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Effectiveness of the quantum-mechanical formalism in cognitive modeling

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Abstract Traditional approaches to cognitive psychology are founded on a classical vision of logic and probability theory. According to this perspective, the probabilistic aspects of human reasoning can be formalized in a Kolmogorovian probability framework and reveal underlying Boolean-type logical structures. This vision has been seriously challenged by various discoveries in experimental psychology in the last three decades. Meanwhile, growing research indicates that quantum theory provides the conceptual and mathematical framework to deal with these classically problematical situations. In this paper we apply a general quantum-based modeling scheme to represent two types of cognitive situations where deviations from classical probability occur in human decisions, namely, 'conceptual categorization' and 'decision making'. We show that our quantum-theoretic modeling faithfully describes different sets of experimental data, explaining the observed deviations from classicality in terms of genuine quantum effects. These results may contribute to the development of applied disciplines where cognitive processes are involved, such as natural language processing, semantic analysis, and information retrieval.

Keywords Quantum theory · Cognitive modeling · Concept combination · Decision theory

1 Introduction

Classical logic and probability theory have exercised a long influence on the way in which scholars formalize and model cognitive processes. These structures are so deeply rooted in cognitive scientists that it is hard even to imagine an alternative. This notwithstanding, empirical evidence has accumulated in cognitive psychology in the last thirty years which indicates that classical

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structures do not probably provide the most general modeling framework when human decisions are at stake.

There are two major domains of cognition where deviations from classical logical and probabilistic structures have been observed.

The first of these two domains is 'concept theory'. Since the work of Eleanor Rosch, cognitive scientists know that concepts are 'graded', or 'fuzzy' notions, that is, humans estimate an item such as *Robin* as more typical of *Bird* than *Stork*. In other words, concepts exhibit 'graded typicality' [1]. A problem arises when one tries to mathematically represent this typicality (or also the 'membership weight'), of the combination of two concepts in terms of the typicality (membership weight) of the component concepts. One is intuitively led to think that the rules of classical (fuzzy set) logic and probability theory apply in such combinations. However, Osherson and Smith discovered in 1981 the 'Guppy effect' in concept conjunction (also known as the 'Pet-Fish problem') [2]. Humans score the typicality of an item such as *Guppy* with respect to the conjunction *Pet-Fish* as higher than the typicality of *Guppy* with respect to both *Pet* and *Fish* separately. One realizes at once that typicality violates rules of classical (fuzzy set) logic. A second set of human experiments on concept combinations were performed by James Hampton. He measured the membership weight, i.e. normalized membership estimation, of several items, e.g., *Apple*, *Broccoli*, *Almond*, etc., with respect to pairs of concepts, e.g., *Fruits*, *Vegetables*, and their conjunction, e.g., *Fruits and Vegetables*, or disjunction, e.g., *Fruits or Vegetables*. These membership weights again showed systematic deviations from classical (fuzzy set) rules for conjunction and disjunction of two concepts [3,4]. The conclusion is simple: if one accepts conceptual gradeness as an empirical fact, then one cannot express such gradeness in a classical (fuzzy) set-theoretic model.

The second set of empirical findings showing unexpected deviations from classicality pertains to 'decision theory' and can traced back to the work of Kahneman and Tversky in the eighties. Their famous experiment on the 'Linda story' revealed that situations exist where human subjects estimate the probability of the conjunction of two events as higher than the probability of one of them, thus violating monotonicity of classical probability (more generally, Bayes' rule) [5]. This 'conjunction fallacy' is an example of a human probability judgment. Another effect, the 'disjunction effect', was observed by Tversky and Shafir in the nineties [6]. In the latter, subjects prefer action A over action B if they know that an event X occurs, and also if they know that X does not occur, but they prefer B over A if they do not know whether X occurs or not. The disjunction effect violates a fundamental principle of rational decision theory, Savage's 'Sure-Thing principle' (more generally, the total probability law of classical probability [7]).

These experimental results induced various scholars to look for alternatives to traditional modeling approaches that could better cope with the effects, fallacies and paradoxes above. A major alternative is constituted by the so called 'quantum cognition approach', which employs the conceptual and mathematical framework of quantum theory to model cognitive processes (see, e.g., [8–12]). In this paper we explicitly apply the quantum-theoretic approach to cognitive psychology that was originated in Brussels [8, 11, 13–19]. This approach was inspired by a two decade research on the conceptual and mathematical foundations of quantum theory, the origins of quantum probability and its connections with contextuality [20, 21], and the detection of genuine quantum aspects outside the microscopic world [22–24].

In our perspective a concept is not a container of instantiations, as in classical (fuzzy set) approaches to concepts but, rather, an 'entity in a specific state that changes under the influence of a context' [13, 14]. There is a deep analogy between quantum and conceptual entities: both are realms of genuine potentialities, not of lack of knowledge of actualities. Indeed, in a quantum measurement process, the measurement context actualizes one outcome among the possible outcomes, thus provoking an indeterministic change of state of the microscopic quantum particle that is measured. Similarly, whenever a subject is asked to estimate the membership of an item with respect to one (or more concepts) and, more generally, in any decision process, contextual influence (of a cognitive type) and a transition from potential to actual occur in which an outcome is actualized from a set of possible outcomes. At variance with classical Kolmogorovian probability, quantum probability enables coping with this kind of contextuality and pure potentiality [8, 16].

The above considerations constituted the theoretic background for the development of a general quantum-based perspective for cognitive processes, which we apply in the present paper.

After briefly reviewing in Sect. 2 the technical aspects that we need to attain our results, we apply in Sect. 3 our quantum modeling approach to the conjunction 'A and B' and the disjunction 'A or B' of two concepts A and B showing, at the same time, that it enables successful modeling of Hampton's experimental data. In the same section we deal with the conjunction of two concepts where the second concept is negated, i.e. 'A and not B', showing that our modeling faithfully represents a large amount of data collected by ourselves on this type of conceptual combinations [25, 26]. Successively, we come to the disjunction effect. We describe two variants of this effect, the 'two-stage gamble' and the 'Hawaii problem', in Sect. 4, showing that in both cases the disjunction effect can be explained in terms of quantum interference and superposition. Finally, we analyse the conjunction fallacy and apply our quantum conceptual scheme to model a recent experiment on the 'Linda problem' (Sect. 5) [27]. Conclusive remarks in Sect. 6 illustrate the potential of application of our quantum cognition approach to computer science, in particular, latent semantic analysis and information retrieval.

Let us conclude this introductory section with a remark. Our quantum-theoretic approach contains a fundamentally novel element, which distinguishes it from other quantum cognition approaches, namely, we explain the occurrence of deviations from classicality in concrete human decisions not as biases of human mind but, rather, as genuine expressions of intrinsic quantum structures, such as contextuality, emergence, interference and superposition. Hence, the aforementioned effects, fallacies, paradoxes and contradictions are natural manifestations of a fundamentally quantum dynamics, without however requiring the existence of microscopic quantum processes in the human brain.[1]

2 Quantum mathematics for conceptual modeling

We illustrate here how the mathematical formalism of quantum theory can be applied to model cognitive situations [29]. For the sake of simplicity, we will limit technicalities to the essential that is needed for our purposes.

When the quantum mechanical formalism is applied for modeling purposes, each considered entity – in our case a cognitive entity – is associated with a complex Hilbert space $\mathcal{H}$, that is, a vector space over the field $\mathbb{C}$ of complex numbers, equipped with an inner product $\langle \cdot | \cdot \rangle$ that maps two vectors $\langle A|$ and $|B\rangle$ onto a complex number $\langle A|B\rangle$. We denote vectors by using the bra-ket notation introduced by Paul Adrien Dirac, one of the pioneers of quantum theory [29]. Vectors can be 'kets', denoted by $|A\rangle$, $|B\rangle$, or 'bras', denoted by $\langle A|$, $\langle B|$. The inner product between the ket vectors $|A\rangle$ and $|B\rangle$, or the bra-vectors $\langle A|$ and $\langle B|$, is realized by juxtaposing the bra vector $\langle A|$ and the ket vector $|B\rangle$, and $\langle A|B\rangle$ is also called a 'bra-ket', and it satisfies the following properties:

(i) $\langle A|A\rangle \geq 0$;

(ii) $\langle A|B\rangle = \langle B|A\rangle^*$, where $\langle B|A\rangle^*$ is the complex conjugate of $\langle A|B\rangle$;

(iii) $\langle A|(z|B\rangle + t|C\rangle) = z\langle A|B\rangle + t\langle A|C\rangle$, for $z, t \in \mathbb{C}$, where the sum vector $z|B\rangle + t|C\rangle$ is called a 'superposition' of vectors $|B\rangle$ and $|C\rangle$ in the quantum jargon.

From (ii) and (iii) follows that inner product $\langle \cdot | \cdot \rangle$ is linear in the ket and anti-linear in the bra, i.e. $(z\langle A| + t\langle B|)|C\rangle = z^*\langle A|C\rangle + t^*\langle B|C\rangle$.

We recall that the 'absolute value' of a complex number is defined as the square root of the product of this complex number times its complex conjugate, that is, $|z| = \sqrt{z^* z}$. Moreover, a complex number z can either be decomposed into its cartesian form $z = x + iy$, or into its polar form $z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$. As a consequence, we have $|\langle A|B\rangle| = \sqrt{\langle A|B\rangle\langle B|A\rangle}$. We define the 'length' of a ket (bra) vector $|A\rangle$ ($\langle A|$) as $|||A\rangle|| = ||\langle A||| = \sqrt{\langle A|A\rangle}$. A vector of unitary length is called a 'unit vector'. We say that the ket vectors $|A\rangle$ and $|B\rangle$ are 'orthogonal' and write $|A\rangle \perp |B\rangle$ if $\langle A|B\rangle = 0$.

We have now introduced the necessary mathematics to state the first modeling rule of quantum theory, as follows.

First quantum modeling rule: A state A of an entity – in our case a cognitive entity – modeled by quantum theory is represented by a ket vector $|A\rangle$ with length 1, that is $\langle A|A\rangle = 1$.

[1] It is worth to mention that our quantum conceptual approach shares some common aspects with the 'epistemic quantum computational structures' recently developed by some authors (see, e.g., [28]), where emergent conceptual properties are formalized, and an holistic meaning of the sentence is considered instead of a compositional one. Notwithstanding their similarities, the technical developments of the two approaches are different.

An orthogonal projection M is a linear operator on the Hilbert space, that is, a mapping $M : \mathcal{H} \to \mathcal{H}, |A\rangle \mapsto M|A\rangle$ which is Hermitian and idempotent. The latter means that, for every $|A\rangle, |B\rangle \in \mathcal{H}$ and $z, t \in \mathbb{C}$, we have:

(i) $M(z|A\rangle + t|B\rangle) = zM|A\rangle + tM|B\rangle$ (linearity);

(ii) $\langle A|M|B\rangle = \langle B|M|A\rangle^*$ (hermiticity);

(iii) $M \cdot M = M$ (idempotency).

The identity operator $\mathbb{1}$ maps each vector onto itself and is a trivial orthogonal projection. We say that two orthogonal projections M_k and M_l are orthogonal operators if each vector belonging to $M_k(\mathcal{H})$ is orthogonal to each vector contained in $M_l(\mathcal{H})$, and we write $M_k \perp M_l$, in this case. The orthogonality of the projection operators M_k and M_l can also be expressed by $M_k M_l = 0$, where 0 is the null operator. A set of orthogonal projection operators $\{M_k \ |k = 1, \ldots, n\}$ is called a 'spectral family' if all projectors are mutually orthogonal, that is, $M_k \perp M_l$ for $k \neq l$, and their sum is the identity, that is, $\sum_{k=1}^n M_k = \mathbb{1}$.

The above definitions give us the necessary mathematics to state the second modeling rule of quantum theory, as follows.

Second quantum modeling rule: A measurable quantity Q of an entity – in our case a cognitive entity – modeled by quantum theory, and having a set of possible real values $\{q_1, \ldots, q_n\}$ is represented by a spectral family $\{M_k \ |k = 1, \ldots, n\}$ in the following way. If the entity – in our case a cognitive entity – is in a state represented by the vector $|A\rangle$, then the probability of obtaining the value q_k in a measurement of the measurable quantity Q is $\langle A|M_k|A\rangle = ||M_k|A\rangle||^2$. This formula is called the 'Born rule' in the quantum jargon. Moreover, if the value q_k is actually obtained in the measurement, then the initial state is changed into a state represented by the vector

$$|A_k\rangle = \frac{M_k|A\rangle}{||M_k|A\rangle||} \tag{1}$$

This change of state is called 'collapse' in the quantum jargon.

This formalism can be extended to model more complex situations by allowing states to be represented by 'density operators' and measurements to be represented by 'positive operator valued measures'. Density operators would be used when the conceptual state is not completely defined, while positive operators would be used to describe state transformations in non-ideal cognitive measurements. This extension is however not necessary for our purposes.

3 Quantum modeling combinations of two concepts

We present our quantum modeling approach in Fock space for the combination of two concepts using the notions and symbols defined in Sect. 2. In the case of two combining entities, a Fock space $\mathcal{F}$ consists of two sectors: 'sector 1' is a Hilbert space $\mathcal{H}$, while 'sector 2' is a tensor product $\mathcal{H} \otimes \mathcal{H}$ of two isomorphic versions of $\mathcal{H}$.

Hampton identified in his experiments systematic deviations from classical (fuzzy) set conjunctions [3] and disjunctions [4]. More explicitly, if the membership weight of an item x with respect to the conjunction 'A and B' of two concepts A and B is higher than the membership weight of x with respect to one concept (both concepts), we say that the membership weight of x is 'overextended' ('double overextended') with respect to the conjunction (we briefly say that x is overextended with respect to the conjunction, in this case). If the membership weight of an item x with respect to the disjunction 'A or B' of two concepts A and B is less than the membership weight of x with respect to one concept, we say that the membership weight of x is 'underextended' with respect to the disjunction (we briefly say that x is underextended with respect to the disjunction, in this case).

A large part of Hampton's data on concept conjunctions cannot be modeled in a classical probability space satisfying the axioms of Kolmogorov [8]. For example, the membership weight of the item *Razor* with respect to the concepts *Weapons*, *Tools* and their conjunction *Weapons and Tools* were estimated in [3] as 0.63, 0.68 and 0.83, respectively. Thus, the item *Razor* is overextended with respect to the conjunction *Weapons and Tools* of the concepts *Weapons* and *Tools*. These data cannot be represented in a single classical probability space. Indeed, the membership weights $\mu(A), \mu(B)$ and $\mu(A \text{ and } B)$ of the item x with respect to concepts A and B and their conjunction 'A and B', respectively, can be represented in a classical Kolmogorovian probability model if and only if they satisfy the following inequalities [8,25]

$$\mu(A \text{ and } B) - \min(\mu(A), \mu(B)) \leq 0 \tag{2}$$

$$\mu(A) + \mu(B) - \mu(A \text{ and } B) \leq 1 \tag{3}$$

A similar situation occurs in the case of concept disjunctions. A large part of Hampton's data cannot be modeled in a classical Kolmogorovian probability space [8]. For example, the membership weight of the item *Curry* with respect to the concepts *Spices*, *Herbs* and their disjunction *Spices or Herbs* were estimated in [4] as 0.9, 0.4 and 0.76, respectively. Thus, the item *Curry* is underextended with respect to the disjunction *Spices or Herbs* of the concepts *Spices* and *Herbs*. These data cannot be represented in a single classical probability space. Indeed, the membership weights $\mu(A), \mu(B)$ and $\mu(A \text{ or } B)$ of the item x with respect to concepts A and B and their disjunction 'A or B', respectively, can be represented in a classical Kolmogorovian probability model if and only if they satisfy the following inequalities [8]

$$\max(\mu(A), \mu(B)) - \mu(A \text{ or } B) \leq 0 \tag{4}$$

$$0 \leq \mu(A) + \mu(B) - \mu(A \text{ or } B) \tag{5}$$

Let us construct our quantum model in Fock space for the conjunction of two concepts [8,16]. Let x denote an item and $\mu(A)$, $\mu(B)$ and $\mu(A \text{ and } B)$ denote the membership weights of x with respect to the concepts A, B and 'A and B', respectively. Let $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ be the Fock space where we represent conceptual entities. The states of the concepts A, B and 'A and B' are represented by the unit vectors $|A\rangle, |B\rangle \in \mathcal{H}$ and $|A \text{ and } B\rangle \in \mathcal{F}$, respectively, where

$$|A \text{ and } B\rangle = me^{i\lambda}|A\rangle \otimes |B\rangle + ne^{i\nu}\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \tag{6}$$

The superposition vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ describes 'A and B' as a new emergent concept, while the product vector $|A\rangle \otimes |B\rangle$ describes 'A and B' in terms of concepts A and B. The weights m and n are such that $m^2 + n^2 = 1$. The decision measurement of a subject who estimates the membership of the item x with respect to the concept 'A and B' is represented by the orthogonal projection operator $M \oplus (M \otimes M)$ on $\mathcal{F}$, where M is an orthogonal projection operator on $\mathcal{H}$. Hence, the membership weight of x with respect to 'A and B' is given by

$$\begin{aligned}\mu(A \text{ and } B) &= \langle A \text{ and } B|M \oplus (M \otimes M)|A \text{ and } B\rangle \\ &= m^2\mu(A)\mu(B) + n^2\left(\frac{\mu(A) + \mu(B)}{2} + \Re\langle A|M|B\rangle\right)\end{aligned} \tag{7}$$

where $\mu(A) = \langle A|M|A\rangle$ and $\mu(B) = \langle B|M|B\rangle$. The term $\Re\langle A|M|B\rangle$ is the usual 'interference term' of quantum theory. A solution of Eq. (7) exists in the Fock space $\mathcal{F} = \mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$, where M is the subspace of $\mathbb{C}^3$ generated by the vectors $(1, 0, 0)$ and $(0, 1, 0)$ ($\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is the canonical basis of $\mathbb{C}^3$), the interference term is given by

$$\begin{aligned}\Re\langle A|M|B\rangle &= \sqrt{1 - a(A)}\sqrt{1 - b(B)}\cos\theta \\ &= \begin{cases} \sqrt{1 - \mu(A)}\sqrt{1 - \mu(B)}\cos\theta & \text{if } \mu(A) + \mu(B) > 1 \\ \sqrt{\mu(A)}\sqrt{\mu(B)}\cos\theta & \text{if } \mu(A) + \mu(B) \le 1 \end{cases}\end{aligned} \tag{8}$$

(θ is the 'interference angle for the conjunction'), and the unit vectors $|A\rangle, |B\rangle \in \mathbb{C}^3$ are given by

$$|A\rangle = \left(\sqrt{a(A)}, 0, \sqrt{1 - a(A)}\right) \tag{9}$$

$$|B\rangle = e^{i\theta}\left(\sqrt{\frac{(1 - a(A))(1 - b(B))}{a(A)}}, \sqrt{\frac{a(A) + b(B) - 1}{a(A)}}, -\sqrt{1 - b(B)}\right) \tag{10}$$

if $a(A) \neq 0$, and

$$|B\rangle = e^{i\theta}(0, 1, 0) \tag{11}$$

if $a(A) = 0$. For example, the item *Razor* has a Fock space representation in $\mathcal{F} = \mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with $\theta = 35.77$, $m^2 = 0.24$, $n^2 = 0.76$, $|A\rangle = (0.79, 0, 0.61)$ and $|B\rangle = e^{i35.77^\circ}(0.36, 0.81, -0.47)$.

Conceptual disjunctions can be modeled in a similar way. Let x denote an item and let $\mu(A)$, $\mu(B)$ and $\mu(A \text{ or } B)$ denote the membership weights of x with respect to the concepts A, B and 'A or B', respectively. We again represent conceptual entities in the Fock space $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$. The concepts

A, B and 'A or B' are represented by the unit vectors $|A\rangle, |B\rangle \in \mathcal{H}$ and $|A \text{ or } B\rangle \in \mathcal{F}$, respectively, where

$$|A \text{ or } B\rangle = me^{i\lambda}|A\rangle \otimes |B\rangle + ne^{i\nu}\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \tag{12}$$

The superposition vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ describes 'A or B' as a new emergent concept, while the product vector $|A\rangle \otimes |B\rangle$ describes 'A or B' in terms of concepts A and B. The positive numbers m and n are such that $m^2 + n^2 = 1$, and they estimate the 'degree of participation' of sectors 2 and 1, respectively, in the disjunction case. The decision measurement of a subject who estimates the membership of the item x with respect to the concept 'A or B' is represented by the orthogonal projection operator $M \oplus (M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M)$ on $\mathcal{F}$, where M has been introduced above. We observe that

$$M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M = \mathbb{1} - (\mathbb{1} - M) \otimes (\mathbb{1} - M) \tag{13}$$

that is, in the transition from conjunction to disjunction we have applied de Morgan's laws of logic in sector 2 of Fock space. The membership weight of x with respect to 'A or B' is given by

$$\begin{aligned} \mu(A \text{ or } B) &= \langle A \text{ or } B|M \oplus (M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M)|A \text{ or } B\rangle \\ &= m^2\left(\mu(A) + \mu(B) - \mu(A)\mu(B)\right) + n^2\left(\frac{\mu(A) + \mu(B)}{2} + \Re\langle A|M|B\rangle\right) \end{aligned} \tag{14}$$

where $\mu(A) = \langle A|M|A\rangle$, $\mu(B) = \langle B|M|B\rangle$ and $\Re\langle A|M|B\rangle$ is the interference term. A solution of Eq. (14) exists in the Fock space $\mathcal{F} = \mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ by choosing the projection operator M, the interference term $\Re\langle A|M|B\rangle$ and the unit vectors $|A\rangle$ and $|B\rangle \in \mathbb{C}^3$ as in the case of conjunction (see Eqs. (8)–(11)). For example, the item *Curry* has a Fock space representation in $\mathcal{F} = \mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with $\theta = 97.07$, $m^2 = 0.44$, $n^2 = 0.56$, $|A\rangle = (0.95, 0, 0.32)$ and $|B\rangle = e^{i97.07^\circ}(0.26, 0.58, -0.77)$.

The quantum mathematics above admits the following interpretation. Whenever a subject is asked to estimate whether a given item x belongs to the concepts A, B, 'A and B' ('A or B'), two mechanisms act simultaneously and in superposition in the subject's thought. A 'quantum logical reasoning', which is a probabilistic version of classical logical reasoning, where the subject considers two copies of item x and estimates whether the first copy belongs to A and (or) the second copy of x belongs to B, and further the probabilistic version of the conjunction (disjunction) is applied to both estimates. But also a 'quantum conceptual reasoning' acts, where the subject estimates whether the exemplar x belongs to the newly emergent concept 'A and B' ('A or B'). The place whether these superposed processes can be suitably structured is Fock space. The conceptual reasoning process occurs in sector 1, and the logical reasoning process occurs in sector 2, while the weights m^2 and n^2 measure the 'degree of participation' of sectors 2 and 1, respectively, in the case of conjunction (disjunction). In both examples of *Razor* and *Curry*, the combination process mainly occurs in sector 1 of Fock space, i.e. n^2 is higher, which means that emergence aspects prevail over logical aspects in the reasoning process.

We have recently extended this analysis by performing a new cognitive test to study how human subjects estimate the membership weights of specific items with respect to the conjunctions 'A and B' and 'A and not B' of the concepts A and B, where 'not B' denotes the negation of B, e.g., *Fruits and not Vegetables*, or *Spices and not Herbs* [25,26]. The data collected on 'A and B' systematically showed overextension, thus confirming the patterns observed in [3]. Our quantum modeling for the conjunction faithfully represents almost all the collected data. For example, the item *Olive* was double overextended with respect to *Fruits and Vegetables*, since its membership weight with respect to *Fruits* and *Vegetables* was 0.53 and 0.63, respectively, while its membership weight with respect to *Fruits and Vegetables* was 0.65 [25]. *Olive* can be modeled in the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with an interference angle $\theta = 60.48°$, a weight $m^2 = 0.30$ in sector 2, and a weight $n^2 = 0.70$ in sector 1. The concepts *Fruits* and *Vegetables* are respectively represented by the unit vectors $|A\rangle = (0.73, 0, 0.68)$ and $|B\rangle = e^{i60.48°}(0.69, 0.55, -0.61)$ in the canonical basis of $\mathbb{C}^3$. More, the item *Goldfish* showed a big overextension with respect to *Pets and Farmyard Animals*, since it scored 0.93 with respect to *Pets*, 0.17 with respect to *Farmyard Animals*, and 0.43 with respect to *Pets and Farmyard Animals*. *Goldfish* can be modeled in $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with $\theta = 99.22°$, $m^2 = 0.23$ and $n^2 = 0.77$. The concept *Pets* is represented by $|A\rangle = (0.96, 0, 0.27)$, while the concept *Farmyard Animals* is represented by $|B\rangle = e^{i99.22°}(0.38, 0.32, -0.91)$.

Analogously, the data on the conjunction 'A and not B' showed systematic deviations from classicality, which were of two types:

(i) overextension in conceptual conjunction, i.e. items such that

$$\mu(A \text{ and not } B) > \min(\mu(A), \mu(\text{not } B)) \tag{15}$$

(ii) deviation from Kolmogorovness in conceptual negation, i.e. items such that

$$\mu(B) + \mu(\text{not } B) \neq 1 \tag{16}$$

Our Fock space modeling for the conjunction 'A and B' can be naturally extended to the conjunction 'A and not B' by representing the latter concept by the unit vector

$$|A \text{ and not } B\rangle = me^{i\lambda}|A\rangle \otimes |\text{not } B\rangle + ne^{i\nu}\frac{1}{\sqrt{2}}(|A\rangle + |\text{not } B\rangle) \tag{17}$$

A solution exists in $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ where the membership weight of the item x with respect to 'A and not B' is given by [25,26]

$$\begin{aligned} \mu(A \text{ and not } B) &= m^2\mu(A)\mu(\text{not } B) \\ +n^2 \Big(\tfrac{\mu(A)+\mu(\text{not } B)}{2} &+ \sqrt{1-a(A)}\sqrt{1-b(\text{not } B)}\cos\theta\Big) \end{aligned} \tag{18}$$

Symbols in Eq. (18) are defined as in Eq. (8), and formulas analogous to Eqs. (9)–(11) hold also in this case.

Let us consider a couple of examples. The item *Prize Bull* was double overextended with respect to *Pets and not Farmyard Animals*, since it scored 0.13 with respect to *Pets*, 0.26 with respect to *not Farmyard Animals* and 0.28 with respect to *Pets and not Farmyard Animals*. The exemplar *Prize Bull* can be modeled in Fock space with an interference angle $\theta = 45.11^\circ$ and weights $m^2 = 0.18$ for sector 2 of Fock space and $n^2 = 0.82$ for sector 1. The concepts *Pets* and *not Farmyard Animals* are represented by $|A\rangle = (0.93, 0, 0.36)$ and $|\text{not } B\rangle = e^{i45.11^\circ}(0.2, 0.84, -0.51)$ with respect to the item *Prize Bull*. The item *Wall Mirror* had membership weight 0.45 with respect to *Furniture*, 0.76 with respect to *not Furniture*, hence negation showed deviation from Kolmogorovness. *Wall Mirror* can be modeled in $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ for $\theta = 74.90^\circ$, $m^2 = 0.23$, $n^2 = 0.77$, $|A\rangle = (0.96, 0, 0.30)$ and $|\text{not } B\rangle = e^{i74.90^\circ}(0.23, 0.63, -0.74)$.

The quantum mathematics above can again be interpreted in terms of quantum logical and quantum conceptual reasoning. Two superposed mechanisms act when a subject is asked to estimate whether a given item x belongs to the concepts A, 'not B', 'A and not B': a quantum logical reasoning, where the subject considers two copies of x and estimates whether the first copy belongs to A and the second copy of x does not belong to B, and also a quantum conceptual reasoning, where the subject estimates whether the exemplar x belongs to the newly emergent concept 'A and not B'. Both processes can be simultaneously represented in Fock space.

As we can see, quantum structures can describe how the human mind combines two concepts, and genuine quantum aspects, i.e. contextuality, emergence, interference, superposition, can account for the observed divergences from classical structures.

4 Nonclassicality in the disjunction effect

A whole set of findings in other domains of cognitive science point to a deviation of classical logical reasoning in concrete human decisions, as mentioned in Sect. 1. In behavioral economics, this deviation is manifest in the 'Ellsberg paradox' [30] and 'Machina paradox' [31], which violate the so-called 'Savage's Sure-Thing principle' of expected utility theory [7]. We discussed these paradoxical situations in a recent paper [18]. However, we do not deal with them here, for the sake of brevity. We instead illustrate the violation of the Sure-Thing principle that is observed in decision theory.

Savage stated this principle by means of the following story [7].

'A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant. So, to clarify the matter to himself, he asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say.'

Tversky and Shafir tested the Sure-Thing principle in an experiment where they presented a group of students with a 'two-stage gamble', that is, a gamble which can be played twice [6, 10]. At each stage the decision consisted in whether or not playing a gamble that has an equal chance of winning, say \$200, or losing, say \$100. The key result is based on the decision for the second bet, after finishing the first bet. The experiment included three situations: (i) the students were informed that they had already won the first gamble; (ii) the students were informed that they had lost the first gamble; (iii) the students did not know the outcome of the first gamble. Tversky and Shafir found that 69%, i.e. the majority, of the students who knew they had won the first gamble chose to play again, 59%, i.e. the majority, of the students who knew they had lost the first gamble, chose to play again; but only 36% of the students who did not know whether they had won or lost chose to play again (equivalently, 64%, i.e. the majority, decided not to play in the second gamble).

This two-stage gamble experiment violates Savage's Sure-Thing Principle: students generally prefer to play again if they know they won, and they also prefer to play again if they know they lost, but they generally prefer not to play again when they do not know whether they won or lost. More generally, the experiment performed by Tversky and Shafir violates the total law of classical probability. If we denote by $p(P)$ the total probability that a student decides to play again without knowing whether he/she has won or lost in the first gamble, by $p(W)$ and $p(L)$ the probability that the student wins or loses, respectively, by $p(P|W)$ the conditional probability that the student decides to play again when he/she knows he/she has won, and by $p(P|L)$ the conditional probability that the student decides to play again when he/she knows he/she has lost, then it is not possible to find any value of $p(W)$ and $p(L) = 1 - p(W)$ such that $p(P|W) = 0.69$ and $p(P|L) = 0.59$, $p(P) = 0.36$ and the law of total probability

$$p(P) = p(W)p(P|W) + p(L)p(P|L) \tag{19}$$

is satisfied. This violation of the laws of classical probability is called the 'disjunction effect'.

An equivalent formulation of the disjunction effect is known as the 'Hawaii problem', and it is again due to Tversky and Shafir [6]. Consider the following situations.

'Disjunctive version'. Imagine that you have just taken a tough qualifying examination. It is the end of the fall quarter, you feel tired and run-down, and you are not sure that you passed the exam. In case you failed you have to take the exam again in a couple of months after the Christmas holidays. You now have an opportunity to buy a very attractive 5-day Christmas vacation package to Hawaii at an exceptionally low price. The special offer expires tomorrow, while the exam grade will not be available until the following day. Would you: x buy the vacation package; y not buy the vacation package; z pay a \$5 non-refundable fee in order to retain the rights to buy the vacation package at the same exceptional price the day after tomorrow after you find out whether or not you passed the exam?

'Pass/fail version'. Imagine that you have just taken a tough qualifying examination. It is the end of the fall quarter, you feel tired and run-down, and you find out that you passed the exam (failed the exam. You will have to take it again in a couple of months after the Christmas holidays). You now have an opportunity to buy a very attractive 5-day Christmas vacation package to Hawaii at an exceptionally low price. The special offer expires tomorrow. Would you: x buy the vacation package; y not buy the vacation package: z pay a \$5 non-refundable fee in order to retain the rights to buy the vacation package at the same exceptional price the day after tomorrow.

In this experiment, Tversky and Shafir experienced the same pattern of the two-stage gamble situation. Indeed, more than half of the subjects chose option x (buy the vacation package) if they knew the outcome of the exam (54% in the pass condition and 57% in the fail condition), whereas only 32% chose option x (buy the vacation package) if they did not know the outcome of the exam. The Hawaii problem clearly shows a violation of the Sure-Thing principle: subjects generally prefer option x (buy the vacation package) when they know that they passed the exam, and they also prefer x when they know that they failed the exam, but they refuse x (or prefer z) when they don't know whether they passed or failed the exam. Moreover, as in the two-stage gamble experiment, also the Hawaii problem violates the total law of classical probability.

A seemingly plausible explanation, which is also given in the Ellsberg paradox, is that the origin of the violation of the Sure-Thing principle in the Hawaii problem is 'uncertainty aversion', that is, subjects prefer to buy the vacation package in both cases where they have certainty about the outcome of the exam, while they refuse to buy the package when they do not yet know whether they passed or failed the exam and hence lack this certainty.

We now work out a quantum-theoretic model for these two experiments, where the above mentioned deviation is described in terms of genuine quantum effects.

The disjunction effect in decision theory is an example of a situation that can be described in the quantum modeling scheme that we have elaborated in Sect. 3 [8]. Let us firstly consider the Hawaii problem and denote by A the conceptual situation in which the subject has passed the exam, and by B the conceptual situation in which the subject has failed the exam. The disjunction of both conceptual situations, denoted by 'A or B', is the conceptual situation in which the subject 'has passed or failed the exam'. The subject needs to make a decision whether to buy the vacation package – positive outcome, or not to buy it – negative outcome.

We introduce the notion of state of a concept, as in Sect. 3 [8,13,14]. Thus, each conceptual situation above is described by a conceptual state and represented by a unit vector in a complex Hilbert space. More explicitly, we represent A by a unit vector $|A\rangle$ and B by a unit vector $|B\rangle$ in a complex Hilbert space $\mathcal{H}$, respectively. We assume that $|A\rangle$ and $|B\rangle$ are orthogonal, that is, $\langle A|B\rangle = 0$, and represent the disjunction 'A or B' by means of the normalized superposition state vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$. The decision to be made is 'to buy the vacation package' or 'not to buy the vacation package'. This decision is represented by an orthogonal projection operator M of the Hilbert space $\mathcal{H}$ in our modeling scheme. The probability of the outcome 'yes', i.e. 'buy the package', in the 'pass' situation, i.e. state vector $|A\rangle$, is 0.54, and we denote it by $\mu(A) = 0.54$. The probability of the outcome 'yes', i.e. buy the package, in the 'fail' situation, i.e. state vector $|B\rangle$, is 0.57, and we denote it by $\mu(B) = 0.57$. The probability of the outcome 'yes', i.e. buy the package, in the 'pass or fail' situation, i.e. state vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$, is 0.32, and we denote it by $\mu(A \text{ or } B) = 0.32$.

In accordance with standard quantum rules (see Sect. 2), we have

$$\mu(A) = \langle A|M|A\rangle \tag{20}$$
$$\mu(B) = \langle B|M|B\rangle \tag{21}$$
$$\mu(A \text{ or } B) = \tfrac{1}{2}(\langle A| + \langle B|)M(|A\rangle + |B\rangle) \tag{22}$$

By applying the linearity of Hilbert space and the hermiticity of M, that is, $\langle B|M|A\rangle^* = \langle A|M|B\rangle$, we then get

$$\begin{aligned}\mu(A \text{ or } B) &= \frac{1}{2}(\langle A|M|A\rangle + \langle A|M|B\rangle + \langle B|M|A\rangle + \langle B|M|B\rangle) \\ &= \frac{\mu(A) + \mu(B)}{2} + \Re\langle A|M|B\rangle \end{aligned} \tag{23}$$

where $\Re\langle A|M|B\rangle$ is the real part of the complex number $\langle A|M|B\rangle$, i.e. the typical interference term of quantum theory. Its presence allows to produce a deviation from the average value $\frac{1}{2}(\mu(A)+\mu(B))$, which would be the outcome in absence of interference. Note that, also in this disjunction effect situation, we have applied two key quantum features, namely, 'superposition', in taking $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ to represent 'A or B', and 'interference', as the effect appearing in Eq. (23).

Our quantum model can be realized in the three-dimensional complex Hilbert space $\mathbb{C}^3$ [8], as follows. Let us distinguish two cases:

(i) if $\mu(A) + \mu(B) \le 1$, we put $a(A) = 1 - \mu(A)$, $b(B) = 1 - \mu(B)$ and $\gamma = \pi$;

(ii) if $\mu(A) + \mu(B) > 1$, we put $a(A) = \mu(A)$, $b(B) = \mu(B)$ and $\gamma = 0$.

Moreover, we choose

$$|A\rangle = (\sqrt{a(A)}, 0, \sqrt{1 - a(A)}) \tag{24}$$

$$|B\rangle = \begin{cases} e^{i(\beta+\gamma)}\left(\sqrt{\frac{(1-a(A))(1-b(B))}{a(A)}}, \sqrt{\frac{a(A)+b(B)-1}{a(A)}}, -\sqrt{1-b(B)}\right) & \text{if } a(A) \neq 0 \\ e^{i\beta}(0,1,0) & \text{if } a(A) = 0 \end{cases} \tag{25}$$

$$\beta = \begin{cases} \arccos\left(\frac{2\mu(A \text{ or } B) - \mu(A) - \mu(B)}{2\sqrt{(1-a(A))(1-b(B))}}\right) & \text{if } \quad a(A) \neq 1, b(B) \neq 1 \\ \text{arbitrary} & \text{if } a(A) = 1 \text{ or } b(B) = 1 \end{cases} \tag{26}$$

If $\mu(A)+\mu(B) \leq 1$, we take $M(\mathbb{C}^3)$ as the ray spanned by the vector $(0,0,1)$, that is, $M = |0,0,1\rangle\langle 0,0,1|$. If $\mu(A)+\mu(B) > 1$, we take $M(\mathbb{C}^3)$ as the subspace of $\mathbb{C}^3$ spanned by the vectors $(1,0,0)$ and $(0,1,0)$, that is, $M = |1,0,0\rangle\langle 1,0,0| + |0,1,0\rangle\langle 0,1,0|$.

One can verify that this construction gives rise to a quantum mechanical representation of the Hawaii problem situation with probabilities $\mu(A), \mu(B)$ and $\mu(A \text{ or } B)$. In particular, the interference term in Eq. (23) is given by

$$\Re\langle A|M|B\rangle = \sqrt{(1-a(A))(1-b(B))}\cos\beta \tag{27}$$

where β is the 'interference angle for the disjunction'. We refer to [8] for a more detailed technical analysis of this quantum-theoretic model.

Equations (23) and (27) can be used to represent the Hawaii problem situation. If we put $\mu(A) = 0.54$, $\mu(B) = 0.57$ and $\mu(A \text{ or } B) = 0.32$, and observe that $\mu(A)+\mu(B) = 1.11 > 1$, then we have $a(A) = 0.54$, $b(B) = 0.57$ and $\gamma = 0$. After making the calculations of Eqs. (24), (25) and (26), we obtain $|A\rangle = (0.73, 0, 0.68)$, $|B\rangle = e^{i121.90^\circ}(0.61, 0.45, -0.66)$ and we take $M(\mathbb{C}^3)$ the subspace of $\mathbb{C}^3$ spanned by the vectors $(1,0,0)$ and $(0,1,0)$. One verifies at once that this model indeed yields the correct numerical outcomes.

Let us now come to the two-stage gamble situation. Here, we have $\mu(A) = 0.69$, $\mu(B) = 0.59$ and $\mu(A \text{ or } B) = 0.36$, hence $\mu(A)+\mu(B) = 1.28 > 1$, $a(A) = 0.69$, $b(B) = 0.59$ and $\gamma = 0$. Equations (23) and (27) can be solved for $\beta = 141.76^\circ$. In addition, Eqs. (24), (25) and (26) can be solved for $|A\rangle = (0.83, 0, 0.56)$, $|B\rangle = e^{i141.76^\circ}(0.43, 0.64, -0.64)$ and $M(\mathbb{C}^3)$ is the subspace of $\mathbb{C}^3$ spanned by vectors $(1,0,0)$ and $(0,1,0)$. Also in this case, one easily verifies that our quantum model yields the correct numerical outcomes.

We have thus provided a quantum-theoretic model which successfully represents the disjunction effect occurring in the experiments by Tversky and Shafir [6]. It is important to observe that the observed deviations from classical Kolmogorovian probability are not interpreted as biases of human mind in our approach but, rather, as the deepest expressions of pure quantum effects, namely, contextuality, interference and superposition. It is also worth noticing the fundamental role that complex numbers play in our construction, since they make it possible to have a non-null interference term in Eq. (23).

5 Nonclassicality in the conjunction fallacy

An important deviation from classicality that occurs in decision theory and is similar to over- and under- extensions occurring in concept combinations is the 'conjunction fallacy'. Tversky and Kahneman discovered this fallacy in an experiment that is known in the literature as the 'Linda problem' [5,10].

In the experiment, subjects were presented with the following story about a woman named 'Linda'.

'Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.'

Then, subjects were asked to check which of the two alternatives was more probable:

(a) Linda is a bank teller;

(b) Linda is a bank teller and is active in the feminist movement.

Tversky and Kahneman found that overall 85% of the respondents indicated that option (b) – Linda is a bank teller and is active in the feminist movement – is more likely than option (a) – Linda is a bank teller. If one denotes by $p(B)$ the probability that the sentence "Linda is a bank teller" is true, and by $p(B \text{ and } F)$ the probability that the sentence "Linda is a bank teller and is active in the feminist movement" is true, then one expects that $p(B \text{ and } F) \leq p(B)$ in classical Kolmogorovian probability. This experimental violation exactly expresses the conjunction fallacy.

There is now a large empirical literature in cognitive psychology confirming the results found by Tversky and Kahneman in the Linda problem. In particular, an interesting experiment was performed by Morier and Borgida [10, 27]. They used the Linda story and asked subjects to rank the likelihood of the following events:

(a) Linda is a feminist;

(b) Linda is a bank teller;

(c) Linda is a feminist and a bank teller;

(d) Linda is a feminist or a bank teller.

Morier and Borgida found that the mean probability judgements were ordered as p(feminist)=0.83 > p(feminist or bank teller)=0.60 > p(feminist and bank teller)=0.36 > p(bank teller)=0.26. We have seen above that the conjunction fallacy occurs when option (c) is estimated to be more likely than option (b). A 'disjunction fallacy' instead occurs when option (a) is judged to be more likely than option (d). One can immediately recognize similarities between conjunction fallacy and conceptual overextension on one side, and between disjunction fallacy and conceptual underextension on the other side. Both types of fallacy are present in Morier and Borgida's experiment [27].

Various approaches have been put forward to provide an alternative explanation of the conjunction fallacy. In particular, a recurring explanation suggests that these deviations from classical probabilistic rules in human decisions should be considered as 'biases' of human thought, whence the locutions 'fallacy', 'effect' or 'paradox'. We instead show that our quantum modeling approach in Hilbert space also enables faithful representation of both the conjunction and disjunction fallacies.[2]

The conjunction fallacy can be modeled in our quantum-theoretic framework by following similar procedures to the ones adopted in Sects. 3 and 4. To this end we denote by A the conceptual situation where Linda is a feminist, and by B the conceptual situation where Linda is a bank teller. The

[2] The disjunction fallacy introduced here must be distinguished from the disjunction effect discussed in Sect. 4. The latter is classified as a 'decision making error', the former as a 'probability judgement error'. Notwithstanding their conceptual differences, however, both effects can be described in terms of quantum interference effects (see also [10]).

conjunction 'A and B' corresponds to the conceptual situation where 'Linda is a feminist and a bank teller'.

The conceptual situations A and B are represented by the unit vectors $|A\rangle$ and $|B\rangle$, respectively, in a complex Hilbert space $\mathcal{H}$. We assume that $|A\rangle$ and $|B\rangle$ are orthogonal, i.e. $\langle A|B\rangle = 0$, and represent the conjunction 'A and B' by the normalized superposition state vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$. The decision measurement of a subject estimating whether 'Linda is a feminist and a bank teller' is represented by the orthogonal projection operator M of the Hilbert space $\mathcal{H}$. Let us denote by $\mu(A)$, $\mu(B)$ and $\mu(A \text{ and } B)$ the probabilities that 'Linda is a feminist', 'Linda is a bank teller' and 'Linda is a feminist and a bank teller', respectively. Note that these probabilities can also be interpreted as membership weights of the item *Linda* with respect to the concepts *Feminist*, *Bank Teller* and *Feminist and Bank Teller*, respectively, in accordance with the analysis in Sect. 3. These probabilities are equal to $\mu(A) = 0.83$, $\mu(B) = 0.26$ and $\mu(A \text{ and } B) = 0.36$ in [27].

To represent the experimental situation above we follow standard quantum rules (see Sect. 2), as follows.

$$\begin{aligned}\mu(A \text{ and } B) &= \frac{1}{2}(\langle A|M|A\rangle + \langle A|M|B\rangle + \langle B|M|A\rangle + \langle B|M|B\rangle) \\ &= \frac{\mu(A)+\mu(B)}{2} + \Re\langle A|M|B\rangle \end{aligned} \tag{28}$$

where $\Re\langle A|M|B\rangle$ is the interference term. Our quantum model can be realized in the three-dimensional complex Hilbert space $\mathbb{C}^3$ [8], as follows. We distinguish two cases:

(i) if $\mu(A)+\mu(B) \le 1$, we put $a(A) = 1-\mu(A)$, $b(B) = 1-\mu(B)$ and $\gamma = \pi$;

(ii) if $\mu(A)+\mu(B) > 1$, we put $a(A) = \mu(A)$, $b(B) = \mu(B)$ and $\gamma = 0$.

Moreover, we choose

$$|A\rangle = (\sqrt{a(A)}, 0, \sqrt{1-a(A)}) \tag{29}$$

$$|B\rangle = \begin{cases} e^{i(\alpha+\gamma)}\left(\sqrt{\frac{(1-a(A))(1-b(B))}{a(A)}}, \sqrt{\frac{a(A)+b(B)-1}{a(A)}}, -\sqrt{1-b(B)}\right) & \text{if } a(A) \neq 0 \\ e^{i\alpha}(0,1,0) & \text{if } a(A) = 0 \end{cases} \tag{30}$$

$$\alpha = \begin{cases} \arccos\left(\frac{2\mu(A \text{ and } B)-\mu(A)-\mu(B)}{2\sqrt{(1-a(A))(1-b(B))}}\right) & \text{if } \quad a(A) \neq 1, b(B) \neq 1 \\ \text{arbitrary} & \text{if } a(A) = 1 \text{ or } b(B) = 1 \end{cases} \tag{31}$$

As in Sect. 4, if $\mu(A)+\mu(B) \le 1$, we take $M(\mathbb{C}^3)$ as the ray spanned by the vector $(0,0,1)$, if $\mu(A)+\mu(B) > 1$, we take $M(\mathbb{C}^3)$ as the subspace of $\mathbb{C}^3$ spanned by the vectors $(1,0,0)$ and $(0,1,0)$. One can verify that this construction gives rise to a quantum-mechanical representation of the conjunction fallacy situation with probabilities $\mu(A), \mu(B)$ and $\mu(A \text{ and } B)$ [8,25]. In particular, the interference term in Eq. (23) is given by

$$\Re\langle A|M|B\rangle = \sqrt{(1-a(A))(1-b(B))}\cos\alpha \tag{32}$$

where α is the 'interference angle for the conjunction'.

Equations (28) and (32) can be used to represent the conjunction fallacy situation. If we put $\mu(A) = 0.83$, $\mu(B) = 0.26$ and $\mu(A \text{ and } B) = 0.36$, and observe that $\mu(A) + \mu(B) = 1.09 > 1$, then we have $a(A) = 0.83$, $b(B) = 0.26$ and $\gamma = 0$. After making the calculations of Eqs. (29), (30) and (31), we obtain $|A\rangle = (0.91, 0, 0.41)$, $|B\rangle = e^{i121.44^\circ}(0.39, 0.33, -0.86)$ and we take $M(\mathbb{C}^3)$ the subspace of $\mathbb{C}^3$ spanned by the vectors $(1, 0, 0)$ and $(0, 1, 0)$. It is easy to verify that this model indeed yields the correct numerical outcomes.

Interestingly enough, we can also provide a quantum representation for the disjunction fallacy occurring in [27]. Indeed, by following the procedure in Sect. 4 and solving Eq. (27) with respect to the interference angle β for the disjunction, with $\mu(A) = 0.83$, $\mu(B) = 0.26$ and $\mu(A \text{ or } B) = 0.60$, we find $\beta = 81.08^\circ$. Hence, the conceptual situation 'Linda is a feminist', 'Linda is a bank teller' and 'Linda is a feminist or a bank teller' are represented by the unit vectors $|A\rangle$, $|B\rangle$ and $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$, where $|A\rangle = (0.91, 0, 0.41)$ (Eq. (24)), $|B\rangle = e^{i81.08^\circ}(0.39, 0.33, -0.86)$ (Eq. (25)) in $\mathbb{C}^3$.

Conjunction and disjunction fallacies can be respectively interpreted as the decision theory counterparts of overextension and underextension of concept theory (see Sect. 3). The quantum aspects of contextuality, interference and superposition can again naturally account for the observed deviations from classical structures.

6 Possible applications to natural language processing

The application of techniques and procedures of the formalism of quantum theory to domains such as information retrieval (IR) and natural language processing (NLP) has produced various interesting results. These quantum-based approaches mainly integrate standard methods in IR and NLP. Roughly speaking, one considers 'documents' and 'terms' as basic ingredients, focusing on the 'document-term matrix' which contains as entries the number of times that a specific term appears in a specific document. Both terms and documents are represented by vectors in a suitable (Euclidean) semantic space, and the scalar product between these vectors is a measure of the similarity of the corresponding documents and terms. This approach has extended to latent semantic analysis (LSA), hyperspace analogue to language (HAL), latent Dirichlet allocation (LDA), etc. Search engines on the World Wide Web, though introducing on top additional procedures, e.g., page ranking, mostly rely on this linear space technique to determine a basic set of relevant documents. Notwithstanding its success, this procedure meets several difficulties, including high computational costs and lack of incremental updates, which limits its applicability.

Inspired by our general quantum modeling approach to cognition illustrated in this paper, we have recently put forward the first steps of a possible conceptually new perspective for IR and NLP [32]. In this approach, we replace terms by 'entities of meaning' as primary notions, which can be concepts or combinations of concepts. Such entities of meaning can be in different states and change under the influence of the 'meaning landscape', or 'conceptual landscape', or 'conceptual context'. In addition, documents are not regarded as collection of words, but as traces, i.e. more concrete states, of these entities of meaning, or concepts, or combinations of concepts. This means that a document is considered to be a collapse of full states of different entities of meaning, each entity leaving a trace in the document. Words are only spots of these traces and they are not the main meaning carriers. The technical focus of our approach consists in trying to reconstruct the full states of the different entities of meaning from experiments that can only spot their traces, i.e. that can only look at words in documents.

We believe that aspects of our quantum cognition perspective will help in formulating and making technically operational this 'inverse problem', consisting in 'reconstructing the full states of the different entities of meaning, starting from their collapsed states as traces of word spots in documents'. We plan to develop these preliminary aspects in the next future, thus inquiring more deeply into this fascinating problem.

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