Let us map the range of integration in (8.9) onto (0,1) by means of

$$t = 1/s, \qquad x = 1/y,$$

and denote

$$H(1/t) = h(t).$$

Then we obtain

$$\frac{y(1-y)^{1/2}}{\pi}\int_0^1 \frac{sh'(s)\,ds}{(1-s)^{1/2}(s-y)} = \lambda h(y) + 1 - (1-y)^{1/2}, \quad (8.12)$$

where $0 < y < 1$, and

$$h(1) = -1/\lambda, \quad h(0) = 0, \quad h'(1) = 0, \quad h'(0) \text{ finite.}$$

Writing

$$R(y) = y^{1/2}(1-y)^{1/2},$$

then (8.12) becomes

$$\frac{R(y)}{\pi}\int_0^1 \frac{s^{3/2}h'(s)}{R(s)\,(s-y)}\,ds = f(y), \quad (8.13)$$

in which

$$f(y) = \{\lambda h(y) + 1 - (1-y)^{1/2}\}/y^{1/2},$$

and assuming that

$$\lim_{y\to 0} \{h(y)/y^{1/2}\} = 0,$$

then it is seen that $f(y)$ vanishes at both end points of the range. By reference to the inversion formulae of section 2., in the case of a single arc, it is apparent that (8.13) can be regarded as the inverse of an equation, in the class $h(0,1)$. This requires that the solubility condition is satisfied by the free term of the original equation, and to accommodate this condition, we introduce the constant C, unspecified at present, by writing (8.13) in the following way

$$\frac{R(y)}{\pi}\int_0^1 \frac{s^{3/2}h'(s) - CR(s)}{R(s)\,(s-y)}\,ds = f(y) - \frac{C}{\pi}R(y)\int_0^1 \frac{ds}{s-y}$$

$$= f(y) - \frac{C}{\pi} R(y) \log\{(1 - y)/y\}.$$

The right hand side of this equation is zero at both ends of the interval, and the equation can be regarded as the inverse of

$$y^{3/2}h'(y) - C\,R(y) = -\frac{1}{\pi}\int_0^1 \frac{ds}{s - y}\left\{f(s) - \frac{C}{\pi} R(s) \log\{(1 - s)/s\}\right\},$$

bounded at both ends (and zero there), subject to the solubility condition being satisfied. This is

$$\int_0^1 \frac{y^{3/2}h'(y) - C\,R(y)}{R(y)}\,dy = 0,$$

or

$$\int_0^1 \frac{yh'(y)\,dy}{(1 - y)^{1/2}} = C.$$

We must ultimately select C in accordance with this equation. Evaluating the following Cauchy principal values

$$\frac{1}{\pi}\int_0^1 \frac{1 - (1 - s)^{1/2}}{s^{1/2}(s - y)}\,ds = \frac{y^{-1/2}}{\pi}\log\left(\frac{1 - y^{1/2}}{1 + y^{1/2}}\right) + 1,$$

$$\frac{1}{\pi}\int_0^1 \frac{R(s)}{s - y}\log\{(1 - s)/s\}\,ds = 1 - \pi R(y),$$

then the inverted equation becomes

$$y^{3/2}h'(y) = -\frac{\lambda}{\pi}\int_0^1 \frac{h(s)\,ds}{s^{1/2}(s - y)} - \frac{y^{-1/2}}{\pi}\log\left(\frac{1 - y^{1/2}}{1 + y^{1/2}}\right) - \left(1 - \frac{C}{\pi}\right). \quad (8.14)$$

This represents a simple case of the differential equation stage in the general theory, and the usual procedure would be to divide through by $y^{3/2}$ and integrate with respect to y. This, however, leads to terms which are unbounded at y = 0, and we can avoid this by the following method. It can be shown that

$$\int_0^s \frac{dt}{t^{1/2}(t - y)} = y^{-1/2}\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right|,$$

for all s > y, s < y, and so, integrating by parts,

$$-\frac{\lambda}{\pi}\int_0^1 \frac{h(s)\,ds}{s^{1/2}(s - y)} = \frac{y^{-1/2}}{\pi}\log\left(\frac{1 - y^{1/2}}{1 + y^{1/2}}\right) + \frac{\lambda}{\pi}\int_0^1 \frac{h'(s)}{y^{1/2}}\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds.$$

Therefore (8.14) becomes

$$y^2 h'(y) = \frac{\lambda}{\pi}\int_0^1 h'(s)\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds - y^{1/2}\{1 - C/\pi\}, \qquad (8.15)$$

an integral equation for $h'(y)$ with a weakly singular kernel. On returning to the original variables, this becomes

$$H'(x) = \frac{\lambda}{\pi}\int_1^\infty H'(t)\log\left|\frac{t^{1/2} - x^{1/2}}{t^{1/2} + x^{1/2}}\right| dt + x^{-1/2}\{1 - C/\pi\}. \qquad (8.16)$$

We examine this equation later.

In order to determine an integral equation with a regular kernel, we transform (8.14) to the original variables, giving

$$H'(x) = -\frac{\lambda}{\pi}\int_1^\infty \frac{x^{1/2}H(t)\,dt}{t^{1/2}(t-x)} + F_0(x), \tag{8.17}$$

where

$$F_0(x) = \frac{1}{\pi}\log\left(\frac{x^{1/2}-1}{x^{1/2}+1}\right) + x^{-1/2}\{1 - C/\pi\}. \tag{8.18}$$

We rearrange (8.17) in the form

$$H'(x) = -\frac{\lambda}{\pi}\int_1^\infty \frac{H(t)\,dt}{t-x} - \frac{\lambda}{\pi}\int_1^\infty H(t)\,R_1(x,t)\,dt + F_0(x), \tag{8.19}$$

in which

$$R_1(x,t) = \frac{x^{1/2}}{t-x}\left\{\frac{x^{1/2}-t^{1/2}}{x^{1/2}t^{1/2}}\right\} = -\frac{1}{t^{1/2}(t^{1/2}+x^{1/2})}$$

is regular. So is

$$R_2(x,t) = \frac{\partial R_1(x,t)}{\partial x} = \frac{1}{2t^{1/2}x^{1/2}(t^{1/2}+x^{1/2})^2}.$$

Integrating the singular term in (8.19) by parts, and differentiating the resulting equation with respect to x, we easily obtain

$$H''(x) = -\frac{\lambda}{\pi}\int_1^\infty \frac{H'(t)\,dt}{t-x} + \frac{1}{\pi(1-x)} - \frac{\lambda}{\pi}\int_1^\infty H(t)\,R_2(x,t)\,dt + F_0'(x),$$

and, since we can write

$$-R_3(x,t) = \int^t R_2(x,s)\,ds = \frac{1}{x^{1/2}(t^{1/2}+x^{1/2})},$$

then this becomes

$$H''(x) = -\frac{\lambda}{\pi}\int_1^\infty \frac{H'(t)\,dt}{t-x} - \frac{\lambda}{\pi}\int_1^\infty H'(t)\,R_3(x,t)\,dt + F_1(x), \qquad (8.20)$$

where

$$F_1(x) = F_0'(x) + \frac{1}{\pi(1-x)} + \frac{1}{\pi x^{1/2}(1+x^{1/2})}$$

$$= -\tfrac{1}{2}\,x^{-3/2}\{1 - C/\pi\}.$$

Returning to (8.9), it can be written in the form

$$\frac{1}{\pi}\int_1^\infty K(x,t)\,\frac{H'(t)\,dt}{t-x} = \lambda H(x) + 1 - \sqrt{\frac{x-1}{x}}$$

and rearranging this

$$\frac{1}{\pi}\int_1^\infty H'(t)\,R_4(x,t)\,dt + \frac{1}{\pi}\int_1^\infty \frac{H'(t)\,dt}{t-x} = \lambda H(x) + 1 - \sqrt{\frac{x-1}{x}} \qquad (8.21)$$

where

$$K(x,t) = \sqrt{\frac{t(x-1)}{x(t-1)}}, \qquad K(x,x) = 1,$$

and

$$R_4(x,t) = \frac{K(x,t) - K(x,x)}{t-x} = \frac{t^{1/2}(x-1)^{1/2} - x^{1/2}(t-1)^{1/2}}{x^{1/2}(t-1)^{1/2}(x-t)}.$$

Multiplying (8.21) by λ, and eliminating the singular integral between it and (8.20), we obtain

$$H''(x) + \lambda^2 H(x) = \frac{\lambda}{\pi}\int_1^\infty H'(t)\,Q(x,t)\,dt + E(x), \qquad (8.22)$$

where

$$Q(x,t) = R_4(x,t) - R_3(x,t)$$

$$= \frac{-1}{(t-1)^{1/2} + (x-1)^{1/2}} \sqrt{\frac{t}{x(t-1)}} ,$$

and

$$E(x) = -\tfrac{1}{2}\, x^{-3/2}\{1 - C/\pi\} - \lambda \left\{1 - \sqrt{\frac{x-1}{x}}\right\} .$$

If we denote the right hand side of (8.22) by $\lambda f(x)$, then

$$H''(x) + \lambda^2 H(x) = \lambda f(x)$$

has solution

$$H(x) = C_1 \cos \lambda x + C_2 \sin \lambda x + \int_1^x \sin[\lambda(x-t)]\, f(t)\, dt,$$

and using the boundary conditions

$$H(1) = -1/\lambda, \qquad H'(1) = 0,$$

we find that

$$H(x) = -(1/\lambda) \cos[\lambda(x-1)] + \int_1^x \sin[\lambda(x-t)]\, f(t)\, dt,$$

or

$$H'(x) = \sin[\lambda(x-1)] + \int_1^x \cos[\lambda(x-t)]\, \lambda f(t)\, dt.$$

Replacing $\lambda f(t)$ in this expression, we arrive at a regular integral equation for $H'(x)$, which may be written

$$H'(x) = \frac{\lambda}{\pi}\int_1^\infty H'(t)\ P(x,t)\ dt + g(x) \qquad (8.23)$$

where

$$P(x,t) = \int_1^x \cos[\lambda(x - s)]\ Q(s,t)\ ds$$

and

$$g(x) = \int_1^x \cos[\lambda(x - s)]\ E(s)\ ds + \sin[\lambda(x - 1)].$$

<u>The equation (8.10)</u>

This equation for $G(x)$ can be dealt with in a similar way, the only difference being in the term on the right hand side. In the interval $(0,1)$ the equation is, writing $g(x) = G(1/x)$,

$$\frac{R(y)}{\pi}\int_0^1 \frac{s^{3/2}g'(s)\ ds}{R(s)(s - y)} = \lambda\{g(y) + y^{5/2} + y(1 - y)^{1/2}/\pi - \lambda F_1(y)/\pi\}/y^{1/2},$$

where

$$F_1(y) = F(1/y) = \tfrac{2}{3}\left\{(1 - y^{3/2})\log\left(\frac{1 - y^{1/2}}{y^{1/2}}\right) - (1 + y^{3/2})\log\left(\frac{1 + y^{1/2}}{y^{1/2}}\right)\right\}$$
$$+ (4/3)y^{1/2}.$$

$g(x)$ is subject to the boundary conditions

$$g(0) = 0,\quad g(1) = -1/\lambda + (4/3\pi)\{1 - \log 2\},\quad g'(1) = 0,$$
$$g'(0) \text{ is finite.}$$

We note that the right hand side of the above equation vanishes at both ends of the interval, enabling us to employ the method indicated in the case of equation (8.9) to obtain the inverse equation. It is for this reason that we made the particular choice of representing the logarithmic term, Spence's representation giving rise to a divergent term at the lower limit in the equivalent equation. The inverted equation is

$$y^{3/2}g'(y) = -\frac{\lambda}{\pi}\int_0^1 \frac{g(s)ds}{s^{1/2}(s-y)} - \frac{1}{\pi}\int_0^1 \frac{ds}{s^{1/2}(s-y)}\{s^{5/2} + s(1-s)^{1/2} - $$

$$-\lambda F_1(s)/\pi\} + D/\pi,$$

where D is determined by

$$D = \int_0^1 \frac{s\ g(s)\ ds}{(1-s)^{1/2}} \quad .$$

On integrating by parts and inserting the appropriate boundary conditions we arrive at the counterpart of (8.15)

$$y^2 g'(y) = \frac{\lambda}{\pi}\int_0^1 g'(s)\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds + Dy^{1/2}/\pi + I(y)/\pi, \qquad (8.24)$$

where

$$I(y) = \int_0^1 \log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| \left\{5s^{3/2}/2 + (1-s)^{1/2}/\pi - \frac{s}{2\pi(1-s)^{1/2}} + \right.$$

$$\left. + \frac{\lambda}{\pi}s^{1/2}\log\{(1-s)/s\}\right\} ds. \qquad (8.25)$$

We can proceed to determine the regular integral equation for $G'(x)$

in exactly the same manner as previously, the only difference being in the free term, and we find that

$$G'(x) = \frac{\lambda}{\pi}\int_1^\infty G'(t)\, P(x,t)\, dt + g_1(x), \tag{8.26}$$

where $P(x,t)$ is the same as before, and the free term differs in that $E(x)$ is replaced by $E_1(x)$, where $E_1(x)$ takes the rather complicated form

$$E_1(x) = \lambda^2 F(x)/\pi - (D/2\pi)\, x^{-3/2} - (4\lambda/3\pi^2)\,\frac{1 - \log 2}{x^{1/2}(x-1)} + \frac{3}{2\pi x} - \frac{\lambda}{x^{5/2}}$$

$$- \frac{\lambda(x-1)^{1/2}}{\pi x^{3/2}} + \frac{5}{2\pi x^3}\log(x-1) - \frac{\lambda}{2\pi^2 x^2}\log^2(x-1) +$$

$$+ (1 - \lambda\pi/4)/\pi x^2.$$

The more involved free terms in the equation for $G(x)$ are, of course, due to the subtracting out of $F(x)$ from the solution initially. Comparatively straightforward free terms are obtained if one deals with (8.3) and the boundary conditions (8.4) directly, but in any subsequent approximate solution of the corresponding regular or quasi-regular equations, the convergence is likely to be poor and the accuracy lessened.

The quasi-regular integral equations.

We examine further the quasi-regular equations (8.15) and (8.24)

$$y^2 h'(y) = \frac{\lambda}{\pi}\int_0^1 h'(s)\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds + Cy^{1/2}/\pi - y^{1/2}, \qquad [(8.15)]$$

$$y^2 g'(y) = \frac{\lambda}{\pi}\int_0^1 g'(s)\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds + Dy^{1/2}/\pi + I(y)/\pi, \qquad [(8.24)]$$

where the constants C and D are defined by

$$C = \int_0^1 \frac{x\, h'(x)\, dx}{(1 - x)^{1/2}}, \qquad D = \int_0^1 \frac{x\, g'(x)\, dx}{(1 - x)^{1/2}},$$

Let us consider the equation

$$y^2 \psi'(y) = \frac{\lambda}{\pi}\int_0^1 \psi'(s)\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds + Ky^{1/2}/\pi + f(y) \qquad (8.27)$$

where

$$K = \int_0^1 \frac{x\, \psi'(x)\, dx}{(1 - x)^{1/2}}, \qquad (8.28)$$

which enables us to investigate (8.15) and (8.24) by ultimately specifying f(y). Writing

$$y\psi'(y) = \Psi(y),$$

then (8.27) becomes

$$\Psi(y) = \frac{\lambda}{\pi}\int_0^1 \frac{\Psi(s)}{sy}\log\left|\frac{s^{1/2} - y^{1/2}}{s^{1/2} + y^{1/2}}\right| ds + f(y)/y + Ky^{-1/2}/\pi \qquad (8.29)$$

in which the kernel is symmetric. We assume that $f(y)/y$ is integrable.

Carleman [11] proved that Fredholm's theorems can be applied directly to an integral equation with a weakly singular kernel $k(s,y)$ provided that

$$\iint_{\Omega} |k(s,y)|^2 \, ds \, dy < \infty,$$

that is, provided the kernel is square integrable. It is seen that this requirement is not satisfied by the kernel in (8.29), and so this equation cannot be resolved by the Fredholm techniques. However we shall see that by expanding the kernel in terms of orthogonal functions and by assuming a solution of a suitable form, we can deduce a system of linear equations for certain coefficients, somewhat similar to the system evolving from the Fredholm method.

We first make a change of variable in (8.29) by

$$s = \cos^2\tfrac{1}{2}\theta, \qquad y = \cos^2\tfrac{1}{2}\sigma,$$

and write

$$\Psi(\cos^2\tfrac{1}{2}\sigma) = \Psi(\sigma), \quad f(\cos^2\tfrac{1}{2}\sigma) = f(\sigma),$$

to give

$$\Psi(\sigma)\cos\tfrac{1}{2}\sigma = \frac{\lambda}{\pi}\int_0^{\pi} \frac{\sin\frac{1}{2}\theta\,\Psi(\theta)}{\cos\frac{1}{2}\theta\cos\frac{1}{2}\sigma} \log\left|\frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}\sigma}{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\sigma}\right| d\theta + K/\pi + f(\sigma)/\cos\tfrac{1}{2}\sigma, \tag{8.30}$$

where

$$K = \int_0^{\pi} \Psi(\theta)\cos\tfrac{1}{2}\theta \, d\theta, \tag{8.31}$$

and writing $\psi(\cos^2\frac{1}{2}\sigma) = \psi(\sigma)$,

$$\psi'(\sigma) = -\tan\tfrac{1}{2}\sigma\ \Psi(\sigma). \tag{8.32}$$

Now

$$\frac{1}{\cos\frac{1}{2}\theta\ \cos\frac{1}{2}\sigma}\log\left|\frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}\sigma}{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\sigma}\right| = -4\sum_{n=0}^{\infty}\alpha_n\ \phi_n(\theta)\ \phi_n(\sigma),$$

where

$$\phi_n(\theta) = \cos(n+\tfrac{1}{2})\theta\ /\ \cos\tfrac{1}{2}\theta, \qquad \alpha_n = 1\ /(2n+1).$$

The functions $\phi_n(\theta)$ are, according to [12], related to the Jacobi polynomials by

$$P_n^{(-1/2,1/2)}(\cos\ \theta) = (2n)!\ \phi_n(\theta)\ /\ (n!)^2,$$

and are orthonormal with weight function

$$\omega(\theta) = \{1 + \cos\ \theta\}/\pi$$

in $(0,\pi)$.

Using this expansion, the integral equation becomes

$$\Psi(\sigma)\ \cos\tfrac{1}{2}\sigma + \frac{4\lambda}{\pi}\sum_{n=0}^{\infty}\alpha_n\ \phi_n(\sigma)\int_0^{\pi}\Psi(\theta)\ \sin\tfrac{1}{2}\theta\ \phi_n(\theta)\ d\theta$$

$$= K/\pi + f(\sigma)/\cos\tfrac{1}{2}\sigma. \tag{8.33}$$

Let us assume that $\Psi(\theta)$ has the form

$$\Psi(\theta) = \cos\tfrac{1}{2}\theta\ \{1 + \cos\ \theta\}\sum_{m=0}^{\infty}A_m\ \phi_m(\theta). \tag{8.34}$$

The factor outside the sum is chosen so that the resulting integral in (8.33) exists, and also eliminates the half angles from the integrand.

We define this integral to be

$$b_{nm} = \int_0^{\pi} \phi_m(\theta)\ \phi_n(\theta)\ \sin \tfrac{1}{2}\theta\ \cos \tfrac{1}{2}\theta\ \{1 + \cos\ \theta\}\ d\theta$$

and also express

$$\cos \tfrac{1}{2}\theta\ \Psi(\theta) = \sum_{m=0}^{\infty} B_m\ \phi_m(\theta) = \tfrac{1}{2}\{1 + \cos\ \theta\}^2 \sum_{m=0}^{\infty} A_m\ \phi_m(\theta),$$

where

$$B_m = \frac{1}{\pi} \sum_{n=0}^{\infty} A_n\ a_{mn}, \qquad a_{mn} = \tfrac{1}{2} \int_0^{\pi} \phi_n(\theta)\ \phi_m(\theta)\ \{1 + \cos\ \theta\}^3\ d\theta.$$

Obviously $a_{mn} = a_{nm}$ and $b_{mn} = b_{nm}$, and using these expressions in (8.33) we obtain

$$\sum_{m=0}^{\infty} \phi_m(\sigma) \sum_{n=0}^{\infty} A_n\{a_{nm}/\pi + 4\lambda\ \alpha_m\ b_{nm}/\pi\} = K/\pi + f(\sigma)/\cos \tfrac{1}{2}\sigma. \qquad (8.35)$$

But from (8.31), on substituting for $\Psi(\theta)$,

$$K = \sum_{n=0}^{\infty} A_n\ d_n,$$

where

$$d_n = \tfrac{1}{2} \int_0^{\pi} \{1 + \cos\ \theta\}^2\ \phi_n(\theta)\ d\theta.$$

We assume that

$$\frac{f(\sigma)}{\cos \frac{1}{2}\sigma} = \sum_{m=0}^{\infty} C_m\ \phi_m(\sigma). \qquad (8.36)$$

Bearing in mind that $\phi_0(\sigma) = 1$, and using these expressions in (8.35), we easily obtain a system of equations for the A_n,

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} A_n \{a_{n0} - d_n + 4\lambda\, \alpha_0\, b_{n0}\} = \pi C_0, \\ &\sum_{n=0}^{\infty} A_n \{a_{nm} + 4\lambda\, \alpha_m\, b_{nm}\} = \pi C_m, \quad m \geqslant 1. \end{aligned}\right\} \qquad (8.37)$$

It is not difficult to show that

$$b_{nm} = \frac{-1}{(m-n)^2 - 1}, \qquad \text{n,m both odd or both even,}$$

$$= \frac{-1}{(m+n+1)^2 - 1}, \qquad \text{n odd, m even or vice versa,}$$

$$b_{mm} = 1,$$

and

$$a_{00} = 5\pi/4, \qquad a_{01} = 5\pi/8, \qquad a_{02} = \pi/8.$$

Otherwise

$$a_{mm} = 3\pi/4, \quad a_{m,m+1} = a_{m,m-1} = \pi/2, \quad a_{m,m+2} = a_{m,m-2} = \pi/8,$$

all other a_{nm} being zero. Finally, the only non-zero d_n are

$$d_0 = 3\pi/4, \qquad d_1 = \pi/4,$$

and the equations are therefore, on writing for brevity

$$a'_{nm} = a_{nm}, \quad m \geqslant 1,$$

$$a'_{n0} = a_{n0} - d_n,$$

$$\sum_{n=0}^{\infty} A_n\, a'_{nm} - \frac{4\lambda}{(2m+1)} \sum_{n=0}^{\infty} \left\{ \frac{A_{2n}}{(m-2n)^2 - 1} + \frac{A_{2n-1}}{(m+2n+2)^2 - 1} \right\} = \pi\, C_m$$

$$\sum_{n=0}^{\infty} A_n\, a'_{nm} - \frac{4\lambda}{(2m+1)} \sum_{n=0}^{\infty} \left\{ \frac{A_{2n}}{(m+2n+1)^2 - 1} + \frac{A_{2n-1}}{(m-2n-1)^2 - 1} \right\} = \pi\, C_m \tag{8.38}$$

the first of these holding for even values of m, the second for odd values of m.

If we put $A_{nm} = a'_{nm} + \mu\, \alpha_m\, b_{nm}$, where $\mu = 4\lambda$, then the eigenvalues of (8.27) are determined by

$$|A_{nm}| = |a'_{nm} + \mu\, \alpha_m\, b_{nm}| = 0. \tag{8.39}$$

It is seen that the terms a'_{nm} give non-zero contributions only to the diagonal and the first two sub- and super-diagonals. The determinantal equation for the eigenvalues λ_m of a Fredholm equation is of the form (see, for example, [14], Chapter 8),

$$|\lambda\, \gamma_{nm} - \delta_{nm}| = 0,$$

the δ_{nm} contributing, of course, only to the diagonal terms. To attempt to reduce (8.39) to this form, we consider the determinant $|a'_{nm}|$. Performing allowable operations on this determinant, it can easily be reduced to a diagonal form. However, when these operations are used on $|A_{nm}|$ as a whole, it is found that the $\alpha_m\, b_{nm}$ combine in such a way as to give unbounded terms. Hence (8.39) cannot be reduced to the above form. This is of course consistent with the fact that (8.27) was shown to be not of the Fredholm type, and its eigenvalues must be found from (8.39) as it stands.

We do not determine the eigenvalues of (8.27) here, but it is suggested that their sign is governed by the diagonal elements of $|A_{nm}|$. That is, the eigenvalues are perturbations of the approximate values given by the diagonal terms

$$\mu_m = - a'_{mm}/\alpha_m b_{mm} ,$$

and are of the same sign as these approximate values, namely negative, since $a'_{mm} > 0$, $\alpha_m b_{mm} > 0$. This tentative hypothesis requires further investigation, as does the determinantal equation of the type (8.39) in general, when the reduction to 'normal form' is singular.

The leading terms of (A_{nm}) are shown in Fig. 2.

We now consider the determination of the C_m's on the right hand side of (8.37). In the case of (8.15) for $h'(y)$, this is straightforward since by (8.27), (8.36) we must have

$$\frac{f(\sigma)}{\cos \frac{1}{2}\sigma} = - 1 = \sum_{m=0}^{\infty} C_m \, \phi_m(\sigma),$$

hence

$$C_0 = -1, \qquad C_m = 0, \quad m \geqslant 1. \tag{8.41}$$

The case of (8.24) for $g'(y)$ is not so simple. By reference to (8.25) we have

$$\frac{f(\sigma)}{\cos \frac{1}{2}\sigma} = \frac{I(\sigma)}{\pi \cos \frac{1}{2}\sigma}$$

$$= \frac{1}{\pi \cos \frac{1}{2}\sigma} \int_0^{\pi} \log\left|\frac{\cos \frac{1}{2}\theta - \cos \frac{1}{2}\sigma}{\cos \frac{1}{2}\theta + \cos \frac{1}{2}\sigma}\right| \left\{5\cos^3\tfrac{1}{2}\theta/2 + \sin \tfrac{1}{2}\theta /\pi - \frac{\cos^2\frac{1}{2}\theta}{2\pi \sin \frac{1}{2}\theta}\right.$$

$$\left. + \frac{\lambda}{\pi} \cos \tfrac{1}{2}\theta \log\left(\frac{1 - \cos^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta}\right)\right\} \cos \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \, d\theta$$

$$= I_1(\sigma) + \frac{\lambda}{\pi} I_2(\sigma) ,$$

where

$$I_1(\sigma) = \frac{1}{\pi \cos \frac{1}{2}\sigma} \int_0^{\pi} \log\left|\frac{\cos \frac{1}{2}\theta - \cos \frac{1}{2}\sigma}{\cos \frac{1}{2}\theta + \cos \frac{1}{2}\sigma}\right| \left\{5\cos^3\tfrac{1}{2}\theta /2 + \sin \tfrac{1}{2}\theta /\pi - \right.$$

$$\left. - \frac{\cos^2\frac{1}{2}\theta}{2\pi \sin \frac{1}{2}\theta}\right\} \cos \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \, d\theta$$

$$= - \frac{4}{\pi} \sum_{m=0}^{\infty} \alpha_m \phi_m(\sigma) \int_0^{\pi} \left\{5\cos^3\tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta /\pi - \frac{\cos^2\frac{1}{2}\theta}{2\pi \sin \frac{1}{2}\theta}\right\} \times$$

$$\times \cos(m + \tfrac{1}{2})\theta \cos \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \, d\theta$$

$$= - \frac{4}{\pi} \sum_{m=0}^{\infty} \alpha_m R_m \phi_m(\sigma) ,$$

where, on performing the integrations, it is found that

$$R_m = \frac{-5(m - 1)}{4m(m^2 - 4)} , \quad m \text{ odd}, \geqslant 3, \quad R_m = \frac{-5(m + 2)}{4(m + 3)(m^2 - 1)} , \; m \text{ even}, \geqslant 2, \tag{8.42}$$

$$R_0 = 37/48, \qquad R_1 = -3/16.$$

Also, we have

$$I_2(\sigma) = \frac{1}{\pi \cos \frac{1}{2}\sigma} \int_0^\pi \log\left|\frac{\cos \frac{1}{2}\theta - \cos \frac{1}{2}\sigma}{\cos \frac{1}{2}\theta + \cos \frac{1}{2}\sigma}\right| \log\left(\frac{1 - \cos \theta}{1 + \cos \theta}\right) \cos^2\tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \times$$

$$\times \, d\theta$$

$$= -\frac{4}{\pi} \sum_{m=0}^{\infty} \alpha_m \phi_m(\sigma) \int_0^\pi \log\left(\frac{1 - \cos \theta}{1 + \cos \theta}\right) \cos(m + \tfrac{1}{2})\theta \cos^2\tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \times$$

$$\times \, d\theta$$

$$= -\frac{4}{\pi} \sum_{m=0}^{\infty} \alpha_m T_m \phi_m(\sigma),$$

in which

$$T_m = \int_0^\pi \log\left(\frac{1 - \cos \theta}{1 + \cos \theta}\right) \cos(m + \tfrac{1}{2})\theta \cos^2\tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \, d\theta$$

$$= -4 \sum_{n=0}^{\infty} \alpha_n \int_0^\pi \cos(2n + 1)\theta \cos(m + \tfrac{1}{2})\theta \cos^2\tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta \, d\theta$$

$$= \sum_{n=0}^{\infty} \alpha_n K_{nm} ,$$

where

$$\left.\begin{aligned} K_{nm} &= \frac{1}{(2n + m + 2)^2 - 1} + \frac{1}{(2n - m)^2 - 1}, \quad m \text{ even}, \\ &= \frac{1}{(2n + m + 1)^2 - 1} + \frac{1}{(2n - m + 1)^2 - 1}, \quad m \text{ odd}. \end{aligned}\right\} \qquad (8.43)$$

Now,

$$\sum_{n=0}^{\infty} \frac{1}{(2n + 1)} \left\{ \frac{1}{(2n + m + 2)^2 - 1} + \frac{1}{(2n - m)^2 - 1} \right\} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{1}{[(2n+m+2)^2 - 1]} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \frac{1}{[(2n-m-2)^2 - 1]}$$

$$= 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{1}{[(2n+m+2)^2 - 1]} -$$

$$- \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)} \frac{1}{[(2n+m+2)^2 - 1]} .$$

The second sum in this last expression is easily seen to vanish for $m > 0$, for it may be written as

$$\frac{1}{2m} \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{(2n+1)} - \frac{1}{(2n+m+1)} \right\} - \frac{1}{2(m+2)} \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{(2n+1)} - \frac{1}{(2n+m+3)} \right\}$$

in which each fraction is zero over the doubly infinite range. Hence

$$\left.\begin{aligned} T_m &= 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{1}{[(2n+m+2)^2 - 1]} , \quad m \text{ even}, > 0, \\ &\text{and similarly} \\ T_m &= 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{1}{[(2n+m+1)^2 - 1]} , \quad m \text{ odd}, > 1, \end{aligned}\right\} \quad (8.44)$$

which are, of course, more quickly convergent than the forms given directly by (8.43). Using results from Jolley [15], we can show that

$$T_0 = T_1 = -\tfrac{1}{2} , \qquad T_2 = T_3 = 1/6.$$

Finally,

$$I(\sigma) = -\frac{4}{\pi} \sum_{m=0}^{\infty} \alpha_m \, \phi_m(\sigma) \, \{R_m + \lambda T_m/\pi\},$$

and so

$$C_m = -\frac{4\alpha_m}{\pi}\{R_m + \lambda T_m/\pi\}. \tag{8.45}$$

We see that

$$R_m \approx -5/4m^2, \quad m \geqslant 2,$$

while the largest term in T_m for $m \geqslant 2$ is $O(m^{-2})$. Hence the C_m are quickly convergent

$$C_m \approx O(m^{-3})\{1 + \lambda\}.$$

The solutions for $h'(\theta)$ and $g'(\theta)$ are indicated by (8.32) and (8.34),

$$h'(\theta) = -\sin\theta \sum_{m=0}^{\infty} h_m \cos(m + \tfrac{1}{2})\theta,$$

$$g'(\theta) = -\sin\theta \sum_{m=0}^{\infty} g_m \cos(m + \tfrac{1}{2})\theta,$$

where h_m and g_m are the solutions of the system of equations (8.37) subject to the right hand sides given by (8.41) and (8.45) respectively. In the case when λ is large, if we consider as a first approximation only the terms involving λ, then we immediately have

$$\left.\begin{aligned} h_m &= 0 + O(\lambda^{-1}) \\ g_{2m} = g_{2m+1} &= \frac{1}{\pi(2m+1)} + O(\lambda^{-1}) \end{aligned}\right\} m \geqslant 0. \tag{8.46}$$

In the case of $g(\theta)$, this is seen by taking T_m in the form suggested by (8.43).

The total solution (8.7) of (8.3) is given by

$$\phi(x) = -\frac{2\tau}{\pi} F(x) + 2\alpha\, h(\theta) + 2\tau\, g(\theta),$$

on $(1,\infty)$, where

$$x = 1/\cos^2\tfrac{1}{2}\theta.$$

From (8.5), the vorticity distribution on (0,1) is given by

$$f(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{x}}\int_1^\infty \sqrt{\frac{t}{t-1}}\,\frac{\{-2\tau F'(t)/\pi + 2\alpha H'(t) + 2\tau G'(t)\}}{t-x}\,dt +$$

$$+ 2\alpha\sqrt{\frac{1-x}{x}}$$

$$= -\frac{2\tau}{\pi}\{x^{-5/2}\log(1-x) + x^{-3/2}(1-x)^{1/2}\} + 2\alpha\sqrt{\frac{1-x}{x}} +$$

$$+ \frac{2}{\pi}\sqrt{\frac{1-x}{x}}\int_0^\pi \frac{\{\alpha h'(\theta) + \tau g'(\theta)\}}{1/\cos^2\tfrac{1}{2}\theta - x}\,\frac{d\theta}{\sin\tfrac{1}{2}\theta},$$

and so

$$\lim_{x\to 0} x^{1/2} f(x) = \frac{2\tau}{\pi} + 2\alpha + \frac{2}{\pi}\int_0^\pi \{\alpha h'(\theta) + \tau g'(\theta)\}\,\frac{\cos^2\tfrac{1}{2}\theta\, d\theta}{\sin\tfrac{1}{2}\theta}$$

$$= \frac{2\tau}{\pi} + 2\alpha - \frac{2}{\pi}\{\alpha C + \tau D\},$$

using (8.31) and (8.32) for the two specific functions $h(\theta)$ and $g(\theta)$.

From Spence, the lift coefficient is

$$C_\ell = 2\int_1^\infty \sqrt{\frac{t}{t-1}}\,\phi'(t)\,dt + 2\pi\alpha,$$

$$= 4\tau - 4\pi\{\alpha h_0 + \tau g_0\} + 2\pi\alpha,$$

on mapping the range of integration onto $(0,\pi)$, and substituting for $h'(\theta)$ and $g'(\theta)$ from above.

Spence shows that, if we define the coefficient of suction at the leading edge to be

$$C_s = \frac{\pi}{2}\left\{ \lim_{x\to 0} x^{1/2} f(x) \right\}^2 ,$$

then for small α and τ, the following relation holds,

$$C_s = \alpha C_\ell - 2\{\alpha^2 - \tau^2\}/\lambda .$$

Using the above results to substitute into this expression for C_s and C_ℓ, and separately equating terms in α^2, τ^2 and $\alpha\tau$, we obtain,

$$\left.\begin{aligned} (1 - D)^2 &= \pi/\lambda, \\ (1 - C/\pi)^2 &= 1 - 2h_0 - 1/\pi\lambda, \\ (1 - D)(1 - C/\pi) &= 1 - \pi g_0. \end{aligned}\right\} \qquad (8.47)$$

We also have, from the general result

$$K = (3A_0 + A_1)\pi/4$$

for (8.30), the relations

$$C = \pi(3h_0 + h_1)/4, \qquad D = \pi(3g_0 + g_1)/4. \qquad (8.48)$$

Together, (8.47) and (8.48) serve as a check on the leading coefficients, and since they arise from physical requirements of the problem, ensure that the solutions of the quasi-regular integral equations are sound. In other words, they serve to establish equivalence between the quasi-regular equations and those from which they were derived.

We recall that the slope of the jet is related to $\phi(x)$ by

$$y'(x) = -\tfrac{1}{2}\lambda\phi(x)$$
$$= \tau\{\lambda F(x)/\pi - \lambda G(x)\} - \alpha\lambda H(x).$$

Using (8.9) and (8.10) to substitute for terms on the right hand side, and mapping the variable x onto $(0,\pi)$ by $x = 1/\cos^2\tfrac{1}{2}\theta$, we can show that the slope of the jet is

$$\frac{dy}{d\sigma} = \tau\{\sin\tfrac{1}{2}\sigma\cos^2\tfrac{1}{2}\sigma + \sin^2\tfrac{1}{2}\sigma/\pi\cos\tfrac{1}{2}\sigma\} + \alpha\sin\tfrac{1}{2}\sigma\{1 - \sin\tfrac{1}{2}\sigma\}/\cos^3\tfrac{1}{2}\sigma$$
$$- \frac{\sin^2\tfrac{1}{2}\sigma}{2\cos\tfrac{1}{2}\sigma}\{1 + 2\cos^2\tfrac{1}{2}\sigma\}\{\tau g_0 + \alpha h_0\} - \tfrac{1}{2}\sin\sigma\sum_{m=1}^{\infty}\{\tau g_m + \alpha h_m\}\sin(m + \tfrac{1}{2})\sigma,$$

the derivation of which makes use of the Cauchy principal values

$$\int_0^{\pi}\frac{\cos m\theta\, d\theta}{\cos\theta - \cos\sigma} = \frac{\pi\sin m\sigma}{\sin\sigma}, \quad m \geq 0.$$

Solution of the system of equations.

The equations can easily be solved numerically using standard techniques. We replace the infinite set of equations for A_0, A_1, A_2, .. by the first M equations, retaining only the terms in A_0, A_1, ... A_{M-1} on the left hand side. To justify this, let us assume that the A_m converge. Then we have assumed that the terms

$$(a'_{M+k\,m} + \mu\alpha_m b_{M+k\,m})\, A_{M+k}, \quad k=0, 1, \ldots. \quad m=0,\ldots. (M - 1),$$

in the first M equations are effectively zero, and that the remaining infinite set of equations are identically satisfied. That is, all the

terms contained in them are effectively zero, these being

$$(a'_{nm} + \mu\alpha_m b_{nm})\ A_n\ ,\ C_m\ ,\quad n=0,\ 1,\ \ldots\ ,\ m=M,\ M+1,\ \ldots$$

We have seen that the $\alpha_m b_{nm}$ decrease in magnitude as $(n - m)$ increases (as we move away from the diagonal of the matrix) and as m increases. Also, in the case of (8.15), all C_m are zero but the first, while for (8.24) the C_m fall off like m^{-3}. Hence, for moderate values of $\mu = 4\lambda$, we see that the above approximation will provide a reasonably accurate solution for a suitably chosen value of M.

In fact we solve the equations for M = 10 and M = 20, the variation in the values of the coefficients between the two cases being of course a measure of the accuracy of the method. We employ the technique and computer program due to Bowdler et al. (see Appendix I, page 274), to evaluate the h_m and g_m for several values of λ. As a check on the leading coefficients we make use of (8.47), and find that in all cases they are satisfied accurately.

The values of the first four coefficients are displayed in Table 1. for $h(\theta)$ and Table 2. for $g(\theta)$, for both values of M. It is seen that there is little difference between the solutions for M = 10 and M = 20, the maximum variation being in h_2, h_3, g_2 and g_3 for the higher values of C_j, while for the smaller C_j the variation is minimal.

There is, of course, little to be gained by reducing integro-differential equations, if one still has to resort to computer techniques to resolve the problem, and the chief advantage of the method

is the possibility of an approximate manual solution. That the present system of equations is open to such an approach can be seen by retaining only the following terms,

$$h_0 - \tfrac{1}{3}h_1 - \tfrac{1}{3}h_2 - h_3/15 = -\pi C_0/\mu,$$

$$-h_0/9 + \tfrac{1}{3}h_1 - h_2/45 - h_3/9 = -\pi C_1/\mu,$$

$$-h_0/15 + h_2/5 = -\pi C_2/\mu,$$

$$-h_1/21 + h_3/7 = -\pi C_3/\mu,$$

this approximation being valid for values of μ such that the a'_{nm} are small compared with the $\mu\alpha_m b_{nm}$, but not so large that the coefficients are given by the exact limiting values (8.46). Substituting for the C_m from the above results, we find that

$$h_0 = -\pi/3\lambda, \qquad h_1 = 3h_0/8,$$

$$h_2 = h_0/3, \qquad h_3 = h_1/3,$$

and

$$g_0 = -7/10\lambda + 4/5\pi, \qquad g_1 = 7/100\lambda + 4/5\pi,$$

$$g_2 = g_0/3 + 1/3\lambda - 1/6\pi, \qquad g_3 = g_1/3 + 1/6\lambda - 1/6\pi.$$

Evaluating these for appropriate values of λ, we obtain the results shown in Table 3., the leading terms comparing favourably with the computed values. We expect the h_2, h_3, g_2 and g_3 to be less accurate than the dominant coefficients, since we ignored comparatively large terms in them above.

This crude approximation indicates that the system is open to more rigorous approximate techniques, such as those discussed by

Kantorovich and Krylov [16], which will give values of the coefficients for moderate values of λ, as accurate as one is likely to require.

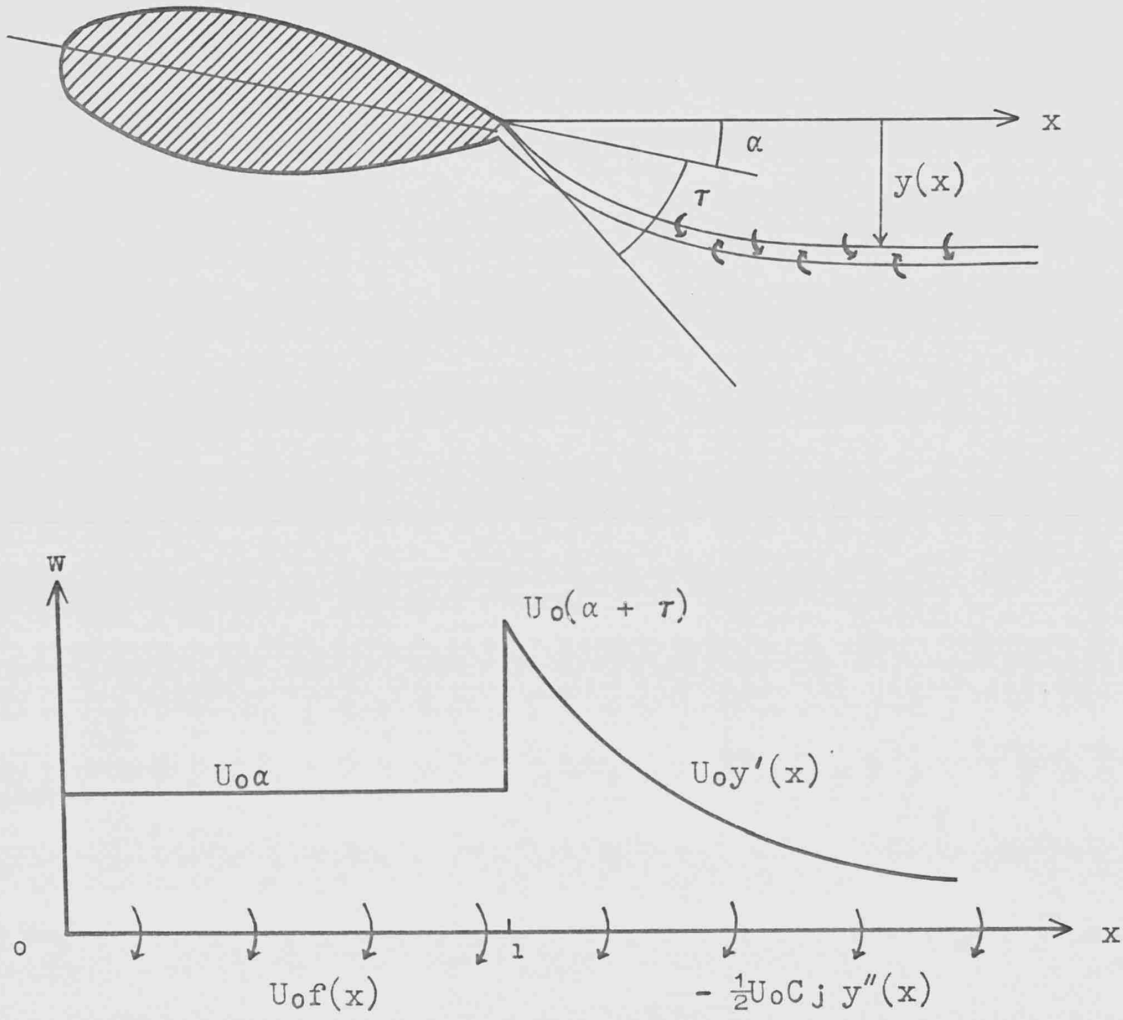


FIG. 1 : The jet-flapped aerofoil, showing the physical variables and the vorticity and downwash distributions on the x axis.

$$
\begin{bmatrix}
\frac{\pi}{2}+\mu & \frac{3\pi}{8}-\frac{1}{3}\mu & \frac{\pi}{8}-\frac{1}{3}\mu & -\frac{1}{15}\mu & -\frac{1}{15}\mu & -\frac{1}{35}\mu & -\frac{1}{35}\mu & -\frac{1}{63}\mu & \cdots \\
\frac{5\pi}{8}-\frac{1}{9}\mu & \frac{3\pi}{4}+\frac{1}{3}\mu & \frac{\pi}{2}-\frac{1}{45}\mu & \frac{\pi}{8}-\frac{1}{9}\mu & -\frac{1}{105}\mu & -\frac{1}{45}\mu & -\frac{1}{189}\mu & -\frac{1}{105}\mu & \cdot\cdot \\
\frac{\pi}{8}-\frac{1}{15}\mu & \frac{\pi}{2}-\frac{1}{75}\mu & \frac{3\pi}{4}+\frac{1}{5}\mu & \frac{\pi}{2}-\frac{1}{175}\mu & \frac{\pi}{8}-\frac{1}{15}\mu & -\frac{1}{315}\mu & -\frac{1}{75}\mu & -\frac{1}{495}\mu & \cdot\cdot \\
-\frac{1}{105}\mu & \frac{\pi}{8}-\frac{1}{21}\mu & \frac{\pi}{2}-\frac{1}{245}\mu & \frac{3\pi}{4}+\frac{1}{7}\mu & \frac{\pi}{2}-\frac{1}{441}\mu & \frac{\pi}{8}-\frac{1}{21}\mu & -\frac{1}{693}\mu & -\frac{1}{105}\mu & \cdot\cdot \\
-\frac{1}{135}\mu & -\frac{1}{315}\mu & \frac{\pi}{8}-\frac{1}{27}\mu & \frac{\pi}{2}-\frac{1}{567}\mu & \frac{3\pi}{4}+\frac{1}{9}\mu & \frac{\pi}{2}-\frac{1}{891}\mu & \frac{\pi}{8}-\frac{1}{27}\mu & -\frac{1}{1287}\mu & \cdot\cdot \\
-\frac{1}{385}\mu & -\frac{1}{165}\mu & -\frac{1}{693}\mu & \frac{\pi}{8}-\frac{1}{33}\mu & \frac{\pi}{2}-\frac{1}{1089}\mu & \frac{3\pi}{4}+\frac{1}{11}\mu & \frac{\pi}{2}-\frac{1}{1573}\mu & \frac{\pi}{8}-\frac{1}{33}\mu & \cdot \\
-\frac{1}{455}\mu & -\frac{1}{819}\mu & -\frac{1}{195}\mu & -\frac{1}{1287}\mu & \frac{\pi}{8}-\frac{1}{39}\mu & \frac{\pi}{2}-\frac{1}{1869}\mu & \frac{3\pi}{4}+\frac{1}{13}\mu & \frac{\pi}{2}-\frac{1}{2535}\mu & \\
-\frac{1}{845}\mu & -\frac{1}{525}\mu & -\frac{1}{1485}\mu & -\frac{1}{225}\mu & -\frac{1}{2145}\mu & \frac{\pi}{8}-\frac{1}{45}\mu & \frac{\pi}{2}-\frac{1}{2925}\mu & \frac{3\pi}{4}+\frac{1}{15}\mu & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
$$

FIG. 2. The leading terms in the matrix of coefficients.

μ	C_j	10 X 10 case				20 X 20 case			
		h_0	h_1	h_2	h_3	h_0	h_1	h_2	h_3
1.6	10.00	-1.258	1.171	- .8108	.5185	-1.259	1.175	- .8185	.5297
3.2	5.00	- .7016	.4383	- .2498	.1415	- .7021	.4397	- .2522	.1449
16.0	1.00	- .1848	.0109	- .0296	.0096	- .1848	.0111	- .0299	.0100
32.0	0.50	- .1039	- .0114	- .0211	.0001	- .1039	- .0113	- .0212	.0003
40.0	0.40	- .0862	- .0134	- .0193	- .0016	- .0862	- .0133	- .0193	- .0015
80.0	0.20	- .0481	- .0142	- .0142	- .0044	- .0481	- .0143	- .0144	- .0044
160.0	0.10	- .0266	- .0097	- .0098	- .0046	- .0267	- .0098	- .0098	- .0045
320.0	0.05	- .0146	- .0063	- .0063	- .0036	- .0147	- .0064	- .0063	- .0036
1600.0	0.01	- .0033	- .0018	- .0018	- .0012	- .0035	- .0019	- .0019	- .0013

TABLE 1.

μ	C_j	10 X 10 case				20 X 20 case			
		g_0	g_1	g_2	g_3	g_0	g_1	g_2	g_3
1.6	10.00	-1.151	1.161	-.6919	.4481	-1.153	1.165	-.6938	.4564
3.2	5.00	-.5745	.4746	-.1871	.1184	-.5750	.4753	-.1893	.1215
16.0	1.00	-.0018	.1510	-.0268	.0201	-.0018	.1512	-.0273	.0207
32.0	0.50	.1027	.1670	-.0142	.0165	.1029	.1673	-.0145	.0170
40.0	0.40	.1278	.1761	-.0095	.0172	.1279	.1764	-.0098	.0178
80.0	0.20	.1872	.2072	.0095	.0253	.1874	.2073	.0096	.0256
160.0	0.10	.2272	.2352	.0303	.0384	.2274	.2354	.0304	.0387
320.0	0.05	.2548	.2580	.0490	.0529	.2549	.2581	.0491	.0531
1600.0	0.01	.2909	.2910	.0791	.0798	.2909	.2912	.0794	.0800

TABLE 2.

λ	400	80	40	20
$-h_0$	.003	.014	.026	.052
$-h_1$	.001	.005	.010	.020
$-h_2$	.001	.005	.009	.018
$-h_3$	.000	.002	.003	.007
g_0	.249	.242	.223	.216
g_1	.251	.251	.252	.254
g_2	.031	.032	.030	.029
g_3	.031	.031	.035	.047

TABLE 3.

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CHAPTER V

WAVE PROPAGATION IN A TWO DIMENSIONAL, INFINITE REGULAR ARRAY OF RIGID ICE FLOES.

1. Introduction and formulation of the problem.

We investigate the conditions under which a wave can propagate through a regular, infinite system of ice floes by constructing the following mathematical model. Suppose that the floes occupy positions on the surface of an infinitely deep expanse of water, and assume that they are displaced so little by the fluid motion that their vertical velocities can be ignored. We consider the two dimensional situation, taking the fluid to occupy the half plane $y < 0$, and its undisturbed surface to coincide with the x axis. The fluid motion is described by the velocity potential $\Phi(x,y,t)$, which must be a solution of

$$\Phi_{xx} + \Phi_{yy} = 0,$$

assuming the fluid to be incompressible and irrotational.

The presence of the floes is dealt with by imposing

$$\Phi_y = 0, \qquad x \in \sum_k a_k b_k, \quad y = 0,$$

subject to the periodicity condition

$$a_{k+1} - a_k = 2\ell, \qquad \text{all } k,$$

while the relation

$$b_k - a_k = 2L, \qquad \text{all } k, \qquad L < \ell,$$

ensures that the floes are of equal dimension.

On the remainder of $y = 0$, there exists a free surface, and from the linearised versions of the kinematic condition and the Bernoulli equation, we obtain the usual requirement on such a surface,

$$\Phi_{tt} + g\Phi_y = 0, \qquad x \in \sum_k b_k\, a_{k+1}, \quad y = 0.$$

The problem thus posed is an extension of the finite dock situation, which has been discussed by Rubin [1] and Sparenberg [2], to the case of infinitely many docks.

Assuming the the velocity potential contains the time harmonically,

$$\Phi(x,y,t) = \exp(i\omega t)\ \phi(x,y),$$

then $\phi(x,y)$ is defined by

$$\phi_{xx} + \phi_{yy} = 0, \qquad y < 0, \tag{1.1}$$

$$\left.\begin{aligned} \phi_y &= 0, \qquad x \in \sum_k a_k\, b_k, \quad y = 0, \\ \phi_y &= \omega^2\phi/g, \qquad x \in \sum_k b_k\, a_{k+1}, \quad y = 0. \end{aligned}\right\} \tag{1.2}$$

(See Fig. 1 (a)).

This boundary value problem can be approached in two ways. First, by making use of automorphic function theory in relation to the Riemann-Hilbert problem, we can deduce an integral equation for $\phi(x,0)$ on the free surface, and second, by posing two further independent potential problems, the solutions of which can be combined in a particular way to provide the solution of the problem stated above. It will be shown that these two methods lead to essentially the same

conclusion.

Automorphic functions offer a powerful means of determining periodic solutions of boundary value problems, when there is periodicity in the boundary condition. In the following section we state some of the simplest facts about such functions with the aid of which we can discuss periodic solutions of the Riemann-Hilbert problem.

2. The Riemann-Hilbert problem for automorphic functions.

Consider the group Γ of bilinear transformations

$$\omega_k(z) = \frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}, \quad \alpha_k \delta_k - \beta_k \gamma_k \neq 0, \tag{2.1}$$

Then the single-valued analytic function $F(z)$ is called automorphic with respect to Γ if it is invariant under the transformations of this group

$$F[\omega_k(z)] = F(z). \tag{2.2}$$

Only the so called strictly discontinuous groups of bilinear substitutions are taken into account in the theory of automorphic functions, and for each such group in the z plane, there exists a domain which does not contain two distinct points equivalent to each other, but which does contain points equivalent to all other points in the plane, relative to the group. Such a domain is called the fundamental domain of the group. If the group does not contain integral substitutions, that is if no γ_k vanishes, then the fundamental domain is the exterior

of all the isometric circles of the group, these being given by

$$|\gamma_k z + \delta_k| = 0,$$

assuming the normalisation $\alpha_k\delta_k - \beta_k\gamma_k = 1$.

If the group contains an infinite set of substitutions, then the centres of the isometric circles form an infinite set having one or more points of condensation, or limit points. All points other than limit points are called regular points.

The regular points of a strictly discontinuous group either form one connected domain S, or are divided by the limit points into several domains S_1, S_2, ..., and such a group is called functional if one of these connected domains is mapped into itself. Two of the most important types of functional groups are elementary groups and Fuchsian groups. The former category have been rigorously investigated, and the greatest interest in the present context lies in the singly periodic elementary group generated by the substitution $\omega(z) = z + \Omega$, for some constant Ω, having one limit point at infinity.

Returning to the discussion of automorphic functions, then since these take identical values at equivalent points, it is sufficient to study them only in the fundamental domain of the group. For each functional group there exists an automorphic function which does not reduce to a constant, and for such a function each limit point of the group is an essential singularity. Further, an automorphic function which is not identically zero, has in its fundamental domain one and

the same number of zeros and poles. We call the function $f(z)$, with a single simple pole in the fundamental domain, the fundamental automorphic function of the group.

The fundamental theorem of automorphic functions, which is utilised in the solution of the Riemann-Hilbert problem, can be stated as follows: any automorphic function belonging to the same group as $f(z)$, and having the same domain of existence, is a rational function of $f(z)$.

We are now in a position to formulate the Riemann-Hilbert problem for automorphic functions.

Let L be a smooth open or closed contour situated in one of the domains S (possibly touching the boundary of S at a finite number of points) of a functional group Γ of bilinear transformations $\omega_k(z)$, and let L_k ($k = 1, 2, \ldots$) be the contours generated by $\omega_k(z)$. We assume that the lines L_k are distinct and intersect each other and L in only a finite number of points. Further, let $\Phi(z)$ be a single-valued function of the points z of the domain S, such that

(i) $\Phi(z)$ is automorphic relative to the group Γ,

$$\Phi[\omega_k(z)] = \Phi(z), \quad k = 1, 2, \ldots \tag{2.3}$$

(ii) All the essentially singular points of $\Phi(z)$ are limit points of the group, and in any fundamental domain it has a finite number of poles.

(iii) $\Phi(z)$ is continuously extendable to each point of L (and hence L_k, $k = 1, 2, \ldots$) except near the end points of L, where

however

$$|\Phi(z)| < \frac{\text{const.}}{|z - c|^{\alpha}}, \quad \alpha < 1,$$

and the points of intersection of L and L_k, near which it is almost bounded.

Then the function $\Phi(z)$ is a sectionally-meromorphic automorphic function belonging to the group Γ, and has a line of jump discontinuity L. The Riemann-Hilbert problem can be stated: to determine $\Phi(z)$, having prescribed poles in S, satisfying the boundary condition

$$\Phi_+(t) = V(t)\,\Phi_-(t) + g(t), \quad t \in L, \tag{2.4}$$

where $V(t)$ and $g(t)$ are known and satisfy the Holder condition on L. It is assumed that $V(t)$ is nowhere zero on L.

The contours L_k ($k = 1, 2, \ldots$) are also lines of jump discontinuity for the function $\Phi(z)$, and it is not difficult to show that on them we require

$$\Phi_+(t) = V[\omega_j(t)]\,\Phi_-(t) + g[\omega_j(t)], \quad t \in L_j. \tag{2.5}$$

The boundary conditions (2.4) and (2.5) are equivalent to the condition (2.4) and the relations (2.3), and so in solving for $\Phi(z)$ it is sufficient to satisfy (2.4), ensuring that $\Phi(z)$ is automorphic with respect to the group Γ.

The solution of this Riemann-Hilbert problem in the case when Γ is a finite group has been derived by Gakhov and Chibrikova [3]. The extension of these results to another type of elementary group, the infinite group, has been carried out in a paper by Chibrikova [4],

whose work we now cite.

Consider the particular case when $V(t) = 1$, and

$$\Phi_+(t) - \Phi_-(t) = g(t), \quad t \in L. \tag{2.6}$$

The ordinary Cauchy integral does not provide the solution of this, since the requirement of automorphicity is not satisfied. Let $f(z)$ be the fundamental automorphic function having a simple pole at z_0 in the fundamental domain R. Then the counterpart of the Cauchy integral in automorphic functions is

$$\Phi_1(z) = \frac{1}{2\pi i}\int_L g(\tau)\,\frac{f'(\tau)\,d\tau}{f(\tau) - f(z)} \tag{2.7}$$

and defines a sectionally holomorphic automorphic function belonging to the group Γ, having a line of discontinuity L and vanishing at z_0 and all equivalent points. Formulae similar to those of Plemelj can be deduced as $z \to t \in L$ from both sides,

$$\Phi_{1\pm}(t) = \pm\tfrac{1}{2}g(t) + \frac{1}{2\pi i}\int_L g(\tau)\,\frac{f'(\tau)\,d\tau}{f(\tau) - f(t)}.$$

Hence (2.7) yields the solution of the jump problem (2.6). Further, the problem of zero jump which is zero at z_0 vanishes identically by virtue of analytic continuation and the properties of automorphic functions, so that (2.7) is unique.

By the fundamental theorem of automorphic functions, the general sectionally meromorphic solution is

$$\Phi(z) = \frac{1}{2\pi i}\int_L g(\tau)\,\frac{f'(\tau)\,d\tau}{f(\tau) - f(z)} + \frac{P_v[f(z)]}{[f(z) - f(z_i)]^{v_i}}\,, \qquad (2.8)$$

having poles of orders $v_1, \ldots\ldots, v_m$ at $z_1, \ldots\ldots, z_m$, where $P_v(z)$ is a polynomial in z of degree not greater than $v = v_1 + \ldots\ldots + v_m$, containing (v+1) arbitrary constants.

A solution of (2.6) having a pole of order κ at z_0 is

$$\Phi(z) = \frac{1}{2\pi i}\int_L g(\tau)\,\frac{f'(\tau)\,d\tau}{f(\tau) - f(z)} + P_\kappa[f(z)], \qquad (2.9)$$

and the general sectionally holomorphic solution is

$$\Phi(z) = \frac{1}{2\pi i}\int_L g(\tau)\,\frac{f'(\tau)\,d\tau}{f(\tau) - f(z)} + C, \qquad (2.10)$$

where C is an arbitrary constant. If $\Phi(z)$ is to vanish at z_0 we must place $C = 0$, and return to the unique solution (2.7).

It is a simple jump problem of the type (2.6) which will be of interest to us shortly, but for the sake of completeness, we state the general solution of the boundary condition (2.4), which can be deduced from the above. Defining the index of V(t) to be

$$\nu = [\arg V(t)]_L/2\pi,$$

then (2.4) admits the solution

$$\Phi(z) = X(z)\,\{\Psi(z) + P_\nu(f)\},$$

where

$$X(z) = \{f(z) - f(t_0)\}^{-\nu} e^{\gamma(z)},$$

t_0 being the initial point of the line L,

$$\gamma(z) = \frac{1}{2\pi i} \int_L \log V(\tau) \frac{f'(\tau)\, d\tau}{f(\tau) - f(z)},$$

and

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{X_+(\tau)} \frac{f'(\tau)\, d\tau}{f(\tau) - f(z)}.$$

$P_\nu(z)$ is an arbitrary polynomial of degree ν in z. If the solution is subject to the additional condition $\Phi(z_0) = 0$, then $P_\nu(z)$ must be replaced by $P_{\nu-1}(z)$, and if $\nu \leq 0$, then we must place $P_{\nu-1}(z) = 0$. Further, if $\nu < 0$, the solubility conditions

$$\int_L \frac{g(\tau)}{X_+(\tau)} [f(\tau)]^{j-1} f'(\tau)\, d\tau = 0, \quad j = 1, 2, \ldots, -\nu,$$

must be satisfied.

The foregoing theory is valid under the assumption that g(t) is Holder continuous. However, Gakhov has shown [5] in the case of the ordinary Riemann-Hilbert problem that the results based on this postulate also hold when g(t) has a finite number of discontinuities of the first kind. In such a case, the solution may have logarithmic singularities at the points of discontinuity ultimately. The reasoning on which this conclusion is based carries over to the Riemann-Hilbert problem for automorphic functions immediately.

3. The solution of the boundary value problem of Section 1. using automorphic functions.

We recall that we require the function $\phi(x,y)$ to be periodic with period 2ℓ and satisfy the conditions (1.2) on $y = 0$. We must also impose the boundedness of $\phi(x,y)$ as $x \to \pm\infty$, and the requirement

$$\lim_{y\to -\infty} \phi(x,y) = 0. \tag{3.1}$$

For definiteness let us choose the domain D bounded by

$$x = \tfrac{1}{2}(a_k + b_k) = -\ell,$$

$$x = \tfrac{1}{2}(a_{k+1} + b_{k+1}) = +\ell,$$

$$y = 0, \quad y = -\infty,$$

in which to discuss the problem, and define

$$F(z) = \phi(x,y) + i\psi(x,y), \quad \operatorname{Im} z \leqslant 0, \tag{3.2}$$

and

$$G(z) = \frac{dF}{dz} = \frac{\partial\phi}{\partial x} - i\,\frac{\partial\phi}{\partial y}, \quad \operatorname{Im} z \leqslant 0. \tag{3.3}$$

We can extend the definition of $G(z)$ to the whole plane, by writing

$$G(z) = \overline{G(\bar{z})}, \qquad \operatorname{Im} z > 0,$$

so that $G(z)$ is sectionally holomorphic, having a line of discontinuity on the real axis, on which we define

$$G_-(x) = \lim_{y\to 0-} G(z),$$

so that $$G_+(x) = \lim_{y\to 0+} G(z) = \lim_{y\to 0+} \overline{G(\bar{z})} = \overline{G_-(x)}\ .$$

Hence

$$G_+(x) - G_-(x) = -\,2i \operatorname{Im} G(x). \tag{3.4}$$

By virtue of (3.3) and (3.4), the boundary conditions on $y = 0$ can therefore be written

$$\left.\begin{aligned} G_+(x) - G_-(x) &= 0, \qquad x \in \sum_k a_k\, b_k\,, \\ G_+(x) - G_-(x) &= 2i\omega^2\phi(x,0)/g, \quad x \in \sum_k b_k\, a_{k+1}. \end{aligned}\right\} \quad (3.5)$$

Now the group of transformations

$$\omega(z) = z + 2\ell, \qquad (3.6)$$

has a fundamental domain R bounded by

$$x = \pm\, \ell, \qquad y = \pm\, \infty,$$

and a fundamental automorphic function

$$f(z) = \exp\{\pi i z/\ell\}, \qquad (3.7)$$

which has a simple pole at $z = z_0 = -i\infty$ (and a zero at $z = \bar{z}_0 = i\infty$). The transformations defined by (3.6) generate the boundary condition on the whole x axis from that prescribed on $-\ell < x < \ell$, and so we can formulate a Riemann-Hilbert problem for automorphic functions to replace the boundary relations (3.5) by

$$\left.\begin{aligned} G_+(x) - G_-(x) &= 0, \quad -\ell < x < -\ell_0,\ \ell_0 < x < \ell, \\ G_+(x) - G_-(x) &= 2i\omega^2\phi(x,0)/g, \quad -\ell_0 < x < \ell_0, \end{aligned}\right\} \quad (3.8)$$

together with the requirement that $G(z)$ be invariant with respect to the group of transformations (3.6), and that

$$G(z) = \overline{G(\bar{z})}. \qquad (3.9)$$

Here we have written $b_k = -\ell_0$ and $a_{k+1} = \ell_0$ to complete the definition of the coordinate system. Since we require the fluid velocities to

vanish at great depth, we must also impose the requirement that $G(z)$ vanish at the pole z_0 of $f(z)$, and hence at all equivalent points.

The boundary condition (3.8) is discontinuous, but by the remarks made at the close of Section2. we can make use of the results of that section, bearing in mind that logarithmic singularities may occur in the solution as a consequence. We see that the solution of the present problem is parallel to (2.7), and is therefore

$$G(z) = \frac{1}{2\pi i}\int_{-\ell_0}^{\ell_0} \frac{2i\omega^2}{g}\phi(\tau,0)\ \frac{f'(\tau)\ d\tau}{f(\tau) - f(z)},$$

and using (3.7)

$$G(z) = \frac{\omega^2}{g\pi}\ \frac{\pi}{2\ell}\int_{-\ell_0}^{\ell_0} \phi(\tau,0)\ \{\cot\frac{\pi}{2\ell}(\tau - z) + i\}\ d\tau,$$

from which condition (3.9) easily gives

$$\int_{-\ell_0}^{\ell_0} \phi(\tau,0)\ d\tau = 0. \tag{3.10}$$

We discuss this equation later.

Now $\phi(x,y) = R\ell\ F(z)$, $\operatorname{Im} z < 0$, and so by (3.3),

$$\phi(x,y) = R\ell\left\{\int^{z} \frac{\omega^2}{g\pi}\ \frac{\pi}{2\ell}\left[\int_{-\ell_0}^{\ell_0} \phi(\tau,0)\ \{\cot\frac{\pi}{2\ell}(\tau - \zeta) + i\}\ d\tau\right] d\zeta\right\} + C,$$

where C is a real constant of integration. Since we can reverse the order of integration

$$\phi(x,y) = R\ell\left\{-\frac{\omega^2}{g\pi}\int_{-\ell_0}^{\ell_0} \phi(\tau,0)\ \{\log 2\sin\frac{\pi}{2\ell}(\tau - z) - \frac{\pi i}{2\ell}\ z\}\ d\tau\right\} + C$$

$$= -\frac{\omega^2}{g\pi}\int_{-\ell_0}^{\ell_0} \phi(\tau,0)\ \{\log 2\left|\sin\frac{\pi}{2\ell}(\tau - z)\right| + \frac{\pi y}{2\ell}\}\ d\tau + C.$$

Let us non-dimensionalise by means of the variable change

$$\frac{\pi z}{2\ell} = \frac{\pi}{2} - \Theta_0, \quad \Theta_0 = \theta_0 + i\eta_0, \qquad \frac{\pi\tau}{2\ell} = \frac{\pi}{2} - \theta,$$

then

$$\phi(\theta_0,\eta_0) = -\lambda\int_{\alpha}^{\pi-\alpha} \phi(\theta,0)\ \{\log 2\left|\sin(\theta - \Theta_0)\right| - \eta_0\}\ d\theta + C, \qquad (3.11)$$

where we have introduced the dimensionless parameters

$$\alpha = \frac{\pi}{2} - \frac{\pi\ell_0}{2\ell}, \qquad \lambda = \frac{2\omega^2\ell}{g\pi^2}, \qquad (3.12)$$

and $0 < \alpha < \pi/2$.

It can be easily shown that

$$\log 2\left|\sin(\theta - \Theta_0)\right| = \eta_0 - \sum_{n=1}^{\infty} \frac{e^{-2n\eta_0}\cos 2n(\theta - \theta_0)}{n}, \qquad (3.13)$$

and so as $\eta_0 \to \infty$ (or $y \to -\infty$), the kernel in (3.11) tends to zero. To satisfy (3.1) we must therefore choose $C = 0$, and

$$\phi(\theta_0,\eta_0) = -\lambda\int_{\alpha}^{\pi-\alpha} \{\log 2\left|\sin(\theta - \Theta_0)\right| - \eta_0\}\ \phi(\theta)\ d\theta, \qquad (3.14)$$

where $\phi(\theta) = \phi(\theta,0)$. Letting $\eta_0 \to 0$ we obtain an integral equation for $\phi(\theta)$,

$$\phi(\theta_0) = -\lambda\int_{\alpha}^{\pi-\alpha} \log 2\left|\sin(\theta - \theta_0)\right|\ \phi(\theta)\ d\theta. \qquad (3.15)$$

The velocity potential for all η_0 can be determined from the solution of (3.15) simply by using (3.14).

Substituting (3.13) for $\eta_0 = 0$ into (3.14), we obtain

$$\phi(\theta_0) = \lambda \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \cos 2n\theta_0 \int_{\alpha}^{\pi-\alpha} \phi(\theta) \cos 2n\theta \, d\theta + \right.$$

$$\left. + \sin 2n\theta_0 \int_{\alpha}^{\pi-\alpha} \phi(\theta) \sin 2n\theta \, d\theta \right\}$$

$$= \sum_{n=1}^{\infty} \frac{C_{2n}}{\sqrt{n}} \cos 2n\theta_0 + \sum_{n=1}^{\infty} \frac{S_{2n}}{\sqrt{n}} \sin 2n\theta_0, \tag{3.16}$$

where

$$C_{2n} = \frac{\lambda}{\sqrt{n}} \int_{\alpha}^{\pi-\alpha} \phi(\theta) \cos 2n\theta \, d\theta, \quad S_{2n} = \frac{\lambda}{\sqrt{n}} \int_{\alpha}^{\pi-\alpha} \phi(\theta) \sin 2n\theta \, d\theta. \tag{3.17}$$

On substituting $\phi(\theta)$ from (3.16) into the two equations (3.17), then since integrals of the type

$$\int_{\alpha}^{\pi-\alpha} \cos 2n\theta \sin 2m\theta \, d\theta$$

are identically zero for all m and n, we arrive at

$$\left.\begin{aligned} C_{2n} &= \frac{\lambda}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{C_{2m}}{\sqrt{m}} \int_{\alpha}^{\pi-\alpha} \cos 2n\theta \cos 2m\theta \, d\theta, \\ S_{2n} &= \frac{\lambda}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{S_{2m}}{\sqrt{m}} \int_{\alpha}^{\pi-\alpha} \sin 2n\theta \sin 2m\theta \, d\theta, \end{aligned}\right\} n = 1, 2, \ldots \tag{3.18}$$

two independent systems of equations for the C_{2n} and S_{2n}.

Now (3.15) represents a homogeneous integral equation with a weakly singular kernel. As was indicated in Chapter IV, section 8., Carleman showed that integral equations having weakly singular, but square integrable, kernels can be regarded as Fredholm equations. It can be verified that

$$\int_{\alpha}^{\pi-\alpha}\int \left|\log 2\left|\sin(\theta - \theta_0)\right|\right|^2 \, d\theta \, d\theta_0 < \infty,$$

hence (3.15) is essentially of the Fredholm variety. Further, it is solvable only for discrete values of λ, which are provided by (3.18) if we impose the necessary condition that these be consistent systems of equations. However since we are presented with two independent sets of equations, then it seems likely that we shall obtain two differing sequences of eigenvalues. This also suggests that the solution of the boundary value problem can be developed as the aggregate of the solutions of two further problems. We shall investigate this remark in the following section.

The condition (3.10) which has arisen out of the formulation can easily be recognised as the counterpart of a well known result, in automorphic functions, for it can be written as

$$\int_{-\ell_0}^{\ell_0} \phi_y(\tau,0)\, d\tau = \int_{-\ell}^{\ell} \phi_y(\tau,0)\, d\tau = 0,$$

using the boundary condition on the interval $-\ell < x < \ell$. Expressed in this way, it requires that the integral of the normal derivative of $\phi(x,y)$ vanishes on $-\ell < x < \ell$, and hence, by the automorphic property, on all equivalent arcs. Therefore the integral of the normal derivative is required to vanish on the whole real axis, a result which is an immediate consequence of Green's theorem for functions harmonic in a domain (see, for example, [6]).

Finally, we observe that other classes of solutions can be obtained in the above manner, if we regard the boundary condition on $y = 0$ as being periodic with period $2k\ell$, for some integer k. For such cases we must define the groups

$$\omega(z) = z + 2k\ell,$$

and the fundamental automorphic functions

$$\exp\{\pi i z/k\ell\}.$$

The solutions will then be periodic with period $2k\ell$. In particular, we can find the class of solutions with period 4ℓ, and it is reasonable to suppose that the appropriate formulation will yield a sub-class of solutions of period 2ℓ in addition, as a special case.

Thus the method evolved above is effective in determining all possible solutions of the boundary value problem we have posed.

4. An alternative method of solving the boundary value problem of Section 1.

Due to the periodic nature of the boundary condition (1.2), then if we are able to solve for the velocity potential in the domain D, defined in Section 3. , we can construct the solution in all identical domains, and resolve the problem in the whole half plane $y < 0$.

In isolating the domain D, there are two possibilities for the boundary condition on $x = \pm \ell$, since a point on these lines may be either an antinode or a node of the standing waves which can exist in D. Mathematically, the corresponding conditions are $\phi_x = 0$ and $\phi = 0$ respectively, and the two resulting potential problems are depicted in Fig. 1. (b) and (c). For ease of reference, we shall denote the problem shown in (b) by Problem A, and that in (c) by Problem B.

The dissecting of the half plane $y < 0$ into an infinity of equivalent domains imposes a restriction on the type of solution we can obtain, since in the limit when the ice floes vanish, the longest standing wave will have length 4ℓ. This approach also establishes an eigenvalue problem, so that in the above limit when $y = 0$ is a totally free surface, the solutions are still subject to the discrete values of a parameter. It is of course of fundamental interest to learn how these eigenvalues change as the ice cover varies.

As we have indicated, the solutions of Problems A and B will

be standing waves and we have still to evolve a method of replacing the time dependence and combining these solutions to obtain progressive wave solutions of the originally posed problem.

We can turn the potential problems into integral equations by establishing appropriate Green's functions $G(z,z_0)$ ($z = x + iy$, $z_0 = x_0 + iy_0$), and applying Green's theorem

$$\phi(x_0,y_0) = \frac{1}{2\pi}\oint_C \left\{ G\,\frac{\partial\phi(x,y)}{\partial n} - \phi(x,y)\,\frac{\partial G}{\partial n} \right\} dc, \qquad (4.1)$$

to D. Here C denotes the boundary of D, and n is the outward normal across C.

Dealing first with Problem A, the boundary conditions are

$$\phi_y = 0, \quad y = 0, \quad -\ell < x < -\ell_0, \ \ell_0 < x < \ell,$$

$$\phi_y = \omega^2\phi/g, \quad y = 0, \quad -\ell_0 < x < \ell_0,$$

$$\phi_x = 0, \quad -\infty < y < 0, \quad x = \pm\ell,$$

$$\lim_{y\to-\infty} \phi_y(x,y) = 0.$$

We can also impose

$$\lim_{y\to-\infty} \phi(x,y) = 0.$$

These requirements (assuming for the moment that $\omega^2\phi(x,0)/g$ is known) define a Neumann problem and lead us to seek a Green's function which is harmonic in D, except at (x_0,y_0) where it is logarithmically singular, and is such that

$$G_y = 0, \quad y = 0, \quad -\ell < x < \ell,$$

$$G_x = 0, \quad -\infty < y < 0, \quad x = \pm \ell,$$

$$G_y \text{ bounded, as } y \to -\infty.$$

The Green's function for the Neumann conditions is well known in the half plane, and by effecting a conformal mapping onto D we find that

$$\begin{aligned} G(z|z_0) &= G(x,y|x_0,y_0) \\ &= \log \left|\sin \frac{\pi z}{2\ell} - \sin \frac{\pi z_0}{2\ell}\right| + \log \left|\sin \frac{\pi z}{2\ell} - \sin \frac{\pi \bar{z}_0}{2\ell}\right| + \\ &\qquad + 2 \log 2c, \end{aligned} \tag{4.2}$$

where c is an arbitrary constant. It is clear that $G(z|z_0)$ can vary by an additive constant without violating the conditions imposed upon it, and so cannot be uniquely defined.

Applying (4.1), we obtain a contribution to the integral only on $y = 0$, $-\ell_0 < x < \ell_0$, and

$$\phi(x_0,y_0) = -\frac{\omega^2}{2\pi g}\int_{-\ell_0}^{\ell_0} G(x,0|x_0,y_0) \;\; \phi(x,0)\, dx. \tag{4.3}$$

We shall subsequently see how the arbitrariness in this equation, resulting from the Green's function, can be resolved, but first we derive similar equations for Problem B, which, on denoting the velocity potential by $\psi(x,y)$ to avoid confusion, is subject to

$$\psi_y = 0, \quad y = 0, \quad -\ell < x < -\ell_0, \quad \ell_0 < x < \ell,$$

$$\psi_y = \omega^2\psi/g, \quad y = 0, \quad -\ell_0 < x < \ell_0,$$

$$\psi = 0, \quad -\infty < y < 0, \quad x = \pm \ell,$$

$$\lim_{y\to-\infty} \psi_y(x,y) = 0.$$

Assuming temporarily that $\omega^2\psi(x,0)/g$ is known, the Green's function for the mixed boundary problem is required to be harmonic in D, apart from a logarithmic singularity at (x_0,y_0), and satisfy

$$H_y = 0, \quad y = 0, -\infty, \quad -\ell < x < \ell,$$

$$H = 0, \quad -\infty < y < 0, \quad x = \pm\ell.$$

$H(z|z_0)$ is most readily found by taking the limit of the similar function in the finite domain bounded by $y = 0, -h, x = \pm\ell$, as $h \to \infty$. This latter function can be determined in the manner suggested in [7], solving the Poisson equation

$$\nabla^2 H = 2\pi\,\delta(x - x_0)\,\delta(y - y_0),$$

$\delta(x)$ being the Dirac function, in terms of a Fourier series. On doing this and taking the limit as indicated, we find that

$$H(x,y|x_0,y_0) = -4\sum_{m=1}^{\infty} \frac{\sin\frac{\pi m}{2\ell}(x+\ell)\,\sin\frac{\pi m}{2\ell}(x_0+\ell)}{m} \times$$

$$\times \begin{cases} \exp\{\frac{\pi m y}{2\ell}\}\cosh\{\frac{\pi m y_0}{2\ell}\}, & y_0 > y \\ \exp\{\frac{\pi m y_0}{2\ell}\}-\cosh\{\frac{\pi m y}{2\ell}\}, & y_0 < y \end{cases} \tag{4.4}$$

which can be summed to give

$$\begin{aligned} H(x,y|x_0,y_0) &= \log\left|\frac{e^{i\gamma z} - e^{i\gamma z_0}}{e^{i\gamma z} + e^{-i\gamma z_0}}\right| + \log\left|\frac{e^{i\gamma z} - e^{i\gamma \bar{z}_0}}{e^{i\gamma z} + e^{-i\gamma \bar{z}_0}}\right|, \quad y_0 > y, \\ &= \log\left|\frac{e^{i\gamma z_0} - e^{i\gamma z}}{e^{i\gamma z_0} + e^{-i\gamma z}}\right| + \log\left|\frac{e^{i\gamma z_0} - e^{i\gamma \bar{z}}}{e^{i\gamma z_0} + e^{-i\gamma \bar{z}}}\right|, \quad y_0 < y, \end{aligned} \tag{4.5}$$

where $\gamma = \pi/2\ell$. On applying Green's theorem, we easily deduce that

$$\psi(x_0,y_0) = -\frac{\omega^2}{2\pi g}\int_{-\ell_0}^{\ell_0} H(x,0|x_0,y_0)\ \psi(x,0)\ dx. \tag{4.6}$$

Performing the variable change consistent with Section 3.,

$$\frac{\pi z}{2\ell} = \frac{\pi}{2} - \Theta = \frac{\pi}{2} - (\theta + i\eta), \quad \frac{\pi z_0}{2\ell} = \frac{\pi}{2} - \Theta_0 = \frac{\pi}{2} - (\theta_0 + i\eta_0),$$

then equations (4.3) and (4.6) become

$$\phi(\theta_0,\eta_0) = -\lambda\int_{\alpha}^{\pi-\alpha} G(\theta,\theta_0,\eta_0)\ \phi(\theta,0)\ d\theta, \tag{4.7}$$

$$\psi(\theta_0,\eta_0) = -\lambda\int_{\alpha}^{\pi-\alpha} H(\theta,\theta_0,\eta_0)\ \psi(\theta,0)\ d\theta, \tag{4.8}$$

where

$$G(\theta,\theta_0,\eta_0) = \log 2c|\cos\Theta_0 - \cos\theta|$$

$$= \eta_0 + \log c - \sum_{m=1}^{\infty}\frac{2\cos m\theta\ \cos m\theta_0\ e^{-m\eta_0}}{m}, \tag{4.9}$$

and

$$H(\theta,\theta_0,\eta_0) = \log\left|\frac{e^{i\Theta_0} - e^{i\theta}}{e^{i\Theta_0} - e^{-i\theta}}\right|$$

$$= -\sum_{m=1}^{\infty}\frac{2\sin m\theta\ \sin m\theta_0\ e^{-m\eta_0}}{m}. \tag{4.10}$$

The parameters λ and α are those defined previously. Letting $\eta_0 \to 0$

in (4.7) and (4.8) we obtain integral equations for $\phi(\theta) = \phi(\theta,0)$ and $\psi(\theta) = \psi(\theta,0)$,

$$\phi(\theta_0) = -\lambda \int_{\alpha}^{\pi-\alpha} \log 2c\left|\cos\theta - \cos\theta_0\right| \phi(\theta)\, d\theta, \qquad (4.11)$$

$$\psi(\theta_0) = -\lambda \int_{\alpha}^{\pi-\alpha} \log\left|\frac{e^{i\theta} - e^{i\theta_0}}{e^{i\theta} - e^{-i\theta_0}}\right| \psi(\theta)\, d\theta. \qquad (4.12)$$

We can also obtain these equations in a different form by taking the limit $y_0 \to 0$ in (4.4) and (4.6), and making the alternative change of variables

$$s = \sin(\pi x/2\ell), \qquad s_0 = \sin(\pi x_0/2\ell),$$

and writing

$$\tau = \sin(\pi\ell_0/2\ell) = \cos\alpha \qquad (4.13)$$

to give

$$\phi(s_0) = -\lambda \int_{-\tau}^{\tau} \log 2c\left|s - s_0\right| \frac{\phi(s)\, ds}{\sqrt{\{1 - s^2\}}}$$

$$\psi(s_0) = -\lambda \int_{-\tau}^{\tau} \log\left|\frac{(1-s)^{1/2}(1+s_0)^{1/2} - (1+s)^{1/2}(1-s_0)^{1/2}}{(1-s)^{1/2}(1+s_0)^{1/2} + (1+s)^{1/2}(1-s_0)^{1/2}}\right| \times \qquad (4.14)$$

$$\times \frac{\psi(s)\, ds}{\sqrt{\{1 - s^2\}}}.$$

Differentiating these with respect to s_0 gives the singular integro-differential equations

$$\left.\begin{aligned} \phi'(s_0) &= \lambda \int_{-T}^{T} \frac{\phi(s)\, ds}{(s - s_0)\sqrt{\{1 - s^2\}}}\,, \\ \sqrt{\{1 - s_0{}^2\}}\, \psi'(s_0) &= \lambda \int_{-T}^{T} \frac{\psi(s)\, ds}{(s - s_0)}\,. \end{aligned}\right\} -T < s_0 < T. \quad (4.15)$$

We see that the integral equations (4.11) and (4.12) both have kernels which are weakly singular and symmetric, and as these can be shown to be square integrable, they can therefore be regarded as Fredholm equations. Further, as the equations are homogeneous, they possess non-trivial solutions only for certain values of the parameter λ. Due to the symmetry of the kernels we can invoke some well known results concerning these eigenvalues and their corresponding eigenvectors which are conveniently stated in [8] (section 12). These indicate that the eigenvalues are all real, and that the eigenfunctions form a mutually orthogonal set. Also, we deduce from a theorem of Pogorzelski [9] (page 132) that the integral equations each admit an infinity of eigenvalues. This follows from the fact that the kernels are expressible as a sum of orthogonal functions over an infinite range as we have seen.

The solution of an integral equation with a weakly singular kernel can be reduced to the problem of solving an integral equation with a regular kernel by the method of iteration, which is elucidated in, for example [9]. Iterating the logarithimic kernels is not an easy

proposition, and an alternative approach would seem to be the use of the methods of Chapter IV to reduce the equations (4.15) to regular integral equations. This has been done, but the resulting solutions of the regular equations were too involved algebraically to be a practical proposition, though theoretically the technique was sound.

It is found that the generalised Riemann-Hilbert equations corresponding to (4.15) offer closed solutions only in the case $\tau = 1$, when the terms containing the factors $\sqrt{\{1 - s_0^2\}}$ clearly fall into the category of functions described in Chapter IV, Section 6. As this corresponds to the case when no ice is present, there is no value in pursuing it, since solutions are readily found directly from (4.11) and (4.12) when $\alpha = 0$.

We are therefore led to deal with the Fredholm equations as they stand.

In Fig. 2. is displayed the variation in ice cover with α and τ; the fraction of the surface covered by ice, ρ, is simply $(\ell_0/\ell + 1)^{-1}$ and hence we have

$$\rho = 2\alpha/\pi = \frac{2}{\pi} \cos^{-1}\tau.$$

We recall that in the previous section a certain condition evolved, equivalent to the well known corollary of the Green theorem that if a function is harmonic in a certain domain, then the integral of its normal derivative over the boundary of this domain is zero. In

the case of the Neumann problem, this theorem manifests itself as a solubility condition, requiring that the prescribed values of the normal derivate on the boundary are such that the corollary holds, and applying it to Problem A, with respect to the domain D, and changing to the dimensionless variable, this solubility condition becomes

$$\int_{\alpha}^{\pi-\alpha} \phi(\theta)\, d\theta = 0. \tag{4.16}$$

Writing (4.7) explicitly,

$$\phi(\theta_0,\eta_0) = \lambda \sum_{m=1}^{\infty} \frac{2 \cos m\theta_0\, e^{-m\eta_0}}{m} \int_{\alpha}^{\pi-\alpha} \phi(\theta) \cos m\theta\, d\theta -$$

$$- \lambda\{\eta_0 + \log c\} \int_{\alpha}^{\pi-\alpha} \phi(\theta)\, d\theta,$$

and so the imposing of (4,16) removes both the arbitrary constant and the term in η_0, which is unbounded as $\eta_0 \to \infty$ (or $y \to -\infty$), violating the condition placed on the solution. Also in the limit $\eta_0 \to 0$, (4.16) eliminates from the solution of the integral equation an arbitrary constant, and we obtain

$$\phi(\theta_0) = \sum_{m=1}^{\infty} C_m \sqrt{\frac{2}{m}} \cos m\theta_0, \tag{4.17}$$

where

$$C_m = \lambda \sqrt{\frac{2}{m}} \int_{\alpha}^{\pi-\alpha} \phi(\theta) \cos m\theta\, d\theta. \tag{4.18}$$

Although we have elected to impose (4.16) at this stage, we must bear in mind that any subsequent solutions will still be expected to satisfy it to ensure their validity.

In a similar manner we find from (4.12) that, on using the expression (4.10) for $\eta_0 = 0$,

$$\psi(\theta_0) = \sum_{m=1}^{\infty} S_m \sqrt{\frac{2}{m}} \sin m\theta_0, \tag{4.19}$$

where

$$S_m = \lambda \sqrt{\frac{2}{m}} \int_{\alpha}^{\pi-\alpha} \phi(\theta) \sin m\theta \, d\theta. \tag{4.20}$$

A similar condition to that above is also relevant to Problem B (and indeed to all harmonic boundary value problems), but it is not necessary to use it explicitly.

By virtue of the above expression for $\phi(\theta_0,\eta_0)$ and an equivalent equation for $\psi(\theta_0,\eta_0)$, we can retrieve the velocity potentials in the whole domain simply by

$$\left.\begin{aligned} \phi(\theta_0,\eta_0) &= \sum_{m=1}^{\infty} C_m \sqrt{\frac{2}{m}} \, e^{-m\eta_0} \cos m\theta_0, \\ \psi(\theta_0,\eta_0) &= \sum_{m=1}^{\infty} S_m \sqrt{\frac{2}{m}} \, e^{-m\eta_0} \sin m\theta_0. \end{aligned}\right\} \tag{4.21}$$

Returning to (4.17) and using it in (4.18), we obtain

$$C_m = \lambda \sum_{n=1}^{\infty} \frac{2}{\sqrt{mn}} C_n \, c_{mn},$$

or

$$\sum_{n=1}^{\infty} C_n \left\{ \frac{2\lambda c_{mn}}{\sqrt{mn}} - \delta_{mn} \right\} = 0, \quad m = 1, 2, \ldots. \tag{4.22}$$

where

$$c_{mn} = c_{nm} = \int_{\alpha}^{\pi-\alpha} \cos m\theta \cos n\theta \, d\theta$$

$$= -\frac{\sin(m+n)\alpha}{m+n} - \frac{\sin(m-n)\alpha}{m-n}, \qquad m, n \text{ both even or both odd, } m \neq n,$$

$$= \tfrac{1}{2}(\pi - 2\alpha) - \frac{\sin 2n\alpha}{2n}, \qquad m = n,$$

$$= 0, \qquad \text{otherwise.}$$

The introduction of the square roots in (4.17) is now seen to result in the symmetry of the matrix of coefficients in (4.22), which is advantageous. The vanishing of the c_{mn} when m is odd and n is even and vice versa, implies that the system of equations (4.22) is equivalent to a set of equations for the C_{2n} and an independent system for the C_{2n-1}. Defining

$$c_{omn} = \frac{2c_{2m-1,2n-1}}{(2m-1)^{1/2}(2n-1)^{1/2}}, \quad c_{emn} = \frac{2c_{2m,2n}}{2m^{1/2}n^{1/2}},$$

we can write

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} C_{2n-1}\,\{\mathrm{co}_{mn} - \mu\delta_{mn}\} = 0, \qquad \text{(a)} \\ &\sum_{n=1}^{\infty} C_{2n}\,\{\mathrm{ce}_{mn} - \mu\delta_{mn}\} = 0, \qquad \text{(b)} \end{aligned}\right\} \quad m = 1, 2, \ldots. \tag{4.23}$$

where $\mu = 1/\lambda$. The eigenvalue problem is posed by

$$\left.\begin{aligned} &|\mathrm{co}_{mn} - \mu\delta_{mn}| = 0, \qquad \text{(a)} \\ &|\mathrm{ce}_{mn} - \mu\delta_{mn}| = 0, \qquad \text{(b)} \end{aligned}\right\} \tag{4.24}$$

the first of which gives the μ_{2n-1} $(n = 1, 2, \ldots.)$ and requires that the C_{2n} be placed equal to zero, and the second provides the μ_{2n} $(n = 1, 2, \ldots.)$, giving only the trivial solution for all the C_{2n-1}. Hence the eigensystem of Problem A is

$$\left.\begin{aligned} &\mu_{2n-1}, \qquad \phi_{2n-1}(\theta_0) = \sqrt{2}\sum_{m=1}^{\infty} \frac{C_{2m-1}}{(2m-1)^{1/2}} \cos(2m-1)\theta_0, \\ &\mu_{2n}, \qquad \phi_{2n}(\theta_0) = \sum_{m=1}^{\infty} \frac{C_{2m}}{\sqrt{m}} \cos 2m\theta_0, \end{aligned}\right\} \tag{4.25}$$

Similarly, on writing in Problem B,

$$\mathrm{so}_{mn} = \frac{2s_{2m-1,\,2n-1}}{(2m-1)^{1/2}(2n-1)^{1/2}}, \quad \mathrm{se}_{mn} = \frac{2s_{2m,2n}}{2m^{1/2}n^{1/2}},$$

we obtain

$$\sum_{n=1}^{\infty} S_{2n-1}\,\{\mathrm{so}_{mn} - \kappa\delta_{mn}\} = 0, \quad m = 1, 2, \ldots. \quad \text{(a)} \tag{4.26}$$

and

$$\sum_{n=1}^{\infty} S_{2n} \{se_{mn} - \kappa\delta_{mn}\} = 0, \; m = 1, 2, \ldots \qquad \text{(b)} \qquad (4.26)$$

where we have now written $\kappa = 1/\lambda$ for clarity, and the eigensystem is

$$\left.\begin{aligned} &\kappa_{2n-1}, \quad \psi_{2n-1}(\theta_0) = \sqrt{2} \sum_{m=1}^{\infty} \frac{S_{2m-1}}{(2m-1)^{1/2}} \sin(2m-1)\theta_0, && \text{(a)} \\ &\kappa_{2n}, \quad \psi_{2n}(\theta_0) = \sum_{m=1}^{\infty} \frac{S_{2m}}{\sqrt{m}} \sin 2m\theta_0, && \text{(b)} \end{aligned}\right\} \qquad (4.27)$$

where the κ_n are provided by setting equal to zero the determinants of coefficients in (4.26) (a) and (b).

We are now in a position to compare the present solutions with the formulation of Section 3. It is at once clear from (3.16) - (3.18) that the approach via the Riemann-Hilbert equation provides the eigenvalues λ_{2n} and κ_{2n} and the corresponding eigenfunctions $\phi_{2n}(\theta_0)$ and $\psi_{2n}(\theta_0)$, the whole solution (3.16) being simply the sum of these two. Since the functions $\phi_{2n-1}(\theta_0)$ and $\psi_{2n-1}(\theta_0)$ have period 4ℓ in the initial notation, these will result from further use of the automorphic function method suggested at the close of Section 3., with k = 2.

The leading terms of the matrices of coefficients are shown in Fig. 3. For ease of reference we identify the matrices of (4.23) (a) and (b) and (4.26) (a) and (b) by AO, AE, BO, and BE respectively.

5. Perturbation solutions for small floes.

It is not difficult to see that, in the limit $\alpha = 0$, corresponding to no ice, the normalised solutions of Problems A and B are

$$\left.\begin{aligned} \mu_n &= \pi/n, \quad \phi_n(\theta_0) = \sqrt{(2/\pi)}\ \cos n\theta_0, \\ \kappa_n &= \pi/n, \quad \psi_n(\theta_0) = \sqrt{(2/\pi)}\ \sin n\theta_0. \end{aligned}\right\} \quad n = 1, 2, \ldots. \tag{5.1}$$

These follow at once from (4.17) - (4.20) since the two sequences $\cos n\theta$ and $\sin n\theta$ are orthogonal in $(0,\pi)$, In view of (5.1) we choose to replace the time dependence by

$$\begin{aligned} \Phi_n(\theta_0,\eta_0,t) &= \cos \omega_n t\ \phi_n(\theta_0,\eta_0) + \sin \omega_n t\ \psi_n(\theta_0,\eta_0) \\ &= \sqrt{(2/\pi)}\ e^{-n\eta_0} \cos(n\theta_0 - \omega_n t), \end{aligned} \tag{5.2}$$

where

$$\omega_n^2 = \frac{g\pi^2}{2\ell\mu_n} = \frac{g\pi n}{2\ell} \tag{5.3}$$

give the allowable frequencies, for a fixed value of ℓ. When α is different from zero, the construction of a progressive wave may be more involved, since μ_n and κ_n may not coincide, as they do in this simple case.

To investigate the case when α is small, let us first turn to the secular equations (4.24), and the corresponding equations for the κ parameter. The tridiagonal determinant of order n defined by

$$\Delta_n = \begin{vmatrix} f_1 & a_1 & & & 0 \\ a_1 & f_2 & a_2 & & \\ & a_2 & \cdot & & \\ & & & \cdot & a_{n-1} \\ & 0 & & a_{n-1} & f_n \end{vmatrix} \tag{5.4}$$

can be shown, on expanding by minors, to satisfy the recurrence relation

$$\left.\begin{aligned} \Delta_n &= f_n\,\Delta_{n-1} - a_{n-1}^2\,\Delta_{n-2}, \quad n \geqslant 2, \\ \Delta_0 &= 1. \end{aligned}\right\} \qquad (5.5)$$

Further, if we introduce the next adjacent sub- and superdiagonal elements b_n in (5.4) to form a 5-diagonal array, we can show that the additional terms resulting in (5.5) are of order b_n^4 and $a_n^2 b_n^2$. Observing that the off-diagonal elements in our present determinantal equations are of $O(\alpha)$, ($\{\sin k\alpha/k\} \leqslant \alpha$, all k,α), we can immediately determine the eigenvalues to $O(\alpha)$, since we have

$$\Delta_n = f_n\,\Delta_{n-1} + O(\alpha^2).$$

Hence

$$\Delta_\infty = \prod_{n=1}^{\infty} f_n + O(\alpha^2),$$

and the solution of $\Delta_\infty = 0$ is simply $f_n = 0$, for all n. Therefore

$$\mu_{2n-1} = \mathrm{co}_n, \qquad \mu_{2n} = \mathrm{ce}_n,$$

$$\kappa_{2n-1} = \mathrm{so}_n, \qquad \kappa_{2n} = \mathrm{se}_n,$$

or

$$\left.\begin{aligned} \mu_n &= \frac{1}{n}\left\{\pi - 2\alpha - \frac{2\sin 2n\alpha}{2n}\right\} + O(\alpha^2) \\ \kappa_n &= \frac{1}{n}\left\{\pi - 2\alpha + \frac{2\sin 2n\alpha}{2n}\right\} + O(\alpha^2) \end{aligned}\right\} \; n = 1, 2, \ldots \qquad (5.6)$$

It is not so straightforward to determine the corresponding eigenvectors to the same order, but we are able to develop a perturbation technique to achieve this, by returning to the integral equations. Since the eigenfunctions are known for $\alpha = 0$, their determination as

α moves from zero can be regarded as a classical perturbation of boundary problem. In view of (5.1) let us seek perturbed solutions of the form

$$\left.\begin{aligned} \phi_n(\theta_0) &= \sqrt{(2/\pi)}\cos n\theta_0 + \alpha\phi_n^{(1)}(\theta_0) + O(\alpha^2), \\ \mu_n^{-1} &= n/\pi + \alpha\mu_n^{(1)} + O(\alpha^2), \end{aligned}\right\} \qquad (5.7)$$

$$\left.\begin{aligned} \psi_n(\theta_0) &= \sqrt{(2/\pi)}\sin n\theta_0 + \alpha\psi_n^{(1)}(\theta_0) + O(\alpha^2), \\ \kappa_n^{-1} &= n/\pi + \alpha\kappa_n^{(1)} + O(\alpha^2). \end{aligned}\right\} \qquad (5.8)$$

It is convenient at this stage to state the solution of the non-homogeneous Fredholm equation

$$\Omega(x_0) - \lambda\int_a^b K(x,x_0)\,\Omega(x)\,dx = f(x_0), \qquad (5.9)$$

which, in the case when $K(x,x_0)$ is symmetric, can be found explicitly using the well known Hilbert-Schmidt theorem. Let the normalised eigen-solutions of the homogeneous equation corresponding to (5.9) be $\Omega_1(x)$, $\Omega_2(x)$, ... , λ_1, λ_2, ... Then, following Mikhlin [8] (section19), the solution of (5.9) can be written

$$\Omega(x_0) = f(x_0) + \lambda\sum_{n=1}^{\infty}\frac{f_n}{\lambda_n - \lambda}\,\Omega_n(x_0), \qquad (5.10)$$

in which

$$f_n = \int_a^b f(x)\,\overline{\Omega_n(x)}\,dx.$$

If λ coincides with an eigenvalue λ_m of the kernel, then we require $f_m = 0$. In other words, (5.9) is soluble for $\lambda = \lambda_m$ if and only if the free term $f(x)$ is orthogonal to the conjugate of the corresponding

eigenfunction $\phi_m(x)$. If this is satisfied then the indeterminate coefficient of $\phi_m(x)$ in the series (5.10) can be replaced by an arbitrary constant.

Returning to the perturbation problem, let us deal first with the integral equation (4.11) which can be written, on extracting the constant c as indicated,

$$\phi_n(\theta_0) = -\frac{1}{\mu_n}\int_0^{\pi} \log 2|\cos\theta - \cos\theta_0|\ \phi_n(\theta)\ d\theta + \frac{1}{\mu_n} f_n(\theta_0), \quad (5.11)$$

where

$$f_n(\theta_0) = \int_0^{\alpha} \{\log 2|\cos\theta - \cos\theta_0|\ \phi_n(\theta) + \\ + \log 2|\cos\theta + \cos\theta_0|\ \phi_n(\pi - \theta)\}\ d\theta$$

$$= \int_0^{\alpha} \{\ \log 2|\cos\theta - \cos\theta_0|\ [\sqrt{(2/\pi)}\ \cos n\theta + \alpha\phi_n^{(1)}(\theta)] + \\ + \log 2|\cos\theta + \cos\theta_0|\ [\sqrt{(2/\pi)}\ (-1)^n \cos n\theta + \alpha\phi_n^{(1)}(\pi - \theta)]\}d\theta$$

$$= -\frac{\sqrt{2}}{\sqrt{\pi}}\left\{\sum_{m=1}^{\infty}{}' [1 + (-1)^{m+n}]\ \frac{\cos m\theta_0}{m}\left[\frac{\sin(m+n)\alpha}{m+n} + \frac{\sin(m-n)\alpha}{m-n}\right]\right. \\ \left. + \frac{2}{n}\cos n\theta_0\left[\alpha + \frac{\sin 2n\alpha}{2n}\right]\right\} + \alpha\ R_n(\theta_0), \qquad (5.12)$$

on using the series form of the kernel, where

$$R_n(\theta_0) = \int_0^{\alpha} \{\log 2|\cos\theta - \cos\theta_0|\ \phi_n^{(1)}(\theta) + \log 2|\cos\theta + \cos\theta_0|\ \phi_n^{(1)}(\pi - \\ - \theta)\}\ d\theta.$$

The accent associated with the summation denotes that the term $m = n$ is omitted. The leading term in (5.12) is essentially of order α, and since we anticipate $\phi_n^{(1)}(\theta)$ to be a linear combination of cosines, we conclude that $R_n(\theta)$ is also $O(\alpha)$, so that the final term in (5.12) is $O(\alpha^2)$ and can be neglected in the present approximation. Substituting the trial solution (5.7) into (5.11) and equating the terms in α, we obtain

$$\phi_n^{(1)}(\theta_0) = -\frac{n}{\pi}\int_0^{\pi} \log 2|\cos\theta - \cos\theta_0|\, \phi_n^{(1)}(\theta)\, d\theta + g_n(\theta_0), \qquad (5.13)$$

in which

$$g_n(\theta_0) = \sqrt{\frac{2}{\pi}}\left\{ \frac{\pi}{n}\mu_n^{(1)} \cos n\theta_0 - \frac{n}{\pi}\sum_{m=1}^{\infty}{}' [1 + (-1)^{m+n}]\, \frac{\cos m\theta_0}{m}\left[\frac{\sin(m+n)\alpha}{m+n}\right.\right.$$

$$\left.\left. + \frac{\sin(m-n)\alpha}{m-n}\right] - \frac{2}{\pi}\left[1 + \frac{\sin 2n\alpha}{2n}\right]\cos n\theta_0 \right\}. \qquad (5.14)$$

In order to solve (5.13) let us consider

$$\zeta(\theta_0) = \lambda\int_0^{\pi} \log 2|\cos\theta - \cos\theta_0|\, \zeta(\theta)\, d\theta + G(\theta_0), \qquad (5.15)$$

the homogeneous part of which has the known solutions

$$\zeta_m(\theta_0) = \sqrt{(2/\pi)}\cos m\theta_0, \qquad \lambda_m = -m/\pi, \qquad m \geqslant 1.$$

Therefore, by the theory quoted above, (5.15) is satisfied by

$$\zeta(\theta_0) = G(\theta_0) + \lambda\sum_{k=1}^{\infty} \frac{G_k}{-k/\pi - \lambda}\, \zeta_m(\theta_0),$$

where

$$G_k = \int_0^{\pi} G(\theta)\, \overline{\zeta_k(\theta)}\, d\theta.$$

Comparing (5.13) with (5.15), the former has solution

$$\phi_n^{(1)}(\theta_0) = g_n(\theta_0) + \frac{n}{\pi} \sum_{k=1}^{\infty}{}' \frac{g_{kn}}{k/\pi - n/\pi} \sqrt{\frac{2}{\pi}} \cos k\theta_0 + \sqrt{\frac{2}{\pi}}\, C_n \cos n\theta_0, \tag{5.16}$$

in which

$$g_{kn} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} g_n(\theta) \cos k\theta\, d\theta,$$

provided that g_{nn} vanishes.

We have replaced the coefficient of $\sqrt{(2/\pi)} \cos n\theta_0$ by C_n, an arbitrary constant. It is easily seen that

$$g_{nn} = \left\{ \frac{\pi}{n} \mu_n^{(1)} - \frac{2}{\pi}\left[1 + \frac{\sin 2n\alpha}{2n\alpha}\right] \right\},$$

and the requirement that it be zero yields the eigenvalue perturbation

$$\mu_n^{(1)} = \frac{2n}{\pi^2}\left[1 + \frac{\sin 2n\alpha}{2n\alpha}\right]. \tag{5.17}$$

We also find that, for $k \neq n$,

$$g_{kn} = -\frac{n}{\pi} \frac{[1 + (-1)^{k+n}]}{k} \left\{ \frac{\sin(k+n)\alpha}{k+n} + \frac{\sin(k-n)\alpha}{k-n} \right\},$$

which, on substituting into (5.16), and rearranging the summations, gives

$$\phi_n^{(1)}(\theta_o) = \sqrt{\frac{2}{\pi}}\left\{C_n \cos n\theta_o - \frac{n}{\pi}\sum_{m=1}^{\infty}{}' \frac{[1 + (-1)^{m+n}]}{m - n}\left[\frac{\sin(m + n)\alpha}{(m + n)\alpha} + \right.\right.$$

$$\left.\left. + \frac{\sin(m - n)\alpha}{(m - n)\alpha}\right] \cos m\theta_o \right\}. \qquad (5.18)$$

We can utilise an exactly similar procedure in the case of Problem B, and the integral equation (4.12), and setting the perturbed solutions as indicated in (5.8), we find that

$$\kappa_n^{(1)} = \frac{2n}{\pi^2}\left[1 - \frac{\sin 2n\alpha}{2n\alpha}\right], \qquad (5.19)$$

and

$$\psi_n^{(1)}(\theta_o) = \sqrt{\frac{2}{\pi}}\left\{S_n \sin n\theta_o - \frac{n}{\pi}\sum_{m=1}^{\infty}{}' \frac{[1 + (-1)^{m+n}]}{m - n}\left[\frac{\sin(m - n)\alpha}{(m - n)\alpha} - \right.\right.$$

$$\left.\left. - \frac{\sin(m + n)\alpha}{(m + n)\alpha}\right] \sin m\theta_o \right\}, \qquad (5.20)$$

in which S_n is arbitrary. Hence we have

$$\frac{1}{\mu_n} = \frac{n}{\pi} + \frac{2n\alpha}{\pi^2}\left\{1 + \frac{\sin 2n\alpha}{2n\alpha}\right\} + O(\alpha^2),$$

giving

$$\left.\begin{aligned} \mu_n &= \frac{\pi}{n}\left\{1 - \frac{2\alpha}{\pi}\left[1 + \frac{\sin 2n\alpha}{2n\alpha}\right]\right\} + O(\alpha^2), \\ &\text{while} \\ \kappa_n &= \frac{\pi}{n}\left\{1 - \frac{2\alpha}{\pi}\left[1 - \frac{\sin 2n\alpha}{2n\alpha}\right]\right\} + O(\alpha^2). \end{aligned}\right\} \qquad (5.21)$$

These expressions coincide, of course, with (5.6), which were deduced directly from the secular equations.

Comparing (5.18) with (5.20), we observe that the terms cos $m\theta_0$ in $\phi_n^{(1)}(\theta_0)$, other than cos $n\theta_0$, are proportional to

$$\frac{\sin(m + n)\alpha}{m + n} + \frac{\sin(m - n)\alpha}{m - n},$$

while the sin $m\theta_0$ in $\psi_n^{(1)}(\theta_0)$, apart from sin $n\theta_0$, are proportional to

$$\frac{\sin(m - n)\alpha}{m - n} - \frac{\sin(m + n)\alpha}{m + n},$$

Hence the coefficients of the perturbation terms in $\phi_n(\theta_0)$ are always larger than those in $\psi_n(\theta_0)$, both tending to $\sin(m - n)\alpha/(m - n)$, as n increases.

We also remark that, due to the factor $(m - n)^{-1}$ in $\phi_n^{(1)}(\theta_0)$, the magnitude of the coefficient of cos $m\theta_0$ decreases as m moves away from n in either direction. There will be also a change of sign of the coefficients as m moves through the value n, though it is possible for some n and α that the circular functions of α, which will produce a change of sign regularly in the coefficients, may disguise this phenomenon. An exactly similar argument applies in the case of $\psi_n^{(1)}(\theta_0)$.

It is also clear from (5.21) that the eigenvalues μ_n are more greatly perturbed than the κ_n, for initial values of n, the κ_n hardly varying from their total free surface value of π/n. Hence μ_n and κ_n

differ for the smaller n, and tend to the common asymptotic value

$$\frac{\pi}{n}\{1 - 2\alpha/\pi\} + O(n^2) \tag{5.22}$$

with increasing n.

Let us try to construct a progressive wave by combining the two solutions as we did in the free surface case. By virtue of (5.7) and (5.18), (5.8) and (5.19), let us write the eigenfunctions in the general form

$$\phi_n(\theta_o) = \sum_m (X_m + Y_m) \cos m\theta_o,$$

$$\psi_n(\theta_o) = \sum_m (X_m - Y_m) \sin m\theta_o,$$

omitting the n dependence in the coefficients for simplicity. As we stated above, the X_m are the dominant terms, and the Y_m tend to zero with increasing n. For a fixed ℓ, the allowable frequencies of the two problems are

$$\omega_n^2 = \frac{g\pi^2}{2\ell\mu_n}, \qquad \nu_n^2 = \frac{g\pi^2}{2\ell\kappa_n},$$

and by reference to (5.21), we can write, to order α,

$$\omega_n = \Omega_n + \epsilon_n, \qquad \nu_n = \Omega_n - \epsilon_n,$$

where

$$\Omega_n = \sqrt{\frac{g\pi^2}{2\ell}\frac{n}{\pi}}\left\{1 + \frac{2\alpha}{\pi}\right\}, \qquad \epsilon_n = \sqrt{\frac{g\pi^2}{2\ell}\frac{n}{\pi}}\ \frac{\sin 2n\alpha}{\pi n},$$

ϵ_n tending to zero as n becomes large.

Now

$$\Phi_n(\theta_o,0,t), = \cos \omega_n t\, \phi_n(\theta_o) + \sin \nu_n t\, \psi_n(\theta_o)$$

$$= \sum_m (X_m + Y_m) \cos \omega_n t \cos m\theta_o + \sum_m (X_m - Y_m) \sin \nu_n t \sin m\theta_o,$$

$$= \sum_m \{X_m \cos \epsilon_n t \cos(m\theta_o - \Omega_n t) + Y_m \sin \epsilon_n t \sin(m\theta_o - \Omega_n t)\}$$

$$- \sum_m \{X_m \sin \epsilon_n t \sin(m\theta_o + \Omega_n t) - Y_m \cos \epsilon_n t \cos(m\theta_o + \Omega_n t)\} \tag{5.23}$$

on rearranging the terms. This equation represents two sets of waves travelling in opposite directions, and having time dependent amplitudes, giving a complicated group effect. Thus the construction of a simple progressing wave is no longer possible by this means, the combined velocity potential being an involved superposition of waves, in addition to a wave group travelling in the opposite direction. It is difficult to interpret this latter effect and we suggest that it represents a net reflection from the floes, which is always present. As n becomes large, then in view of the limits mentioned above

$$\Phi_n(\theta_o,0,t) \to \sum_m X_m \cos(m\theta_o - \Omega_n t),$$

that is, the higher frequency waves tend to a simpler wave superposition having no reflected part, and no time dependence in its amplitude.

We are thus led to enquire whether these conclusions are peculiar to the case when α is small, or whether they can be generalised for all α. We must turn to a numerical investigation to answer this question.

6. Numerical determination of the eigensystems.

To study the eigensystems for arbitrary α we must utilise some approximation to replace the infinite sets of equations by finite systems in the leading coefficients. We discuss such a step and its validity later, and for the moment assess the problem of finding the characteristic values and functions of a symmetric matrix A_1 of order n.

There are two widely used methods of approaching this classical eigenvalue problem numerically. The first is due to Jacobi, and consists of reducing the matrix A_1 to a similar diagonal matrix, by means of pre- and post-multiplication with selected orthogonal matrices. The second is to find a tridiagonal matrix satisfying a similarity relationship with A_1 from which, by virtue of the recurrence formula (5.5) and Sturm's theorem (see, for example [10]), the eigenvalues can be found. The method we elect to use in the present case is essentially a combination of these two techniques, in the sense that we reduce A_1 to a tridiagonal matrix, for which we in turn derive a similar diagonal array, the roots of which are simply given by the diagonal elements. The reason for this amalgamation of techniques is purely one of economy: the Jacobi method is an "n^3" process, while the Givens reduction to tridiagonal form involves $\frac{1}{2}(n-1)(n-2)$ matrix multiplications. These two devices are discussed in [11]. We make use of Householder's method to achieve tridiagonalisation, and it is due to the fact that it employs only (n-2) orthogonal transformations that this is thought superior to

the other possibilities.

To elucidate the Householder technique, let us define matrices A_i by the recurrence relation

$$A_{i+1} = P_i \, A_i \, P_i, \qquad i = 1, 2, \ldots, (n-2),$$

in which P_i are the orthogonal matrices

$$P_i = I - 2\omega_i \, \omega_i^T,$$

$$\omega_i^T = [\omega_{i,1}, \omega_{i,2}, \ldots, \omega_{i,n-i}, 0, 0, \ldots, 0]$$

$$\omega_i^T \omega_i = 1,$$

for some $\omega_{i,k}$. I is the identity matrix of order n and A_1 the initial symmetric matrix.

Let us assume that A_i is tridiagonal in its last (i-1) rows and columns. Then since P_i will have zero elements in its last i rows and columns, apart from those on the diagonal which are unity, the operation $P_i \, A_i \, P_i$ will leave the last (i-1) rows and columns of A_i unaltered, and we can choose the $\omega_{i,1}, \omega_{i,2}, \ldots, \omega_{i,n-i}$ so that the elements of A_{i+1} in the positions

$$(n-i+1,1), (n-i+1,2), \ldots, (n-i+1,n-i-1),$$

are zero. Hence since all the A_i will be symmetric, A_{i+1} becomes tridiagonal in its last i rows and columns. Thus (n-2) such operations reduce A_1 to the tridiagonal matrix A_{n-1} given by

$$A_{n-1} = P_{n-2} \, P_{n-1} \ldots P_1 \, A_1 \, P_1 \ldots P_{n-1} \, P_{n-2},$$

so that A_{n-1} and A_1 are similar and have coincident eigenvalues. If X is an eigenfunction of A_{n-1}, then the corresponding eigenfunction of

A_1 is easily found by

$$P_1 P_2 \ldots P_{n-2} X.$$

Writing $\ell = n-i+1$, then if the elements of A_i are denoted by $a_{j,k}$ and we define

$$\sigma_i = a_{\ell,1}^2 + a_{\ell,2}^2 + \ldots + a_{\ell,\ell-1}^2 ,$$

and

$$H_i = \sigma_i \pm a_{\ell,\ell-1} \sigma_i^{1/2},$$

it can be shown that the choice of $\omega_{i,k}$ which has the desired effect of eliminating the appropriate elements of A_{i+1} is

$$\omega_{i,\ell-1} = \{a_{\ell,\ell-1} \pm \sigma_i^{1/2}\}/(2H_i)^{1/2},$$

$$\omega_{i,j} = a_{\ell,j}/(2H_i)^{1/2}, \quad j = 1, 2, \ldots (\ell-2),$$

in which the sign in $\omega_{i,\ell-1}$ is ultimately chosen to give numerical stability.

Procedures for utilising this reduction computationally have been put forward by Wilkinson [13], whose notation we have used above, and these have subsequently been refined by Martin et al. [14]. These procedures are valid for arbitrary symmetric matrices with real elements, and we elect to use for our present purpose the variant identified by Martin as TRED 2. In practise it is convenient to tridiagonalise only the lower triangular array and preserve the original elements of the matrix in the remaining upper triangle, or vice versa. This is done in TRED 2, which also provides the array H, being the orthogonal matrix formed by the product of the Householder transformation matrices,

$$H = P_1 P_2 \ldots . P_{n-2},$$

so that the net transformation is

$$A_{n-1} = H' A_1 H.$$

We must now discuss the determination of the eigenvalues of A_{n-1}, which we achieve using the algorithm known as TQL 2, developed by Bowdler et al. [15]. It is natural that we should adopt this course, since the common factors in the author-ship of TRED 2 and TQL 2 have obviously led to the two programs being used conveniently in conjunction if desired. TQL 2 is a version, for use with tridiagonal arrays, of the so-called QL method, which is based on the general result that if

$$B = QL, \qquad C = LQ,$$

where Q is unitary ($Q^{-1} = Q^{+} = \bar{Q}'$) and L is lower triangular, then

$$C = LQ = Q^{+} B\, Q.$$

That is, B and C are conjunctive or unitarily similar, and thus have coincident eigenvalues. Starting from a given matrix B_1 the matrices B_ℓ can be successively constructed by the recurrence relations

$$B_\ell = Q_\ell L_\ell, \qquad B_{\ell+1} = L_\ell Q_\ell = Q_\ell^{+} B_\ell Q_\ell,$$

and in general B_ℓ tends to lower triangular form. In particular, if B_1 is real and symmetric, then the Q_ℓ are simply orthogonal transformations and the B_ℓ are all real and symmetric, and in general tend to diagonal form.

Following Bowdler, the recurrence relations for use on a tridiagonal form are more conveniently written as

$$Q_\ell(B_\ell - k_\ell I) = L_\ell, \qquad B_{\ell+1} = L_\ell\, Q_\ell',$$

so that B_ℓ is similar to $B_1 - \Sigma k_i I$ instead of B_1. The k_ℓ are subtracted from the diagonal elements of B_ℓ to accelerate the convergence of the process, for if k_ℓ is chosen to be close to an eigenvalue, then clearly the off-diagonal elements in the row associated with this eigenvalue will decrease sharply in magnitude. In practice, if $d_r^{(\ell)}$ are the diagonal elements, and $e_r^{(\ell)}$ the off-diagonal elements, at the ℓth iteration, k_ℓ is taken to be that root of

$$\begin{vmatrix} d_1^{(\ell)} & e_1^{(\ell)} \\ e_1^{(\ell)} & d_2^{(\ell)} \end{vmatrix} = 0$$

which is closer to $d_1^{(\ell)}$. Bowdler has found this to be an effective means of determining the shift k_ℓ, with regard to speed of convergence. Also if and $e_r^{(\ell)}$ at the ℓth iteration is effectively zero, the matrix can be partitioned into the product of two smaller tridiagonal matrices, and as this results in considerable economy, it is performed in TQL 2.

Finally we remark that the procedure TQL 2 is designed to find the eigenfunctions of the original matrix, input to TRED 2, directly, without the intermediate stage of locating the proper functions of the tridiagonal matrix. This is achieved from a knowledge of the Householder transformation matrix H which is input to TQL 2.

Having briefly outlined the theory on which the procedures TRED 2 and TQL 2 are based, we shall not discuss in detail the precise routines of these programs and their numerical properties. This would

merely be duplication of the papers cited above, and as we use the Algol programs in the exact forms of Martin and Bowdler, there is no variation from these to be recorded.

We now return to the problem in hand, having the four matrices AO, AE, BO, and BE to deal with. As these are all of a similar structure we can adopt the same procedure for each one. For definiteness, let us discuss AO, the array $(com_{mn} - \mu\delta_{mn})$.

We observe that the com_{mn} diminish with increasing m and n, due both to the factor $(mn)^{1/2}$ and the coefficients of the sinusoidal terms in α. Although this decay is not particularly fast, let us consider the effect of curtailing AO by ignoring all its off-diagonal terms other than those in the first N rows and columns. The secular equation for the μ_{2n-1} can then be expressed as the product of an N X N symmetric determinant, which provides the first N eigenvalues, and an infinite diagonal determinant, which gives the remainder of the eigenvalue sequence. This is, amalgamating similar results for all the arrays,

$$\left.\begin{aligned} \mu_n &= \{\pi - 2\alpha - \frac{2 \sin 2n\alpha}{2n}\}/n, \\ \kappa_n &= \{\pi - 2\alpha + \frac{2 \sin 2n\alpha}{2n}\}/n, \end{aligned}\right\} n = N+1, \ldots. \quad (6.1)$$

or,

$$\mu_n = \kappa_n = \{\pi - 2\alpha\} + O(n^{-2}), \quad n = N+1, \ldots. \quad (6.2)$$

To ensure that we are investigating the appropriate end of the eigen-

value spectrum, we have from (3.11), replecing λ by $1/\mu$,

$$\mu_n = \frac{g\pi^2 T_n^2}{2\ell}. \tag{6.3}$$

That is, μ_n is proportional to the square of the allowable time period when ℓ is fixed. We are principally interested in the longer period waves, and therefore require a knowledge of the larger eigenvalues, and as these are associated with the leading terms of A0, they are obtained from the Nth order array and should therefore be the most accurate.

The abridging of the matrix A0 in the above manner will provide a valid means of determining the characteristic values when the off-diagonal terms are small compared with the dominant influence on the eigenvalues, namely the terms $\{\pi - 2\alpha\}/n$ in the diagonal elements. As α increases and these latter terms approach the same order of magnitude as the off-diagonal terms, we can expect some loss of accuracy. The eigenvalues μ_n as $n \to N$ will be unreliable for all α, since the terms off the diagonal which influence these quantities are increasingly ignored. However, for moderate values of α we anticipate that the eigenvalue sequences will tend to the values (6.1) with increasing n, and any departure from this trend will be discernable and due to a perturbation enforced by the approximation.

The crucial test of the validity of this method is a comparison between the eigenvalues obtained for two differing values of N, and presently we take, somewhat arbitrarily, N = 20, and N = 40. Those eigenvalues which remain unaltered to a certain accuracy as N is

increased from 20 to 40 can be regarded as the exact eigenvalues of A0 to this accuracy. Further, by observing the tendency of the eigenvalues to approach the sequences (6.1) and any variation in this tendency as N changes, we can deduce the remaining eigenvalues to a good approximation.

Turning to the eigenfunctions $\phi_{2n-1}(\theta_0)$ of A0, we note that we have effectively set equal to zero the terms in the system of equations (4.22) (a), other than

$$\left.\begin{aligned} \sum_{n=1}^{\infty} C_{2n-1}\ \{\mathrm{com}n - \mu\delta mn\} &= 0, \quad m = 1, 2, \ldots. N \\ C_{2n-1}\ \{\mathrm{con}n - \mu\} &= 0, \quad n = N+1, N+2, \ldots. \end{aligned}\right\} \quad (6.4)$$

From the perturbation solution of Section 5., we expect the dominant term in $\phi_{2n-1}(\theta_0)$ to be $\cos(2n - 1)\theta_0$, and the coefficients of the other terms, $\cos(2m - 1)\theta_0$, to decrease in magnitude as m moves away from n. It follows that, due to the falling off of the com$n$ and the anticipated decay of the C_{2n-1}, we shall obtain reasonable approximations to the first few eigenfunctions $\phi_1(\theta_0)$, $\phi_3(\theta_0)$, ... but that the later proper functions, for which we are ignoring terms of larger magnitude in (6.4), may not be so accurate. Again, if the coefficients remain the same to a certain accuracy as N changes from 20 to 40, then we conclude that these give the eigenfunctions of A0 to this accuracy. We remark that, by what was said in the solution of the Riemann-Hilbert problem, and from a physical point of view, the eigenfunctions may contain logarithmic singularities at $\theta_0 = \alpha$ and $\theta_0 = \pi-\alpha$. Such singularities

will be concealed in the coefficients and we shall not attempt to extract them, though it is possible that their form can be deduced from the perturbation solutions of Section 5.

It is clear that, since the eigenvalues of the two problems have a common asymptotic behaviour and the elements of AO and BO, co_{mn} and so_{mn} are almost identical for large m and n, then the eigenfunctions $\phi_{2n-1}(\theta_0)$ and $\psi_{2n-1}(\theta_0)$ will be mutually asymptotic also. Similarly for $\phi_{2n}(\theta_0)$ and $\psi_{2n}(\theta_0)$.

We finally remark that the end of the spectrum corresponding to great ice cover, for which, as we have seen, the above method of solution is likely to become increasingly inaccurate, can be readily investigated by means of the integral equations in the forms (4.14). We see by reference to Fig. 2. that the appropriate ice cover corresponds to τ small.

An immediately apparent method of proceeding in the case of $\phi(s)$ is to map the interval of integration onto $(0,\pi)$ by means of $s = \tau\cos\omega$, $s_0 = \tau\cos\omega_0$. By expanding the term $(1 - \tau^2\cos^2\omega)^{-1/2}$ in powers of τ^2 we arrive at a similar, but more complex, system of equations to that investigated above, for which the same type of numerical procedure is applicable. However we do not pursue this line of thought further in the present discussion.

7. Numerical results and conclusion.

We evaluate the eigensystems, and also the diagonal elements, of the four matrices for the two values of N stated above.

Returning to the condition (4.16) imposed on the Neumann problem (A), we see, using (4.25), that it is identically satisfied by the $\phi_{2n-1}(\theta_0)$, but is only satisfied by the $\phi_{2n}(\theta_0)$ provided that

$$\sum_{m=1}^{\infty} \frac{C_{2m}}{\sqrt{m}} \sin 2m\alpha = 0.$$

We therefore evaluate, for each of the functions $\phi_{2n}(\theta_0)$, the expression

$$e_{2n} = \sum_{m=1}^{N} \frac{C_{2m}}{\sqrt{m}} \sin 2m\alpha , \qquad (7.1)$$

and find that it is non-zero to our prescribed accuracy. We conclude that the $\phi_{2n}(\theta_0)$ are not valid solutions of the problem and, moreover, that no solution exists having a wavelength 2ℓ. This is probably due to the fact that the orthogonal set $\cos 2\theta_0$, $\cos 4\theta_0$, ... in terms of which $\phi_{2n}(\theta_0)$ is expressed, is incomplete, the constant term being inadmissable as we saw earlier.

In fact it is found that

$$e_{2n} < \alpha, \quad \text{all } n, \text{ all } \alpha,$$

and that e_{2n} decreases in absolute magnitude with increasing n, so that the later eigenfunctions can be thought of as solutions of the problem, valid to a certain accuracy.

We apply the procedures of Section 6. in turn for $\alpha = \pi/36$, $\pi/18$, $\pi/9$, $\pi/6$, $\pi/4$, $\pi/3$ and $5\pi/12$. The final values are not expected to yield such accurate results as the smaller α, but we tentatively try them. In each case we compare the eigenvalues provided by the N = 20 and N = 40 arrays, and establish the trend of these towards the diagonal elements. In this way we are able to provide a sufficiently accurate picture of the eigenvalues for all the α examined, with the exception of $\alpha = \pi/3$ and $\alpha = 5\pi/12$, for which only the leading characteristic roots can be established.

The eigenvalues are depicted graphically in Figs. 4 - 9, two different scales being chosen for clarity. For this reason we also draw curves through the values, though they are of course, discrete points. We notice at once that the μ_{2n} curves are distorted somewhat for the initial values of n, compared with the other graphs, but soon tend to the same sort of regular pattern exhibited by the other eigenvalues. This is presumably a consequence of the invalidity of the $\phi_{2n}(\theta_0)$ discussed above.

The conclusions we drew from the perturbation expressions (5.21) are seen to be substantiated by the graphs: the μ_n visibly depart from their free surface values as soon as α moves from zero; the κ_n are virtually coincident with their free surface values when $\alpha = \pi/36$ (and are for this reason not shown explicitly), and are barely distinct from them when $\alpha = \pi/18$.

Restricting our remarks to the admissable values κ_{2n-1} and μ_{2n-1}, we observe that $\kappa_{2n-1} > \mu_{2n-1}$ for the initial values of n, and all α, while the two sets of eigenvalues have common asymptotes with increasing n.

We have therefore established that the difference in the initial eigenvalues of Problems A and B and the mutually asymptotic behaviour of these eigenvalues as n increases, which was found for small α, is true for the whole α spectrum.

As regards the eigenfunctions $\phi_{2n-1}(\theta_0)$ and $\psi_{2n-1}(\theta_0)$, display is more difficult and we restrict it to the leading vectors given by n = 1 and n = 2, discussing the general trend of the functions. The dominant coefficients of $\phi_1(\theta_0)$, $\psi_1(\theta_0)$, $\phi_3(\theta_0)$ and $\psi_3(\theta_0)$ are shown in Tables 1 and 2, being given by

$$c_{2m-1} = C_{2m-1}\sqrt{\{2/(2m-1)\}}, \qquad s_{2m-1} = S_{2m-1}\sqrt{\{2/(2m-1)\}},$$

by virtue of the assumed forms of the solutions.

Again, qualitative conclusions of the perturbation method follow through to all values of α: the coefficients in $\phi_{2n-1}(\theta_0)$ are greater in absolute magnitude than those in $\psi_{2n-1}(\theta_0)$; these coefficients undergo a sign change as m passes through the value n, and oscillate, changing sign with a regularity which is more marked as α increases.

The decrease in magnitude of c_{2m-1} and s_{2m-1} as m moves away from n becomes less marked as α increases, and for larger values of ice cover, c_{2n-1} and s_{2n-1} cease to be the dominant terms in $\phi_{2n-1}(\theta_0)$

and $\psi_{2n-1}(\theta_0)$. This is first observed for n = 2, and is more apparent as n increases. It is noted that the two sets of coefficients tend towards common values as n increases, confirming the deduction we made above.

Thus the combining of the solutions is a complicated procedure, and the conclusions at the close of Section 5. would seem to hold for the whole range of ice cover. It is of course out of the question to investigate the possibility of several eigenfunctions superimposing in some particular way to form a simple progressing wave..

Qualitatively, we may summarise some of the above results:

(i) Corresponding to each standing wave of an allowable period in the total free surface case, there is, in the presence of the floes, a more complex standing wave of smaller period, this period decreasing as the floe size increases.

(ii) As the ice cover increases, the dominance of cos $n\theta_0$ and sin $n\theta_0$ in $\phi_n(\theta_0)$ and $\psi_n(\theta_0)$ for the leading values of n diminishes, the eigenfunctions departing more and more from their total free surface states. A possible, but unproven, consequence of this is that a combination of the type (5.23) will more closely resemble a simple progressive wave, with a lesser reflected part, when α is small.

(iii) The higher frequency waves combine in a simpler way than those with longer periods, being a superposition of progressive waves

and having a small reflected component, as we saw for α small and which has been shown to follow for all α. These waves are such that their energies are dissipated by various mechanisms which we have not accounted for here, but it is interesting to note that they are permissible solutions of our present mathematical model.

The present formulation has dealt only with a restricted class of solutions, having periods 4ℓ, and it would be interesting to compare the results of a general class of solution with those obtained here. However we do not follow this immediate extension further at this stage.

This problem was conceived with a view to obtaining possible progressive wave solutions, for use in a transmission-reflection investigation in the situation of a semi-infinite free surface and a semi-infinite regular array of rigid floes. This latter part has not proved feasible, due to the complexity of obtaining the basic wave solutions, which has highlighted the difficulty of solving the more realistic mathematical interpretations of the wave - ice interaction.

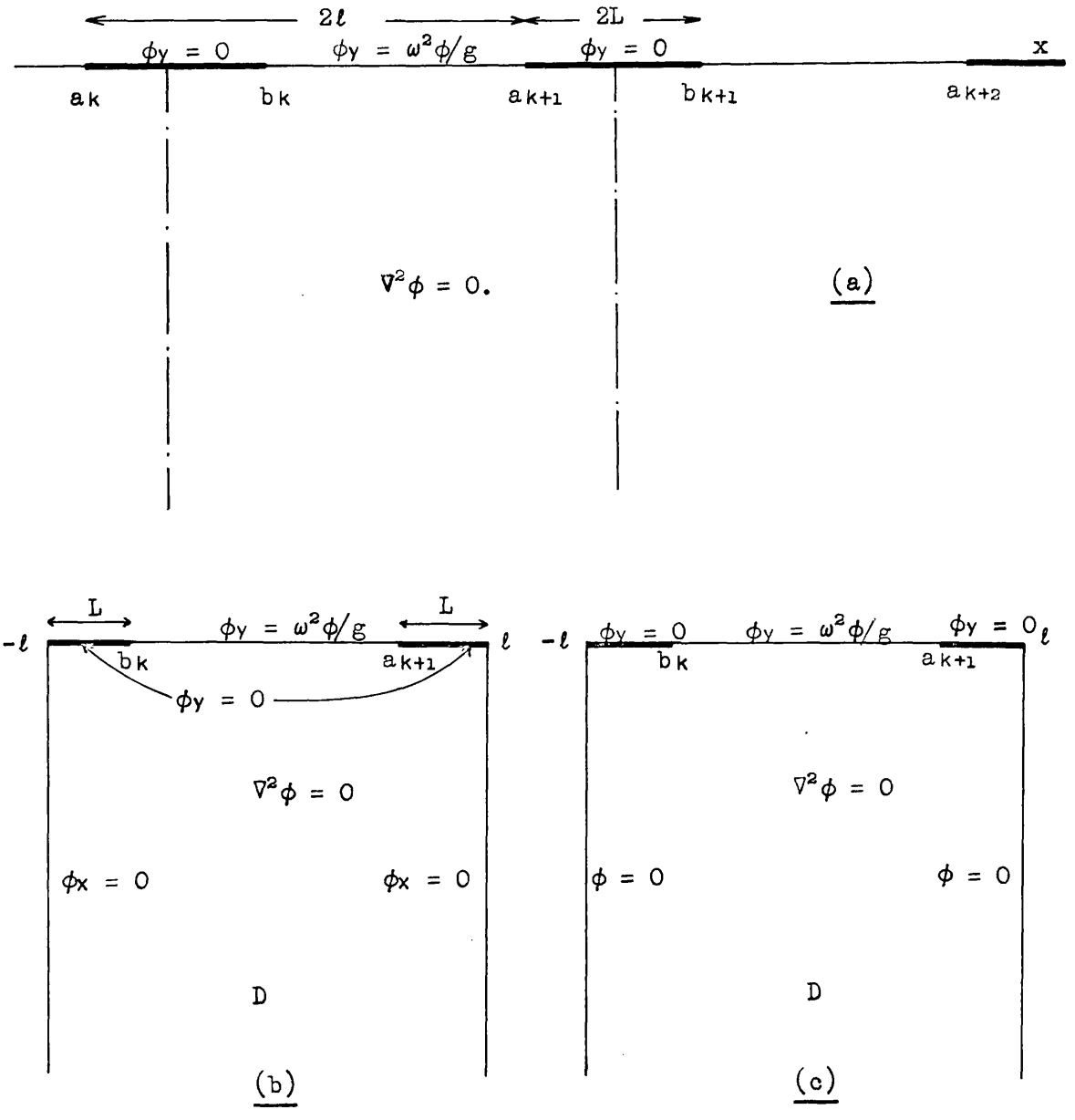
2ℓ
2L
φy = 0
φy = ω²φ/g
φy = 0
x
ak
bk
ak+1
bk+1
ak+2
∇²φ = 0.
(a)
L
L
-ℓ
ℓ
φy = ω²φ/g
bk
ak+1
φy = 0
∇²φ = 0
φx = 0
φx = 0
D
(b)
-ℓ
ℓ
φy = 0
φy = ω²φ/g
φy = 0
bk
ak+1
∇²φ = 0
φ = 0
φ = 0
D
(c)

FIG. 1.

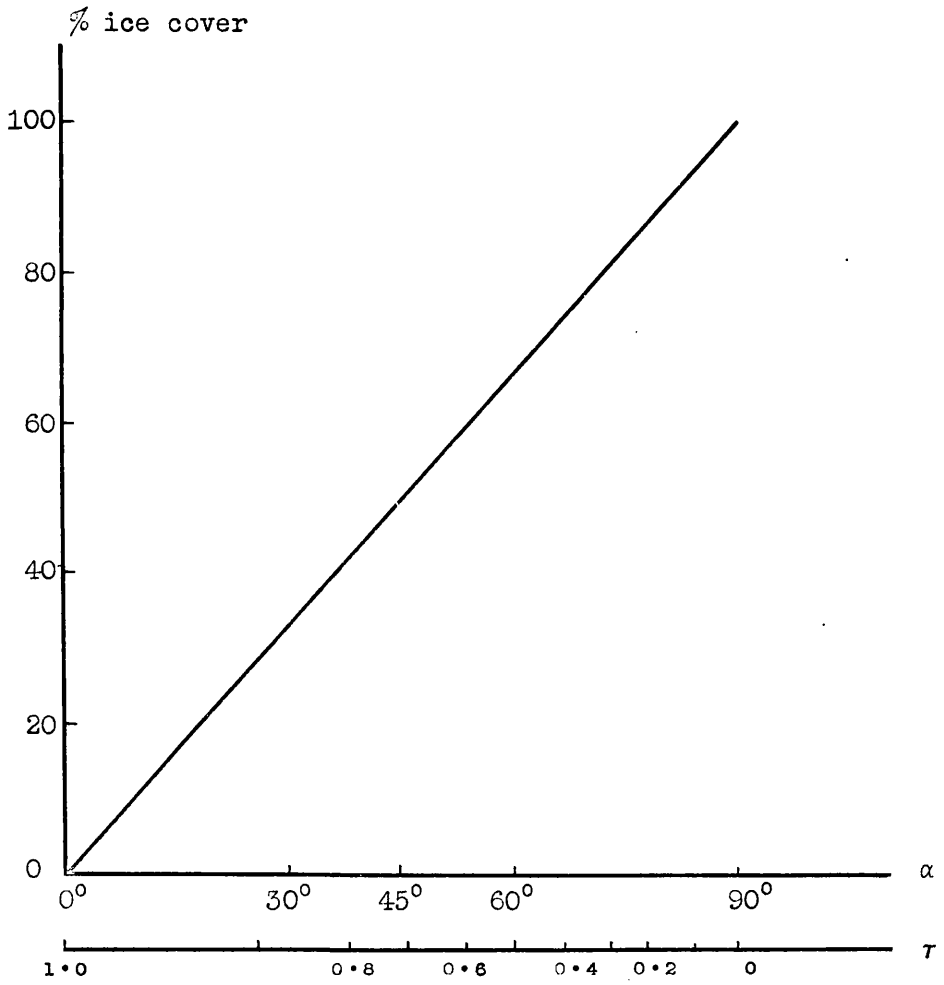

FIG. 2 : Graph of ice cover vs. α and τ.

$$
\begin{bmatrix}
2\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 2\alpha}{2}\right\} & \frac{2}{\sqrt{3}}\left\{\pm \frac{\sin 4\alpha}{4} - \frac{\sin 2\alpha}{2}\right\} & \frac{2}{\sqrt{5}}\left\{\pm \frac{\sin 6\alpha}{6} - \frac{\sin 4\alpha}{4}\right\} & \frac{2}{\sqrt{7}}\left\{\pm \frac{\sin 8\alpha}{8} - \frac{\sin 6\alpha}{6}\right\} \\
 & \frac{2}{3}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 6\alpha}{6}\right\} & \frac{2}{\sqrt{15}}\left\{\pm \frac{\sin 8\alpha}{8} - \frac{\sin 2\alpha}{2}\right\} & \frac{2}{\sqrt{21}}\left\{\pm \frac{\sin 10\alpha}{10} - \frac{\sin 4\alpha}{4}\right\} \\
 & & \frac{2}{5}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 10\alpha}{10}\right\} & \frac{2}{\sqrt{35}}\left\{\pm \frac{\sin 12\alpha}{12} - \frac{\sin 2\alpha}{2}\right\} \\
 & & & \frac{2}{7}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 14\alpha}{14}\right\}
\end{bmatrix}
$$

\- sign gives AO elements

\+ sign gives BO elements

$$
\begin{bmatrix}
\frac{2}{2}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 4\alpha}{4}\right\} & \frac{2}{\sqrt{8}}\left\{\pm \frac{\sin 6\alpha}{6} - \frac{\sin 2\alpha}{2}\right\} & \frac{2}{\sqrt{12}}\left\{\pm \frac{\sin 8\alpha}{8} - \frac{\sin 4\alpha}{4}\right\} & \frac{2}{\sqrt{16}}\left\{\pm \frac{\sin 10\alpha}{10} - \frac{\sin 6\alpha}{6}\right\} \\
 & \frac{2}{4}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 8\alpha}{8}\right\} & \frac{2}{\sqrt{24}}\left\{\pm \frac{\sin 10\alpha}{10} - \frac{\sin 2\alpha}{2}\right\} & \frac{2}{\sqrt{32}}\left\{\pm \frac{\sin 12\alpha}{12} - \frac{\sin 4\alpha}{4}\right\} \\
 & & \frac{2}{6}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 12\alpha}{12}\right\} & \frac{2}{\sqrt{48}}\left\{\pm \frac{\sin 14\alpha}{14} - \frac{\sin 2\alpha}{2}\right\} \\
 & & & \frac{2}{8}\left\{\frac{(\pi-2\alpha)}{2} \pm \frac{\sin 16\alpha}{16}\right\}
\end{bmatrix}
$$

\- sign gives AE elements

\+ sign gives BE elements

FIG. 3 : The leading terms of the symmetric matrices AO, BO, AE and BE. The diagonal terms $-\mu$, $-\kappa$ are omitted.

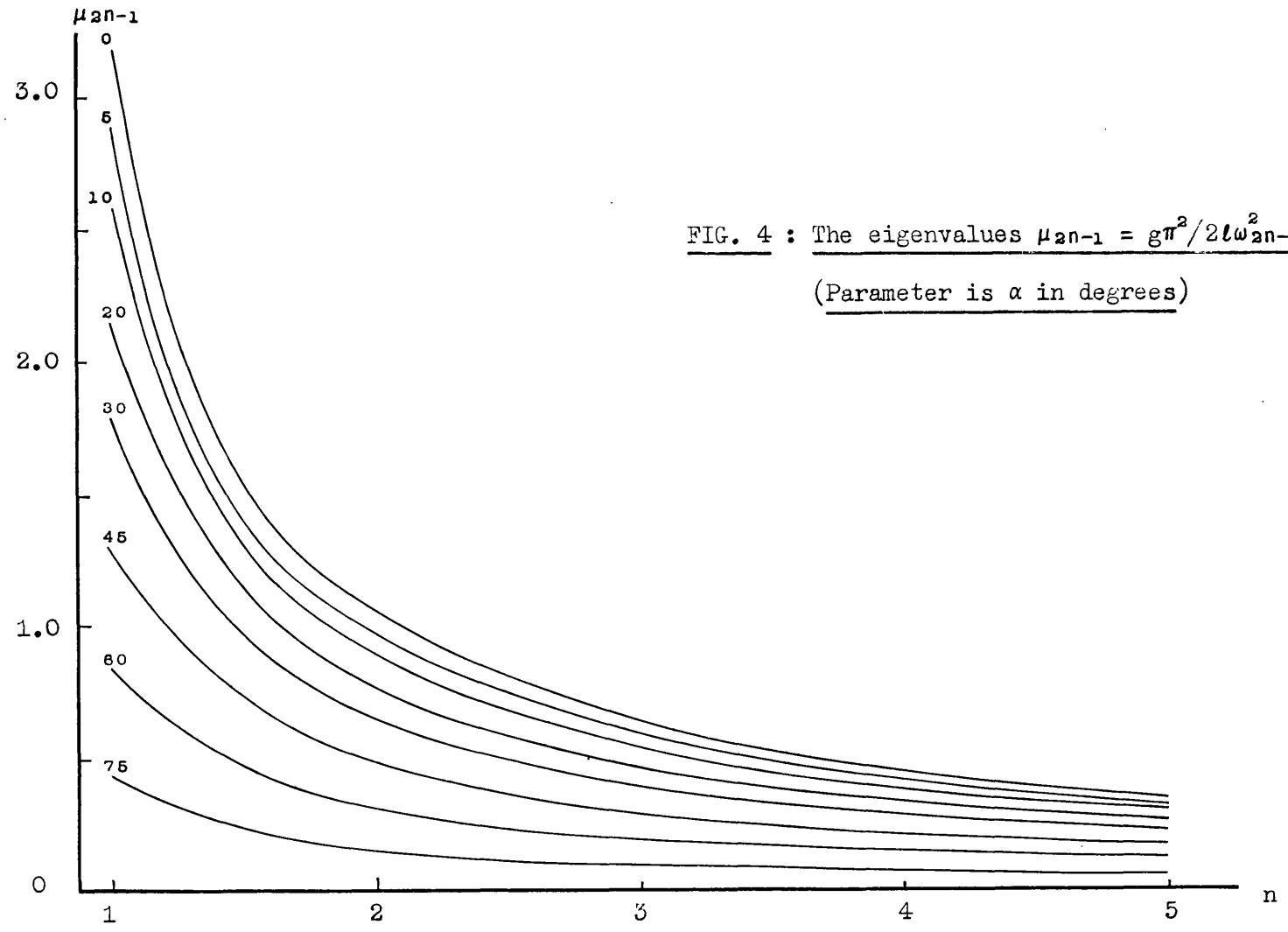

FIG. 4 : The eigenvalues $\mu_{2n-1} = g\pi^2/2\ell\omega^2_{2n-1}$.

(Parameter is α in degrees)

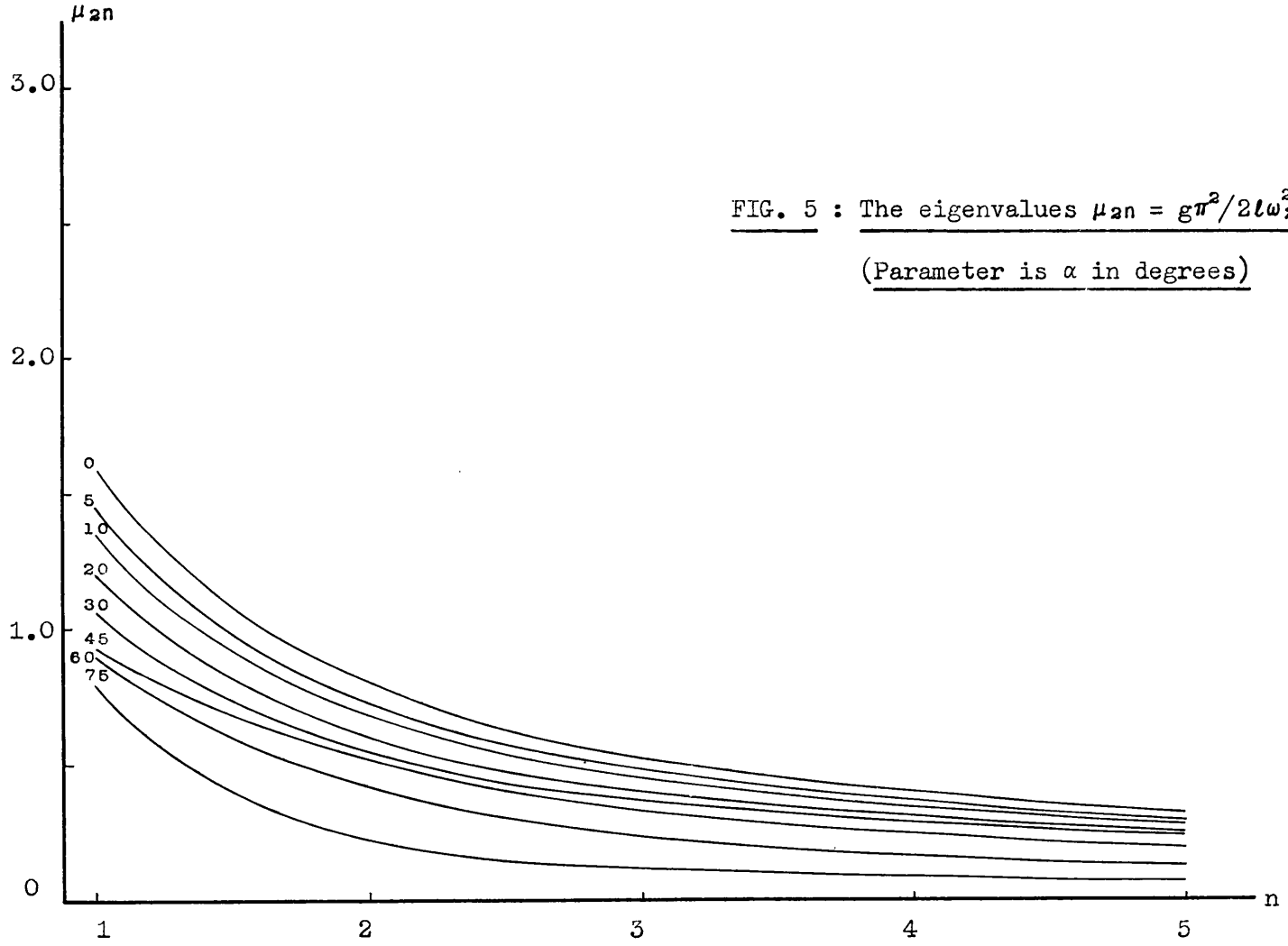

FIG. 5 : The eigenvalues $\mu_{2n} = g\pi^2/2\ell\omega_{2n}^2$.

(Parameter is α in degrees)

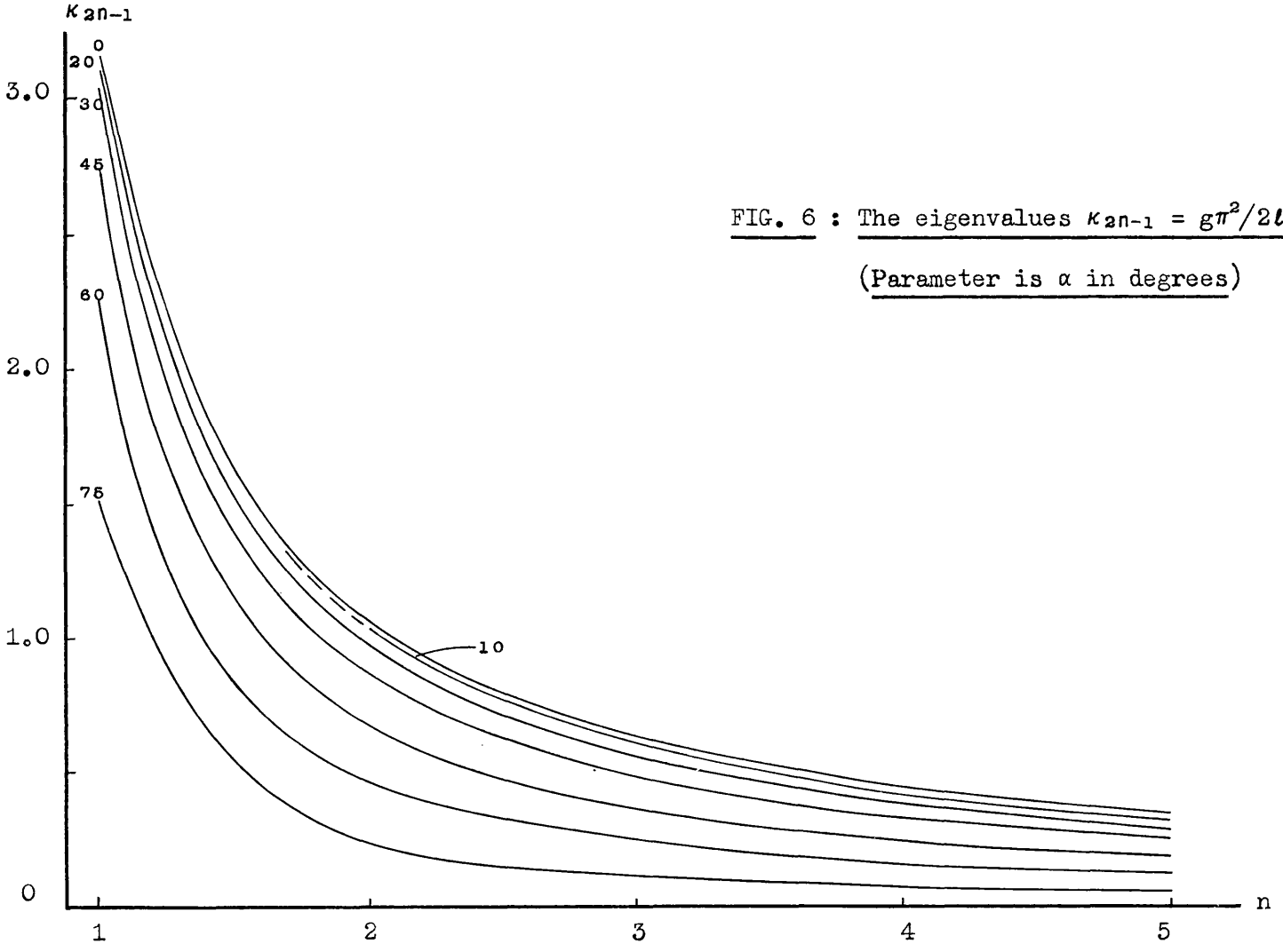

FIG. 6 : The eigenvalues $\kappa_{2n-1} = g\pi^2/2\ell\nu_{2n-1}^2$

(Parameter is α in degrees)

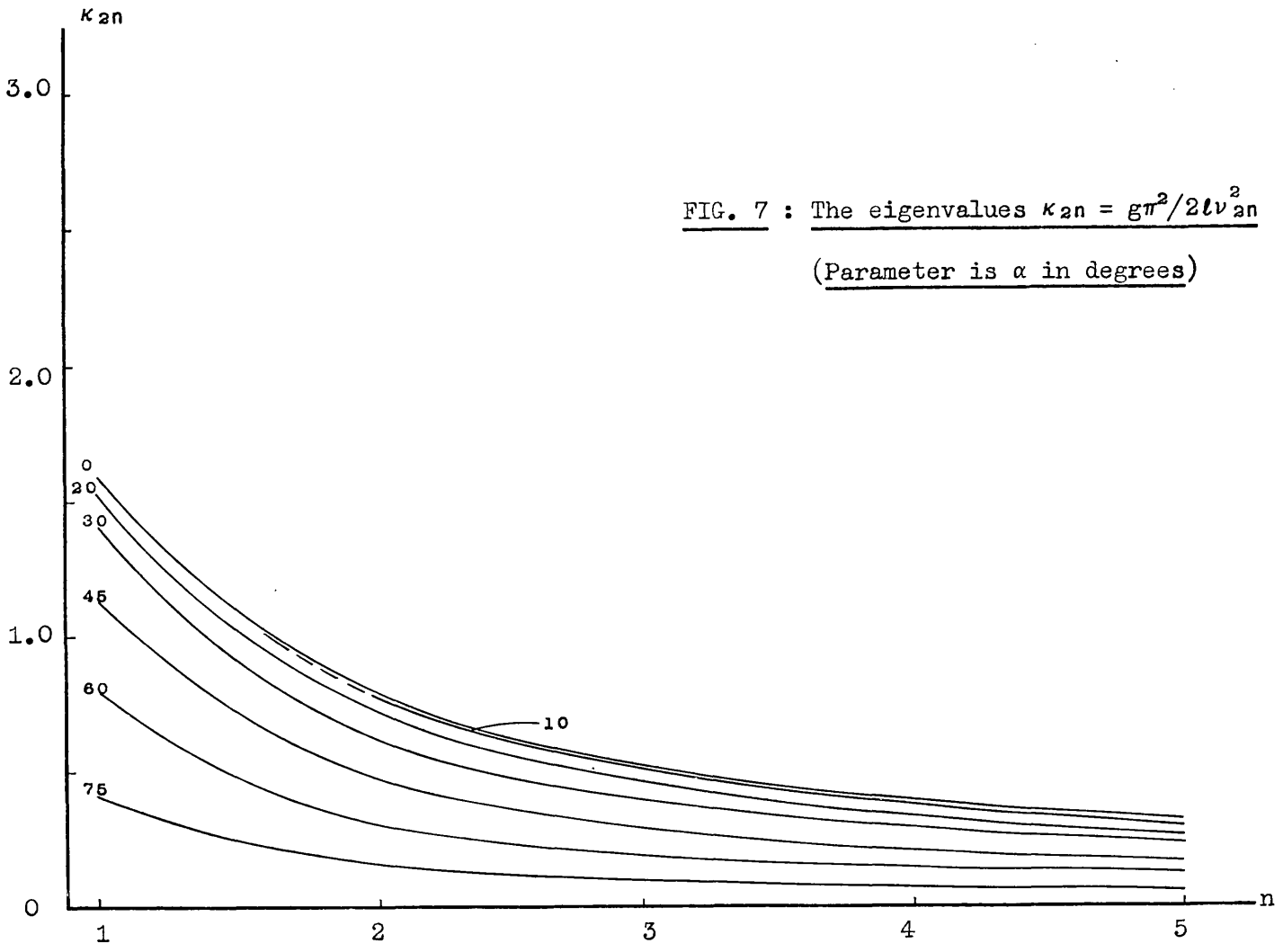

FIG. 7 : The eigenvalues $\kappa_{2n} = g\pi^2/2\ell\nu_{2n}^2$

(Parameter is α in degrees)

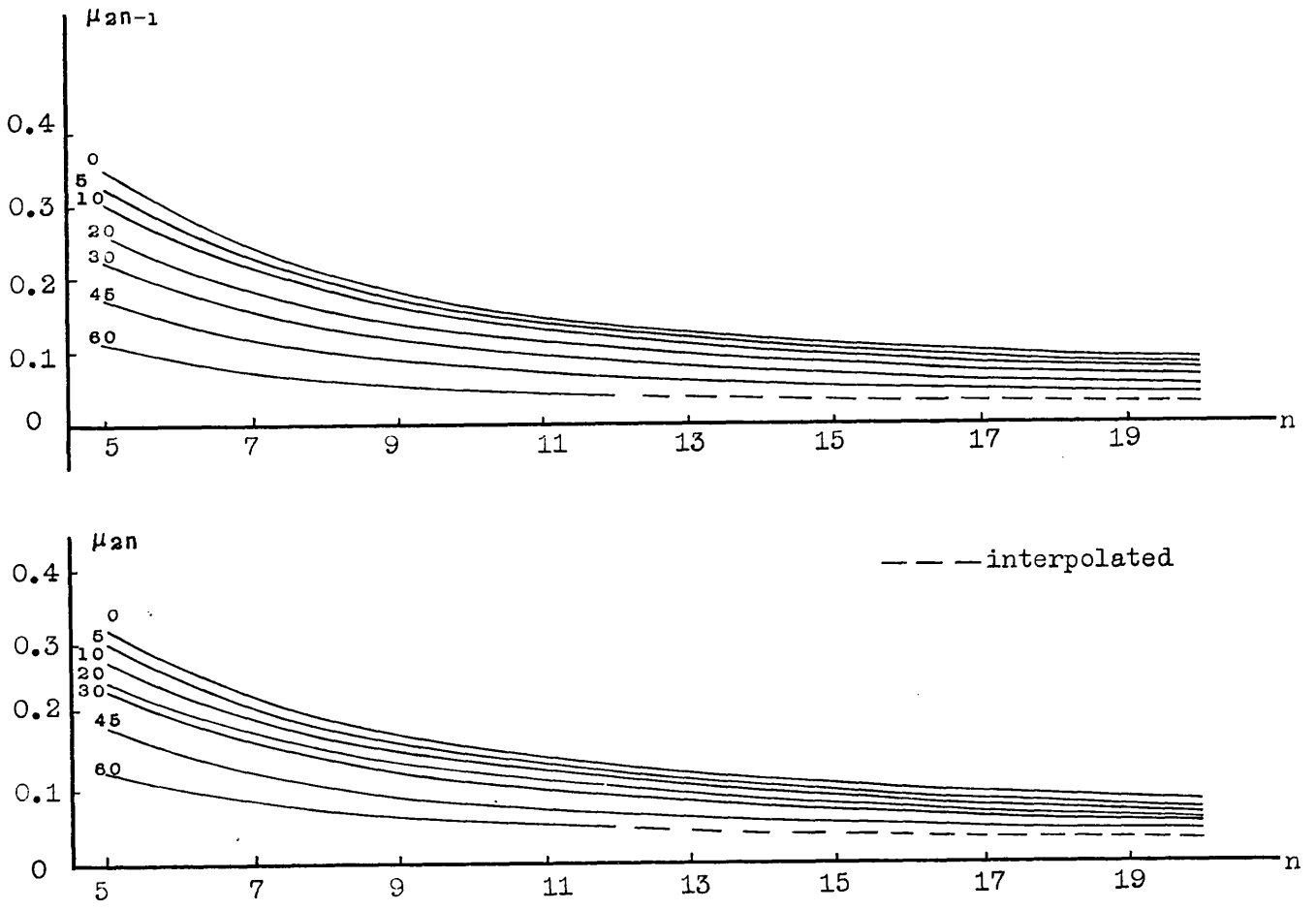

FIG. 8 : The eigenvalues μ_n.

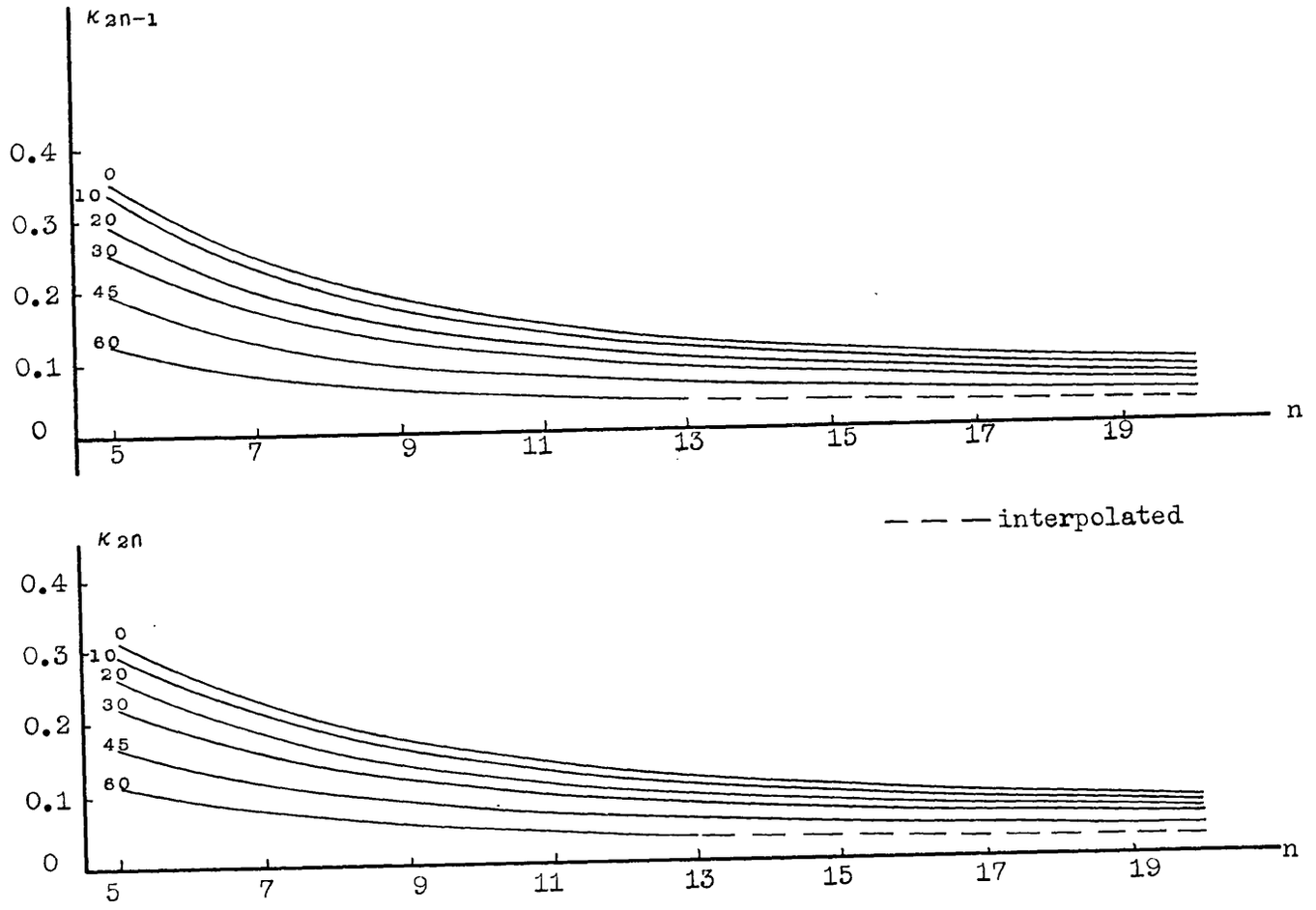

FIG. 9 : The eigenvalues κ_n.

α	$\pi/36$	$\pi/18$	$\pi/9$	$\pi/6$	$\pi/4$	$\pi/3$
c_1	1.401	1.372	1.282	1.158	.922	.640
c_3	- .078	- .158	- .318	- .458	- .572	- .524
c_5	- .037	- .068	- .092	- .039	.162	.337
c_7	- .024	- .039	- .025	.034	.042	- .150
c_9	- .017	- .024	.003	.031	- .037	.020
c_{11}	- .013	- .014	.013	.008	- .016	.034
c_{13}	- .010	- .008	.012	- .009	.016	- .030
c_{15}	- .008	- .004	.008	- .011	.009	.005
c_{17}	- .006	- .001	.002	- .003	- .009	.013
c_{19}	- .005	.001	- .002	.004	- .005	- .013
c_{21}	- .004	.002	- .004	.005	.006	.002
c_{23}	- .003	.003	- .004	.002	.004	.007
c_{25}	- .002	.003	- .002	- .002	- .004	- .007
s_1	1.414	1.414	1.412	1.404	1.371	1.300
s_3	- .001	- .004	- .029	- .080	- .192	- .303
s_5	.000	- .003	- .019	- .039	- .025	.070
s_7		- .003	- .013	- .012	.024	.012
s_9		- .003	- .007	.003	.009	- .025
s_{11}		- .002	- .003	.007	- .009	.011
s_{13}		- .002	.000	.004	- .004	.004
s_{15}		- .001	.000	- .001	.005	- .009
s_{17}		- .001	.002	- .003	.003	.004
s_{19}		- .001	.002	- .002	- .003	.002
s_{21}		.000	.001	.000	- .002	- .004
s_{23}			.000	.001	.002	.002
s_{25}				.001	.001	.001

TABLE 1 : The eigenfunctions $\phi_1 = \sum_m c_{2m-1} \cos(2m-1)\theta_0$

and $\psi_1 = \sum_m s_{2m-1} \sin(2m-1)\theta_0$

α	$\pi/36$	$\pi/18$	$\pi/9$	$\pi/6$	$\pi/4$	$\pi/3$
c_1	.097	.160	.240	.275	.268	.208
c_3	.788	.728	.556	.351	.071	− .089
c_5	− .124	− .243	− .441	− .524	− .379	− .085
c_7	− .055	− .089	− .051	.129	.371	.229
c_9	− .035	− .046	.013	.072	− .140	− .279
c_{11}	− .024	− .025	.027	.012	− .034	.228
c_{13}	− .018	− .013	.023	− .019	.039	− .122
c_{15}	− .014	− .005	.013	− .019	.015	.022
c_{17}	− .011	− .001	.002	− .005	− .019	.030
c_{19}	− .008	.002	− .005	.008	− .009	− .031
c_{21}	− .006	.004	− .007	.009	.011	.007
c_{23}	− .005	.005	− .006	.003	.006	.013
c_{25}	− .004	.005	− .003	− .004	− .007	− .015
s_1	.001	.006	.031	.071	.134	.177
s_3	.816	.814	.785	.703	.486	.223
s_5	− .004	− .029	− .146	.038	− .455	− .348
s_7	− .003	− .018	− .067	.053	.167	.322
s_9	− .003	− .014	− .032	.017	.032	− .206
s_{11}	− .002	− .011	− .010	.028	− .040	.076
s_{13}	− .002	− .009	.002	.013	− .014	.011
s_{15}	− .002	− .007	.008	− .004	.018	− .037
s_{17}	− .002	− .005	.008	− .010	.008	.020
s_{19}	− .002	− .004	.005	− .006	− .010	.005
s_{21}	− .001	− .002	.001	.001	− .005	− .016
s_{23}	− .001	− .001	− .002	.005	.007	.009
s_{25}	− .001	.000	− .003	.004	.004	.003

TABLE 2. : The eigenfunctions $\phi_3 = \sum_m c_{2m-1} \cos(2m-1)\theta_0$

and $\psi_3 = \sum_m s_{2m-1} \sin(2m-1)\theta_0$.

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CONCLUSION

The preceding chapters have attempted to provide a survey of the present knowledge of the water wave - ice floe interaction, and to extend certain aspects of this knowledge by investigating a number of hitherto unexplored problems. In addition, the contents of Chapter IV have served in a limited way to assimilate and develop the theory of singular integro-differential equations, the orientation of this survey being towards specific methods of solving such equations.

Extensions to the shallow water problems of Chapters II and III are immediate, a primary objective being the derivation of solutions for an arbitrary water depth. The possibility of achieving this on an analytic basis in the case of the flexible ice sheet of Chapter II seems remote, due to the complexity of the boundary condition which describes the ice.

The problem of the variable thickness ice sheet of Chapter III is more promising in this respect, and is similar to that of Weitz and Keller* mentioned in Chapter I. On prescribing a sinusoidal ice thickness, as in the shallow water formulation, a Wiener-Hopf integral equation can be derived. Without dwelling on the detail of such equations,

* Weitz, M., and Keller, J.B., "Reflection of water waves from floating ice in water of finite depth", Communications on Pure and Applied Mathematics, vol. 3, (1950).

a solution was attempted using the standard technique until a deviation from the Wiener-Hopf theory evolved. No way was found of proceeding at the time, but it is possible that the problem is tractable.

An alternative extension to this problem is the three dimensional case in which the ice thickness is allowed to vary with respect to both the horizontal variables. An approach along these lines established, by a separation of variables, that a Mathieu equation arises in each of these variables. A qualitative discussion of the problem is immediately made complex, since there are now two stability diagrams in independent parameters, and the solution was therefore taken no further.

The final Chapter emphasises the difficulty of solving a mathematical model in which the ice floes are regarded as rigid blocks. Nevertheless, it is felt that this is the direction in which the overall problem must proceed, and that perhaps a knowledge of the three dimensional situation of a single rigid disc floating on an expanse of fluid would be a suitable stepping stone towards this end.

Also the question of viscous effects in the form of boundary layers under the ice sheet is yet to be resolved, and as was made apparent by the experimental approaches mentioned in Chapter I, this may indeed be a significant mechanism.

APPENDIX I : The numerical solution of a system of linear equations.

An efficient method of solving the system of linear equations

$$A\,X = B,$$

for the n x r matrix X, where A is an n x n array and B is an n x r matrix comprising the elements of r different right hand sides, has been developed by Bowdler et al.*, together with various computer programs based on it. Referring to Bowdler's notation, we employ in the present case the variants designated UNSYMDET and UNSYMMACCSOLVE.

The first of these gives the Crout factorisation of the matrix A, based on the fact that a non-singular square matrix can in general be expressed in the form

$$A = L\,U,$$

where L is lower triangular and U upper triangular. Writing this as

$$A = (L\,D)(D^{-1}\,U),$$

the Crout choice of diagonal matrix D is such that $D^{-1}\,U$ is unit upper triangular. Without going into detail, the factorisation involves n major steps, each representable by a permutation matrix P_r operating on A, so that

$$P_n \;....\; P_2\,P_1\,A = L\,U.$$

* Bowdler, H.J., Martin, R.S., Peters, G. and Wilkinson, J.H.,"Solution of real and complex systems of linear equations", Numerische Matematik, vol. 8, pp. 217 - 234, (1966).

Thus for one of the right hand sides b, and an unknown vector x, we can write

$$P_n \ldots P_2 P_1 A x = L U x = P_n \ldots P_1 P_2 b = b_0,$$

say, so that in effect we must solve

$$L y = b_0, \qquad U x = y.$$

That the solution of the first of these can be found by repeated substitution is clear, and from a knowledge of the vector y, x can be determined with similar ease. These latter processes are accomplished by UNSYMMACCSOLVE, which also invloves an iterative procedure to ensure sufficient accuracy in the solution.

Further detail here is superfluous since the Algol programs are used in the exact form proposed by Bowdler, in whose paper, cited above, can be found all the notational and procedural details of the method.