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Robust Explicit Model Predictive Control via Regular Piecewise-Affine Approximation

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This paper proposes an explicit model predictive control design approach for regulation of linear time-invariant systems subject to both state and control constraints, in the presence of additive disturbances. The proposed control law is implemented as a piecewise-affine function defined on a regular simplicial partition, and has two main positive features. Firstly, the regularity of the simplicial partition allows one to efficiently implement the control law on digital circuits, thus achieving extremely fast computation times. Moreover, the asymptotic stability (or the convergence to a set including the origin) of the closed-loop system can be enforced a-priori, rather than checked a-posteriori via Lyapunov analysis.

Keywords: Model predictive control, uncertain systems, piecewise-affine functions.

Model predictive control (MPC) is becoming increasingly popular both in academia and in industry due to its ability to solve control problems while satisfying constraints on state and control variables (Rawlings and Mayne 2009). The main drawback of MPC is the computation

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time required for solving an optimization problem on line, which has historically prevented its application to fast processes. To circumvent this problem, two main research directions were pursued in the last decade (we limit our overview to the control of linear time-invariant (LTI) systems, that are the subject of this paper). The first relates to fast algorithms for on-line optimization (Ferreau *et al.* 2008, Wang and Boyd 2010, Richter *et al.* 2011, Patrinos and Bemporad 2014, Rubagotti *et al.* 2014). The second regards computing the control law off-line as an explicit piecewise-affine (PWA) function of the state vector (Bemporad *et al.* 2002): the off-line computation employs a multiparametric programming solver, and leads to the same solution obtained by solving the optimization problem on-line. The on-line computation in explicit MPC relies on determining the region of the PWA partition where the current state is located (usually referred to as the *point location* problem, which typically takes a high percentage of the overall on-line computation time), and then on evaluating an affine function from a pre-stored lookup table. To simplify the complexity of explicit MPC controllers, *approximate* explicit MPC, in which optimality is sacrificed for a control law defined over a smaller number of regions, has been considered in the last decade (see, e.g., Grieder *et al.* (2005), Jones and Morari (2010), Kvasnica *et al.* (2011), Kvasnica and Fikar (2012), and the references therein).

In a recent work (Bemporad *et al.* 2011), an approximate MPC controller for LTI systems was proposed, based on a special class of functions, hereafter referred to as *piecewise-affine simplicial (PWAS) functions*, proposed by Julián *et al.* (2000). The choice of PWAS functions leads to a regular partition, so that the point-location problem is solved with a negligible effort compared to explicit MPC defined on generic PWA partitions (the reader is also referred to Oliveri *et al.* (2012) for the practical implementation). The control law proposed by Bemporad *et al.* (2011) presents feasibility and local optimality properties, but the asymptotic stability of the origin of the closed-loop system and the evaluation of its domain of attraction can be determined only a posteriori (see, e.g., Rubagotti *et al.* (2013) and the references therein). We would like to remark that PWAS functions are not the only choice for approximation of explicit MPC aimed at hardware implementation: for example, two different approaches based on the use of PWA hyper-rectangular partitions have been recently proposed by Genuit *et al.* (2011) and Lu *et al.* (2011). In all of these approaches (Bemporad *et al.* 2011, Genuit *et al.* 2011, Lu *et al.* 2011), the possible presence of disturbance terms acting on the system is not taken into account. Note that all the proposed techniques for approximation of explicit MPC lead to a reduction of the computation time, but are applicable only to relatively small-size problems, which is an inherent limitation of explicit MPC.

In this paper, we propose an approximation method for explicit MPC based on PWAS functions, which can be implemented on digital circuits as in Bemporad *et al.* (2011). However, in addition to that, we guarantee *a-priori* the convergence to a minimal set including the origin for the resulting closed-loop system (also obtaining the domain of attraction in which hard constraints on state and input variables are satisfied), in the presence of external disturbances.

More specifically, two different methods are hereafter proposed to design a robust MPC control law $u^*(x)$, based on tightened constraints: an approximation procedure is carried out, in order to find an approximate PWAS control law $u(x)$, such that the approximation error $u(x) - u^*(x)$ satisfies the previously-defined bounds. As a drawback, $u^*(x)$ must be explicitly computed in order to obtain $u(x)$. Also, the proposed method, like all explicit MPC techniques, can only be applied to small-sized problems, due to the exponential increase of the problem complexity as the prediction horizon or the number of states/inputs increases. However, we can obtain a considerable decrease in the time needed to compute the control law if compared to directly applying $u^*(x)$, mainly due to the strong simplification of the point-location problem. A preliminary version of the theoretical development in this paper is presented in Rubagotti *et al.* (2012), where one of the two synthesis methods here considered is proposed in the case of systems without disturbances.

The paper is organized as follows: the main notation used throughout the paper and the formulation of the control problem are introduced in Sections 1 and 2, respectively, while Section

3 describes the structure of the PWAS control law. In Section 4, the synthesis of the robustly stabilizing MPC control law is described, while Section 5 deals with the approximation procedure leading to the stabilizing PWAS control law. In Section 6, two simulation examples are shown. Finally, conclusions are drawn in Section 7.

1 Notation

Let $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$, $\mathbb{R}_{>0}$ denote the sets of positive integers, non-negative integers, real, and positive real numbers, respectively. Given a set $\mathcal{A} \subset \mathbb{R}^n$, its interior is referred to as $\mathrm{int}(\mathcal{A})$. Given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \oplus \mathcal{B} \triangleq \{a + b \ : \ a \in \mathcal{A},\ b \in \mathcal{B}\}$ and $\mathcal{A} \sim \mathcal{B} = \{a \ : \ a + b \in \mathcal{A},\ \forall b \in \mathcal{B}\}$ are their Minkowski addition and Pontryagin difference, respectively. Also, given $\lambda \in \mathbb{R}_{\geq 0}$, we define $\lambda\mathcal{A} \triangleq \{x \in \mathbb{R}^n \ : \ x = \lambda a,\ a \in \mathcal{A}\}$. We denote by $\|v\|_1$ and $\|v\|_\infty$ the 1-norm and the ∞-norm of v, respectively. Given two vectors $u, v \in \mathbb{R}^n$, the notation $u \leq v$ refers to component-wise inequalities. Given a square matrix $H \in \mathbb{R}^{n\times n}$, its trace is $\mathrm{tr}(H)$, its Cholesky factor is $H^{\frac{1}{2}}$, and its positive definiteness is referred to as $H \succ 0$. The symbol $\mathbb{I}_n$ represents the identity matrix in $\mathbb{R}^{n\times n}$. Given a vector $v \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n\times n}$, $\|v\|_M^2 \triangleq v'Mv$. Given a matrix $H \in \mathbb{R}^{n\times m}$ and a compact set $\mathcal{W} \subset \mathbb{R}^m$, the product $H\mathcal{W}$ denotes the image of $\mathcal{W}$ under the mapping defined by H, $H\mathcal{W} \triangleq \{v \in \mathbb{R}^n \ : \ v = Hw,\ \forall\, w \in \mathcal{W}\}$. When convenient, the explicit dependence on time of the dynamic variables will be omitted for the sake of readability.

2 Problem statement

The controlled plant is described by the following discrete-time LTI state space model

$$x(t+1) = Ax(t) + Bu(t) + d(t) \tag{1}$$

where $t \in \mathbb{Z}_{\geq 0}$, $x, d \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. The whole state vector x is available for feedback, while u and d represent the control input and an unknown and unmeasurable disturbance term, respectively. The state and input values are required to satisfy

$$x \in \mathcal{X},\ \mathcal{X} \triangleq \{x \in \mathbb{R}^n \ : \ C_x x \leq g_x\} \tag{2}$$

$$u \in \mathcal{U}, \mathcal{U} \triangleq \{u \in \mathbb{R}^m \ : \ C_u u \leq g_u\} \tag{3}$$

with $C_x \in \mathbb{R}^{s_x \times n}$, $C_u \in \mathbb{R}^{s_u \times m}$, $g_x \in \mathbb{R}^{s_x}$, $g_u \in \mathbb{R}^{s_u}$, while the disturbance term is assumed to be such that

$$d \in \mathcal{D},\ \mathcal{D} \triangleq \{d \in \mathbb{R}^n \ : \ C_d d \leq g_d\} \tag{4}$$

with $C_d \in \mathbb{R}^{s_d \times n}$, $g_d \in \mathbb{R}^{s_d}$.

Assumption 2.1 The following holds for system (1):

(i) the pair (A, B) is stabilizable;

(ii) $\mathcal{X}$ and $\mathcal{U}$ are nonempty, compact, and contain the origin in their interiors;

(iii) $\mathcal{D}$ is nonempty, compact, and contains the origin.

The objective of the control law is to solve a regulation problem to the smallest possible set containing the origin, without violating the constraints (2)-(3). The control variable $u(x)$ is a state-feedback control law defined on a PWAS partition, whose structure is described in the next section.

3 Control law on a simplicial partition

The function $u(x)$ is defined on a closed hyper-rectangle $\mathcal{S} = \{x \in \mathbb{R}^n : \ x_{min} \leq x \leq x_{max}\}$, which is partitioned as $\mathcal{S} = \bigcup_{i=0}^{L_S-1} S_i$, where $\{S_i\}_{i=0}^{L-1}$ are simplices, i.e., polytopes given by the convex hull of their $n+1$ vertices $x_i^0, x_i^1, \ldots, x_i^n \in \mathbb{R}^n$. The partitioning of $\mathcal{S}$ is performed as follows:

(1) Every dimensional component x_j of $\mathcal{S}$ is divided into p_j subintervals of length $(x_{max,j} - x_{min,j})/p_j$. These intervals define a number $\prod_{j=1}^n p_j$ of hyper-rectangles, and $\mathcal{S}$ contains

$N_v \triangleq \prod_{j=1}^{n}(p_j+1)$ vertices v_k, collected into a set named $\mathcal{V}_S$.

(2) Every rectangle is partitioned into $n!$ simplices with non-overlapping interiors. The set $\mathcal{S}$ contains $L_S \triangleq n! \prod_{j=1}^{n} p_j$ simplices S_i, such that $\mathcal{S} = \cup_{i=0}^{L_S-1} S_i$ and $\mathrm{int}(S_i) \cap \mathrm{int}(S_j) = \emptyset$, $\forall i, j = 0, \ldots, L_S - 1$.

Note that, since the partitioning of the hyper-rectangles into simplices is univocally determined, the resulting number of simplices is determined by $p_1, ..., p_n$. After defining the sets S_i, it is possible to introduce the related PWAS function. We choose to define each component of $u(x)$, namely $u_j(x)$, $j = 1, , m$, as the weighted sum of N_v linearly independent α-basis functions (Julián *et al.* 2000). Every element of the j-th basis is affine over each simplex and satisfies

$$\alpha_{j,k}(v_h) = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k. \end{cases}$$

After ordering the functions of the α-basis, we can consider them as an N_v-length vector $\phi(x)$. Then, each component of $u(x)$, namely $u_j(x)$, is a scalar PWAS function defined as

$$u_j(x) \triangleq \sum_{k=1}^{N_v} \theta_{j,k}\, \phi_k(x) = \phi(x)'\theta_j \tag{5}$$

where $\theta_j = [\theta_{j,1} \; \ldots \; \theta_{j,N_v}]' \in \mathbb{R}^{N_v}$ is the weight vector. Note that the coefficients $\theta_{j,k}$ coincide with the values of the PWAS function $u_j(x)$ at the vertices of the simplicial partition. The PWAS vector function $u : \mathbb{R}^n \to \mathbb{R}^m$ is defined by the weight vector $\theta = [\theta_1' \; \theta_2' \; \ldots \; \theta_m']' \in \mathbb{R}^{mN_v}$, as

$$u(x) = \begin{bmatrix} u_1(x) \\ \vdots \\ u_m(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi(x)'\theta_1 \\ \vdots \\ \phi(x)'\theta_m \end{bmatrix} = \begin{bmatrix} \phi'(x) & 0 & \cdots & 0 \\ 0 & \phi'(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi'(x) \end{bmatrix} \theta = \Phi(x)\theta. \tag{6}$$

The main reason for defining $u(x)$ as in (6) is that PWAS functions can be implemented in digital circuits using linear interpolators. In fact, by exploiting the regularity of the partition, the point location problem becomes much easier than for the case of generic PWA partitions. The value of $u(x)$ can be obtained, for any $x \in \mathcal{S}$, as a linear interpolation of the values of u at the $n+1$ vertices $x_{i,0}, ..., x_{i,n}$ of the simplex S_i that contains x. For a summary of the actual FPGA implementation (also employed to design virtual sensors in Poggi *et al.* (2012)), the interested reader is referred to Storace and Poggi (2010).

4 Robustly stabilizing optimal MPC

The next step is to obtain a function $u(x)$ as in (6) using a procedure that leads to asymptotic convergence to a set containing the origin for the closed-loop system. The proposed approach consists of expressing the control variable $u(x)$ as

$$u(x) = u^*(x) + w(x)$$

where $u^*(x)$ is an optimal control law which satisfies

$$u^* \in \mathcal{U}, \tag{7}$$

while $w(x)$ represents an approximation error (a priori unknown), and is considered as a bounded disturbance. System (1) can therefore be expressed as

$$x(t+1) = Ax(t) + Bu^*(t) + Bw(t) + d(t) \tag{8}$$

4.1 *Definition of the auxiliary control laws*

In order to formulate the MPC control law $u^*(x)$, we first need to define an auxiliary control law, for which we introduce two alternative choices.

The first control law is synthesized on the nominal system, as follows:

Statement 4.1 The auxiliary control law is defined as $u^*(x) = K_n x$, where K_n is the solution for the nominal system

$$x(t+1) = Ax(t) + Bu^*(t) \tag{9}$$

of the infinite-horizon linear quadratic regulator (IH-LQR), given the weight matrices $Q = Q' \in \mathbb{R}^{n\times n}$ on the state and $R = R' \in \mathbb{R}^{m\times m}$ on the input, with $Q, R \succ 0$.

Remark 1 : Note that, by classical results of LQR theory, the closed-loop system obtained by imposing $u^*(x) = K_n x$ in (9) is asymptotically stable.

The second choice concerns an auxiliary control law which is robustly stabilizing for

$$x(t+1) = Ax(t) + B\left(u^*(t) + w(x(t))\right), \tag{10}$$

where each component w_i of w is such that

$$|w_i(x)| \leq \alpha \|x\|_1, \;\; i = 1, ..., m \tag{11}$$

and α is a tuning parameter. This formulation of the uncertainty can be shown to be a *structured feedback uncertainty*, as in Kothare *et al.* (1996). To this purpose, let $\mathbf{1} \in \mathbb{R}^{m\times n}$ be a matrix of ones, and $\Delta = diag(\delta_1, \delta_2, ..., \delta_n)$ be a matrix of uncertain parameters such that $|\delta_i| \leq 1$ for all $i \in 1, ..., n$. Then, (11) can be equivalently formulated as $w(x) = \alpha \mathbf{1}\Delta x$. More precisely, this latter expression is equivalent to $w_i = \alpha\left(\sum_{j=1}^n \delta_j x_j\right)$ for all $i = 1, ..., m$, which leads to (11).

Statement 4.2 The auxiliary control law is defined as $u^*(x) = K_p x$, where $K_p = Y\Pi^{-1}$,

$\Pi = \Pi' \succ 0$ and Y are the solution of the following semidefinite program:

$$\min_{\gamma,\Lambda,\Pi,Y} \quad \gamma \tag{12a}$$

$$\text{s.t.} \quad \Lambda > 0 \tag{12b}$$

$$\mathrm{tr}(\Pi) = 1 \tag{12c}$$

$$\begin{bmatrix} \Pi & Y'R^{\frac{1}{2}} & \Pi Q^{\frac{1}{2}} & \Pi & (A\Pi + BY)' \\ R^{\frac{1}{2}}Y & \gamma I & 0 & 0 & 0 \\ Q^{\frac{1}{2}}\Pi & 0 & \gamma I & 0 & 0 \\ \Pi & 0 & 0 & \Lambda & 0 \\ A\Pi + BY & 0 & 0 & 0 & \Pi - B_p\Lambda B_p' \end{bmatrix} \succeq 0 \tag{12d}$$

where $B_p \triangleq \alpha B\mathbf{1}$ and $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$.

Remark 2 : The control law obtained in Statement 4.2 is related to the result in (Kothare *et al.* 1996, Th. 1). While in Kothare *et al.* (1996) a semidefinite program is solved on-line, we here fix the gain K_p off line. Also, as $\alpha \to 0$ (i.e., the dynamics of the uncertain system (10) approaches the nominal dynamics (9)), the gain K_p tends to the gain K_n (with the same weight matrices Q, R) defined in Statement 4.1 (Kothare *et al.* 1996, Rem. 4). Every possible realization of matrix $A + B(K_p + \alpha\mathbf{1}\Delta)$ has eigenvalues strictly inside the unit circle, which means that the closed-loop system (10) with $u^* = K_p x$ is *absolutely asymptotically stable* as defined by Gurvits (1995).

In conclusion, two different auxiliary control laws have been defined, both assuming no external disturbance d, the first assuming no approximation error w, and the second assuming that w is vanishing as x approaches the origin as described in (11). By defining either $K \triangleq K_n$ or $K \triangleq K_p$, the resulting closed-loop system is

$$x(t+1) = A_\kappa x(t) + Bw(t) + d(t), \tag{13}$$

where $A_\kappa \triangleq A + BK$. Both of these control laws will be used in the remainder of the paper as baseline to design the MPC controller.

4.2 *Preliminary concepts for the definition of the MPC control law*

A robust MPC control law is now described, which leads to robust convergence of the state to the origin without violating constraints (2) and (7). Let $\bar{w} \in \mathbb{R}_{\geq 0}$ be a fixed scalar such that

$$w(x) \in \mathcal{W} \triangleq \{w \in \mathbb{R}^m : \|w\|_\infty \leq \bar{w}\}, \ \forall x \in \mathcal{X} \tag{14}$$

which represents a requirement on the maximum approximation error. At this point, two additive disturbances are present in the system. We define

$$\xi(t) \triangleq Bw(t) + d(t)$$

from which it follows that $\xi \in \Xi = B\mathcal{W} \oplus \mathcal{D}$, and rewrite system (8) as

$$x(t+1) = Ax(t) + Bu^*(t) + \xi(t). \tag{15}$$

The following standard definition is used in the following:

Definition 4.3: A set $\mathcal{P}$ is *robust positively invariant* (RPI) for system (15), if $x(0) \in \mathcal{P}$ implies $x(t) \in \mathcal{P}$ for all $\xi(t) \in \Xi$ and for all $t \in \mathbb{Z}_{\geq 0}$.

First of all, we define

$$\mathcal{R}_k \triangleq \bigoplus_{i=0}^{k-1} A_\kappa^i \Xi \tag{16}$$

which is the set of states reachable by system (13) in k time steps from the origin. Then, we compute the minimal RPI set $\mathcal{R}_\infty$ for the closed-loop system (13), assuming that (14) holds. The minimal RPI set for system (13) with $\xi \in \Xi$ is defined as $\mathcal{R}_\infty \triangleq \lim_{k\to\infty} \mathcal{R}_k$. Considering that this set can be computed exactly only under very restrictive assumptions, one usually needs to compute a polytopic over-approximation (not necessarly RPI) $\hat{\mathcal{R}}_\infty$ such that $\mathcal{R}_\infty \subseteq \hat{\mathcal{R}}_\infty$. Details on the characterization and computation of $\mathcal{R}_\infty$ and $\hat{\mathcal{R}}_\infty$ as compact polytopes are given in

Appendix A.

Referring to a generic gain K, which can be determined equal to K_n or to K_p, let the MPC control law acting on system (1) be

$$u^*(x) \triangleq Kx + \mu^*(x), \tag{17}$$

so that (1) becomes $x(t+1) = A_\kappa x(t) + B\mu^*(t) + \xi(t)$. Note from (17) that $\mu^*(x)$ represents the difference between the MPC control move and the baseline linear control law Kx. In the following, we will make use of tightened constraints on the nominal evolution of (15) to ensure the fulfillment of the actual constraints for the perturbed system. Starting from the initial condition $x(t) = x$ at time t, the nominal evolution of (15) at time $t+k$ is denoted by $\hat{x}(t+k|t)$, while the evolution of the actual system with the same initial condition by $x(t+k|t)$. Both evolutions are obtained by applying the corresponding control sequence denoted by $\mu^*(t|t), \ldots, \mu^*(t+k-1|t)$. It is well known from the set-theoretical analysis in Chisci *et al.* (2001) and Kolmanovsky and Gilbert (1998), that, given $\mathcal{X}_k \triangleq \mathcal{X} \sim \mathcal{R}_k$ and $\mathcal{U}_k \triangleq \mathcal{U} \sim K\mathcal{R}_k$, one has that, for all $k \in \mathbb{Z}_{\geq 0}$, $\hat{x}(t+k|t) \in \mathcal{X}_k \Leftrightarrow x(t+k|t) \in \mathcal{X}$ and $K\hat{x}(t+k|t) \in \mathcal{U}_k \Leftrightarrow Kx(t+k|t) \in \mathcal{U}$, for all $\xi \in \Xi$.

The next step is to find the maximal output admissible robust set for system (13), defined as

$$\mathcal{X}_f \triangleq \{x(0) \in \mathbb{R}^n : x(k|0) \in \mathcal{X},\ Kx(k|0) \in \mathcal{U},\ \forall k \in \mathbb{Z}_{\geq 0},\ \forall \xi \in \Xi\}. \tag{18}$$

Details on computating $\mathcal{X}_f$ as a compact polytope are given in Appendix A.

Assumption 4.4 It is supposed that $0 \in \text{int}(\mathcal{X} \sim \mathcal{R}_\infty)$ and $0 \in \text{int}(\mathcal{U} \sim K\mathcal{R}_\infty)$ (which ensures the computability of $\mathcal{X}_f$, see Appendix A). Moreover, we assume that $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{X}_f)$.

Remark 3 : The condition $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{X}_f)$ represents only a slightly stronger requirement with respect to condition $\mathcal{R}_\infty \subseteq \mathcal{X}_f$, which always holds. Note that, if $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{X}_f)$, being $\hat{\mathcal{R}}_\infty$ a closed set, any state trajectory that converges to $\hat{\mathcal{R}}_\infty$ asymptotically, converges to $\mathcal{X}_f$ in finite time.

Recalling the sets S_i defined in Section 3, we introduce the set

$$\mathcal{S}_f \triangleq \bigcup S_i \ : \ S_i \subseteq \mathcal{X}_f, \ i = 0, ..., L-1 \tag{19}$$

which will be useful to formulate the subsequent results. Being $\mathcal{X}_f$ a convex set, $\mathcal{S}_f$ is connected, but not necessarily convex.

4.3 *MPC with tightened constraints*

For the proposed robust MPC strategies, the prediction of the system trajectory on the finite *prediction horizon* $N \in \mathbb{Z}_{>0}$ will make use of the nominal model of the system and of tightened constraints, as in Chisci *et al.* (2001). The vector of optimization variables (inputs) to be determined at time t is $M \triangleq [\mu'(t|t) \ \cdots \ \mu'(t|t+N-1)]' \in \mathbb{R}^{mN}$. The definition of the optimal sequence $\mu^*(x)$ is based on the solution of the following finite-horizon optimal control problem (FHOCP) at each time t, with $x(t) = x$:

$$M^*(x) = \arg\min_{M} \sum_{k=0}^{N-1} \|\mu(k)\|_{\Psi}^2, \ \ \Psi = \Psi' \succ 0 \tag{20a}$$

$$\text{s.t. } \hat{x}(k) \in \mathcal{X}_k, \qquad\qquad k = 0, \ldots, N-1 \tag{20b}$$

$$K\hat{x}(k) + \mu(k) \in \mathcal{U}_k, \ k = 0, \ldots, N-1 \tag{20c}$$

$$\hat{x}(N) \in \mathcal{X}_f \sim \mathcal{R}_N \tag{20d}$$

For ease of notation, implying that the solution of the FHOCP is computed at time t, we set $\mu(k) \triangleq \mu(t+k|t)$ and $\hat{x}(k) \triangleq \hat{x}(t+k|t)$. Note that (20b) and (20c) lead to the fulfillment of (2) and (7), respectively, along the prediction horizon. Finally, (20d) guarantees that $x(t+k|t) \in \mathcal{X}_f$ for all possible disturbance sequences.

The FHOCP (20) is quadratic with respect to the decision variable M, and is subject to linear constraints. Also, the current state x can be considered as a parameter. Therefore, (20) can be recast as a multi-parametric quadratic program (mpQP), where the set of parameters x for which a feasible solution exists is called $\mathcal{F}_N$. Since $\mathcal{X}$, $\mathcal{U}$ and Ξ are convex polyhedra,

$\mathcal{F}_N$ is a convex polytope and can be easily computed using linear programming and projections (Chisci *et al.* 2001). Also, an increase of the prediction horizon leads to a larger set $\mathcal{F}_N$, i.e. $\mathcal{F}_N \supseteq \mathcal{F}_{N-1} \supseteq \ldots \supseteq \mathcal{F}_1 \supseteq \mathcal{X}_f$. The nominal case (i.e., $\Xi = 0$) can be seen as a limit of the robust case, and $\mathcal{F}_N$ is always included in the corresponding set obtained for the nominal case.

Recalling Remark 3 in Chisci *et al.* (2001), matrix Ψ can be chosen such that (20) coincides with the solution of the constrained IH-LQR associated to the weight matrices Q and R.

The application of the *receding horizon* principle leads to defining the MPC control law $\mu^*(x)$ as $\mu^*(x) \triangleq [I\ 0\ \ldots\ 0]M^*(x)$. Following the development in Bemporad *et al.* (2002), explicit expressions for the optimal value of the cost function in (20a), namely $J^*(x)$, and for $M^*(x)$, can be obtained solving an mpQP. In particular, both $J^*(x)$ and $M^*(x)$ are Lipschitz continuous, and more precisely $J^*(x)$ is piecewise-quadratic, while $M^*(x)$ is piecewise-affine. This implies that also $\mu^*(x)$ and $u^*(x)$ are piecewise-affine function defined in $\mathcal{F}_N$. The set $\mathcal{F}_N$ is then partitioned as $\mathcal{F}_N = \bigcup_{i=0}^{L_F-1} F_i$, where $\{F_i\}_{i=0}^{L_F-1}$ are polytopes (not necessarily simplices) with non-overlapping interiors.

Next, define the following two sets: $\mathcal{R}_\infty^n$, the minimal RPI set for the closed-loop system

$$x(t+1) = (A + BK_n)x(t) + d(t) \tag{21}$$

and $\mathcal{R}_\infty^p$, the minimal RPI set for the closed-loop system

$$x(t+1) = (A + BK_p + \alpha B\mathbf{1}\Delta)x(t) + d(t). \tag{22}$$

In both cases, the presence of the disturbance $w(t)$ is not taken into account. The computation of sets $\mathcal{R}_\infty^n$ and $\mathcal{R}_\infty^p$, and of their over-approximations $\hat{\mathcal{R}}_\infty^n$ and $\hat{\mathcal{R}}_\infty^p$ are described in Appendix A.

We are now ready to state the first main result of the paper.

Theorem 4.5 : *Let Assumptions 2.1 and 4.4 hold for system* (8) *with* $\xi \in \Xi$, *and let* $u^*(x)$ *be defined in* (17).

(I) Let the MPC control law in (17) *be designed with* $K = K_n$*,* $w(x)$ *such that*

$$w(x) = 0, \ \forall \ x \in \mathcal{S}_f, \tag{23}$$

with $0 \in \text{int}(\mathcal{S}_f)$ *(this latter being defined in* (19)*), and* $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{S}_f)$*. Then, for all possible realizations of the disturbance term* $d(t)$*, if* $x(0) \in \mathcal{F}_N$ *then* $x(t) \in \mathcal{X}$ *and* $u^*(t) \in \mathcal{U}$ *for all* $t \geq 0$*, and moreover* $x(t) \to \mathcal{R}^n_\infty \subseteq \mathcal{R}_\infty$ *as* $t \to \infty$*.*

(II) Let the MPC control law in (17) *be designed with* $K = K_p$ *for a given* $\alpha > 0$*. Moreover,*

$$|w_i(x)| \leq \alpha \|x\|_1, \ \forall i = 1, ..., m, \ \forall x \in \mathcal{X}_f \tag{24}$$

i.e., condition (11) *be satisfied for all* $x \in \mathcal{X}_f$*. Then, for all possible realizations of the disturbance term* $d(t)$*, if* $x(0) \in \mathcal{F}_N$ *then* $x(t) \in \mathcal{X}$ *and* $u^*(t) \in \mathcal{U}$ *for all* $t \geq 0$*, and moreover* $x(t) \to \mathcal{R}^p_\infty \cap \mathcal{R}_\infty$ *as* $t \to \infty$*.*

In both cases, if $\mathcal{D} = \{0\}$*, then* $\mathcal{R}^n_\infty = \mathcal{R}^p_\infty = \{0\}$*, i.e., the origin is an asymptotically stable equilibrium for system* (1)*, with domain of attraction* $\mathcal{F}_N$*.*

Proof: See Appendix B.1.

5 PWAS approximation

In this section, we describe how to obtain the control law $u(x)$ defined on a PWAS partition as in (6) approximating the control law $u^*(x)$ in (17), in order to obtain asymptotic stability and constraints satisfaction for system (1).

5.1 *Approximation procedure*

Assume that a control law $u^*(x)$ has been computed for system (1) with domain of attraction $\mathcal{F}_N$. Let $\mathcal{S}$ be defined as the smallest hyper-rectangle such that $\mathcal{F}_N \subseteq \mathcal{S}$, as described in Section 3. Then, we partition the (not necessarily convex) set $\mathcal{S} \setminus \mathcal{F}_N$ as $\mathcal{S} \setminus \mathcal{F}_N = \bigcup_{i=0}^{\hat{L}_F - 1} F_i$, where $\{F_i\}_{i=0}^{\hat{L}_F - 1}$ are polytopes with non-overlapping interiors. In this way, a generic partition of $\mathcal{S}$ as

$\mathcal{S} = \bigcup_{i=0}^{\tilde{L}_F-1} F_i$ is obtained, where $\tilde{L}_F \triangleq L_F + \hat{L}_F$, while we denote its set of vertices as $\tilde{\mathcal{V}}_F$. In order to introduce the used approximation procedure, we use the concept of *mixed partition* (see, e.g., Bemporad *et al.* (2011)), as the partition of $\mathcal{S}$ induced by the facets of both simplicial (S_i) and generic (F_i) partitions. As a result, $\mathcal{S}$ is further partitioned into convex polytopes, and the partition is completely defined by the sets of vertices $\mathcal{V}_S$, $\tilde{\mathcal{V}}_F$ and $\mathcal{V}_M$, the latter representing the set of vertices given by the intersection of the two partitions and belonging neither to $\mathcal{V}_S$ nor to $\tilde{\mathcal{V}}_F$. Finally, let $\mathcal{V}_I \triangleq \left\{ v \in \left(\mathcal{V}_S \cup \tilde{\mathcal{V}}_F \cup \mathcal{V}_M \right) \ : \ v \in \mathcal{F}_N \right\}$, and note that $\mathcal{F}_N$ is the convex hull of all $v \in \mathcal{V}_I$.

Let $u(x)$ be defined as the control law that minimizes the maximum discrepancy with respect to $u^*(x)$ for all $x \in \mathcal{F}_N$ (note that $u^*(x)$ is not defined on $\mathcal{S} \setminus \mathcal{F}_N$), that is

$$F_\infty \triangleq \max_{j=1,\ldots,m} \sup_{x \in \mathcal{F}_N} \left\{ \left| u_j(x) - u_j^*(x) \right| \right\} \tag{25}$$

When minimizing F_∞ in (25), some constraints have to be imposed for all $x \in \mathcal{F}_N$. Since the minima and maxima of the PWA function $w(x) = u(x) - u^*(x)$ on any of the regions of the mixed partition are on vertices, it is sufficient to impose constraints only on the vertices of $\mathcal{V}_I$. In particular:

(1) The control law $u(x)$ must satisfy the constraint (3), which is already satisfied by $u^*(x)$. This can be done imposing $C_u u(v) \leq g_u$ for all $v \in \mathcal{V}_I$, which implies $C_u u(x) \leq g_u$ for all $x \in \mathcal{F}_N$.

(2) The value of $u(x)$ must be computed such that $\|u(x) - u^*(x)\|_\infty \leq \bar{w}$, in order for system (1) to satisfy (14). This can be obtained by simply imposing $\|u(v) - u^*(v)\|_\infty \leq \bar{w}$ for all $v \in \mathcal{V}_I$;

(3)
- If $K = K_n$, in order to obtain (23), we impose that $u(v) = u^*(v)$ for all $v \in \mathcal{V}_I \cap \mathcal{S}_f$;
- If $K = K_p$, in order for system (1) to fulfill (24) we require that $|w_i(x)| \leq \alpha \|x\|_1$ for all $i = 1, ..., m$ and all $x \in \mathcal{F}_N$, which is obtained by forcing $|u_i(v) - u_i^*(v)| \leq \alpha \|v\|_1$ for all $v \in \mathcal{V}_I \cap \mathcal{X}_f$.

Therefore, after recalling the relationship between vector θ and the control variable $u(x)$ in (5)-(6), we obtain $u(x)$ by solving the following linear program:

$$\min_{\theta,\eta} \quad \eta \tag{26a}$$

$$\text{s.t.} \quad \eta \geq \pm \left(\phi(v)'\theta_j - u_j^*(v)\right), \; v \in \mathcal{V}_I, \; j = 1, \ldots, m \tag{26b}$$

$$C_u \Phi(v)\theta \leq g_u, \; v \in \mathcal{V}_I \tag{26c}$$

$$\begin{cases} \Phi(v)\theta = u^*(v), & v \in \mathcal{V}_I \cap \mathcal{S}_f, \text{ if } K = K_n \\ |u_i(v) - u_i^*(v)| \leq \alpha \|v\|_1, \; i = 1, ..., m, \, v \in \mathcal{V}_I \cap \mathcal{X}_f, & \text{if } K = K_p \end{cases} \tag{26d}$$

$$\eta \leq \bar{w} \tag{26e}$$

The formulation of the cost function (26a) together with the constraint (26b) leads to finding the vector θ that minimizes the maximum difference between $u_j(x)$ and $u_j^*(x)$ for all $x \in \mathcal{F}_N$ and all components j. Conditions (26c) and (26d) lead to the fulfillment of (3) and (23) (or (24)), respectively. Condition (26e) ensures the fulfillment of (14). Once a feasible solution to (26) has been found, vector θ determines the control law $u(x)$ for all $x \in \mathcal{S}$.

5.2 *Properties of the PWAS control law*

The following result holds when the approximate control law $u(x)$ is applied to system (1).

Theorem 5.1 : *Let* (14) *and Assumptions 2.1, 4.4 hold for system* (1). *Assume that a feasible solution for the FHOCP* (20) *exists, and define* $u^*(x)$ *as in* (17). *Finally, suppose that one of the following holds:*

(i) The MPC control law in (17) *is designed with* $K = K_n$, *all the assumptions in case* (*I*) *of Theorem 4.5 are satisfied, and there exists a realization of* $u(x)$ *obtained through a feasible solution of* (26).

(ii) The MPC control law in (17) *is designed with* $K = K_p$. *Moreover, all the assumptions in case* (*II*) *of Theorem 4.5 are satisfied, and there exists a realization of* $u(x)$ *obtained*

through a feasible solution of (26).

Then, if $x(0) \in \mathcal{F}_N$*, one has* $x(t) \in \mathcal{X}$ *and* $u(t) \in \mathcal{U}$ *for all* $t \geq 0$*. Moreover the state is asymptotically driven to* $\mathcal{R}_\infty^n$ *in case* (i)*, or to* $\mathcal{R}_\infty^p \cap \mathcal{R}_\infty$ *in case* (ii)*. Finally, if* $\mathcal{D} = \{0\}$*, in both cases* (i) *and* (ii) *the origin is an asymptotically stable equilibrium point for system* (1)*, with domain of attraction equal to* $\mathcal{F}_N$*.*

Proof: See Appendix B.2.

Remark 4: Due to the properties of the α-basis chosen to formulate $u(x)$, (26) imposes conditions only on a subset of the components of θ related to the vertices $v \in \mathcal{V}_S$. In particular, if $v \in S_i$ with $S_i \cap \mathcal{F}_N = \emptyset$, then any value assigned to the corresponding component of θ is not influencing the solution of (26), because their values do not affect $u(x)$ in $\mathcal{F}_N$.

5.3 *Parameter tuning*

Considering that the feasibility of (12), (20) and (26) is not guaranteed a priori, we give some guidelines on choosing the design parameters of the proposed approach. We assume that the number of vertices N_v of the simplicial partition is fixed, which fixes the memory occupation and latency time on the digital circuit implementing the control law, since these quantities only depend on the structure of the chosen PWAS structure, and not on its values. Given the sets $\mathcal{X}$ and $\mathcal{U}$, the tuning parameters on which the designer can act to design $u^*(x)$ are $\bar{w}$ (if $K = K_n$) or both $\bar{w}$ and α (if $K = K_p$). In case $K = K_n$, we can fix a value of $\bar{w}$, compute $\hat{\mathcal{R}}_\infty$ and $\mathcal{X}_f$ checking if Assumption 2.1 is satisfied, check if $0 \in \text{int}(\mathcal{S}_f)$ and $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{S}_f)$, and then solve (20). If (20) is feasible and all the required assumptions are satisfied for a given $\bar{w} = \bar{w}_1$, then the same will hold for any $\bar{w} \leq \bar{w}_1$. Then, one can find by bisection the maximum feasible value of $\bar{w}$, namely $\bar{w}_{max}$, and then (20) will be feasible for all $\bar{w}$ such that $0 \leq \bar{w} \leq \bar{w}_{max}$.

In case $K = K_p$ one can choose a sufficiently small value for the parameter α (such that (12) be feasible), and compute K_p. Then, one would act on the value of $\bar{w}$ as in the previous case, but without checking the condition relative to the set $\mathcal{S}_f$, since the equality constraints in (26d) are not imposed if $K = K_p$.

In any case, we know that, once all the other parameters are fixed, a smaller value of $\bar{w}$ leads to a larger set $\mathcal{F}_N$. On the other hand, a small value of $\bar{w}$ could impose a too tight approximation in problem (26), making it infeasible. In conclusion, the designer can start obtaining a feasible realization of the PWAS control for a value of $\bar{w}$ close to $\bar{w}_{max}$. Then, this value can be decreased in order to enlarge the set $\mathcal{F}_N$ and obtain the desired performance.

6 Simulation examples

6.1 *Example 1*

As a first example, we consider the problem of regulating to the origin the LTI discrete-time system proposed in Bemporad *et al.* (2011), where system (1) is defined by

$$A = \begin{bmatrix} 1.2 & 1 \\ 0 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{27}$$

with the sets in (2)-(4) defined as $\mathcal{X} = \{x \in \mathbb{R}^2 \ : \ \|x\|_\infty \leq 2\}$, $\mathcal{U} = \{u \in \mathbb{R}^2 \ : \ |u_1| \leq 0.5, |u_2| \leq 0.6\}$, $\mathcal{D} = \{d \in \mathbb{R}^2 \ : \ \|d\|_\infty \leq 5 \cdot 10^{-2}\}$. In this case, we decide to design the auxiliary control law with $K = K_n = \left[\begin{smallmatrix} 0.9337 & -0.1540 \\ -1.0333 & -0.9373 \end{smallmatrix}\right]$, obtained using the weight matrices $Q = \mathbb{I}_2$ and $R = 0.1\mathbb{I}_2$, and we set $\bar{w} = 0.1$. The MPC control law $u^*(x)$ in (17) is computed with $\Psi = \mathbb{I}_2$ and $N = 4$, and its domain of attraction $\mathcal{F}_N$ is shown in Fig. B1, together with sets $\hat{\mathcal{R}}_\infty$, $\mathcal{X}_f$ and $\hat{\mathcal{R}}^n_\infty$. The control law $u^*(x)$ is composed of 83 irregular regions, and the point location problem (see, e.g., Bemporad *et al.* (2002)) uses a binary search tree with 427 nodes, and a depth between 6 and 9. The approximate control law $u(x)$ is computed with $p_1 = p_2 = 50$ (defined in Section 3), obtaining $N_v = 2601$ vertices and $L_s = 5000$ simplices, with a maximum approximation error $\eta = 0.0373$. In Fig. B1 the set $\mathcal{S}_f$ is also shown, and it is possible to verify that all the assumptions required in case (i) of Theorem 5.1 are satisfied. The PWAS control law so obtained is shown in Fig. B2. In Fig. B1 the set $\mathcal{F}'_N$ of feasible states using the optimal MPC control law (designed with $\mathcal{W} = \{0\}$) is shown, and one can note a reasonably contained reduction of the region of attraction with respect to the direct employment of $u^*(x)$.

6.2 *Example 2*

As a second example, we design the approximate MPC controller for the same system in form (1) described by matrices A and B in (27), in case $\mathcal{D} = \{0\}$. In this case, we set $\alpha = 0.05$, and design the auxiliary control law with $K = K_p = \left[\begin{smallmatrix} 0.9385 & -0.1696 \\ -1.0387 & -0.9570 \end{smallmatrix}\right]$, which is obtained using the same weight matrices as in Example 1. The MPC control law $u^*(x)$ in (17) is computed with $\Psi = \mathbb{I}_2$ and $N = 4$, and its domain of attraction $\mathcal{F}_N$ is shown in Fig. B3, together with the other sets related to this example. The approximate control law $u(x)$ is computed with $p_1 = p_2 = 50$, obtaining $N_v = 2601$ vertices and $L_s = 5000$ simplices, with a maximum approximation error $\eta = 0.0297$. In this case the control law $u^*(x)$ is composed of 104 irregular regions, thus the point location problem uses a binary search tree with 701 nodes, and a depth between 7 and 10. Since all the conditions required in case (*ii*) of Theorem 5.1 are satisfied, the asymptotic stability of the origin is guaranteed for all initial conditions in $\mathcal{F}_N$. In Fig. B4, the time evolution of the state and control variables are shown starting from the initial condition $x(0) = \begin{bmatrix} 0.88 & -0.2 \end{bmatrix}'$.

6.3 *Circuit performance comparison*

In order to test the performance of the proposed control laws on real circuits, we used a Xilinx Spartan 3 FPGA (xc3s200) board to implement the PWAS law of Example 2, coding the state variables (circuit inputs) with 12 bits words. The employment of architecture B (serial) in Storace and Poggi (2010) for the simplicial approximation uses 7.8 KB of RAM, 165 slices, and one multiplier, allowing the control law computation to occur in 12 clock cycles. The simplified circuit design allows an effective circuit frequency of 80MHz, which leads to a sampling time interval of 150 ns. Note that implementing the PWAS control law of Example 1 requires exactly the same circuit specification on the FPGA.

The optimal MPC control law, which is a generic PWA function described in (17), in case $\mathcal{W} = \{0\}$ (i.e., no approximation error) and serial implementation uses 1.012 KB of RAM, 1684 slices, and one multiplier, allowing the computation to occur in 49 clock cycles. Using the parallel architecture, the circuit uses 1.012 KB of RAM, 1267 slices, and two multipliers, allowing the computation to occur in 25 clock cycles. Both architectures can push the circuit frequency to 60 MHz, leading to latencies of 813 ns and 415 ns for serial and parallel implementations, respectively.

As one would expect given the more involved hardware architecture, the generic PWA implementations have greater computation latency. Moreover, the number of used slices is increased by a factor of 10 with respect to the simplicial approximation, which, however, requires more RAM to store data relative to the greater number of regions. Notice, however, that the ongoing trend in computer hardware technology is not, as few decades ago, to push on frequency, but to increase the number of processing units and RAM memory. Evidently, an upper limit to the execution efficiency of hardly parallelizable algorithms (such as optimization) is reached, suggesting the manufacturers to invest in quantity rather than in pure speed. As a consequence, the tradeoff between time and space resources has changed, making RAM occupation an increasingly negligible issue when compared to on-line computation power (as required by on-line MPC). The parameters of the described circuits are summed up in Table 1.

Control law	$PWAS$	$PWA(serial)$	$PWA(parallel)$
RAM (kB)	7.800	1.012	1.012
# Slices	165	1684	1267
# Multipliers	1	1	2
# Clock cycles	12	49	25
Frequency (MHz)	80	60	60
Latency (ns)	150	813	415

Table 1. Parameters relative to the FPGA implementation of the described control laws

7 Conclusions

In this paper, an approximate MPC control law for uncertain LTI systems based on PWAS functions has been proposed, which can be efficiently implemented on digital hardware. The proposed synthesis methods guarantees a-priori the asymptotic convergence of the closed-loop system to a terminal set (or its asymptotic stability in case no external disturbance is present). In particular, the approach with $K = K_n$ does not require the introduction of the additional tuning parameter α, but can be applied only if the simplicial partition is dense enough to obtain a non-empty set $\mathcal{S}_f$. The approach with $K = K_p$, instead, requires the introduction of α, but can be applied also with a coarser simplicial partition. The applicability of the proposed control strategy is effective for the case of small-sized systems, similarly to standard explicit MPC. The theoretical properties of the control law have been proved based on robust MPC synthesis, and the simulation results have confirmed the expected results, both for the theoretical properties of the PWAS controller and for the performance of the related FPGA implementation.

Appendix A: Characterization and computation of RPI sets

Relying for instance on (Blanchini and Miani 2008, Proposition 6.9), one can prove that $\mathcal{R}_\infty$ is a polytope in our case. Nonetheless, an explicit computation of $\mathcal{R}_\infty$ is in general impossible (apart from the very specific case of A_κ nilpotent, as stated by Mayne and Schroeder (1997)). The (not necessarily RPI) polytopic over-approximation $\hat{\mathcal{R}}_\infty$ can be computed using various numerical algorithms: the reader is referred to (Blanchini 1999, Sec. 6.4-6.5), and Rakovic *et al.* (2005) for an overview (an implementation of the algorithm described in the latter paper is also available, see Riverso *et al.* (2013)). The same procedure, in the particular case $\mathcal{W} = 0$, leads to the computation of $\mathcal{R}^n_\infty$ and $\hat{\mathcal{R}}^n_\infty$. Analogous considerations are valid for the characterization of $\mathcal{R}^p_\infty$ and for the computation of $\hat{\mathcal{R}}^p_\infty$, which can be obtained as a polytope after a finite number of iterations of the numerical algorithm described by Kouramas *et al.* (2005) (note that the system with structured feedback uncertainty (22) is equivalent to a linear time-varying system, as highlighted by Kothare *et al.* (1996)).

The set $\mathcal{X}_f$ in (18) can also be conveniently expressed, using tightened constraints, as

$$\mathcal{X}_f = \{x : A_\kappa^k x \in \mathcal{X}_k,\ KA_\kappa^k x \in \mathcal{U}_k,\ \forall k \in \mathbb{Z}_{\geq 0}\} \tag{A1}$$

and can be computed by Algorithm 6.1 in Kolmanovsky and Gilbert (1998) using linear programming. In particular, exploiting the results in Theorems 6.2 and 6.3 in Kolmanovsky and Gilbert (1998), $\mathcal{X}_f$ is finitely generated, if $0 \in \text{int}(\mathcal{X} \sim \mathcal{R}_\infty)$ and $0 \in \text{int}(\mathcal{U} \sim K\mathcal{R}_\infty)$. If $\mathcal{R}_\infty$ is not computable, one can use the above mentioned over-approximation $\hat{\mathcal{R}}_\infty$ instead. Efficient methods for the computation of $\mathcal{X}_f$ are implemented in the MPT Toolbox for MATLAB (Herceg *et al.* 2013).

Appendix B: Proofs

B.1 *Proof of Theorem 4.5*

The first part of the proof holds for both choices of K. We recall that Assumptions (A1)-(A5) in Chisci *et al.* (2001) are automatically satisfied if Assumptions 2.1 and 4.4 hold, together with (14). Therefore, according to Lemma 7 and Theorem 8 in Chisci *et al.* (2001), recursive feasibility is ensured if $x(0) \in \mathcal{F}_N$. Therefore, $x(t) \in \mathcal{X}$ and $u^*(t) \in \mathcal{U}$ for all $t \in \mathbb{Z}_{\geq 0}$. Also, $x(t) \to \mathcal{R}_\infty$ as $t \to \infty$, for all choices of K that are stabilizing for the nominal system (i.e., both K_n or K_p). On the other hand, according to the expression of $\mathcal{X}_f$ in (A1), the evolution of the nominal system given by $\hat{x}(k)$ with initial condition $x \in \mathcal{X}_f$ and $\mu(t+k|t) = 0,\ \forall k = 1, ..., N-1$, fulfills the constraints (20b)-(20c). Also, as noticed in Chisci *et al.* (2001), the constraints $\hat{x}(k) \in \mathcal{X}_k$ and $K\hat{x}(k) \in \mathcal{U}_k$ for $k \geq N$ are equivalent to the terminal constraint (20d). Then, we conclude that $V = [0\ \cdots\ 0]'$ is a feasible solution for (20) whenever $x \in \mathcal{X}_f$, and is the minimizer of (20), since it is the global minimum of the objective function, i.e., $x \in \mathcal{X}_f \Longrightarrow M^*(x) = [0\ \cdots\ 0]'$.

Consider now case (I). Since $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{S}_f)$, then there exists $\epsilon \in \mathbb{R}_{>0}$ arbitrary small, such that $(1+\epsilon)\hat{\mathcal{R}}_\infty \subseteq \text{int}(\mathcal{S}_f)$. Considering that $\hat{\mathcal{R}}_\infty$ is a RPI set for system (13), it is a RPI set for system (21) as well. Therefore, by linearity of the system, $(1+\epsilon)\hat{\mathcal{R}}_\infty$ is also a RPI set for (21). Considering now the actual dynamics (8), from the trivial relation $\hat{\mathcal{R}}_\infty \subset (1+\epsilon)\hat{\mathcal{R}}_\infty$ it follows that, for all initial conditions $x(0) \in \mathcal{F}_N$, there exists $t_1 \in \mathbb{Z}_{\geq 0}$ such that $x(t_1) \in (1+\epsilon)\hat{\mathcal{R}}_\infty$. Since it is assumed that $w(x) = 0$ for all $x \in \mathcal{S}_f$, and $(1+\epsilon)\hat{\mathcal{R}}_\infty$ is positively invariant for the system (21), one has that the system dynamics is given by (21) for all $t \geq t_1$, which leads to the asymptotic convergence of the state of system (8) to $\mathcal{R}^n_\infty$ for all $x(0) \in \mathcal{F}_N$.

Consider now case (*II*). By Assumption 4.4, for any initial condition $x(0) \in \mathcal{F}_N$ there exists $t_2 \in \mathbb{Z}_{\geq 0}$ such that, applying dynamics (8), $x(t_2) \in \mathcal{X}_f$. Considering that $\mathcal{X}_f$ is by definition an RPI set for system (13), we get $u^*(x) = K_p x$ for all $t \geq t_2$. As a consequence, since, given $x(0) \in \mathcal{X}_f$, both $x(t) \to \mathcal{R}_\infty$ and $x(t) \to \mathcal{R}^p_\infty$ as $t \to \infty$, it follows that $x(t) \to \mathcal{R}_\infty \cap \mathcal{R}^p_\infty$ as $t \to \infty$ for all $x(0) \in \mathcal{F}_N$.

In both cases (*I*) and (*II*), if $\mathcal{D} = \{0\}$, it is immediate to see that the asymptotic stability of systems (21) or (22) implies $\mathcal{R}^n_\infty = \mathcal{R}^p_\infty = \{0\}$. Therefore, the origin would be an asymptotically stable equilibrium point with region of attraction equal to $\mathcal{F}_N$. ■

B.2 *Proof of Theorem 5.1*

In both cases (*i*) and (*ii*), conditions (26d)-(26e) allow one to consider $w(x) = u(x) - u^*(x)$ as a disturbance term that satisfies all the requirements to synthesize $u^*(x)$ in (17). Therefore, by application of Theorem 4.5, all the mentioned properties are proved. ■

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List of captions

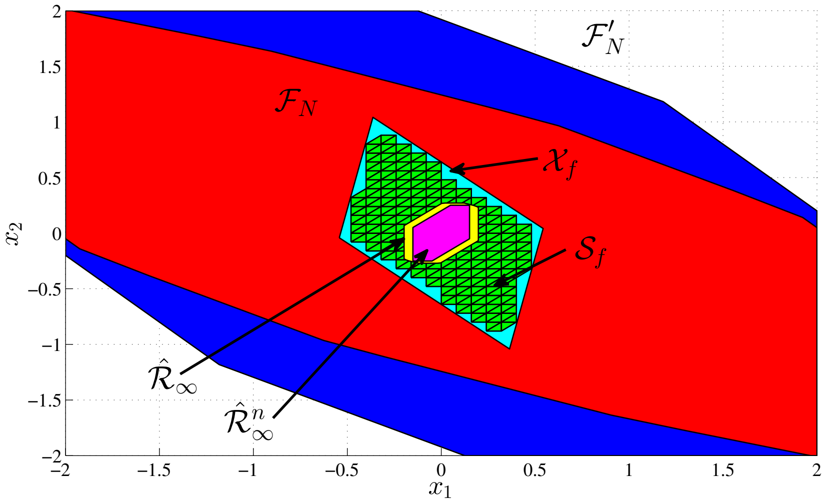

Figure B1. Sets $\mathcal{F}'_N$, $\mathcal{F}_N$, $\mathcal{X}_f$, $\hat{\mathcal{R}}_\infty$, $\hat{\mathcal{R}}^n_\infty$ for the obtained robust MPC control law $u^*(x)$ in Example 1

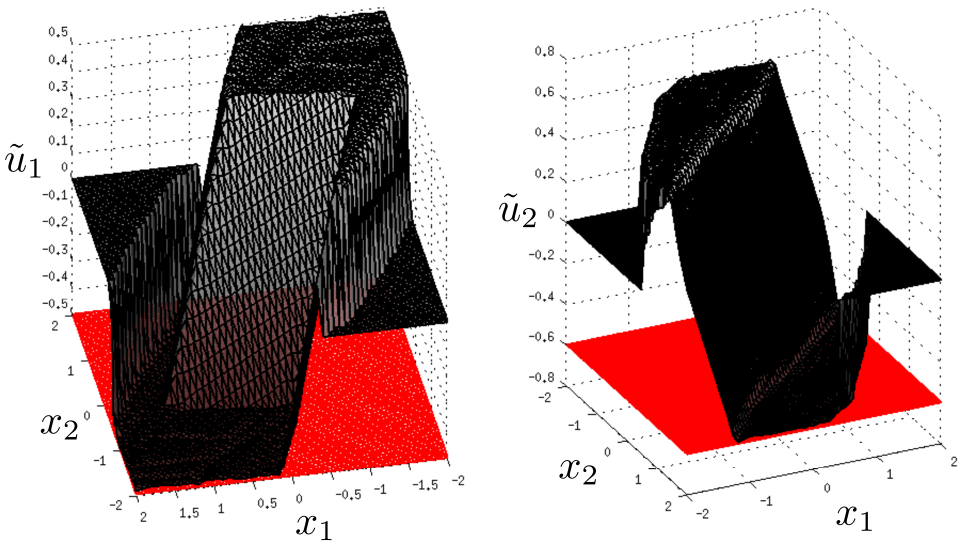

Figure B2. Control function $u(x)$ on the simplicial partition of the set $\mathcal{S}$ in Example 1

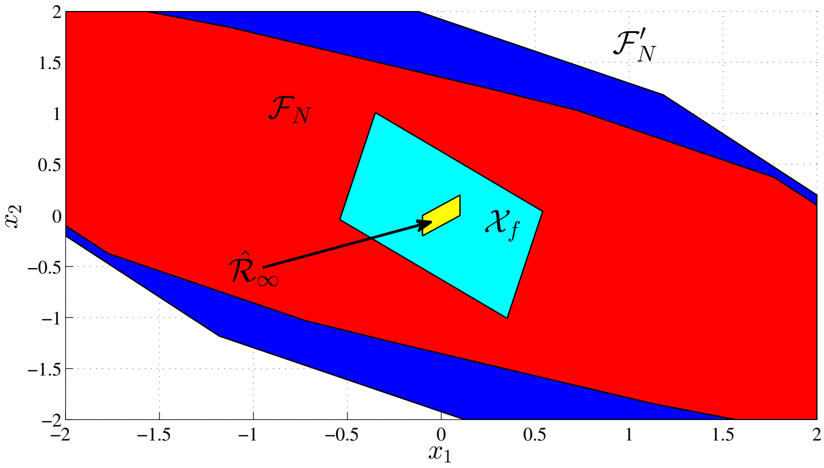

Figure B3. Sets $\mathcal{F}'_N$, $\mathcal{F}_N$, $\mathcal{X}_f$, $\hat{\mathcal{R}}_\infty$ for the obtained robust MPC control law $u^*(x)$ in Example 2

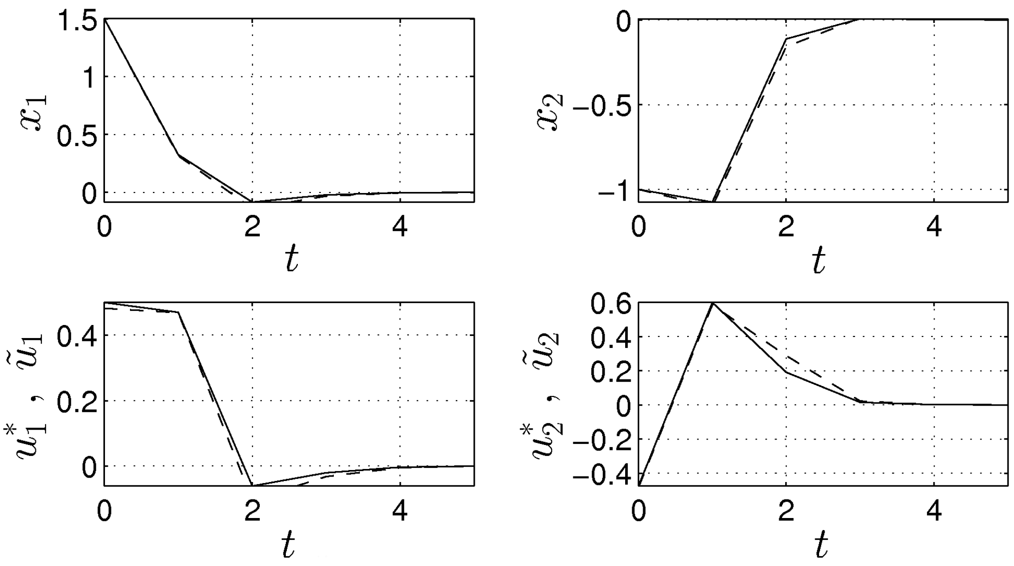

Figure B4. Time evolution of the state and control variables in Example 2 (solid line for optimal values, dashed line for approximate PWAS solution)