OPERATIONS ON GENERALIZED FUNCTIONS

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

by

Emin Özçağ BSc, MSc (Hacettepe Uni.)
Department of Mathematics and Computer Science
University of Leicester

7501799754

Dedicated to my parents and my brothers

and

Zehra Ocaker

CONTENTS

ACKNOWLEDGEMENTS

I wish to express my thanks to my supervisor Dr. B. Fisher for his help and encouragement and Dr. Jeremy Levesley for his helpful comments. I also wish to thank Hacettepe University (Turkey) for their financial support and the mathematics department of Leicester University for their hospitality.

ABSTRACT

In Chapter 1, we give some properties of distributions and introduce the notions of neutrix and neutrix limit with examples, in order to study the problem of defining the convolution product and the product of distributions.

The problem of defining the distribution $x_+^\lambda \ln^s x_+$ such that the ordinary derivative formula is satisfied for all λ and $s = 0, 1, 2, \ldots$ is studied in Chapter 2.

In Chapter 3, we define the Beta function $B_{p,q}(\lambda, \mu)$ using the neutrix limit and prove that this neutrix limit exists for all λ, μ.

In Chapter 4 we let f and g be distributions and let $f_n(x) = f(x)\tau_n(x)$, where $\tau_n(x)$ is a certain function which converges to the identity function as n tends to infinity. We then define the neutrix convolution product $f \circledast g$ as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that $\mathrm{N-lim}_{n\to\infty}\langle f_n * g, \phi\rangle = \langle h, \phi\rangle$ for all ϕ in $\mathcal{D}$. The neutrix convolution products $\ln x_- \circledast x_+^\mu$, $x_-^{-r} \circledast x_+^\mu$, $\ln x_- \circledast \ln x_+$, $\ln x_- \circledast x_+^{-s}$ and $x_-^{-r} \circledast x_+^{-s}$ are evaluated, from which other neutrix convolution products are deduced.

The neutrix convolution product of distributions in Chapter 4 is not commutative. Therefore, in Chapter 5, we consider the commutative neutrix convolution product of distributions, $\boxed{*}$, and also evaluate the neutrix convolution product $x_-^\lambda \boxed{*}\, x_+^{r-\lambda}$.

The problem of defining the product of ultradistributions is considered in Chapter 6, and the neutrix product $(Ff)\Box(Fg)$ in $\mathscr{Z}'$, where F denotes the Fourier transform, is defined as the neutrix limit of $\{F(f\tau_n).F(g\tau_n)\}$. Later, we prove that the exchange formula holds.

We finally define the neutrix product $F(f)\circ G(g)$ of $F(f)$ and $G(g)$, where F and G are distributions and f and g are locally summable functions. It is proved that if

f is infinitely differentiable function with $f'(x) > 0$ and if the neutrix product $F \circ G$ exists and equals H, then the neutrix product $F(f) \circ G(f)$ exists and equals $H(f)$. We also give an alternative approach to the form $F(f(x))$ in $\mathcal{D}'$, where F and f are distributions.

CHAPTER I

DISTRIBUTIONS AND THE NEUTRIX CALCULUS

The problems of defining the convolution product and the product of distributions are well-known. Various definitions of the product of distributions have been considered: one by regularization and passage to the limit (Hirata-Ogata [25], Mikusinski [33], Itano [27], Fisher [5],...) and another one (Hörmander [26]) by means of the Fourier transform. It has been shown that these definitions of product of distributions are not equivalent; see Colombeau [4].

In the case of the convolution product of distributions, the primary definitions are given by Schwartz [34], Shiraishi [35] and more recently by Jones [29], Fisher [7] and Kaminski [30]. Despite many efforts from mathematicians, there are still problems in defining the convolution product and the product of distributions for some cases.

Our main purpose is to extend the convolution product and the product to larger classes of distributions. Therefore, in this chapter, we give some basic properties of distributions and introduce the concepts of neutrix and of neutrix limit.

DISTRIBUTIONS.

The *support* of a function ϕ is the closure of the set on which $\phi(x) \neq 0$. An infinitely differentiable function with compact support is called a test function. The vector space of all test functions is denoted by $\mathcal{D}$.

As an example of test function, consider

$$\phi(x) = \begin{cases} e^{(x-b)^{-1}-(x-a)^{-1}}, & a < x < b, \\ 0, & x \leq a, \ x \geq b. \end{cases}$$

This function ϕ is infinitely differentiable and its support contained in the closed interval $[a, b]$.

We note that the product of an infinitely differentiable function f and a test function ϕ is also a test function.

A sequence $\{\phi_n\}$ of test functions is said to *converge to zero* in $\mathcal{D}$ if all these functions vanish outside some bounded region independent of n and converge uniformly to zero together with the derivatives of any order.

Let an infinitely differentiable function $\phi(x, a)$ be defined as follows:

$$\phi(x,a) = \begin{cases} e^{-a^2(a^2-x^2)^{-1}}, & |x| < a, \\ 0, & |x| \geq a. \end{cases}$$

Then $\{n^{-1}\phi(x,a)\}$ converges to zero in $\mathcal{D}$, but $\{n^{-1}\phi(n^{-1}x,a)\}$ does not, since there exists no common bounded region outside which all these functions vanish.

A functional f on $\mathcal{D}$ satisfying the following conditions is a called a distribution:

(i) For any two real (or complex) numbers α_1 and α_2 and any two functions ϕ_1 and ϕ_2 in $\mathcal{D}$ we have

$$\langle f, \alpha_1\phi_1 + \alpha_2\phi_2 \rangle = \alpha_1\langle f, \phi_1\rangle + \alpha_2\langle f, \phi_2\rangle$$

(ii) If the sequence $\{\phi_n\}$ converges to zero in $\mathcal{D}$, then the sequence $\{\langle f, \phi_n\rangle\}$ converges to zero.

For instance, let f be absolutely integrable in every bounded region of R^n (we call such a function locally summable). By means of such a function we can associate every ϕ in $\mathcal{D}$ with

$$\langle f, \phi\rangle = \int_{R^n} f(x)\phi(x)\, dx \tag{1}$$

where the integral is actually over the support of ϕ. It is easily verified that conditions (i) and (ii) are satisfied for the functional f.

Equation (1) represents a very special kind of continuous linear functional on $\mathcal{D}$. Other kinds of functionals are easily shown to exist. The functional which associates with every $\phi(x)$ its value at $x_0 = 0$ is obviously linear and continuous and cannot be written in the form of (1) with any locally summable function.

Functionals defined by equation (1) will be called *regular* and all others will be called *singular.*

The space of all continuous linear functionals on $\mathcal{D}$ will be denoted by $\mathcal{D}'$; see Schwartz [34].

A distribution f is said to vanish in a neighbourhood $\mathcal{U}$ of x_0 if $\langle f, \phi \rangle = 0$ for all functions ϕ in $\mathcal{D}$ having their support in $\mathcal{U}$. If f is a locally summable function and if f vanishes in a neighborhood $\mathcal{U}$ of x_0 as a distribution, then f vanishes almost everywhere in this neighborhood as a function.

The Dirac-delta function $\delta(x - x_1)$ defined by

$$\langle \delta(x - x_1), \phi(x) \rangle = \phi(x_1)$$

for all ϕ in $\mathcal{D}$, is singular and vanishes in a neighborhood of every point $x \neq x_1$.

If f is a distribution which fails to vanish in any neighborhood of x_0, then x_0 is called an *essential point* of the distribution f. The set of all essential points of a distribution f is called its *support.*

The support of the regular distribution f corresponding to the continuous function f is the closure of the set on which $f(x) \neq 0$, i.e. the support of f .

In order to define the derivative of the distribution, we first of all consider a continuous function f of a single variable, having a continuous first derivative. Then

$$\begin{aligned} \langle f', \phi \rangle &= \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx \\ &= \Big[f(x)\phi(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,dx \\ &= -\langle f, \phi' \rangle, \qquad (2) \end{aligned}$$

for all ϕ in $\mathcal{D}$. If f is now an arbitrary distribution, then the functional g, defined by

$$\langle g, \phi \rangle = -\langle f, \phi' \rangle,$$

will be called the derivative of f and be denoted by f' or df/dx. It can be easily shown that g is also a continuous linear functional on $\mathcal{D}$. Since differentiation of a distribution yields again a distribution, the process may be continued. Thus all distributions have derivatives of all orders.

Let H be the Heaviside function, defined by

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We will denote a distribution corresponding to the Heaviside function by H aswell. Then

$$\begin{aligned} \langle H'(x-x_1), \phi(x) \rangle &= -\langle H(x-x_1), \phi'(x) \rangle = -\int_{x_1}^{\infty} \phi'(x)\, dx \\ &= \phi(x_1) = \langle \delta(x-x_1), \phi(x) \rangle, \end{aligned}$$

for all ϕ in $\mathcal{D}$. In particular, the n-th derivative of δ is defined by

$$\langle H^{(n+1)}(x-x_1), \phi(x) \rangle = \langle \delta^{(n)}(x-x_1), \phi(x) \rangle = (-1)^n \phi^{(n)}(x_1).$$

Let us find the derivative of the locally summable function x_+^λ $(\lambda > -1)$ defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0. \end{cases}$$

If $\lambda > 0$ its derivative is the locally summable function $\lambda x_+^{\lambda-1}$, but, if $-1 < \lambda < 0$, $x_+^{\lambda-1}$ is not a locally summable function. However, we will still denote the derivative of x_+^λ by $\lambda x_+^{\lambda-1}$ on any interval containing the origin, but it must be defined by

$$\langle (x_+^\lambda)', \phi \rangle = \lambda \int_0^\infty x^{\lambda-1} \left[\phi(x) - \phi(0)\right] dx.$$

Thus, if $-2 < \lambda < -1$, we have defined x_+^λ by

$$\langle x_+^\lambda, \phi \rangle = \int_0^\infty x^\lambda \left[\phi(x) - \phi(0)\right] dx.$$

In general, we define x_+^λ inductively by

$$\langle x_+^\lambda, \phi \rangle = -(\lambda+1)^{-1} \langle x_+^{\lambda+1}, \phi' \rangle,$$

for $-r-1 < \lambda < -r$, where r is an integer. It can be proved by induction that if $-r-1 < \lambda < -r$, then

$$\langle x_+^\lambda, \phi \rangle = \int_0^\infty x^\lambda \left[\phi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!} \phi^{(i)}(0) \right] dx$$

It can be proved that any distribution f defined on the bounded interval (a,b) is the r-th derivative of a continuous function F on the interval (a,b); see Halperin [24].

DEFINITION 1.1. *The product of a distribution f by an infinitely differentiable function g is defined by*

$$\langle fg, \phi \rangle = \langle f, g\phi \rangle$$

for ϕ in $\mathcal{D}$.

This is well-defined since $g\phi$ is in $\mathcal{D}$ for all ϕ in $\mathcal{D}$. It follows that if f is the r-th derivative of a ordinary summable function F on the interval (a, b) it can be proved that

$$fg = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[Fg^{(i)} \right]^{(r-i)},$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!};$$

see Halperin [24] or Fisher [5].

This suggests the following definition.

DEFINITION 1.2. *Let f be rth derivative of an ordinary summable function F in $L^p(a, b)$ and $g^{(r)}$ be an ordinary summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product fg on the interval (a, b) is defined by*

$$fg = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[Fg^{(i)} \right]^{(r-i)}.$$

DEFINITION 1.3. *Let f and g be functions. Then the convolution product $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt$$

for all points x for which the integral exists.

It follows easily from the definition that if $(f * g)(x)$ exists then $(g * f)(x)$ exists and

$$(f * g)(x) = (g * f)(x) \tag{3}$$

and if $(f * g)'(x)$ and $(f * g')(x)$ (or $(f' * g)(x)$) exists, then

$$(f * g)'(x) = (f * g')(x) \quad (\text{or } (f' * g)(x)). \tag{4}$$

If f and g are functions in $L^p(-\infty, \infty)$ and $L^q(-\infty, \infty)$ respectively, where $1/p + 1/q = 1$, then the convolution product $(f * g)(x)$ exists for all values of x. The following definition for the convolution product of certain distributions f and g in $\mathcal{D}'$, was given by Gel'fand and Shilov [23].

DEFINITION 1.4. *Let f and g be distributions satisfying either of the following conditions:*

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Then the convolution $f * g$ is defined by

$$\langle (f * g)(x), \phi(x) \rangle = \langle g(y), \langle f(x), \phi(x+y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$.

Note that with this definition, if f has bounded support, then $\langle f(x), \phi(x+y) \rangle$ is in $\mathcal{D}$ and it is therefore meaningful to apply $g(y)$ to it. If, on the other hand, $g(y)$ has bounded support while $f(x)$ may not, $\langle f(x), \phi(x+y) \rangle$ is infinitely differentiable. The above equation remains valid since $g(y)$ has bounded support and can therefore be applied meaningfully to this function. If the supports of f and g are bounded on the same side, then the intersection of the supports of $g(y)$ and $\langle f(x), \phi(x+y)\rangle$ is bounded and so $\langle g(y), \langle f(x), \phi(x+y)\rangle\rangle$ is again meaningful. It follows that if the convolution $f * g$ exists by this definition, then equations (3) and (4) always holds.

A sequence $\{f_n\}$ of distributions is defined to converge to the distribution f if

$$\lim_{n\to\infty} \langle f_n, \phi\rangle = \langle f, \phi\rangle$$

for every ϕ in $\mathcal{D}$.

One important property of the space $\mathcal{D}'$ is its completeness with respect to convergence as defined above. In other words, if the sequence $\{f_n\}$ is such that for every ϕ in $\mathcal{D}$ the number sequence $\langle f_n, \phi\rangle$ has a limit, this limit is again a continuous linear functional on $\mathcal{D}$; see Gel'fand and Shilov [23].

In the forthcoming chapters, we often use the property that every distribution is the limit of a sequence of distributions with support contained in bounded sets; see Gel'fand and Shilov [23] or Jones [29].

A sequence of functions, $\{f_n\}$, is said to be *regular* if

(i) f_n is infinitely differentiable,

(ii) $\langle f_n, \phi\rangle$ converges, for each test function ϕ, to a limit, say $L(\phi)$,

(iii) $L(\phi)$ is continuous in ϕ, in the sense that

$$L(\phi_m) \longrightarrow 0$$

for any sequence $\{\phi_m\}$ of test functions which converges to zero in $\mathcal{D}$; see Temple [36].

There are many ways to construct a regular sequence. In the following, we are going to give a specific example of regular sequence.

Let ρ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

We could for example take ρ to be the function defined by

$$\rho(x) = \begin{cases} k.e^{-(1-x^2)^{-1}}, & -1 < x < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where $k^{-1} = \int_{-1}^{1} e^{-(1-x^2)^{-1}}\, dx$.

We now define the function δ_n by

$$\delta_n(x) = n\rho(nx) \quad \text{for } n = 1, 2, \ldots.$$

It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac-delta function δ.

Now let f be an arbitrary distribution and define f_n by

$$f_n(x) = (f * \delta_n)(x) = \langle f(x-t), \delta_n(t) \rangle.$$

Then $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution f.

NEUTRIX CALCULUS.

The essential use of the neutrix limit is to extract an appropriate finite part from a divergent quantity as one has usually done to subtract the the divergent terms via rather complicated procedures in the renormalization theory. In the neutrix calculus each limit, if properly defined, always exists.

The following two definitions were given by Van der Corput [3].

DEFINITION 1.5. *Let N' be a non-empty set and let N be a commutative additive group of functions mapping N' into a commutative additive group N''. If N has the property that the only constant function in N is the zero function, then N is said to be a neutrix and the functions in N are said to be negligible.*

The property asserts that if ν is in N and $\nu(\epsilon) = \gamma$ for all ϵ in N', then $\gamma = 0$.

EXAMPLE 1.1. Let N' be the closed interval $[0,1]$ and let N be the set of all functions defined on N' of the form $a\sin\epsilon + b\epsilon^2$, where a and b are arbitrary real numbers.

Then N is a neutrix, since if

$$a\sin\epsilon + b\epsilon^2 = c$$

for all ϵ in N', then $a = b = c = 0$.

EXAMPLE 1.2. Let N' be the open domain $\{\epsilon : 0 < \epsilon < 1\}$ and let N be the set of all functions of the form

$$a\epsilon^{-\frac{1}{2}} + b(\log\log\frac{1}{\epsilon})^2 + O(\epsilon),$$

where $O(\epsilon)$ is any function which converges to zero as ϵ tends to zero. Then N is a neutrix, since if

$$a\epsilon^{-\frac{1}{2}} + b(\log\log\frac{1}{\epsilon})^2 + O(\epsilon) = c,$$

then $a = b = c = 0$.

DEFINITION 1.6. *Let N' be a set contained in a topological space with a limit point b which is not in N'. Let N'' be the real (or complex) numbers and let N be a commutative additive group of functions mapping N' into N'' with the property that if N contains a function $\nu(\epsilon)$ which converges to a finite limit c as ϵ tends to b, then $c = 0$. Then N is a neutrix.*

If now $f(\epsilon)$ is a real (or complex) valued function defined on N' and it is possible to find a constant β such that $f(\epsilon) - \beta$ is negligible in N, then β is called the neutrix limit of $f(\epsilon)$ as ϵ tends to b and we write

$$\mathrm{N}\!-\!\lim_{\epsilon\to b} f(\epsilon) = \beta.$$

Note that in this definition N is in fact a neutrix, since if f is in N and $f(\epsilon) = c$ for all ξ in N', then $f(\xi)$ converges to the finite limit c as ϵ tends to b and so $c = 0$.

Also note that if a neutrix limit β exists then it is unique since if $f(\epsilon) - \beta$ and $f(\epsilon) - \beta'$ are in N, then the constant function $\beta - \beta'$ is also in N and so $\beta = \beta'$.

EXAMPLE 1.3. Let N be the neutrix with domain $\mathcal{Z}^+$, the positive integers and having negligible functions $a \log \epsilon + O(\epsilon)$, where $O(\epsilon)$ converges to zero as ϵ tends to ∞. Then,

$$\operatorname*{N-lim}_{\epsilon \to \infty} \sum_{n=1}^{\epsilon} \frac{1}{n} = \gamma,$$

where γ denotes Euler's constant. The assertion follows from the relation

$$\sum_{n=1}^{\epsilon} \frac{1}{n} = \log \epsilon + \Gamma + O(\epsilon).$$

EXAMPLE 1.4. Let N' be the open domain $\{\epsilon : 0 < \epsilon < \infty\}$, let $b = 0$ and let N be finite linear sums of the functions,

$$\epsilon^{\lambda} \ln^{r-1} \epsilon, \quad \ln^r \epsilon, \quad O(\epsilon),$$

where $\lambda < 0$, $r = 1, 2, \ldots$ and $O(\epsilon)$ is any function which converges to zero as ϵ tends to zero. The gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$$

for $x > 0$, and in general we have

$$\Gamma(x) = \operatorname*{N-lim}_{\epsilon \to 0} \int_{\epsilon}^{\infty} t^{x-1} e^{-t}\, dt$$

for $x < 0$ and $x \neq -1, -2, \ldots$; see [21].

EXAMPLE 1.5. Let N' be the open domain $\{\epsilon : 0 < \epsilon < 1/2\}$, let $b = 0$ and let N be as in Example 2. The Beta function $B(\lambda, \mu)$ is defined by

$$B(\lambda, \mu) = \int_0^1 t^{\lambda-1} (1-t)^{\mu-1}\, dt$$

for $\lambda, \mu > 0$ and by

$$\begin{aligned} B(\lambda,\mu) &= \int_0^{1/2} t^{\lambda-1}\left[(1-t)^{\mu-1} - \sum_{i=0}^{r}\frac{(-1)^i\Gamma(\mu)}{i!\Gamma(\mu-i)}t^i\right] dt + \\ &+ \sum_{i=0}^{r}\frac{(-1)^i\Gamma(\mu)}{2^{\lambda+i}i!\Gamma(\mu-i)(\lambda+i)} + \\ &+ \int_{1/2}^{1}(1-t)^{\mu-1}\left[t^{\lambda-1} - \sum_{i=0}^{s}\frac{(-1)^i\Gamma(\lambda)}{i!\Gamma(\lambda-i)}(1-t)^i\right] dt + \\ &+ \sum_{i=0}^{s}\frac{(-1)^i\Gamma(\lambda)}{2^{\mu+i}i!\Gamma(\lambda-i)(\mu+i)} \end{aligned}$$

for $\lambda > -r$, $\mu > -s$, $\lambda \neq 0, 1, 2, \ldots, -r+1$ and $\mu \neq 0, 1, 2, \ldots, -s+1$.

It can be shown that

$$B(\lambda,\mu) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} t^{\lambda-1}(1-t)^{\mu-1}\, dt \tag{5}$$

for $\lambda, \mu \neq 0, 1, 2, \ldots$. More generally we have

$$\frac{\partial^{p+q}}{\partial\lambda^p\partial\mu^q}B(\lambda,\mu) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} t^{\lambda-1}\ln^p t(1-t)^{\mu-1}\ln^q(1-t)\, dt \tag{6}$$

for $p, q = 0, 1, 2, \ldots$ and $\lambda, \mu \neq 0, 1, 2, \ldots$.

As we shall see in Chapter 3, equations (5) and (6) can be used to define $B(\lambda,\mu)$ and $B_{p,q}(\lambda,\mu)$ respectively, for all values of λ, μ.

In the next example, the neutrix N is the one defined in Example 1.4.

EXAMPLE 1.6.

$$\langle x_+^\lambda, \phi(x)\rangle = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^\infty x^\lambda\phi(x)\, dx$$

for $\lambda \neq -1, -2, \ldots$ and arbitrary test function ϕ in $\mathcal{D}$, where the distribution x_+^λ is the locally summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0, \end{cases}$$

for $\lambda > -1$ and is defined inductively by the equation

$$x_+^\lambda = \frac{(x_+^{\lambda+1})'}{\lambda+1}$$

for $-n-1<\lambda<-n$ and $n=1,2,\ldots$. More generally, it can be proved that

$$\langle x_+^\lambda \ln^r x_+, \phi(x)\rangle = \operatorname*{N-lim}_{\epsilon\to 0} \int_\epsilon^\infty x^\lambda \ln^r x \phi(x)\, dx$$

for $\lambda \neq -1,-2,\ldots$, $r=0,1,2,\ldots$ and ϕ in $\mathcal{D}$. These results were proved in [11].

Note that the negligible functions in the neutrix N given in Example 1.4 are selected because these are the functions that occur in mathematics and physics.

CHAPTER II

ON DEFINING THE DISTRIBUTION $x_+^{-r} \ln^s x_+$

In the following we are going to redefine the distribution $x_+^{-r} \ln^s x_+$. For $\lambda > -1$, the distribution x_+^λ is a locally summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0. \end{cases}$$

When $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$, the distribution x_+^λ is defined inductively by the equation

$$(x_+^{\lambda+1})' = (\lambda+1)x_+^\lambda. \tag{1}$$

It follows that if $-r-1 < \lambda < -r$, then

$$\begin{aligned}\langle x_+^\lambda, \phi(x)\rangle &= \int_0^\infty x^\lambda \left[\phi(x) - \sum_{i=0}^{r-1} \frac{\phi^{(i)}(0)}{i!} x^i\right] dx \\ &= \int_0^\infty x^\lambda \left[\phi(x) - \sum_{i=0}^{r-2} \frac{\phi^{(i)}(0)}{i!} x^i - \frac{\phi^{(r-1)}(0)}{(r-1)!} H(1-x) x^{r-1}\right] dx + \frac{\phi^{(r-1)}(0)}{(r-1)!(\lambda+r)},\end{aligned}$$

for an arbitrary test function ϕ in the space $\mathcal{D}$ of infinitely differentiable functions with compact support, where H denotes Heaviside's function. Note that if $r = 1$, then $\sum_{i=0}^{-1}$ is understood to mean an empty sum.

Gel'fand and Shilov [23] define the distribution $F_{-r}(x_+, \lambda)$, when $-r-1 < \lambda < -r$, by the equation

$$\langle F_{-r}(x_+, \lambda), \phi(x)\rangle = \int_0^\infty x^\lambda \left[\phi(x) - \sum_{i=0}^{r-2} \frac{\phi^{(i)}(0)}{i!} x^i - \frac{\phi^{(r-1)}(0)}{(r-1)!} H(1-x) x^{r-1}\right] dx,$$

for arbitrary ϕ in $\mathcal{D}$.

They then define the distribution x_+^{-r} by

$$x_+^{-r} = F_{-r}(x_+, -r) \tag{2}$$

for $r = 1, 2, \ldots$. We will now denote $F_{-r}(x_+, -r)$ simply by $F(x_+, -r)$ and it follows easily that

$$\frac{d}{dx}F(x_+, -r) = -rF(x_+, -r-1) + \frac{(-1)^r}{r!}\delta^{(r)}(x).$$

Thus with x_+^{-r} defined by equation (2), equation (1) is not satisfied with $r = -2, -3, \ldots$.

This seems to be rather unfortunate and so an alternative definition of x_+^{-r} was given in [12] by letting $\ln x_+$ be the locally summable function defined by

$$\ln x_+ = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0, \end{cases}$$

then defining x_+^{-1} by the equation

$$(\ln x_+)' = x_+^{-1} \tag{3}$$

and more generally defining x_+^{-r} inductively by the equation

$$(x_+^{-r+1})' = -(r-1)x_+^{-r} \tag{4}$$

for $r = 2, 3, \ldots$. With this definition of x_+^{-r} preserves the derivative rule, but not product behaviour:

$$\begin{aligned} \langle x.x_+^{-2}, \phi(x)\rangle &= \langle x_+^{-2}, x\phi(x)\rangle = \langle (x_+^{-1})', -x\phi(x)\rangle \\ &= \langle x_+^{-1}, \frac{d}{dx}\{x\phi(x)\}\rangle = \langle x_+^{-1}, \phi(x)\rangle - \phi'(0) \\ &= \langle x_+^{-1}, \phi(x)\rangle + \langle \delta', \phi(x)\rangle \end{aligned}$$

It can be proved easily that

$$x_+^{-1} = F(x_+, -1)$$

and it then follows by induction that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r}{(r-1)!}\psi(r-1)\delta^{(r-1)}(x) \tag{5}$$

for $r = 1, 2, \ldots$, where

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^{r} 1/i, & r \geq 1. \end{cases}$$

The distribution $x_+^\lambda \ln^s x_+$ is defined by

$$\frac{\partial^s}{\partial \lambda^s} x_+^\lambda = x_+^\lambda \ln^s x_+$$

for $\lambda \neq -1, -2, \ldots$ and $s = 1, 2, \ldots$. Then $x_+^\lambda \ln^s x_+$ is a locally summable function for $\lambda > -1$ and

$$\langle x_+^\lambda \ln^s x_+, \phi(x)\rangle = \int_0^\infty x^\lambda \ln^s x \left[\phi(x) - \sum_{i=0}^{r-1} \frac{\phi^{(i)}(0)}{i!} x^i\right] dx$$

$$= \int_0^\infty x^\lambda \ln^s x \left[\phi(x) - \sum_{i=0}^{r-2} \frac{\phi^{(i)}(0)}{i!} x^i - \frac{\phi^{(r-1)}(0)}{(r-1)!} H(1-x) x^{r-1}\right] dx +$$

$$+ \frac{(-1)^s s! \phi^{(r-1)}(0)}{(r-1)!(\lambda + r)^{s+1}}$$

for $-r - 1 < \lambda < -r$, $s = 1, 2, \ldots$ and arbitrary ϕ in $\mathcal{D}$.

It follows easily from the definition that

$$(x_+^\lambda \ln^s x_+)' = \lambda x_+^{\lambda-1} \ln^s x_+ + s x_+^{\lambda-1} \ln^{s-1} x_+, \tag{6}$$

for $\lambda \neq -1, -2, \ldots$ and $s = 0, 1, 2, \ldots$. Although the distribution $x_+^\lambda \ln^s x_+$ is considered as a single entity and not as a product of the distribution x_+^λ and the locally summable function $\ln^s x_+$, equation (6) shows us that differentiation of $x_+^\lambda \ln^s x_+$ acts as if it were such a product.

We now consider the problem of defining $x_+^{-r} \ln^s x_+$ so that equation (6) is satisfied for all λ and $s = 0, 1, 2, \ldots$. Gel'fand and Shilov [23] define $x_+^{-r} \ln^s x_+$ by the equation

$$\frac{\partial^s}{\partial \lambda^s} F_{-r}(x_+, \lambda)\Big]_{\lambda = -r} = x_+^{-r} \ln^s x_+,$$

for $r, s = 1, 2, \ldots$. From now on, we will denote this distribution by

$$F(x_+, -r) \ln^s x_+,$$

so that

$$\langle F(x_+,-r)\ln^s x_+,\phi(x)\rangle$$
$$= \int_0^\infty x^{-r}\ln^s x\left[\phi(x)-\sum_{i=0}^{r-2}\frac{\phi^{(i)}(0)}{i!}x^i-\frac{\phi^{(r-1)}(0)}{(r-1)!}H(1-x)x^{r-1}\right]dx,$$

for arbitary ϕ in $\mathcal{D}$.

THEOREM 2.1.

$$[F(x_+,-r)\ln^s x_+]' = -rF(x_+,-r-1)\ln^s x_+ + sF(x_+,-r-1)\ln^{s-1} x_+$$

for $r,s=1,2,\ldots$.

PROOF. For arbitrary ϕ in $\mathcal{D}$ we have

$$\langle [F(x_+,-r)\ln^s x_+]',\phi(x)\rangle = -\langle F(x_+,-r)\ln^s x_+,\phi'(x)\rangle$$
$$= -\int_0^\infty x^{-r}\ln^s x\left[\phi'(x)-\sum_{i=0}^{r-2}\frac{\phi^{(i+1)}(0)}{i!}x^i-\frac{\phi^{(r)}(0)}{(r-1)!}H(1-x)x^{r-1}\right]dx$$
$$= \int_0^\infty x^{-r-1}\ln^{s-1} x(-r\ln x+s)\left[\phi(x)-\sum_{i=0}^{r-1}\frac{\phi^{(i)}(0)}{i!}x^i-\frac{\phi^{(r)}(0)}{r!}H(1-x)x^{r}\right]dx$$
$$= \langle -rF(x_+,-r-1)\ln^s x_+ + sF(x_+,-r-1)\ln^{s-1} x_+,\phi(x)\rangle,$$

for $r,s=1,2,\ldots$. □

It follows from the theorem that with Gel'fand and Shilov's definition of the distribution $x_+^{-r}\ln^s x_+$, equation (6) is satisfied for all λ and $s=1,2,\ldots$, even though it is not satisfied for $r=-1,-2,\ldots$ when $s=0$.

In order to define $x_+^{-r}\ln^s x_+$ so that equation (6) is satisfied for all λ and $s=0,1,2,\ldots$, we first of all define $x_+^{-1}\ln^s x_+$ by the equation

$$(\ln^{s+1} x_+)' = (s+1)x_+^{-1}\ln^s x_+$$

for $s=0,1,2,\ldots$, so that equation (6) is satisfied with $\lambda=0$ and $s=1,2,\ldots$.

THEOREM 2.2.

$$x_+^{-1}\ln^s x_+ = F(x_+,-1)\ln^s x_+$$

for $s = 0, 1, 2, \ldots$.

PROOF. We have

$$\begin{aligned}
(s+1)\langle x_+^{-1} \ln^s x_+, \phi(x)\rangle &= -\langle \ln^{s+1} x_+, \phi'(x)\rangle \\
&= -\int_0^1 \ln^{s+1} x \, d[\phi(x) - \phi(0)] - \int_1^\infty \ln^{s+1} x \, d\phi(x) \\
&= (s+1)\int_0^\infty x^{-1} \ln^s x [\phi(x) - \phi(0)H(1-x)]\, dx \\
&= (s+1)\langle F(x_+, -1) \ln^s x_+, \phi(x)\rangle
\end{aligned}$$

for $s = 0, 1, 2, \ldots$ and arbitrary ϕ in $\mathcal{D}$. □

More generally we now define $x_+^{-r} \ln^s x_+$ by the equation

$$x_+^{-r} \ln^s x_+ = F(x_+, -r) \ln^s x_+ + \frac{(-1)^r}{(r-1)!} \psi_s(r-1) \delta^{(r-1)}(x)$$

for $r, s = 1, 2, \ldots$, where

$$\psi_s(r) = \begin{cases} 0, & r = 0, \\ s \displaystyle\sum_{i=1}^{r} \frac{\psi_{s-1}(i)}{i}, & r \geq 1 \end{cases}$$

for $s = 1, 2, \ldots$, with the particular case $\psi_0(r)$ being equal to $\psi(r)$ defined above. Note that in the particular case $r = 1$, $x_+^{-1} \ln^s x_+$ is in agreement with Theorem 2.2.

THEOREM 2.3.

$$(x_+^{-r} \ln^s x_+)' = -r x_+^{-r-1} \ln^s x_+ + s x_+^{-r-1} \ln^{s-1} x_+$$

for $r, s = 1, 2, \ldots$.

PROOF. Using the definition of $x_+^{-r} \ln^s x_+$ and Theorem 2.1 we have

$$\begin{aligned}
(x_+^{-r} \ln^s x_+)' &= -rF(x_+, -r-1) \ln^s x_+ + sF(x_+, -r-1) \ln^{s-1} x_+ \\
&\quad + \frac{(-1)^r}{(r-1)!} \psi_s(r-1) \delta^{(r)}(x) \\
&= -r \left[F(x_+, -r-1) \ln^s x_+ + \frac{(-1)^{r+1}}{r!} \psi_s(r) \delta^{(r)}(x) \right] + \\
&\quad + s \left[F(x_+, -r-1) \ln^{s-1} x_+ + \frac{(-1)^{r+1}}{r!} \psi_{s-1}(r) \delta^{(r)}(x) \right] + \\
&\quad + \frac{(-1)^r}{(r-1)!} \left[\psi_s(r-1) - \psi_s(r) + \frac{s}{r} \psi_{s-1}(r) \right] \\
&= -r x_+^{-r-1} \ln^s x_+ + s x_+^{-r-1} \ln^{s-1} x_+,
\end{aligned}$$

for $r, s = 1, 2, \ldots$. □

It follows that with this definition of $x_+^{-r} \ln^s x_+$, equation (6) is satisfied for all λ and $s = 0, 1, 2, \ldots$.

The distribution $x_-^\lambda \ln^s x_-$ is defined by replacing x by $-x$ in the distribution $x_+^\lambda \ln^s x_+$ for $\lambda \neq -1, -2, \ldots$ and $s = 0, 1, 2, \ldots$ and the distribution $F(x_-, -r) \ln^s x_-$ is defined by replacing x by $-x$ in the distribution $F(x_+, -r) \ln^s x_+$ for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. We therefore define the distribution $x_-^{-r} \ln^s x_-$ by replacing x by $-x$ in the distribution $x_+^{-r} \ln^s x_-$ for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. It follows that

$$x_-^{-r} \ln^s x_- = F(x_-, -r) \ln^s x_- - \frac{1}{(r-1)!} \psi_s(r-1) \delta^{(r-1)}(x)$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ and that

$$(x_-^\lambda \ln^s x_-)' = -\lambda x_-^{\lambda - 1} \ln^s x_- - s x_-^{-\lambda - 1} \ln^{s-1} x_-$$

for all λ and $s = 0, 1, 2, \ldots$.

We finally define the distribution $x^{-r} \ln^s |x|$ by

$$x^{-r} \ln^s |x| = x_+^{-r} \ln^s x_+ + (-1)^r x_-^{-r} \ln^s x_-$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. It follows that

$$x^{-r} \ln^s |x| = F(x_+, -r) \ln^s x_+ + (-1)^r F(x_-, -r) \ln^s x_-$$

so that this definition of $x^{-r} \ln^s |x|$ is in agreement with Gel'fand and Shilov's definition. We then of course have

$$(x^{-r} \ln^s |x|)' = -r x^{-r-1} \ln^s |x| + s x^{-r-1} \ln^{s-1} |x|$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$.

CHAPTER III

ON PARTIAL DERIVATIVES OF THE BETA FUNCTION

The Beta function $B(\lambda,\mu)$ is usually defined by

$$B(\lambda,\mu) = \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}\,dt$$

for $\lambda,\mu > 0$. It then follows that

$$B(\lambda,\mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)},$$

where Γ denotes the Gamma function, and this expression is then used to define the Beta function for $\lambda,\mu < 0$ and $\lambda,\mu \neq -1,-2,\ldots$.

It can then be shown, see Gel'fand and Shilov [23], that

$$\begin{aligned} B(\lambda,\mu) &= \int_0^{1/2} t^{\lambda-1}\left[(1-t)^{\mu-1} - \sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!}t^i\right]dt + \\ &+\sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{2^{\lambda+i}i!(\lambda+i)} + \\ &+\int_{1/2}^{1}(1-t)^{\mu-1}\left[t^{\lambda-1} - \sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{i!}(1-t)^i\right]dt + \\ &+\sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{2^{\mu+i}i!(\mu+i)}, \end{aligned}$$

for $\lambda > -r$, $\mu > -s$, $\lambda \neq 0,-1,\ldots -r+1$ and $\mu \neq 0,-1,\ldots,-s+1$, where

$$(\lambda)_i = \begin{cases} 1, & i=0, \\ \prod_{j=0}^{i-1}(\lambda-j), & i \geq 1. \end{cases}$$

In [20], it was shown that

$$\begin{aligned} \int_\epsilon^{1-\epsilon} t^{\lambda-1}(1-t)^{\mu-1}\,dt &= \int_\epsilon^{1/2} t^{\lambda-1}\left[(1-t)^{\mu-1} - \sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!}t^i\right]dt + \\ &+\sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!(\lambda+i)}(2^{-\lambda-i}-\epsilon^{\lambda+i}) + \\ &+\int_{1/2}^{1-\epsilon}(1-t)^{\mu-1}\left[t^{\lambda-1} - \sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{i!}(1-t)^i\right]dt + \\ &+\sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{i!(\mu+i)}(2^{-\mu-i}-\epsilon^{\mu+i}) \end{aligned}$$

for $\lambda > -r$, $\mu > -s$, $\lambda \neq 0, -1, \ldots, -r+1$ and $\mu \neq 0, -1, \ldots, -s+1$, so that

$$B(\lambda, \mu) = \operatorname*{N-lim}_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} t^{\lambda-1}(1-t)^{\mu-1}\, dt,$$

where N is the neutrix having domain $N' = \{\epsilon : 0 < \epsilon < \frac{1}{2}\}$ with negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \ \ln^r \epsilon \quad (\lambda < 0, \ r = 1, 2, \ldots)$$

and all functions of ϵ which converge to zero in the usual sense as ϵ tends to zero; see van der Corput [3].

This suggests the following definition, given in [20], for

$$B_{p,q}(\lambda, \mu) = \frac{\partial^{p+q}}{\partial^p \lambda \partial^q \mu} B(\lambda, \mu)$$

for all values of λ, μ and $p, q = 0, 1, 2, \ldots$:

DEFINITION 3.1. *The function $B_{p,q}(\lambda, \mu)$ is defined by*

$$B_{p,q}(\lambda, \mu) = \operatorname*{N-lim}_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} t^{\lambda-1} \ln^p t\, (1-t)^{\mu-1} \ln^q(1-t)\, dt \tag{1}$$

for $p, q = 0, 1, 2, \ldots$ and all λ, μ.

It is not immediately obvious that the neutrix limit in equation (1) exists and it was proved in [20] that this neutrix limit existed for the case $p = q = 0$. In the following, we prove that this neutrix limit exists for $p, q = 0, 1, 2, \ldots$ and all λ, μ so that $B_{p,q}(\lambda, \mu)$ is well defined.

We first of all need the following lemma:

LEMMA 3.1. *The neutrix limits, as ϵ tends to zero, of the functions*

$$\int_\epsilon^{1/2} t^\lambda \ln^p t \ln^q(1-t)\, dt, \quad \int_{1/2}^{1-\epsilon} (1-t)^\lambda \ln^p t \ln^q(1-t)\, dt$$

exist for $p, q = 0, 1, 2, \ldots$ and all λ.

PROOF. Suppose first of all that $p = q = 0$. Then

$$\int_\epsilon^{1/2} t^\lambda\, dt = \begin{cases} \dfrac{2^{-\lambda-1} - \epsilon^{\lambda+1}}{\lambda+1}, & \lambda \neq -1, \\ -\ln 2 - \ln \epsilon, & \lambda = -1, \end{cases}$$

and so

$$\underset{\epsilon\to 0}{\text{N}-\lim}\int_\epsilon^{1/2} t^\lambda\,dt$$

exists for all λ.

Now suppose that $q = 0$ and that

$$\underset{\epsilon\to 0}{\text{N}-\lim}\int_\epsilon^{1/2} t^\lambda \ln^p t\,dt$$

exists for some positive integer p and all λ. Then

$$\int_\epsilon^{1/2} t^\lambda \ln^{p+1} t\,dt = \begin{cases} \dfrac{-2^{-\lambda-1}\ln^{p+1}2 - \epsilon^{\lambda+1}\ln^{p+1}\epsilon}{\lambda+1} - \dfrac{p+1}{\lambda+1}\displaystyle\int_\epsilon^{1/2} t^\lambda \ln^p t\,dt, & \lambda \neq -1, \\ \dfrac{(-1)^p \ln^{p+2}2 - \ln^{p+2}\epsilon}{p+2}, & \lambda = -1 \end{cases}$$

and it follows by induction that

$$\underset{\epsilon\to 0}{\text{N}-\lim}\int_\epsilon^{1/2} t^\lambda \ln^p t\,dt$$

exists for $p = 0, 1, 2, \ldots$ and all λ.

Finally we note that we can write

$$\ln^q(1-t) = \sum_{i=q}^{\infty} \alpha_{iq} t^i$$

for $q = 1, 2, \ldots$, the expansion being valid for $|t| < 1$. Choosing a positive integer k such that $\lambda + k > -1$, we have

$$\int_\epsilon^{1/2} t^\lambda \ln^p t \ln^q(1-t)\,dt = \sum_{i=1}^{k-1}\alpha_{iq}\int_\epsilon^{1/2} t^{\lambda+i}\ln^p t\,dt + \sum_{i=k}^{\infty}\alpha_{iq}\int_\epsilon^{1/2} t^{\lambda+i}\ln^p t\,dt.$$

It follows from what we have just proved that

$$\underset{\epsilon\to 0}{\text{N}-\lim}\sum_{i=1}^{k-1}\alpha_{iq}\int_\epsilon^{1/2} t^{\lambda+i}\ln^p t\,dt$$

exists and further

$$\begin{aligned} \underset{\epsilon\to 0}{\text{N}-\lim}\sum_{i=k}^{\infty}\alpha_{iq}\int_\epsilon^{1/2} t^{\lambda+i}\ln^p t\,dt &= \lim_{\epsilon\to 0}\sum_{i=k}^{\infty}\alpha_{iq}\int_\epsilon^{1/2} t^{\alpha+i}\ln^p t\,dt \\ &= \sum_{i=k}^{\infty}\alpha_{iq}\int_0^{1/2} t^{\lambda+i}\ln^p t\,dt, \end{aligned}$$

proving that

$$\mathrm{N-}\lim_{\epsilon\to 0}\int_\epsilon^{1/2} t^\lambda \ln^p t \ln^q(1-t)\,dt$$

exists for $p, q = 0, 1, 2, \ldots$ and all λ.

Making the substitution $1 - t = u$ in

$$\int_{1/2}^{1-\epsilon}(1-t)^\lambda \ln^p t \ln^q(1-t)\,dt,$$

it follows that

$$\mathrm{N-}\lim_{\epsilon\to 0}\int_{1/2}^{1-\epsilon}(1-t)^\lambda \ln^p t \ln^q(1-t)\,dt$$

also exists for $p, q = 0, 1, 2, \ldots$ and all λ. □

THEOREM 3.1. *The function $B_{p,q}(\lambda,\mu)$ exists for $p, q = 0, 1, 2, \ldots$ and all λ, μ.*

PROOF. Choose positive integers r, s such that $\lambda > -r$ and $\mu > -s$. Then we can write

$$\int_\epsilon^{1-\epsilon} t^{\lambda-1}\ln^p t\,(1-t)^{\mu-1}\ln^q(1-t)\,dt$$

$$\begin{aligned}
= & \int_\epsilon^{1/2} t^{\lambda-1}\ln^p t\ln^q(1-t)\left[(1-t)^{\mu-1} - \sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!}t^i\right]dt + \\
&+ \sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!}\int_\epsilon^{1/2} t^{\lambda+i-1}\ln^p t\ln^q(1-t)\,dt + \\
&+ \int_{1/2}^{1-\epsilon}\ln^p t(1-t)^{\mu-1}\ln^q(1-t)\left[t^{\lambda-1} - \sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{i!}(1-t)^i\right]dt + \\
&+ \sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{i!}\int_{1/2}^{1-\epsilon}(1-t)^{\mu+i-1}\ln^p t\ln^q(1-t)\,dt.
\end{aligned}$$

We have

$$\lim_{\epsilon\to 0}\int_\epsilon^{1/2} t^{\lambda-1}\ln^p t\ln^q(1-t)\left[(1-t)^{\mu-1} - \sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!}t^i\right]dt$$

$$= \int_0^{1/2} t^{\lambda-1}\ln^p t\ln^q(1-t)\left[(1-t)^{\mu-1} - \sum_{i=0}^{r}\frac{(-1)^i(\mu-1)_i}{i!}t^i\right]dt$$

and

$$\lim_{\epsilon\to 0}\int_{1/2}^{1-\epsilon}\ln^p t(1-t)^{\mu-1}\ln^q(1-t)\left[t^{\lambda-1} - \sum_{i=0}^{s}\frac{(-1)^i(\lambda-1)_i}{i!}(1-t)^i\right]dt$$

$$= \int_{1/2}^{1} \ln^p t(1-t)^{\mu-1} \ln^q(1-t) \left[t^{\lambda-1} - \sum_{i=0}^{s} \frac{(-1)^i(\lambda-1)_i}{i!}(1-t)^i \right] dt,$$

the integrals being convergent. Further, from Lemma 3.1 we see that the neutrix limit of the function

$$\sum_{i=0}^{r} \frac{(-1)^i(\mu-1)_i}{i!} \int_{\epsilon}^{1/2} t^{\lambda+i-1} \ln^p t \ln^q(1-t)\, dt +$$
$$+ \sum_{i=0}^{s} \frac{(-1)^i(\lambda-1)_i}{i!} \int_{1/2}^{1-\epsilon} (1-t)^{\mu+i-1} \ln^p t \ln^q(1-t)\, dt$$

exists, implying that

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{1-\epsilon} t^{\lambda-1} \ln^p t(1-t)^{\mu-1} \ln^q(1-t)\, dt$$

exists. This proves the existence of the function $B_{p,q}(\lambda,\mu)$ for $p,q = 0,1,2,\dots$ and all λ,μ. □

THEOREM 3.2.

$$B_{p,q}(\lambda,\mu) = B_{q,p}(\mu,\lambda)$$

for $p,q = 0,1,2,\dots$ *and all* λ,μ.

The proof of this theorem is trivial.□

In the following, we now evaluate some particular values of $B_{p,q}(\lambda,\mu)$. In order to simplify the proofs, we note that

$$B_{p,q}(\lambda,\mu) = \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{1} t^{\lambda-1} \ln^p t(1-t)^{\mu-1} \ln^q(1-t)\, dt$$

if $\mu > 0$, since the integral is then convergent in the neighbourhood of the point $t = 1$.

THEOREM 3.3.

$$B_{p,0}(0,1) = 0$$

for $p = 1,2,\dots$.

PROOF. We have

$$\int_{\epsilon}^{1} t^{-1} \ln^p t\, dt = -\frac{\ln^{p+1}\epsilon}{p+1}$$

and so

$$B_{p,0}(0,1) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^1 t^{-1}\ln^p t\,dt = 0$$

for $p = 1, 2, \ldots$. $\square$

THEOREM 3.4.

$$B_{p,0}(0, r+1) = \sum_{i=1}^{r}\binom{r}{i}\frac{(-1)^{p+i}p!}{i^{p+1}}$$

for $p, r = 1, 2, \ldots$, *where*

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}.$$

PROOF. We have

$$\int_\epsilon^1 t^{-1}\ln^p t(1-t)^r\,dt = \int_\epsilon^1 t^{-1}\ln^p t\,dt + \sum_{i=1}^{r}(-1)^i\binom{r}{i}\int_\epsilon^1 t^{i-1}\ln^p t\,dt$$

and so

$$\begin{aligned} B_{p,0}(0,r+1) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^1 t^{-1}\ln^p t(1-t)^r\,dt \\ &= B_{p,0}(0,1) + \sum_{i=1}^{r}(-1)^i\binom{r}{i}\int_0^1 t^{i-1}\ln^p t\,dt \\ &= \sum_{i=1}^{r}\binom{r}{i}\frac{(-1)^{p+i}p!}{i^{p+1}}, \end{aligned}$$

for $p, r = 1, 2, \ldots$, since it is easily proved that

$$\int_0^1 t^i \ln^p t\,dt = \frac{(-1)^p p!}{(i+1)^{p+1}}, \tag{2}$$

for $i = 0, 1, 2, \ldots$. $\square$

THEOREM 3.5.

$$B_{p,0}(-n, 1) = -\frac{p!}{n^{p+1}} \tag{3}$$

for $p, n = 1, 2, \ldots$.

PROOF. It is enough to prove for $p = 1$. Integrating by parts we have

$$\int_\epsilon^1 t^{-n-1}\ln t\,dt = n^{-1}\epsilon^{-n}\ln\epsilon + n^{-1}\int_\epsilon^1 t^{-n-1}\,dt$$

and so

$$B_{1,0}(-n,1) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^1 t^{-n-1}\ln t\,dt = -n^{-2}$$

proving equation (3) for $p=1$ and $n=1,2,\dots$. □

More generally we have

THEOREM 3.6.

$$B_{p,0}(-n,r+1) = \begin{cases} \displaystyle\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{i+1}p!}{(n-i)^{p+1}}, & r<n, \\ \displaystyle\sum_{i=0}^{n-1}\binom{r}{i}\frac{(-1)^{i+1}p!}{(n-i)^{p+1}}, & r=n, \end{cases} \tag{4}$$

for $p,n=1,2,\dots$ and $r=0,1,\dots,n$ and

$$B_{p,0}(-n,r+1) = \sum_{\substack{i=0,\\ i\neq n}}^{r}\binom{r}{i}\frac{(-1)^{i+1}p!}{(n-i)^{p+1}}, \tag{5}$$

for $p,n=1,2,\dots$ and $r=n+1,n+2,\dots$.

PROOF. We have

$$\int_\epsilon^1 t^{-n-1}\ln^p t(1-t)^r\,dt = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\int_\epsilon^1 t^{i-n-1}\ln^p t\,dt \tag{6}$$

and so

$$\begin{aligned} B_{p,0}(-n,r+1) &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}\,\mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^1 t^{i-n-1}\ln^p t\,dt \\ &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}B_{p,0}(i-n,1) \end{aligned}$$

for $r=0,1,\dots,n$. Equation (4) follows on using Theorem 3.3 and Theorem 3.5.

When $r\geq n+1$, equation (6) again holds, but this time we have

$$B_{p,0}(-n,r) = \sum_{i=0}^{n}(-1)^i\binom{r}{i}B_{p,0}(i-n,1) + \sum_{i=n+1}^{r}(-1)^i\binom{r}{i}\int_0^1 t^{i-n-1}\ln^p t\,dt$$

and equation (5) follows on using Theorem 3.3 and Theorem 3.5 and equation (2).

□

THEOREM 3.7.

$$B_{p,0}(0,0) = B_{p,0}(1,0) = (-1)^p p!\zeta(p+1), \tag{7}$$

$$B_{p,0}(-1,0) = -p! + (-1)^p p!\zeta(p+1), \tag{8}$$

for $p = 1, 2, \ldots$, *where*

$$\zeta(p) = \sum_{i=1}^{\infty} i^{-p}, \quad p > 1,$$

denotes the zeta function.

PROOF. We have

$$\int_{\epsilon}^{1-\epsilon} t^{-1} \ln^p t (1-t)^{-1}\, dt = \int_{\epsilon}^{1-\epsilon} [t^{-1} + (1-t)^{-1}] \ln^p t\, dt,$$

and so

$$B_{p,0}(0,0) = B_{p,0}(0,1) + B_{p,0}(1,0) = B_{p,0}(1,0),$$

on using Theorem 3.3.

Further,

$$\begin{aligned}
B_{p,0}(1,0) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_0^{1-\epsilon} (1-t)^{-1} \ln^p t\, dt \\
&= \sum_{i=0}^{\infty} \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_0^{1-\epsilon} t^i \ln^p t\, dt \\
&= \sum_{i=0}^{\infty} \frac{(-1)^p p!}{(i+1)^{p+1}} \\
&= (-1)^p p! \zeta(p+1),
\end{aligned}$$

on using equation (2), proving equation (7).

To prove equation (8), we see that

$$\begin{aligned}
B_{p,0}(-1,0) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{1-\epsilon} t^{-2} \ln^p t (1-t)^{-1}\, dt \\
&= \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{1-\epsilon} [t^{-2} + t^{-1} + (1-t)^{-1}] \ln^p t\, dt \\
&= B_{p,0}(-1,1) - B_{p,0}(0,1) + B_{p,0}(1,0) \\
&= -p! + (-1)^p p! \zeta(p+1),
\end{aligned}$$

on using Theorem 3.3 and Theorem 3.5 and equation (7). □

THEOREM 3.8.

$$B_{0,1}(-n,r+1) = \begin{cases} \displaystyle\sum_{i=0}^{r} \frac{(-1)^i}{n-i}\binom{r}{i}\left[\psi(n-i) - \frac{2}{n-i}\right], & r < n, \\ \displaystyle\sum_{i=0}^{n-1} \frac{(-1)^i}{n-i}\binom{r}{i}\left[\psi(n-i) - \frac{2}{n-i}\right] - (-1)^n\zeta(2), & r = n, \end{cases} \tag{9}$$

for $n = 1,2,\ldots$ *and* $r = 0,1,\ldots,n$ *and*

$$\begin{aligned} B_{0,1}(-n,r+1) &= \sum_{i=0}^{n-1} \frac{(-1)^i}{n-i}\binom{r}{i}\left[\psi(n-i) - \frac{2}{n-i}\right] - (-1)^n\binom{r}{n}\zeta(2) + \\ &\quad - \sum_{i=n+1}^{r} \frac{(-1)^i}{i-n}\binom{r}{i}\psi(i-n), \end{aligned} \tag{10}$$

for $n = 1,2,\ldots$ *and* $r = n+1, n+2, \ldots$, *where the function* ψ *is defined as in Chapter 2 .*

PROOF. We have

$$\int_\epsilon^1 t^{-n-1}\ln(1-t)\,dt = n^{-1}\epsilon^{-n}\ln(1-\epsilon) - n^{-1}\int_\epsilon^1 t^{-n}(1-t)^{-1}\,dt$$

and it follows that

$$\begin{aligned} B_{0,1}(-n,1) &= -n^{-2} - n^{-1}B(-n+1,0) \\ &= -n^{-2} + n^{-1}\psi(n-1) \\ &= n^{-1}[\psi(n) - 2n^{-1}], \end{aligned} \tag{11}$$

since it was proved in [20] that

$$B(-n,-r) = -\frac{(n+r)!}{n!r!}[\psi(n) + \psi(r) - 2\psi(n+r)], \tag{12}$$

for $n, r = 0,1,2,\ldots$. Equation (9) is therefore proved for the case $r = 0$ and $n = 1,2,\ldots$.

More generally we have

$$\int_\epsilon^1 t^{-n-1}\ln(1-t)(1-t)^r\,dt = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\int_\epsilon^1 t^{i-n-1}\ln(1-t)\,dt$$

and it follows that

$$B_{0,1}(-n, r+1) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} B_{0,1}(-n+i, 1). \tag{13}$$

Equation (9) now follows on using equations (7) and (11).

To prove equation (10) we note that

$$\begin{aligned} B_{0,1}(s,1) &= \frac{\partial}{\partial\mu}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}\Big]_{\substack{\lambda=s,\\ \mu=1}} \\ &= \frac{\Gamma'(1)}{s} - \frac{\Gamma'(s+1)}{ss!} \\ &= -s^{-1}\psi(s) \end{aligned}$$

and so

$$B_{0,1}(-n+i, 1) = -(i-n)^{-1}\psi(i-n),$$

for $i = n+1, n+2, \ldots$. Equation (10) now follows from equation (13). □

THEOREM 3.9.

$$B_{1,0}(-n, 0) = -\sum_{i=1}^{n} i^{-2} - \zeta(2), \tag{14}$$

$$B_{1,0}(-n+1, -1) = -n\sum_{i=1}^{n} i^{-2} - n\zeta(2) - 1 + \psi(n), \tag{15}$$

for $n = 1, 2, \ldots$.

PROOF. We have

$$\int_{\epsilon}^{1-\epsilon} t^{-2}\ln t(1-t)^{-1}\,dt = \int_{\epsilon}^{1-\epsilon} t^{-1}\ln t[t^{-1} + (1-t)^{-1}]\,dt$$

and so

$$\begin{aligned} B_{1,0}(-1,0) &= B_{1,0}(-1,1) + B_{1,0}(0,0) \\ &= -1 - \zeta(2), \end{aligned}$$

on using equations (3) and (7). Equation (14) is therefore proved for the case $n = 1$.

Now assume that equation (14) holds for some positive n. Then

$$\int_{\epsilon}^{1-\epsilon} t^{-n-2}\ln t(1-t)^{-1}\,dt = \int_{\epsilon}^{1-\epsilon} t^{-n-1}\ln t[t^{-1} + (1-t)^{-1}]\,dt$$

and so

$$\begin{aligned} B_{1,0}(-n-1,0) &= B_{1,0}(-n-1,1)+B_{1,0}(-n,0) \\ &= -\sum_{i=1}^{n+1} i^{-2}-\zeta(2), \end{aligned}$$

on using equation (4) and our assumption. Equation (14) now follows by induction for $n = 1, 2, \ldots$.

To prove equation (15) we note that

$$\begin{aligned} \int_\epsilon^{1-\epsilon} t^{-n-1}\ln t(1-t)^{-1}\,dt &= -n^{-1}\int_\epsilon^{1-\epsilon} \ln t(1-t)^{-1}\,dt^{-n} \\ &= -n^{-1}\left[t^{-n}\ln t(1-t)^{-1}\right]_\epsilon^{1-\epsilon}+ \\ &\quad +n^{-1}\int_\epsilon^{1-\epsilon}[t^{-1}(1-t)^{-1}+\ln t(1-t)^{-2}]t^{-n}\,dt \end{aligned}$$

and so

$$\begin{aligned} B_{1,0}(-n,0) &= n^{-1}[1+B(-n,0)+B_{1,0}(-n+1,-1)] \\ &= n^{-1}[1-\psi(n)+B_{1,0}(-n+1,-1)] \end{aligned}$$

on using equation (12). Equation (15) follows on using equation (14). □

THEOREM 3.10.

$$\begin{aligned} B_{1,0}(-n,-r) = -\frac{1}{n!r!}\sum_{j=0}^{r-1}\sum_{i=0}^{j}\frac{(n+r+i-j-1)!j!}{(j-i+1)i!}+ \\ -\frac{(n+r)!}{n!r!}\left\{\sum_{i=n+1}^{n+r}\frac{\psi(i)}{i}+\sum_{i=0}^{r-1}\frac{\psi(i)}{n+r-i}-2\psi(n+r)[\psi(n+r)-\psi(n)]+\right. \\ \left. +\sum_{i=1}^{n+r} i^{-2}+\zeta(2)\right\} \end{aligned} \tag{16}$$

for $n, r = 1, 2, \ldots$.

PROOF. We note first of all that equation (16) holds for $r = 1$ and $n = 1, 2, \ldots$ by equation (15). We therefore assume that equation (16) holds for some $r-1$ and $n = 1, 2, \ldots$.

We have

$$\begin{aligned}
\int_\epsilon^{1-\epsilon} t^{-n-1}\ln t(1-t)^{-r-1}\,dt &= r^{-1}\int_\epsilon^{1-\epsilon} t^{-n-1}\ln t\,d(1-t)^{-r} \\
&= r^{-1}\left[t^{-n-1}\ln t(1-t)^{-r}\right]_\epsilon^{1-\epsilon}+ \\
&\quad -r^{-1}\int_\epsilon^{1-\epsilon}[t^{-n-2}-(n+1)t^{-n-2}\ln t](1-t)^{-r}\,dt,
\end{aligned}$$

where

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0}(1-\epsilon)^{-n-1}\ln(1-\epsilon)\epsilon^{-r} = -\sum_{i=0}^{r-1}\frac{(n+i)!}{(r-i)i!n!},$$

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0}\epsilon^{-n-1}\ln\epsilon(1-\epsilon)^{-r} = 0$$

and so

$$\begin{aligned}
B_{1,0}(-n,-r) &= -\sum_{i=0}^{r-1}\frac{(n+i)!}{r(r-i)i!n!} - r^{-1}B(-n-1,-r+1)+ \\
&\quad +\frac{n+1}{r}B_{1,0}(-n-1,-r+1) \\
&= -\sum_{i=0}^{r-1}\frac{(n+i)!}{r(r-i)i!n!} - \frac{(n+r)!}{r!(n+1)!}[\psi(n+1)+\psi(r-1)-2\psi(n+r)]+ \\
&\quad -\frac{1}{n!r!}\sum_{j=0}^{r-2}\sum_{i=0}^{j}\frac{(n+r+i-j+1)!j!}{(j-i+1)i!}+
\end{aligned}$$

$$-\frac{(n+r)!}{n!r!}\left\{\sum_{i=n+2}^{n+r}\frac{\psi(i)}{i}+\sum_{i=0}^{r-2}\frac{\psi(i)}{n+r-i}-2\psi(n+r)[\psi(n+r)-\psi(n+1)]+\right.$$
$$\left.+\sum_{i=1}^{n+r}i^{-2}+\zeta(2)\right\}$$

on using our assumption. This equation can be rearranged to give equation (16) which now follows by induction. □

CHAPTER IV

THE NON-COMMUTATIVE CONVOLUTION PRODUCT OF DISTRIBUTIONS

The convolution product of distributions is a very important tool in the theory of integral equations and differential equations. It exists under certain conditions given in Definition 1.3 and Definition 1.4 . However, these definitions are very restrictive and can only be used for a small class of distributions.

In this chapter we shall consider the neutrix convolution product of distributions f and g which extends the classical definition of the convolution product of functions and Gel'fand and Shilov's definition of the convolution product of distributions. This neutrix convolution product is denoted by $f \circledast g$ and is in general non-commutative.

In order to extend the convolution product to a larger class of distributions, Jones [29] gave the following definition.

DEFINITION 4.1. *Let f and g be distributions and let τ be an infinitely differentiable function satisfying the following properties:*

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leq \tau(x) \leq 1$,

(iii) $\tau(x) = 1$ *for* $|x| \leq \frac{1}{2}$,

(iv) $\tau(x) = 0$ *for* $|x| \geq 1$.

Let

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

for $n = 1, 2, \ldots$. Then the convolution $f * g$ is defined as the limit of the sequence $\{f_n * g_n\}$ providing the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all test functions ϕ in $\mathcal{D}$.

In this definition the convolution $f_n * g_n$ exists in the sense of Definition 1.4 since f_n and g_n both have bounded supports.

It is also clear that if the limit of the sequence $\{f_n * g_n\}$ exists, so that the convolution $f * g$ exists, then equation (3) of Chapter 1 holds. However, equation (4) of Chapter 1 need not necessarily hold since Jones proved that

$$1 * \operatorname{sgn} x = x = \operatorname{sgn} x * 1$$

and

$$(1 * \operatorname{sgn} x)' = 1, \quad 1' * \operatorname{sgn} x = 0, \quad 1 * (\operatorname{sgn} x)' = 2.$$

An alternative extension of Definition 1.3 and Definition 1.4 was given in [7] as follows.

DEFINITION 4.2. *Let f and g be distributions and let f_n be defined as in Definition 4.1. Then the convolution $f * g$ is defined as the limit of the sequence $\{f_n * g\}$, providing the limit h exists in the sence that*

$$\lim_{n \to \infty} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle$$

for all test functions ϕ in $\mathcal{D}$.

In this definition the convolution $f_n * g$ is, again in the sense of Definition 1.4, the distribution f_n having bounded support.

We also note that because of the lack of symmetry in this definition the convolution of two distributions is not always commutative.

In the following we give another non-commutative extension of Definition 1.3 and Definition 1.4. This definition is also possibly an extension of Definition 4.2 since not only are all the results proved in [7] in agreement with the new definition but further convolutions exist which are not defined by Definition 4.2.

DEFINITION 4.3. *Let f and g be distributions and let τ_n be the infinitely differentiable function defined by*

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

where τ is defined as in Definition 4.1. Let

$$f_n(x) = f(x)\tau_n(x)$$

for $n = 1, 2, \ldots$ and let N be the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers with negligible functions, finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity. Then the neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, providing the limit h exists in the sense that

$$\underset{n \to \infty}{\mathrm{N-lim}} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle$$

for all test functions ϕ in $\mathcal{D}$.

From now on, we will let N be the neutrix given above.

The convolution $f_n * g$ in this definition is again in the sense of the Definition 1.4, the distribution f_n having bounded support, since the support of τ_n is contained in the interval $(-n - n^{-n}, n + n^{-n})$.

We now give some results on the neutrix convolution product of distributions that we often refer in forthcoming chapters. These were proved in [10].

THEOREM 4.1. *Let f and g be functions in $L^p(-\infty, \infty)$ and $L^q(-\infty, \infty)$ respectively, where $1/p + 1/q = 1$. Then the convolution $f \circledast g$ exists and*

$$f \circledast g = f * g$$

This theorem shows that Definition 4.3 is an extension of Definition 1.3. The next theorem shows that Definition 4.3 is also an extension of Definition 1.4.

THEOREM 4.2. *Let f and g be distributions satisfying either condition (a) or condition (b) of Definition 1.4. Then the convolution $f \circledast g$ exists and*

$$f \circledast g = f * g;$$

see [10].

The convolution in the following example exists in the sense of Definition 4.3, but not in the sense of Definition 4.1 and Definition 4.2.

EXAMPLE 4.1.

$$x^2 \circledast (x^2 + \epsilon^2)^{-1} = \frac{\pi}{\epsilon}(x^2 - \epsilon^2).$$

PROOF. We put

$$(x^2)_n = x^2 \tau_n(x), \quad f_\epsilon(x) = (x^2 + \epsilon^2)^{-1}.$$

Then the convolution $(x^2)_n * f_\epsilon(x)$ exists by Definition 3.1 and

$$\begin{aligned}(x^2)_n * f_\epsilon(x) &= \int_{-\infty}^{\infty} \frac{(x-y)^2 \tau_n(x-y)}{y^2 + \epsilon^3}\, dy \\ &= \int_{-n+x}^{n+x} \frac{(x-y)^2}{y^2+\epsilon^2}\, dy + \int_{n+x}^{n+n^{-n}+x} \frac{(x-y)^2 \tau(x-y)}{y^2+\epsilon^2}\, dy + \\ &\quad + \int_{-n-n^{-n}+x}^{-n+x} \frac{(x-y)^2 \tau(x-y)}{y^2+\epsilon^2}\, dy.\end{aligned}$$

Now,

$$\begin{aligned}\int_{-n+x}^{n+x} \frac{(x-y)^2}{y^2+\epsilon^2}\, dy &= \int_{-n+x}^{n+x} \left[\frac{x^2-\epsilon^2}{y^2+\epsilon^2} - \frac{2xy}{y^2+\epsilon^2} + 1\right] dy \\ &= \frac{x^2-\epsilon^2}{\epsilon}\left[\tan^{-1}\frac{n+x}{\epsilon} - \tan^{-1}\frac{x-n}{\epsilon}\right] - x \ln\frac{(n+x)^2+\epsilon^2}{(n-x)^2+\epsilon^2} + 2n,\end{aligned}$$

$$\left|\int_{n+x}^{n+n^{-n}+x} \frac{(x-y)^2\tau(x-y)}{y^2+\epsilon^2}\, dy\right| \le \frac{(n+n^{-n})^2 n^{-n}}{(n+x)^2+\epsilon^2} = O(n^{-n}),$$

and similarly

$$\int_{-n-n^{-n}+x}^{-n+x} \frac{(x-y)^2\tau(x-y)}{y^2+\epsilon^2}\, dy = O(n^{-n}).$$

It follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle (x^2)_n * f_\epsilon(x), \phi(x)\rangle = \langle \frac{\pi}{\epsilon}(x^2-\epsilon^2), \phi(x)\rangle$$

for arbitrary test function ϕ in $\mathcal{D}$. □

Whether or not there exist distributions f and g which give different results for the convolution defined by Definition 4.2, or for which the convolution $f * g$ exists in the sense of Definition 4.2 but not by Definition 4.3, are open questions.

The next result holds for the convolutions given by Definitions 4.2 and 4.3. However, this result does not hold in general for the convolution given by Definition 4.1.

THEOREM 4.3. *Let f and g be distributions and suppose that the convolution $f \circledast g$ exists. Then the convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'. \tag{1}$$

Note however that equation (4) of Chapter 1 does not necessarily hold for the neutrix convolution product and that *$(f \circledast g)'$ is not necessarily equal to $f' \circledast g$* since

$$\begin{aligned} (\operatorname{sgn} x)' \circledast 1 &= 2\delta \circledast 1 = 2 \\ &\neq 0 = (\operatorname{sgn} x \circledast 1)' \\ &\neq 1 = (\operatorname{sgn} x \circledast 1)'; \end{aligned}$$

see [10].

So far we have described the neutrix convolution in connection with distribution theory. The applicability of this concept to particular problems such as the convolution product of x_-^{-r} and x_+^{μ} and of x_-^{-r} and x_+^{-s} is of great interest. This requires attention since the extraction of the finite part from the divergent terms has to be done properly.

However, the following two theorems were proved in [10] and [15] respectively:

THEOREM 4.4. *The neutrix convolution product $x_-^\lambda \circledast x_+^s$ exists and*

$$x_-^\lambda \circledast x_+^s = x_+^s \circledast x_-^\lambda = (-1)^{s+1} B(\lambda+1, s+1) x_-^{\lambda+s+1}$$

for all $\lambda \neq -2, -3, \ldots$ and $s = 0, 1, 2, \ldots$, where B denotes the Beta function.

THEOREM 4.5. *The neutrix convolution product $x_-^\lambda \circledast x_+^{s-\lambda}$ exists and*

$$\begin{aligned} x_-^\lambda \circledast x_+^{s-\lambda} &= (-1)^{s+1} B(-s-1, s+1-\lambda) x^{s+1} + \\ &\quad + \frac{(-1)^{s+1}(\lambda)_{s+1}}{(s+1)!}[\pi \cot(\pi\lambda) x_+^{s+1} - x^{s+1} \ln|x|], \end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = -1, 0, 1, 2, \ldots$.

The next theorem is an extension of Theorem 4.5 and was proved in [16].

THEOREM 4.6. *The neutrix convolution product $x_-^\lambda \circledast x_+^{-s-\lambda}$ exists and*

$$x_-^\lambda \circledast x_+^{-s-\lambda} = \frac{\pi \cot(\pi\lambda)}{(-1-\lambda)_{s-1}} \delta^{(s-2)}(x) - \frac{(-1)^s (s-2)!}{(-1-\lambda)_{s-1}} x^{-s+1},$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 2, 3, \ldots$.

The next theorem was proved in [17].

THEOREM 4.7. *The neutrix convolution product $x_-^\lambda \circledast x_+^\mu$ exists and*

$$\begin{aligned} x_-^\lambda \circledast x_+^\mu &= B(-\lambda-\mu-1, \mu+1) x_-^{\lambda+\mu+1} + \\ &\quad + B(-\lambda-\mu-1, \lambda+1) x_+^{\lambda+\mu+1}, \end{aligned}$$

for $\lambda, \mu, \lambda+\mu \neq 0, \pm 1, \pm 2, \ldots$.

In the following we are going to consider the neutrix convolution products $x_-^{-r} \circledast x_+^\mu$ and $x_-^\mu \circledast x_+^{-r}$, where x_+^{-r} is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^{(r)}$$

and x_-^{-r} is defined by $x_-^{-r} = (-x)_+^{-r}$. First though we prove

THEOREM 4.8. *The neutrix convolution products* $\ln x_- \circledast x_+^\mu$ *and* $x_-^\mu \circledast \ln x_+$ *exist and*

$$\ln x_- \circledast x_+^\mu = -(\mu+1)^{-1} x_+^{\mu+1} \ln x_+ + \frac{\gamma + \Psi(-\mu-1)}{\mu+1} x_+^{\mu+1}, \tag{2}$$

$$x_-^\mu \circledast \ln x_+ = -(\mu+1)^{-1} x_-^{\mu+1} \ln x_- + \frac{\gamma + \Psi(-\mu-1)}{\mu+1} x_-^{\mu+1} \tag{3}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$, *where* γ *denotes Euler's constant,* $\Psi = \Gamma'/\Gamma$ *and* Γ *denotes the Gamma function.*

PROOF. We will suppose first of all that $\mu > -1$ and $\mu \neq 0, 1, 2, \ldots$ so that x_+^μ is a locally summable function. Put

$$(\ln x_-)_n = \ln x_- \tau_n(x).$$

Then

$$\begin{aligned}
\langle (\ln x_-)_n * x_+^\mu, \phi(x) \rangle &= \langle (\ln y_-)_n, \langle x_+^\mu, \phi(x+y) \rangle \rangle \\
&= \int_{-n-n^{-n}}^0 \ln(-y) \tau_n(y) \int_a^b (x-y)_+^\mu \phi(x)\, dx\, dy \\
&= \int_a^b \phi(x) \int_{-n}^0 \ln(-y)(x-y)_+^\mu\, dy\, dx + \\
&\quad + \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y) \tau_n(y) (x-y)^\mu\, dy\, dx
\end{aligned} \tag{4}$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$ with support of ϕ contained in the interval $[a, b]$.

When $x < 0$, we have, on making the substitution $y = xu^{-1}$,

$$\begin{aligned}
\int_{-n}^0 \ln(-y)(x-y)_+^\mu\, dy &= \int_{-n}^x \ln(-y)(x-y)^\mu\, dy \\
&= (-x)^{\mu+1} \ln(-x) \int_{-x/n}^1 u^{-\mu-2}(1-u)^\mu\, du \\
&\quad -(-x)^{\mu+1} \int_{-x/n}^1 u^{-\mu-2} \ln u (1-u)^\mu\, du \\
&= I_{1n} - I_{2n}.
\end{aligned}$$

Choosing an integer $r > \mu + 1$, we have

$$\int_{-x/n}^{1} u^{-\mu-2}(1-u)^{\mu}\,du = \int_{-x/n}^{1} u^{-\mu-2}\left[(1-u)^{\mu} - \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!}u^i\right] + \\ + \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!(i-\mu-1)}[1-(x/n)^{i-\mu-1}]$$

and it follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_{1n} = B(-\mu-1,\mu+1)(-x)^{\mu+1}\ln(-x) = 0.$$

Further,

$$\int_{-x/n}^{1} u^{-\mu-2}\ln u(1-u)^{\mu}\,du = \int_{-x/n}^{1} u^{-\mu-2}\ln u\left[(1-u)^{\mu} - \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!}u^i\right]du+ \\ -\sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!(i-\mu-1)^2}[(i-\mu-1)(-x/n)^{i-\mu-1}\ln(-x/n)+1-(-x/n)^{i-\mu-1}]$$

and it follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_{2n} = B_{10}(-\mu-1,\mu+1)(-x)^{\mu+1} = 0.$$

Thus,

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-n}^{0}\ln(-y)(x-y)_+^{\mu}\,dy = 0. \tag{5}$$

When $x > 0$, we have, on making the substitution $y = x(1-u^{-1})$,

$$\begin{aligned}\int_{-n}^{0}\ln(-y)(x-y)_+^{\mu}\,dy &= \int_{-n}^{0}\ln(-y)(x-y)^{\mu}\,dy \\ &= x^{\mu+1}\ln x\int_{\frac{x}{x+n}}^{1}u^{-\mu-2}\,du + x^{\mu+1}\int_{\frac{x}{x+n}}^{1}u^{-\mu-2}\ln(1-u)\,du+ \\ &\qquad -x^{\mu+1}\int_{\frac{x}{x+n}}^{1}u^{-\mu-2}\ln u\,du \\ &= I_{3n}+I_{4n}-I_{5n}.\end{aligned}$$

Also,

$$x^{\mu+1}\ln x\int_{\frac{x}{x+n}}^{1}u^{-\mu-2}\,du = -\frac{x^{\mu+1}\ln x}{\mu+1} + \frac{n^{\mu+1}}{\mu+1}(1+x/n)^{\mu+1}\ln x$$

and it follows that

$$\mathrm{N-}\lim_{n\to\infty} I_{3n} = -\frac{x^{\mu+1}\ln x}{\mu+1}.$$

Making the substitution $u = 1 - v$ we have

$$\int_{\frac{x}{x+n}}^{1} u^{-\mu-2}\ln(1-u)\,du = \int_0^{\frac{n}{x+n}} \ln v(1-v)^{-\mu-2}\,dv$$

$$= \int_0^{\frac{n}{x+n}} \ln v\left[(1-v)^{-\mu-2} - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}v^i\right] dv +$$

$$+\sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}\left[\frac{(1+x/n)^{-i-1}\ln(1+x/n)}{i+1} + \frac{(1+x/n)^{-i-1}}{(i+1)^2}\right],$$

and it follows that

$$\begin{aligned}\mathrm{N-}\lim_{n\to\infty}\int_{\frac{x}{x+n}}^{1} u^{-\mu-2}\ln(1-u)\,du &= \int_0^1 \ln v\left[(1-v)^{-\mu-2} - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}v^i\right] dv \\ &\quad -\sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!(i+2)^2}(1+x/n)^{-i-1} \\ &= B_{10}(1,-\mu-1).\end{aligned}$$

Thus,

$$\mathrm{N-}\lim_{n\to\infty} I_{4n} = B_{10}(1,-\mu-1)x^{\mu+1}.$$

Next, we have

$$\int_{\frac{x}{x+n}}^{1} u^{-\mu-2}\ln u\,du = \frac{(x+n)^{\mu+1}[\ln x + \ln(x+n)]}{(\mu+1)x^{\mu+1}} - \frac{1}{(\mu+1)^2} + \frac{(x+n)^{\mu+1}}{(\mu+1)^2x^{\mu+1}},$$

and it follows that

$$\mathrm{N-}\lim_{n\to\infty} I_{5n} = -(\mu+1)^{-2}x^{\mu+1}.$$

Now it is easily proved that the two following equations are true:

$$B_{10}(1,\mu) = \frac{-\gamma - \Psi(1+\mu)}{\mu}$$

and

$$\mu^{-1} + \Psi(\mu) = \Psi(\mu+1).$$

Thus,

$$B_{10}(1,-\mu-1) + (\mu+1)^{-2} = \frac{\gamma + \Psi(-\mu-1)}{\mu+1}.$$

Thus,

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-n}^{0}\ln(-y)(x-y)_+^{\mu}\,dy = -\frac{x^{\mu+1}\ln x}{\mu+1}+\frac{\gamma+\Psi(-\mu-1)}{\mu+1}x^{\mu+1}. \tag{6}$$

Further, with $a \le x \le b$ and $n > -a$, we have

$$\left|\int_{-n-n^{-n}}^{-n}\ln(-y)\tau_n(y)(x-y)^{\mu}\,dy\right| = O(n^{\mu-n}\ln n)$$

and so

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}\ln(-y)\tau_n(y)(x-y)^{\mu}\,dy = 0. \tag{7}$$

It now follows from equations (4), (5), (6) and (7) that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle(\ln x_-)_n * x_+^{\mu},\phi(x)\rangle =$$

$$= \langle -(\mu+1)^{-1}x_+^{\mu+1}\ln x_+ + [\gamma+\Psi(-\mu-1)](\mu+1)^{-1}x_+^{\mu+1},\phi(x)\rangle$$

and equation (2) follows for $\mu > -1$ and $\mu \ne 0,1,2,\dots$.

Now assume that equation (2) holds for $-k < \mu < -k+1$, where k is some positive integer. This is certainly true when $k = 1$. Then by Theorem 4.2, the neutrix convolution product $\ln x_- \circledast x_+^{\mu-1}$ exists and

$$\begin{aligned}\mu\ln x_- \circledast x_+^{\mu-1} &= -x_+^{\mu}\ln x_+ - (\mu+1)^{-1}x_+^{\mu} + [\gamma+\Psi(-\mu-1)]x_+^{\mu}\\ &= -x_+^{\mu}\ln x_+ + [\gamma+\Psi((-\mu)]x_+^{\mu},\end{aligned}$$

since

$$\psi(-\mu-1)-(\mu+1)^{-1} = \Psi(-\mu).$$

Equation (2) follows by induction for $\mu \ne 0,\pm1,\pm2,\dots$.

To prove equation (3) we will, again, suppose first of all that $\mu > -1$ and $\mu \ne 0,1,2,\dots$, so that x_-^{μ} is a locally summable function. Put

$$(x_-^{\mu})_n = x_-\tau_n(x).$$

Then

$$\begin{aligned}
\langle (x_-^\mu)_n * \ln x_+, \phi(x)\rangle &= \langle (y_-^\mu)_n, \langle \ln x_+, \phi(x+y)\rangle\rangle \\
&= \int_{-n-n^{-n}}^{0} (-y)^\mu \tau_n(x) \int_a^b \ln(x-y)_+ \phi(x)\, dx\, dy \\
&= \int_a^b \phi(x) \int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy\, dx + \\
&\quad + \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\mu \tau_n(y) \ln(x-y)\, dy\, dx, \qquad (8)
\end{aligned}$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$ with support of ϕ contained in the interval $[a, b]$.

When $x < 0$, we have, on making the substitution $y = xu^{-1}$,

$$\begin{aligned}
\int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy &= \int_{-n}^{x} (-y)^\mu \ln(x-y)\, dy \\
&= -(-x)^{\mu+1} \int_{-x/n}^{1} u^{-\mu-2} \ln u\, du + \\
&\quad +(-x)^{\mu+1} \ln(-x) \int_{-x/n}^{1} u^{-\mu-2}\, du + \\
&\quad +(-x)^{\mu+1} \int_{-x/n}^{1} u^{-\mu-2} \ln(1-u)\, du \\
&= -J_{1n} + J_{2n} + J_{3n}.
\end{aligned}$$

We have

$$\int_{-x/n}^{1} u^{-\mu-2} \ln u\, du = \frac{(-n/x)^{\mu+1}}{\mu+1} - \frac{1-(-n/x)^{\mu+1}}{(\mu+1)^2}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} J_{1n} = -(\mu+1)^{-2}(-x)^{\mu+1}.$$

Next we have

$$\int_{-x/n}^{1} u^{-\mu-2}\, du = -\frac{1-(-n/x)^{\mu+1}}{\mu+1}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} J_{2n} = -\frac{(-x)^{\mu+1}\ln(-x)}{\mu+1}.$$

Making the substitution $u = 1 - v$, we have

$$\int_{-x/n}^{1} u^{-\mu-2} \ln(1-u)\, du = \int_0^{\frac{x+n}{n}} \ln v (1-v)^{-\mu-2}\, dv$$

$$= \int_0^{\frac{x+n}{n}} \ln v \left[(1-v)^{-\mu-2} - \sum_{i=0}^{r} \frac{(-1)^i(\mu+2)_i}{i!} v^i \right] dv +$$

$$+ \sum_{i=0}^{r} \frac{(-1)^i(\mu+2)_i}{i!} \left[\frac{(1+x/n)^{i+1}\ln(1+x/n)}{i+1} - \frac{(1+x/n)^{i+1}}{(i+1)^2} \right]$$

and it follows, as above, that

$$\operatorname*{N-lim}_{n\to\infty} J_{3n} = B_{10}(1, -\mu-1)(-x)^{\mu+1}.$$

Thus,

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} \int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy &= -\frac{(-x)^{\mu+1}\ln(-x)}{\mu+1} + B_{10}(1,-\mu-1)(-x)^{\mu+1} + \\ &\quad + \frac{(-x)^{\mu+1}}{(\mu+1)^2} \\ &= -\frac{(-x)^{\mu+1}\ln(-x)}{\mu+1} + \frac{\gamma + \Psi(-\mu-1)}{\mu+1}(-x)^{\mu+1}, \end{aligned} \tag{9}$$

as above.

When $x > 0$, we have, on making the substitution $y = x(1-u^{-1})$,

$$\begin{aligned} \int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy &= \int_{-n}^{0} (-y)^\mu \ln(x-y) \, dy \\ &= x^{\mu+1}\ln x \int_{\frac{x}{x+n}}^{1} u^{-\mu-2}(1-u)^\mu \, du - x^{\mu+1} \int_{\frac{x}{x+n}}^{1} u^{-\mu-2} \ln u (1-u)^\mu \, du \\ &= J_{4n} - J_{5n}. \end{aligned}$$

It follows, as above, that

$$\operatorname*{N-lim}_{n\to\infty} J_{4n} = B(-\mu-1, \mu+1)x^{\mu+1}\ln x = 0$$

and

$$\operatorname*{N-lim}_{n\to\infty} J_{5n} = B_{10}(-\mu-1, \mu+1)[x/(x+n)]^{\mu+1} = 0.$$

Thus,

$$\operatorname*{N-lim}_{n\to\infty} \int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy = 0. \tag{10}$$

Further, with $a \le x \le b$ and $n > -a$, we have

$$\left| \int_{-n-n^{-n}}^{-n} (-y)^\mu \tau_n(y) \ln(x-y) \, dy \right| = O(n^{\mu-n}\ln n)$$

and so

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}(-y)^{\mu}\tau_n(y)\ln(x-y)\,dy=0. \tag{11}$$

It now follows from equations (8), (9), (10) and (11) that

$$\operatorname*{N-lim}_{n\to\infty}\langle (x_-^{\mu})_n * \ln x_+, \phi(x)\rangle =$$

$$= \langle -(\mu+1)^{-1}x_-^{\mu+1}\ln x_- + [\gamma+\Psi(-\mu-1)](\mu+1)^{-1}x_-^{\mu+1}, \phi(x)\rangle$$

and equation (3) follows for $\mu > -1$ and $\mu \neq 0,1,2,\dots$.

Finally, assume that equation (3) holds for $-k<\mu<-k+1$. This is certainly true when $k=1$. The convolution product $(x_-^{\mu})_n * \ln x_+$ exists by Definition 1.3 and so equation (4) in Chapter 1 holds. Thus, if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a,b]$,

$$\begin{aligned}\langle [(x_-^{\mu})_n * \ln x_+]', \phi(x)\rangle &= -\langle (x_-^{\mu})_n * \ln x_+, \phi'(x)\rangle \\ &= -\mu\langle (x_-^{\mu-1})_n * \ln x_+, \phi(x)\rangle + \langle [x_-^{\mu}\tau_n'(x)] * \ln x_+, \phi(x)\rangle\end{aligned}$$

and so

$$\mu\langle (x_-^{\mu-1})_n * \ln x_+, \phi(x)\rangle = \langle (x_-^{\mu})_n * \ln x_+, \phi'(x)\rangle + \langle [x_-^{\mu}\tau_n'(x)] * \ln x_+, \phi(x)\rangle.$$

The support of $x_-^{\mu}\tau_n'(x)$ is contained in the interval $[-n-n^{-n},-n]$ and so, with $n>-a$, it follows, as above, that

$$\langle [x_-^{\mu}\tau_n'(x)] * \ln x_+, \phi(x)\rangle = \int_a^b \phi(x)\int_{-n-n^{-n}}^{-n}(-y)^{\mu}\tau_n'(y)\ln(x-y)\,dy\,dx,$$

where, on the domain of integration, $(-y)^{\mu}$ and $\ln(x-y)$ are locally summable functions. Integrating by parts, it follows that

$$\begin{aligned}\int_{-n-n^{-n}}^{-n}(-y)^{\mu}\tau_n'(y)\ln(x-y)\,dy &= n^{\mu}\ln(x+n) + \int_{-n-n^{-n}}^{-n}[\mu(-y)^{\mu-1}\ln(x-y)+ \\ &\quad +(-y)^{\mu}(x-y)^{-1}]\tau_n(y)\,dy.\end{aligned}$$

Choosing a positive integer r greater than μ, we see that

$$n^{\mu}\ln(x+n) = n^{\mu}\ln n + n^{\mu}\sum_{i=1}^{r}\frac{(-1)^i x^i}{in^i} + O(1/n)$$

and so, since μ is not an integer,

$$\mathrm{N}-\lim_{n\to\infty} n^{\mu}\ln(x+n) = 0.$$

Further, it follows as in the proof of equation (7) that

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}[\mu(-y)^{\mu-1}\ln(x-y)+(-y)^{\mu}(x-y)^{-1}]\tau_n(y)\,dy = 0.$$

Thus,

$$\begin{aligned}
\mathrm{N}-\lim_{n\to\infty}\mu\langle(x_-^{\mu-1})_n * \ln x_+, \phi(x)\rangle &= \mathrm{N}-\lim_{n\to\infty}\langle(x_-^{\mu})_n * \ln x_+, \phi'(x)\rangle \\
&= \langle x_-^{\mu} \circledast \ln x_+, \phi'(x)\rangle,
\end{aligned}$$

by assumption. This proves that the neutrix convolution product $x_-^{\mu-1}\circledast\ln x_+$ exists and

$$\begin{aligned}
\mu x_-^{\mu-1} \circledast \ln x_+ &= -(x_-^{\mu} \circledast \ln x_+)' \\
&= -x_-^{\mu}\ln x_- - (\mu+1)^{-1}x_-^{\mu} + [\gamma+\Psi(-\mu-1)]x_-^{\mu} \\
&= -x_-^{\mu}\ln x_- + [\gamma+\Psi(-\mu)]x_-^{\mu},
\end{aligned}$$

as above. Equation (3) now follows by induction for $\mu \neq 0, \pm1, \pm2, \ldots$.□

COROLLARY 4.1. *The neutrix convolution products* $\ln x_+\circledast x_-^{\mu}$, $x_+^{\mu}\circledast\ln x_-$, $\ln|x|\circledast x_+^{\mu}$, $x_+^{\mu}\circledast\ln|x|$, $\ln|x|\circledast x_-^{\mu}$, $x_-^{\mu}\circledast\ln|x|$, $\ln|x|\circledast|x|^{\mu}$ *and* $|x|^{\mu}\circledast\ln|x|$ *exist and*

$$\begin{aligned}
\ln x_+ \circledast x_-^{\mu} &= -(\mu+1)^{-1}x_-^{\mu+1}\ln x_- + \frac{\gamma+\Psi(-\mu-1)}{\mu+1}x_-^{\mu+1}, \\
x_+^{\mu} \circledast \ln x_- &= -(\mu+1)^{-1}x_+^{\mu+1}\ln x_+ + \frac{\gamma+\Psi(-\mu-1)}{\mu+1}x_+^{\mu+1}, \\
\ln|x| \circledast x_+^{\mu} &= \frac{\pi\cot\pi\mu}{\mu+1}x_+^{\mu+1} && (12)\\
&= x_+^{\mu}\circledast\ln|x|, && (13)\\
\ln|x| \circledast x_-^{\mu} &= \frac{\pi\cot\pi\mu}{\mu+1}x_-^{\mu+1} && (14)\\
&= x_-^{\mu}\circledast\ln|x|, && (15)\\
\ln|x| \circledast |x|^{\mu} &= \frac{\pi\cot\pi\mu}{\mu+1}|x|^{\mu} \\
&= |x|^{\mu}\circledast\ln|x|,
\end{aligned}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$.

PROOF. The first two equations follow immediately on replacing x by $-x$ in equations (2) and (3).

The convolution product $\ln x_+ * x_+^\mu$ exists in the sense of Definition 1.4 and it is easily proved that

$$\begin{aligned}
\ln x_+ * x_+^\mu &= (\mu+1)^{-1} x_+^{\mu+1} \ln x_+ + B_{10}(1, \mu+1) x_+^{\mu+1} \\
&= (\mu+1)^{-1} x_+^{\mu+1} \ln x_+ - \frac{\gamma + \Psi(\mu+2)}{\mu+1} x_+^{\mu+1} \\
&= \ln x_+ \circledast x_+^\mu .
\end{aligned}$$

Since the neutrix convolution product is clearly distributive with respect to addition, it follows that

$$\begin{aligned}
\ln x_- \circledast x_+^\mu + \ln x_+ \circledast x_+^\mu &= \ln|x| \circledast x_+^\mu \\
&= \frac{\Psi(-\mu-1) - \Psi(\mu+2)}{\mu+1} x_+^{\mu+1} \\
&= \frac{\pi \cot \pi\mu}{\mu+1} x_+^{\mu+1},
\end{aligned}$$

since it can be easily proved that

$$\Psi(-\mu-1) - \Psi(\mu+2) = \pi \cot \pi\mu .$$

This proves equation (12).

Equation (13) follows on noting that the neutrix convolution products of $\ln x_-$ and $\ln x_+$ with x_+^μ are commutative.

Replacing x by $-x$ in equations (12) and (13) gives us equations (14) and (15).

The last two equations follow from equations (12), (13), (14) and (15) on noting that

$$|x|^\mu = x_+^\mu + x_-^\mu . \square$$

THEOREM 4.9. *The neutrix convolution products* $x_-^{-r} \circledast x_+^\mu$ *and* $x_-^\mu \circledast x_+^{-r}$ *exist and*

$$\begin{aligned} x_-^{-r} \circledast x_+^\mu &= \frac{(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma + \Psi(-\mu+r-1)]x_+^{\mu-r+1}\}, & (16) \\ x_-^\mu \circledast x_+^{-r} &= \frac{(\mu)_{r-1}}{(r-1)!}\{x_-^{\mu-r+1}\ln x_- - [\gamma + \Psi(-\mu+r-1)]x_-^{\mu-r+1}\}, & (17) \end{aligned}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$ *and* $r = 1, 2, \ldots$.

PROOF. We put

$$(x_-^{-r})_n = x_-^{-r}\tau_n(x)$$

for $r = 1, 2, \ldots$, so that

$$(\ln x_-)'_n = -(x_-^{-1})_n + \ln x_- \tau'_n(x).$$

Then, if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, we have, from above

$$\begin{aligned} \langle [(\ln x_-)_n * x_+^\mu]', \phi(x)\rangle &= -\langle (\ln x_-)_n * x_+^\mu, \phi'(x)\rangle \\ &= -\langle (x_-^{-1})_n * x_+^\mu, \phi(x)\rangle + \langle \ln x_- \tau'_n(x), \phi(x)\rangle \end{aligned}$$

and so

$$\langle (x_-^{-1})_n * x_+^\mu, \phi(x)\rangle = \langle (\ln x_-)_n * x_+^\mu, \phi'(x)\rangle + \langle [\ln x_- \tau'_n(x)] * x_+^\mu, \phi(x)\rangle.$$

The support of $\ln x_- \tau'_n(x)$ is contained in the interval $[-n - n^{-n}]$ and so, with $n > -a$, it follows that

$$\langle [\ln x_- \tau'_n(x)] * x_+^\mu, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau'_n(y)(x-y)^\mu \, dy \, dx,$$

where on the domain of integration $\ln(-y)$ and $(x-y)^\mu$ are locally summable functions. Integrating by parts, it follows, as above, that

$$\operatorname*{N-lim}_{n\to\infty} \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau'_n(y)(x-y)^\mu \, dy \, dx = 0.$$

Thus,

$$\begin{aligned}
\underset{n\to\infty}{\mathrm{N-lim}}\langle (x_-^{-1})_n * x_+^\mu, \phi(x)\rangle &= \underset{n\to\infty}{\mathrm{N-lim}}\langle (\ln x_-)_n * x_+^\mu, \phi'(x)\rangle \\
&= \langle \ln x_- \circledast x_+^\mu, \phi'(x)\rangle,
\end{aligned}$$

by assumption. This proves that the neutrix convolution product $x_-^{-1} \circledast x_+^\mu$ exists and

$$\begin{aligned}
x_-^{-1} \circledast x_+^\mu &= -(\ln x_- \circledast x_+^\mu)' \\
&= x_+^\mu \ln x_+ - [\gamma + \Psi(-\mu)]x_+^\mu,
\end{aligned}$$

as above, for $\mu \neq 0, \pm 1, \pm 2, \ldots$. Equation (16) is therefore proved for the case $r = 1$.

Now assume that equation (16) holds for some $r \geq 1$. The convolution product $(x_-^{-r})_n * x_+^\mu$ exists in the sense of Definition 1.4 and so equation (4) in Chapter 1 holds. Thus, if ϕ is an arbitrary function in $\mathcal{D}$,

$$\begin{aligned}
\langle [(x_-^{-r})_n * x_+^\mu]', \phi(x)\rangle &= -\langle (x_-^{-r})_n * x_+^\mu, \phi'(x)\rangle \\
&= r\langle (x_-^{-r-1})_n * x_+^\mu, \phi(x)\rangle + \langle [x_-^{-r}\tau_n'(x)] * x_+^\mu, \phi(x)\rangle
\end{aligned}$$

and so

$$r\langle (x_-^{-r-1})_n * x_+^\mu, \phi(x)\rangle = -\langle (x_-^{-r})_n * x_+^\mu, \phi'(x)\rangle - \langle [x_-^{-r}\tau_n'(x)] * x_+^\mu, \phi(x)\rangle.$$

It follows, as above, that

$$\underset{n\to\infty}{\mathrm{N-lim}}\langle [x_-^{-r}\tau_n'(x)] * x_+^\mu, \phi(x)\rangle = 0$$

and so

$$\begin{aligned}
\underset{n\to\infty}{\mathrm{N-lim}}\; r\langle (x_-^{-r-1})_n * x_+^\mu, \phi(x)\rangle &= -\underset{n\to\infty}{\mathrm{N-lim}}\langle (x_-^{-r})_n * x_+^\mu, \phi'(x)\rangle \\
&= -\langle x_-^{-r} \circledast x_+^\mu, \phi'(x)\rangle,
\end{aligned}$$

by assumption. Thus $x_-^{-r-1} \circledast x_+^\mu$ exists and

$$x_-^{-r-1} \circledast x_+^\mu = r^{-1}(x_-^{-r} \circledast x_+^\mu)'$$

$$= \frac{(\mu)_{r-1}}{r!}\{(\mu-r+1)x_+^{\mu-r}\ln x_+ + x_+^{\mu-r} - (\mu-r+1)[\gamma+\Psi(-\mu+r-1)]x_+^{\mu-r}\}$$

$$= \frac{(\mu)_r}{r!}\{x_+^{\mu-r}\ln x_+ - [\gamma+\Psi(-\mu+r)]x_+^{\mu-r}\}.$$

Equation (16) now follows by induction for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

Using Theorem 4.3, it follows from equation (3), that

$$\begin{aligned}(x_-^\mu \circledast \ln x_+)' &= x_-^\mu \circledast x_+^{-1} \\ &= x_-^\mu \ln x_- - [\gamma + \Psi(-\mu)]x_-^\mu\end{aligned}$$

and equation (17) follows for the case $r = 1$.

Assuming equation (17) holds for some r, it again follows from Theorem 4.3 that

$$(x_-^\mu \circledast x_+^{-r})' = -rx_-^\mu \circledast x_+^{-r-1}$$

$$= \frac{(\mu)_{r-1}}{(r-1)!}\{-(\mu-r+1)x_-^{\mu-r}\ln x_- - x_-^{\mu-r} + (\mu-r+1)[\gamma+\Psi(-\mu+r-1)]x_-^{\mu-r}$$

$$= -\frac{(\mu)_r}{(r-1)!}\{x_-^{\mu-r}\ln x_- - [\gamma+\Psi(-\mu+r)]x_-^{\mu-r}\}.$$

Equation (17) now follows by induction for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.□

COROLLARY 4.2. *The neutrix convolution products $x_+^{-r} \circledast x_-^\mu$, $x_+^\mu \circledast x_-^{-r}$, $x^{-r} \circledast x_+^\mu$, $x_+^\mu \circledast x^{-r}$, $x^{-r} \circledast x_-^\mu$, $x_-^\mu \circledast x^{-r}$ and $x^{-r} \circledast |x|^\mu$ exist and*

$$\begin{aligned}
x_+^{-r} \circledast x_-^\mu &= \frac{(\mu)_{r-1}}{(r-1)!}\{x_-^{\mu-r+1}\ln x_- - [\gamma+\Psi(-\mu+r-1)]x_-^{\mu-r+1}\}, \\
x_+^\mu \circledast x_-^{-r} &= \frac{(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma+\Psi(-\mu+r-1)]x_+^{\mu-r+1}\}, \\
x^{-r} \circledast x_+^\mu &= \frac{(-1)^{r-1}(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}x_+^{\mu-r+1} && (18)\\
&= x_+^\mu \circledast x^{-r}, && (19)\\
x^{-r} \circledast x_-^\mu &= \frac{-(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}x_-^{\mu-r+1} && (20)\\
&= x_-^\mu \circledast x^{-r}, && (21)\\
x^{-r} \circledast |x|^\mu &= \begin{cases} -\dfrac{(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}|x|^{\mu-r+1}, & \text{even } r, \\ \dfrac{(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}\operatorname{sgn} x.|x|^{\mu-r+1}, & \text{odd } r, \end{cases}
\end{aligned}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$ *and* $r = 1, 2, \ldots$.

PROOF. Replacing x by $-x$ in equations (16) and (17) gives us first two results of the corollary.

The convolution product $x_+^{-r} * x_+^{\mu}$ exists by Definition 1.4 and it is easily proved that

$$\begin{aligned} x_+^{-r} * x_+^{\mu} &= \frac{(-1)^{r-1}(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1} \ln x_+ - [\gamma + \Psi(\mu - r + 2)]x_+^{\mu-r+1}\} \\ &= x_+^{-r} \circledast x_+^{\mu}. \end{aligned}$$

Since

$$x^{-r} = x_+^{-r} + (-1)^r x_-^{-r},$$

we have

$$\begin{aligned} x_+^{-r} \circledast x_+^{\mu} + (-1)^r x_-^{-r} \circledast x_+^{\mu} &= x^{-r} \circledast x_+^{\mu} \\ &= \frac{(-1)^r(\mu)_{r-1}}{(r-1)!}[\Psi(\mu - r + 2) - \Psi(-\mu + r - 1)]x_+^{\mu-r+1} \\ &= \frac{(-1)^{r-1}(\mu)_{r-1} \cot \pi\mu}{(r-1)!} x_+^{\mu-r+1}, \end{aligned}$$

because

$$\Psi(\mu - r + 2) - \Psi(-\mu + r - 1) = -\cot \pi(\mu - r) = -\cot \pi\mu.$$

This proves equation (18).

Equation (19) follows on noting that the neutrix convolution products of x_-^{-r} and x_+^{-r} with x_+^{μ} are commutative.

Replacing x by $-x$ in equations (18) and (19) gives us equations (20) and (21).

The last two results follow from equations (18), (19), (20) and (21) on noting that

$$|x|^{\mu} = x_+^{\mu} + x_-^{\mu}, \quad \operatorname{sgn} x.|x|^{\mu} = x_+^{\mu} - x_-^{\mu}.\square$$

In the following we will consider the neutrix convolution products $\ln x_- \circledast \ln x_+$ and $x_-^{-r} \circledast x_+^{-s}$ for $r, s = 1, 2, \ldots$. First of all, we have

THEOREM 4.10. *The neutrix convolution products* $\ln x_- \circledast \ln x_+$ *and* $\ln x_+ \circledast \ln x_-$ *exist and*

$$\begin{aligned}\ln x_- \circledast \ln x_+ &= -(\pi^2/6+1)|x| + |x|\ln|x| - \tfrac{1}{2}|x|\ln^2|x| \qquad (22)\\ &= \ln x_+ \circledast \ln x_-.\end{aligned}$$

PROOF. Putting

$$(\ln x_-)_n = \ln x_- \tau_n(x),$$

we have

$$\begin{aligned}\langle(\ln x_-)_n * \ln x_+, \phi(x)\rangle &= \langle(\ln y_-)_n, \langle \ln x_+, \phi(x+y)\rangle\rangle\\ &= \int_{-n-n^{-n}}^{0} \ln(-y)\tau_n(y)\int_a^b \ln(x-y)_+\phi(x)\,dx\,dy\\ &= \int_a^b \phi(x)\int_{-n}^{0} \ln(-y)\ln(x-y)_+\,dy\,dx +\\ &\quad + \int_a^b \phi(x)\int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)\ln(x-y)\,dy\,dx, \qquad (23)\end{aligned}$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$, with support contained in the interval $[a, b]$.

When $x < 0$, we have, on making the substitution $y = xu^{-1}$,

$$\begin{aligned}\int_{-n}^{0} \ln(-y)\ln(x-y)_+\,dy &= \int_{-n}^{x} \ln(-y)\ln(x-y)\,dy\\ &= (-x)\int_{-x/n}^{1} [\ln^2 u - \ln u\ln(1-u) + \ln(-x)\ln(1-u) +\\ &\quad -2\ln(-x)\ln u + \ln^2(-x)]u^{-2}\,du\\ &= I_{1n} - I_{2n} + I_{3n} - I_{4n} + I_{5n}. \qquad (24)\end{aligned}$$

We first of all note that

$$\begin{aligned}\int u^{-2}\ln u\,du &= -u^{-1} - u^{-1}\ln u, \qquad (25)\\ \int u^{-2}\ln^2 u\,du &= -2u^{-1} - 2u^{-1}\ln u - u^{-1}\ln^2 u, \qquad (26)\\ \int u^{-2}\ln(1-u)\,du &= -(u^{-1}-1)\ln(1-u) - \ln u, \qquad (27)\\ \int u^{-2}\ln u\ln(1-u)\,du &= -\ln u - \tfrac{1}{2}\ln^2 u - (u^{-1}-1)\ln(1-u) +\\ &\quad -(u^{-1}-1)\ln u\ln(1-u) + \sum_{i=1}^{\infty}\frac{u^i}{i^2}. \qquad (28)\end{aligned}$$

Using equation (26) we have

$$\int_{-x/n}^{1} u^{-2}\ln^2 u\,du = -2-2nx^{-1}-2nx^{-1}[\ln(-x)-\ln n]-nx^{-1}[\ln(-x)-\ln n]^2$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} I_{1n} = -2(-x). \tag{29}$$

Using equation (28), and noting that $\sum_{i=1}^{\infty} i^{-2} = \frac{\pi^2}{6}$, we have

$$\begin{aligned}\int_{-x/n}^{1} u^{-2}\ln u\ln(1-u)\,du &= \pi^2/6+\ln(-x)-\ln n+\tfrac{1}{2}[\ln(-x)-\ln n]^2\\ &\quad -(nx^{-1}+1)\ln(1+xn^{-1})[1+\ln(-x)-\ln n]+\\ &\quad -\sum_{i=1}^{\infty}\frac{(-x)^i}{i^2n^i}\end{aligned}$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} I_{2n} = (\pi^2/6-1)(-x)+\tfrac{1}{2}(-x)\ln^2(-x). \tag{30}$$

Using equation (27) we have

$$\int_{-x/n}^{1} u^{-2}\ln(1-u)\,du = -(nx^{-1}+1)\ln(1+xn^{-1})+\ln(-x)-\ln n$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} I_{3n} = -(-x)\ln(-x)+(-x)\ln^2(-x). \tag{31}$$

Using equation (25) we have

$$\int_{-x/n}^{1} u^{-2}\ln u\,du = -1-nx^{-1}-nx^{-1}[\ln(-x)-\ln n]$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} I_{4n} = -2(-x)\ln(-x). \tag{32}$$

Finally, we have

$$\int_{-x/n}^{1} u^{-2}\,du = -1-nx^{-1}$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} I_{5n} = -(-x)\ln^2(-x). \tag{33}$$

It now follows from equations (24), (29), (30), (31), (32) and (33) that

$$\int_{-n}^{0} \ln(-y)\ln(x-y)_+\,dx = -(\pi^2/6+1)(-x) + (-x)\ln(-x) + -\tfrac{1}{2}(-x)\ln^2(-x). \tag{34}$$

When $x > 0$, we have on, making the substitution $y = x(1-u^{-1})$,

$$\int_{-n}^{0} \ln(-y)\ln(x-y)_+\,dy = \int_{-n}^{0} \ln(-y)\ln(x-y)\,dy$$

$$\begin{aligned} &= x\int_{\frac{x}{x+n}}^{1} [\ln^2 u - \ln u \ln(1-u) + \ln x \ln(1-u) - 2\ln x \ln u + \ln^2 x]u^{-2}\,du \\ &= J_{1n} - J_{2n} + J_{3n} - J_{4n} + J_{5n}. \end{aligned} \tag{35}$$

Using equation (26) we have

$$\begin{aligned} \int_{\frac{x}{x+n}}^{1} u^{-2}\ln^2 u\,du &= -2 + 2(1+nx^{-1}) + 2(1+nx^{-1})[\ln x - \ln n - \ln(1+xn^{-1})] + \\ &+(1+nx^{-1})[\ln x - \ln n - \ln(1+xn^{-1})]^2 \end{aligned}$$

and it follows that

$$\mathrm{N}\text{-}\lim_{n\to\infty} J_{1n} = -2x + x\ln^2 x. \tag{36}$$

Using equation (28) we have

$$\begin{aligned} \int_{\frac{x}{x+n}}^{1} u^{-2}\ln u \ln(1-u)\,du &= \pi^2/6 + \ln x - \ln n - \ln(1+xn^{-1}) + \\ &+\tfrac{1}{2}[\ln x - \ln n - \ln(1+xn^{-1})]^2 - nx^{-1}\ln(1+xn^{-1}) \\ &-nx^{-1}[\ln x - \ln n - \ln(1+xn^{-1})]\ln(1+xn^{-1}) + \\ &-\sum_{i=1}^{\infty} \frac{x^i}{i^2(x+n)^i} \end{aligned}$$

and it follows that

$$\mathrm{N}\text{-}\lim_{n\to\infty} J_{2n} = (\pi^2/6 - 1)x + \tfrac{1}{2}x\ln^2 x. \tag{37}$$

Using equation (27) we have

$$\int_{\frac{x}{x+n}}^{1} u^{-2}\ln(1-u)\,du = -nx^{-1}\ln(1+xn^{-1}) + \ln x - \ln n - \ln(1+xn^{-1})$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} J_{3n} = -x\ln x + x\ln^2 x. \tag{38}$$

Using equation (25) we have

$$\int_{\frac{x}{x+n}}^{1} u^{-2}\ln u\,du = -1 + 1 + nx^{-1} + (1 + nx^{-1})[\ln x - \ln n - \ln(1 + xn^{-1})]$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} J_{4n} = -2x\ln x + 2x\ln^2 x. \tag{39}$$

Finally, we have

$$\int_{\frac{x}{x+n}}^{1} u^{-2}\,du = -1 + 1 + nx^{-1}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} J_{5n} = 0. \tag{40}$$

It now follows from equations (35), (36), (37), (38), (39) and (40) that

$$\int_{-n}^{0} \ln(-y)\ln(x-y)_+\,dy = -(\pi^2/6+1)x + x\ln x - \tfrac{1}{2}x\ln^2 x. \tag{41}$$

Further, with $a \le x \le b$ and $n > -a$, we have

$$\left|\int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)\ln(x-y)\,dy\right| = O(n^{-n}\ln^2 n)$$

and so

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)\ln(x-y)\,dy = 0. \tag{42}$$

It now follows from equations (24), (34), (41) and (42) that

$$\operatorname*{N-lim}_{n\to\infty}\langle(\ln x_-)_n * \ln x_+, \phi(x)\rangle = \langle -(\pi^2/6+1)|x| + |x|\ln|x| - \tfrac{1}{2}|x|\ln^2|x|, \phi(x)\rangle$$

and equation (22) follows.□

COROLLARY 4.3. *The neutrix convolution products* $\ln|x| \circledast \ln x_+$, $\ln x_+ \circledast \ln|x|$, $\ln|x| \circledast \ln x_-$, $\ln x_- \circledast \ln|x|$ *and* $\ln|x| \circledast \ln|x|$ *exist and*

$$\begin{aligned}
\ln|x| \circledast \ln x_+ &= -(\pi^2/3-1)x - \frac{\pi^2}{2}x_- - x\ln|x| + \tfrac{1}{2}x\ln^2|x| && (43)\\
&= \ln x_+ \circledast \ln|x|, && (44)\\
\ln|x| \circledast \ln x_- &= (\pi^2/3-1)x - \frac{\pi^2}{2}x_+ + x\ln|x| - \tfrac{1}{2}x\ln^2|x| && (45)\\
&= \ln x_- \circledast \ln|x|, && (46)\\
\ln|x| \circledast \ln|x| &= -\frac{\pi^2}{2}|x|. && (47)
\end{aligned}$$

PROOF. The convolution product $\ln x_+ * \ln x_+$ exists in the sense of Definition 1.4 and it is easily proved that

$$\begin{aligned}
\ln x_+ * \ln x_+ &= (2-\pi^2/6)x_+ - 2x_+\ln x_+ + x_+\ln^2 x_+ && (48)\\
&= \ln x_+ \circledast \ln x_+
\end{aligned}$$

Since the neutrix convolution product is clearly distributive with respect to addition, it follows that

$$\ln x_- \circledast \ln x_+ + \ln x_+ \circledast \ln x_+ = \ln|x| \circledast \ln x_+.$$

Equation (43) now follows from equations (22) and (48). Equation (44) follows on noting that the neutrix convolution product of $\ln x_-$ and $\ln x_+$ is commutative. Equations (45) and (46) follow from equations (43) and (44) respectively on replacing x by $-x$. Equation (47) follows from equations (43) and (45) on noting that

$$\ln|x| \circledast \ln x_+ + \ln|x| \circledast \ln x_- = \ln|x| \circledast \ln|x|.\square$$

THEOREM 4.11. *The neutrix convolution products* $\ln x_- \circledast x_+^{-s}$, $x_-^{-s} \circledast \ln x_+$, $\ln x_+ \circledast x_-^{-s}$ *and* $x_+^{-s} \circledast \ln x_-$ *exist and*

$$\ln x_- \circledast x_+^{-1} = -\frac{\pi^2}{6}\operatorname{sgn} x - \tfrac{1}{2}\operatorname{sgn} x \ln^2|x| \qquad (49)$$

$$
\begin{aligned}
&= -x_-^{-1} \circledast \ln x_+ && (50)\\
&= -\ln x_+ \circledast x_-^{-1} && (51)\\
&= x_+^{-1} \circledast \ln x_-, && (52)\\
(s-1)\ln x_- \circledast x_+^{-s} &= \frac{(-1)^s\pi^2}{3(s-2)!}\delta^{(s-2)}(x) - \psi(s-2)[x_+^{-s+1} + (-1)^s x_-^{-s+1}] + \\
&\quad + x_+^{-s+1}\ln x_+ + (-1)^s x_-^{-s+1}\ln x_- && (53)\\
&= (-1)^s(s-1)x_-^{-s} \circledast \ln x_+, && (54)\\
(s-1)\ln x_+ \circledast x_-^{-s} &= \frac{\pi^2}{3(s-2)!}\delta^{(s-2)}(x) - \psi(s-2)[(-1)^s x_+^{-s+1} + x_-^{-s+1}] + \\
&\quad + (-1)^s x_+^{-s+1}\ln x_+ + x_-^{-s+1}\ln x_- && (55)\\
&= (-1)^s(s-1)x_+^{-s} \circledast \ln x_-, && (56)
\end{aligned}
$$

for $s = 2, 3, \ldots$, *where*

$$
\begin{aligned}
(\ln^2 x_+)' &= 2x_+^{-1}\ln x_+,\\
(x_+^{-s+1}\ln x_+)' &= x_+^{-s} - (s-1)x_+^{-s}\ln x_+,
\end{aligned}
$$

for $s = 2, 3, \ldots$.

PROOF. Using Theorem 4.3 and equation (22) we have

$$
\begin{aligned}
(\ln x_- \circledast \ln x_+)' &= \ln x_- \circledast x_+^{-1}\\
&= -(\pi^2/6 + 1)\operatorname{sgn} x + \operatorname{sgn} x \ln|x| + \operatorname{sgn} x + \\
&\quad -\tfrac{1}{2}\operatorname{sgn} x \ln^2|x| - \operatorname{sgn} x \ln|x|\\
&= -\frac{\pi^2}{6}\operatorname{sgn} x - \tfrac{1}{2}\operatorname{sgn} x \ln^2|x|,
\end{aligned}
$$

giving equation (49).

Replacing x by $-x$ in equation (49) gives equation (51).

The convolution product $(\ln x_-)_n * \ln x_+$ exists in the sense of Definition 1.4 and so equation (4) of Chapter 1 holds. Thus, if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$,

$$
\begin{aligned}
\langle[(\ln x_-)_n * \ln x_+]', \phi(x)\rangle &= -\langle(\ln x_-)_n * \ln x_+, \phi'(x)\rangle\\
&= -\langle(x_-^{-1})_n * \ln x_+, \phi(x)\rangle + \langle[\ln x_- \tau_n'(x)] * \ln x_+, \phi(x)\rangle,
\end{aligned}
$$

and so

$$\langle (x_-^{-1})_n * \ln x_+, \phi(x)\rangle = \langle (\ln x_-)_n * \ln x_+, \phi'(x)\rangle + \langle [\ln x_- \, \tau_n'(x)] * \ln_+, \phi(x)\rangle.$$

The support of $\ln x_- \, \tau_n'(x)$ is contained in the interval $[-n-n^{-n}, -n]$ and so, with $n > -a$, it follows, as above, that

$$\langle [\ln x_- \, \tau_n'(x)] * \ln x_+, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\, \tau_n'(y) \ln(x-y)\, dy\, dx.$$

Integrating by parts we have

$$\begin{aligned}\int_{-n-n^{-n}}^{-n} \ln(-y)\, \tau_n'(y) \ln(x-y)\, dy &= \ln n \ln(x+n) + \\ &- \int_{-n-n^{-n}}^{-n} [y^{-1} \ln(x-y) - (x-y)^{-1} \ln(-y)]\, \tau_n(y)\, dy.\end{aligned}$$

Now,

$$\ln n \ln(x+n) = \ln^2 n + O(n^{-1} \ln n)$$

and so

$$\mathrm{N}\!-\!\lim_{n\to\infty}(\ln n \ln(x+n)) = 0.$$

Further,

$$\int_{-n-n^{-n}}^{-n} [y^{-1} \ln(x-y) - (x-y)^{-1} \ln(-y)] \tau_n(y)\, dy = O(n^{-1} \ln n)$$

and so

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} [y^{-1} \ln(x-y) - (x-y)^{-1} \ln(-y) \tau_n(y)]\, dy = 0.$$

Thus,

$$\begin{aligned}\mathrm{N}\!-\!\lim_{n\to\infty} \langle (x_-^{-1})_n * \ln x_+, \phi(x)\rangle &= \mathrm{N}\!-\!\lim_{n\to\infty} \langle (\ln x_-)_n * \ln x_+, \phi'(x)\rangle \\ &= \langle \ln x_- \circledast \ln x_+, \phi'(x)\rangle,\end{aligned}$$

proving that the convolution product $x_-^{-1} \circledast \ln x_+$ exists and

$$x_-^{-1} \circledast \ln x_+ = -(\ln x_- \circledast \ln x_+)'.$$

Equation (50) now follows as above.

Equation (52) follows from equation (50) on replacing x by $-x$.

Using Theorem 4.3 and equation (49) we have

$$-\ln x_- \circledast x_+^{-2} = -\frac{\pi^2}{3}\delta(x) - x_+^{-1}\ln x_+ - x_-^{-1}\ln x_-,$$

and equation (53) follows for the case $s = 2$.

Now assume that equation (53) holds for some s. Then, using Theorem 4.3, we have

$$\begin{aligned}
-s(s-1)\ln x_- \circledast x_+^{-s-1} &= \frac{(-1)^s\pi^2}{3(s-2)!}\delta^{(s-1)}(x) + (s-1)\psi(s-2)[x_+^{-s} - (-1)^s x_-^{-s}]+ \\
&\quad + x_+^{-s} - (s-1)x_+^{-s}\ln x_+ - (-1)^s x_-^{-s} + (-1)^s(s-1)x_-^{-s}\ln x_- \\
&= \frac{(-1)^s\pi^2}{3(s-2)!}\delta^{(s-1)}(x) + (s-1)\psi(s-2)[x_+^{-s} + (-1)^{s+1}x_-^{-s}] \\
&\quad -(s-1)[x_+^{-s}\ln x_+ + (-1)^{s+1}x_-^{-s}\ln x_-],
\end{aligned}$$

and equation (53) follows for the case $s+1$. Equation (53) now follows by induction.

Equation (55) follows from equation (53) on replacing x by $-x$.

The convolution product $(x_-^{-1})_n * \ln x_+$ exists in the sense of Definition 1.4. Thus, if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$,

$$\begin{aligned}
\langle [(x_-^{-1})_n * \ln x_+]', \phi(x)\rangle &= -\langle (x_-^{-1})_n * \ln x_+, \phi'(x)\rangle \\
&= \langle (x_-^{-2})_n * \ln x_+, \phi(x)\rangle + \langle [x_-^{-1}\tau_n'(x)] * \ln x_+, \phi(x)\rangle
\end{aligned}$$

and so

$$\langle (x_-^{-2})_n * \ln x_+, \phi(x)\rangle = -\langle (x_-^{-1})_n * \ln x_+, \phi'(x)\rangle - \langle [x_-^{-1}\tau_n'(x)] * \ln x_+, \phi(x)\rangle.$$

With $n > -a$, it follows, as above, that

$$\langle [x_-^{-1}\tau_n'(x)] * \ln x_+, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-1}\tau_n'(y)\ln(x-y)\,dy\,dx.$$

Integrating by parts we have

$$\int_{-n-n^{-n}}^{-n} (-y)^{-1}\tau_n'(y)\ln(x-y)\,dy = n^{-1}\ln(x+n)+$$

$$-\int_{-n-n^{-n}}^{-n}[y^{-2}\ln(x-y)+y^{-1}(x-y)^{-1}]\tau_n(y)\,dy.$$

Clearly,

$$\begin{aligned}\lim_{n\to\infty} n^{-1}\ln(x+n) &= 0\\ &= \lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}[y^{-2}\ln(x-y)+y^{-1}(x-y)^{-1}]\tau_n(y)\,dy\end{aligned}$$

and so

$$\lim_{n\to\infty}\langle[x_-^{-1}\,\tau_n'(x)]*\ln x_+,\phi(x)\rangle = 0.$$

Thus,

$$\begin{aligned}\mathrm{N}-\lim_{n\to\infty}\langle(x_-^{-2})_n*\ln x_+,\phi(x)\rangle &= -\,\mathrm{N}-\lim_{n\to\infty}\langle(x_-^{-1})_n*\ln x_+,\phi'(x)\rangle\\ &= -\langle x_-^{-1}\circledast\ln x_+,\phi'(x)\rangle,\end{aligned}$$

proving that the neutrix convolution product $x_-^{-2}\circledast\ln x_+$ exists and

$$x_-^{-2}\circledast\ln x_+ = (x_-^{-1}\circledast\ln x_+)'.$$

Equation (54) follows as above for the case $s=2$.

Now assume that equation (54) holds for some s. Then,

$$\begin{aligned}\langle[(x_-^{-s})_n*\ln x_+]',\phi(x)\rangle &= -\langle(x_-^{-s})_n*\ln x_+,\phi'(x)\rangle\\ &= s\langle(x_-^{-s-1})_n*\ln x_+,\phi(x)\rangle+\langle[x_-^{-s}\,\tau_n'(x)]*\ln x_+,\phi(x)\rangle,\end{aligned}$$

where it follows, as above, that

$$\lim_{n\to\infty}\langle[x_-^{-s}\,\tau_n'(x)]*\ln x_+,\phi(x)\rangle = 0,$$

and so

$$\begin{aligned}\mathrm{N}-\lim_{n\to\infty} s\langle(x_-^{-s-1})_n*\ln x_+,\phi(x)\rangle &= -\,\mathrm{N}-\lim_{n\to\infty}\langle(x_-^{-s})_n*\ln x_+,\phi'(x)\rangle\\ &= -\langle x_-^{-s}\circledast\ln x_+,\phi'(x)\rangle.\end{aligned}$$

This proves the existence of the neutrix convolution product $x_-^{-s-1}\circledast\ln x_+$ and

$$sx_-^{-s-1}\circledast\ln x_+ = (x_-^{-s}\circledast\ln x_+)'.$$

Equation (54) follows as above for the case $s+1$. Equation (54) now follows by induction.

Equation (56) follows from equation (54) on replacing x by $-x$.□

COROLLARY 4.4. *The neutrix convolution products* $\ln|x| \circledast x_+^{-s}$, $x_+^{-s} \circledast \ln|x|$, $\ln x_+ \circledast x^{-s}$, $x^{-s} \circledast \ln x_+$, $\ln|x| \circledast x_-^{-s}$, $x_-^{-s} \circledast \ln|x|$, $\ln x_- \circledast x^{-s}$, $x^{-s} \circledast \ln x_-$, $\ln|x| \circledast x^{-s}$ *and* $x^{-s} \circledast \ln|x|$ *exist, and*

$$\begin{aligned}
\ln|x| \circledast x_+^{-1} &= \frac{\pi^2}{6}[1-3H(x)] + \tfrac{1}{2}\ln^2|x| \\
&= x_+^{-1} \circledast \ln|x| \\
&= \ln x_+ \circledast x^{-1} \\
&= x^{-1} \circledast \ln x_+, \\
\ln|x| \circledast x_-^{-1} &= \frac{\pi^2}{6}[3H(x)-2] + \tfrac{1}{2}\ln^2|x| \\
&= x_-^{-1} \circledast \ln|x| \\
&= -\ln x_- \circledast x^{-1} \\
&= -x^{-1} \circledast \ln x_-, \\
\ln|x| \circledast x^{-1} &= -\frac{\pi^2}{2}\operatorname{sgn} x \\
&= x^{-1} \circledast \ln|x|, \\
(s-1)\ln|x| \circledast x_+^{-s} &= \frac{(-1)^s\pi^2}{3(s-2)!}\delta^{(s-2)}(x) - \psi(s-2)[-x_+^{-s+1} + (-1)^s x_-^{-s+1}] + \\
&\quad -x_+^{-s+1}\ln x_+ + (-1)^s x_-^{-s+1}\ln x_- \\
&= (s-1)x_+^{-s} \circledast \ln|x| \\
&= (s-1)\ln x_+ \circledast x^{-s} \\
&= (s-1)x^{-s} \circledast \ln x_+, \\
(s-1)\ln|x| \circledast x_-^{-s} &= \frac{\pi^2}{3(s-2)!}\delta^{(s-2)}(x) - \psi(s-2)[(-1)^s x_+^{-s+1} - x_-^{-s+1}] + \\
&\quad +(-1)^s x_+^{-s+1}\ln x_+ - x_-^{-s+1}\ln x_- \\
&= (s-1)x_-^{-s} \circledast \ln|x| \\
&= (-1)^s(s-1)\ln x_- \circledast x^{-s}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^s(s-1)x^{-s} * \ln x_-, \\
\ln|x| \circledast x^{-s} &= \frac{(-1)^s\pi^2}{(s-1)!}\delta^{(s-2)}(x) \\
&= x^{-s} \circledast \ln|x|,
\end{aligned}
$$

for $s = 2, 3, \dots$, where H denotes Heaviside's function.

PROOF. Differentiating equation (48) we get

$$\ln x_+ * x_+^{-1} = -\frac{\pi^2}{6}H(x) + \ln^2 x_+$$

and it can be proved by induction that

$$(s-1)\ln x_+ * x_+^{-s} = \frac{(-1)^s\pi^2}{6(s-2)!}\delta^{(s-2)}(x) + 2\psi(s-2)x_+^{-s+1} - 2x_+^{-s+1}\ln x_+, \quad (57)$$

for $s = 2, 3, \dots$. The results of the theorem now follow on noting that

$$x^{-s} = x_+^{-s} + (-1)^s x_-^{-s},$$

for $s = 1, 2, \dots$.□

THEOREM 4.12. *The neutrix convolution products $x_-^{-r} \circledast x_+^{-s}$ and $x_+^{-r} \circledast x_-^{-s}$ exist, and*

$$
\begin{aligned}
\frac{(-1)^{r+1}(r-1)!(s-1)!}{(r+s-2)!}x_-^{-r} \circledast x_+^{-s} &= \frac{(-1)^{r+s}\pi^2}{3(r+s-2)!}\delta^{(r+s-2)}(x) + \\
&- \psi(r+s-2)[x_+^{-r-s+1} + (-1)^{r+s}x_-^{-r-s+1}] + \\
&+ x_+^{-r-s+1}\ln x_+ + (-1)^{r+s}x_-^{-r-s+1}\ln x_-, \quad (58) \\
\frac{(-1)^{r+1}(r-1)!(s-1)!}{(r+s-2)!}x_+^{-r} \circledast x_-^{-s} &= \frac{\pi^2}{3(r+s-2)!}\delta^{(r+s-2)}(x) + \\
&- \psi(r+s-2)[(-1)^{r+s}x_+^{-r-s+1} + x_-^{-r-s+1}] + \\
&+ (-1)^{r+s}x_+^{-r-s+1}\ln x_+ + x_-^{-r-s+1}\ln x_-, \quad (59)
\end{aligned}
$$

for $r, s = 1, 2, \dots$.

PROOF. It follows, as above, that

$$
\begin{aligned}
(\ln x_- \circledast x_+^{-s})^{(r)} &= (\ln x_-)^{(r)} \circledast x_+^{-s} \\
&= -(r-1)!x_-^{-r} \circledast x_+^{-s} \\
&= \frac{(-1)^r(r+s-1)!}{(s-2)!}\ln x_- \circledast x_+^{-r-s},
\end{aligned}
$$

and equation (58) follows from equation (53). Equation (59) now follows on replacing x by $-x$ in equation (58).□

COROLLARY 4.5. *The neutrix convolution products* $x^{-r} \circledast x_+^{-s}$, $x_+^{-r} \circledast x^{-s}$, $x_-^{-r} \circledast x^{-s}$, $x^{-r} \circledast x_-^{-s}$ *and* $x^{-r} \circledast x^{-s}$ *exist and*

$$\begin{aligned}
\frac{(r-1)!(s-1)!}{(r+s-2)!} x^{-r} \circledast x_+^{-s} &= \frac{(-1)^{r+s+1}\pi^2}{2(r+s-2)!}\delta^{(r+s-2)}(x) + \\
&\quad -\psi(r+s-2)[x_+^{-r-s+1} - (-1)^{r+s}x_-^{-r-s+1}] + \\
&\quad +x_+^{-r-s+1}\ln x_+ - (-1)^{r+s}x_-^{-r-s+1}\ln x_- \\
&= \frac{(r-1)!(s-1)!}{(r+s-2)!} x_+^{-r} \circledast x^{-s}, \\
\frac{(-1)^r(r-1)!(s-1)!}{(r+s-2)!} x^{-r} \circledast x_-^{-s} &= -\frac{\pi^2}{2(r+s-2)!}\delta^{(r+s-2)}(x) + \\
&\quad -\psi(r+s-2)[x_-^{-r-s+1} - (-1)^{r+s}x_+^{-r-s+1}] + \\
&\quad +x_-^{-r-s+1}\ln x_- - (-1)^{r+s}x_+^{-r-s+1}\ln x_+ \\
&= \frac{(-1)^s(r-1)!(s-1)!}{(r+s-2)!} x_-^{-r} \circledast x^{-s}, \\
x^{-r} \circledast x^{-s} &= \frac{(-1)^{r+s+1}\pi^2}{(r-1)!(s-1)!}\delta^{(r+s-2)}(x),
\end{aligned}$$

for $r, s = 1, 2, \dots$.

PROOF. Differentiating equation (57) r times we get

$$(r-1)!(s-1)!x_+^{-r} * x_+^{-s} = -(r+s-1)!\ln x_+ * x_+^{-r-s}$$

and it follows that

$$\begin{aligned}
\frac{(r-1)!(s-1)!}{(r+s-2)!} x_+^{-r} * x_+^{-s} &= \frac{(-1)^{r+s+1}\pi^2}{6(r+s-2)!}\delta^{(r+s-2)}(x) + \\
&\quad - 2[\psi(r+s-2)x_+^{-r-s+1} - x_+^{-r-s+1}\ln x_+].
\end{aligned}$$

The results of the corollary now follow as above.

CHAPTER V

THE COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF DISTRIBUTIONS

In this chapter, we consider the commutative neutrix convolution product of distributions f and g which extends both Definitions 1.3 and 1.4 and Definition 4.1. We will denote the commutative convolution product of distributions f and g by $f \boxed{*} g$ to distinguish it from the non-commutative neutrix convolution product. If the condition (a) or (b) of Definition 1.4 is satisfied for the distributions f and g, then the commutative and the non-commutative neutrix convolution products are equal. They might be equal for some cases. But, in general, they are not.

DEFINITION 5.1. *Let f and g be distributions and let τ_n be defined as in Definition 3 in Chapter 4. Let $f_n(x) = f(x)\tau_n(x)$ and $g_n(x) = g(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that*

$$\operatorname*{N-lim}_{n\to\infty}\langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is again the neutrix defined in Definition 4.3.

The convolution $f_n * g_n$ in this definition is again in the sense of Definition 1.4 and, since $f_n * g_n = g_n * f_n$, the neutrix convolution product $f \boxed{*} g$ is clearly commutative.

The next theorem, proved in [19], shows that this definition generalizes Definition 1.4 and Definition 4.1.

THEOREM 5.1. *Let f and g be distributions satisfying condition (a) or (b) of Definition 1.4 so that the convolution product $f * g$ exists. Then the neutrix convolution product $f \boxed{*} g$ exists and*

$$f \boxed{*} g = f * g.$$

However, equation (4) of Chapter 1 does not hold in the sense of Definition 5.1 since, in general, $f_n'(x) \neq (f_n(x))'$.

The following theorem was also proved in [19]:

THEOREM 5.2. *The neutrix convolution product* $x_-^\lambda \boxed{*} x_+^\mu$ *exists and*

$$x_-^\lambda \boxed{*} x_+^\mu = B(-\lambda-\mu-1, \mu+1)x_-^{\lambda+\mu+1} + B(-\lambda-\mu-1, \lambda+1)x_+^{\lambda+\mu+1},$$

for $\lambda, \mu, \lambda+\mu \neq 0, \pm 1, \pm 2, \ldots$, *where* B *denotes the Beta function.*

We now prove the following extension of Theorem 5.2:

THEOREM 5.3. *The neutrix convolution product* $x_-^\lambda \boxed{*} x_+^{r-\lambda}$ *exists and*

$$\begin{aligned} x_-^\lambda \boxed{*} x_+^{r-\lambda} &= B(-r-1, r+1-\lambda)x_-^{r+1} + B(-r-1, \lambda+1)x_+^{r+1} \\ &\quad + \frac{(-1)^r(\lambda)_{r+1}}{(r+1)!}x^{r+1}\ln|x|, \end{aligned} \tag{1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *and* $r = -1, 0, 1, 2, \ldots$.

In [20] it was, in particular, proved that

$$B(-r, \lambda) = \frac{(-1)^r(\lambda-1)_r}{r!}\left[\psi(r) - \gamma - \frac{\Gamma'(\lambda-r)}{\Gamma(\lambda-r)}\right], \tag{2}$$

for $r = 0, 1, 2, \ldots$ and $\lambda \neq 0, \pm 1, \pm 2, \ldots$, where Γ denotes the Gamma function and γ denotes Euler's constant.

PROOF. We will first of all suppose that $\lambda, r-\lambda > -1$ so that x_-^λ and $x_+^{r-\lambda}$ are locally summable functions. Put

$$(x_-^\lambda)_n = x_-^\lambda \tau_n(x), \qquad (x_+^{r-\lambda})_n = x_+^{r-\lambda}\tau_n(x).$$

Then the convolution product $(x_-^\lambda)_n * (x_+^{r-\lambda})_n$ exists in the sense of Definition 1.4 and

$$\begin{aligned} \langle (x_-^\lambda)_n * (x_+^{r-\lambda})_n, \phi(x)\rangle &= \langle (y_-^\lambda)_n, \langle (x_+^{r-\lambda})_n, \phi(x+y)\rangle\rangle \\ &= \int_{-n-n^{-n}}^0 (-y)^\lambda \tau_n(y) \int_a^b (x-y)_+^{r-\lambda}\tau_n(x-y)\phi(x)\, dx\, dy \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \phi(x)\int_{-n}^0 (-y)^\lambda (x-y)_+^{r-\lambda}\tau_n(x-y)\,dy\,dx\,+ \\
&+\int_a^b \phi(x)\int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n(y)(x-y)_+^{r-\lambda}\tau_n(x-y)\,dy\,dx \qquad (3)
\end{aligned}
$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$, with support of ϕ contained in the interval $[a, b]$.

When $x < 0$ and $-n \le y \le 0$, $\tau_n(x-y) = 1$ on the support of ϕ. Thus, with $x < 0$ and $-n \le y \le 0$, we have, on making the substitution $y = xu^{-1}$,

$$
\begin{aligned}
\int_{-n}^0 (-y)^\lambda (x-y)_+^{r-\lambda}\tau_n(x-y)\,dy &= \int_{-n}^x (-y)^\lambda (x-y)^{r-\lambda}\,dy \\
&= (-x)^{r+1}\int_{-x/n}^1 u^{-r-2}(1-u)^{r-\lambda}\,du
\end{aligned}
$$

$$
\begin{aligned}
&= (-x)^{r+1}\int_{-x/n}^1 u^{-r-2}\left[(1-u)^{r-\lambda} - \sum_{i=0}^{r+1}\frac{(-1)^i(r-\lambda)_i}{i!}u^i\right]du+ \\
&\quad +(-x)^{r+1}\sum_{i=0}^{r}\frac{(-1)^i(r-\lambda)_i}{i!(i-r-1)}[1-(-x/n)^{i-r-1}]+ \\
&\quad +\frac{(-1)^r(r-\lambda)_{r+1}}{(r+1)!}(-x)^{r+1}[\ln(-x)-\ln n].
\end{aligned}
$$

It follows that

$$
\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-n}^0 (-y)^\lambda (x-y)_+^{r-\lambda}\tau_n(x-y)\,dy =
$$

$$
\begin{aligned}
&= B(-r-1, r+1-\lambda)(-x)^{r+1} + \frac{(-1)^r(r-\lambda)_{r+1}}{(r+1)!}(-x)^{r+1}\ln(-x) \\
&= B(-r-1, r+1-\lambda)(-x)^{r+1} - \frac{(\lambda)_{r+1}}{(r+1)!}(-x)^{r+1}\ln(-x); \qquad (4)
\end{aligned}
$$

see [20].

When $x > 0$ and $-n \le y \le 0$, we have

$$
\begin{aligned}
\int_{-n}^0 (-y)^\lambda (x-y)_+^{r-\lambda}\tau_n(x-y)\,dy &= \int_{x-n}^0 (-y)^\lambda (x-y)^{r-\lambda}\,dy\,+ \\
&+ \int_{x-n-n^{-n}}^{x-n} (-y)^\lambda (x-y)^{r-\lambda}\tau_n(x-y)\,dy. \quad (5)
\end{aligned}
$$

On making the substitution $y = x(1-u^{-1})$, we have

$$
\int_{x-n}^0 (-y)^\lambda (x-y)^{r-\lambda}\,dy = x^{r+1}\int_{x/n}^1 u^{-r-2}(1-u)^\lambda\,du
$$

and it follows, as above, that

$$\underset{n\to\infty}{\mathrm{N-lim}} \int_{x-n}^{0} (-y)^{\lambda}(x-y)^{r-\lambda}\, dy \;=\; B(-r-1,\lambda+1)x^{r+1} + \\ + \frac{(-1)^r(\lambda)_{r+1}}{(r+1)!}x^{r+1}\ln x. \qquad (6)$$

Further, with $n > 2x$,

$$\left|\int_{x-n-n^{-n}}^{x-n} (-y)^{\lambda}(x-y)^{r-\lambda}\tau_n(x-y)\, dy\right|$$

$$\leq \int_{n}^{n+n^{-n}} (y-x)^{\lambda}y^{r-\lambda}\, dy = \int_{n}^{n+n^{-n}} y^r(1-x/y)^{\lambda}\, dy$$
$$\leq \begin{cases} (n+n^{-n})^r n^{-n}, & \lambda > 0, \\ 2^{-\lambda}(n+n^{-n})^r n^{-n}, & -1 < \lambda < 0, \end{cases}$$

and so

$$\lim_{n\to\infty} \int_{x-n-n^{-n}}^{x-n} (-y)^{\lambda}(x-y)^{r-\lambda}\tau_n(x-y)\, dy = 0. \qquad (7)$$

It now follows from equations (5), (6) and (7) that

$$\underset{n\to\infty}{\mathrm{N-lim}} \int_{-n}^{0} (-y)^{\lambda}(x-y)_+^{r-\lambda}\tau_n(x-y)\, dy \;=\; B(-r-1,\lambda+1)x^{r+1} + \\ \frac{(-1)^r(\lambda)_{r+1}}{(r+1)!}x^{r+1}\ln x. \qquad (8)$$

Next, with $-\frac{1}{2}n < a \leq x \leq b < \frac{1}{2}n$, we have

$$\left|\int_{-n-n^{-n}}^{-n} (-y)^{\lambda}\tau_n(y)(x-y)^{r-\lambda}\tau_n(x-y)\, dy\right|$$
$$\leq \int_{-n-n^{-n}}^{-n} (-y)^r(1-x/y)^{r-\lambda}\, dy$$
$$\leq \begin{cases} 2^{r-\lambda}(n+n^{-n})^r n^{-n}, & r-\lambda > 0, \\ 2^{\lambda-r}(n+n^{-n})^r n^{-n}, & -1 < r-\lambda < 0, \end{cases}$$

and so

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} (-y)^{\lambda}\tau_n(y)(x-y)^{r-\lambda}\tau_n(x-y)\, dy = 0. \qquad (9)$$

It now follows from equations (3), (4), (8) and (9) that

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle (x_-^{\lambda})_n * (x_+^{r-\lambda})_n, \phi(x)\rangle =$$

$$= \left\langle B(-r-1, r+1-\lambda)x_-^{r+1} + B(-r-1, \lambda+1)x_+^{r+1} + \frac{(-1)^r}{(r+1)!}x^{r+1}\ln|x|, \phi(x)\right\rangle$$

and equation (1) follows for $\lambda, r-\lambda > -1$ and $\lambda \neq 0, 1, 2, \ldots$.

Now assume that equation (1) holds for $-k < \lambda < -k+1$ and $r - \lambda > -1$, where k is some positive integer. This is certainly true when $k = 1$. The convolution product $(x_-^\lambda)_n * (x_+^{r-\lambda})_n$ exists in the sense of Definition 1.4 and so equation (4) holds. Thus, if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, where we may suppose that $a < 0 < b$,

$$\begin{aligned} \langle [(x_-^\lambda)_n * (x_+^{r-\lambda})_n]', \phi(x)\rangle &= -\langle (x_-^\lambda)_n * (x_+^{r-\lambda})_n, \phi'(x)\rangle \\ &= -\lambda\langle (x_-^{\lambda-1})_n * (x_+^{r-\lambda})_n, \phi(x)\rangle + \langle [x_-^\lambda \tau_n'(x)] * (x_+^{r-\lambda})_n, \phi(x)\rangle \end{aligned}$$

and so

$$\begin{aligned} \lambda\langle (x_-^{\lambda-1})_n * (x_+^{r-\lambda})_n, \phi(x)\rangle &= \langle (x_-^\lambda)_n * (x_+^{r-\lambda})_n, \phi'(x)\rangle + \\ &\quad + \langle [x_-^\lambda \tau_n'(x)] * (x_+^{r-\lambda})_n, \phi(x)\rangle. \qquad (10) \end{aligned}$$

The support of $x_-^\lambda \tau_n'(x)$ is contained in the interval $[-n - n^{-n}, -n]$ and so, with $n > -a > n^{-n}$, it follows, as above, that

$$\begin{aligned} &\langle [x_-^\lambda \tau_n'(x)] * (x_+^{r-\lambda})_n, \phi(x)\rangle = \\ &= \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{r-\lambda}\tau_n(x-y)\, dy\, dx \\ &= \int_a^0 \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{r-\lambda}\, dy\, dx + \\ &\quad - \int_{-n^{-n}}^0 \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{r-\lambda}\, dy\, dx + \\ &\quad + \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{r-\lambda}\tau_n(x-y)\, dy\, dx, \qquad (11) \end{aligned}$$

where, on the domain of integration, $(-y)^\lambda$ and $(x-y)^{r-\lambda}$ are locally summable functions.

Putting $M = \sup\{|\tau'(x)|\}.\sup\{|\phi(x)|\}$, we have

$$\begin{aligned} &\left| \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{r-\lambda}\tau_n(x-y)\, dy\, dx \right| \\ &\qquad \le Mn^n \int_{-n^{-n}}^{n^{-n}} \int_{-n-n^{-n}}^{-n} (-y)^r(1 - x/y)^{r-\lambda}\, dy\, dx \\ &\qquad \le \begin{cases} 2^{r+1-\lambda}M(n+n^{-n})^r n^{-n}, & r-\lambda > 0, \\ 2^{1-r+\lambda}M(n+n^{-n})^r n^{-n}, & -1 < r-\lambda < 0, \end{cases} \end{aligned}$$

and it follows that

$$\lim_{n\to\infty}\int_{-n^{-n}}^{n^{-n}}\phi(x)\int_{-n-n^{-n}}^{-n}(-y)^{\lambda}\tau_n'(y)(x-y)^{r-\lambda}\tau_n(x-y)\,dy\,dx=0. \quad (12)$$

Similarly,

$$\lim_{n\to\infty}\int_{-n^{-n}}^{0}\phi(x)\int_{-n-n^{-n}}^{-n}(-y)^{\lambda}\tau_n'(y)(x-y)^{r-\lambda}\,dy\,dx=0. \quad (13)$$

Integrating by parts, we have

$$\int_{-n-n^{-n}}^{-n}(-y)^{\lambda}\tau_n'(y)(x-y)^{r-\lambda}\,dy=n^{\lambda}(x+n)^{r-\lambda}+$$
$$+\int_{-n-n^{-n}}^{-n}[\lambda(-y)^{\lambda-1}(x-y)^{r-\lambda}+(r-\lambda)(-y)^{\lambda}(x-y)^{r-1-\lambda}]\tau_n(y)\,dy. \quad (14)$$

Now,

$$n^{\lambda}(x+n)^{r-\lambda}=n^r\sum_{i=0}^{r}\frac{(r-\lambda)_i x^i}{i!n^i}+O(1/n),$$

and so

$$\mathrm{N}\!-\!\lim_{n\to\infty}n^{\lambda}(x+n)^{r-\lambda}=\frac{(r-\lambda)_r}{r!}x^r. \quad (15)$$

As in equation (9)

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}[\lambda(-y)^{\lambda-1}(x-y)^{r-\lambda}+(r-\lambda)(-y)^{\lambda}(x-y)^{r-1-\lambda}]\tau_n(y)\,dy=0. \quad (16)$$

It follows, from equations (11) to (16), that

$$\begin{aligned}\mathrm{N}\!-\!\lim_{n\to\infty}\langle[x_-^{\lambda}\tau_n'(x)]*(x_+^{r-\lambda})_n,\phi(x)\rangle &= \frac{(r-\lambda)_r}{r!}\int_a^0 x^r\phi(x)\,dx\\ &= \frac{(-1)^r(r-\lambda)_r}{r!}\int_a^b x_-^r\phi(x)\,dx\\ &= \frac{(-1)^r(r-\lambda)_r}{r!}\langle x_-^r,\phi(x)\rangle. \quad (17)\end{aligned}$$

It now follows, from equations (10) and (17), that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\lambda\langle(x_-^{\lambda-1})_n*(x_+^{r-\lambda})_n,\phi(x)\rangle=$$
$$=\mathrm{N}\!-\!\lim_{n\to\infty}\langle(x_-^{\lambda})_n*(x_+^{r-\lambda})_n,\phi'(x)\rangle+\frac{(-1)^r(r-\lambda)_r}{r!}\langle x_-^r,\phi(x)\rangle$$
$$=\langle x_-^{\lambda}\boxed{*}\,x_+^{r-\lambda},\phi'(x)\rangle+\frac{(-1)^r(r-\lambda)_r}{r!}\langle x_-^r,\phi(x)\rangle$$

by our assumption. This proves that the neutrix product $x_-^{\lambda-1} \boxed{*}\, x_+^{r-\lambda}$ exists and that

$$\begin{aligned}
x_-^{\lambda-1} \boxed{*}\, x_+^{r-\lambda} &= -\frac{(x_-^{\lambda} \boxed{*}\, x_+^{r-\lambda})'}{\lambda} + \frac{(-1)^r(r-\lambda)_r}{\lambda r!}x_-^r \\
&= \frac{(r+1)B(-r-1,r+1-\lambda)}{\lambda}x_-^r - \frac{(r+1)B(-r-1,\lambda+1)}{\lambda}x_+^r + \\
&\quad -\frac{(-1)^r(\lambda-1)_r}{(r+1)!}[(r+1)x^r\ln|x| + x^r] + \frac{(-1)^r(r-\lambda)_r}{\lambda r!}x_-^r \\
&= \frac{(-1)^r(r-\lambda)_r}{r!}\left[\psi(r+1) - \gamma - \frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)}\right]x_-^r + \\
&\quad -\frac{(\lambda-1)_r}{(r+1)!}x_-^r + \frac{(-1)^r(r-\lambda)_r}{\lambda r!}x_-^r + \\
&\quad +\frac{(-1)^r(\lambda-1)_r}{r!}\left[\psi(r+1) - \gamma - \frac{\Gamma'(\lambda-r)}{\Gamma(\lambda-r)}\right]x_+^r + \\
&\quad -\frac{(-1)^r(\lambda-1)_r}{(r+1)!}x_+^r - \frac{(-1)^r(\lambda-1)_r}{r!}x^r\ln|x| \\
&= B(-r,r+1-\lambda)x_-^r + B(-r,\lambda)x_+^r + \frac{(-1)^{r-1}(\lambda-1)_r}{r!}x^r\ln|x|,
\end{aligned}$$

on using equation (2) and the equations

$$\frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} - \frac{1}{\lambda} = \frac{\Gamma'(1-\lambda)}{\Gamma(1-\lambda)},$$

$$(r-\lambda)_r = (-1)^r(\lambda-1)_r.$$

Equation (1) now follows by induction for $\lambda \neq 0, \pm1, \pm2, \dots$, $r - \lambda > -1$ and $r = -1, 0, 1, 2, \dots$.

Finally, assume that equation (1) holds for $-k < r - \lambda < -k + 1$ and $\lambda \neq 0, \pm1, \pm2, \dots$. This is certainly true when $k = 1$. Then, since

$$(x_-^{\lambda})_n * (x_+^{r-\lambda})_n = (x_+^{r-\lambda})_n * (x_-^{\lambda})_n,$$

an argument similar to that given above shows us that equation (1) follows by induction for $\lambda \neq 0, \pm1, \pm2, \dots$ and $r = -1, 0, 1, 2, \dots$. This completes the proof of the theorem.□

THEOREM 5.4. *The neutrix convolution product $x_-^{\lambda} \boxed{*}\, x_+^{-r-\lambda}$ exists and*

$$x_-^{\lambda} \boxed{*}\, x_+^{-r-\lambda} = \frac{\pi\cot(\pi\lambda)}{(-1-\lambda)_{r-1}}\delta^{(r-2)}(x) - \frac{(-1)^r(r-2)!}{(-1-\lambda)_{r-1}}x^{-r+1}, \tag{18}$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ *and* $r = 2, 3, \dots$ *. In particular*

$$x_-^{s-\frac{1}{2}} \boxed{*} x_+^{-r-s+\frac{1}{2}} = -\frac{(-1)^r (r-2)!}{(-\frac{1}{2}-s)_{r-1}} x^{-r+1}, \tag{19}$$

for $s = 0, \pm 1, \pm 2, \dots$ *and* $r = 2, 3, \dots$ *.*

PROOF. It follows from Theorem 5.3, with $r = -1$, that

$$\operatorname*{N-lim}_{n\to\infty} \langle (x_-^\lambda)_n * (x_+^{-1-\lambda})_n, \phi(x) \rangle =$$

$$= \langle B(0,-\lambda)[1 - H(x)] + B(0, \lambda+1) H(x) - \ln|x|, \phi(x) \rangle,$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$, where H denotes Heaviside's function. Thus

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} \langle [(x_-^\lambda)_n * (x_+^{-1-\lambda})_n]', \phi(x) \rangle \\
&\quad = \operatorname*{N-lim}_{n\to\infty} \langle -\lambda (x_-^{\lambda-1})_n * (x_+^{-1-\lambda})_n + x_-^\lambda \tau_n'(x) * (x_+^{-1-\lambda})_n, \phi(x) \rangle \\
&\quad = \langle -B(0,-\lambda)\delta(x) + B(0,\lambda+1)\delta(x) - x^{-1}, \phi(x) \rangle \\
&\quad = \langle \pi \cot(\pi\lambda) \delta(x) - x^{-1}, \phi(x) \rangle,
\end{aligned} \tag{20}$$

using equation (2) and the equation

$$\frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} - \frac{\Gamma'(\lambda+1)}{\Gamma(\lambda+1)} = \pi \cot(\pi\lambda).$$

Equation (11) still holds for the case $r = -1$. It is easily seen, from equations (12) and (13), that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{-1-\lambda} \tau_n(x-y)\, dy\, dx \\
&= \lim_{n\to\infty} \int_{-n^{-n}}^{0} \phi(x) \int_{-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{-1-\lambda}\, dy\, dx = 0
\end{aligned}$$

and

$$\int_a^0 \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{-1-\lambda}\, dy\, dx = O(n^{-1}).$$

Thus,

$$\lim_{n\to\infty} \langle [x_-^\lambda \tau_n'(x)] * (x_+^{-1-\lambda})_n, \phi(x) \rangle = 0.$$

It now follows from equation (20) that

$$
\begin{aligned}
-\operatorname*{N-lim}_{n\to\infty} \lambda\langle (x_-^{\lambda-1})_n * (x_+^{-1-\lambda})_n, \phi(x)\rangle &= \langle \pi\cot(\pi\lambda)\delta(x) - x^{-1}, \phi(x)\rangle \\
&= -\lambda\langle x_-^{\lambda-1} \boxed{*}\, x_+^{-2-(\lambda-1)}, \phi(x)\rangle,
\end{aligned}
$$

proving equation (18) for the case $r = 2$.

Now assume that equation (18) holds for some $r > 2$. Then,

$$
\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} \langle (x_-^{\lambda})_n * (x_+^{-r-\lambda})_n, \phi(x)\rangle \\
&= \frac{1}{(-1-\lambda)_{r-1}} \langle \pi\cot(\pi\lambda)\delta^{(r-2)}(x) - (-1)^r (r-2)! x^{-r+1}, \phi(x)\rangle
\end{aligned}
$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$. Thus,

$$
\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} \langle [(x_-^{\lambda})_n * (x_+^{-r-\lambda})_n]', \phi(x)\rangle \\
&= \operatorname*{N-lim}_{n\to\infty} \langle -\lambda(x_-^{\lambda-1})_n * (x_+^{-r-\lambda})_n + [x_-^{\lambda}\tau_n'(x)] * (x_+^{-r-\lambda})_n, \phi(x)\rangle \\
&= \frac{1}{(-1-\lambda)_{r-1}} \langle \pi\cot(\pi\lambda)\delta^{(r-1)}(x) - (-1)^{r+1}(r-1)! x^{-r}, \phi(x)\rangle. \qquad (21)
\end{aligned}
$$

It follows, as above, that

$$
\lim_{n\to\infty} \langle [x_-^{\lambda}\tau_n'(x)] * (x_+^{-r-\lambda})_n, \phi(x)\rangle = 0,
$$

and so, from equation (21), we have

$$
\begin{aligned}
&-\operatorname*{N-lim}_{n\to\infty} \lambda\langle (x_-^{\lambda-1})_n * (x_+^{-r-\lambda})_n, \phi(x)\rangle = \\
&= \frac{1}{(-1-\lambda)_{r-1}} \langle \pi\cot(\pi\lambda)\delta^{(r-1)}(x) - (-1)^{r+1}(r-1)! x^{-r}, \phi(x)\rangle \\
&= -\lambda\langle x_-^{\lambda-1} \boxed{*}\, x_+^{(-r-1)-(\lambda-1)}, \phi(x)\rangle,
\end{aligned}
$$

proving equation (18) for the case $r+1$. Equation (18) now follows by induction, and equation (19) follows easily.□

CHAPTER VI

COMMUTATIVE NEUTRIX PRODUCT OF ULTRADISTRIBUTIONS AND THE EXCHANGE FORMULA

The problems of defining the product of distributions and ultradistributions in the dual spaces $\mathcal{D}'$ of $\mathcal{D}$ and $\mathcal{Z}'$ of $\mathcal{Z}$ (see below), respectively, are well-known. The object of this chapter is to define the neutrix product $(\mathcal{F}f)\Box(\mathcal{F}g)$ in $\mathcal{Z}'$, where $\mathcal{F}$ denotes the Fourier transform, to be the neutrix limit of the sequence $\{\mathcal{F}(f\tau_n).\mathcal{F}(g\tau_n)\}$. Later, we prove that the exchange formula holds. The product in $\mathcal{D}'$ will be considered in the next chapter.

As in [23], we define the Fourier transform of a function ϕ in $\mathcal{D}$ by

$$\mathcal{F}(\phi)(\sigma) = \tilde{\phi}(\sigma) = \int_{-\infty}^{\infty} \phi(x)e^{ix\sigma}\, dx.$$

Here $\sigma = \sigma_1 + i\sigma_2$ is a complex variable and it is well known that $\tilde{\phi}(\sigma)$ is an entire analytic function with the property

$$|\sigma|^q|\tilde{\phi}(\sigma)| \leq C_q e^{a|\sigma_2|}, \tag{1}$$

for some constants C_q and a depending on $\tilde{\phi}$. The set $\mathcal{Z}$, of all analytic functions with property (1), is, in fact, the space

$$\Im(\mathcal{D}) = \{\psi : \exists\phi \in \mathcal{D}, \mathcal{F}(\phi) = \psi\}.$$

The definition of convergence in $\mathcal{Z}$ can be carried over from $\mathcal{D}$. That is, the sequence of functions $\psi_\nu(\sigma)$ converges to zero in $\mathcal{Z}$ if the sequence of their inverse images (inverse Fourier transforms) $\phi_\nu(x)$ converges to zero in $\mathcal{D}$. We say that a sequence $\psi_\nu(\sigma)$ converges to zero in $\mathcal{Z}$ if for each function in this sequence we have

$$|\sigma^q\psi_\nu(\sigma)| \leq C_q e^{a|\sigma_2|}.$$

The Fourier transform, $\tilde{f}$, of a distribution f in $\mathcal{D}'$, is an ultradistribution in $\mathcal{Z}'$, i.e. a continuous linear functional on $\mathcal{Z}$. It is defined by *Parseval's equation*:

$$\langle \tilde{f}, \tilde{\phi} \rangle = 2\pi \langle f, \phi \rangle.$$

The *exchange formula* is the equality

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g). \tag{2}$$

It is well known that the exchange formula holds for all convolution products of distributions f and g, provided f and g both have compact support; see for example Treves [37].

We now consider the problem of defining multiplication in $\mathcal{Z}'$. To do this we need the Fourier transform $\mathcal{F}(\tau_n)$ of τ_n, defined as in Definition 4.3, and write

$$\delta_n(\sigma) = \frac{1}{2\pi}\mathcal{F}(\tau_n),$$

which is a function in $\mathcal{Z}$. Putting $\psi = \tilde{\phi}$, we have, from *Parseval's equation*,

$$\langle \tau_n, \phi \rangle = \frac{1}{2\pi}\langle \mathcal{F}(\tau_n), \mathcal{F}(\phi) \rangle = \langle \delta_n, \psi \rangle.$$

Since

$$\lim_{n\to\infty} \langle \tau_n, \phi \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \tau_n(x)\phi(x)\, dx = \int_{-\infty}^{\infty} \phi(x)\, dx = \langle 1, \phi \rangle,$$

for all ϕ in $\mathcal{D}$, and since $\mathcal{F}(1) = 2\pi\delta$, we obtain

$$\lim_{n\to\infty} \langle \delta_n, \psi \rangle = \langle \delta, \psi \rangle,$$

for all ψ in $\mathcal{Z}$. Thus $\{\delta_n\}$ is a sequence in $\mathcal{Z}'$ converging to the *Dirac delta function* δ.

If f is an arbitrary distribution in $\mathcal{D}'$, then, since δ_n is a function in $\mathcal{Z}$, the convolution product $\tilde{f} * \delta_n$ is defined by

$$\langle (\tilde{f} * \delta_n)(\sigma), \psi(\sigma) \rangle = \langle \tilde{f}(\nu), \langle \delta_n(\sigma), \psi(\sigma + \nu) \rangle \rangle, \tag{3}$$

for arbitrary ψ in $\mathcal{Z}$. If $\psi = \tilde{\phi}$, we have

$$\psi(\sigma + \nu) = \mathcal{F}[e^{ix\nu}\phi(x)]$$

and it follows from *Parseval's equation* that

$$\begin{aligned} \langle \delta_n(\sigma), \psi(\sigma+\nu)\rangle &= \frac{1}{2\pi}\langle \mathcal{F}(\tau_n)(\sigma), \mathcal{F}(e^{ix\nu}\phi)(\sigma)\rangle = \langle \tau_n(x), e^{ix\nu}\phi(x)\rangle \\ &= \int_{-\infty}^{\infty} \tau_n(x) e^{ix\nu}\phi(x)\,dx \qquad (4) \\ &\to \int_{-\infty}^{\infty} e^{ix\nu}\phi(x)\,dx = \psi(\nu). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \langle \tilde{f} * \delta_n, \psi\rangle = \langle \tilde{f}, \psi\rangle,$$

for arbitrary ψ in $\mathcal{Z}$, and it follows that $\{\tilde{f} * \delta_n\}$ is a regular sequence of infinitely differentiable functions converging to $\tilde{f}$ in $\mathcal{Z}'$.

This leads us to the following definition:

DEFINITION 6.1. *Let f and g be distributions in $\mathcal{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathcal{Z}'$ and let $\tilde{f}_n = \tilde{f} * \delta_n$ and $\tilde{g}_n = \tilde{g} * \delta_n$. Then the neutrix product $\tilde{f}\Box\tilde{g}$ is defined to be the neutrix limit of the sequence $\{\tilde{f}_n.\tilde{g}_n\}$, provided the limit $\tilde{h}$ exists, in the sense that*

$$\mathop{\mathrm{N-lim}}_{n\to\infty} \langle \tilde{f}_n.\tilde{g}_n, \psi\rangle = \langle \tilde{h}, \psi\rangle,$$

for all ψ in $\mathcal{Z}$.

In this definition we use $\tilde{f}\Box\tilde{g}$ to denote the neutrix product of $\tilde{f}$ and $\tilde{g}$ to distinguish it from the usual definition of the product, $\tilde{f}_n.\tilde{g}_n$, of two infinitely differentiable functions $\tilde{f}_n$ and $\tilde{g}_n$. If

$$\lim_{n\to\infty} \langle \tilde{f}_n.\tilde{g}_n, \psi\rangle = \langle \tilde{h}, \psi\rangle,$$

for all ψ in $\mathcal{Z}$, we simply say that the *product* $\tilde{f}.\tilde{g}$ exists and equals $\tilde{h}$. We then, of course, have

$$\tilde{f}\Box\tilde{g} = \tilde{f}.\tilde{g}.$$

It is immediately obvious that if the neutrix product $\tilde{f}\square\tilde{g}$ exists then the neutrix product is commutative.

The product of ultradistributions in $\mathcal{Z}'$ also has the following property:

THEOREM 6.1. *Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $\mathcal{Z}'$ and suppose that the neutrix products $\tilde{f}\square\tilde{g}$ and $\tilde{f}\square\tilde{g}'$ (or $\tilde{f}'\square\tilde{g}$) exist. Then the neutrix product $\tilde{f}'\square\tilde{g}$ (or $\tilde{f}\square\tilde{g}'$) exists and*

$$(\tilde{f}\square\tilde{g})' = \tilde{f}'\square\tilde{g} + \tilde{f}\square\tilde{g}'. \tag{5}$$

PROOF. Let ψ be an arbitrary function in $\mathcal{Z}$. Then,

$$\langle \tilde{f}\square\tilde{g}, \psi\rangle = \operatorname*{N-lim}_{n\to\infty}\langle \tilde{f}_n.\tilde{g}_n, \psi\rangle, \quad \langle \tilde{f}\square\tilde{g}', \psi\rangle = \operatorname*{N-lim}_{n\to\infty}\langle \tilde{f}_n.\tilde{g}_n', \psi\rangle.$$

Further,

$$\begin{aligned}
\langle (\tilde{f}\square\tilde{g})', \psi\rangle &= -\langle \tilde{f}\square\tilde{g}, \psi'\rangle = -\operatorname*{N-lim}_{n\to\infty}\langle \tilde{f}_n.\tilde{g}_n, \psi'\rangle \\
&= -\operatorname*{N-lim}_{n\to\infty}\langle \tilde{g}_n, (\tilde{f}_n.\psi)' - \tilde{f}_n'.\psi\rangle \\
&= \operatorname*{N-lim}_{n\to\infty}\langle \tilde{g}_n', \tilde{f}_n.\psi\rangle + \operatorname*{N-lim}_{n\to\infty}\langle \tilde{g}_n, \tilde{f}_n'.\psi\rangle
\end{aligned}$$

and so

$$\operatorname*{N-lim}_{n\to\infty}\langle \tilde{f}_n'.\tilde{g}_n, \psi\rangle = \langle (\tilde{f}\square\tilde{g})', \psi\rangle - \langle \tilde{f}\square\tilde{g}', \psi\rangle.$$

Hence the neutrix product $\tilde{f}'\square\tilde{g}$ exists and equation (5) follows.

It follows similarly that if $\tilde{f}'\square\tilde{g}$ exists then $\tilde{f}\square\tilde{g}'$ exists.□

We can now prove the exchange formula.

THEOREM 6.2. *Let f and g be distributions in $\mathcal{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathcal{Z}'$. Then the neutrix convolution product $f \boxed{*} g$ exists in $\mathcal{D}'$ if and only if the neutrix product $\tilde{f}\square\tilde{g}$ exists in $\mathcal{Z}'$, and the exchange formula*

$$\mathcal{F}(f \boxed{*} g) = \tilde{f}\square\tilde{g}$$

is then satisfied.

PROOF. We have from equation (4) that

$$\langle \delta_n(\sigma), \psi(\sigma+\nu)\rangle = \mathcal{F}(\tau_n\phi)$$

and then, from equation (3), that

$$\begin{aligned}\langle \tilde{f}_n, \psi\rangle &= \langle \tilde{f} * \delta_n, \psi\rangle = \langle \tilde{f}, \mathcal{F}(\tau_n\phi)\rangle = 2\pi\langle f, \tau_n\phi\rangle \\ &= 2\pi\langle f_n, \phi\rangle = \langle \mathcal{F}(f_n), \psi\rangle\end{aligned}$$

on using *Parseval's equation* twice. It follows that $\mathcal{F}(f_n) = \tilde{f}_n$. Similarly, we have $\mathcal{F}(g_n) = \tilde{g}_n$. Now, since f_n, g_n both have compact support, the convolution product $f_n * g_n$ exists in the sense of Definition 1.4 and so

$$\mathcal{F}(f_n * g_n) = \mathcal{F}(f_n).\mathcal{F}(g_n) = \tilde{f}_n.\tilde{g}_n$$

and so, on using *Parseval's equation* again,

$$2\pi\langle f_n * g_n, \phi\rangle = \langle \mathcal{F}(f_n * g_n), \psi\rangle = \langle \tilde{f}_n.\tilde{g}_n, \psi\rangle.$$

Suppose the neutrix convolution product $f \boxed{*} g$ exists. Then

$$\begin{aligned}2\pi\langle f \boxed{*} g, \phi\rangle &= \mathop{\mathrm{N-lim}}_{n\to\infty} 2\pi\langle f_n * g_n, \phi\rangle = \mathop{\mathrm{N-lim}}_{n\to\infty}\langle \mathcal{F}(f_n * g_n), \psi\rangle \\ &= \mathop{\mathrm{N-lim}}_{n\to\infty}\langle \tilde{f}_n.\tilde{g}_n, \psi\rangle = \langle \tilde{f}.\tilde{g}, \psi\rangle\end{aligned}$$

for arbitrary ϕ in $\mathcal{D}$ and $\mathcal{F}\phi$ in $\mathcal{Z}$, proving the existence of the neutrix product $\tilde{f}\square\tilde{g}$ and the exchange formula.

Conversely, if the neutrix product $\tilde{f}\square\tilde{g}$ exists then the argument can be reversed to prove the existence of the neutrix convolution product $f \boxed{*} g$ and the exchange formula.□

Gel'fand and Shilov define the distributions $(x+i0)^\lambda$ and $(x-i0)^\lambda$ as follows:

$$\begin{aligned}(x+i0)^\lambda &= x_+^\lambda + e^{i\lambda\pi}x_-^\lambda, \\ (x-i0)^\lambda &= x_+^\lambda + e^{-i\lambda\pi}x_-^\lambda;\end{aligned}$$

see [23]. We now prove the following theorem.

THEOREM 6.3. *The products $(\sigma + i0)^\lambda.(\sigma + i0)^\mu$ and $(\sigma - i0)^\lambda.(\sigma - i0)^\mu$ exist and*

$$(\sigma + i0)^\lambda.(\sigma + i0)^\mu = (\sigma + i0)^{\lambda+\mu}, \tag{6}$$

$$(\sigma - i0)^\lambda.(\sigma - i0)^\mu = (\sigma - i0)^{\lambda+\mu}, \tag{7}$$

for all λ and μ .

PROOF. It is easy to show

$$x_+^\lambda * x_+^\mu = B(\lambda+1, \mu+1)x_+^{\lambda+\mu+1}, \tag{8}$$

for $\lambda, \mu, \lambda + \mu + 1 \neq -1, -2, \dots$.

Further (see [23]),

$$\mathcal{F}(x_+^\lambda) = ie^{i\lambda\pi/2}\Gamma(\lambda+1)(\sigma + i0)^{-\lambda-1}, \tag{9}$$

for $\lambda \neq -1, -2, \dots$. On using the exchange formula, it follows, from equations (8) and (9), that

$$\begin{aligned} -e^{i(\lambda+\mu)\pi/2}\Gamma(\lambda+1)\Gamma(\mu+1)(\sigma + i0)^{-\lambda-1}.(\sigma + i0)^{-\mu-1} = \\ = B(\lambda+1, \mu+1)ie^{i(\lambda+\mu+1)\pi/2}\Gamma(\lambda+\mu+2)(\sigma + i0)^{-\lambda-\mu-2}, \end{aligned}$$

for $\lambda, \mu, \lambda + \mu \neq 0, 1, 2, \dots$, the product $(\sigma + i0)^{-\lambda-1}.(\sigma + i0)^{-\mu-1}$ existing since the convolution product $x_+^\lambda * x_+^\mu$ exists. Equation (6) now follows for $\lambda, \mu, \lambda + \mu \neq 0, 1, 2, \dots$.

Now suppose that $\lambda, \mu, \lambda + \mu > -1$ and put

$$(\sigma + i0)_n^\lambda = (\sigma + i0)^\lambda * \delta_n(\sigma).$$

Then, since

$$(\sigma + i0)^\lambda = \sigma_+^\lambda + e^{i\lambda\pi}\sigma_-^\lambda$$

(see [23]), it follows that $\{(\sigma + i0)^{\lambda}_n.(\sigma + i0)^{\mu}_n\}$ is a sequence of locally summable functions which converges to the locally summable function $(\sigma + i0)^{\lambda+\mu}$. Equation (6) follows for $\lambda, \mu, \lambda + \mu > -1$.

Now suppose that equation (6) holds when $-k-1 < \lambda < -k$, for some positive integer k, and $\lambda + \mu = 0, \pm 1, \pm 2, \ldots$. This is certainly true when $k = 0$. Then,

$$\lim_{n\to\infty} (\sigma + i0)^{\lambda}_n.(\sigma + i0)^{\mu}_n = (\sigma + i0)^{\lambda+\mu},$$

by our assumption when $-k-1 < \lambda < -k$. It follows that

$$\begin{aligned}
&\lim_{n\to\infty} [(\sigma + i0)^{\lambda}_n.(\sigma + i0)^{\mu}_n]' \\
&\quad = \lim_{n\to\infty} [\lambda(\sigma + i0)^{\lambda-1}_n.(\sigma + i0)^{\mu}_n + \mu(\sigma + i0)^{\lambda}_n.(\sigma + i0)^{\mu-1}_n] \\
&\quad = (\lambda + \mu)(\sigma + i0)^{\lambda+\mu-1}
\end{aligned}$$

and so

$$\lim_{n\to\infty} (\sigma + i0)^{\lambda-1}_n.(\sigma + i0)^{\mu}_n = (\sigma + i0)^{\lambda+\mu-1}.$$

Equation (6) follows by induction for $\lambda \neq -1, -2, \ldots$ and $\lambda + \mu = 0, \pm 1, \pm 2, \ldots$.

We are finally left to prove equation (6) for the case $\lambda = r = -1, -2, \ldots$ and $\mu = s = 0, 1, 2, \ldots$. Since (see [23]),

$$\ln(\sigma + i0) = \ln|\sigma| + i\pi H(-\sigma)$$

and

$$(\sigma + i0)^s = \sigma^s,$$

for $s = 0, 1, 2, \ldots$ are locally summable functions, it follows, as above, that if

$$\ln(\sigma + i0)_n = \ln(\sigma + i0) * \delta_n(\sigma),$$

then the sequence $\{\ln(\sigma + i0)_n.(\sigma + i0)^s_n\}$ converges to the locally summable function $(\sigma + i0)^s \ln(\sigma + i0)$. Thus, as in [23],

$$\begin{aligned}
&\lim_{n\to\infty} [\ln(\sigma + i0)_n.(\sigma + i0)^s_n]' \\
&\quad = \lim_{n\to\infty} [(\sigma + i0)^{-1}_n.(\sigma + i0)^s_n + s\ln(\sigma + i0)(\sigma + i0)^{s-1}_n] \\
&\quad = [(\sigma + i0)^s \ln(\sigma + i0)]' \\
&\quad = s(\sigma + i0)^{s-1} \ln(\sigma + i0) + (\sigma + i0)^{s-1},
\end{aligned}$$

and so

$$\lim_{n\to\infty}(\sigma+i0)_n^{-1}.(\sigma+i0)_n^s = (\sigma+i0)^{s-1}.$$

Equation (6) follows for $\lambda = -1$ and $\mu = 0,1,2,\ldots$. Another induction argument shows that equation (6) holds for $\lambda = -1,-2,\ldots$ and $\mu = 0,1,2,\ldots$.□

COROLLARY 6.1.

$$\sigma^{-r}.\sigma^s = \sigma^{s-r} \tag{10}$$

$$\delta^{(r-1)}(\sigma).\sigma^s = \begin{cases} 0, & s \geq r, \\ \dfrac{(-1)^s(r-1)!}{(r-s-1)!}\delta^{(r-s-1)}(\sigma), & r > s \end{cases} \tag{11}$$

$$\sigma^{-r}.\delta^{(r-1)}(\sigma) = \frac{(-1)^r(r-1)!}{2(2r-1)!}\delta^{(2r-1)}(\sigma), \tag{12}$$

for $r = 1,2,\ldots$ *and* $s = 0,1,2,\ldots$.

$$\sigma_+^{-r-1/2}.\sigma_-^{-r-1/2} = \frac{(-1)^r\pi}{2(2r)!}\delta^{2r}(\sigma), \tag{13}$$

for $r = 0,1,2,\ldots$.

PROOF. Since

$$(\sigma+i0)^s = \sigma^s,$$

for $s = 0,1,2,\ldots$, and

$$(\sigma+i0)^{-r} = \sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma),$$

for $r = 1,2,\ldots$, see [23], it follows from equation (6) that

$$\begin{aligned}(\sigma+i0)^{-r}.\sigma^s &= \begin{cases} \sigma^{s-r}, & s \geq r, \\ \sigma^{s-r} + \dfrac{i\pi(-1)^{r+s}}{(r-s-1)!}\delta^{(r-s-1)}(\sigma), & r > s \end{cases} \\ &= \sigma^{-r}.\sigma^s + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma).\sigma^s,\end{aligned}$$

the product clearly being distributive with respect to addition. Equating real and imaginary parts, equations (10) and (11) follow.

It follows from equation (6), that

$$\begin{aligned}
(\sigma+i0)^{-r}.(\sigma+i0)^{-r} &= (\sigma+i0)^{-2r} \\
&= \left[\sigma^{-r}+\frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma)\right].\left[\sigma^{-r}+\frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma)\right] \\
&= \sigma^{-2r}+\frac{i\pi}{(2r-1)!}\delta^{(2r-1)}(\sigma),
\end{aligned}$$

for $r=1,2,\ldots$. Expanding and equating imaginary parts gives equation (12).

Again from equation (6), it follows that

$$\begin{aligned}
(\sigma+i0)^{-r-1/2}.(\sigma+i0)^{-r-1/2} &= (\sigma+i0)^{-2r-1} \\
&= \left[\sigma_+^{-r-1/2}-i(-1)^r\sigma_-^{-r-1/2}\right].\left[\sigma_+^{-r-1/2}-i(-1)^r\sigma_-^{-r-1/2}\right] \\
&= \sigma^{-2r-1}-\frac{i\pi}{(2r)!}\delta^{(2r)}(\sigma),
\end{aligned}$$

for $r=0,1,2,\ldots$. Expanding and equating the imaginary parts gives equation (13).□

THEOREM 6.4. *The neutrix product* $\sigma_+^\lambda \Box \delta^{(s)}(\sigma)$ *exists and*

$$\sigma_+^\lambda \Box \delta^{(s)}(\sigma)=0, \tag{14}$$

for real $\lambda \neq 0,\pm1,\pm2,\ldots$ *and* $s=0,1,2,\ldots$.

PROOF. It was proved in [19] that

$$x_+^\lambda \boxed{*}\, x^s=0, \quad x_-^\lambda \boxed{*}\, x^s=0,$$

for real $\lambda \neq 0,\pm1,\pm2,\ldots$ and $s=0,1,2,\ldots$. Thus,

$$(x-i0)^\lambda \boxed{*}\, x^s=(x_+^\lambda+e^{-i\lambda\pi}x_-^\lambda)\boxed{*}\, x^s=0,$$

for real $\lambda \neq 0,\pm1,\pm2,\ldots$ and $s=0,1,2,\ldots$. On applying the exchange formula to this equation we get

$$\sigma_+^{-\lambda-1}\Box\delta^{(s)}(\sigma)=0,$$

for real $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $s = 0, 1, 2, \dots$, since

$$\mathcal{F}[(x - i0)^\lambda] = \frac{2\pi e^{-i\lambda\pi/2}}{\Gamma(-\lambda)}\sigma_+^{-\lambda-1},$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and

$$\mathcal{F}(x^s) = 2(-i)^s\pi\delta^{(s)}(\sigma)$$

for $s = 0, 1, 2, \dots$; see [23]. Equation (14) follows immediately.□

COROLLARY 6.2. *The neutrix product* $\sigma_-^\lambda \Box \delta^{(s)}(\sigma)$ *exists and*

$$\sigma_-^\lambda \Box \delta^{(s)}(\sigma) = 0,$$

for real $\lambda \neq 0, \pm 1, \pm 2, \dots$ *and* $s = 0, 1, 2, \dots$.

PROOF. The result follows immediately from equation (14) on replacing σ by $-\sigma$ in equation (14).

THEOREM 6.5. *The neutrix product* $(\sigma - i0)^\lambda \Box (\sigma + i0)^\mu$ *exists and*

$$(\sigma - i0)^\lambda \Box (\sigma + i0)^\mu = \sigma_+^{\lambda+\mu} + e^{i(\mu-\lambda)\pi}\sigma_-^{\lambda+\mu}, \tag{15}$$

for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$.

PROOF. It was proved in [19] that

$$x_+^{-\lambda-1} \boxed{*}\, x_-^{-\mu-1} = B(\lambda + \mu + 1, -\mu)x_-^{-\lambda-\mu-1} + B(\lambda + \mu + 1, -\lambda)x_+^{-\lambda-\mu-1},$$

for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$. Applying the exchange formula to this equation, using equation (9) and

$$\mathcal{F}(x_-^\lambda) = -ie^{-i\lambda\pi/2}\Gamma(\lambda + 1)(\sigma - i0)^{-\lambda-1},$$

we get

$$\begin{aligned} e^{i(\lambda-\mu)\pi/2}\Gamma(-\lambda)\Gamma(-\mu)(\sigma - i0)^\lambda \Box (\sigma + i0)^\mu = \\ = e^{i(\lambda+\mu)\pi/2}B(\lambda + \mu + 1, -\mu)\Gamma(-\lambda - \mu)(\sigma - i0)^{\lambda+\mu} + \\ + e^{-i(\lambda+\mu)\pi/2}B(\lambda + \mu + 1, -\lambda)\Gamma(-\lambda - \mu)(\sigma + i0)^{\lambda+\mu}, \end{aligned}$$

and so

$$
\begin{aligned}
(\sigma - i0)^{\lambda} \square (\sigma + i0)^{\mu} &= e^{i\mu\pi} \sin(\lambda\pi) \operatorname{cosec}[(\lambda+\mu)\pi](\sigma - i0)^{\lambda+\mu} + \\
&\quad + e^{-i\lambda\pi} \sin(\mu\pi) \operatorname{cosec}[(\lambda+\mu)\pi](\sigma + i0)^{\lambda+\mu} \\
&= \sigma_{+}^{\lambda+\mu} + e^{i(\mu-\lambda)\pi} \sigma_{-}^{\lambda+\mu},
\end{aligned}
$$

proving equation (15) for real $\lambda, \mu \neq 0, \pm1, \pm2, \dots$.$\square$

CHAPTER VII

THE COMPOSITION OF DISTRIBUTIONS

There are two methods of defining the product of distributions: one by regularization and passage to the limit (Hirata-Ogata [25], Tillmann [28] and Kaminski [30]) and another one (Hörmander [26]) by means of the Fourier transform. These two methods are compared in [4]. The definitions of product of distributions given by Mikusinski, Hirata-Ogata, Tillmann and Kaminski are not equivalent since the Delta sequences considered by each author were different;see for example [31] or [28]. In this chapter, we will use symmetric model sequences, whilst in some of above mentioned work, non-symmetric delta sequences were used. However, we shall not examine the relation betweeen our product and the other products.

We begin this chapter by considering another extension of the product of distributions in $\mathcal{D}'$, so that we will be able to study the substitution of infinitely differentiable functions in the product of distributions. In [2], the composition of a distribution and an infinitely differentiable function is extended to distributions by continuity provided the derivative of the infinitely differentiable function is different from zero. Later, in [13] and [14], Fisher defined the composition of a distribution F and a summable function f which has a single simple root in the open interval (a, b), and it was recently generalized in [32] by allowing f to be a distribution. This generalization is also an extension of the definition of the composition of distributions given in recent paper by Antosik; see [1]. In this chapter we give another alternative approach.

Here, we let $\{\delta_n(x)\}$ be a regular sequence of infinitely differentiable functions defined as in Chapter 1. Then the following definition was given in [8].

DEFINITION 7.1. *Let f and g be arbitrary distributions and let $g_n = g * \delta_n$. We say that the product $f.g$ of f and g exists and is equal to h on the open interval (a,b) if*

$$\lim_{n\to\infty}\langle fg_n, \phi\rangle = \lim_{n\to\infty}\langle f, g_n\phi\rangle = \langle h, \phi\rangle$$

for all test functions ϕ with compact support contained in the interval (a,b).

Since this definition of the product is not symmetric, the product $f.g$ is not necessarily commutative. However, many such products are in fact commutative as is seen from the following theorem, which was proved in [8].

THEOREM 7.1. *Let f and g be distributions. If the product fg exists on the open interval (a,b) in the sense of Definition 1.2, then the products $f.g$ and $g.f$ exist and*

$$f.g = g.f = f.g$$

on this interval.

Thus, Definition 7.1 is also an extension of Definition 1.1. In the following, a definition for the product of two distributions extends Definition 7.1 to an even wider class of distributions.

DEFINITION 7.2. *Let f and g be arbitrary distributions and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a,b) if*

$$\underset{n\to\infty}{\mathrm{N-lim}}\langle fg_n, \phi\rangle = \underset{n\to\infty}{\mathrm{N-lim}}\langle f, g_n\phi\rangle = \langle h, \phi\rangle, \tag{1}$$

for all test functions ϕ with compact support contained in the interval (a,b).

Note that if we put $f_m = f * \delta_m$, we have

$$\langle fg_n, \phi\rangle = \underset{m\to\infty}{\mathrm{N-lim}}\langle f_m g_n, \phi\rangle,$$

and so the equation (1) could be replaced by the equation

$$\underset{n\to\infty}{\mathrm{N-lim}}\left[\underset{m\to\infty}{\mathrm{N-lim}}\langle f_m g_n, \phi\rangle\right] = \langle h, \phi\rangle. \tag{2}$$

It was proved in [8] that if the product $f.g$ exists in the sense of Definition 7.1, then the neutrix product $f \circ g$ exists and defines the same distribution. It was also proved that if the neutrix products $f \circ g$ and $f \circ g'$ exist on the open interval (a, b), then the neutrix product $f' \circ g$ exists on the interval (a, b) and

$$(f \circ g)' = f' \circ g + f \circ g'.$$

Some properties of the above neutrix product were proved in [18].

The definition of composition of the delta function and an infinitely differentiable function is as follows:

$$\delta(f(x)) = \sum_n \frac{1}{|f'(x_n)|} \delta(x - x_n),$$

where f has n simple roots $x_1, x_2, \ldots, x_n$ and $f' > 0$ at these roots.

In general, by formal differentiation, $\delta^{(k)}(f(x))$ is defined by

$$\delta^{(k)}(f(x)) = \sum_n \frac{1}{|f'(x_n)|} \Big(\frac{1}{f'(x)} \frac{d}{dx}\Big)^k \delta(x - x_n);$$

see Gel'fand and Shilov [23].

The following definition for the change of variable in distributions is an extension of the definition above and was given in [13].

DEFINITION 7.3. *Let f be an infinitely differentiable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the interval (a, b) if*

$$\operatorname*{N-lim}_{n \to \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\phi(x)\, dx = \langle h(x), \phi(x) \rangle,$$

for all test functions with compact support contained in the interval (a, b).

An extension of Definition 7.3 was given in [14] as follows:

DEFINITION 7.4. *Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to the distribution H on the interval (a, b) if*

$$\operatorname*{N-lim}_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x))\phi(x)\, dx = \langle H, \phi \rangle,$$

for all test functions ϕ *in* $\mathcal{D}$ *with support contained in the interval* (a,b)*, where* $F_n(x) = (F * \delta_n)(x)$.

The following theorem was, however, proved in [22].

THEOREM 7.2. *Let* F *be a distribution in* $\mathcal{D}'$ *and let* f *be an infinitely differentiable function with* $f'(x) > 0$*, (or* < 0*), for all* x *in the interval* (a,b)*. Then the distribution* $F(f(x))$ *exists on the interval* (a,b)*.*

Further, if F *is the p-th derivative of a locally summable function* $F^{(-p)}$ *on the interval* $(f(a), f(b))$ *(or* $f(b), f(a)$*), then*

$$\begin{aligned}\langle F(f(x)), \phi(x)\rangle &= (-1)^p \int_{f(a)}^{f(b)} F^{(-p)}(x)[g'(x)\phi(g(x))]^{(p)}\,dx \qquad (3)\\ &= (-1)^p \int_{-\infty}^{\infty} F^{(-p)}(f(x))|f'(x)| \left[\frac{1}{f'(x)}\frac{d}{dx}\right]^p \left[\frac{\phi(x)}{f'(x)}\right] dx, \qquad (4)\end{aligned}$$

for all ϕ *in* $\mathcal{D}$ *with support contained in the interval* (a,b)*.*

Using equation (3), it was proved that, if f has a single simple zero at the point $x = x_1$ in the interval (a,b), then

$$\delta^{(s)}(f(x)) = \frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^s \delta(x - x_1), \qquad (5)$$

on the interval (a,b), for $s = 0, 1, 2, \ldots$, showing that Definition 7.4 is in agreement with the definition of $\delta^{(s)}(f(x))$ given by Gel'fand and Shilov see [13].

The problem of defining the product $F(f) \circ G(g)$ was considered in [14]. Putting $F(f) = F_1$ and $G(g) = G_1$ the product $F_1 \circ G_1 = H_1$ is, of course, defined by the equation

$$\operatorname*{N-lim}_{n\to\infty}\left[\operatorname*{N-lim}_{m\to\infty}\langle F_{1m}G_{1n}, \phi\rangle\right] = \langle H_1, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where $F_{1m} = F_1 * \delta_m$ and $G_{1n} = G_1 * \delta_n$.

However, it has been pointed out (see [14]), that since the distributions $F(f)$ and $G(g)$ were defined by the sequences $\{F_m\}$ and $\{G_n\}$ the product $F(f) \circ G(g)$ should be defined by these sequences, leading to the following definition.

DEFINITION 7.5. *Let F and G be distributions in $\mathcal{D}'$, f and g be locally summable functions, and $F_m = F * \delta_m$ and $G_n = G * \delta_n$. We say that the neutrix product $F(f) \circ G(g)$, of $F(f)$ and $G(g)$ exists and is equal to the distribution H on the interval (a,b), if $F_m(f)G_n(g)$ is a locally summable function on the interval (a,b), and*

$$\operatorname*{N-lim}_{n\to\infty}\left[\operatorname*{N-lim}_{m\to\infty}\langle F_m(f)G_n(g), \phi\rangle\right] = \langle H, \phi\rangle,$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a,b).

The following two examples were given in [14] and show that the neutrix product $F(f) \circ G(g)$ can be equal to, but is not necessarily equal to, the neutrix product $F_1 \circ G_1$.

EXAMPLE 7.1. Let $F = x_+^{1/2}$, $G = \delta'(x)$, $f = x_+^2$ and $g = x_+$. Then

$$F(f) = F_1 = x_+, \quad G(g) = G_1 = \frac{1}{2}\delta'(x)$$

and

$$F(f) \circ G(g) = -\frac{1}{2}\delta(x) = F_1 \circ G_1.$$

EXAMPLE 7.2. Let $F = x_+^{-1/2}$, $G = \delta(x)$, $f = x$ and $g = x_+^{1/2}$. Then

$$F(f) = F_1 = x_+^{-1/2}, \quad G(g) = G_1 = 0$$

and

$$F(f) \circ G(g) = \delta(x) \neq 0 = F_1 \circ G_1.$$

The following theorem was also proved in [14].

THEOREM 7.3. *Let F and G be distributions in $\mathcal{D}'$, f be a locally summable function and g be an infinitely differentiable function. If the distributions $F(f) = F_1$ and $G(g) = G_1$ exist, and the neutrix product $F(f) \circ G(g)$ exists on the interval (a,b), then*

$$F(f) \circ G(g) = F_1 \circ G(g)$$

on the interval (a,b). *In particular, if* $g(x) = x$, *then*

$$F(f) \circ G(g) = F_1 \circ G_1$$

on the interval (a,b).

In this theorem, $F_1 \circ G(g)$ was used to denote the distribution defined by

$$\mathrm{N-}\lim_{n\to\infty} \langle F_1 G_n(g), \phi \rangle.$$

We now prove the following theorem.

THEOREM 7.4. *Let* F *and* G *be distributions in* $\mathcal{D}'$ *and* f *be an infinitely differentiable function with* $f'(x) > 0$ *(or* < 0*), for all* x *in the interval* (a,b). *If the neutrix product* $F \circ G$ *exists and is equal to* H *on the interval* $(f(a), f(b))$ *(or* $(f(b), f(a))$*), then*

$$F(f) \circ G(f) = H(f)$$

on the interval (a,b).

PROOF. Note first of all that the distributions $F(f)$ and $G(f)$ exist on the interval $(f(a), f(b))$ (or $(f(b), f(a))$), by Theorem 7.2.

We will suppose that $f'(x) > 0$ and that g is the inverse of f on the interval (a,b). Letting ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval (a,b), and making the substitution $t = f(x)$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} F_m(f(x)) G_n(f(x)) \phi(x)\, dx &= \int_{-\infty}^{\infty} F_m(t) G_n(t) \phi(g(t)) g'(t)\, dt \\ &= \int_{-\infty}^{\infty} F_m(t) G_n(t) \psi(t)\, dt, \end{aligned}$$

where $\psi(t) = \phi(g(t))g'(t)$ is a function in $\mathcal{D}$ with support contained in the interval $(f(a), f(b))$. It follows that

$$\mathrm{N-}\lim_{n\to\infty} \left[\mathrm{N-}\lim_{m\to\infty} \langle F_m(f) G_n(f), \phi \rangle \right] = \langle H, \psi \rangle,$$

for all ϕ or ψ.

Further, on making the substitution $t = f(x)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} H_n(t)\psi(t)\,dt &= \int_{-\infty}^{\infty} H_n(t)\phi(g(t))g'(t)\,dt \\ &= \int_{-\infty}^{\infty} H_n(f(x))\phi(x)\,dx,\end{aligned}$$

and so

$$\operatorname*{N-lim}_{n\to\infty}\langle H_n, \psi\rangle = \langle H(f), \phi\rangle.\square$$

THEOREM 7.5. *Let F and G be distributions in $\mathcal{D}'$ and f be an infinitely differentiable function with $f'(x) > 0$ (or < 0) for all x in the interval (a,b). If the neutrix products $F \circ G$ and $F \circ G'$ (or $F' \circ G$), exist on the interval $(f(a), f(b))$ (or $(f(b), f(a))$), then*

$$[F(f) \circ G(f)]' = [F(f)]' \circ G(f) + F(f) \circ [G(f)]'$$

on the interval (a,b).

PROOF. The usual law

$$(F \circ G)' = F' \circ G + F \circ G'$$

for the differentiation of a product holds and so the result of the theorem follows immediately from Theorem 7.4.

THEOREM 7.6. *Let f be an infinitely differentiable function with $f'(x) > 0$ (or < 0), for all x in the interval (a,b), and having a simple zero at the point $x = x_1$ in the interval (a,b). Then the neutrix products $(f(x))_+^r \circ \delta^{(s)}(f(x))$ and $\delta^{(s)}(f(x)) \circ (f(x))_+^r$ exist and*

$$(f(x))_+^r.\delta^{(s)}(f(x)) = \delta^{(s)}(f(x)).(f(x))_+^r = 0 \tag{6}$$

for $s = 0, 1, \ldots, r-1$ and $r = 1, 2, \ldots$. Further,

$$\begin{aligned}(f(x))_+^r \circ \delta^{(s)}(f(x)) &= \delta^{(s)}(f(x)) \circ (f(x))_+^r \\ &= \frac{(-1)^r s!}{2(s-r)!}\frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^{s-r}\delta(x - x_1),\end{aligned} \tag{7}$$

for $r = 0, 1, \ldots, s$, *and* $s = 0, 1, 2, \ldots$, *on the interval* (a, b).

PROOF. If g is an s times continuously differentiable function at the origin, then the product $g.\delta^{(s)} = \delta^{(s)}.g$ is given by

$$g(x).\delta^{(s)}(x) = \delta^{(s)}(x).g(x) = \sum_{i=0}^{s}(-1)^{s+i}\binom{s}{i}g^{(s-i)}(0)\delta^{(i)}(x).$$

It follows that

$$x_+^r.\delta^{(s)}(x) = \delta^{(s)}(x).x_+^r = 0,$$

for $s = 1, 2, \ldots, r-1$ and $r = 1, 2, \ldots$. Equation (6) follows immediately on using Theorem 7.4.

It was proved in [9] that

$$x_+^r \circ \delta^{(s)}(x) = \delta^{(s)}(x) \circ x_+^r = \frac{(-1)^r s!}{2(s-r)!}\delta^{(s-r)}(x),$$

for $r = 0, 1, 2, \ldots, s$ and $s = 0, 1, 2, \ldots$. Using Theorem 7.4, it follows that

$$(f(x))_+^r \circ \delta^{(s)}(f(x)) = \delta^{(s)}(f(x)) \circ (f(x))_+^r = \frac{(-1)^r s!}{2(s-r)!}\delta^{(s-r)}(f(x)),$$

for $r = 0, 1, 2, \ldots, s$ and $s = 0, 1, 2, \ldots$. Equation (7) follows immediately on using equation (5).□

EXAMPLE 7.3. For all $x \in R$,

$$\begin{aligned}(x+x^2)_+^r \circ \delta^{(r)}(x+x^2) &= \delta^{(r)}(x+x^2) \circ (x+x^2)_+^r \\ &= \tfrac{1}{2}(-1)^r r![\delta(x) + \delta(x+1)], \end{aligned} \tag{8}$$

$$\begin{aligned}(x+x^2)_+^r \circ \delta^{(r+1)}(x+x^2) &= \delta^{(r+1)}(x+x^2) \circ (x+x^2)_+^r \\ &= \tfrac{1}{2}(-1)^r (r+1)![\delta'(x) + 2\delta(x) - \delta'(x+1) + \\ &\quad +2\delta(x+1)] \end{aligned} \tag{9}$$

for $r = 0, 1, 2, \ldots$.

PROOF. The function $f(x) = x + x^2$ has simple zeros at the points $x = 0, -1$. It follows from equations (5) and (7) that

$$\begin{aligned}
(x+x^2)_+^r \circ \delta^{(r)}(x+x^2) &= \delta^{(r)}(x+x^2) \circ (x+x^2)_+^r \\
&= \tfrac{1}{2}(-1)^r r! \delta(x+x^2) \\
&= \tfrac{1}{2}(-1)^r r! [\delta(x) + \delta(x+1)],
\end{aligned}$$

proving equation (8) for $r = 0, 1, 2, \ldots$.

It again follows from equations (5) and (7) that

$$\begin{aligned}
(x+x^2)_+^r \circ \delta^{(r+1)}(x+x^2) &= \delta^{(r+1)}(x+x^2) \circ (x+x^2)_+^r \\
&= \tfrac{1}{2}(-1)^r (r+1)! \frac{1}{1+2x}[\delta'(x) + \delta'(x+1)] \\
&= \tfrac{1}{2}(-1)^r (r+1)! [\delta'(x) + 2\delta(x) - \delta'(x+1) + 2\delta(x+1)],
\end{aligned}$$

proving equation (9) for $r = 0, 1, 2, \ldots$.□

THEOREM 7.7. *Let f be an infinitely differentiable function with $f'(x) > 0$ (or < 0), for all x in the interval (a, b), having a simple zero at the point $x = x_1$ in the interval (a, b). Then the neutrix products $(f(x))^{-r} \circ \delta^{(s)}(f(x))$ and $\delta^{(s)}(f(x)) \circ (f(x))^{-r}$ exist and*

$$(f(x))^{-r} \circ \delta^{(s)}(f(x)) = \frac{(-1)^r s!}{(r+s)!} \frac{1}{|f'(x_1)|} \left[\frac{1}{f'(x)} \frac{d}{dx}\right]^{r+s} \delta(x - x_1), \tag{10}$$

$$\delta^{(s)}(f(x)) \circ (f(x))^{-r} = 0, \tag{11}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$, on the interval (a, b).

PROOF. It was proved in [9] that

$$x^{-r} \circ \delta^{(s)}(x) = \frac{(-1)^r s!}{(r+s)!} \delta^{(r+s)}(x),$$

$$\delta^{(s)}(x) \circ x^{-r} = 0,$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. Equations (10) and (11) follow immediately as in the proof of Theorem 7.6.□

EXAMPLE 7.4. For all $x \in R$,

$$(x^2-1)^{-1} \circ \delta(x^2-1) = -\tfrac{1}{4}[\delta'(x-1)+\delta(x-1)-\delta'(x+1)+\delta(x+1)], \qquad (12)$$

$$\delta^{(s)}(x^2-1) \circ (x^2-1)^{-r} = 0, \qquad (13)$$

for $r = 1, 2, \ldots,$ and $s = 0, 1, 2, \ldots$.

PROOF. The function $f(x) = x^2 - 1$ has simple zeros at the points $x = \pm 1$. It follows from equations (5) and (10) that

$$\begin{aligned}(x^2-1)^{-1} \circ \delta(x^2-1) &= -\frac{1}{4x}[\delta'(x-1)+\delta'(x+1)] \\ &= -\tfrac{1}{4}[\delta'(x-1)+\delta(x-1)-\delta'(x+1)+\delta(x+1)],\end{aligned}$$

proving equation (12).

Equation (13) follows immediately from equations (5) and (11) for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$.

THEOREM 7.8. *Let f be an infinitely differentiable function with $f'(x) > 0$ (or < 0), for all x in the interval (a, b), having a simple zero at the point $x = x_1$ in the interval (a, b). Then the neutrix products $(f(x))_+^\lambda \circ (f(x))_-^{-\lambda-r}$ and $(f(x))_-^{-\lambda-r} \circ (f(x))_+^\lambda$ exist and*

$$\begin{aligned}(f(x))_+^\lambda \circ (f(x))_-^{-\lambda-r} &= (f(x))_-^{-\lambda-r} \circ (f(x))_+^\lambda \\ &= -\frac{\pi\operatorname{cosec}(\pi\lambda)}{2(r-1)!}\frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x_1)}\frac{d}{dx}\right]^{r-1}\delta(x-x_1), \qquad (14)\end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots,$ on the interval (a, b).

PROOF. It was proved in [9] that

$$x_+^\lambda \circ x_-^{-\lambda-r} = x_-^{-\lambda-r} \circ x_+^\lambda = -\frac{\pi\operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}(x),$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$. Equation (14) follows immediately as in the proof of Theorem 7.6.□

EXAMPLE 7.5. Let $f(x) = t$ be the inverse of the function $g(t) = t + t^3 = x$. Then, for all $x \in R$,

$$\begin{aligned}
(f(x))_+^\lambda \circ (f(x))_-^{-\lambda-1} &= (f(x))_-^{-\lambda-1} \circ (f(x))_+^\lambda \\
&= -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta(x), \qquad (15)
\end{aligned}$$

$$\begin{aligned}
(f(x))_+^\lambda \circ (f(x))_-^{\lambda-2} &= (f(x))_-^{\lambda-2} \circ (f(x))_+^\lambda \\
&= -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)[\delta'(x) + \delta(x)], \qquad (16)
\end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

PROOF. Since

$$g'(t) = 1 + 3t^2 > 0$$

for all t, it follows that $f'(x) > 0$ for all x and so, on using equation (3) with $p = 1$, we have, for all ϕ in $\mathcal{D}$,

$$\begin{aligned}
\langle \delta(f(x)), \phi(x) \rangle &= -\int_{-\infty}^{\infty} H(x) d[(1 + 3x^2)\phi(x + x^3)] \\
&= -\int_0^{\infty} d[(1 + 3x^2)\phi(x + x^3)] = \phi(0).
\end{aligned}$$

It follows that

$$\delta(f(x)) = \delta(x). \qquad (17)$$

Using equation (3), again with $p = 2$, we have, for all ϕ in $\mathcal{D}$,

$$\begin{aligned}
\langle \delta'(f(x)), \phi(x) \rangle &= \int_0^{\infty} d[(1 + 3x^2)\phi(x + x^3)]' \\
&= -\phi'(0) - \int_0^{\infty} d[(1 + 3x^2)\phi(x + x^3)] \\
&= -\phi'(0) + \phi(0).
\end{aligned}$$

It follows that

$$\delta'(f(x)) = \delta'(x) + \delta(x). \qquad (18)$$

It now follows from equations (14) and (17), that

$$\begin{aligned}(f(x))_+^{\lambda} \circ (f(x))_-^{-\lambda-1} &= (f(x))_-^{-\lambda-1} \circ (f(x))_+^{\lambda} \\ &= -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta(f(x)) \\ &= -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta(x),\end{aligned}$$

proving equation (15) for $\lambda \neq 0, \pm 1, \pm 2, \dots$.

It again follows, from equations (14) and (18), that

$$\begin{aligned}(f(x))_+^{\lambda} \circ (f(x))_-^{-\lambda-2} &= (f(x))_-^{-\lambda-2} \circ (f(x))_+^{\lambda} \\ &= -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta'(f(x)) \\ &= \tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)[\delta'(x) + \delta(x)],\end{aligned}$$

proving equation (16) for $\lambda \neq 0, \pm 1, \pm 2, \dots$.□

In the following, we consider another alternative definition of composition which extends Definition 7.4.

DEFINITION 7.6. *Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $F(f(x))$ exists and is equal to the distribution $h(x)$ in $\mathcal{D}'$, on the interval (a, b), if*

$$\underset{m\to\infty}{\text{N}-\lim}\left[\underset{n\to\infty}{\text{N}-\lim}\int_a^b F_n(f_m(x))\phi(x)\,dx\right] = \langle h(x), \phi(x)\rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b), where

$$F_n(x) = (F * \delta_n)(x), \quad f_m(x) = (f * \delta_m)(x).$$

An alternative generalization was considered in [32], where the order in which the neutrix limits were taken in Definition 7.6 were reversed.

THEOREM 7.9. *Let F be a bounded, continuous function on the real line. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = F(0),$$

for $s = 0, 1, 2, \ldots$.

PROOF. Let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval (a, b). Then, since F is a continuous function, it follows that

$$\begin{aligned}\underset{n\to\infty}{\text{N}-\lim} \left\langle F_n(\delta_m^{(s)}(x)), \phi(x)\right\rangle &= \lim_{n\to\infty} \int_a^b F_n(\delta_m^{(s)}(x))\phi(x)\,dx \\ &= \int_a^b F(\delta_m^{(s)}(x))\phi(x)\,dx. \qquad (19)\end{aligned}$$

Now,

$$\begin{aligned} F(\delta_m^{(s)}(x)) &= F(0), \quad |x| \geq 1/m, \\ \left|F(\delta_m^{(s)}(x))\right| &\leq K, \quad |x| \leq 1/m, \end{aligned}$$

where

$$K = \sup\{|F(x)|\} < \infty,$$

since F is bounded. Thus,

$$\left|\int_a^b \left[F(\delta_m^{(s)}(x)) - F(0)\right] \phi(x)\,dx\right| \leq \int_{-1/m}^{1/m} [K + |F(0)|]\,|\phi(x)|\,dx,$$

which tends to zero as m tends to infinity. It now follows, from equation (19), that

$$\underset{m\to\infty}{\text{N}-\lim} \left[\underset{n\to\infty}{\text{N}-\lim} \left\langle F_n(\delta_m^{(s)}(x)), \phi(x)\right\rangle\right] = \langle F(0), \phi(x)\rangle. \square$$

THEOREM 7.10. *The distribution* $H(\delta^{(s)}(x))$ *exists on the real line and*

$$H(\delta^{(s)}(x)) = \tfrac{1}{2},$$

for $s = 0, 1, 2, \ldots$ *, where* H *denotes Heaviside's function.*

PROOF. We put

$$H_n(x) = (H * \delta_n)(x),$$

for $n = 1, 2, \ldots$, so that

$$H_n(x) = \begin{cases} 1, & x > 1/n, \\ \displaystyle\int_{-1/n}^{x} \delta_n(t)\,dt, & |x| \leq 1/n, \\ 0, & x < -1/n, \end{cases}$$

$$0 \leq H_n(x) \leq 1,$$

$$H_n(0) = \tfrac{1}{2},$$

for $n = 1, 2, \ldots$. Thus

$$H_n(\delta_m^{(s)}(x)) = H_n(0) = \tfrac{1}{2},$$

for $|x| \geq 1/m$ and

$$|H_n(\delta_m^{(s)}(x))| \leq 1,$$

for $m, n = 1, 2, \ldots$.

Now, let ϕ be an arbitrary function in $\mathcal{D}$. Then,

$$\begin{aligned}\left\langle H_n(\delta_m^{(s)}(x)), \phi(x)\right\rangle &= \int_{|x|\geq 1/m} H_n(\delta_m^{(s)}(x))\phi(x)\,dx + \int_{|x|\leq 1/m} H_n(\delta_m^{(s)}(x))\phi(x)\,dx \\ &= \int_{|x|\geq 1/m} \tfrac{1}{2}\phi(x)\,dx + \int_{|x|\leq 1/m} H_n(\delta_m^{(s)}(x))\phi(x)\,dx,\end{aligned}$$

and so

$$\left|\left\langle H_n(\delta_m^{(s)}(x)) - \tfrac{1}{2}, \phi(x)\right\rangle\right| = \left|\int_{|x|\leq 1/m} \left[H_n(\delta_m^{(s)}(x)) - \tfrac{1}{2}\right] \phi(x)\,dx\right|.$$

Choosing an arbitrary $\epsilon > 0$, there exists an M such that $m\epsilon > 1$ for $m > M$. Then, with $m > M$, we have

$$H_n(\delta_m^{(s)}(x)) = H_n(0) = \tfrac{1}{2},$$

for $|x| > \epsilon$ and $n = 1, 2, \ldots$. It follows that, for $m > M$,

$$\left|\left\langle H_n(\delta_m^{(s)}(x)) - \tfrac{1}{2}, \phi(x)\right\rangle\right| = \left|\int_{|x|\leq \epsilon} \left[H_n((\delta_m^{(s)}(x)) - \tfrac{1}{2}\right] \phi(x)\,dx\right|,$$

for $n = 1, 2, \ldots$. Thus

$$\lim_{m\to\infty} \left[\lim_{n\to\infty} \left\langle H_n(\delta_m^{(s)}(x)), \phi(x)\right\rangle\right] = \langle \tfrac{1}{2}, \phi(x)\rangle. \square$$

THEOREM 7.11. *Let F be a bounded, locally summable function on the real line which is continuous everywhere except for a simple discontinuity at the origin. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = \tfrac{1}{2}[F(0+) + F(0-)],$$

for $s = 0, 1, 2, \ldots$.

PROOF. Let

$$F(0+) - F(0-) = c.$$

Then, the function G, defined by

$$G(x) = F(x) - cH(x),$$

satisfies the conditions of Theorem 7.9. Thus,

$$G(\delta^{(s)}(x)) = G(0) = F(0-),$$

and so

$$\begin{aligned} G(\delta^{(s)}(x)) + cH(\delta^{(s)}(x)) &= F(0-) + \tfrac{1}{2}[F(0+) - F(0-)] \\ &= \tfrac{1}{2}[F(0+) + F(0-)], \end{aligned}$$

for $s = 0, 1, 2, \ldots$.□

THEOREM 7.12. *Let* $F_+(x, \lambda)$ *be the continuous function* x_+^λ, *where* $\lambda > 0$. *Then, the distribution* $F_+(\delta^{(s)}(x), \lambda)$ *exists on the real line and*

$$F_+(\delta^{(s)}(x), \lambda) = 0, \tag{20}$$

for $s\lambda + \lambda \neq 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots$, *also*

$$F_+(\delta^{(s)}(x), \lambda) = \frac{(-1)^{s\lambda+\lambda-1} c(\rho, s, \lambda)}{(s\lambda + \lambda - 1)!} \delta^{(s\lambda+\lambda-1)}(x), \tag{21}$$

for $s\lambda + \lambda = 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots$, *where*

$$c(\rho, s, \lambda) = \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda) y^{s\lambda+\lambda-1}\, dy.$$

PROOF. We put

$$F_{+n}(x) = (F_+ * \delta_n)(x),$$

for $n = 1, 2, \ldots$ .Then, since F_+ is a continuous function,

$$\lim_{n \to \infty} F_{+n}(\delta_m^{(s)}(x), \lambda) = F_+(\delta_m^{(s)}(x), \lambda),$$

for $m = 1, 2, \ldots$. Further,

$$F_+(\delta_m^{(s)}(x), \lambda) = m^{s\lambda+\lambda} F_+(\rho^{(s)}(mx), \lambda),$$

and in particular

$$F_+(\delta_m^{(s)}(x), \lambda) = m^{s\lambda+\lambda} F_+(0) = 0,$$

for $|x| \geq 1/m$. Thus, if ϕ is an arbitrary function in $\mathcal{D}$, then

$$\left\langle F_+(\delta_m^{(s)}(x), \lambda), \phi(x) \right\rangle = m^{s\lambda+\lambda} \int_{-1/m}^{1/m} F_+(\rho^{(s)}(mx), \lambda)\phi(x)\, dx. \tag{22}$$

On making the substitution $mx = y$, we have

$$\begin{aligned}\int_{-1/m}^{1/m} F_+(\rho^{(s)}(mx), \lambda)\phi(x)\, dx &= m^{-1} \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda)\phi(y/m)\, dy \\ &= \sum_{i=0}^{k} \frac{\phi^{(i)}(0)}{m^{i+1} i!} \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda) y^i\, dy + \\ &\quad + \frac{1}{m^{k+2}(k+1)!} \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda)\phi^{(k+1)}(\xi y) y^{k+1}\, dy,\end{aligned}$$

where $0 < \xi < 1$ and k is the smallest integer greater than or equal to $s\lambda + \lambda - 1$. It follows, from equation (22), that

$$\begin{aligned}\left\langle F_+(\delta_m^{(s)}(x), \lambda), \phi(x) \right\rangle &= \sum_{i=0}^{k} \frac{m^{s\lambda+\lambda-i-1}\phi^{(i)}(0)}{i!} \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda) y^i\, dy + \\ &\quad + \frac{m^{s\lambda+\lambda-k-2}}{(k+1)!} \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda)\phi^{(k+1)}(\xi y) y^{k+1}\, dy.\end{aligned}$$

Now,

$$\begin{aligned}&\left| \frac{m^{s\lambda+\lambda-k-2}}{(k+1)!} \int_{-1}^{1} F_+(\rho^{(s)}(y), \lambda)\phi^{(k+1)}(\xi y) y^{k+1}\, dy \right| \leq \\ &\qquad \leq \frac{2m^{s\lambda+\lambda-k-2}}{(k+1)!} \sup_{\{|y| \leq 1\}} \left\{ \left| F_+(\rho^{(s)}(y), \lambda) y^{k+1} \right| \right\} \cdot \sup_{\{|y| \leq 1\}} \{ |\phi(y)| \},\end{aligned}$$

which tends to 0 as m tends to infinity. Thus,

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{m\to\infty}\left\langle F_+(\delta_m^{(s)}(x),\lambda),\phi(x)\right\rangle = \\
&\qquad = \mathrm{N}\!-\!\lim_{m\to\infty}\sum_{i=0}^{k}\frac{m^{s\lambda+\lambda-i-1}\phi^{(i)}(0)}{i!}\int_{-1}^{1}F_+(\rho^{(s)}(y),\lambda)y^i\,dy \\
&\qquad = 0,
\end{aligned}$$

for $s\lambda+\lambda\neq 1,2,\dots$ and $s=0,1,\dots$, proving equation (20). For $s\lambda+\lambda=1,2,\dots$ and $s=0,1,2,\dots$

$$\frac{\phi^{(s\lambda+\lambda-1)}(0)}{(s\lambda+\lambda-1)!}=\frac{(-1)^{s\lambda+\lambda-1}}{(s\lambda+\lambda-1)!}\left\langle\delta^{(s\lambda+\lambda-1)}(x),\phi(x)\right\rangle,$$

proving equation (21).□

COROLLARY 7.1 *Let $F_-(x,\lambda)$, $F(x,\lambda)$ and $G(x,\lambda)$ be the continuous functions*

$$x_-^\lambda,\quad x_+^\lambda+x_-^\lambda,\quad x_+^\lambda-x_-^\lambda$$

respectively, where $\lambda>0$. Then the distributions $F_-(\delta^{(s)}(x),\lambda)$, $F(\delta^{(s)}(x),\lambda)$ and $G(\delta^{(s)}(x),\lambda)$ exist on the real line and

$$F_-(\delta^{(s)}(x),\lambda)=0, \tag{23}$$

$$F(\delta^{(s)}(x),\lambda)=G(\delta^{(s)}(x),\lambda)=0, \tag{24}$$

for $s\lambda+\lambda\neq 1,2,\dots$ and $s=0,1,2,\dots$. Further,

$$F_-(\delta^{(s)}(x),\lambda) = \frac{(-1)^{s\lambda+\lambda-1}b(\rho,s,\lambda)}{(s\lambda+\lambda-1)!}\delta^{(s\lambda+\lambda-1)}(x), \tag{25}$$

$$F(\delta^{(s)}(x),\lambda) = \frac{(-1)^{s\lambda+\lambda-1}[c(\rho,s,\lambda)+b(\rho,s,\lambda)]}{(s\lambda+\lambda-1)!}\delta^{(s\lambda+\lambda-1)}(x), \tag{26}$$

$$G(\delta^{(s)}(x),\lambda) = \frac{(-1)^{s\lambda+\lambda-1}[c(\rho,s,\lambda)-b(\rho,s,\lambda)]}{(s\lambda+\lambda-1)!}\delta^{(s\lambda+\lambda-1)}(x) \tag{27}$$

for $s\lambda+\lambda=1,2,\dots$ and $s=0,1,2,\dots$, where

$$b(\rho,s,\lambda)=\int_{-1}^{1}F_-(\rho^{(s)}(y),\lambda)y^{s\lambda+\lambda-1}\,dy.$$

In particular,

$$F(\delta^{(s)}(x),\lambda)=\left|\delta^{(s)}(x)\right|^\lambda=0, \tag{28}$$

for $s\lambda + \lambda = 2, 4, \dots$ *and* $s = 0, 1, 2, \dots$.

PROOF. The proofs of equations (23) and (25) are similar to the proofs of equations (20) and (21). Equations (24), (26) and (27) follow immediately.

Further,

$$c(\rho, s, \lambda) + b(\rho, s, \lambda) = \int_{-1}^{1} \left|\rho^{(s)}(y)\right|^{\lambda} y^{s\lambda+\lambda-1}\, dy = 0,$$

and equation (28) follows.□

THEOREM 7.13. *Let f be a locally summable function on the real line and suppose that* $\inf\{f(x) : -\infty < x < \infty\} = c > 0$. *Then, the distribution $\delta^{(r)}(f(x))$ exists and*

$$\delta^{(r)}(f(x)) = 0,$$

for $r = 0, 1, 2, \dots$.

PROOF. We have

$$|f_m(x)| = |(f * \delta)_m(x)| \geq c,$$

for $m = 1, 2, \dots$ and all x. Choosing $K > c^{-1}$, we have

$$nf_m(x) \geq nc \geq 1,$$

for $m = 1, 2, \dots$, all x, and $n > K$. It follows that

$$\delta_n^{(r)}(f_m(x)) = 0,$$

for $n > K$, and so

$$\operatorname*{N-lim}_{m\to\infty}\left[\operatorname*{N-lim}_{n\to\infty}\left\langle \delta_n^{(r)}(f_m(x)), \phi(x)\right\rangle\right] = 0.\square$$

COROLLARY 7.2. *Let f be a locally summable function on the real line and suppose that* $\sup\{f(x) : -\infty < x < \infty\} = c < 0$. *Then, the distribution $\delta^{(r)}(f(x))$ exists, and*

$$\delta^{(r)}(f(x)) = 0,$$

for $r = 0, 1, 2, \ldots$.

PROOF. Defining $g(x) = -f(x)$, we have

$$\inf\{g(x) : -\infty < x < \infty\} = -c > 0.$$

It follows from the theorem that

$$\delta^{(r)}(g(x)) = 0 = (-1)^r \delta^{(r)}(f(x)),$$

for $r = 1, 2, \ldots$, proving the corollary.

The neutrix product in Definition 6.1 was defined for ultradistributions. If f and g are distributions in $\mathcal{D}'$ then the neutrix product of f and g is similarly defined as follows:

DEFINITION 7.8. *Let f and g be distributions in $\mathcal{D}'$ and let*

$$f_n(x) = (f * \delta_n)(x), \quad g_n(x) = (g * \delta_n)(x).$$

Then, the product $f.g$ is defined to exist and be equal to the distribution h on the interval (a, b) if

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle f_n(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle,$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

We note that with this definition of the product of two distributions, the definition of the distribution f^2 as the composition of the function x^2 and the distribution f, if it exists, is distinct from the definition of the product $f.f$, if it exists. However, the following theorem holds:

THEOREM 7.14. *Let f be a distribution in $\mathcal{D}'$. Then the distribution f^2 exists on the interval (a, b) if and only if the distribution $f.f$ exists on the interval (a, b). Then,*

$$f^2 = f.f$$

on the interval (a, b).

PROOF. We have

$$(x^2)_n = x^2 * \delta_n(x) = \int_{-1/n}^{1/n} (x-t)^2 \delta_n(t)\,dt = \int_{-1/n}^{1/n} t^2 \delta_n(t)(t)\,dt + x^2,$$

where

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} t^2 \delta_n(t)\,dt = 0.$$

Thus,

$$\left[(f_m(x))^2\right]_n = \int_{-1/n}^{1/n} t^2 \delta_n(t)\,dt + [f_m(x)]^2.$$

It follows that f^2 exists on the interval (a, b), if and only if

$$\operatorname*{N-lim}_{m\to\infty} \left[\operatorname*{N-lim}_{n\to\infty} \left\langle \left[(f_m(x))^2\right]_n, \phi(x)\right\rangle\right] \tag{29}$$

exists for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b). But

$$\operatorname*{N-lim}_{n\to\infty} \left\langle \left[(f_m(x))^2\right]_n, \phi(x)\right\rangle = \lim_{n\to\infty} \left\langle \left[(f_m(x))^2\right]_n, \phi(x)\right\rangle = \langle f_m(x) f_m(x), \phi(x)\rangle$$

certainly exists, and so (29) exists if and only if $f.f$ exists. □