OPTIMUM HEATING AND OPTIMUM SHAPE PROBLEMS IN DISTRIBUTED PARAMETER CONTROL THEORY

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BY

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SUMMARY

In Part I, the problem of heating a thin plate or material travelling through a furnace, in which the system is described by first order linear partial differential equations, is introduced as an example of optimal control theory in distributed parameter systems. The variational technique in a fixed domain is used to obtain the necessary conditions for optimality. Many cases of the problem with the state equation described by first order linear partial differential equations are discussed, in which the control function enters into the state equation in different positions. The problems are analysed and solved by making use of characteristic curves.

In Part II, we have studied the variation of a functional defined on a variable domain, and we apply it to the problem of finding the optimum shape of the domain in which some performance criterion has an extremum. The problem in which the state equation is Laplace's equation defined on the variable domain of an annular shape with given boundary conditions is discussed and completely solved for the case when the inner boundary of the domain is only a small departure from a circle. We also introduce the method of logarithmic potential of a single layer to solve the boundary value problem of Laplace's equation with mixed boundary conditions and two simple examples are solved by using this method which leads to coupled integral equations.

PART
I

OPTIMUM HEATING PROBLEM

INTRODUCTION

Optimal control problems for systems with lumped parameters, whose model structure is described by ordinary differential equations, were the first problems to be investigated. The theory has been extensively developed and the problems are solved by many methods, namely, Pontryagin's maximum principle [1], Bellman's method of dynamic programming [2] or variational methods [3,4].

Quite a number of physical problems encountered in industry have systems governed by partial differential equations, integral equations, integro-differential equations or more generally functional equations. These systems are called distributed parameter systems. It is not possible to reduce all systems with distributed parameters into systems with lumped parameters, therefore it is necessary to study separately the systems with distributed parameters. We may say that optimal control theory of distributed parameter systems was first considered by Butkovskii and Lerner in 1960, [5]. The later papers by Butkovskii [6,7] developed a maximum principle for systems described by integral equations. It is analogous to the maximum principle of Pontryagin for lumped parameter systems, but expressed in the form of integral equations. Instead of changing the systems described by partial differential equations into integral equations and applying Butkovskii's maximum principle, the necessary optimality conditions can be obtained directly from the partial differential equations by using the methods of calculus of variations. These methods have been used by many authors, namely, Egorov [8,9], Kim and Gajwani [10], Sirazetdinov [11], and Degtyarev and Sirazetdinov [12].

In this thesis we discuss two problems:

(a). Optimum heating problem,

and (b). Optimum shape problem.

In each of these problems, a variational calculus approach has been used to derive the conditions for optimality.

The introduction for optimum shape problem will be mentioned separately on page 95 .

The motivation for the heating problem is the following:

Consider a "thin" plate being heated by moving it through a continuous furnace of length L with velocity $v > 0$, as in Fig. 1. The state equation which represents this process is expressed in the form

$$\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = k\left\{w - \phi\right\}, \qquad 0 \leq x \leq L \ ; \ 0 \leq t \leq T \ , \tag{1}$$

with the initial condition and the boundary condition at the entrance defined as follows:

$$\left.\begin{aligned} \phi(0,x) &= \phi_0(x), \qquad 0 \leq x \leq L \ , \\ \phi(t,0) &= \phi_1(t), \qquad 0 \leq t \leq T \ , \end{aligned}\right\} \tag{2}$$

where $\phi_0(x)$ and $\phi_1(t)$ are given functions and satisfy $\phi_0(0) = \phi_1(0)$,

$\phi(t,x)$ is the temperature of the heated plate,

w is the temperature of the furnace, and, $k = \frac{\alpha}{c\rho s}$ where

α is the coefficient of heat transfer from the furnace to the plate,

c is the heat capacity of the plate, ρ is specific gravity and

s is the thickness of the plate.

If v , k and w are given then the system of (1) and (2) is uniquely determined for $\phi(t,x)$, by using the known method of characteristics, in the region S as in Fig. 2.

Obviously, the temperature of the plate at the exit depends on the temperature variation in the furnace. It may also depend on the velocity of the plate through the furnace and also on the thickness δ of the plate.

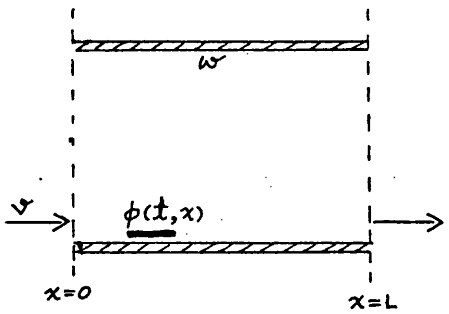


Fig. 1

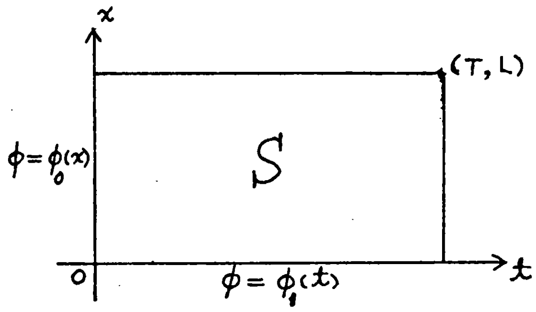


Fig. 2

Here, however, we consider a different class of problems.

In general we wish to determine a control function u, which can be the temperature of the furnace or the velocity of the plate, in order that the exit temperature of the plate should be as close as possible to the prescribed temperature $\phi^*(t)$ and at the same time the control function is minimised; i.e.,

find u which minimises a functional of the form

$$I = \int_{t=0}^{t=T} \frac{1}{2}\left\{\phi(t,L) - \phi^*(t)\right\}^2 dt + \iint_S \frac{1}{2} m^2 u^2 \, dt \, dx , \tag{3}$$

where m (say) is a constant.

We generalise this problem in Chapter 1 by writing the state equation (1) and the performance criterion (3) in the vector form as follows:

$$\frac{\partial \underset{\sim}{\phi}}{\partial t} = \underset{\sim}{g}(t, x, \underset{\sim}{\phi}, \frac{\partial \underset{\sim}{\phi}}{\partial x}, \underset{\sim}{u}) , \qquad (t,x) \in S \tag{4}$$

and

$$I = \iint_S F(t,x,\underset{\sim}{\phi}, \frac{\partial \underset{\sim}{\phi}}{\partial x}, \underset{\sim}{u}) \, dt \, dx + \oint_{\Gamma_2} \left[p(t,x,\underset{\sim}{\phi}) \, dx + q(t,x,\underset{\sim}{\phi}) \, dt \right] , \tag{5}$$

where $\underset{\sim}{\phi} \in E_n$, $\underset{\sim}{u} \in E_r$, S is a fixed domain bounded by a closed curve $C = \Gamma_1 \cup \Gamma_2$, and the given functions $\underset{\sim}{g}$, F, p and q are assumed to be continuous in t and x and twice continuously differentiable with respect to the remaining arguments. Furthermore it is assumed that (4) is a hyperbolic system.

The necessary conditions for optimality are formally derived for the following cases, which depend on the constraints imposed on the control $\underset{\sim}{u}$,

(i). $\underset{\sim}{u} = \underset{\sim}{u}(t,x)$ and $\underset{\sim}{u}$ is continuous;

(ii). $\underset{\sim}{u} = \underset{\sim}{u}(t)$ and $\underset{\sim}{u}$ is continuous. (6)

The Pontryagin Maximum Theorem is assumed in the piecewise continuous case when $\underset{\sim}{A}_1 \leq \underset{\sim}{u}(t,x) \leq \underset{\sim}{A}_2$, where $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are given constants.

In Chapter 2 we consider the optimal control problem when the state equation (4) is in the form

$$\frac{\partial \phi}{\partial t} = -a\frac{\partial \phi}{\partial x} - b\phi + u, \qquad 0 \leq x \leq L \; ; \; 0 \leq t \leq T, \tag{7}$$

with the same conditions as in (2) and the performance criterion I as in (3). Here a and b are constants and u is of the different types described in (6). The state equation (7) is similar to (1) in which the velocity of the plate $v \equiv a > 0$, $k \equiv b$ and $k\omega \equiv u$. Also in Chapter 2, in the case $u = u(t)$ with $u(t)$ continuous, we discuss the problem of heating a "thin" plate in a furnace which is divided into n parts.

In Chapter 3 we modify the state equation to the form

$$\frac{\partial \phi}{\partial t} = -a\frac{\partial \phi}{\partial x} + u\phi - c, \qquad 0 \leq x \leq L \; ; \; 0 \leq t \leq T, \tag{8}$$

and using the same conditions as in (2), the functional I as in (3).

In Chapter 4 the velocity of the plate acts as the control function and we modify the state equation to the form

$$\frac{\partial \phi}{\partial t} = -u(t)\frac{\partial \phi}{\partial x} - b\phi - c , \qquad 0 \le x \le L \ ; \ 0 \le t \le T , \tag{9}$$

where a, b and c are constants and the other conditions are unaltered.

CHAPTER 1

OPTIMUM CONTROL IN A GENERAL FIRST ORDER HEATING PROBLEM: $\underset{\sim}{\phi}_t = \underset{\sim}{g}(t, x, \underset{\sim}{\phi}, \underset{\sim}{\phi}_x, \underset{\sim}{u})$.

We shall discuss here the derivation of the conditions of optimality for the controlled system described by a set of n partial differential equations in the form

$$\frac{\partial \underset{\sim}{\phi}}{\partial t} = \underset{\sim}{g}\left(t, x, \underset{\sim}{\phi}, \frac{\partial \underset{\sim}{\phi}}{\partial x}, \underset{\sim}{u}\right), \qquad (t,x) \in S, \tag{1.1}$$

where $\underset{\sim}{\phi}(t,x) \in E_n$ is an n-vector function of variables t and x which characterises a state of the system, $\underset{\sim}{u} \in E_r$ is an r-vector characterises the domain control and $\underset{\sim}{g} \in E_n$ is a given vector function of the variables t, x, $\underset{\sim}{\phi}$, $\frac{\partial \underset{\sim}{\phi}}{\partial x}$ and $\underset{\sim}{u}$. Unless otherwise stated the functions $\underset{\sim}{\phi}(t,x)$, $\underset{\sim}{u}(t,x)$ and their partial derivatives up to the second order will be assumed to be continuous.

Here S is a simple and fixed domain in (t,x) plane bounded by a closed curve C where $C = \Gamma_1 \cup \Gamma_2$. We shall assume that C is divided into two parts Γ_1 and Γ_2 at the points A and B in such a way that, for increasing t, each family of characteristics associated with the set of n partial differential equations (1.1) enter the domain S along the arc Γ_1 and leave S along the arc Γ_2, and all characteristics passing through A and B must not cut through the domain S. Therefore, it is clear that for case $n=1$ or when a set of n partial differential equations (1.1) have the same family of characteristics, the domain S is arbitrary but if (1.1) have many sets of families of characteristics the domain S must have corners at A and B, as shown in Fig. 3 and Fig. 4 respectively.

We shall first consider the case $\underset{\sim}{u} = \underset{\sim}{u}(t,x)$, a vector function of variables t and x, and $\underset{\sim}{u}(t,x)$ is continuous. The cases of $\underset{\sim}{u} = \underset{\sim}{u}(t)$, a continuous function of t only, and of a piecewise continuous control function will be discussed later.

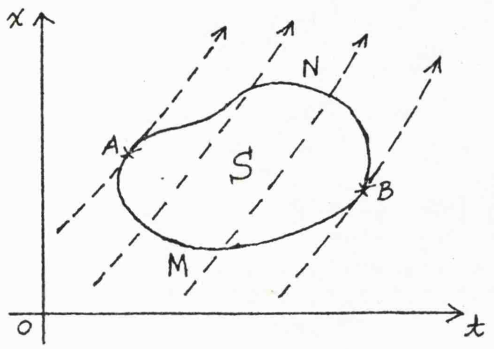


Fig. 3: $C = \Gamma_1 \cup \Gamma_2$ where Γ_1 is the arc AMB and Γ_2 is the arc BNA. The characteristics are the dashed curves.

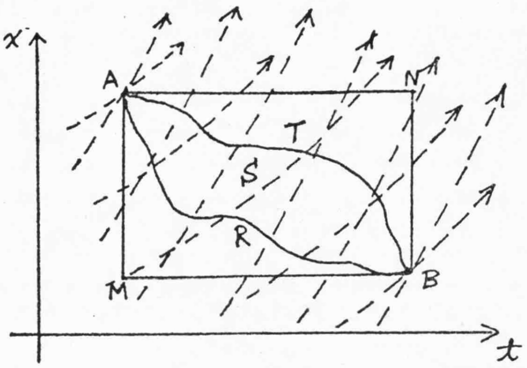


Fig. 4: The characteristics are the dashed curves. Γ_1 can be AMB or any curve ARB inside $AMBNA$ and Γ_2 can be any curve BTA inside $AMBNA$ or BNA.

We shall assume that $\underset{\sim}{\phi}(t,x)$ is known on the arc Γ_1 and defined as follows:

$$\underset{\sim}{\phi}(t,x) = \underset{\sim}{h}(t,x), \qquad \text{on } \Gamma_1, \tag{1.2}$$

hence, if the control $\underset{\sim}{u}$ is given then the solution $\underset{\sim}{\phi}(t,x)$ of (1.1) is uniquely determined.

The problem of optimum control can now be stated as follows:

Find the control $\underset{\sim}{u}(t,x)$ which minimises the functional of the form

$$I = \iint_S F(t,x,\underset{\sim}{\phi},\frac{\partial \underset{\sim}{\phi}}{\partial x},\underset{\sim}{u})\,dt\,dx + \int_{\Gamma_2} \left[\underset{\sim}{p}(t,x,\underset{\sim}{\phi})\,dx + \underset{\sim}{q}(t,x,\underset{\sim}{\phi})\,dt \right], \tag{1.3}$$

where $\underset{\sim}{\phi}(t,x)$ satisfies (1.1) and (1.2), the functions F, p and q are given and we assume that they are continuous with respect to t and x and twice continuously differentiable with respect to the remaining variables. The direction of Γ_2 is in a positive sense.

To find the optimality conditions, we introduce a domain Lagrange multiplier vector $\underset{\sim}{\lambda}(t,x) \in E_n$ and denote $\underset{\sim}{\lambda}^T(t,x)$ as its transpose. Consider the modified performance criterion J defined in the form

$$J = I + \iint_S \underset{\sim}{\lambda}^T(t,x)\left\{ \underset{\sim}{g}(t,x,\underset{\sim}{\phi},\frac{\partial \underset{\sim}{\phi}}{\partial x},\underset{\sim}{u}) - \frac{\partial \underset{\sim}{\phi}}{\partial t}\right\} dt\,dx. \tag{1.4}$$

Define the Hamiltonian H as follows:

$$H \equiv H(t,x,\underset{\sim}{\phi},\frac{\partial \underset{\sim}{\phi}}{\partial x},\underset{\sim}{u},\underset{\sim}{\lambda}) = F(t,x,\underset{\sim}{\phi},\frac{\partial \underset{\sim}{\phi}}{\partial x},\underset{\sim}{u}) + \underset{\sim}{\lambda}^T(t,x)\,\underset{\sim}{g}(t,x,\underset{\sim}{\phi},\frac{\partial \underset{\sim}{\phi}}{\partial x},\underset{\sim}{u}). \tag{1.5}$$

Thus we can write (1.4) in the form

$$J = \iint_S \left(H - \underset{\sim}{\lambda}^T \frac{\partial \underset{\sim}{\phi}}{\partial t}\right) dt\,dx + \oint_{\Gamma_2}\left[p(t,x,\underset{\sim}{\phi})\,dx + q(t,x,\underset{\sim}{\phi})\,dt\right]. \tag{1.6}$$

Let $\underset{\sim}{u}(t,x)$ be the optimum control vector which provides I the minimum value and $\underset{\sim}{\phi}(t,x)$ be the corresponding optimum state vector. Let $\underset{\sim}{u}(t,x) + \varepsilon\,\underset{\sim}{\eta}(t,x)$ and $\underset{\sim}{\phi}(t,x) + \varepsilon\,\underset{\sim}{\xi}(t,x)$ be the modified control and modified state vector respectively, ε being a small numerical parameter, $\underset{\sim}{\eta}(t,x)$ and $\underset{\sim}{\xi}(t,x)$ are continuous vector functions. Similarly, $\underset{\sim}{\lambda}(t,x)$ is the optimum value of Lagrange multiplier corresponding to $\underset{\sim}{u}(t,x)$ and $\underset{\sim}{\phi}(t,x)$. The modified value is $\underset{\sim}{\lambda}(t,x) + \varepsilon\,\underset{\sim}{\zeta}(t,x)$ where $\underset{\sim}{\zeta}(t,x)$ is also a continuous vector function.

The value of J in (1.6) corresponding to the modified state, control and Lagrange multiplier variables will be as follows:

$$\begin{aligned} J(\varepsilon) \equiv J + \Delta J = \iint_S \Big[& H\Big(t,x,\underset{\sim}{\phi}+\varepsilon\underset{\sim}{\xi},\frac{\partial \underset{\sim}{\phi}}{\partial x}+\varepsilon\frac{\partial \underset{\sim}{\xi}}{\partial x},\underset{\sim}{u}+\varepsilon\underset{\sim}{\eta},\underset{\sim}{\lambda}+\varepsilon\underset{\sim}{\zeta}\Big) - \\ & - (\underset{\sim}{\lambda}^T + \varepsilon\underset{\sim}{\zeta}^T)\Big(\frac{\partial \underset{\sim}{\phi}}{\partial t} + \varepsilon\frac{\partial \underset{\sim}{\xi}}{\partial t}\Big)\Big]\,dt\,dx\; + \\ & + \oint_{\Gamma_2}\left[p(t,x,\underset{\sim}{\phi}+\varepsilon\underset{\sim}{\xi})\,dx + q(t,x,\underset{\sim}{\phi}+\varepsilon\underset{\sim}{\xi})\,dt\right]. \end{aligned} \tag{1.7}$$

It follows by using Taylor's theorem and retaining only the first degree of ε that

$$\delta J = \iint_S \left[\varepsilon \left\{ \underset{\sim}{\xi}^T \frac{\partial H}{\partial \underset{\sim}{\phi}} + \left(\frac{\partial \underset{\sim}{\xi}}{\partial x}\right)^T \frac{\partial H}{\partial \underset{\sim}{\phi}_x} + \underset{\sim}{\eta}^T \frac{\partial H}{\partial \underset{\sim}{u}} + \underset{\sim}{\zeta}^T \frac{\partial H}{\partial \underset{\sim}{\lambda}} - \underset{\sim}{\zeta}^T \frac{\partial \underset{\sim}{\phi}}{\partial t} - \underset{\sim}{\lambda}^T \frac{\partial \underset{\sim}{\xi}}{\partial t} \right\} \right] dt\,dx$$
$$+ \oint_{\Gamma_2} \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \frac{\partial p}{\partial \underset{\sim}{\phi}} dx + \frac{\partial q}{\partial \underset{\sim}{\phi}} dt \right\} \right], \tag{1.8}$$

where we use the notations δJ for the principal linear part (in ε) of ΔJ and,

$$\frac{\partial H}{\partial \underset{\sim}{\phi}} \equiv \begin{bmatrix} \frac{\partial H}{\partial \phi_1} \\ \vdots \\ \frac{\partial H}{\partial \phi_n} \end{bmatrix}, \qquad \frac{\partial H}{\partial \underset{\sim}{\phi}_x} \equiv \begin{bmatrix} \frac{\partial H}{\partial\left(\frac{\partial \phi_1}{\partial x}\right)} \\ \vdots \\ \frac{\partial H}{\partial\left(\frac{\partial \phi_n}{\partial x}\right)} \end{bmatrix}, \qquad \frac{\partial H}{\partial \underset{\sim}{u}} \equiv \begin{bmatrix} \frac{\partial H}{\partial u_1} \\ \vdots \\ \frac{\partial H}{\partial u_r} \end{bmatrix},$$

and similarly for a vector $\frac{\partial H}{\partial \underset{\sim}{\lambda}}$.

Performing an integration by parts with respect to "x" on the term $\varepsilon\left(\frac{\partial \underset{\sim}{\xi}}{\partial x}\right)^T \frac{\partial H}{\partial \underset{\sim}{\phi}_x}$ and with respect to a variable "t" on the term $\underset{\sim}{\lambda}^T \frac{\partial \underset{\sim}{\xi}}{\partial t}$ in (1.8), we then have

$$\delta J = \iint_S \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \frac{\partial H}{\partial \underset{\sim}{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \underset{\sim}{\phi}_x}\right) + \frac{\partial \underset{\sim}{\lambda}}{\partial t} \right\} + \varepsilon \underset{\sim}{\eta}^T \frac{\partial H}{\partial \underset{\sim}{u}} + \varepsilon \underset{\sim}{\zeta}^T \left(\frac{\partial H}{\partial \underset{\sim}{\lambda}} - \frac{\partial \underset{\sim}{\phi}}{\partial t} \right) \right] dt\,dx +$$
$$+ \iint_S \varepsilon \left[\frac{\partial}{\partial x}\left(\underset{\sim}{\xi}^T \frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) - \frac{\partial}{\partial t}\left(\underset{\sim}{\xi}^T \underset{\sim}{\lambda} \right) \right] dt\,dx + \oint_{\Gamma_2} \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \frac{\partial p}{\partial \underset{\sim}{\phi}} dx + \frac{\partial q}{\partial \underset{\sim}{\phi}} dt \right\} \right].$$

Using Green's theorem in two dimensions in the form

$$\iint_S \left(\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x} \right) dt\,dx = \oint_C \left(P\,dt + Q\,dx \right),$$

we have

$$\iint_S \varepsilon \left[\frac{\partial}{\partial x}\left(\underset{\sim}{\xi}^T \frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) - \frac{\partial}{\partial t}\left(\underset{\sim}{\xi}^T \underset{\sim}{\lambda} \right) \right] dt\,dx = -\oint_C \left[\varepsilon \underset{\sim}{\xi} \left(\underset{\sim}{\lambda}\,dx + \frac{\partial H}{\partial \underset{\sim}{\phi}_x} dt \right) \right],$$

where $C = \Gamma_1 \cup \Gamma_2$.

Therefore, we can write δJ in the form

$$\delta J = \iint_S \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \frac{\partial H}{\partial \underset{\sim}{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) + \frac{\partial \underset{\sim}{\lambda}}{\partial t} \right\} + \varepsilon \underset{\sim}{\eta}^T \frac{\partial H}{\partial \underset{\sim}{u}} + \varepsilon \underset{\sim}{\zeta}^T \left(\frac{\partial H}{\partial \underset{\sim}{\lambda}} - \frac{\partial \underset{\sim}{\phi}}{\partial t} \right) \right] dt\, dx -$$

$$- \oint_{\Gamma_1} \left[\varepsilon \underset{\sim}{\xi}^T \left(\underset{\sim}{\lambda}\, dx + \frac{\partial H}{\partial \underset{\sim}{\phi}_x} dt \right) \right] + \oint_{\Gamma_2} \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \left(\frac{\partial p}{\partial \underset{\sim}{\phi}} - \underset{\sim}{\lambda} \right) dx + \left(\frac{\partial q}{\partial \underset{\sim}{\phi}} - \frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) dt \right\} \right].$$

Since in (1.2) we assume that

$$\underset{\sim}{\phi}(t,x) = \underset{\sim}{h}(t,x) \qquad \text{on } \Gamma_1 ,$$

$$\therefore \quad \underset{\sim}{\xi}(t,x) = 0 \qquad \text{on } \Gamma_1 ,$$

hence

$$\delta J = \iint_S \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \frac{\partial H}{\partial \underset{\sim}{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) + \frac{\partial \underset{\sim}{\lambda}}{\partial t} \right\} + \varepsilon \underset{\sim}{\eta}^T \frac{\partial H}{\partial \underset{\sim}{u}} + \varepsilon \underset{\sim}{\zeta}^T \left(\frac{\partial H}{\partial \underset{\sim}{\lambda}} - \frac{\partial \underset{\sim}{\phi}}{\partial t} \right) \right] dt\, dx +$$

$$+ \oint_{\Gamma_2} \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \left(\frac{\partial p}{\partial \underset{\sim}{\phi}} - \underset{\sim}{\lambda} \right) dx + \left(\frac{\partial q}{\partial \underset{\sim}{\phi}} - \frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) dt \right\} \right]. \tag{1.9}$$

To establish the necessary conditions for a minimum value of I ~~subjects~~ subject to the constraints (1.1), we set the first variation, δJ equal to zero, i.e.,

$$\iint_S \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \frac{\partial H}{\partial \underset{\sim}{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) + \frac{\partial \underset{\sim}{\lambda}}{\partial t} \right\} + \varepsilon \underset{\sim}{\eta}^T \frac{\partial H}{\partial \underset{\sim}{u}} + \varepsilon \underset{\sim}{\zeta}^T \left(\frac{\partial H}{\partial \underset{\sim}{\lambda}} - \frac{\partial \underset{\sim}{\phi}}{\partial t} \right) \right] dt\, dx +$$

$$+ \oint_{\Gamma_2} \left[\varepsilon \underset{\sim}{\xi}^T \left\{ \left(\frac{\partial p}{\partial \underset{\sim}{\phi}} - \underset{\sim}{\lambda} \right) dx + \left(\frac{\partial q}{\partial \underset{\sim}{\phi}} - \frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) dt \right\} \right] = 0 . \tag{1.10}$$

Using the standard arguments of variational calculus, the following conditions must then be satisfied:

$$\frac{\partial H}{\partial \underset{\sim}{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \underset{\sim}{\phi}_x} \right) + \frac{\partial \underset{\sim}{\lambda}}{\partial t} = 0 , \qquad (t,x) \in S , \tag{1.11}$$

$$\frac{\partial H}{\partial \underset{\sim}{\lambda}} - \frac{\partial \underset{\sim}{\phi}}{\partial t} = 0 , \qquad (t,x) \in S , \tag{1.12}$$

$$\frac{\partial H}{\partial \underset{\sim}{u}} = 0 , \qquad (t,x) \in S , \qquad (1.13)$$

and

$$\left(\frac{\partial P}{\partial \underset{\sim}{\phi}} - \underset{\sim}{\lambda}\right) dx + \left(\frac{\partial g}{\partial \underset{\sim}{\phi}} - \frac{\partial H}{\partial \underset{\sim}{\phi}_x}\right) dt = 0 , \qquad \text{on } \Gamma_2 , \qquad (1.14)$$

where H is defined in (1.5).

We note that the equation (1.12) is the state equation (1.1). The conditions (1.11) to (1.14) are the necessary conditions for the functional I to have a minimum value. We have $(2n+r)$ equations in (1.11)-(1.13) to solve for $(2n+r)$ unknown $\underset{\sim}{\phi}(t,x)$, $\underset{\sim}{u}(t,x)$ and $\underset{\sim}{\lambda}(t,x)$. The boundary conditions on Γ_1 and Γ_2 are defined in (1.2) and (1.14) respectively. The condition (1.14) is known as the natural boundary condition.

Let us discuss two special cases as follows:

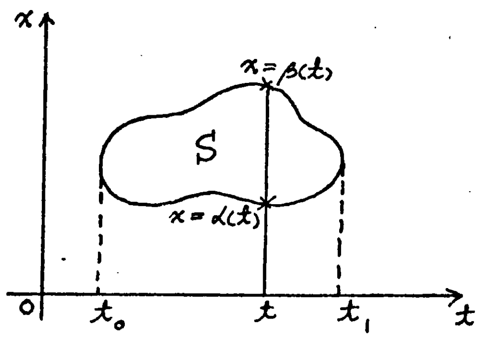
<u>Special case 1.</u> $\underset{\sim}{u} = \underset{\sim}{u}(t)$, control is a continuous function of t only.

The modified control will be $\underset{\sim}{u}(t) + \varepsilon \underset{\sim}{\eta}(t)$ where $\underset{\sim}{\eta}$ is also a function of t only. Thus (1.10) can be written in the form

$$\iint_S \left[\varepsilon \underset{\sim}{\xi}^T \left\{\frac{\partial H}{\partial \underset{\sim}{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \underset{\sim}{\phi}_x}\right) + \frac{\partial \underset{\sim}{\lambda}}{\partial t}\right\} + \varepsilon \underset{\sim}{\zeta}^T \left(\frac{\partial H}{\partial \underset{\sim}{\lambda}} - \frac{\partial \underset{\sim}{\phi}}{\partial t}\right)\right] dt\, dx + \int_{t_0}^{t_1} \varepsilon \underset{\sim}{\eta}^T(t) \left(\int_{x=\alpha(t)}^{x=\beta(t)} \frac{\partial H}{\partial \underset{\sim}{u}}\, dx\right) dt +$$

$$+ \oint \left[\varepsilon \underset{\sim}{\xi}^T \left\{\left(\frac{\partial P}{\partial \underset{\sim}{\phi}} - \underset{\sim}{\lambda}\right) dx + \left(\frac{\partial g}{\partial \underset{\sim}{\phi}} - \frac{\partial H}{\partial \underset{\sim}{\phi}_x}\right) dt\right\}\right] = 0 ,$$

where t_0 , t_1 , $\alpha(t)$ and $\beta(t)$ are shown as in Fig. 5, and then the condition (1.13) will be replaced by the condition



<u>Fig.5</u>

$$\int_{x=\alpha(t)}^{x=\beta(t)} \frac{\partial H}{\partial \underset{\sim}{u}}\, dx = 0 , \qquad (1.15)$$

Special case 2. The control is a piecewise continuous control.

Suppose that the control is bounded and satisfies $\underset{\sim}{A}_1 \le \underset{\sim}{u}(t,x) \le \underset{\sim}{A}_2$. The condition (1.13) or (1.15) is no longer applied. We shall state without proof that the optimum control must satisfy the maximum principle as follows:

For I to have a minimum (maximum) value, the control $\underset{\sim}{u}$ must be chosen to minimise (maximise) the Hamiltonian H, where

$$H = F + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{g} \quad .$$

These statements can be found in the books of Butkovskiy [13] or Sage [4] or in the paper of Sirazetdinov [11].

Let us consider when $n=1$, $r=1$, i.e., $\phi(t,x)$ and u are the optimum state and optimum control respectively and $\lambda(t,x)$ is the optimum Lagrange multiplier. The cost function will be

$$I = \iint_S F(t,x,\phi,\phi_x,u)\,dt\,dx + \oint_{\Gamma_2}\left[p(t,x,\phi)\,dx + q(t,x,\phi)\,dt\right] . \tag{1.16}$$

The conditions (1.11) - (1.15) are rewritten as follows:

$$\frac{\partial\lambda}{\partial t} + \frac{\partial H}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial\phi_x}\right) = 0 , \qquad (t,x)\in S , \tag{1.17}$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial H}{\partial\lambda} , \qquad (t,x)\in S . \tag{1.18}$$

When $u = u(t,x)$ is a continuous control, the optimality condition is

$$\frac{\partial H}{\partial u} = 0 , \qquad (t,x)\in S . \tag{1.19}$$

When $u = u(t)$, a continuous function of t only, the optimality condition is

$$\int_{x=\alpha(t)}^{x=\beta(t)} \frac{\partial H}{\partial u}\,dx = 0 , \qquad (t,x)\in S . \tag{1.20}$$

When $A_1 \leq u(t,x) \leq A_2$, the control $u = u(t,x)$ is chosen so that to minimise the Hamiltonian H, where

$$H = F + \lambda(t,x) g ,$$

and A_1, A_2 are given constants. (1.21)

The boundary conditions are

$$\phi(t,x) = h(t,x) , \qquad \text{on } \Gamma_1 , \tag{1.22}$$

and $$\left(\frac{\partial p}{\partial \phi} - \lambda\right) dx + \left(\frac{\partial q}{\partial \phi} - \frac{\partial H}{\partial \phi_x}\right) dt = 0 , \qquad \text{on } \Gamma_2 , \tag{1.23}$$

where H is the Hamiltonian defined as

$$H = F(t,x,\phi,\phi_x,u) + \lambda(t,x) g(t,x,\phi,\phi_x,u) , \tag{1.24}$$

and $h(t,x)$, g, p, q and F are known functions.

In the next three chapters we shall discuss the problems associated with linear first order partial differential equation in the form

$$\frac{\partial \phi}{\partial t} = g(t,x,\phi,\phi_x,u) \equiv A(t,x,u)\frac{\partial \phi}{\partial x} + B(t,x,u)\phi + C(t,x,u) , \tag{1.25}$$

where $(t,x) \in S$ and S is the rectangular region $0 \leq t \leq T$, $0 \leq x \leq L$ and the functional to be minimised is defined as follows:

$$I = \int_0^T \frac{1}{2}\left\{\phi(t,L) - \phi^*(t)\right\}^2 dt + \iint_S \frac{1}{2} m^2 u^2 \, dt\, dx , \tag{1.26}$$

in other words, we find the control u in order that the function $\phi(t,L)$ is as close as possible to some prescribed function $\phi^*(t)$ and at the same time the control u is minimised, where $\phi(t,x)$ satisfies (1.25) with the given conditions on $t=0$ and on $x=0$.

The problems that we shall discuss are divided into 3 cases which depend on the position of the control u in the state equation.

Case 1. $\frac{\partial\phi}{\partial t} = g(t, x, \phi, \phi_x, u) \equiv -a\phi_x - b\phi + u$,

Case 2. $\frac{\partial\phi}{\partial t} = g(t, x, \phi, \phi_x, u) \equiv -a\phi_x + u\phi - c$,

Case 3. $\frac{\partial\phi}{\partial t} = g(t, x, \phi, \phi_x, u) \equiv -u\phi_x - b\phi - c$,

where a , b and c are constants, u is a control and ϕ is a state.

CHAPTER 2

OPTIMUM CONTROL IN A LINEAR FIRST ORDER HEATING PROBLEM. CASE 1: $q(t,x,\phi,\phi_x,u) \equiv -a\phi_x - b\phi + u$.

We now discuss a linear problem already posed in the introduction which we can restate as follows:

Find a control u which minimises a performance criterion

$$I = \int_{t=0}^{t=T} \frac{1}{2}\left\{\phi(t,L) - \phi^*(t)\right\}^2 dt + \iint_S \frac{1}{2} m^2 u^2 \, dt\, dx \quad , \tag{2.1}$$

where S is a rectangular region $0 \le t \le T$; $0 \le x \le L$, m is a constant and $\phi^*(t)$ is a prescribed function. The function $\phi(t,x)$ must satisfy the linear partial differential equation in the form

$$\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} + b\phi = u \, , \qquad 0 \le t \le T \, ; \; 0 \le x \le L \, , \tag{2.2}$$

with the initial and boundary conditions defined as follows:

$$\phi(0,x) = \phi_0(x) \, , \qquad 0 \le x \le L \, , \tag{2.3}$$

and

$$\phi(t,0) = \phi_1(t) \, , \qquad 0 \le t \le T \, , \tag{2.4}$$

where $a\,(>0)$ and b are constants, $\phi_0(x)$ and $\phi_1(t)$ are given functions satisfying $\phi_0(0) = \phi_1(0)$.

Three special cases are discussed in this chapter, depending on the conditions which are imposed on the control function u.

Special case 1: $u = u(t,x)$, a continuous control.

The Hamiltonian H is defined in (1.24) where in this problem $F \equiv \frac{1}{2}m^2u^2$ and $q \equiv -a\phi_x - b\phi + u$, hence

$$H = \frac{1}{2}m^2u^2 + \lambda(u - a\phi_x - b\phi) \, . \tag{2.5}$$

The necessary conditions for I to have a minimum value are given in (1.17) - (1.23).

The equations (1.17) and (1.18) give us the Lagrange equation and the state equation respectively, as follows:

$$\frac{\partial\lambda}{\partial t} + a\frac{\partial\lambda}{\partial x} = b\lambda\,, \qquad 0 \le x \le L\,;\ 0 \le t \le T\,, \tag{2.6}$$

and
$$\frac{\partial\phi}{\partial t} + a\frac{\partial\phi}{\partial x} = -b\phi + u\,, \qquad 0 \le x \le L\,;\ 0 \le t \le T. \tag{2.7}$$

Since $u = u(t,x)$ is continuous, the equation (1.19) is applied here i.e., $\dfrac{\partial H}{\partial u} = 0\,, \quad 0 \le t \le T\,;\ 0 \le x \le L\,,$

hence
$$m^2 u + \lambda = 0\,, \qquad 0 \le t \le T\,;\ 0 \le x \le L. \tag{2.8}$$

The boundary conditions on $x = L$, $0 \le t \le T$ and on $t = T$, $0 \le x \le L$ are derived from (1.23), i.e.,

$$\left(\frac{\partial p}{\partial \phi} - \lambda\right)dx + \left(\frac{\partial q}{\partial \phi} - \frac{\partial H}{\partial \phi_x}\right)dt = 0\,, \qquad \text{on } \Gamma_2\,,$$

where (t,x) is on the boundaries $x = L$, $0 \le t \le T$ and $t = T$, $0 \le x \le L$. The direction of Γ_2 is in a positive sense as shown in Fig. 6.

In this problem $p \equiv 0$

$$q \equiv -\tfrac{1}{2}\{\phi(t,L) - \phi^*(t)\}^2.$$

On $t = T$, $0 \le x \le L$; $dt = 0$, $dx \ne 0$, hence we have

$$\left(\frac{\partial p}{\partial \phi} - \lambda\right) = 0\,,$$

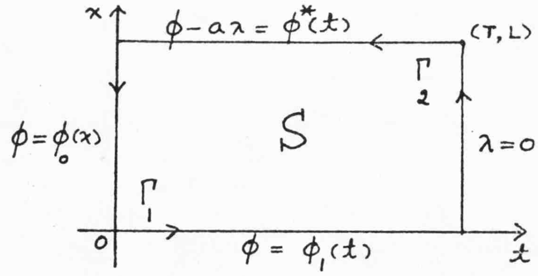


Fig. 6

i.e.,
$$\lambda(T,x) = 0\,, \quad t = T\,,\ 0 \le x \le L. \tag{2.9}$$

On $x = L$, $0 \le t \le T$; $dx = 0$, $dt \ne 0$, hence we have

$$\left(\frac{\partial q}{\partial \phi} - \frac{\partial H}{\partial \phi_x}\right) = 0\,,$$

i.e.,
$$\phi(t,L) - a\lambda(t,L) = \phi^*(t)\,, \qquad x = L\,,\ 0 \le t \le T. \tag{2.10}$$

The characteristics of the linear first order equation (2.6) and (2.7) are the same and given by the integral curves of the differential equation

$$\frac{dt}{1} = \frac{dx}{a}$$

i.e., $x = at + \text{constant}.$

To solve the problem, we introduce two new independent or characteristic variables ξ , η defined as follows:

$$\xi = t \ , \quad \eta = x - at \tag{2.11}$$

It is easy to verify that when we regard λ , u and ϕ as functions of ξ and η , we can write (2.6) and (2.7) in the form

$$\frac{\partial \lambda}{\partial \xi} = b\lambda \ , \tag{2.12}$$

and
$$\frac{\partial \phi}{\partial \xi} = -b\phi + u \ , \tag{2.13}$$

hence
$$\lambda(\xi,\eta) = A(\eta) e^{b\xi} \ , \tag{2.14}$$

$$\phi(\xi,\eta) = B(\eta) e^{-b\xi} + e^{-b\xi} \int u e^{b\xi} d\xi \ , \tag{2.15}$$

where $A(\eta)$ and $B(\eta)$ are arbitrary continuous functions of η .

In this case the optimality condition (2.8) and the equation (2.14) give us the optimum control as

$$u(\xi,\eta) = -\frac{1}{m^2} A(\eta) e^{b\xi} \ ,$$

and then (2.15) becomes

$$\phi(\xi,\eta) = B(\eta) e^{-b\xi} - \frac{A(\eta)}{2bm^2} e^{b\xi} \ , \qquad b \neq 0.$$

When we revert to the original independent variables, using (2.11) we have

$$u(t,x) = -\frac{1}{m^2} A(x-at) e^{bt} \ , \tag{2.16}$$

$$\phi(t,x) = -\frac{A(x-at)}{2bm^2} e^{bt} + B(x-at) e^{-bt} \ , \quad b \neq 0 \ , \tag{2.17}$$

where the arbitrary functions can be found by using the boundary conditions (2.3), (2.4), (2.9) and (2.10).

The solution of the problem depends on the magnitudes of constants a , T and L .

Case (i). $aT < L$.

In this case the characteristics $x = at$ and $x = L + a(t-T)$ will divide the domain S : $0 \le t \le T$; $0 \le x \le L$ into 3 subdomains S_1 , S_2 and S_3 as shown in the diagram (Fig. 7).

In subdomain S_1 : $0 \le x \le at$; $0 \le t \le T$, $u(t,x)$ and $\phi(t,x)$ in (2.16) and (2.17) must satisfy the boundary conditions $\phi = \phi_1(t)$ on $x = 0$, $0 \le t \le T$ and $\lambda = 0$ on $t = T$, $0 \le x \le L$, but since $m^2 u + \lambda = 0$ then $u = 0$ on $t = T$ for all x in $0 \le x \le L$.

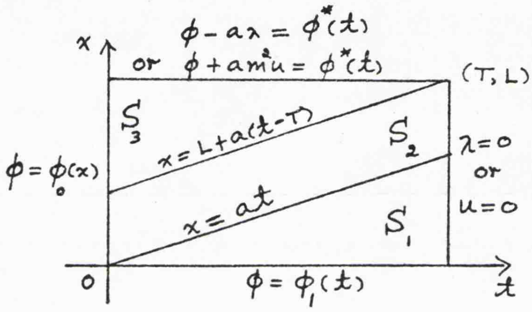


Fig. 7

Using the condition $u = 0$ on $t = T$, $0 \le x \le aT$, we have

$$A(x - aT) = 0 \quad \text{, i.e.,} \quad A(\xi) = 0 \text{, for all } \xi \text{ ,}$$

hence the control $u(t,x)$ in this subdomain is zero, i.e.,

$$u(t,x) = 0 \text{ , in } S_1 : 0 \le x \le at \text{ ; } 0 \le t \le T. \tag{2.18}$$

The equation (2.17) becomes

$$\phi(t,x) = B(x - at)e^{-bt},$$

and by using the condition $\phi = \phi_1(t)$ on $x = 0$, $0 \le t \le T$, we can find that

$$B(\eta) = \phi_1\left(-\frac{\eta}{a}\right)e^{-b\eta/a},$$

hence
$$\phi(t,x) = \phi_1\left(t - \frac{x}{a}\right)e^{-bx/a}, \qquad 0 \le x \le at \text{ ; } 0 \le t \le T \tag{2.19}$$

In subdomain S_2 : $at \le x \le L + a(t-T)$; $0 \le t \le T$, the functions $u(t,x)$ and $\phi(t,x)$ in (2.16) and (2.17) must satisfy the boundary conditions $u=0$ on $t=T$, $aT \leq x \leq L$ and $\phi = \phi_0(x)$ on $t=0$, $0 \leq x \leq L-aT$. The first condition, as in subdomain S_1, gives us

$$u(t,x) = 0 \quad , \text{ in } S_2 : at \leq x \leq L+a(t-T) ; 0 \leq t \leq T. \tag{2.20}$$

The second condition gives us

$$B(x) = \phi_0(x) \qquad , \text{ for all } x ,$$

hence
$$\phi(t,x) = \phi_0(x-at)\, e^{-bt} , \qquad \text{in } S_2 \tag{2.21}$$

In subdomain $S_3 : L+a(t-T) \leq x \leq L ; 0 \leq t \leq T$,
the boundary condition $\phi = \phi_0(x)$ on $t=0$; $L-aT \leq x \leq L$ provides us, by using (2.17), that

$$\phi_0(x) = -\frac{A(x)}{2bm^2} + B(x) , \qquad \text{for all } x. \tag{2.22}$$

By using the condition on $x=L$, $0 \leq t \leq T$, i.e., $\phi + am^2 u = \phi^*(t)$, together with (2.16) and (2.17), we obtain the relation

$$-\left[\frac{1}{2bm^2} + a\right] A(L-at)\, e^{bt} + B(L-at)\, e^{-bt} = \phi^*(t) ,$$

i.e.,
$$-\left[\frac{1}{2bm^2} + a\right] A(\eta)\, e^{b\left(\frac{L-\eta}{a}\right)} + B(\eta)\, e^{-b\left(\frac{L-\eta}{a}\right)} = \phi^*\left(\frac{L-\eta}{a}\right) . \tag{2.23}$$

Solving for $A(\eta)$ and $B(\eta)$ from (2.22) and (2.23), we have

$$A(\eta) = \frac{m^2\left[\phi_0(\eta)\, e^{-b\left(\frac{L-\eta}{a}\right)} - \phi^*\left(\frac{L-\eta}{a}\right)\right]}{\frac{1}{b}\sinh\left\{b\left(\frac{L-\eta}{a}\right)\right\} + am^2 e^{b\left(\frac{L-\eta}{a}\right)}} ,$$

$$B(\eta) = \frac{\left[\frac{1}{2b} + am^2\right]\phi_0(\eta)\, e^{b\left(\frac{L-\eta}{a}\right)} - \frac{1}{2b}\phi^*\left(\frac{L-\eta}{a}\right)}{\frac{1}{b}\sinh\left\{b\left(\frac{L-\eta}{a}\right)\right\} + am^2 e^{b\left(\frac{L-\eta}{a}\right)}} .$$

Hence, in subdomain S_3, it follows from (2.16), (2.17) and the above definitions of $A(\eta)$ and $B(\eta)$ that

$$u(t,x) = \left[\frac{e^{bt}\phi^*\left(\frac{L-x+at}{a}\right) - e^{-b\left(\frac{L-x}{a}\right)}\phi_0(x-at)}{\frac{1}{b}\sinh\left\{\frac{b}{a}(L-x+at)\right\} + am^2 e^{b\left(\frac{L-x+at}{a}\right)}} \right], \tag{2.24}$$

and

$$\phi(t,x) = \left[\frac{\frac{1}{b}\phi^*\left(\frac{L-x+at}{a}\right)\sinh(bt) + \frac{1}{b}\phi_0(x-at)\sinh\left\{b\left(\frac{L-x}{a}\right)\right\} + am^2\phi_0(x-at)\,e^{b\left(\frac{L-x}{a}\right)}}{\frac{1}{b}\sinh\left\{\frac{b}{a}(L-x+at)\right\} + am^2 e^{b\left(\frac{L-x+at}{a}\right)}} \right] \tag{2.25}$$

Case (ii). $aT = L$.

The characteristic $x = at$ will divide the domain S into 2 subdomains S_1 and S_2 as in the diagram (Fig. 8).

The solutions in subdomain S_1 : $0 \le x \le at$; $0 \le t \le T$ are the same as in (2.18) and (2.19) and, in subdomain S_2 : $at \le x \le L$; $0 \le t \le T$ are also the same as in (2.24) and (2.25).

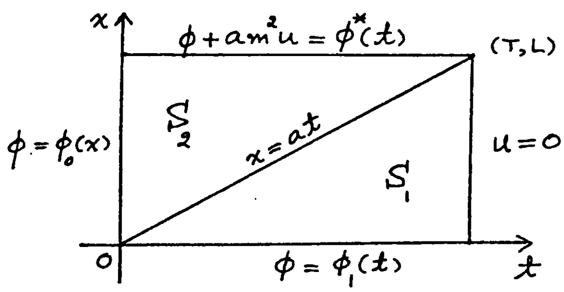


Fig. 8

Case (iii). $aT > L$.

As before, the characteristics $x = at$ and $x = L + a(t-T)$ will divide the domain S into 3 subdomains S_1 , S_2 and S_3 . Three diagrams are possible in this case, depending on the magnitude of aT . If $aT < 2L$ for which $t_N < t_M$, where $t_N = T - \frac{L}{a}$ and $t_M = \frac{L}{a}$, we then get the diagram as in Fig. 9 . When $aT = 2L$, where $t_M = t_N = \frac{L}{a}$, and when $aT > 2L$, where $t_M < t_N$, the diagrams are as shown in Fig. 10 and Fig. 11 respectively.

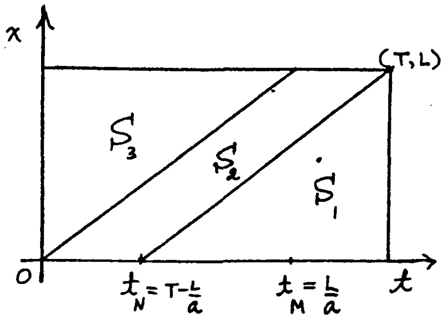


Fig. 9

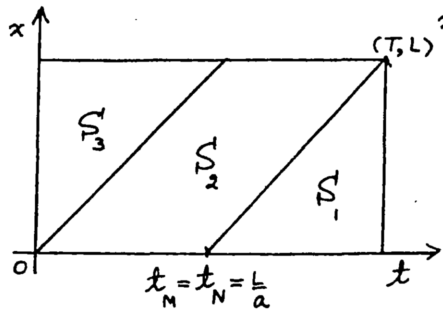


Fig. 10

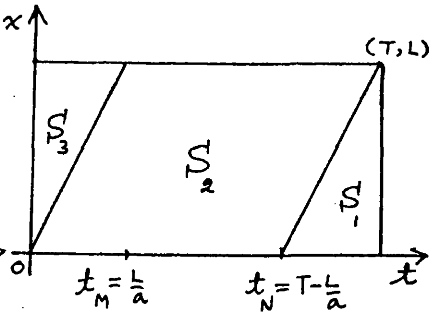


Fig. 11

The solution corresponding to each diagram, for this problem, is the same and can be found as follows:

In subdomain S_1 : $0 \le x \le L + a(t-T)$; $t_N \le t \le T$, and in subdomain S_3 : $at \le x \le L$; $0 \le t \le t_M$, the boundary condition for $\phi(t,x)$ and $u(t,x)$ are the same as in case (i) thus the solutions are the same, i.e., in subdomain S_1 we have $u(t,x)$ and $\phi(t,x)$ as in (2.18) and (2.19) and, in S_3 we shall have $u(t,x)$ and $\phi(t,x)$ as in (2.24) and (2.25) respectively.

For subdomain S_2 , $u(t,x)$ and $\phi(t,x)$ in (2.16) and (2.17) must satisfy the conditions $\phi = \phi_1(t)$ on $x=0$ and $\phi + am^2u = \phi^*(t)$ on $x=L$. We can easily find the arbitrary functions $A(\eta)$ and $B(\eta)$ by using these two boundary conditions and defined as follows:

$$A(\eta) = \frac{-m^2\left[\phi^*\left(\frac{L-\eta}{a}\right)e^{b\eta/a} - \phi_1\left(-\frac{\eta}{a}\right)e^{-b\left(\frac{L-\eta}{a}\right)}\right]}{\frac{1}{b}\sinh\left(\frac{bL}{a}\right) + am^2 e^{bL/a}} ,$$

$$B(\eta) = \frac{\left[\frac{1}{2b} + am^2\right]e^{b\left(\frac{L-\eta}{a}\right)}\phi_1\left(-\frac{\eta}{a}\right) - \frac{1}{2b}\phi^*\left(\frac{L-\eta}{a}\right)e^{-b\eta/a}}{\frac{1}{b}\sinh\left(\frac{bL}{a}\right) + am^2 e^{bL/a}} .$$

Hence, we obtain from (2.16) and (2.17) that

$$u(t,x) = \left[\frac{e^{bx/a}\phi^*\left(\frac{L-x+at}{a}\right) - e^{-b\left(\frac{L-x}{a}\right)}\phi_1\left(t-\frac{x}{a}\right)}{\frac{1}{b}\sinh\left(\frac{bL}{a}\right) + am^2 e^{bL/a}}\right] \qquad (2.26)$$

and

$$\phi(t,x) = \left[\frac{\frac{1}{b}\phi^*\left(\frac{L-x+at}{a}\right)\sinh\left(\frac{bx}{a}\right) + \frac{1}{b}\phi_1\left(t-\frac{x}{a}\right)\sinh\left\{b\left(\frac{L-x}{a}\right)\right\} + a m^2 \phi_1\left(t-\frac{x}{a}\right) e^{b\left(\frac{L-x}{a}\right)}}{\frac{1}{b}\sinh\left(\frac{bL}{a}\right) + a m^2 e^{bL/a}} \right]. \quad (2.27)$$

We have seen that the control function $u(t,x)$ can be evaluated explicitly at any point x of the furnace length L and at any time in the interval $0 \leq t \leq T$.

<u>Special case 2: $u = u(t)$, a continuous control function of time only.</u>

We assume in this case that $u(t)$ and $u'(t)$ are continuous functions of t .

Since the control function depends only on time t and is independent of x , the optimality condition will be derived from (1.20), namely

$$\int_{x=0}^{x=L} \frac{\partial H}{\partial u}\, dx = 0 \quad \text{, where } H \text{ is defined in (2.5).}$$

Hence, we obtain

$$m^2 L\, u(t) = -\int_{x=0}^{x=L} \lambda(t,x)\, dx\,, \quad (2.28)$$

where λ and ϕ are defined in (2.14) and (2.15), in terms of the characteristic variables ξ , η in (2.11), as follows:

$$\lambda(\xi,\eta) = A(\eta) e^{b\xi}$$

$$\phi(\xi,\eta) = B(\eta) e^{-b\xi} + e^{-b\xi} \int u e^{b\xi} d\xi\,.$$

Since $u(t)$ is assumed to be a continuous function it is always possible to express $u(t)$ in the form

$$u(t) = \alpha'(t) e^{-bt} \quad (2.29)$$

where $\alpha(t)$ and $\alpha'(t)$ are continuous functions. The prime $'$ means differentiating with respect to the argument inside the bracket. The inverse expression for $\alpha(t)$ in terms of $u(t)$ is

$$\alpha(t) = \int_0^t u(\sigma)\, e^{b\sigma}\, d\sigma \quad , \qquad \alpha(0) = 0 .$$

There is no loss of generality in taking $\alpha(0)=0$ since in finding the control it is the first derivative of $\alpha(t)$ which is important.

Thus we can write $\phi(\xi,\eta)$ in the form

$$\phi(\xi,\eta) = B(\eta)\, e^{-b\xi} + \alpha(\xi)\, e^{-b\xi} .$$

Reverting to the original independent variables, by using the relations in (2.11), we have

$$\lambda(t,x) = A(x-at)\, e^{bt} \quad , \tag{2.30}$$

and

$$\phi(t,x) = \alpha(t)\, e^{-bt} + B(x-at)\, e^{-bt} , \tag{2.31}$$

where A and B are arbitrary functions which can be found by using the boundary conditions.

As before, the magnitudes of the constants a , T and L are important, gives us the different control function $u(t)$. So we shall consider the following cases.

Case (i). $aT < L$.

The characteristics $x = at$ and $x = L + a(t-T)$ divide the domain S into 3 subdomains S_1 , S_2 and S_3 as shown in Fig.12 .

We note here that from now on we shall find only the control function $u(t)$ or the function $\alpha(t)$. The state function $\phi(t,x)$ will follow from (2.31)

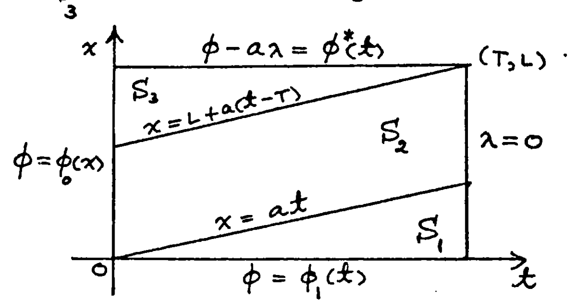


Fig. 12

together with the boundary conditions either $\phi = \phi_1(t)$ on $x=0$ or $\phi = \phi_0(x)$ on $t=0$.

In subdomain S_1 and S_2 , $\lambda(t,x)$ must satisfy the boundary condition $\lambda = 0$ on $t=T$. It follows from (2.30) that

$$A(x-aT) = 0 \quad \text{, i.e., } A(\xi) = 0 \text{ , for all } \xi \text{ ,}$$

hence

$$\lambda(t,x) = 0 \qquad \text{, in } S_1 \text{ and } S_2 . \tag{2.32}$$

In subdomain S_3 , the boundary conditions which have to be satisfied are $\phi = \phi_0(x)$ on $t=0$ and $\phi - a\lambda = \phi^*(t)$ on $x=L$. Since we assume $\alpha(0)=0$ and by using the condition on $t=0$, (2.31) gives us

$$B(x) = \phi_0(x) \qquad \text{, for all } x \text{ ,}$$

hence

$$\phi(t,x) = \alpha(t)e^{-bt} + \phi_0(x-at)e^{-bt} . \tag{2.33}$$

To satisfy the condition on $x=L$, i.e., $\phi - a\lambda = \phi^*(t)$, by using (2.30) and (2.33), we have

$$A(\eta) = \frac{e^{-2b\left(\frac{L-\eta}{a}\right)}}{a}\left[\alpha\left(\frac{L-\eta}{a}\right) + \phi_0(\eta)\right] - \frac{e^{-b\left(\frac{L-\eta}{a}\right)}}{a}\phi^*\left(\frac{L-\eta}{a}\right) , \tag{2.34}$$

thus (2.30) becomes

$$\lambda(t,x) = \frac{e^{-\frac{b}{a}(2L-2x+at)}}{a}\left[\alpha\left(\frac{L-x+at}{a}\right) + \phi_0(x-at)\right] - \frac{e^{-\frac{b}{a}(L-x)}}{a}\phi^*\left(\frac{L-x+at}{a}\right) . \tag{2.35}$$

The optimum control is defined in (2.28) and, by using (2.30), (2.32) and (2.34), we obtain

$$m^2 L u(t) = -\int_{x=L+at-aT}^{x=L} \left[\lambda(t,x)\right]_{S_3} dx$$

$$= - e^{bt} \int_{x=L+at-aT}^{x=L} \left[A(x-at) \right]_{S_3} dx , \qquad 0 \le t \le T .$$

Substituting $u(t) = \alpha'(t) e^{-bt}$, we have

$$m^2 L \alpha'(t) e^{-2bt} = - \int_{L-aT}^{L-at} \left[A(\eta) \right]_{S_3} d\eta . \tag{2.36}$$

Differentiating (2.36) with respect to variable t , we obtain

$$m^2 L e^{-2bt} \left[\alpha''(t) - 2b \alpha'(t) \right] = a \left[A(L-at) \right]_{S_3} ,$$

where $A(\eta)$ in S_3 is defined in (2.34).

Hence, the function $\alpha(t)$ satisfies the differential equation in the form

$$m^2 L \left[\alpha''(t) - 2b \alpha'(t) \right] = \alpha(t) + \phi_0(L-at) - e^{bt} \phi^*(t) , \tag{2.37}$$

with the condition $\alpha(0) = 0$ and it follows from (2.36) that another condition ~~that~~ is $\alpha'(T) = 0$. The solution of (2.37) for $\alpha(t)$ is unique.

The general solution of the linear equation of second order (2.37) can be written in the form

$$\alpha(t) = B e^{\gamma_1 t} + C e^{\gamma_2 t} + \frac{1}{F'(\gamma_1)} \int_0^t e^{-\gamma_1(\tau - t)} \Phi(\tau) d\tau + $$
$$+ \frac{1}{F'(\gamma_2)} \int_0^t e^{-\gamma_2(\tau - t)} \Phi(\tau) d\tau , \qquad 0 \le t \le T , \tag{2.38}$$

where B and C are arbitrary constants,

$$F(D) = D^2 - 2bD - \xi^2 , \quad D \equiv \frac{d}{dt} , \quad \xi^2 = \frac{1}{m^2 L} ,$$

$$\Phi(t) \equiv \frac{1}{m^2 L} \left[\phi_0(L-at) - e^{bt} \phi^*(t) \right] , \tag{2.39}$$

γ_1 and γ_2 are the roots of $m^2 - 2bm - \xi^2 = 0$ and we assume that $b^2 + \xi^2 \neq 0$, hence

$$\left.\begin{aligned} \gamma_1 &= b+\sqrt{b^2+\xi^2}\ , \quad \gamma_2 = b-\sqrt{b^2+\xi^2}\ , \\ F'(\gamma_1) &= 2\sqrt{b^2+\xi^2}\ , \quad F'(\gamma_2) = -2\sqrt{b^2+\xi^2}\ . \end{aligned}\right\} \tag{2.40}$$

After using the conditions $\alpha(0)=0$ and $\alpha'(T)=0$, the arbitrary constants B and c are evaluated, (2.38) becomes

$$\alpha(t) = \frac{\left(e^{\gamma_1 t}-e^{\gamma_2 t}\right)}{\left(\gamma_2 e^{\gamma_2 T}-\gamma_1 e^{\gamma_1 T}\right)}\left[\frac{\gamma_1}{F'(\gamma_1)}\int_0^T e^{-\gamma_1(\tau-T)}\Phi(\tau)\,d\tau + \frac{\gamma_2}{F'(\gamma_2)}\int_0^T e^{-\gamma_2(\tau-T)}\Phi(\tau)\,d\tau\right]+$$

$$+\frac{1}{F'(\gamma_1)}\int_0^t e^{-\gamma_1(\tau-t)}\Phi(\tau)\,d\tau + \frac{1}{F'(\gamma_2)}\int_0^t e^{-\gamma_2(\tau-t)}\Phi(\tau)\,d\tau\ . \tag{2.41}$$

From the assumption that $u(t)=\alpha'(t)\,e^{-bt}$, $\alpha(0)=0$ and using (2.40), we shall have the optimum control $u(t)$ in the form

$$u(t) = -\frac{\left\{\sqrt{b^2+\xi^2}\cosh\left(t\sqrt{b^2+\xi^2}\right)+b\sinh\left(t\sqrt{b^2+\xi^2}\right)\right\}}{\left\{\sqrt{b^2+\xi^2}\cosh\left(T\sqrt{b^2+\xi^2}\right)+b\sinh\left(T\sqrt{b^2+\xi^2}\right)\right\}}\left[\frac{1}{\sqrt{b^2+\xi^2}}\int_0^T\left\{\sqrt{b^2+\xi^2}\cosh\left((\tau-T)\sqrt{b^2+\xi^2}\right)-\right.\right.$$

$$\left.\left.-b\sinh\left((\tau-T)\sqrt{b^2+\xi^2}\right)\right\}e^{-b\tau}\Phi(\tau)\,d\tau\right]+$$

$$+\frac{1}{\sqrt{b^2+\xi^2}}\int_0^t\left\{\sqrt{b^2+\xi^2}\cosh\left((\tau-t)\sqrt{b^2+\xi^2}\right)-b\sinh\left((\tau-t)\sqrt{b^2+\xi^2}\right)\right\}e^{-b\tau}\Phi(\tau)\,d\tau\ , \tag{2.42}$$

where $\xi^2=\frac{1}{m^2L}$ and $\Phi(t)$ is defined in (2.39).

For simplicity, if we put $b=0$, we shall have

$$u(t) = -\frac{\cosh(\xi t)}{\cosh(\xi T)}\int_0^T \Phi(\tau)\cosh\{\xi(\tau-T)\}\,d\tau + \int_0^t \Phi(\tau)\cosh\{\xi(\tau-t)\}\,d\tau\ , \tag{2.43}$$

where $\Phi(t) = \xi^2\left[\phi_0(L-at)-\phi^*(t)\right]$. Moreover, if ϕ_0 and ϕ^* are constants where $K\equiv(\phi_0-\phi^*)$, we have

$$u(t) = \xi K\left[\sinh(\xi t) - \{\tanh(\xi T)\}\cosh(\xi t)\right]$$

i.e., $$u(t) = \frac{\xi K}{\cosh(\xi T)} \cdot \sinh\{\xi(t-T)\}\ , \qquad 0 \le t \le T.$$

It is easily shown in this case using (2.1) that the above solutions for u and ϕ provide the minimum of I .

Case (ii). $aT = L$.

There are only 2 subdomains S_1 and S_2 to be considered in this case (Fig. 13).

In subdomain S_1 , we shall have $\lambda(t,x) = 0$ since $\lambda = 0$ on $t = T$. In subdomain S_2, $\lambda(t,x)$ will be the same as in (2.35).

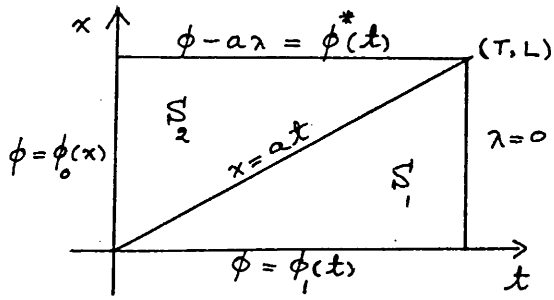


Fig. 13

Hence the optimum control $u(t)$ will be the same as in case (i). $aT < L$, as defined in (2.42) for case $b \ne 0$ and in (2.43) for case $b = 0$.

Case (iii). $aT > L$.

As in Special case 1, when $u = u(t,x)$, we have 3 diagrams to be considered depending on $aT < 2L$, $aT = 2L$ or $aT > 2L$. The solutions of $\lambda(t,x)$ and $\phi(t,x)$ in subdomains S_1 , S_2 and S_3 for each diagram (Fig. 9 - Fig. 11 page 21) will be the same, since S_1 , S_2 and S_3 corresponding to each diagram have the same boundary conditions upon λ and ϕ .

In subdomains S_1 and S_3 , $\lambda(t,x)$ can be found as in case (i) and defined in (2.32) and (2.35) respectively, i.e.,

$$\lambda(t,x) = 0 \qquad , \text{ in } S_1 , \tag{2.44}$$

and

$$\left.\begin{aligned} \lambda(t,x) &= A(x-at)e^{bt} \qquad \text{in } S_3 , \\ \text{where} \quad A(\eta) &= \frac{e^{-2b\left(\frac{L-\eta}{a}\right)}}{a}\left[\alpha\left(\frac{L-\eta}{a}\right)+\phi_0(\eta)\right]-\frac{e^{-b\left(\frac{L-\eta}{a}\right)}}{a}\phi^*\left(\frac{L-\eta}{a}\right). \end{aligned}\right\} \quad (2.45)$$

In subdomain S_2 , $\lambda(t,x)$ and $\phi(t,x)$ in (2.30) and (2.31) have to satisfy the boundary conditions $\phi=\phi_1(t)$ on $x=0$ and $\phi-a\lambda=\phi^*(t)$ on $x=L$. Hence we shall have

$$\lambda(t,x) = A(x-at)e^{bt}$$

where

$$A(\eta) = \frac{e^{-2b\left(\frac{L-\eta}{a}\right)}}{a}\left[\alpha\left(\frac{L-\eta}{a}\right)-\alpha\left(\frac{-\eta}{a}\right)\right]+$$

$$+\frac{e^{-b\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_1\left(\frac{-\eta}{a}\right)e^{-bL/a}-\phi^*\left(\frac{L-\eta}{a}\right)\right], \qquad \text{in } S_2 . \quad (2.46)$$

Next we shall find the optimum control in each diagram. Firstly, let us consider case $L<aT<2L$, for which $t_N<t_M$, since $t_N=T-\frac{L}{a}$ and $t_M=\frac{L}{a}$. The diagram for this case is shown in Fig. 9 .

By using the optimality condition (2.28), we obtain:

When $0<t<t_N$, where $t_N=T-\frac{L}{a}$, we have

$$m^2 L u(t) = -\left[\int_0^{at}\{\lambda\}_{S_2}dx + \int_{x=at}^{x=L}\{\lambda\}_{S_3}dx\right],$$

but $u(t)=\alpha'(t)e^{-bt}$, $\alpha(0)=0$, thus

$$m^2 L\alpha'(t)e^{-2bt} = -\int_{-at}^{0}\left[A(\eta)\right]_{S_2}d\eta - \int_0^{L-at}\left[A(\eta)\right]_{S_3}d\eta , \qquad (2.47)$$

where $A(\eta)$ in S_2 and S_3 are defined in (2.46) and (2.45) respectively.

Differentiating (2.47) with respect to variable t and using (2.45) and (2.46) we obtain the result that $\alpha(t)$ must satisfy the differential-difference equation,

$$m^2L\left[\alpha''(t)-2b\,\alpha'(t)\right]-2e^{-bL/a}\alpha(t)\cosh\left(\tfrac{bL}{a}\right)+e^{-2bL/a}\alpha\left(t+\tfrac{L}{a}\right)=$$

$$=\phi_0(L-at)-e^{-\frac{2bL}{a}+bt}\phi_1(t)-e^{bt}\phi^*(t)+e^{-\frac{bL}{a}+bt}\phi^*\left(t+\tfrac{L}{a}\right), \qquad 0<t<t_N. \tag{2.48}$$

When $t_N<t<t_M$, where $t_N=T-\frac{L}{a}$ and $t_M=\frac{L}{a}$, we have

$$m^2Lu(t)=-\left[\int_0^{L+a(t-T)}\{\lambda\}_{S_1}dx+\int_{L+a(t-T)}^{at}\{\lambda\}_{S_2}dx+\int_{at}^{L}\{\lambda\}_{S_3}dx\right],$$

or
$$m^2Le^{-2bt}\alpha'(t)=-\int_{L-aT}^{0}\left[A(\eta)\right]_{S_2}d\eta-\int_0^{L-at}\left[A(\eta)\right]_{S_3}d\eta, \tag{2.49}$$

where $\lambda=0$ in S_1, $A(\eta)$ in S_2 and S_3 are defined in (2.46) and (2.45) respectively.

Differentiating (2.49) with respect to variable t, we shall get the differential equation in the form

$$m^2L\left[\alpha''(t)-2b\alpha'(t)\right]-\alpha(t)=\phi_0(L-at)-e^{bt}\phi^*(t), \quad t_N<t<t_M. \tag{2.50}$$

Similarly, when $t_M<t<T$, we shall have

$$m^2L\alpha'(t)e^{-bt}=-\left[\int_0^{L+a(t-T)}\{\lambda\}_{S_1}dx+\int_{L+a(t-T)}^{L}\{\lambda\}_{S_2}dx\right],$$

or $$m^2L\,\alpha'(t)\,e^{-2bt} = -\int_{L-aT}^{L-at}[A(\eta)]_{S_2}\,d\eta\ . \tag{2.51}$$

Hence $\alpha(t)$ must satisfy the differential-difference equation in the form

$$m^2L\left[\alpha''(t)-2b\alpha'(t)\right]-\alpha(t)+\alpha(t-\tfrac{L}{a}) = e^{-\frac{bL}{a}+bt}\phi_1(t-\tfrac{L}{a})-e^{bt}\phi^*(t),$$

$$t_M < t < T. \tag{2.52}$$

The conditions imposed on $\alpha(t)$ in the system of equations (2.48), (2.50) and (2.52) are as follows:

$\alpha(0) = 0$,

$\alpha(t)$ is continuous at $t=t_N$ and $t=t_M$,

$\alpha'(t)$ is continuous at $t=t_N$ and $t=t_M$,

$\alpha'(T) = 0$.

We note that the last three conditions follow from (2.47), (2.49) and (2.51).

We shall consider here how to solve the system of equations (2.48), (2.50) and (2.52) only in the simple case when $b=0$.

Let us denote $D^2 \equiv \frac{d^2}{dt^2}$ and $\xi^2 \equiv \frac{1}{m^2L}$, then (2.48), (2.50) and (2.52) can be written in the form

$$\left.\begin{aligned}
(D^2-2\xi^2)\,\alpha(t) &= f_1(t)-\xi^2\alpha(t+\tfrac{L}{a}), && 0<t<t_N;\\
(D^2-\xi^2)\,\alpha(t) &= f_2(t), && t_N<t<t_M;\\
(D^2-\xi^2)\,\alpha(t) &= f_3(t)-\xi^2\alpha(t-\tfrac{L}{a}), && t_M<t<T,
\end{aligned}\right\} \tag{2.53}$$

where

$$\left.\begin{aligned}
f_1(t) &= \xi^2\left[\phi_0(L-at)-\phi_1(t)-\phi^*(t)+\phi^*(t+\tfrac{L}{a})\right],\\
f_2(t) &= \xi^2\left[\phi_0(L-at)-\phi^*(t)\right],\\
f_3(t) &= \xi^2\left[\phi_1(t-\tfrac{L}{a})-\phi^*(t)\right].
\end{aligned}\right\} \tag{2.54}$$

In the case $L < aT < 2L$, there are three time ranges and we define

$$\alpha(t) = \begin{cases} \alpha_1(t), & \text{when } 0 < t < t_N ; \\ \alpha_2(t), & \text{when } t_N < t < t_M ; \\ \alpha_3(t), & \text{when } t_M < t < T . \end{cases} \tag{2.55}$$

When $0 < t < t_N$, then $t_M < t + \frac{L}{a} < T$, (Fig. 14) and by using the definition in (2.55) we can write

$\alpha(t) \equiv \alpha_1(t)$ and $\alpha(t+\frac{L}{a}) \equiv \alpha_3(t+\frac{L}{a})$.

Similarly, when $t_M < t < T$, then $0 < t - \frac{L}{a} < t_N$, (Fig. 14) and we can write

$\alpha(t) \equiv \alpha_3(t)$ and $\alpha(t-\frac{L}{a}) \equiv \alpha_1(t-\frac{L}{a})$.

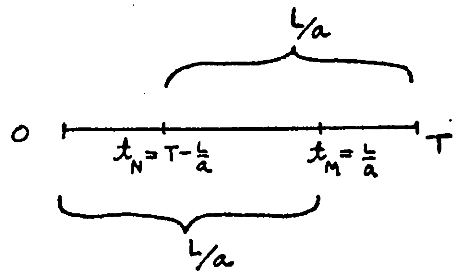


Fig. 14

Hence, the system of equations (2.53) can be written in terms of α_1 , α_2 and α_3 , defined in (2.55), as follows:

$$(D^2 - 2\xi^2)\,\alpha_1(t) = f_1(t) - \xi^2 \alpha_3(t + \tfrac{L}{a}), \qquad 0 < t < t_N ; \tag{2.56}$$

$$(D^2 - \xi^2)\,\alpha_2(t) = f_2(t), \qquad t_N < t < t_M ; \tag{2.57}$$

$$(D^2 - \xi^2)\,\alpha_3(t) = f_3(t) - \xi^2 \alpha_1(t - \tfrac{L}{a}), \qquad t_M < t < T . \tag{2.58}$$

By putting $t + \frac{L}{a}$ instead of t in (2.58), we have

$$(D^2 - \xi^2)\,\alpha_3(t + \tfrac{L}{a}) = f_3(t + \tfrac{L}{a}) - \xi^2 \alpha_1(t), \qquad t_M < t + \tfrac{L}{a} < T \text{ or } 0 < t < t_N . \tag{2.59}$$

Solving the system of equations (2.56) and (2.59) for $\alpha_1(t)$ and $\alpha_3(t + \frac{L}{a})$, where $0 < t < t_N$:

Since the determinant of the coefficients, i.e.,

$$\begin{vmatrix} (D^2-2\xi^2) & \xi^2 \\ \xi^2 & (D^2-\xi^2) \end{vmatrix} = D^4-3\xi^2D^2+\xi^4,$$

is of degree 4 in D , thus the general solution of the system (2.56) and (2.59) will contain exactly 4 arbitrary constants.

Take the operation: $-\frac{1}{\xi^2}(D^2-\xi^2)$(2.56) + (2.59) gives the equivalent system

$$(D^2-2\xi^2)\,\alpha_1(t)+\xi^2\alpha_3(t+\tfrac{L}{a}) = f_1(t)\,, \tag{2.56}$$

$$\left[-\frac{1}{\xi^2}(D^2-\xi^2)(D^2-2\xi^2)+\xi^2\right]\alpha_1(t) = f_3(t+\tfrac{L}{a})-\frac{1}{\xi^2}(D^2-\xi^2)f_1(t)\,. \tag{2.60}$$

The general solution of $\alpha_1(t)$ in (2.60) can be written in the form

$$\alpha_1(t) = Ae^{\gamma_1 t}+Be^{-\gamma_1 t}+Ce^{\gamma_2 t}+Ee^{-\gamma_2 t}-\frac{2}{F'(\gamma_1)}\int_0^t \Phi(\tau)\sinh\{\gamma_1(\tau-t)\}d\tau -$$

$$-\frac{2}{F'(\gamma_2)}\int_0^t \Phi(\tau)\sinh\{\gamma_2(\tau-t)\}d\tau\,, \qquad 0<t<t_N\,, \tag{2.61}$$

where A , B , C , E are arbitrary constants,

$F(D) = D^4-3\xi^2D^2+\xi^4$, $F'(D) = 4D^3-6\xi^2D$,

$\Phi(t) \equiv \xi^2D^2\left[\phi^*(t+\tfrac{L}{a})-\phi_1(t)\right]+\xi^2(D^2-\xi^2)\left[\phi_0(L-at)-\phi^*(t)\right]$,

and $\pm\gamma_1$, $\pm\gamma_2$ are roots of $m^4-3\xi^2m^2+\xi^4=0$,

i.e., $\gamma_1 \sim (1.618)\xi$ and $\gamma_2 \sim (0.618)\xi$.

Since $\alpha_1(t)$, $0<t<t_N$, is evaluated and defined in (2.61), then it follows from (2.56) that

$$\alpha_3(t+\tfrac{L}{a}) = \frac{1}{\xi^2}\left[f_1(t)-(D^2-2\xi^2)\,\alpha_1(t)\right], \qquad 0<t<t_N\,,$$

which is equivalent to

$$\alpha_3(t) = \frac{1}{\xi^2}\left[f_1(t-\tfrac{L}{a}) - (D^2-2\xi^2)\,\alpha_1(t-\tfrac{L}{a})\right], \qquad t_M < t < T \text{ or } 0 < t-\tfrac{L}{a} < t_N, \tag{2.62}$$

where $\alpha_1(t-\frac{L}{a})$ is known and derived from (2.61). The function $f_1(t)$ is defined in (2.54).

<u>When $t_N < t < t_M$</u>, $\alpha(t) \equiv \alpha_2(t)$ and it can be found from (2.57),and can be written in the form

$$\alpha_2(t) = G e^{\xi t} + H e^{-\xi t} - \frac{1}{\xi}\left[\int_{t_N}^{t} f_2(\tau) \sinh\{\xi(\tau-t)\}\,d\tau, \qquad t_N < t < t_M, \tag{2.63}$$

where G and H are arbitrary constants and $f_2(t)$ is defined in (2.54).

Hence the solution of the system of equations (2.56)-(2.58), defined in (2.61) - (2.63), has totally 6 unknown constants A, B, C, E, G and H which can be evaluated by using the following conditions:

$$\alpha_1(0) = 0, \quad \alpha_1(t_N) = \alpha_2(t_N), \quad \alpha_2(t_M) = \alpha_3(t_M),$$

$$\alpha_1'(t_N) = \alpha_2'(t_N), \quad \alpha_2'(t_M) = \alpha_3'(t_M) \text{ and } \alpha_3'(T) = 0 .$$

Therefore, the optimum control $u(t)$ in each time interval will be known from $u(t) = \alpha'(t)$ (for case $b=0$), where $\alpha(t)$ is defined in (2.55) in terms of $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ according to time intervals and α_1, α_2, α_3 are defined in (2.61) - (2.63).

<u>Secondly, let us consider case $L < aT = 2L$</u>. (Fig. 10)

In this case, we have $t_M = t_N = \frac{L}{a}$, the result will be the same as in case $L < aT < 2L$ where we omit equations (2.50) and (2.63).

Thirdly, let us consider case $aT > 2L$, for which $t_M < t_N$, since $t_M = \frac{L}{a}$ and $t_N = T - \frac{L}{a}$. (see Fig. 11)

In this case, when $0 < t < t_M$ and $t_N < t < T$, we have the same results as in (2.48) and (2.52) respectively. i.e., the function $\alpha(t)$ must satisfy the following differential-difference equations:

$$m^2L\left[\alpha''(t) - 2b\alpha'(t)\right] - 2e^{-bL/a}\alpha(t)\cosh\left(\frac{bL}{a}\right) + e^{-2bL/a}\alpha\left(t+\frac{L}{a}\right) =$$

$$= \phi_0(L-at) - e^{-\frac{2bL}{a}+bt}\phi_1(t) - e^{bt}\phi^*(t) + e^{-\frac{bL}{a}+bt}\phi^*\left(t+\frac{L}{a}\right), \quad 0<t<t_M, \tag{2.48}$$

and

$$m^2L\left[\alpha''(t)-2b\alpha'(t)\right] - \alpha(t) + \alpha\left(t-\frac{L}{a}\right) = e^{-\frac{bL}{a}+bt}\phi_1\left(t-\frac{L}{a}\right) - e^{bt}\phi^*(t),$$

$$t_N < t < T. \tag{2.52}$$

Consider when $t_M < t < t_N$, the optimality condition (2.28) implies that

$$m^2L\, u(t) = -\int_{x=0}^{x=L}\left\{\lambda\right\}_{S_2} dx,$$

or
$$m^2L\alpha'(t)e^{-2bt} = -\int_{-at}^{L-at}\left[A(\eta)\right]_{S_2} d\eta, \tag{2.64}$$

where $A(\eta)$ in S_2 is defined in (2.46).

Differentiating (2.64) with respect to variable t, then we shall obtain that $\alpha(t)$ must satisfy the differential-difference equation in the form

$$m^2L\left[\alpha''(t)-2b\alpha'(t)\right]-2\alpha(t)e^{-bL/a}\cosh\left(\frac{bL}{a}\right)+\alpha\left(t-\frac{L}{a}\right)+e^{-2bL/a}\alpha\left(t+\frac{L}{a}\right)=$$

$$=-e^{-2\frac{bL}{a}+bt}\phi_1(t)+e^{-\frac{bL}{a}+bt}\phi_1\left(t-\frac{L}{a}\right)-e^{bt}\phi^*(t)+e^{-\frac{bL}{a}+bt}\phi^*\left(t+\frac{L}{a}\right),$$

$$t_M<t<t_N. \quad (2.65)$$

This system of equations are solved subject to the conditions $\alpha(0)=0$, α and α' are continuous at $t=t_M$ and $t=t_N$ and $\alpha'(T)=0$.

Let us discuss in more detail how to solve the above system of equations in a simple case when $b=0$. The system is in the form

$$\left.\begin{aligned}(D^2-2\xi^2)\,\alpha(t) &= f_1(t)-\xi^2\alpha\left(t+\tfrac{L}{a}\right), & 0<t<t_M;\\ (D^2-2\xi^2)\,\alpha(t) &= f_2(t)-\xi^2\alpha\left(t+\tfrac{L}{a}\right)-\xi^2\alpha\left(t-\tfrac{L}{a}\right), & t_M<t<t_N;\\ (D^2-\xi^2)\,\alpha(t) &= f_3(t)-\xi^2\alpha\left(t-\tfrac{L}{a}\right), & t_N<t<T,\end{aligned}\right\} \quad (2.66)$$

where

$$\left.\begin{aligned}f_1(t) &= \xi^2\left[\phi_0(L-at)-\phi_1(t)-\phi^*(t)+\phi^*\left(t+\tfrac{L}{a}\right)\right],\\ f_2(t) &= \xi^2\left[\phi_1\left(t-\tfrac{L}{a}\right)-\phi_1(t)-\phi^*(t)+\phi^*\left(t+\tfrac{L}{a}\right)\right],\\ f_3(t) &= \xi^2\left[\phi_1\left(t-\tfrac{L}{a}\right)-\phi^*(t)\right],\end{aligned}\right\} \quad (2.67)$$

and $D^2\equiv\dfrac{d^2}{dt^2}$, $\xi^2\equiv\dfrac{1}{m^2L}$.

Assuming that $T=\frac{2L}{a}+\epsilon$, we shall consider here only case $\epsilon<\frac{L}{a}$. (We note that the case $\epsilon=\frac{L}{a}$ will follow from the case $\epsilon<\frac{L}{a}$ by taking $\epsilon=t_M=\frac{L}{a}$ and $T-\epsilon=t_N=T-\frac{L}{a}$. Likewise the case $\epsilon>\frac{L}{a}$ can be handled in the same way).

Let us define that

$$\alpha(t) = \begin{cases} \alpha_1(t) , & \text{when} \quad 0 < t < \epsilon \quad ; \\ \alpha_2(t) , & \text{when} \quad \epsilon < t < t_M \quad ; \\ \alpha_3(t) , & \text{when} \quad t_M < t < t_N \quad ; \\ \alpha_4(t) , & \text{when} \quad t_N < t < T-\epsilon \quad ; \\ \alpha_5(t) , & \text{when} \quad T-\epsilon < t < T \quad . \end{cases} \tag{2.68}$$

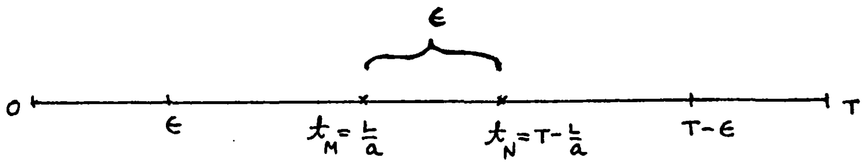


Fig. 15

When $0 < t < \epsilon$, then $t_M < t + \frac{L}{a} < t_N$ (see Fig. 15), and by using the definition in (2.68) we then can write

$$\alpha(t) \equiv \alpha_1(t) \text{ and } \alpha(t+\tfrac{L}{a}) \equiv \alpha_3(t+\tfrac{L}{a}).$$

When $\epsilon < t < t_M$, then $t_N < t + \frac{L}{a} < T-\epsilon$ (see Fig. 15) and we can write

$$\alpha(t) \equiv \alpha_2(t) \text{ and } \alpha(t+\tfrac{L}{a}) \equiv \alpha_4(t+\tfrac{L}{a}) .$$

When $t_M < t < t_N$, then $T-\epsilon < t+\frac{L}{a} < T$ and $0 < t - \frac{L}{a} < \epsilon$ (Fig. 15), hence

$$\alpha(t) \equiv \alpha_3(t), \; \alpha(t+\tfrac{L}{a}) \equiv \alpha_5(t+\tfrac{L}{a}) \text{ and } \alpha(t-\tfrac{L}{a}) \equiv \alpha_1(t-\tfrac{L}{a}).$$

When $t_N < t < T-\epsilon$, then $\epsilon < t - \frac{L}{a} < t_M$ and we can write

$$\alpha(t) \equiv \alpha_4(t) \text{ and } \alpha(t-\tfrac{L}{a}) \equiv \alpha_2(t-\tfrac{L}{a}).$$

Similarly, when $T-\epsilon < t < T$, then $t_M < t - \frac{L}{a} < t_N$ and hence

$$\alpha(t) \equiv \alpha_5(t) \text{ and } \alpha(t-\tfrac{L}{a}) \equiv \alpha_3(t-\tfrac{L}{a}).$$

Using the above arguments, we then can write (2.66) in the form of 2 sets of system of equations as follows:

$$(D^2 - 2\xi^2)\,\alpha_1(t) = f_1(t) - \xi^2 \alpha_3(t+\tfrac{L}{a}), \qquad 0 < t < \epsilon \; ; \tag{2.69}$$

$$(D^2-2\xi^2)\alpha_3(t) = f_2(t)-\xi^2\alpha_5(t+\tfrac{L}{a})-\xi^2\alpha_1(t-\tfrac{L}{a}), \qquad t_M<t<t_N; \tag{2.70}$$

$$(D^2-\xi^2)\alpha_5(t) = f_3(t)-\xi^2\alpha_3(t-\tfrac{L}{a}), \qquad T-\epsilon<t<T, \tag{2.71}$$

and

$$(D^2-2\xi^2)\alpha_2(t) = f_1(t)-\xi^2\alpha_4(t+\tfrac{L}{a}), \qquad \epsilon<t<t_M; \tag{2.72}$$

$$(D^2-\xi^2)\alpha_4(t) = f_3(t)-\xi^2\alpha_2(t-\tfrac{L}{a}), \qquad t_N<t<T-\epsilon. \tag{2.73}$$

First of all, let us consider a set of equations (2.69) - (2.71). The equations (2.70) and (2.71) also can be written in the following forms.

$$(D^2-2\xi^2)\alpha_3(t+\tfrac{L}{a}) = f_2(t+\tfrac{L}{a})-\xi^2\alpha_5(t+\tfrac{2L}{a})-\xi^2\alpha_1(t), \quad t_M<t+\tfrac{L}{a}<t_N \text{ or } 0<t<\epsilon, \tag{2.74}$$

and

$$(D^2-\xi^2)\alpha_5(t+\tfrac{2L}{a}) = f_3(t+\tfrac{2L}{a})-\xi^2\alpha_3(t+\tfrac{L}{a}), \quad T-\epsilon<t+\tfrac{2L}{a}<T \text{ or } 0<t<\epsilon, \tag{2.75}$$

hence we shall solve (2.69), (2.74) and (2.75) for $\alpha_1(t)$, $\alpha_3(t+\frac{L}{a})$ and $\alpha_5(t+\frac{2L}{a})$, where $0<t<\epsilon$.

The system has the determinant of the operators as follows:

$$\begin{vmatrix} (D^2-2\xi^2) & \xi^2 & 0 \\ \xi^2 & (D^2-2\xi^2) & \xi^2 \\ 0 & \xi^2 & (D^2-\xi^2) \end{vmatrix} = D^6-5\xi^2D^4+6\xi^4D^2-\xi^6,$$

which is of degree 6 in D. Hence the general solution of the system will contain 6 arbitrary constants.

Taking the operation: $-\frac{1}{\xi^2}(D^2-\xi^2)(2.74) + (2.75)$, we shall have the equivalent system as

$$(D^2-2\xi^2)\,\alpha_1(t)+\xi^2\alpha_3(t+\tfrac{L}{a}) = f_1(t)\,, \tag{2.69}$$

$$\xi^2\alpha_1(t)+(D^2-2\xi^2)\,\alpha_3(t+\tfrac{L}{a})+\xi^2\alpha_5(t+\tfrac{2L}{a}) = f_2(t+\tfrac{L}{a})\,, \tag{2.74}$$

$$-(D^2-\xi^2)\,\alpha_1(t)-\left[\frac{1}{\xi^2}(D^2-\xi^2)(D^2-2\xi^2)-\xi^2\right]\alpha_3(t+\tfrac{L}{a}) = -\frac{1}{\xi^2}(D^2-\xi^2)\,f_2(t+\tfrac{L}{a})+f_3(t+\tfrac{2L}{a}). \tag{2.76}$$

Next, take another operation: $\frac{1}{\xi^4}(D^4-3\xi^2D^2+\xi^4)(2.69)+(2.76)$ we obtain the system of equations

$$(D^2-2\xi^2)\,\alpha_1(t)+\xi^2\alpha_3(t+\tfrac{L}{a}) = f_1(t)\,, \tag{2.69}$$

$$\xi^2\alpha_1(t)+(D^2-2\xi^2)\,\alpha_3(t+\tfrac{L}{a})+\xi^2\alpha_5(t+\tfrac{2L}{a}) = f_2(t+\tfrac{L}{a})\,, \tag{2.74}$$

$$\left[\frac{1}{\xi^4}(D^4-3\xi^2D^2+\xi^4)(D^2-2\xi^2)-(D^2-\xi^2)\right]\alpha_1(t) = \frac{1}{\xi^4}(D^4-3\xi^2D^2+\xi^4)\,f_1(t)\,- \\ -\frac{1}{\xi^2}(D^2-\xi^2)\,f_2(t+\tfrac{L}{a})+f_3(t+\tfrac{2L}{a})\,, \tag{2.77}$$

where $0<t<\epsilon$.

The solution $\alpha_1(t)$ of (2.77) can be written in the form

$$\alpha_1(t) = A_1e^{\eta_1 t}+B_1e^{-\eta_1 t}+C_1e^{\eta_2 t}+E_1e^{-\eta_2 t}+G_1e^{\eta_3 t}+H_1e^{-\eta_3 t}- \\ -\sum_{i=1}^{3}\frac{2}{F_1'(\eta_i)}\int_0^t \Psi(\tau)\sinh\{\eta_i(\tau-t)\}\,d\tau\,, \qquad 0<t<\epsilon\,, \tag{2.78}$$

where A_1 , B_1 , C_1 , E_1 , G_1 and H_1 are arbitrary constants,

$F_1(D)=D^6-5\xi^2D^4+6\xi^4D^2-\xi^6 \quad \therefore \; F_1'(D)=6D^5-20\xi^2D^3+12\xi^4D\,,$

$\pm\eta_i$ ($i=1,\,2\,,3$) are six roots of $m^6-5\xi^2m^4+6\xi^4m^2-\xi^6=0$,

i.e., $\pm\eta_1\sim\pm0.45\,\xi$, $\pm\eta_2\sim\pm1.25\,\xi$ and $\pm\eta_3\sim\pm1.8\,\xi$,

$$\Psi(t) = \xi^2(D^4-3\xi^2D^2+\xi^4)\left[\phi_0(L-at)-\phi^*(t)\right] -$$

$$-\xi^2D^2(D^2-2\xi^2)\left[\phi_1(t)-\phi^*(t+\tfrac{L}{a})\right]+\xi^4D^2\left[\phi_1(t+\tfrac{L}{a})-\phi^*(t+\tfrac{2L}{a})\right].$$

The solution $\alpha_3(t+\frac{L}{a})$ will then follow from (2.69) that

$$\alpha_3(t+\tfrac{L}{a}) = \frac{1}{\xi^2}\left[f_1(t)-(D^2-2\xi^2)\alpha_1(t)\right], \qquad 0<t<\epsilon,$$

which is equivalent to

$$\alpha_3(t) = \frac{1}{\xi^2}\left[f_1(t-\tfrac{L}{a})-(D^2-2\xi^2)\,\alpha_1(t-\tfrac{L}{a})\right], \quad t_M<t<t_N$$

$$\text{or} \quad 0<t-\tfrac{L}{a}<\epsilon, \qquad (2.79)$$

where $\alpha_1(t-\frac{L}{a})$ can be obtained from (2.78).

Since $\alpha_1(t)$ and $\alpha_3(t)$ are known as in (2.78) and (2.79) then $\alpha_5(t+\frac{2L}{a})$ can be found from (2.74) as

$$\alpha_5(t+\tfrac{2L}{a}) = \frac{1}{\xi^2}\left[f_2(t+\tfrac{L}{a})-\xi^2\alpha_1(t)-(D^2-2\xi^2)\,\alpha_3(t+\tfrac{L}{a})\right], \qquad 0<t<\epsilon,$$

which is equivalent to

$$\alpha_5(t) = \frac{1}{\xi^2}\left[f_2(t-\tfrac{L}{a})-\xi^2\alpha_1(t-\tfrac{2L}{a})-(D^2-2\xi^2)\,\alpha_3(t-\tfrac{L}{a})\right], \quad T-\epsilon<t<T,$$

and by using (2.79), we have

$$\alpha_5(t) = \frac{1}{\xi^2}\left[f_2(t-\tfrac{L}{a})-\frac{(D^2-2\xi^2)}{\xi^2}f_1(t-\tfrac{2L}{a})+\frac{(D^4-4\xi^2D^2+3\xi^4)}{\xi^2}\alpha_1(t-\tfrac{2L}{a})\right],$$

$$T-\epsilon<t<T, \qquad (2.80)$$

where $\alpha_1(t-\frac{2L}{a})$ can be obtained from (2.78) and $f_1(t)$, $f_2(t)$ are defined in (2.67).

The other set of system of equations (2.72) and (2.73) is similar to (2.56) and (2.58) in case $aT<L$. Hence the solution will be the same and can be rewritten as follows:

$$\alpha_2(t) = Ae^{\gamma_1 t} + Be^{-\gamma_1 t} + Ce^{\gamma_2 t} + Ee^{-\gamma_2 t} - \frac{2}{F'(\gamma_1)} \int_0^t \Phi(\tau) \sinh\{\gamma_1(\tau - t)\} d\tau -$$

$$- \frac{2}{F'(\gamma_2)} \int_0^t \Phi(\tau) \sinh\{\gamma_2(\tau - t)\} d\tau , \qquad \epsilon < t < t_M ,$$

(2.81)

and

$$\alpha_4(t) = \frac{1}{\xi^2}\left[f_1(t - \tfrac{L}{a}) - (D^2 - 2\xi^2)\, \alpha_2(t - \tfrac{L}{a})\right] , \qquad t_N < t < T - \epsilon , \tag{2.82}$$

where $\alpha_2(t-\frac{L}{a})$ can be obtained from (2.81), and $\pm\gamma_1$, $\pm\gamma_2$, $F'(D)$ and $\Phi(t)$ are defined as in case $aT < L$.

Hence the solution $\alpha(t)$ of the system (2.66) is defined in (2.78) - (2.82) with altogether 10 unknown arbitrary constants A_1 , B_1 , C_1 , E_1 , G_1 , H_1 , A , B , C and E . These constants can be evaluated by using the following conditions:

$$\alpha_1(0) = 0,\ \alpha_1(\epsilon) = \alpha_2(\epsilon),\ \alpha_2(t_M) = \alpha_3(t_M),\ \alpha_3(t_N) = \alpha_4(t_N),$$
$$\alpha_4(T-\epsilon) = \alpha_5(T-\epsilon),$$
$$\alpha_1'(\epsilon) = \alpha_2'(\epsilon),\ \alpha_2'(t_M) = \alpha_3'(t_M),\ \alpha_3'(t_N) = \alpha_4'(t_N),\ \alpha_4'(T-\epsilon) = \alpha_5'(T-\epsilon)$$

and $\alpha'(T) = 0$.

Therefore, the optimum control $u(t)$ for this case will then follow from the assumption $u(t) = \alpha'(t)\, e^{-bt}$, (for case $b = 0$, $u(t) = \alpha'(t)$) where $\alpha(t)$ is defined in (2.68).

<u>Generalisation of the problem containing n control functions and n parts of the region.</u>

We suppose that the region S : $0 \le t \le T$; $0 \le x \le L$ is divided into n subregions S_i defined as $0 \le t \le T$; $x_i \le x \le x_{i+1}$, $i = 1, 2, \ldots\ldots, n$. where $x_1 = 0$ and $x_{n+1} = L$. (Fig. 16)

Let $u_i(t)$, $i=1,2,\ldots\ldots,n$ be the control in subregion S_i .

Find the controls $u_i(t)$, $i=1,2,\ldots\ldots n$ which minimise the functional

$$I = \sum_{i=1}^{n} \iint_{S_i} F_i(t,x,\phi_i,\frac{\partial\phi_i}{\partial x},u_i)\,dt\,dx + \sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} p(T,x,\phi_i)\,dx + \int_0^T q(t,L,\phi_n)\,dt,$$

where $\phi_i(t,x)$ is the state function in subregion S_i , $i=1,2,\ldots\ldots n$ and satisfies the partial differential equations

$$\frac{\partial\phi_i}{\partial t} = g_i(t,x,\phi_i,\frac{\partial\phi_i}{\partial x},u_i) , \qquad (t,x)\in S_i,\ i=1,2,\ldots n. \quad (2.83)$$

with the initial and boundary conditions defined as

$$\left.\begin{array}{lll} \phi_i(0,x) = \phi_{i0}(x) , & x_i \le x \le x_{i+1} , & i=1,2,\ldots n \\ \phi_1(t,0) = \phi_{11}(t) , & 0\le t\le T . & \end{array}\right\} \quad (2.84)$$

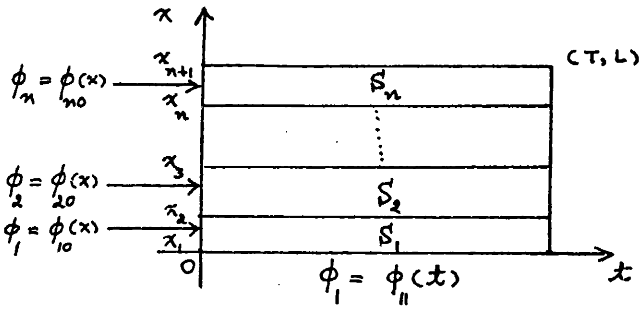


Fig. 16

As before, we set a functional J as follows:

$$J = \sum_{i=1}^{n} \iint_{S_i} [F_i + \lambda_i(g_i - \frac{\partial\phi_i}{\partial t})]\,dt\,dx + \sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} p(T,x,\phi_i)\,dx + \int_0^T q(t,L,\phi_n)\,dt,$$

(2.85)

where $\lambda_i(t,x)$ is Lagrange multiplier corresponding to subregion S_i , ($i=1,2,\ldots\ldots\ldots,n$).

Introducing a Hamiltonian H_i in subregion S_i , ($i=1,2,\ldots,n$) defined as

$$H_i \equiv H_i(t, x, \phi_i, \frac{\partial \phi_i}{\partial x}, \lambda_i, u_i) = F_i + \lambda_i g_i \quad , \quad i = 1, 2, \ldots n. \tag{2.86}$$

We then can write (2.85) in the form

$$J = \sum_{i=1}^{n} \iint_{S_i} \left[H_i - \lambda_i \frac{\partial \phi_i}{\partial t} \right] dt\, dx + \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} p(T, x, \phi_i)\, dx + \int_0^T q(t, L, \phi_n)\, dt. \tag{2.87}$$

Taking variation of (2.87), we can either work it out in a direct method as in Chapter 1 or write it down immediately that

$$\delta J = \sum_{i=1}^{n} \iint_{S_i} \left[\frac{\partial H_i}{\partial \phi_i} \delta \phi_i + \frac{\partial H_i}{\partial (\frac{\partial \phi_i}{\partial x})} \delta\left(\frac{\partial \phi_i}{\partial x}\right) + \frac{\partial H_i}{\partial \lambda_i} \delta \lambda_i + \frac{\partial H_i}{\partial u_i} \delta u_i - \lambda_i \delta\left(\frac{\partial \phi_i}{\partial t}\right) - \frac{\partial \phi_i}{\partial t} \delta \lambda_i \right] dt\, dx +$$

$$+ \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \frac{\partial p}{\partial \phi_i} \delta \phi_i \, dx + \int_0^T \frac{\partial q}{\partial \phi_n} \delta \phi_n \, dt. \tag{2.88}$$

But since $\delta\left(\frac{\partial \phi_i}{\partial x}\right) = \frac{\partial}{\partial x}(\delta \phi_i)$ and $\delta\left(\frac{\partial \phi_i}{\partial t}\right) = \frac{\partial}{\partial t}(\delta \phi_i)$ in a fixed domain we then can write (2.88) in the form

$$\delta J = \sum_{i=1}^{n} \iint_{S_i} \left[(\delta \phi_i) \left\{ \frac{\partial H_i}{\partial \phi_i} - \frac{\partial}{\partial x}\left(\frac{\partial H_i}{\partial (\frac{\partial \phi_i}{\partial x})} \right) + \frac{\partial \lambda_i}{\partial t} \right\} + (\delta \lambda_i)\left(\frac{\partial H_i}{\partial \lambda_i} - \frac{\partial \phi_i}{\partial t} \right) + (\delta u_i) \frac{\partial H_i}{\partial u_i} \right] dt\, dx +$$

$$+ \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \frac{\partial p}{\partial \phi_i} \delta \phi_i \, dx + \int_0^T \frac{\partial q}{\partial \phi_n} \delta \phi_n \, dt + \sum_{i=1}^{n} \iint_{S_i} \left[\frac{\partial}{\partial x} \left\{ \frac{\partial H_i}{\partial (\frac{\partial \phi_i}{\partial x})} \delta \phi_i \right\} - \frac{\partial}{\partial t}(\lambda_i \delta \phi_i) \right] dt\, dx.$$

Applying the Green's theorem in [2], i.e.,

$$\iint_{S_i} \left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} \right) dt\, dx = \oint_{C_i} (u\, dt + v\, dx) ,$$

to the last set of integrals in δJ , we have

$$\delta J = \sum_{i=1}^{n} \iint_{S_i} \left[(\delta\phi_i)\left\{ \frac{\partial H_i}{\partial \phi_i} - \frac{\partial}{\partial x}\left(\frac{\partial H_i}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} \right) + \frac{\partial \lambda_i}{\partial t} \right\} + (\delta u_i)\frac{\partial H_i}{\partial u_i} + (\delta\lambda_i)\left(\frac{\partial H_i}{\partial \lambda_i} - \frac{\partial \phi_i}{\partial t} \right) \right] dt\,dx +$$

$$+ \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \frac{\partial p}{\partial \phi_i}\delta\phi_i\, dx + \int_0^T \frac{\partial q}{\partial \phi_n}\delta\phi_n\, dt - \sum_{i=1}^{n}\left[\int_0^T \left\{ \frac{\partial H_i}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} \cdot \delta\phi_i \right\}_{x=x_{i+1}}^{x=x_i} dt + \right.$$

$$\left. + \int_{x_i}^{x_{i+1}} \left\{ \lambda_i\, \delta\phi_i \right\}_{t=0}^{t=T} dx \right] , \tag{2.89}$$

where we note here that $\left\{ X(t) \right\}_{t=0}^{t=T}$ means $X(T) - X(0)$.

Since $\phi_1(t,0)$ and $\phi_i(0,x)$ are known, as defined in (2.84), hence $\delta\phi_1(t,0)$ and $\delta\phi_i(0,x)$, $i=1, 2, \ldots, n$, are zero. Let us assume the continuity of the functions ϕ_i and ϕ_{i+1} at $x = x_{i+1}$ where $i=1, 2, \ldots, (n-1)$. $\therefore\ \delta\phi_i = \delta\phi_{i+1}$ and then (2.89) becomes

$$\delta J = \sum_{i=1}^{n} \iint_{S_i} \left[(\delta\phi_i)\left\{ \frac{\partial H_i}{\partial \phi_i} - \frac{\partial}{\partial x}\left(\frac{\partial H_i}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} \right) + \frac{\partial \lambda_i}{\partial t} \right\} + (\delta u_i)\frac{\partial H_i}{\partial u_i} + (\delta\lambda_i)\left(\frac{\partial H_i}{\partial \lambda_i} - \frac{\partial \phi_i}{\partial t} \right) \right] dt\,dx +$$

$$+ \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left[(\delta\phi_i)\left\{ \frac{\partial p}{\partial \phi_i} - \lambda_i \right\} \right]_{t=T} dx + \int_0^T \left[\left\{ \frac{\partial q}{\partial \phi_n} + \frac{\partial H_n}{\partial\left(\frac{\partial \phi_n}{\partial x}\right)} \right\} \delta\phi_n \right]_{x=L} dt +$$

$$+ \sum_{i=1}^{n-1} \int_0^T \left[\left\{ \frac{\partial H_i}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} - \frac{\partial H_{i+1}}{\partial\left(\frac{\partial \phi_{i+1}}{\partial x}\right)} \right\} \delta\phi_i \right]_{x=x_{i+1}} dt.$$

The necessary condition for I to attain a minimum value is $\delta J = 0$, which gives us the following conditions:

$$\frac{\partial H_i}{\partial \phi_i} - \frac{\partial}{\partial x}\left(\frac{\partial H_i}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} \right) + \frac{\partial \lambda_i}{\partial t} = 0 , \qquad (t,x) \in S_i ,\ i=1,2,\ldots,n. \tag{2.90}$$

$$\frac{\partial \phi_i}{\partial t} = \frac{\partial H_i}{\partial \lambda_i} \, , \quad \text{i.e.,} \; \frac{\partial \phi_i}{\partial t} = g_i \, , \qquad (t,x) \in S_i \, , \; i=1,2,\ldots,n. \tag{2.91}$$

Since $u_i = u_i(t)$ and then δu_i will be a function of t only, we shall have

$$\int_{x_i}^{x_{i+1}} \frac{\partial H_i}{\partial u_i} \, dx = 0 \, , \qquad i = 1, 2, \cdots n. \tag{2.92}$$

we also note here that if $u_i = u_i(t,x)$, $0 \leq t \leq T$; $x_i \leq x \leq x_{i+1}$, $i=1, 2, \ldots, n$, the condition (2.92) is replaced by $\frac{\partial H_i}{\partial u_i} = 0$.

Boundary conditions

The conditions on $t=0$ and on $x=0$ are given as in (2.84). Since $\delta \phi_i$,$(i=1, 2, \ldots, n)$ are arbitrary on the boundary $t=T$ and also on $x=x_{i+1}$ $(i=1, 2, \ldots, n)$, thus we shall have the following boundary conditions:

$$\frac{\partial p}{\partial \phi_i} - \lambda_i = 0 \, , \quad t=T; \; x_i \leq x \leq x_{i+1} \, , \; i=1,2,\ldots,n. \tag{2.93}$$

$$\frac{\partial q}{\partial \phi_n} + \frac{\partial H_n}{\partial \left(\frac{\partial \phi_n}{\partial x}\right)} = 0 \, , \quad x=L \, ; \; 0 \leq t \leq T, \tag{2.94}$$

and $$\frac{\partial H_i}{\partial \left(\frac{\partial \phi_i}{\partial x}\right)} = \frac{\partial H_{i+1}}{\partial \left(\frac{\partial \phi_{i+1}}{\partial x}\right)} \, , \qquad x=x_{i+1} \, , \; i = 1,2,\ldots,(n-1). \tag{2.95}$$

where ϕ_i and ϕ_{i+1} are assumed to be continuous at $x = x_{i+1}$, $(i=1, 2, \ldots, (n-1))$.

We shall consider in the following section an example of case $n=2$. We have already mentioned a physical problem of this kind in the introduction, but mathematically we can state the problem as follows:

Find control functions $u_1(t)$ and $u_2(t)$, corresponding to subregions S_1 and S_2 respectively, which minimise the functional I , defined as

$$I = \int_0^T \frac{1}{2}\left\{\phi_2(t,L) - \phi^*(t)\right\}^2 dt + \iint_{S_1} \frac{1}{2} m_1^2 u_1^2 \, dt\,dx + \iint_{S_2} \frac{1}{2} m_2^2 u_2^2 \, dt\,dx ,$$

where S_1 : $0 \le x \le \frac{L}{2}$; $0 \le t \le T$ and S_2 : $\frac{L}{2} \le x \le L$; $0 \le t \le T$ are subregions of S : $0 \le x \le L$; $0 \le t \le T$, the function $\phi^*(t)$ is prescribed and m_1, m_2 are constants.

The constraints are the state equations described by the following partial differential equations

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial t} + a\frac{\partial \phi_1}{\partial x} + k\phi_1 &= u_1(t) , \qquad (t,x) \in S_1 , \\ \frac{\partial \phi_2}{\partial t} + a\frac{\partial \phi_2}{\partial x} + k\phi_2 &= u_2(t) , \qquad (t,x) \in S_2 , \end{aligned}\right\} \quad (2.96)$$

where a (>0) and k are constants. The initial condition and the boundary condition on $x=0$ are given as follows:

$$\left.\begin{aligned} \phi_1(0,x) &= \phi_{10}(x) , \qquad 0 \le x \le \tfrac{L}{2} , \\ \phi_2(0,x) &= \phi_{20}(x) , \qquad \tfrac{L}{2} \le x \le L , \\ \text{and} \quad \phi_1(t,0) &= \phi_{11}(t) , \qquad 0 \le t \le T , \end{aligned}\right\} \quad (2.97)$$

where $\phi_{10}(x)$, $\phi_{20}(x)$ and $\phi_{11}(t)$ are given and $\phi_{10}(0) = \phi_{11}(0)$.(Fig. 17)

Introducing the Hamiltonians,

$$H_1 = \frac{1}{2} m_1^2 u_1^2 + \lambda_1\left\{u_1 - k\phi_1 - a\frac{\partial \phi_1}{\partial x}\right\} ,$$

$$H_2 = \frac{1}{2} m_2^2 u_2^2 + \lambda_2\left\{u_2 - k\phi_2 - a\frac{\partial \phi_2}{\partial x}\right\} .$$

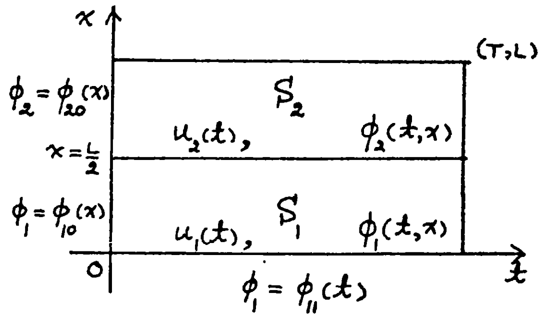


Fig. 17

It then follows from (2.90) that

$$\left.\begin{aligned} \frac{\partial \lambda_1}{\partial t} + a\frac{\partial \lambda_1}{\partial x} &= k\lambda_1 , \qquad (t,x) \in S_1 , \\ \frac{\partial \lambda_2}{\partial t} + a\frac{\partial \lambda_2}{\partial x} &= k\lambda_2 , \qquad (t,x) \in S_2 , \end{aligned}\right\} \quad (2.98)$$

The characteristics of (2.96) and (2.98) are the same and obtained from

$$\frac{dx}{a} = \frac{dt}{1} \quad , \text{ i.e., } \quad x = at + \text{constant}$$

The optimality condition (2.92) gives us that

$$\left.\begin{aligned} &\int_{x=0}^{x=L/2} (m_1^2 u_1 + \lambda_1)\,dx = 0 , \quad \text{i.e., } \tfrac{1}{2} m_1^2 L u_1(t) = -\int_0^{L/2} \lambda_1(t,x)\,dx , \\ &\int_{x=L/2}^{x=L} (m_2^2 u_2 + \lambda_2)\,dx = 0 , \quad \text{i.e., } \tfrac{1}{2} m_2^2 L u_2(t) = -\int_{L/2}^{L} \lambda_2(t,x)\,dx . \end{aligned}\right\} \quad (2.99)$$

The boundary conditions on $t = T$, on $x = L$ and on $x = \frac{L}{2}$ are obtained from (2.93) - (2.95) and can be written down as follows:

Here $p \equiv 0$, $q \equiv \frac{1}{2}\{\phi_2(t,L) - \phi^*(t)\}^2$, hence

$$\left.\begin{aligned} &\lambda_1(T,x) = 0 , && 0 \le x \le \tfrac{L}{2} ; \ t = T , \\ &\lambda_2(T,x) = 0 , && \tfrac{L}{2} \le x \le L ; \ t = T , \\ &\phi_2(t,T) - a\lambda_2(t,L) = \phi^*(t) , && x = L ; \ 0 \le t \le T , \\ &\phi_1(t,\tfrac{L}{2}) = \phi_2(t,\tfrac{L}{2}) , && x = \tfrac{L}{2} ; \ 0 \le t \le T , \\ &\lambda_1(t,\tfrac{L}{2}) = \lambda_2(t,\tfrac{L}{2}) , && x = \tfrac{L}{2} ; \ 0 \le t \le T . \end{aligned}\right\} \quad (2.100)$$

Solving a set of equations (2.96) and (2.98) by introducing new independent or characteristic variables $\xi = t$ and $\eta = x - at$. We also assume as earlier that $u_i(t) = \alpha_i'(t)\,e^{-kt}$, where $\alpha_i(0) = 0$, ($i = 1 , 2$) thus the solutions of (2.96) and (2.98) can be written in the form

$$\left.\begin{aligned} &\lambda_i(t,x) = A_i(x-at)\,e^{kt} , \qquad (t,x) \in S_i , i = 1,2 . \\ &\phi_i(t,x) = \alpha_i(t)\,e^{-kt} + B_i(x-at)\,e^{-kt} , \quad (t,x) \in S_i , \ i = 1,2 . \end{aligned}\right\} \quad (2.101)$$

We shall discuss here only case $aT > 2L$, but for other cases we can work them out in the same method.

The characteristics $x = at$ and $x = L + a(t-T)$ will divide the domain S into 3 subdomains I , II and III as shown in a diagram (Fig. 18).

In <u>subdomain I</u>, by using the boundary conditions $\lambda_1 = 0$ on $t = T$, $0 \le x \le \frac{L}{2}$ and $\lambda_2 = 0$ on $t = T$, $\frac{L}{2} \le x \le L$, it then follows from (2.101) that

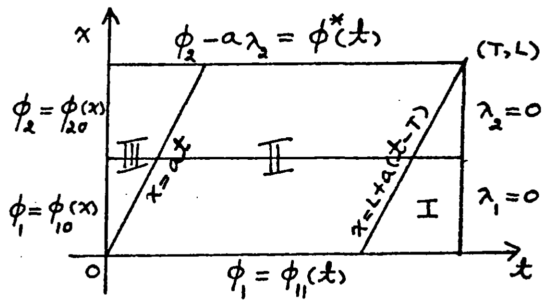


Fig. 18

$A_i(x - aT) = 0$, for all x in S_i and I ,

i.e., $A_i(\xi) = 0$, for all ξ ,

hence

$$\lambda_i(t,x) = 0 \quad , \quad (t,x) \in S_i \ , i = 1, 2 \text{ and in subdomain I} . \tag{2.102}$$

In <u>subdomain II</u>, by using (2.101) and the condition on $x=0$ i.e., $\phi_1(t,0) = \phi_{11}(t)$, we then can write $\phi_1(t,x)$ in the form

$$\phi_1(t,x) = \left[\alpha_1(t) - \alpha_1(t - \tfrac{x}{a})\right] e^{-kt} + \phi_{11}(t - \tfrac{x}{a}) e^{-kx/a} , \qquad (t,x) \in S_1 \text{ in II} ,$$

but from (2.100) we have $\phi_1(t, \frac{L}{2}) = \phi_2(t, \frac{L}{2})$, we then can deduce that

$$B_2(\tfrac{L}{2} - at) = \left[\alpha_1(t) - \alpha_1(t - \tfrac{L}{2a}) - \alpha_2(t)\right] + \phi_{11}(t - \tfrac{L}{2a}) e^{k(t - \frac{L}{2a})} ,$$

or $$B_2(\eta) = \left[\alpha_1(\tfrac{L}{2a} - \tfrac{\eta}{a}) - \alpha_1(-\tfrac{\eta}{a}) - \alpha_2(\tfrac{L}{2a} - \tfrac{\eta}{a})\right] + \phi_{11}(-\tfrac{\eta}{a}) e^{k(-\frac{\eta}{a})} , \text{ for all } \eta ,$$

thus

$$\phi_2(t,x) = \left[\alpha_2(t) - \alpha_2(t + \tfrac{\frac{L}{2} - x}{a}) + \alpha_1(t + \tfrac{\frac{L}{2} - x}{a}) - \alpha_1(t - \tfrac{x}{a})\right] e^{-kt} + \\ + \phi_{11}(t - \tfrac{x}{a}) e^{-kx/a} , \qquad (t,x) \in S_2 \text{ in II} . \tag{2.103}$$

To satisfy the condition $\phi_2 - a\lambda_2 = \phi^*(t)$ on $x = L$, $0 \le t \le T$ and by using (2.101), (2.103), we obtain

$$A_2(\eta) = \frac{e^{-2k\left(\frac{L-\eta}{a}\right)}}{a}\left[\alpha_2\left(\frac{L-\eta}{a}\right) - \alpha_2\left(\frac{L}{2a}-\frac{\eta}{a}\right) + \alpha_1\left(\frac{L}{2a}-\frac{\eta}{a}\right) - \alpha_1\left(-\frac{\eta}{a}\right)\right] +$$

$$+\frac{e^{-k\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_{11}\left(-\frac{\eta}{a}\right)e^{-kL/a} - \phi^*\left(\frac{L-\eta}{a}\right)\right] \quad , \text{ for all } \eta \tag{2.104}$$

From the condition in (2.100), we have $\lambda_1(t, \frac{L}{2}) = \lambda_2(t, \frac{L}{2})$, hence it will follow from (2.101) that

$$A_1\left(\tfrac{L}{2} - at\right) = A_2\left(\tfrac{L}{2} - at\right) \quad , \text{ all } t \; ,$$

i.e.,
$$A_1(\xi) = A_2(\xi) \quad , \text{ all } \xi \; ,$$

thus

$$A_1(\eta) \equiv A_2(\eta) \text{ and defined as in (2.104).}$$

In <u>subdomain III</u>, to satisfy the conditions (2.97) on $t = 0$, the equation (2.101) implies that

$$B_i(x) = \phi_{io}(x) \quad , \text{ for all } x \text{ in } S_i \text{ and III} \; , \; i = 1, 2 \, .$$

hence

$$\phi_i(t,x) = \alpha_i(t)e^{-kt} + \phi_{io}(x - at)e^{-kt}, \qquad (t,x) \in S_i, \text{ and in III}, \quad i = 1, 2. \tag{2.105}$$

By using the condition $\phi_1(t, \frac{L}{2}) = \phi_2(t, \frac{L}{2})$ on $x = \frac{L}{2}$, $0 \le t \le T$ we then can deduce that

$$\phi_{20}(\eta) = \phi_{10}(\eta) + \alpha_1\left(\frac{L}{2a} - \frac{\eta}{a}\right) - \alpha_2\left(\frac{L}{2a} - \frac{\eta}{a}\right), \qquad \text{all } \eta. \tag{2.106}$$

The state function $\phi_2(t,x)$ must satisfy the boundary condition $\phi_2 - a\lambda_2 = \phi^*(t)$ on $x = L$, $0 \le t \le T$, in which we can find $A_2(\eta)$ in the form

$$A_2(\eta) = \frac{e^{-2k\left(\frac{L-\eta}{a}\right)}}{a}\left[\alpha_2\left(\frac{L-\eta}{a}\right) + \phi_{20}(\eta)\right] - \frac{e^{-k\left(\frac{L-\eta}{a}\right)}}{a}\,\phi^*\left(\frac{L-\eta}{a}\right), \quad \text{all } \eta ,$$

and after using the relation in (2.106), we have

$$A_2(\eta) = \frac{e^{-2k\left(\frac{L-\eta}{a}\right)}}{a}\left[\alpha_2\left(\frac{L-\eta}{a}\right) - \alpha_2\left(\frac{L}{2a}-\frac{\eta}{a}\right) + \alpha_1\left(\frac{L}{2a}-\frac{\eta}{a}\right)\right] + \frac{e^{-k\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_{10}(\eta)\,e^{-k\left(\frac{L-\eta}{a}\right)} - \phi^*\left(\frac{L-\eta}{a}\right)\right],$$

$$\text{for all } \eta \,. \qquad (2.107)$$

As before, if we use the condition $\lambda_1(t, \frac{L}{2}) = \lambda_2(t, \frac{L}{2})$, all t, we shall have $A_1(\eta) \equiv A_2(\eta)$ and defined as in (2.107).

Next, let us find the optimum controls $u_1(t)$ and $u_2(t)$ in each time intervals.

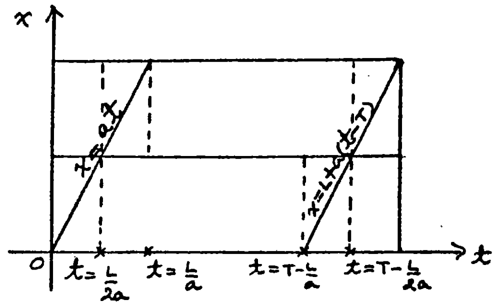


Fig. 19

<u>When $0 < t < \frac{L}{2a}$</u>, by using the optimality conditions in (2.99) and the assumption that $u_i(t) = \alpha_i'(t)\,e^{-kt}$, $(i=1,\ 2)$, $\alpha_i(0) = 0$ we have

$$\left.\begin{aligned} \frac{L}{2}\, m_1^2\, \alpha_1'(t)\, e^{-kt} &= -\int_0^{at} \{\lambda_1\}_{II}\, dx - \int_{at}^{L/2} \{\lambda_1\}_{III}\, dx \\ \frac{L}{2}\, m_2^2\, \alpha_2'(t)\, e^{-kt} &= -\int_{L/2}^{L} \{\lambda_2\}_{III}\, dx \end{aligned}\right\} \qquad (2.108)$$

Substituting $\lambda_i(t,x) = A_i(x-at)\,e^{kt}$, from (2.101), we then can write (2.108) in the form

$$\left.\begin{aligned} \frac{1}{2}\, m_1^2 L\, \alpha_1'(t)\, e^{-2kt} &= -\int_{-at}^{0} \{A_1(\eta)\}_{II}\, d\eta - \int_0^{L/2-at} \{A_1(\eta)\}_{III}\, d\eta \\ \frac{1}{2}\, m_2^2 L\, \alpha_2'(t)\, e^{-2kt} &= -\int_{L/2-at}^{L-at} \{A_2(\eta)\}_{III}\, d\eta \end{aligned}\right\} \qquad (2.109)$$

Differentiating (2.109) with respect to variable t and then using (2.104) and (2.107), we obtain that $\alpha_1(t)$ and $\alpha_2(t)$ must satisfy the differential-difference equations in the form

$$\frac{1}{2}m_1^2 L\left[\alpha_1''(t)-2k\alpha_1'(t)\right]-e^{-kL/a}\left[2\cosh\left(\frac{kL}{2a}\right)\alpha_1(t)e^{-kL/2a}-\alpha_1\left(t+\frac{L}{2a}\right)e^{-kL/a}+\right.$$
$$\left.+2\cosh\left(\frac{kL}{2a}\right)\alpha_2\left(t+\frac{L}{2a}\right)e^{-kL/2a}-\alpha_2\left(t+\frac{L}{a}\right)e^{-kL/a}-\alpha_2(t)\right]=$$
$$=e^{k(t-\frac{L}{a})}\left[\phi_{10}\left(\frac{L}{2}-at\right)e^{-kt}-\phi_{11}(t)e^{-kL/a}-e^{kL/2a}\phi^*\left(t+\frac{L}{2a}\right)+\phi^*\left(t+\frac{L}{a}\right)\right],$$
$$0<t<\frac{L}{2a}, \qquad (2.110)$$

and

$$\frac{1}{2}m_2^2 L\left[\alpha_2''(t)-2k\alpha_2'(t)\right]-\left[2\cosh\left(\frac{kL}{2a}\right)\alpha_2(t)e^{-kL/2a}-\alpha_2\left(t+\frac{L}{2a}\right)e^{-kL/a}-\right.$$
$$\left.-\alpha_2\left(t-\frac{L}{2a}\right)+\alpha_1\left(t-\frac{L}{2a}\right)-\alpha_1(t)e^{-kL/a}\right]=$$
$$=\left[\phi_{10}(L-at)-\phi_{10}\left(\frac{L}{2}-at\right)e^{-kL/a}+e^{kt}\left\{\phi^*\left(t+\frac{L}{2a}\right)e^{-kL/2a}-\phi^*(t)\right\}\right],$$
$$0<t<\frac{L}{2a}. \qquad (2.111)$$

<u>When $\frac{L}{2a}<t<\frac{L}{a}$,</u> since we assume that $u_i(t)=\alpha_i'(t)e^{-kt}$, $(i=1, 2)$ it will follow from (2.99) that

$$\left.\begin{aligned}
&\frac{1}{2}m_1^2 L\alpha_1'(t)e^{-kt}=-\int_0^{L/2}\{\lambda_1\}_{\mathrm{II}}\,dx,\\
\text{and}\quad &\frac{1}{2}m_2^2 L\alpha_2'(t)e^{-kt}=-\int_{L/2}^{at}\{\lambda_2\}_{\mathrm{II}}\,dx-\int_{at}^{L}\{\lambda_2\}_{\mathrm{III}}\,dx,
\end{aligned}\right\} \qquad (2.112)$$

where $\lambda_i(t,x)=A_i(x-at)e^{kt}$, $(i=1, 2)$, or we have

$$\left.\begin{aligned} &\tfrac{1}{2} m_1^2 L\, \alpha_1'(t)\, e^{-2kt} = -\int_{-at}^{L/2-at} \{A_1(\eta)\}_{\mathrm{II}}\, d\eta\,, \\ \text{and}\quad &\tfrac{1}{2} m_2^2 L\, \alpha_2'(t)\, e^{-2kt} = -\int_{L/2-at}^{0} \{A_2(\eta)\}_{\mathrm{II}}\, d\eta - \int_{0}^{L-at} \{A_2(\eta)\}_{\mathrm{III}}\, d\eta\,, \end{aligned}\right\} \quad (2.113)$$

where $A_i(\eta)$ in domain II and III are defined in (2.104) and (2.107).

Differentiating (2.113) with respect to variable t and substituting A_i from (2.104) and (2.107), we then have a set of differential-difference equations satisfied by $\alpha_1(t)$ and $\alpha_2(t)$ as follows:

$$\begin{aligned} &\tfrac{1}{2} m_1^2 L\left[\alpha_1''(t) - 2k\alpha_1'(t)\right] - e^{-kL/a}\Big[2\cosh\left(\tfrac{kL}{2a}\right)\alpha_1(t)\, e^{-kL/2a} - \alpha_1\left(t+\tfrac{L}{2a}\right) e^{-kL/a} - \\ &- \alpha_1\left(t-\tfrac{L}{2a}\right) + 2\cosh\left(\tfrac{kL}{2a}\right)\alpha_2\left(t+\tfrac{L}{2a}\right) e^{-kL/2a} - \alpha_2\left(t+\tfrac{L}{a}\right) e^{-kL/a} - \alpha_2(t)\Big] = \\ &= e^{k(t-\frac{L}{a})}\left[\phi_{11}\left(t-\tfrac{L}{2a}\right) e^{-kL/2a} - \phi_{11}(t)\, e^{-kL/a} - e^{kL/2a}\phi^*\left(t+\tfrac{L}{2a}\right) + \phi^*\left(t+\tfrac{L}{a}\right)\right], \end{aligned}$$

(2.114)

and

$$\begin{aligned} &\tfrac{1}{2} m_2^2 L\left[\alpha_2''(t) - 2k\alpha_2'(t)\right] - \Big[2\cosh\left(\tfrac{kL}{2a}\right)\alpha_2(t)\, e^{-kL/2a} - \alpha_2\left(t+\tfrac{L}{2a}\right) e^{-kL/a} - \\ &- \alpha_2\left(t-\tfrac{L}{2a}\right) + 2\cosh\left(\tfrac{kL}{2a}\right)\alpha_1\left(t-\tfrac{L}{2a}\right) e^{-kL/2a} - \alpha_1(t)\, e^{-kL/a}\Big] = \\ &= e^{k(t-L/a)}\left[\phi_{10}(L-at)\, e^{-k(t-L/a)} - \phi_{11}\left(t-\tfrac{L}{2a}\right) e^{-kL/2a} - \phi^*(t)\, e^{kL/a} + \phi^*\left(t+\tfrac{L}{2a}\right) e^{kL/2a}\right], \end{aligned}$$

(2.115)

where $\frac{L}{2a} < t < \frac{L}{a}$.

<u>When $\frac{L}{a} < t < T-\frac{L}{a}$</u>, the first condition in (2.99) will give the same result as in (2.114). The second condition in (2.99) can be written

in the form

$$\frac{1}{2} m_2^2 L \alpha_2'(t) e^{-kt} = - \int_{L/2}^{L} \{\lambda_2\}_{\mathrm{II}} \, dx ,$$

or

$$\frac{1}{2} m_2^2 L \alpha_2'(t) e^{-2kt} = - \int_{L/2-at}^{L-at} \{A_2(\eta)\}_{\mathrm{II}} \, d\eta , \tag{2.116}$$

where $A_2(\eta)$ in subdomain II is defined in (2.104).

After differentiating (2.116) with respect to variable t and using (2.104), we then have a differential-difference equation as follows:

$$\frac{1}{2} m_2^2 L \left[\alpha_2''(t) - 2k\alpha_2'(t)\right] - \Big[2\cosh\left(\frac{kL}{2a}\right) \alpha_2(t) e^{-kL/2a} - \alpha_2\left(t+\frac{L}{2a}\right) e^{-kL/a} -$$

$$- \alpha_2\left(t-\frac{L}{2a}\right) + 2\cosh\left(\frac{kL}{2a}\right) \alpha_1\left(t-\frac{L}{2a}\right) e^{-kL/2a} - \alpha_1\left(t-\frac{L}{a}\right) - \alpha_1(t) e^{-kL/a}\Big] =$$

$$= e^{k(t-L/a)} \left[\phi_{\mathrm{II}}\left(t-\frac{L}{a}\right) - \phi_{\mathrm{II}}\left(t-\frac{L}{2a}\right) e^{-kL/2a} - \phi^*(t) e^{kL/a} + \phi^*\left(t+\frac{L}{2a}\right) e^{kL/2a}\right], \tag{2.117}$$

where $\frac{L}{a} < t < T - \frac{L}{a}$.

When $T-\frac{L}{a} < t < T-\frac{L}{2a}$, the second condition in (2.99) gives us the same result as in (2.117). Since $\lambda_1 = 0$ in subdomain I , the first condition in (2.99) becomes

$$\frac{1}{2} m_1^2 \alpha_1'(t) e^{-kt} = - \int_{L+at-aT}^{L/2} \{A_1(x-at) e^{kt}\}_{\mathrm{II}} \, dx ,$$

where we assumed $u_i(t) = \alpha_i'(t) e^{-kt}$, or we have

$$\frac{1}{2} m_1^2 \alpha_1'(t) e^{-2kt} = - \int_{L-aT}^{L/2-at} \{A_1(\eta)\}_{\mathrm{II}} \, d\eta \tag{2.118}$$

Differentiating (2.118) with respect to variable t and using (2.104), we obtain

$$\frac{1}{2} m_1^2 L\left[\alpha_1''(t)-2k\alpha_1'(t)\right]-e^{-kL/a}\left[\alpha_1(t)-\alpha_1(t-\tfrac{L}{2a})+\alpha_2(t+\tfrac{L}{2a})-\alpha_2(t)\right]=$$

$$=e^{k(t-\frac{L}{a})}\left[\phi_{11}(t-\tfrac{L}{2a})e^{-kL/2a}-e^{kL/2a}\phi^*(t+\tfrac{L}{2a})\right],\quad T-\tfrac{L}{a}<t<T-\tfrac{L}{2a}$$

(2.119)

<u>When $T-\frac{L}{2a}<t<T$</u>, since $\lambda_1=0$ and $\lambda_2=0$ in subdomain I , as in (2.102), then the optimality conditions in (2.99) become

$$u_1(t) = 0 \quad \text{or} \quad \alpha_1'(t) = 0 \quad , \tag{2.120}$$

and

$$\frac{1}{2} m_2^2 L\, \alpha_2'(t)\, e^{-2kt} = -\int_{L-aT}^{L-at}\{A_2(\eta)\}_{II}\, d\eta \;, \tag{2.121}$$

where we put $u_i(t)=\alpha_i'(t)\,e^{-kt}$, $\lambda_i = A_i(x-at)\,e^{kt}$; $(i=1\,,\,2)$ and $A_2(\eta)$ in (2.121) is defined in (2.104).

As before, after differentiating (2.121) with respect to variable t and using (2.104), we have

$$\frac{1}{2} m_2^2 L\left[\alpha_2''(t)-2k\alpha_2'(t)\right]-\left[\alpha_2(t)-\alpha_2(t-\tfrac{L}{2a})+\alpha_1(t-\tfrac{L}{2a})-\alpha_1(t-\tfrac{L}{a})\right]=$$

$$=e^{kt}\left[\phi_{11}(t-\tfrac{L}{a})e^{-kL/a}-\phi^*(t)\right],\qquad T-\tfrac{L}{2a}<t<T.$$

(2.122)

The conditions upon $\alpha_1(t)$ and $\alpha_2(t)$ are defined as follows:

$\alpha_1(0) = 0$,

$\alpha_1(t)$ and $\alpha_1'(t)$ are continuous at $t=\frac{L}{2a}$; $t=T-\frac{L}{a}$ and $t=T-\frac{L}{2a}$,

$\alpha_1'(T) = 0$,

$\alpha_2(0) = 0$,

$\alpha_2(t)$ and $\alpha_2'(t)$ are continuous at $t = \frac{L}{2a}$; $t = \frac{L}{a}$ and $t = T - \frac{L}{2a}$,
and $\alpha_2'(T) = 0$.

We note here that the above conditions upon $\alpha_1'(t)$ and $\alpha_2'(t)$ follow from (2.109), (2.113), (2.116), (2.118), (2.120) and (2.121).

Special case 3: $u = u(t,x)$, a piecewise continuous control.

Let the constraint imposed on $u(t,x)$ be $-1 \leq u(t,x) \leq 1$.

The Hamiltonian $H = \frac{1}{2}m^2u^2 + \lambda(u - a\phi_x - b\phi)$ which can be rearranged and written in the form

$$H = \frac{1}{2}\left(mu + \frac{\lambda}{m}\right)^2 - \frac{\lambda^2}{2m^2} - \lambda(a\phi_x + b\phi)$$

The control $u(t,x)$ is chosen such as to minimise H and this leads to the following cases:

(1). $m^2u + \lambda = 0$, if $|\lambda| \leq m^2$, (2.123)

(2). $u = -1$, if $\lambda > m^2$, (2.124)

(3). $u = +1$, if $\lambda < -m^2$, (2.125)

The solutions $\lambda(t,x)$ and $\phi(t,x)$ of case $|\lambda| \leq m^2$ in (2.123) have already been solved as in Special case 1. Let us consider when $u = -1$ and $u = +1$, we can write the solutions of (2.6) and (2.7) in the form

$$\left.\begin{aligned} \lambda(t,x) &= C(x-at)e^{bt} \\ \phi(t,x) &= -\frac{1}{b} + D(x-at)e^{-bt} \end{aligned}\right\} u = -1 , \qquad (2.126)$$

and

$$\left.\begin{aligned} \lambda(t,x) &= E(x-at)e^{bt} \\ \phi(t,x) &= \frac{1}{b} + F(x-at)e^{-bt} \end{aligned}\right\} u = +1 , \qquad (2.127)$$

where $C(\eta)$, $D(\eta)$, $E(\eta)$ and $F(\eta)$ are arbitrary functions of η.

We shall discuss how to solve the problem only for case $aT > 2L$, but other cases can be worked out in a similar way.

In subdomain S_1, (Fig. 20) since $\lambda(T,x) = 0$ on the boundary $t = T$ and λ is a continuous function, then there exists a region neighbouring to $t = T$ in which $|\lambda| \le m^2$. Hence (2.123) applies and the solutions of $\lambda(t,x)$ and $\phi(t,x)$ are defined in (2.18) and (2.19) respectively, i.e.,

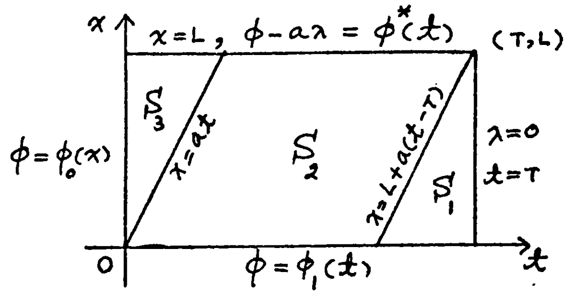


Fig. 20

$$\left.\begin{aligned} & u(t,x) = 0\,, \qquad (t,x) \in S_1 \\ \text{and} \quad & \\ & \phi(t,x) = \phi_1\!\left(t - \frac{x}{a}\right) e^{-bx/a}, \qquad (t,x) \in S_1 \end{aligned}\right\} \tag{2.128}$$

In subdomain S_2, (Fig. 21), since $u(t,x) = 0$ in S_1 and from (2.123) it follows that

$$\lambda(t,x) = 0\,, \qquad (t,x) \in S_1\,,$$

and since $\lambda(t,x)$ is continuous then $\lambda(t,x) = 0$ on the characteristic $x = L + a(t - T)$. Its neighbourhood ABCDA also satisfies $|\lambda| \le m^2$ where on the boundary AD, $|\lambda| = m^2$. The solutions $u(t,x)$ and $\phi(t,x)$ of the region ABCDA are defined in (2.26) and (2.27) respectively, where $u = -\frac{\lambda}{m^2}$, hence $\lambda(t,x)$ in this region can be deduced from (2.26) that

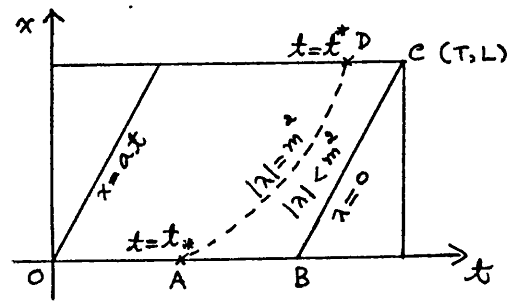


Fig. 21

$$\lambda(t,x) = \frac{e^{bx/a}\left[e^{-bL/a}\phi_1(t-\frac{x}{a}) - \phi^*(\frac{L-x+at}{a})\right]}{\left[ae^{bL/a} + \frac{1}{bm^2}\sinh\left(\frac{bL}{a}\right)\right]} \tag{2.129}$$

Since we have $\lambda(t,x) = 0$ on the characteristic $x = L+a(t-T)$, it will follow from (2.129) that $\phi_1(t)$ and $\phi^*(t)$ are related as follows:

$$\phi_1(T-\tfrac{L}{a}) = e^{bL/a}\phi^*(T),$$

hence $\phi_1(t)$ and $\phi^*(t)$ can not be arbitrarily prescribed.

Let us find a curve AD on which $\lambda(t,x) = m^2$ or $\lambda(t,x) = -m^2$. Suppose that the curve AD meets $x=0$ at $t=t_*$ and $x = L$ at $t = t^*$, it then follows from (2.129) that

$$\pm m^2 = \frac{\left[e^{-bL/a}\phi_1(t_*) - \phi^*(t_*+\frac{L}{a})\right]}{\left[ae^{bL/a} + \frac{1}{bm^2}\sinh\left(\frac{bL}{a}\right)\right]}, \tag{2.130}$$

and

$$\pm m^2 = \frac{\left[\phi_1(t^*-\frac{L}{a}) - e^{bL/a}\phi^*(t^*)\right]}{\left[ae^{bL/a} + \frac{1}{bm^2}\sinh\left(\frac{bL}{a}\right)\right]}, \tag{2.131}$$

Solving (2.130) and (2.131) for each case, i.e., when $\lambda = +m^2$ and $\lambda = -m^2$. The possible results are stated as follows:

(i). If t_* is not in the interval $(0, T-\frac{L}{a})$ and t^* not in $(\frac{L}{a}, T)$ then in the whole subdomain S_2 will satify the condition $|\lambda| < m^2$ and the control $u(t,x)$ is defined as in (2.129) where $u = -\frac{\lambda}{m^2}$.

(ii). If t_* is outside $(0, T-\frac{L}{a})$ and t^* is inside $(\frac{L}{a}, T)$, (Fig. 22) then we can find the solution $u(t,x)$ as follows:

We shall consider when $\lambda = +m^2$ on AD, for the case $\lambda = -m^2$ we can solve the problem in the same way.

In a region $OBCDAO$, it will satisfy the condition $|\lambda| < m^2$ and then the control $u = -\frac{\lambda}{m^2}$ where $\lambda(t,x)$ is defined in (2.129).

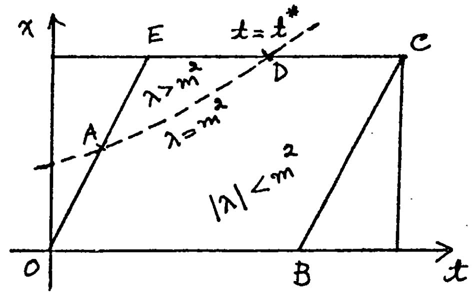


Fig. 22

Let the curve AD be $x = g(t)$. The region $ADEA$ will satisfy the condition $\lambda > m^2$, hence (2.124) applies to this case and the control will be $u = -1$. We then look for the solutions $\lambda(t,x)$ and $\phi(t,x)$ in (2.126) which satisfy the conditions $\phi - a\lambda = \phi^*(t)$ on $x = L$ and the continuity of $\phi(t,x)$ and $\lambda(t,x)$ on the curve $x = g(t)$.

To satisfy $\phi - a\lambda = \phi^*(t)$ on $x = L$, it will follow from (2.126) that in region $ADEA$, we have

$$\phi(t,x) = -\frac{1}{b} + \left[\phi^*\left(\frac{L-x+at}{a}\right) + \frac{1}{b} + aC(x-at)e^{\frac{b}{a}(L-x+at)}\right]e^{b\left(\frac{L-x}{a}\right)},$$

$$\lambda(t,x) = C(x-at)e^{bt}. \tag{2.132}$$

From the condition of the continuity of $\phi(t,x)$ and $\lambda(t,x)$ on $x = g(t)$ and by using (2.129), (2.27) and (2.132), we obtain

$$e^{bt}C\left(g(t)-at\right) = \frac{e^{bg(t)/a}\left[e^{-bL/a}\phi_1\left(t - \frac{g(t)}{a}\right) - \phi^*\left(t + \frac{L-g(t)}{a}\right)\right]}{\left[a\,e^{bL/a} + \frac{1}{bm^2}\sinh\left(\frac{bL}{a}\right)\right]}, \tag{2.133}$$

and

$$-\frac{1}{b} + \left[\phi^*\left(t+\frac{L-g(t)}{a}\right) + \frac{1}{b} + a e^{b\left(t+\frac{L-g(t)}{a}\right)} \cdot C(g(t)-at)\right] e^{b\left(\frac{L-g(t)}{a}\right)} =$$

$$= \frac{a e^{b\left(\frac{L-g(t)}{a}\right)}\phi_1\left(t-\frac{g(t)}{a}\right) + \frac{1}{bm^2}\left[\phi_1\left(t-\frac{g(t)}{a}\right)\sinh\left\{\frac{b(L-g(t))}{a}\right\} + \phi^*\left(t+\frac{L-g(t)}{a}\right)\sinh\left\{\frac{b\,g(t)}{a}\right\}\right]}{\left[a e^{bL/a} + \frac{1}{bm^2}\sinh\left(\frac{bL}{a}\right)\right]}$$

(2.134)

Solving (2.133) and (2.134) for $g(t)$ and then the function $C(\eta)$ will be evaluated from one of these two equations.

(iii). If t_* is inside $(0, T-\frac{L}{a})$ and t^* also inside $(\frac{L}{a}, T)$, (Fig. 23) we can find the solution as follows:

The region $ABCDA$ satisfies the condition $|\lambda| < m^2$, hence $u = -\frac{\lambda}{m^2}$ and $\lambda(t,x)$ is defined in (2.129).

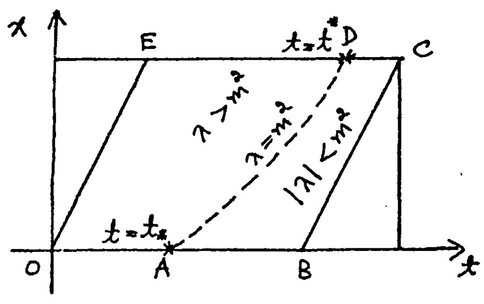


Fig. 23

Let a curve AD be $x = h(t)$ on which $\lambda = m^2$. A region $ADEOA$ will satisfy $\lambda > m^2$ and then $u = -1$ in that region. The solution $\lambda(t,x)$ and $\phi(t,x)$ will follow from (2.126) with the boundary conditions $\phi = \phi_1(t)$ on $x=0$ and $\phi - a\lambda = \phi^*(t)$ on $x = L$. Hence in region $ADEOA$ we have

$$\phi(t,x) = -\frac{1}{b} + \left[\phi_1\left(t-\frac{x}{a}\right) + \frac{1}{b}\right] e^{-bx/a},$$

and (2.135)

$$\lambda(t,x) = \frac{e^{-b\left(\frac{L-x}{a}\right)}}{a}\left[-\frac{1}{b} + \left\{\frac{1}{b} + \phi_1\left(t-\frac{x}{a}\right)\right\} e^{-bL/a} - \phi^*\left(t+\frac{L-x}{a}\right)\right]$$

The curve $x = h(t)$ can be found from the condition of continuity of $\lambda(t,x)$ and $\phi(t,x)$ on $x = h(t)$, as in (ii).

We note that in subdomain S_3 , the solution can be found by a similar method. For case $b = 0$, the curve AD will be one of the characteristics of the system.

CHAPTER 3

OPTIMUM CONTROL IN A LINEAR FIRST ORDER HEATING PROBLEM. CASE 2: $g(t,x,\phi,\phi_x,u) \equiv -a\phi_x + u\phi - c$.

We shall consider in this chapter the case in which the state equation is expressed in the form

$$\frac{\partial\phi}{\partial t} = -a\frac{\partial\phi}{\partial x} + u\phi - c, \qquad 0 \le t \le T\,; \quad 0 \le x \le L, \tag{3.1}$$

where a (>0) and c are constants and u is a control function. The initial and the boundary conditions are given as follows:

$$\left.\begin{aligned} \phi(0,x) &= \phi_0(x), \qquad t=0\,; \; 0 \le x \le L, \\ \phi(t,0) &= \phi_1(t), \qquad x=0\,; \; 0 \le t \le T, \end{aligned}\right\} \tag{3.2}$$

where $\phi_0(x)$ and $\phi_1(t)$ are prescribed and satisfy $\phi_0(0) = \phi_1(0)$.

The problem can be stated as follows:

Find the control u which minimises the performance criterion

$$I = \int_{t=0}^{t=T} \frac{1}{2}\left\{\phi(t,L) - \phi^*(t)\right\}^2 dt + \iint_S \frac{1}{2} m^2 u^2 \, dt\, dx \tag{3.3}$$

where S is a domain $0 \le t \le T$; $0 \le x \le L$, $\phi^*(t)$ is a prescribed function of t and m is a constant.

The necessary conditions for I in (3.3) to have a minimum value have been derived in Chapter 1, as in (1.17) - (1.23); in this case, we have

$$H \equiv \frac{1}{2} m^2 u^2 + \lambda(t,x)\left\{-a\phi_x + u\phi - c\right\},$$

$$p \equiv 0,$$

$$q \equiv -\tfrac{1}{2}\left\{\phi(t,L)-\phi^*(t)\right\}^2.$$

Thus it follows from (1.17) - (1.23) that

$$\frac{\partial\lambda}{\partial t}+a\frac{\partial\lambda}{\partial x} = -\lambda u\,, \qquad (t,x)\in S\,, \tag{3.4}$$

$$\frac{\partial\phi}{\partial t}+a\frac{\partial\phi}{\partial x} = u\phi - c\,, \qquad (t,x)\in S\,. \tag{3.5}$$

When $u = u(t,x)$ is a continuous control, the optimality condition is $\frac{\partial H}{\partial u}=0$ and this becomes

$$m^2 u + \lambda\phi = 0\,, \qquad (t,x)\in S\,. \tag{3.6}$$

When $u = u(t)$ is a continuous control function of t only, the optimality condition is $\int_{x=0}^{x=L}\frac{\partial H}{\partial u}\,dx = 0$, and this becomes

$$m^2 L\, u(t) = -\int_{x=0}^{x=L}\lambda(t,x)\,\phi(t,x)\,dx\,, \qquad (t,x)\in S\,. \tag{3.7}$$

When $A_1 \le u(t,x) \le A_2$, we choose the control $u = u(t,x)$ so as to minimise the Hamiltonian H , where in this problem

$$H \equiv \tfrac{1}{2}m^2u^2+\lambda\left\{-a\phi_x+u\phi-c\right\}.$$

The boundary condition on $x=L$ for all t , and on $t=T$ for all x , can be obtained from (1.23) and these can be expressed as follows:

$$\left.\begin{aligned}&\phi(t,L)-a\lambda(t,L) = \phi^*(t)\,, && x=L\,;\ 0\le t\le T\,,\\ \text{and}\qquad &\lambda(T,x) = 0\,, && t=T\,;\ 0\le x\le L\,,\end{aligned}\right\} \tag{3.8}$$

As in Chapter 2, we solve the equations (3.4) and (3.5) by using the characteristics $x = at +$ constant. By changing the independent variables t and x into new characteristic variables ξ and η defined as $\xi = t$ and $\eta = x - at$, the equations (3.4) and (3.5) become

$$\frac{\partial \lambda}{\partial \xi} = -\lambda u \quad , \tag{3.9}$$

and

$$\frac{\partial \phi}{\partial \xi} = u\phi - c \; . \tag{3.10}$$

<u>Special case 1</u>: <u>$u = u(t,x)$ is a continuous control</u>.

The optimality condition for this case is defined in (3.6) as

$$m^2 u = -\lambda \phi \tag{3.11}$$

From (3.9) and (3.10), we have

$$\frac{\partial}{\partial \xi}(\lambda \phi) = -\lambda c \tag{3.12}$$

The difficulty in this case lies in solving the partial differential equations (3.9) and (3.10) with the condition (3.11) where $u = u(t,x)$.

In order to simplify the problem, let us consider the case when the constant $c = 0$. It then follows from (3.12) that

$$\lambda \phi = A(\eta) \; ,$$

using (3.11), we have

$$u = -\frac{1}{m^2} A(\eta) \; ,$$

i.e.,

$$u(t,x) = -\frac{1}{m^2} A(x - at) \; , \tag{3.13}$$

where $A(\eta)$ is an arbitrary function of η.

Substituting $u(t,x)$ from (3.13) into (3.10) when $c=0$, we then have

$$\frac{\partial\phi}{\partial\xi} = -\frac{1}{m^2} A(\eta)\phi$$

hence $$\phi(t,x) = B(x-at)e^{-t\{A(x-at)\}/m^2} \tag{3.14}$$

and

$$\lambda(t,x) = \frac{A(x-at)}{\phi} = \frac{A(x-at)}{B(x-at)} e^{t\{A(x-at)\}/m^2} \tag{3.15}$$

where $B(\eta)$ is an arbitrary function of η which assumed to be not equal to zero.

We shall investigate the control $u(t,x)$ in each diagram which has been constructed depending on the magnitudes of the constants T, L and a (>0).

Case (i). $aT < L$.

The characteristics $x=at$ and $x=L+a(t-T)$ divide the domain S into 3 subdomains S_1, S_2 and S_3. (Fig. 12 page 23)

In subdomains S_1 and S_2, $\lambda(t,x)$ has to satisfy the boundary condition $\lambda(T,x)=0$, for all x. Thus, from (3.15), we shall have

$$A(\zeta) = 0 \qquad \text{for all } \zeta,$$

hence $$u(t,x) = 0 \quad , \quad (t,x) \in S_1 \text{ and } S_2 \tag{3.16}$$

In subdomain S_3, $\phi(t,x)$ and $\lambda(t,x)$ must satisfy the boundary conditions $\phi=\phi_0(x)$ on $t=0$ and $\phi-a\lambda=\phi^*(t)$ on $x=L$. It then follows from (3.14) that the first of these conditions leads to the result

$$\phi(t,x) = \phi_0(x-at)\,e^{-t\{A(x-at)\}/m^2}$$

and the second condition leads to the identity

$$\phi_0(L-at)\,e^{-t\{A(L-at)\}/m^2} - \frac{a\,A(L-at)}{\phi_0(L-at)}\,e^{t\{A(L-at)\}/m^2} \equiv \phi^*(t),$$

i.e.,

$$A(\eta)\,e^{\left(\frac{L-\eta}{am^2}\right)A(\eta)} \equiv \frac{1}{a}\left[\phi_0^2(\eta)\,e^{-\left(\frac{L-\eta}{am^2}\right)A(\eta)} - \phi^*\left(\frac{L-\eta}{a}\right)\phi_0(\eta)\right]. \tag{3.17}$$

Therefore, it will follow from (3.13) and (3.17) that the control $u(t,x)$ in subdomain S_3 will satisfy the transcendental equation

$$u\,e^{-\left(t+\frac{L-x}{a}\right)u} = \frac{1}{am^2}\left[\phi^*\left(t+\frac{L-x}{a}\right)\phi_0(x-at) - \phi_0^2(x-at)\,e^{\left(t+\frac{L-x}{a}\right)u}\right], \tag{3.18}$$

where $\phi^*(t)$ and $\phi_0(x)$ are prescribed functions.

Case (ii). $aT = L$.

In this case, the domain S is divided into two subdomains S_1 and S_2 ,(Fig. 13 page 27), by the characteristic $x = at$. The optimal control in S_1 and S_2 are the same as in case $aT < L$ and defined in (3.16) and (3.18) respectively.

Case (iii). $aT > L$.

There are 3 diagrams according as $L < aT < 2L$, $aT = 2L$ or $aT > 2L$,(Fig. 9 - Fig. 11 page 21), but each diagram leads to the same solution for $u(t,x)$.

In subdomain S_1 and S_3, the solution will be the same as in the csae $aT<L$ and defined in (3.16) and (3.18) respectively.

In subdomain S_2, $\phi(t,x)$ and $\lambda(t,x)$ must satisfy the boundary condition $\phi=\phi_1(t)$ on $x=0$ and $\phi-a\lambda=\phi^*(t)$ on $x=L$. Using the first condition and (3.14), we shall have

$$\phi_1(t) = B(-at)e^{-t\{A(-at)\}/m^2}$$

i.e., $$B(\eta) = \phi_1\left(-\frac{\eta}{a}\right)e^{-\left(\frac{\eta}{am^2}\right)A(\eta)} \quad , \text{ for all } \eta \; , \tag{3.19}$$

and the second condition with (3.14) and (3.15) lead to

$$B(L-at)e^{-t\{A(L-at)\}/m^2} - \frac{a\,A(L-at)}{B(L-at)}e^{t\{A(L-at)\}/m^2} = \phi^*(t),$$

i.e., $$B(\eta)e^{-\left(\frac{L-\eta}{am^2}\right)A(\eta)} - \frac{aA(\eta)}{B(\eta)}e^{\left(\frac{L-\eta}{am^2}\right)A(\eta)} = \phi^*\left(\frac{L-\eta}{a}\right) \quad , \text{ for all } \eta$$

By using (3.19), we obtain

$$\phi_1\left(-\frac{\eta}{a}\right)e^{-\frac{LA(\eta)}{am^2}} - \frac{aA(\eta)}{\phi_1\left(-\frac{\eta}{a}\right)}e^{\frac{LA(\eta)}{am^2}} \equiv \phi^*\left(\frac{L-\eta}{a}\right) \quad , \text{ for all } \eta \tag{3.20}$$

Hence, by substituting (3.13) into (3.20), we then have that the control $u(t,x)$ will satisfy the transcendental equation

$$ue^{-Lu/a} = \frac{1}{am^2}\left[\phi^*\left(t+\frac{L-x}{a}\right)\phi_1\left(t-\frac{x}{a}\right) - \phi_1^2\left(t-\frac{x}{a}\right)e^{Lu/a}\right], \tag{3.21}$$

where $\phi_1(t)$ and $\phi^*(t)$ are prescribed functions.

Special case 2: $u = u(t)$, a continuous control function of t only.

The optimality condition for a control $u = u(t)$ where u is not bounded is defined in (3.7) as

$$m^2 L u(t) = - \int_{x=0}^{x=L} \lambda(t,x)\phi(t,x)\,dx \tag{3.22}$$

As earlier since $u(t)$ is a continuous function it is always possible to express $u(t)$ in the form

$$u = u(t) = \alpha'(t) , \qquad \alpha(0) = 0 , \tag{3.23}$$

where $\alpha(t)$ and $\alpha'(t)$ are continuous functions of t. We then can write the solutions of (3.9) and (3.10) in the form

$$\lambda = A(\eta)e^{-\alpha(\xi)}$$

and $$\phi = B(\eta)e^{\alpha(\xi)} - ce^{\alpha(\xi)}\int_0^{\xi} e^{-\alpha(\tau)}d\tau ,$$

in which when we revert to the original independent variables t and x, we shall have

$$\lambda(t,x) = A(x-at)e^{-\alpha(t)}, \qquad (t,x) \in S \tag{3.24}$$

and

$$\phi(t,x) = B(x-at)e^{\alpha(t)} - ce^{\alpha(t)}\int_0^{t} e^{-\alpha(\tau)}d\tau, \quad (t,x)\in S, \tag{3.25}$$

where $A(\eta)$ and $B(\eta)$ are arbitrary functions of η and $\alpha(t)$ is a continuous function defined in (3.23).

Case (i). $aT < L$.

The domain S is divided into subdomains S_1, S_2 and S_3

as shown in Fig. 12 page 23 .

In subdomains S_1 and S_2 , $\lambda(t,x)$ must satisfy the condition on $t=T$, i.e., $\lambda(T,x)=0$. It then follows from (3.24) that

$$A(\xi) = 0 \quad , \text{ for all } \xi ,$$

hence

$$\lambda(t,x) = 0 \quad , \quad (t,x) \in S_1 \text{ and } S_2 \quad . \tag{3.26}$$

In subdomain S_3 , $\phi(t,x)$ and $\lambda(t,x)$ must satisfy the conditions $\phi=\phi_0(x)$ on $t=0$ and $\phi - a\lambda = \phi^*(t)$ on $x=L$.
By using (3.24) and (3.25), the first condition leads to the result

$$\phi(t,x) = \phi_0(x-at)e^{\alpha(t)} - ce^{\alpha(t)}\int_0^t e^{-\alpha(\tau)}d\tau , \tag{3.27}$$

where $\alpha(0)=0$, and the second condition leads to

$$\phi^*(t) = \phi_0(L-at)e^{\alpha(t)} - ce^{\alpha(t)}\int_0^t e^{-\alpha(\tau)}d\tau - aA(L-at)e^{-\alpha(t)} ,$$

i.e.,

$$A(\eta) = \frac{e^{\alpha\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_0(\eta)e^{\alpha\left(\frac{L-\eta}{a}\right)} - ce^{\alpha\left(\frac{L-\eta}{a}\right)}\int_0^{\frac{L-\eta}{a}} e^{-\alpha(\tau)}d\tau - \phi^*\left(\frac{L-\eta}{a}\right)\right] , \tag{3.28}$$

In order to simplify the problem, we assume that

$$\int_0^t e^{-\alpha(\tau)}d\tau = \gamma(t) \tag{3.29}$$

where $\gamma(t)$ is a continuous function and $\gamma(0)=0$, hence

$$\gamma'(t) = e^{-\alpha(t)} \quad ; \quad \gamma''(t) = -\alpha'(t)e^{-\alpha(t)} .$$

As earlier since $u(t)$ is a continuous function it is always possible to express $u(t)$ in the form

$$u(t) = \alpha'(t) = -\frac{\gamma''(t)}{\gamma'(t)} , \tag{3.30}$$

where $\gamma'(t) \neq 0$, with the conditions

$$\gamma(0) = 0 \quad \text{and} \quad \gamma'(0) = 1 . \tag{3.31}$$

Since $\lambda(t,x) = 0$ in S_1 and S_2 , as in (3.26), the optimality condition (3.22) can be written in the form

$$m^2 L\, u(t) = -\int_{x=L+a(t-T)}^{L} \{\lambda\phi\}_{S_3}\, dx$$

By using (3.24), (3.27), (3.29) and (3.30), we obtain

$$m^2 L\, \frac{\gamma''(t)}{\gamma'(t)} = \int_{L-aT}^{L-at} A(\eta)\phi_0(\eta)\, d\eta - c\gamma(t)\int_{L-aT}^{L-at} A(\eta)\, d\eta , \tag{3.32}$$

where $0 \leq t \leq T$.

Differentiating (3.32) with respect to the variable t , we obtain

$$m^2 L\, \frac{\gamma'''(t)}{\gamma'(t)} - m^2 L \frac{\{\gamma''(t)\}^2}{\{\gamma'(t)\}^2} = ac\,\gamma(t)\, A(L-at) - c\gamma'(t)\int_{L-aT}^{L-at} A(\eta)\, d\eta - aA(L-at)\,\phi_0(L-at). \tag{3.33}$$

Differentiating (3.33) with respect to the variable t

again, we have

$$m_L^2 \frac{\gamma^{IV}(t)}{\gamma'(t)} - 3m_L^2 \frac{\gamma'''(t)\gamma''(t)}{\{\gamma'(t)\}^2} + 2m_L^2 \frac{\{\gamma''(t)\}^3}{\{\gamma'(t)\}^3} = -c\gamma''(t)\int_{L-aT}^{L-at} A(\eta)\,d\eta -$$

$$-a^2A'(L-at)\left[c\gamma(t)-\phi_0(L-at)\right]+aA(L-at)\left[a\phi_0'(L-at)+2c\gamma'(t)\right].$$

Substituting $\int_{L-aT}^{L-at} A(\eta)\,d\eta$ from the expression in (3.33), thus the function $\gamma(t)$ will satisfy the following differential equation

$$m_L^2\gamma^{IV}(t)\{\gamma'(t)\}^2-4m_L^2\gamma'''(t)\gamma''(t)\gamma'(t)+3m_L^2\{\gamma''(t)\}^3=$$

$$=a^2\{\gamma'(t)\}^3A'(L-at)\left[\phi_0(L-at)-c\gamma(t)\right]+a\{\gamma'(t)\}^3A(L-at)\Big[a\phi_0'(L-at)+$$

$$+2c\gamma'(t)+\frac{\gamma''(t)}{\gamma'(t)}\phi_0(L-at)-\frac{c\gamma(t)\gamma''(t)}{\gamma'(t)}\Big],\qquad 0\le t\le T,$$

$$(3.34)$$

where $A(L-at)$ and $A'(L-at)$ can be obtained from (3.28) and can be expressed in terms of $\gamma(t)$, $\gamma'(t)$ and $\gamma''(t)$ as follows:

$$A(L-at)=\frac{1}{a\{\gamma'(t)\}^2}\left[\phi_0(L-at)-c\gamma(t)-\phi^*(t)\gamma'(t)\right], \qquad (3.35)$$

and

$$A'(L-at)=\frac{1}{a\{\gamma'(t)\}^2}\Big[\phi_0'(L-at)+\frac{c}{a}\gamma'(t)+\frac{1}{a}\phi^{*\prime}(t)\gamma'(t)+$$

$$+\frac{1}{a}\phi^*(t)\gamma''(t)\Big]+\frac{2\gamma''(t)}{a\gamma'(t)}A(L-at). \qquad (3.36)$$

The conditions upon $\gamma(t)$ are defined as follows:

$$\left.\begin{aligned} &\gamma(0) = 0 \quad , \\ &\gamma'(0) = 1 \quad , \\ &\gamma''(T) = 0 \quad , \\ \text{and} \quad &\gamma'''(T) = \frac{a\gamma'(T)}{m^2 L} A(L-aT)\left[c\gamma(T) - \phi_0(L-aT)\right], \\ \text{where} \quad &A(L-aT) = \frac{1}{a\{\gamma'(T)\}^2}\left[\phi_0(L-aT) - c\gamma(T) - \phi^*(T)\gamma'(T)\right]. \end{aligned}\right\} \quad (3.37)$$

We note here that the first two conditions were defined in (3.31) and the last two conditions are obtained from (3.32) and (3.33).

The optimal control $u(t)$ can be found from the assumption

$$u(t) = -\frac{\gamma''(t)}{\gamma'(t)} \; .$$

Case (ii). $aT = L$. (Fig. 13 page 27)

We shall get the same result as in case $aT < L$ since for this case, (3.32) becomes

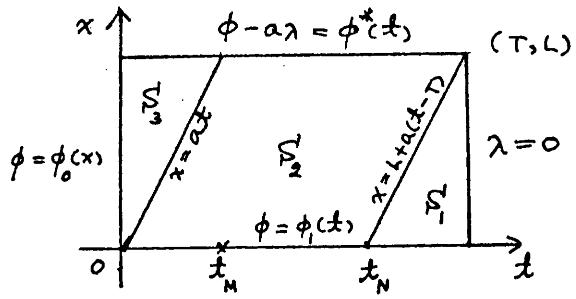
$$m^2 L \frac{\gamma''(t)}{\gamma'(t)} = \int_0^{L-at} A(\eta)\phi_0(\eta)\,d\eta - c\gamma(t)\int_0^{L-at} A(\eta)\,d\eta$$

in which the lower limits are independent of t as well, hence the further procedure will be the same.

Case (iii). $aT > L$. (Fig. 9 - Fig. 11 page 21)

We shall consider in this section only when $aT > 2L$ for which $t_M < t_N$, where $t_M = \frac{L}{a}$ and $t_N = T - \frac{L}{a}$. The solutions for the cases $aT < 2L$ and $aT = 2L$ can be found in the same way.

In subdomains S_1 and S_3 , $\lambda(t,x)$ and $\phi(t,x)$ will be the same as in (3.26), (3.27) and (3.28) and can be rewritten here as follows:



$$\lambda(t,x) = 0, \qquad (t,x) \in S_1 \tag{3.38}$$

$$\left.\begin{aligned} \phi(t,x) &= \phi_0(x-at)e^{\alpha(t)} - ce^{\alpha(t)}\int_0^t e^{-\alpha(\tau)}d\tau, \quad (t,x)\in S_3 \\ \text{and} \quad \lambda(t,x) &= A(x-at)e^{-\alpha(t)}, \qquad (t,x)\in S_3 \\ \text{where} \quad A(\eta) &= \frac{e^{\alpha\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_0(\eta)e^{\alpha\left(\frac{L-\eta}{a}\right)} - ce^{\alpha\left(\frac{L-\eta}{a}\right)}\int_0^{\frac{L-\eta}{a}} e^{-\alpha(\tau)}d\tau - \phi^*\left(\frac{L-\eta}{a}\right)\right] \end{aligned}\right\} \tag{3.39}$$

In <u>subdomain</u> S_2, $\lambda(t,x)$ and $\phi(t,x)$ must satisfy the boundary conditions $\phi = \phi_1(t)$ on $x=0$ and $\phi - a\lambda = \phi^*(t)$ on $x=L$. It then follows from (3.25) and the first condition mentioned above, that

$$B(\eta) = \phi_1\left(-\frac{\eta}{a}\right)e^{-\alpha\left(-\frac{\eta}{a}\right)} + c\int_0^{-\eta/a} e^{-\alpha(\tau)}d\tau \quad , \text{ for all } \eta, \tag{3.40}$$

and the second condition with (3.24) and (3.25) lead to the result

$$A(\eta) = \frac{e^{\alpha\left(\frac{L-\eta}{a}\right)}}{a}\left[B(\eta)e^{\alpha\left(\frac{L-\eta}{a}\right)} - ce^{\alpha\left(\frac{L-\eta}{a}\right)}\int_0^{\frac{L-\eta}{a}} e^{-\alpha(\tau)}d\tau - \phi^*\left(\frac{L-\eta}{a}\right)\right], \tag{3.41}$$

where $B(\eta)$ is defined in (3.40).

For the case $c \neq 0$, we shall assume a new function $\gamma(t)$

as in (3.29) and proceed in the same way as in case $aT < L$, for each time interval $0 \le t < t_M$, $t_M < t < t_N$ and $t_N < t \le T$.

In this section we shall discuss the case $c = 0$ in which (3.38) - (3.41) can be rewritten in the form

$$\left.\begin{aligned} \lambda(t,x) &= 0, & (t,x) &\in S_1 \\ \lambda(t,x) &= A(x-at)e^{-\alpha(t)}, & (t,x) &\in S_2 \text{ and } S_3 \\ \phi(t,x) &= B(x-at)e^{\alpha(t)}, & (t,x) &\in S_2 \text{ and } S_3 \end{aligned}\right\} \quad (3.42)$$

where

$$\left.\begin{aligned} A(\eta) &= \frac{e^{\alpha\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_1\left(-\tfrac{\eta}{a}\right)e^{\alpha\left(\frac{L-\eta}{a}\right)-\alpha\left(-\frac{\eta}{a}\right)} - \phi^*\left(\tfrac{L-\eta}{a}\right)\right], & &\text{in } S_2 \\ B(\eta) &= \phi_1\left(-\tfrac{\eta}{a}\right)e^{-\alpha\left(-\frac{\eta}{a}\right)}, & &\text{in } S_2 \end{aligned}\right\} \quad (3.43)$$

and

$$\left.\begin{aligned} A(\eta) &= \frac{e^{\alpha\left(\frac{L-\eta}{a}\right)}}{a}\left[\phi_0(\eta)e^{\alpha\left(\frac{L-\eta}{a}\right)} - \phi^*\left(\tfrac{L-\eta}{a}\right)\right], & &\text{in } S_3 \\ B(\eta) &= \phi_0(\eta), & &\text{in } S_3 \end{aligned}\right\} \quad (3.44)$$

<u>When $0 < t < t_M$</u>, where $t_M = \frac{L}{a}$, it follows from the optimality condition (3.22) that

$$m^2 L\, u(t) = -\int_{x=0}^{x=at} \{\lambda\phi\}_{S_2}\, dx - \int_{x=at}^{x=L} \{\lambda\phi\}_{S_3}\, dx$$

By using (3.42) and the assumption $u(t) = \alpha'(t)$; $\alpha(0) = 0$, we obtain

$$m^2 L\,\alpha'(t) = -\int_{-at}^{0}\{A(\eta)\,B(\eta)\}_{S_2}\,d\eta - \int_{0}^{L-at}\{A(\eta)\,B(\eta)\}_{S_3}\,d\eta\,. \tag{3.45}$$

Differentiating (3.45) with respect to the variable t and using (3.43) and (3.44), we obtain that $\alpha(t)$ will satisfy the following equation:

$$m^2 L\,\alpha''(t) + \phi_1^2(t)\,e^{2[\alpha(t+\frac{L}{a})-\alpha(t)]} - \phi_0^2(L-at)\,e^{2\alpha(t)} + \phi_0(L-at)\,\phi^*(t)\,e^{\alpha(t)} -$$

$$- \phi_1(t)\,\phi^*(t+\tfrac{L}{a})\,e^{[\alpha(t+\frac{L}{a})-\alpha(t)]} = 0\,, \qquad 0 < t < t_M\,, \tag{3.46}$$

When $t_M < t < t_N$, where $t_M = \frac{L}{a}$, $t_N = T - \frac{L}{a}$, the condition (3.22) implies that

$$m^2 L\,u(t) = -\int_{x=0}^{x=L}\{\lambda\phi\}_{S_2}\,dx\,,$$

or
$$m^2 L\,\alpha'(t) = -\int_{-at}^{L-at}\{A(\eta)\,B(\eta)\}_{S_2}\,d\eta \tag{3.47}$$

After differentiating (3.47) with respect to the variable t and using (3.43), we then obtain the differential equation of the form

$$m^2 L\,\alpha''(t) + \phi_1^2(t)\,e^{2[\alpha(t+\frac{L}{a})-\alpha(t)]} - \phi_1^2(t-\tfrac{L}{a})\,e^{2[\alpha(t)-\alpha(t-\frac{L}{a})]} -$$

$$- \phi_1(t)\,\phi^*(t+\tfrac{L}{a})\,e^{[\alpha(t+\frac{L}{a})-\alpha(t)]} + \phi^*(t)\,\phi_1(t-\tfrac{L}{a})\,e^{[\alpha(t)-\alpha(t-\frac{L}{a})]} = 0\,, \tag{3.48}$$

where $t_M < t < t_N$.

Similarly, when $t_N < t < T$, since $\lambda = 0$ in S_1 the optimality condition (3.22) will be

$$m^2 L \alpha'(t) = - \int_{L-aT}^{L-at} \{ A(\eta)\, B(\eta) \}_{S_2} \, d\eta \, , \tag{3.49}$$

where we used (3.42) and the assumption $u(t) = \alpha'(t)$; $\alpha(0) = 0$.

Differentiating (3.49) with respect to the variable t and using (3.43), we obtain

$$m^2 L \, \alpha''(t) - \phi_1^2(t - \tfrac{L}{a}) e^{2[\alpha(t) - \alpha(t - \frac{L}{a})]} + \phi^*(t) \phi_1(t - \tfrac{L}{a}) e^{[\alpha(t) - \alpha(t - \frac{L}{a})]} = 0 , \tag{3.50}$$

where $t_N < t < T$.

The end-point conditions upon $\alpha(t)$ are defined as follows:

$\alpha(0) = 0$,

$\alpha(t)$ and $\alpha'(t)$ are continuous at $t = t_M$ and $t = t_N$,

and $\alpha'(T) = 0$.

We note that the above conditions concerning $\alpha'(t)$ are derived from (3.45), (3.47) and (3.49).

The equations (3.46), (3.48) and (3.50) have to be solved subject to the above conditions upon $\alpha(t)$. When $\alpha(t)$ in each time interval has been found then the corresponding optimal control $u = u(t)$ will be calculated from $u(t) = \alpha'(t)$.

Special case 3: $u = u(t,x)$, a piecewise continuous control satisfying $-1 \leq u(t,x) \leq 1$.

To minimise the functional I in (3.3) subject to the constraint (3.1) and $-1 \leq u(t,x) \leq 1$, the control $u(t,x)$ is chosen so as to minimise the Hamiltonian H where

$$H = \frac{1}{2} m^2 u^2 + \lambda(t,x)\{-a\phi_x + u\phi - c\}.$$

After rearranging, we can write H in the form

$$H = \frac{1}{2}\left(mu + \frac{\lambda\phi}{m}\right)^2 - \frac{\lambda^2\phi^2}{2m^2} - \lambda a\phi_x - \lambda c.$$

Hence u must be chosen so that

(1). $m^2u + \lambda\phi = 0$, when $|\lambda\phi| \leq m^2$ (3.51)

(2). $u = -1$, when $\lambda\phi > m^2$ (3.52)

(3). $u = +1$, when $\lambda\phi < -m^2$ (3.53)

We shall consider here the case when the constant $c = 0$.

When $|\lambda\phi| \leq m^2$, u is chosen as in (3.51), i.e., $m^2u + \lambda\phi = 0$ which is the same as in Special case 1. Hence we can write $\lambda(t,x)$ and $\phi(t,x)$ in the following forms, as in (3.15) and (3.14) respectively i.e.,

$$\lambda(t,x) = \frac{A(x-at)}{B(x-at)} e^{t\{A(x-at)\}/m^2} \tag{3.54}$$

$$\phi(t,x) = B(x-at) e^{-t\{A(x-at)\}/m^2} \tag{3.55}$$

and

$$u(t,x) = -\frac{1}{m^2} A(x-at) \tag{3.56}$$

where $A(\eta)$ and $B(\eta)$ are arbitrary functions.

Next, consider when $u = +1$ and $u = -1$, the solutions of (3.9) and (3.10) can be written in the form

$$\left.\begin{aligned} \lambda(t,x) &= E(x-at) e^{-t} \\ \phi(t,x) &= G(x-at) e^{t} \end{aligned}\right\} u = +1 , \tag{3.57}$$

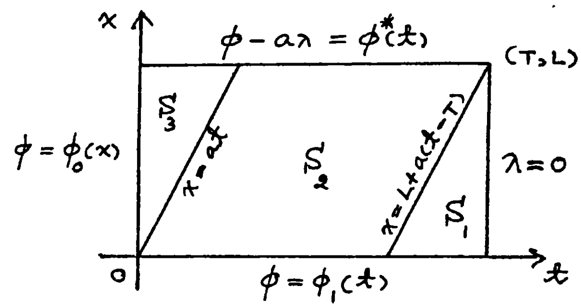
and

$$\left.\begin{aligned} \lambda(t,x) &= H(x-at)e^{t} \\ \phi(t,x) &= K(x-at)e^{-t} \end{aligned}\right\} u=-1 . \qquad (3.58)$$

where $E(\eta)$, $G(\eta)$, $H(\eta)$ and $K(\eta)$ are arbitrary functions of η .

We shall discuss how to solve the problem only for the case $aT>2L$ (see Fig.20 page 55), since other cases can be done by the same method.

First let us consider the subdomain S_1 ; since $\lambda=0$ on the boundary $t=T$ it follows that $\lambda\phi$ will be zero on $t=T$. Hence in some neighbouring region of $t=T$, $|\lambda\phi| \leq m^2$ will be satisfied. Therefore we can commence by looking for the solutions of the form (3.54)-(3.56).



To satisfy the boundary condition $\lambda(T,x)=0$,for all x , we shall have $A(\xi)=0$, for all ξ and then it follows from (3.54) that

$$\lambda(t,x) = 0 , \qquad (t,x) \in S_1$$

and since $u=-\frac{\lambda\phi}{m^2}$ or from (3.56), we have

$$u(t,x) = 0 , \qquad (t,x) \in S_1 \qquad (3.59)$$

The function $\phi(t,x)$ will follow from (3.55) together with the boundary condition on $x=0$, but we shall be interested only in the control function.

In subdomain S_2 , since the Lagrange multiplier $\lambda(t,x)$ is a continuous function and $\lambda(t,x)=0$ in the subdomain S_1 , then

$\lambda = 0$ on the characteristic $x = L + a(t - T)$ which divides the subdomain S_1' from S_2'. Therefore, there exists a region in the neighbourhood of the line $x = L + a(t - T)$ which satisfies $|\lambda\phi| \leq m^2$ and thus we shall look for the solutions of the form (3.54) - (3.56).

Since we have

$$\lambda\phi = A(x - at)$$

and if we introduce ξ and η defined as $\xi = t$, $\eta = x - at$, then

$$\frac{\partial}{\partial\xi}(\lambda\phi) = 0 ,$$

i.e., $\lambda\phi$ = constant along the characteristics η = constant. Hence the boundary of the region which satisfies $|\lambda\phi| \leq m^2$ will be the characteristic line on which $\lambda\phi = +m^2$ or $\lambda\phi = -m^2$.

By using (3.54), (3.55) and the boundary conditions on $x = 0$ and $x = L$, we then have the result similar to (3.20) that $A(\zeta)$ in S_2' must satisfy

$$A(\zeta)\, e^{\frac{LA(\zeta)}{am^2}} = \frac{1}{a}\left[\phi_1^2\left(-\frac{\zeta}{a}\right) e^{-\frac{LA(\zeta)}{am^2}} - \phi^*\left(\frac{L-\zeta}{a}\right)\phi_1\left(-\frac{\zeta}{a}\right)\right] , \qquad \text{for all } \zeta . \tag{3.60}$$

In this case the characteristics $\eta = 0$ and $\eta = L - aT$ are the boundaries of S_2' and since $aT > L$ the domain S_2' is characterised by $-(aT - L) \leq \eta \leq 0$. On the characteristic $\eta = L - aT$ we have $\lambda(t, x) = 0$ and it implies that $A(\eta) = 0$ there, hence $\phi_1(t)$ and $\phi^*(t)$ are related so that

$$\phi_1\left(T - \frac{L}{a}\right) = \phi^*(T) .$$

We now test whether there is any characteristic $\eta = -k_1$ where $k_1 > 0$ which satisfies either of the following equations:

$$m^2 e^{L/a} = \frac{1}{a}\left[\phi_1^2\left(\frac{k_1}{a}\right)e^{-L/a} - \phi^*\left(\frac{L+k_1}{a}\right)\phi_1\left(\frac{k_1}{a}\right)\right], \quad k_1 > 0 \tag{3.61}$$

or

$$-m^2 e^{-L/a} = \frac{1}{a}\left[\phi_1^2\left(\frac{k_1}{a}\right)e^{L/a} - \phi^*\left(\frac{L+k_1}{a}\right)\phi_1\left(\frac{k_1}{a}\right)\right], \quad k_1 > 0 \tag{3.62}$$

If there is no such value of k_1 , then the whole subdomain S_2 must satisfy the condition $|\lambda\phi| \leq m^2$ and the optimal control $u(t,x)$ for that domain will follow from (3.51) that $u = -\frac{\lambda\phi}{m^2}$ which leads to the same result as in (3.21).

If there exists one value of k_1 satisfying (3.61) when $-(aT-L) < \eta < 0$, then $u = -1$ on that characteristic line $\eta = -k_1$ and then the region between $\eta = L-aT$ and $\eta = -k_1$ must have the control $u = -\frac{\lambda\phi}{m^2}$ and defined as in (3.21). The neighbouring region on the left hand side of the line $\eta = -k_1$ will satisfy $\lambda\phi > m^2$. (Fig. 24)

There now exists a neighbourhood $-k_1 < \eta < -k_1+\varepsilon$ in which $\lambda\phi > m^2$ and $u = -1$. Wwithin this region, to satisfy the conditions on $x = 0$ and on $x = L$ and by using (3.58), we obtain

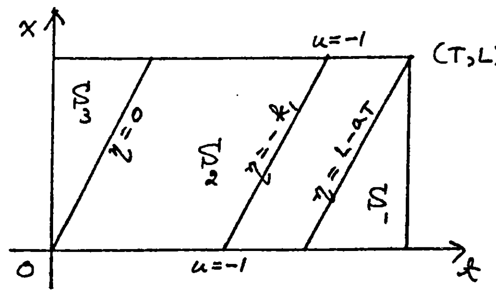


Fig. 24

and

$$\left.\begin{aligned} \phi(t,x) &= \phi_1\left(t-\frac{x}{a}\right)e^{-x/a} \\ \lambda(t,x) &= \frac{e^{-\left(\frac{L-x}{a}\right)}}{a}\left[\phi_1\left(t-\frac{x}{a}\right)e^{-L/a} - \phi^*\left(t+\frac{L-x}{a}\right)\right] \end{aligned}\right\} \tag{3.63}$$

Hence

$$\lambda\phi = \frac{1}{a}\left[\phi_1^2(t-\tfrac{x}{a})e^{-2L/a} - e^{-L/a}\phi^*(t+\tfrac{L-x}{a})\phi_1(t-\tfrac{x}{a})\right]. \tag{3.64}$$

It is clear that $\lambda\phi$ = constant along the characteristics η = constant where η is defined as $\eta = x - at$.

Next we test whether there is any characteristic $\eta = -k_2$, where $k_1 > k_2 > 0$, which satisfies (3.64) when $\lambda\phi = m^2$. i.e.,

$$m^2 = \frac{1}{a}\left[\phi_1^2\left(\frac{k_2}{a}\right)e^{-2L/a} - e^{-L/a}\phi^*\left(\frac{L+k_2}{a}\right)\phi_1\left(\frac{k_2}{a}\right)\right], \quad k_1 > k_2 > 0 \tag{3.65}$$

We note that (3.65) is the same as (3.61), thus if there exists one value of k_2 which satisfies (3.65) it means that there were two values of k_1 satisfying (3.61).

If there is no such value of k_2, then the rest of the subdomain S_2 on the left of the line $\eta = -k_1$ will satisfy $\lambda\phi > m^2$ and the control $u = -1$. If there exists $\eta = -k_2$ where k_2 satisfying (3.65), we shall consider the neighbouring region on the left hand side of $\eta = -k_2$ to satisfy $|\lambda\phi| \le m^2$ and so on.

Similarly, we can deal with the case when k_1 satisfies (3.62) by the same technique but instead of $\lambda\phi > m^2$; $u = -1$ we replace it with $\lambda\phi < -m^2$; $u = +1$ and commence by looking for the solutions for $\lambda(t,x)$ and $\phi(t,x)$ of the form (3.57) with the boundary conditions on $x = 0$ and $x = L$.

In subdomain S_3, we can handle it by the same method.

CHAPTER 4

OPTIMUM CONTROL IN A LINEAR FIRST ORDER HEATING PROBLEM. CASE 3: $g(t,x,\phi,\phi_x,u) = -u(t)\phi_x - b\phi - c$.

We now discuss a problem in which the system is governed by a linear partial differential equation of the form

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} - b\phi - c, \qquad 0 \le t \le T \ ; \ 0 \le x \le L \tag{4.1}$$

where b and c (<0) are constants and u is a control function.

The initial and boundary conditions are given as follows:

$$\left.\begin{aligned} \phi(0,x) &= \phi_0(x) , \qquad 0 \le x \le L , \\ \phi(t,0) &= \phi_1(t) , \qquad 0 \le t \le T , \end{aligned}\right\} \tag{4.2}$$

where $\phi_0(x)$ and $\phi_1(t)$ are prescribed and satisfying $\phi_0(0) = \phi_1(0)$.

As in Chapter 2 and Chapter 3, we want to find a control u which minimises the functional I defined by

$$I = \int_{t=0}^{t=T} \tfrac{1}{2}\left\{\phi(t,L) - \phi^*(t)\right\}^2 dt + \iint_S \tfrac{1}{2} m^2 u^2 \, dt\, dx , \tag{4.3}$$

where m is a constant, $\phi^*(t)$ is a prescribed function and S is a domain $0 \le t \le T$; $0 \le x \le L$.

Referring to the general theory in Chapter 1 and by comparing (4.3) with the form of I in (1.16), namely

$$I = \iint_S F(t,x,\phi,\phi_x,u)\, dt\, dx + \oint_{\Gamma_2} \left[p(t,x,\phi)\, dx + q(t,x,\phi)\, dt\right]$$

we have

$$H \equiv \tfrac{1}{2} m^2 u^2 + \lambda(t,x)\{-u\phi_x - b\phi - e\} ,$$
$$p \equiv 0 ,$$
$$q \equiv -\tfrac{1}{2}\{\phi(t,L) - \phi^*(t)\}^2 .$$

The conditions for I in (4.3) to have a minimum value will follow from (1.17) - (1.23) which can be written down as follows:

$$\frac{\partial \lambda}{\partial t} + u\frac{\partial \lambda}{\partial x} = b\lambda - \lambda\frac{\partial u}{\partial x} , \qquad (t,x) \in S, \qquad (4.4)$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = -b\phi - e , \qquad (t,x) \in S , \qquad \text{as in (4.1).}$$

When $u = u(t,x)$ is a continuous control, the optimality condition is $\frac{\partial H}{\partial u} = 0$ and this becomes

$$m^2 u - \lambda\phi_x = 0 , \qquad (t,x) \in S . \qquad (4.5)$$

When $u = u(t)$, a continuous control function of t only, the optimality condition is $\int_{x=0}^{x=L} \frac{\partial H}{\partial u}\, dx = 0$ and this leads to

$$m^2 L\, u(t) = \int_{x=0}^{x=L} \lambda\phi_x\, dx , \qquad (t,x) \in S . \qquad (4.6)$$

When $u = u(t,x)$ where $A_1 \leq u(t,x) \leq A_2$, a control u is chosen so that the Hamiltonian H has a minimum value.

The boundary conditions on $t = T$ and on $x = L$ follow from (1.23) that

$$\lambda(T,x) = 0 , \qquad 0 \leq x \leq L ; \qquad (4.7)$$

and

$$\phi(t,L) - \lambda(t,L)u = \phi^*(t) , \quad 0 \leq t \leq T . \qquad (4.8)$$

In this chapter we shall restrict attention to the case in which $u = u(t)$ is a continuous control, since the case $u = u(t,x)$ is probably unrealisable in practice.

When $u = u(t)$, the equation (4.4) becomes

$$\frac{\partial \lambda}{\partial t} + u(t)\frac{\partial \lambda}{\partial x} = b\lambda \quad , \qquad (t,x) \in S \, , \tag{4.9}$$

and the state equation (4.1) will be

$$\frac{\partial \phi}{\partial t} + u(t)\frac{\partial \phi}{\partial x} = -b\phi - c \, , \qquad (t,x) \in S \; . \tag{4.10}$$

The characteristics of (4.9) and (4.10) are the same and defined as the integral curves of the differential equation

$$\frac{dt}{1} = \frac{dx}{u(t)}$$

As earlier since $u(t)$ is a continuous function it is possible to express $u(t)$ in the form

$$u(t) = \alpha'(t) \quad , \qquad \alpha(0) = 0 \, , \tag{4.11}$$

where $\alpha(t)$ is a smooth continuous curve for all t in $0 \leqslant t \leqslant T$. Hence the characteristics can be written in the form

$$x = \alpha(t) + \text{constant}$$

We introduce two new variables ξ and η related to t and x as follows: -

$$\xi = t \quad ; \quad \eta = x - \alpha(t) \; . \tag{4.12}$$

By using (4.12), it is easy to verify that the equations (4.9) and (4.10) can be written in the following forms:

$$\left.\begin{aligned} \frac{\partial \lambda}{\partial \xi} &= b\lambda \ , \\ \frac{\partial \phi}{\partial \xi} &= -b\phi - c \ . \end{aligned}\right\} \qquad (4.13)$$

The general solutions of (4.13) will be

$$\lambda = A(\eta)\, e^{b\xi}$$

$$\phi = -\frac{c}{b} + B(\eta)\, e^{-b\xi}$$

which after reverting to the original independent variables t and x , we shall have

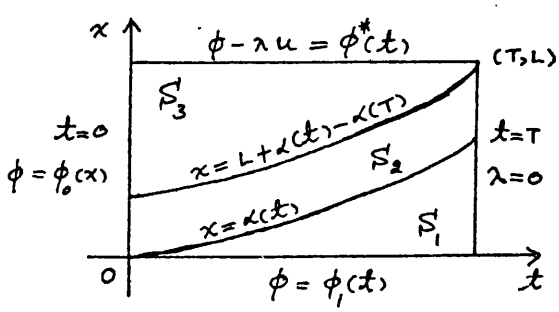
$$\left.\begin{aligned} \lambda(t,x) &= A(x-\alpha(t))\, e^{bt} \\ \phi(t,x) &= -\frac{c}{b} + B(x-\alpha(t))\, e^{-bt} \end{aligned}\right\} \qquad (4.14)$$

where $A(\eta)$ and $B(\eta)$ are arbitrary functions of η .

We shall investigate the optimum control $u(t)$ of this problem in four cases, depending on the values of $\alpha(T)$ and L .

<u>Case 1: when $\alpha(T) < L$</u> .

The characteristics $x = \alpha(t)$ and $x = L + \alpha(t) - \alpha(T)$ will divide the domain S into 3 subdomains S_1 , S_2 and S_3 as shown in a diagram. (Fig. 25)



<u>Fig. 25</u>

We shall find the solutions for ϕ and λ or $A(\eta)$ and $B(\eta)$ in each subdomain.

In <u>subdomains S_1 and S_2</u> , on $t = T$, $\lambda(t,x)$ must satisfy the condition $\lambda(T,x) = 0$ hence it follows from (4.14) that

$$A(x-\alpha(T)) \equiv 0 \quad , \quad \text{for all } x \, ,$$

or

$$A(\xi) \equiv 0 \quad , \quad \text{for all } \xi \, ,$$

$$\therefore \qquad \lambda(t,x) \equiv 0 \quad , \quad (t,x) \in S_1 \text{ and } S_2 . \tag{4.15}$$

The function $\phi(t,x)$ in S_1 and S_2 can be found by using the conditions $\phi = \phi_1(t)$ on $x=0$ and $\phi = \phi_0(x)$ on $t=0$, respectively, to evaluate the arbitrary function $B(\eta)$.

In subdomain S_3, $\phi(t,x)$ must satisfy the condition $\phi = \phi_0(x)$ on $t=0$ hence from (4.14) and since $\alpha(0)=0$ we have

$$B(x) \equiv \phi_0(x) + \frac{c}{b} \qquad , \quad \text{for all } x$$

thus

$$\phi(t,x) = \frac{c}{b}\left(e^{-bt} - 1\right) + \phi_0(x-\alpha(t))e^{-bt} . \tag{4.16}$$

On $x=L$ we must satisfy the condition $\phi - \lambda u = \phi^*(t)$ and by using (4.14) and (4.16), we obtain

$$A\left(L-\alpha(t)\right) \equiv \frac{e^{-bt}}{\alpha'(t)}\left[\phi_0(L-\alpha(t))e^{-bt} - \phi^*(t) + \frac{c}{b}\left(e^{-bt} - 1\right)\right] \qquad , \text{ all } t \tag{4.17}$$

The optimality condition for the case when $u = u(t)$ is defined in (4.6), and by using (4.11), this becomes

$$m^2 L \alpha'(t) = \int_{x=0}^{x=L} \lambda \phi_x \, dx \qquad , \qquad (t,x) \in S \tag{4.18}$$

where $\alpha(0) = 0$.

Since $\lambda(t,x) = 0$ in subdomains S_1 and S_2, as in (4.15), and by using (4.14) and (4.16) we can write (4.18) in the form

$$m^2 L \alpha'(t) = \int_{x=L+\alpha(t)-\alpha(T)}^{x=L} \left\{A(x-\alpha(t))\phi_0'(x-\alpha(t))\right\}_{S_3} dx \quad , \quad 0 \le t \le T . \tag{4.19}$$

In the integral, we let $x-\alpha(t) = L-\alpha(\tau)$ $\therefore$ $dx = -\alpha'(\tau)d\tau$ when $x = L+\alpha(t)-\alpha(T)$ we have $\tau = T$ and when $x = L$ we have $\tau = t$, hence (4.19) becomes

$$m^2 L\,\alpha'(t) = \int_t^T \left\{A(L-\alpha(\tau))\,\phi_0'(L-\alpha(\tau))\right\}_{S_3} \alpha'(\tau)\,d\tau\ , \tag{4.20}$$

and using (4.17), we can write (4.20) in the form

$$m^2 L\,\alpha'(t) = \int_t^T \left[e^{-b\tau}\phi_0'(L-\alpha(\tau))\left\{\phi_0(L-\alpha(\tau))e^{-b\tau} - \phi^*(\tau) - \frac{c}{b}(1-e^{-b\tau})\right\}\right]d\tau \tag{4.21}$$

By differentiating (4.21) with respect to t , we obtain the second order non-linear differential equation of the form

$$m^2 L\,\alpha''(t) + e^{-2bt}\phi_0(L-\alpha(t))\,\phi_0'(L-\alpha(t)) - e^{-bt}\phi^*(t)\,\phi_0'(L-\alpha(t)) - e^{-bt}\cdot\frac{c}{b}(1-e^{-bt})\,\phi_0'(L-\alpha(t)) = 0\ , \qquad 0 \leq t \leq T. \tag{4.22}$$

The end-point conditions upon $\alpha(t)$ follow from (4.11)and (4.19) that

$$\alpha(0) = 0 \qquad \text{and} \qquad \alpha'(T) = 0\ . \tag{4.23}$$

The optimum control $u(t)$ will be known at once when $\alpha(t)$ is solved from (4.22) and (4.23), since $u(t) = \alpha'(t)$.

The existence and uniqueness of $\alpha(t)$ are difficult to establish since (4.22) is a non-linear differential equation. It is possible that no solution exists for $\alpha(t)$ and this is the case when no smooth control exists but this remains an unsolved problem.

Case 2: when $\alpha(T) = L$.

In this case, the characteristic $x = \alpha(t)$ will divide the domain S into subdomains S_1 and S_2 as shown in Fig. 26

As in the previous case, we can show that

$$\lambda(t,x) = 0 \,, \quad (t,x) \in S_1$$

and $\lambda(t,x) = A(x-\alpha(t))\, e^{-bt}$, in S_2

where $A(\eta)$ is defined in (4.17) for $\eta = L-\alpha(t)$.

$\phi(t,x)$ in S_2 is also the same as in (4.16).

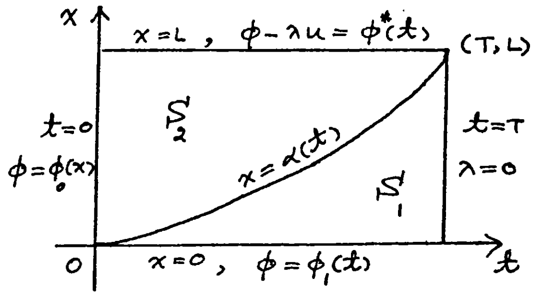


Fig. 26

Hence, the optimality condition (4.18) for this case can be written in the form

$$m^2 L\, \alpha'(t) = \int_{x=\alpha(t)}^{x=L} \left\{ A(x-\alpha(t))\, \phi_0'(x-\alpha(t)) \right\}_{S_2} dx \,, \qquad 0 \le t \le T \qquad (4.24)$$

In the integral, we put $x-\alpha(t) = X$; $dx = dX$, when $x = \alpha(t)$ we have $X = 0$ and when $x = L$ we have $X = L-\alpha(t)$ hence (4.24) becomes

$$m^2 L\, \alpha'(t) = \int_0^{L-\alpha(t)} \left\{ A(X)\, \phi_0'(X) \right\}_{S_2} dX \,, \qquad 0 \le t \le T. \qquad (4.25)$$

Differentiating (4.25) with respect to the variable t , we obtain

$$m^2 L\, \alpha''(t) = -\alpha'(t)\, \phi_0'(L-\alpha(t)) \left\{ A(L-\alpha(t)) \right\}_{S_2}$$

After substituting $A(L-\alpha(t))$ from (4.17) we obtain the second order non-linear differential equation

$$m^2 L\,\alpha''(t) + e^{-2bt}\phi_0(L-\alpha(t))\,\phi_0'(L-\alpha(t)) - e^{-bt}\phi^*(t)\,\phi_0'(L-\alpha(t)) -$$

$$- e^{-bt}\cdot\frac{c}{b}\left(1-e^{-bt}\right)\phi_0'(L-\alpha(t)) = 0, \qquad 0 \le t \le T. \tag{4.26}$$

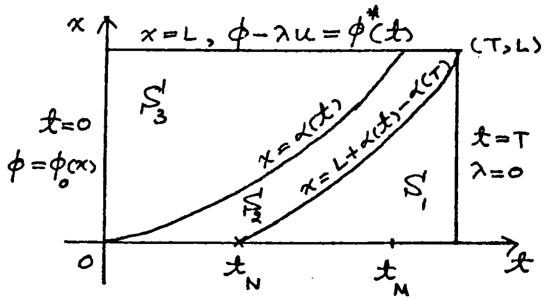
which is the same as in (4.22) as we expected. The conditions upon $\alpha(t)$ are $\alpha(0)=0$ and $\alpha'(T)=0$.

<u>Case 3: when $\alpha(T) > L$ and $t_N \le t_M$ where $\alpha(t_N) = \alpha(T) - L$,</u>

<u>and $\alpha(t_M) = L$</u>.

The domain S is divided into 3 subdomains S_1, S_2 and S_3 by two characteristics $x = \alpha(t)$ and $x = L + \alpha(t) - \alpha(T)$, as shown in Fig. 27.

The solutions for $\lambda(t,x)$ and $\phi(t,x)$ in <u>subdomains</u> S_1 <u>and</u> S_3 are the same as in case 1: when $\alpha(T) < L$. These are defined in (4.14) - (4.17).



<u>Fig. 27</u>

In <u>subdomain</u> S_2, $\phi(t,x)$ and $\lambda(t,x)$ in (4.14) have to satisfy the conditions $\phi = \phi_1(t)$ on $x=0$ and $\phi - \lambda u = \phi^*(t)$ on $x = L$. The first condition leads to

$$B(-\alpha(t)) \equiv e^{bt}\left\{\phi_1(t) + \frac{c}{b}\right\}, \tag{4.27}$$

and we obtain from the second condition that

$$A(L-\alpha(t)) \equiv \frac{e^{-bt}}{\alpha'(t)}\left[B(L-\alpha(t))e^{-bt} - \phi^*(t) - \frac{c}{b}\right]. \tag{4.28}$$

The optimality condition is defined in (4.18) as

$$m^2 L\,\alpha'(t) = \int_{x=0}^{x=L} \lambda \phi_x \, dx \quad , \qquad (t,x) \in S'$$

where $u(t) = \alpha'(t)$ and $\alpha(0) = 0$.

<u>When $0 \le t < t_N$ where $\alpha(t_N) = \alpha(T) - L$</u>, the optimality condition can be written in the form

$$m^2 L\,\alpha'(t) = \int_{x=0}^{x=\alpha(t)} \{\lambda \phi_x\}_{S_2} dx + \int_{x=\alpha(t)}^{x=L} \{\lambda \phi_x\}_{S_3} dx \, ,$$

and by using (4.14) and (4.16), we have

$$m^2 L\,\alpha'(t) = \int_{-\alpha(t)}^{0} \{A(\eta) B'(\eta)\}_{S_2} d\eta + \int_{0}^{L-\alpha(t)} \{A(\eta) \phi_0'(\eta)\}_{S_3} d\eta \, . \tag{4.29}$$

Differentiating (4.29) with respect to the variable t , we obtain

$$m^2 L\,\alpha''(t) = \alpha'(t)\{A(-\alpha(t)) B'(-\alpha(t))\}_{S_2} - \alpha'(t)\{A(L-\alpha(t)) \phi_0'(L-\alpha(t))\}_{S_3} . \tag{4.30}$$

But since $B'(-\alpha(t)) = -\frac{1}{\alpha'(t)} \frac{d}{dt}\{B(-\alpha(t))\}$ and by using (4.17) and (4.27) we then obtain that $\alpha(t)$, $0 \le t < t_N$ must satisfy the following second order non-linear differential equation

$$m^2 L\alpha''(t) + \{A(-\alpha(t))\}_{S_2} \frac{d}{dt}\{\phi_1(t)e^{bt} + \frac{c}{b}e^{bt}\} + e^{-2bt}\phi_0(L-\alpha(t))\phi_0'(L-\alpha(t)) -$$

$$- e^{-bt}\phi_0'(L-\alpha(t))\{\phi^*(t) + \frac{c}{b}(1-e^{-bt})\} = 0 \ , \quad 0 \le t < t_N \ ,$$

(4.31)

where $\{A(L-\alpha(t))\}_{S_2}$ and $\{B(-\alpha(t))\}_{S_2}$ are defined in (4.28) and (4.27) as

$$A(L-\alpha(t)) \equiv \frac{e^{-bt}}{\alpha'(t)}\left[B(L-\alpha(t))e^{-bt} - \phi^*(t) - \frac{c}{b}\right] \qquad \text{in } S_2$$

and $\quad B(-\alpha(t)) \equiv \phi_1(t)e^{bt} + \frac{c}{b}e^{bt} \qquad$ in S_2.

When $t_N < t < t_M$ where $\alpha(t_N) = \alpha(T) - L$ and $\alpha(t_M) = L$, since from (4.15) we have $\lambda(t,x) = 0$ in S_1 and by using (4.14) and (4.16) the optimality condition (4.18) can be written in the form

$$m^2 L\alpha'(t) = \int_{x=L+\alpha(t)-\alpha(T)}^{x=\alpha(t)} \{A(x-\alpha(t))B'(x-\alpha(t))\}_{S_2} dx + \int_{x=\alpha(t)}^{x=L} \{A(x-\alpha(t))\phi_0'(x-\alpha(t))\}_{S_3} dx$$

or

$$m^2 L\alpha'(t) = \int_{L-\alpha(T)}^{0} \{A(\eta)B'(\eta)\}_{S_2} d\eta + \int_{0}^{L-\alpha(t)} \{A(\eta)\phi_0'(\eta)\}_{S_3} d\eta. \qquad (4.32)$$

After differentiating (4.32) with respect to the variable t and using (4.17) we then obtain the non-linear differential equation of second order satisfied by $\alpha(t)$, $t_N < t < t_M$ as

$$m^2 L\,\alpha''(t) + e^{-2bt}\phi_0(L-\alpha(t))\,\phi_0'(L-\alpha(t)) - e^{-bt}\phi_0'(L-\alpha(t))\left\{\phi^*(t)+\frac{c}{b}(1-e^{-bt})\right\} = 0.$$

(4.33)

<u>When $t_M < t \leq T$, where $\alpha(t_M) = L$</u>, since $\lambda(t,x) = 0$ in S_1

then the optimality condition (4.18) becomes

$$m^2 L\,\alpha'(t) = \int_{x=L+\alpha(t)-\alpha(T)}^{x=L} \{\lambda\phi_x\}_{S_2'}\,dx$$

where $u(t) = \alpha'(t)$.

As before, by using (4.14) we can write the above equation in the form

$$m^2 L\,\alpha'(t) = \int_{L-\alpha(T)}^{L-\alpha(t)} \{A(\eta)\,B'(\eta)\}_{S_2'}\,d\eta \qquad (4.34)$$

Differentiating (4.34) with respect to t and using (4.28), we obtain

$$m^2 L\,\alpha''(t) + e^{-2bt}B(L-\alpha(t))\,B'(L-\alpha(t)) - e^{-bt}B'(L-\alpha(t))\left\{\phi^*(t)+\frac{c}{b}\right\} = 0\,,$$

(4.35)

where

$$B(-\alpha(t)) = \left\{\phi_1(t) + \frac{c}{b}\right\}e^{bt}$$

The differential equations (4.31), (4.33) and (4.35) which are satisfied by $\alpha(t)$ in each time interval, must be solved subject to the following conditions:

$\alpha(0) = 0$,

$\alpha(t)$ and $\alpha'(t)$ are continuous at $t = t_M$ and $t = t_N$,

$\alpha'(T) = 0$.

We note here that the above conditions upon $\alpha'(t)$ follow from (4.29), (4.32) and (4.34).

The optimal control $u(t)$ in each time interval will be calculated from the assumption $u(t) = \alpha'(t)$, as soon as $\alpha(t)$ is known.

Case 4: when $\alpha(T) > L$ and $t_M \leq t_N$ where $\alpha(t_M) = L$, and $\alpha(t_N) = \alpha(T) - L$.

The diagram of this case is shown as in Fig. 28 .

When $0 \leq t < t_M$ where $\alpha(t_M) = L$ and when $t_N < t \leq T$ where $\alpha(t_N) = \alpha(T) - L$, the solutions will be the same as in case 3, and defined in (4.31) and (4.35) respectively.

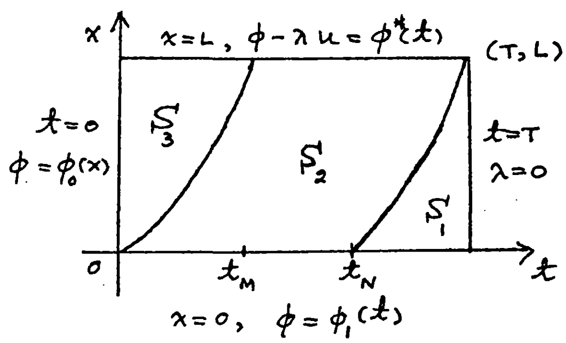


Fig. 28

Let us consider when $t_M < t < t_N$, the domain in this time interval consists of only a part of the subdomain S_2 , thus the optimality condition (4.18) can be written in the form

$$m^2 L\,\alpha'(t) = \int_{x=0}^{x=L} \{\lambda \phi_x\}_{S_2} dx \ ,$$

or

$$m^2 L\,\alpha'(t) = \int_{-\alpha(t)}^{L-\alpha(t)} \{A(\eta)\,B'(\eta)\}_{S_2} d\eta \ , \qquad (4.36)$$

by using (4.14) and changing the variable of integration.

After differentiating (4.36) with respect to the variable t and using (4.27) and (4.28) we then obtain the differential equation satisfied by $\alpha(t)$, $t_M < t < t_N$ of the form

$$m^2 L\,\alpha''(t) + A(-\alpha(t))\,\frac{d}{dt}\left\{\phi_1(t)\,e^{bt} + \frac{c}{b}\,e^{bt}\right\} + e^{-2bt}\,B(L-\alpha(t))\,B'(L-\alpha(t)) -$$

$$- e^{-bt}\,B'(L-\alpha(t))\left\{\phi^*(t) + \frac{c}{b}\right\} = 0\ , \qquad t_M < t < t_N\ ,$$

(4.37)

where the arbitrary functions $A(\eta)$ and $B(\eta)$ are defined in the following forms:

$$A(L-\alpha(t)) = \frac{e^{-bt}}{\alpha'(t)}\left[B(L-\alpha(t))\,e^{-bt} - \phi^*(t) - \frac{c}{b}\right]\ ,$$

and

$$B(-\alpha(t)) = e^{bt}\left\{\phi_1(t) + \frac{c}{b}\right\}\ .$$

The conditions upon $\alpha(t)$ will be the same as in case 3,

i.e., $\alpha(0) = 0$,

$\alpha(t)$ and $\alpha'(t)$ are continuous at $t = t_M$ and $t = t_N$,

$\alpha'(T) = 0$.

As soon as the equations (4.31), (4.37) and (4.35) are solved the corresponding optimum control will be known from the assumption $u(t) = \alpha'(t)$.

The solution of the various non-linear differential equations for $\alpha(t)$ which have arisen in (4.22), (4.31), (4.33), (4.35) and (4.37) has not been attempted. It is not certain that solutions exist and further work is necessary in this area.

CONCLUSION OF PART I

The methods of the classical calculus of variation have been applied to finding the conditions of optimality for the case of a continuous unrestricted control, and when the control is bounded or piecewise continuous the maximum principle has been applied, [11,13].

The approach has been wholly analytical and in the hyperbolic first order partial differential equation which has been studied, the method of characteristics has been used extensively. No elliptic partial differential equation has been studied and it is likely that the analytical attack in this case will be more difficult.

The result of the optimum control problem of minimising the functional

$$I = \frac{1}{2}\int_0^T \{\phi(t,L) - \phi^*(t)\}^2 dt + \int_0^T\int_0^L \frac{1}{2} m^2 u^2 \, dt \, dx$$

where the state $\phi(t,x)$ and the control u satisfy a state equation of the form of a hyperbolic first order linear partial differential equation, is quite different depending on the position and the type of the control u in the state equation. It also depends on the magnitudes of T, L and also upon the coefficient of $\frac{\partial \phi}{\partial x}$ in the state equation. The following table summarises the results attained and indicates a classification of the control problem for this limited system. When the control $u = u(t,x)$ is bounded, the optimum control can be determined with the aid of the maximum principle. Since only analytical methods have been used in this thesis much numerical work needs to be done in order to attain detailed solutions but more work is required to resolve the problems in this area.

State equation	$u=u(t,x)$ and continuous		$u=u(t)$ and continuous		Remark
	$aT \le L$	$aT > L$	$aT \le L$	$aT > L$	
$\frac{\partial\phi}{\partial t}+a\frac{\partial\phi}{\partial x}+b\phi=u$	u is determined explicitly in S.	u is determined explicitly in S.	u is determined explicitly in S.	$u=\alpha'(t)e^{-bt}$ and $\alpha(t)$ satisfies a set of linear differential - difference equations of second order.	$S: 0 \le t \le T$, $0 \le x \le L$; $a>0$ and b are constants.
$\frac{\partial\phi}{\partial t}+a\frac{\partial\phi}{\partial x}+c=u\phi$	For case $c=0$ u satisfies the transcendental equation.	For case $c=0$ u satisfies the transcendental equation.	For case $c\neq 0$ $u(t)=-\frac{\gamma''(t)}{\gamma'(t)}$ and $\gamma(t)$ satisfies the non-linear differential equation of fourth order.	For case $c=0$ $u(t)=\alpha'(t)$ and $\alpha(t)$ satisfies a set of non-linear differential - difference equations of second order.	$a>0$ and c are constants.
$\frac{\partial\phi}{\partial t}+u(t)\frac{\partial\phi}{\partial x}+b\phi+c=0$	———	———	Case $\alpha(T)\le L$: $u(t)=\alpha'(t)$ and $\alpha(t)$ satisfies a non-linear differential equation of second order.	Case $\alpha(T)>L$: $u(t)=\alpha'(t)$ and $\alpha(t)$ satisfies a set of non-linear differential equations of second order.	$u(t)=\alpha'(t)$; b and c are constants.

PART II

OPTIMUM SHAPE PROBLEM

INTRODUCTION OF PART II

All of the problems we have discussed up to this point in distributed parameter control theory have assumed that the process occurs in a domain which is fixed or known in advance, for example the re-heating process occurs in a rectangular region $0 \leq x \leq L$; $0 \leq t \leq T$. However there are problems in which the shape of the domain is unknown and needs to be determined in order to minimise or maximise some performance criterion, for example the problem of designing the most efficient body for extracting the energy from incident sea waves has been recently discussed by Salter, [14]. This problem may be interpreted as the problem of finding the optimum shape of a floating body which minimises the reflection and transmission of the incident wave. Some problems have the boundary of the domain depending on time. This kind of problem in which the system is governed by a parabolic equation of the heat conduction type has been considered by Degtyarev [15] and its necessary conditions for optimality were obtained. Another problem of a similar kind is that of finding the optimum movement of a piston bounding a compressible fluid in order to achieve a given density distribution in fluid, this is discussed by Davies, [16].

Here we apply the variational technique to solve the problem of optimum shape. The theory of maximising or minimising a functional defined on a variable domain is more difficult than the one with the fixed domain since in addition to the type of problem already encountered we also have to consider a transversality condition. Forsyth and Gelfand/Fomin have discussed this theory in their text books Calculus of Variations [17,18] but they have produced no examples to illustrate the theory.

In Part II of this thesis we shall give an example to illustrate the theory by considering a problem in which the state function satisfies a two dimensional Laplace's equation with given boundary conditions and the unknown boundary of the domain acts as a control.

In Chapter 5, we derive the variation of the functional which contains the second partial derivatives defined on the variable domain, by using the method which has been used by Gelfand/Fomin in their text book, [18].

In Chapter 6, we apply the theory which has been derived in Chapter 5 to solve the problem of finding the optimum shape of the domain which gives an extremum of some performance criterion subject to some constraints. One of the constraints is Laplace's equation which the state function must satisfy. The necessary conditions including the transversality condition are obtained. It is also shown in this chapter that the necessary conditions of the problem can be obtained by using the theory of variation of the functional containing more than one dependent variables with their first order partial derivatives.

In Chapter 7, we discuss a particular problem of finding the shape of the inner boundary of an annular region which gives an extremum of the functional

$$I = \iint_{S} \left[\phi_r^2 + \frac{1}{r^2} \phi_\theta^2 \right] r\,dr\,d\theta$$

subject to the given area constraint, i.e.,

$$\iint_{S} r\,dr\,d\theta = K$$

where S is a domain bounded by two closed curves C_1 and C_2.

The unknown curve C_1 is the inner boundary and the fixed given circle C_2 is the outer boundary of the domain. The state function ϕ is a harmonic function which has to satisfy the given boundary conditions on C_1 and C_2, and the shape of C_1 is shown to depend on the boundary conditions.

In Chapter 8, we study the method of logarithmic potential of a single layer and use it to solve the boundary value problems which have arisen in Chapter 7.

CHAPTER 5

THE VARIATION OF A FUNCTIONAL CONTAINING SECOND ORDER PARTIAL DERIVATIVES DEFINED ON A VARIABLE DOMAIN.

In this chapter we shall extend the method which has been used in section 37 of Gelfand and Fomin 's " Calculus of Variations " text book [18] , to derive the first variation of a functional of the form

$$J[\phi] = \iint_R F(x,y,\phi,p,q,r,s,t)\,dx\,dy \tag{5.1}$$

where R is a domain in xy -plane which is varied as well as the dependent variable ϕ and its derivatives p , q , r , s and t . Here we denote $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$, $\frac{\partial^2\phi}{\partial x^2}$, $\frac{\partial^2\phi}{\partial x\partial y}$ and $\frac{\partial^2\phi}{\partial y^2}$ by p , q , r , s and t respectively. The function F is assumed to have continuous first and second derivatives with respect to all its variables.

Let us assume that the surface S , with the equation

$$\phi = \phi(x,y), \qquad (x,y) \in R,$$

is transferred into another surface S^* , with the equation

$$\phi^* = \phi^*(x^*,y^*), \qquad (x^*,y^*) \in R^*,$$

by the following family of transformations:

$$\left.\begin{aligned} x^* &= X(x,y,\phi,p,q,r,s,t;\varepsilon), \\ y^* &= Y(x,y,\phi,p,q,r,s,t;\varepsilon), \\ \phi^* &= \Phi(x,y,\phi,p,q,r,s,t;\epsilon), \end{aligned}\right\} \tag{5.2}$$

depending on the small quantity parameter ε . The functions X , Y and Φ are assumed to be differentiable with respect to ε , and when $\varepsilon = 0$ the following identity transformations are obtained:

$$\left.\begin{aligned} X(x,y,\phi,p,q,r,s,t;0) &\equiv x, \\ Y(x,y,\phi,p,q,r,s,t;0) &\equiv y, \\ \Phi(x,y,\phi,p,q,r,s,t;0) &\equiv \phi. \end{aligned}\right\} \quad (5.3)$$

The transformation (5.2) also carries the functional $J[\phi(x,y)]$ in (5.1) into $J[\phi^*(x^*,y^*)]$ defined as follows:

$$J[\phi^*(x^*,y^*)] = \iint_{R^*} F(x^*,y^*,\phi^*,p^*,q^*,r^*,s^*,t^*)\,dx^*dy^* , \quad (5.4)$$

where R^* is a new domain, $\phi^* = \phi^*(x^*,y^*)$, $p^* = \frac{\partial \phi^*}{\partial x^*}$, $q^* = \frac{\partial \phi^*}{\partial y^*}$,

$r^* = \frac{\partial^2 \phi^*}{\partial x^{*2}}$, $s^* = \frac{\partial^2 \phi^*}{\partial x^* \partial y^*}$ and $t^* = \frac{\partial^2 \phi^*}{\partial y^{*2}}$.

Our aim here is to find δJ which is the principal linear part relative to ε of

$$J[\phi^*(x^*,y^*)] - J[\phi(x,y)] , \quad (5.5)$$

where $J[\phi^*(x^*,y^*)]$ and $J[\phi(x,y)]$ are defined in (5.4) and (5.1) respectively.

Before we calculate δJ , let us first calculate the variations δx , δy , $\delta\phi$, δp , δq , δr , δs and δt .

By applying the Taylor's theorem to (5.2), we obtain

$$x^* = X(x,y,\phi,p,q,r,s,t;0) + \varepsilon \left.\frac{\partial X(x,y,\phi,p,q,r,s,t;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} + o(\varepsilon^2)$$

$$y^* = Y(x,y,\phi,p,q,r,s,t;0) + \varepsilon \left.\frac{\partial Y(x,y,\phi,p,q,r,s,t;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} + o(\varepsilon^2)$$

$$\phi^* = \Phi(x,y,\phi,p,q,r,s,t;0) + \varepsilon \left.\frac{\partial\Phi(x,y,\phi,p,q,r,s,t;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} + o(\varepsilon^2)$$

and by using (5.3) we have

$$\left.\begin{aligned} x^* &= x + \varepsilon X_1(x,y,\phi,p,q,r,s,t) + o(\varepsilon^2), \\ y^* &= y + \varepsilon Y_1(x,y,\phi,p,q,r,s,t) + o(\varepsilon^2), \\ \phi^* &= \phi + \varepsilon \Phi_1(x,y,\phi,p,q,r,s,t) + o(\varepsilon^2), \end{aligned}\right\} \quad (5.6)$$

where

$$\left.\begin{aligned} X_1(x,y,\phi,p,q,r,s,t) &= \left.\frac{\partial X(x,y,\phi,p,q,r,s,t;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \\ Y_1(x,y,\phi,p,q,r,s,t) &= \left.\frac{\partial Y(x,y,\phi,p,q,r,s,t;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \\ \Phi_1(x,y,\phi,p,q,r,s,t) &= \left.\frac{\partial \Phi(x,y,\phi,p,q,r,s,t;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \end{aligned}\right\} \quad (5.7)$$

Since a given surface S has the equation $\phi = \phi(x,y)$ then (5.6) gives us the increments

$$\left.\begin{aligned} \Delta x &= x^* - x = \varepsilon X_1(x,y) + o(\varepsilon^2), \\ \Delta y &= y^* - y = \varepsilon Y_1(x,y) + o(\varepsilon^2), \\ \Delta\phi &= \phi^*(x^*,y^*) - \phi(x,y) = \varepsilon \Phi_1(x,y) + o(\varepsilon^2), \end{aligned}\right\} \quad (5.8)$$

where $X_1(x,y)$, $Y_1(x,y)$ and $\Phi_1(x,y)$ are defined in (5.7) with ϕ and its derivatives expressed in terms of x and y .

Let us denote the principal linear parts relative to ε of the increments Δx , Δy and $\Delta\phi$ by δx , δy and $\delta\phi$ respectively. Hence it follows from (5.8) that

$$\left.\begin{aligned} \delta x &= \varepsilon X_1(x,y) \\ \delta y &= \varepsilon Y_1(x,y) \\ \delta\phi &= \varepsilon \Phi_1(x,y) \end{aligned}\right\} \quad (5.9)$$

The increment $\Delta\phi$ in (5.8) expresses the change in ϕ-coordinate as the point $(x, y, \phi(x,y))$ on the surface S is moved to the point $(x^*, y^*, \phi^*(x^*, y^*))$ on the new surface S^* by the transformation (5.2). Let us also consider the increment

$$\overline{\Delta\phi} = \phi^*(x,y) - \phi(x,y)$$

which expresses the change in ϕ-coordinate as the point $(x, y, \phi(x,y))$ is moved to the point $(x, y, \phi^*(x,y))$ on the new surface S^* but with the same coordinates x and y. We shall introduce a new function $\Phi_2(x,y)$ and the corresponding variation $\overline{\delta\phi}$ defined as follows:

$$\overline{\Delta\phi} = \phi^*(x,y) - \phi(x,y) = \varepsilon\Phi_2(x,y) + o(\varepsilon^2),$$

and (5.10)

$$\overline{\delta\phi} = \varepsilon\Phi_2(x,y).$$

The new function $\Phi_2(x,y)$ or the variation $\overline{\delta\phi}$ is related to $\Phi_1(x,y)$ or $\delta\phi$ which is defined in (5.9), and we can find that relation as follows:

$$\begin{aligned}
\because \quad \Delta\phi &= \phi^*(x^*,y^*) - \phi(x,y) \\
&= [\phi^*(x^*,y^*) - \phi^*(x,y)] + [\phi^*(x,y) - \phi(x,y)] \\
&= \frac{\partial\phi^*}{\partial x}(x^*-x) + \frac{\partial\phi^*}{\partial y}(y^*-y) + \overline{\delta\phi} + o(\varepsilon^2) \\
&= \frac{\partial\phi^*}{\partial x}\delta x + \frac{\partial\phi^*}{\partial y}\delta y + \overline{\delta\phi} + o(\varepsilon^2).
\end{aligned} \tag{5.11}$$

Since it follows from (5.6) that

$$\frac{\partial\phi^*}{\partial x} = \frac{\partial\phi}{\partial x} + \varepsilon\frac{\partial\Phi_1}{\partial x} + o(\varepsilon^2),$$

$$\frac{\partial\phi^*}{\partial y} = \frac{\partial\phi}{\partial y} + \varepsilon\frac{\partial\Phi_1}{\partial y} + o(\varepsilon^2),$$

thus

$$\left.\begin{aligned} \frac{\partial \phi^*}{\partial x}\delta x &= \frac{\partial \phi}{\partial x}\delta x + o(\varepsilon^2) \\ \frac{\partial \phi^*}{\partial y}\delta y &= \frac{\partial \phi}{\partial y}\delta y + o(\varepsilon^2) \end{aligned}\right\} \quad (5.12)$$

where δx and δy are of order 1 in ε and defined in (5.9). By substituting (5.12) into (5.11), we have

$$\Delta \phi = \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \overline{\delta \phi} + o(\varepsilon^2) .$$

Hence

$$\overline{\delta \phi} = \delta \phi - p\,\delta x - q\,\delta y , \quad (5.13)$$

where $\delta \phi$, as usual, is the principal linear part relative to ε of $\Delta \phi$. By using (5.9) and (5.10) we also can write (5.13) in the form

$$\Phi_2(x,y) = \Phi_1(x,y) - pX_1(x,y) - qY_1(x,y) . \quad (5.14)$$

Next we shall calculate the variations δp, δq, δr, δs and δt.

Since x^* and y^* defined in (5.6) are functions of x and y, we can find their derivatives as follows:

$$\frac{\partial x^*}{\partial x} = 1 + \varepsilon \frac{\partial X_1}{\partial x} + o(\varepsilon^2) \; ; \quad \frac{\partial x^*}{\partial y} = \varepsilon \frac{\partial X_1}{\partial y} + o(\varepsilon^2)$$

$$\frac{\partial y^*}{\partial x} = \varepsilon \frac{\partial Y_1}{\partial x} + o(\varepsilon^2) \; ; \quad \frac{\partial y^*}{\partial y} = 1 + \varepsilon \frac{\partial Y_1}{\partial y} + o(\varepsilon^2). \quad (5.15)$$

Hence, it then follows from (5.15) that

$$\frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x}\cdot\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial x}\cdot\frac{\partial}{\partial y^*}$$

$$= \left\{1+\varepsilon\frac{\partial X_1}{\partial x}+o(\varepsilon^2)\right\}\cdot\frac{\partial}{\partial x^*} + \varepsilon\frac{\partial Y_1}{\partial x}\cdot\frac{\partial}{\partial y^*} + o(\varepsilon^2)$$

$$\therefore \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} + \varepsilon\left(\frac{\partial X_1}{\partial x}\cdot\frac{\partial}{\partial x^*} + \frac{\partial Y_1}{\partial x}\cdot\frac{\partial}{\partial y^*}\right) + o(\varepsilon^2). \tag{5.16}$$

Similarly, we have

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y^*} + \varepsilon\left(\frac{\partial X_1}{\partial y}\cdot\frac{\partial}{\partial x^*} + \frac{\partial Y_1}{\partial y}\cdot\frac{\partial}{\partial y^*}\right) + o(\varepsilon^2) \tag{5.17}$$

Now we write

$$\Delta p \equiv \Delta\phi_x = \frac{\partial\phi^*(x^*,y^*)}{\partial x^*} - \frac{\partial\phi(x,y)}{\partial x}$$

$$= \frac{\partial[\phi^*(x^*,y^*)-\phi(x^*,y^*)]}{\partial x^*} + \frac{\partial[\phi(x^*,y^*)-\phi(x,y)]}{\partial x} +$$

$$+ \left(\frac{\partial}{\partial x^*}-\frac{\partial}{\partial x}\right)\phi(x^*,y^*). \tag{5.18}$$

and analyse each term on the right hand side of (5.18) as follows:

By using (5.6), (5.10), (5.16) and (5.17), it is easy to verify that

$$\frac{\partial}{\partial x^*}\left[\phi^*(x^*,y^*)-\phi(x^*,y^*)\right] = \frac{\partial}{\partial x^*}\left[\varepsilon\,\Phi_2(x^*,y^*)\right] + o(\varepsilon^2)$$

$$= \varepsilon\frac{\partial}{\partial x}\left[\Phi_2(x,y)\right] + o(\varepsilon^2), \tag{5.19}$$

also

$$\frac{\partial}{\partial x}[\phi(x^*,y^*)-\phi(x,y)] = \varepsilon\frac{\partial}{\partial x}\left[\frac{\partial\phi}{\partial x}X_1(x,y) + \frac{\partial\phi}{\partial y}Y_1(x,y)\right] + o(\varepsilon^2)$$

$$= \varepsilon\left[\frac{\partial^2\phi}{\partial x^2}X_1(x,y) + \frac{\partial^2\phi}{\partial x\partial y}Y_1(x,y) + \frac{\partial\phi}{\partial x}\cdot\frac{\partial X_1}{\partial x} + \right.$$

$$\left. + \frac{\partial\phi}{\partial y}\cdot\frac{\partial Y_1}{\partial x}\right] + o(\varepsilon^2), \tag{5.20}$$

and

$$\left(\frac{\partial}{\partial x^*} - \frac{\partial}{\partial x}\right)\phi(x^*, y^*) = \left(\frac{\partial}{\partial x^*} - \frac{\partial}{\partial x}\right)\left[\phi(x,y) + \varepsilon p X_1(x,y) + \varepsilon q Y_1(x,y)\right] + o(\varepsilon^2)$$

$$= -\varepsilon\left[\frac{\partial X_1}{\partial x}\cdot\frac{\partial\phi}{\partial x^*} + \frac{\partial Y_1}{\partial x}\cdot\frac{\partial\phi}{\partial y^*}\right] + o(\varepsilon^2).$$

(5.21)

Substituting (5.19) - (5.21) back into (5.18) and using (5.16) and (5.17) once more, we obtain

$$\Delta p \equiv \Delta\phi_x = \varepsilon\frac{\partial}{\partial x}\left[\Phi_2(x,y)\right] + \varepsilon X_1(x,y)\frac{\partial^2\phi}{\partial x^2} + \varepsilon Y_1(x,y)\frac{\partial^2\phi}{\partial x\partial y} + o(\varepsilon^2).$$

(5.22)

Since δp or $\delta\phi_x$ is the principal linear part relative to ε of Δp or $\Delta\phi_x$, and by using (5.9) and (5.10) we can write (5.22) in the form

$$\delta p \equiv \delta\phi_x = \frac{\partial}{\partial x}(\overline{\delta\phi}) + r\,\delta x + s\,\delta y\,, \tag{5.23}$$

where $r \equiv \frac{\partial^2\phi}{\partial x^2}$ and $s \equiv \frac{\partial^2\phi}{\partial x\partial y}$.

Similarly, by starting with

$$\Delta q \equiv \Delta\phi_y = \frac{\partial\phi^*(x^*, y^*)}{\partial y^*} - \frac{\partial\phi(x,y)}{\partial y}\,,$$

and using a similar method, we obtain

$$\delta q \equiv \delta\phi_y = \frac{\partial}{\partial y}(\overline{\delta\phi}) + s\,\delta x + t\,\delta y\,, \tag{5.24}$$

where $s \equiv \frac{\partial^2\phi}{\partial x\partial y}$ and $t \equiv \frac{\partial^2\phi}{\partial y^2}$.

We now proceed to find δr , δs and δt which are the principal linear parts relative to ε of $\Delta\phi_{xx}$, $\Delta\phi_{xy}$ and $\Delta\phi_{yy}$ respectively.

From (5.16) and (5.17), we can write

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^{*2}} + 2\varepsilon\left(\frac{\partial X_1}{\partial x}\cdot\frac{\partial^2}{\partial x^{*2}} + \frac{\partial Y_1}{\partial x}\cdot\frac{\partial^2}{\partial x^*\partial y^*}\right) + \varepsilon\left(\frac{\partial^2 X_1}{\partial x^2}\cdot\frac{\partial}{\partial x^*} + \frac{\partial^2 Y_1}{\partial x^2}\cdot\frac{\partial}{\partial y^*}\right) + o(\varepsilon^2), \tag{5.25}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y^{*2}} + 2\varepsilon\left(\frac{\partial X_1}{\partial y}\cdot\frac{\partial^2}{\partial x^*\partial y^*} + \frac{\partial Y_1}{\partial y}\cdot\frac{\partial^2}{\partial y^{*2}}\right) + \varepsilon\left(\frac{\partial^2 X_1}{\partial y^2}\cdot\frac{\partial}{\partial x^*} + \frac{\partial^2 Y_1}{\partial y^2}\cdot\frac{\partial}{\partial y^*}\right) + o(\varepsilon^2), \tag{5.26}$$

and

$$\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial x^*\partial y^*} + \varepsilon\left(\frac{\partial X_1}{\partial x}\cdot\frac{\partial^2}{\partial x^*\partial y^*} + \frac{\partial Y_1}{\partial x}\cdot\frac{\partial^2}{\partial y^{*2}}\right) + \varepsilon\left(\frac{\partial X_1}{\partial y}\cdot\frac{\partial^2}{\partial x^{*2}} + \frac{\partial Y_1}{\partial y}\cdot\frac{\partial^2}{\partial x^*\partial y^*}\right) +$$
$$+ \varepsilon\left(\frac{\partial^2 X_1}{\partial x\partial y}\cdot\frac{\partial}{\partial x^*} + \frac{\partial^2 Y_1}{\partial x\partial y}\cdot\frac{\partial}{\partial y^*}\right) + o(\varepsilon^2). \tag{5.27}$$

As before, we write

$$\begin{aligned}\Delta r \equiv \Delta\phi_{xx} &= \frac{\partial^2\phi(x^*,y^*)}{\partial x^{*2}} - \frac{\partial^2\phi(x,y)}{\partial x^2} \\ &= \frac{\partial^2[\phi^*(x^*,y^*)-\phi(x^*,y^*)]}{\partial x^{*2}} + \frac{\partial^2[\phi(x^*,y^*)-\phi(x,y)]}{\partial x^2} + \\ &\quad + \left(\frac{\partial^2}{\partial x^{*2}} - \frac{\partial^2}{\partial x^2}\right)\phi(x^*,y^*) ,\end{aligned} \tag{5.28}$$

and analyse those terms on the right hand side of (5.28) as follows:

By using (5.6), (5.10) and (5.25), we have

$$\frac{\partial^2}{\partial x^{*2}}\left[\phi^*(x^*,y^*)-\phi(x^*,y^*)\right] = \frac{\partial^2}{\partial x^{*2}}\left[\varepsilon\,\Phi_2(x^*,y^*)\right] + O(\varepsilon^2)$$

$$= \varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi(x,y)\right] + O(\varepsilon^2)$$

(5.29)

$$\frac{\partial^2}{\partial x^2}\left[\phi(x^*,y^*)-\phi(x,y)\right] = \frac{\partial^2}{\partial x^2}\left[\frac{\partial\phi}{\partial x}.\varepsilon X_1(x,y)+\frac{\partial\phi}{\partial y}.\varepsilon Y_1(x,y)\right]+O(\varepsilon^2)$$

$$= \varepsilon\left[\frac{\partial^3\phi}{\partial x^3}X_1(x,y)+\frac{\partial^3\phi}{\partial x^2\partial y}Y_1(x,y)+\frac{\partial\phi}{\partial x}.\frac{\partial^2 X_1}{\partial x^2}+\right.$$

$$\left.+\frac{\partial\phi}{\partial y}.\frac{\partial^2 Y_1}{\partial x^2}+2\left\{\frac{\partial^2\phi}{\partial x^2}.\frac{\partial X_1}{\partial x}+\frac{\partial^2\phi}{\partial x\partial y}.\frac{\partial Y_1}{\partial x}\right\}\right]+O(\varepsilon^2)$$

(5.30)

and

$$\left(\frac{\partial^2}{\partial x^{*2}}-\frac{\partial^2}{\partial x^2}\right)\phi(x^*,y^*) = \left(\frac{\partial^2}{\partial x^{*2}}-\frac{\partial^2}{\partial x^2}\right)\left[\phi(x,y)+\varepsilon pX_1(x,y)+\varepsilon qY_1(x,y)\right]+O(\varepsilon^2)$$

$$= -2\varepsilon\left[\frac{\partial X_1}{\partial x}.\frac{\partial^2\phi}{\partial x^{*2}}+\frac{\partial Y_1}{\partial x}.\frac{\partial^2\phi}{\partial x^*\partial y^*}\right]-$$

$$-\varepsilon\left[\frac{\partial^2 X_1}{\partial x^2}.\frac{\partial\phi}{\partial x^*}+\frac{\partial^2 Y_1}{\partial x^2}.\frac{\partial\phi}{\partial y^*}\right]+O(\varepsilon^2).$$

(5.31)

By adding (5.29), (5.30) and (5.31) and using (5.16), (5.17), (5.25) and (5.27) we obtain

$$\Delta r \equiv \Delta\phi_{xx} = \varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi_2(x,y)\right]+\varepsilon\left[\frac{\partial^3\phi}{\partial x^3}X_1(x,y)+\frac{\partial^3\phi}{\partial x^2\partial y}Y_1(x,y)\right]+O(\varepsilon^2)$$

Hence

$$\delta r \equiv \delta\phi_{xx} = \frac{\partial^2}{\partial x^2}\left(\overline{\delta\phi}\right)+\frac{\partial r}{\partial x}\delta x+\frac{\partial s}{\partial x}\delta y\,, \qquad (5.32)$$

where $r \equiv \frac{\partial^2\phi}{\partial x^2}$, $s \equiv \frac{\partial^2\phi}{\partial x\partial y}$, $\overline{\delta\phi} = \varepsilon\,\Phi_2(x,y)$, $\delta x = \varepsilon\,X_1(x,y)$

and $\delta y = \varepsilon\,Y_1(x,y)$.

Similarly, by starting with

$$\Delta t \equiv \Delta\phi_{yy} = \frac{\partial^2\phi^*(x^*,y^*)}{\partial y^{*2}} - \frac{\partial^2\phi(x,y)}{\partial y^2} ,$$

$$\Delta s \equiv \Delta\phi_{xy} = \frac{\partial^2\phi^*(x^*,y^*)}{\partial x^*\partial y^*} - \frac{\partial^2\phi(x,y)}{\partial x\partial y} ,$$

and using the same technique as finding δr , but making use of (5.26) and (5.27) instead of (5.25) we find that

$$\delta t \equiv \delta\phi_{yy} = \frac{\partial^2}{\partial y^2}(\overline{\delta\phi}) + \frac{\partial s}{\partial y}\delta x + \frac{\partial t}{\partial y}\delta y , \tag{5.33}$$

and

$$\delta s \equiv \delta\phi_{xy} = \frac{\partial^2}{\partial x\partial y}(\overline{\delta\phi}) + \frac{\partial s}{\partial x}\delta x + \frac{\partial s}{\partial y}\delta y , \tag{5.34}$$

where $s \equiv \frac{\partial^2\phi}{\partial x\partial y}$ and $t \equiv \frac{\partial^2\phi}{\partial y^2}$.

Next we calculate the difference ΔJ defined in (5.5), i.e.,

$$\Delta J = J[\phi^*(x^*,y^*)] - J[\phi(x,y)]$$

$$= \iint_{R^*} F(x^*,y^*,\phi^*,p^*,q^*,r^*,s^*,t^*)\,dx^*dy^* - \iint_{R} F(x,y,\phi,p,q,r,s,t)\,dx\,dy \tag{5.35}$$

Changing the variables of integration x^* and y^* in (5.35) into x and y by using the transformation (5.6), we obtain

$$\Delta J = \iint_R \left[F(x^*, y^*, \phi^*, p^*, q^*, r^*, s^*, t^*) \frac{\partial(x^*, y^*)}{\partial(x,y)} - F(x, y, \phi, p, q, r, s, t) \right] dx\,dy, \tag{5.36}$$

where $\dfrac{\partial(x^*, y^*)}{\partial(x,y)}$ is the Jacobian of the transformation from the variables x , y to the variables x^* , y^* and by using (5.6) we find that

$$\frac{\partial(x^*, y^*)}{\partial(x,y)} = \begin{vmatrix} 1+\varepsilon\dfrac{\partial X_1}{\partial x} & \varepsilon\dfrac{\partial X_1}{\partial y} \\ \varepsilon\dfrac{\partial Y_1}{\partial x} & 1+\varepsilon\dfrac{\partial Y_1}{\partial y} \end{vmatrix} = 1 + \varepsilon\frac{\partial X_1}{\partial x} + \varepsilon\frac{\partial Y_1}{\partial y} + o(\varepsilon^2).$$

Hence, ΔJ in (5.36) can be written in the form

$$\Delta J = \iint_R \left[F(x^*, y^*, \phi^*, p^*, q^*, r^*, s^*, t^*) \left\{ 1 + \varepsilon\frac{\partial X_1}{\partial x} + \varepsilon\frac{\partial Y_1}{\partial y} \right\} - F(x, y, \phi, p, q, r, s, t) \right] dx\,dy + \\ + o(\varepsilon^2). \tag{5.37}$$

Expanding the integrand of (5.37) by using Taylor's theorem and neglecting the terms of higher order than 1 relative to ε , we obtain

$$\delta J = \iint_R \left[\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial r}\delta r + \right. \\ \left. + \frac{\partial F}{\partial s}\delta s + \frac{\partial F}{\partial t}\delta t + \varepsilon F\frac{\partial X_1}{\partial x} + \varepsilon F\frac{\partial Y_1}{\partial y} \right] dx\,dy \tag{5.38}$$

But since it follows from (5.9) that $\dfrac{\partial}{\partial x}(\delta x) = \varepsilon\dfrac{\partial X_1}{\partial x}$ and $\dfrac{\partial}{\partial y}(\delta y) = \varepsilon\dfrac{\partial Y_1}{\partial y}$ thus the last two terms of the integrand in (5.38) can be written in the following forms:

$$\varepsilon F \frac{\partial X_1}{\partial x} = F \frac{\partial}{\partial x}(\delta x) = \frac{\partial}{\partial x}(F\delta x) - \delta x \Big\{ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \phi} \cdot p + \frac{\partial F}{\partial p} \cdot r + \frac{\partial F}{\partial q} \cdot s + \\ + \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial F}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial F}{\partial t} \cdot \frac{\partial s}{\partial y} \Big\}, \tag{5.39}$$

and

$$\varepsilon F \frac{\partial Y_1}{\partial y} = F \frac{\partial}{\partial y}(\delta y) = \frac{\partial}{\partial y}(F\delta y) - \delta y \Big\{ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial \phi} \cdot q + \frac{\partial F}{\partial p} \cdot s + \frac{\partial F}{\partial q} \cdot t + \\ + \frac{\partial F}{\partial r} \cdot \frac{\partial s}{\partial x} + \frac{\partial F}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial F}{\partial t} \cdot \frac{\partial t}{\partial y} \Big\}, \tag{5.40}$$

where $\frac{\partial s}{\partial y} \equiv \frac{\partial t}{\partial x}$ and $\frac{\partial s}{\partial x} \equiv \frac{\partial r}{\partial y}$.

By adding (5.39) and (5.40), and using (5.23), (5.24), (5.32), (5.33), (5.34) and (5.13) we have

$$\varepsilon F \frac{\partial X_1}{\partial x} + \varepsilon F \frac{\partial Y_1}{\partial y} = \frac{\partial}{\partial x}(F\delta x) + \frac{\partial}{\partial y}(F\delta y) - \frac{\partial F}{\partial x}\delta x - \frac{\partial F}{\partial y}\delta y - \frac{\partial F}{\partial \phi}(\delta\phi - \overline{\delta\phi}) - \\ - \frac{\partial F}{\partial p}\Big[\delta p - \frac{\partial}{\partial x}(\overline{\delta\phi})\Big] - \frac{\partial F}{\partial q}\Big[\delta q - \frac{\partial}{\partial y}(\overline{\delta\phi})\Big] - \frac{\partial F}{\partial r}\Big[\delta r - \frac{\partial^2}{\partial x^2}(\overline{\delta\phi})\Big] - \\ - \frac{\partial F}{\partial s}\Big[\delta s - \frac{\partial^2}{\partial x \partial y}(\overline{\delta\phi})\Big] - \frac{\partial F}{\partial t}\Big[\delta t - \frac{\partial^2}{\partial y^2}(\overline{\delta\phi})\Big]. \tag{5.41}$$

By substituting (5.41) into (5.38), we obtain

$$\delta J = \iint_R \Big[\frac{\partial F}{\partial \phi} \cdot \overline{\delta\phi} + \frac{\partial F}{\partial p} \cdot \frac{\partial}{\partial x}(\overline{\delta\phi}) + \frac{\partial F}{\partial q} \cdot \frac{\partial}{\partial y}(\overline{\delta\phi}) + \frac{\partial F}{\partial r} \cdot \frac{\partial^2}{\partial x^2}(\overline{\delta\phi}) + \frac{\partial F}{\partial s} \cdot \frac{\partial^2}{\partial x \partial y}(\overline{\delta\phi}) + \\ + \frac{\partial F}{\partial t} \cdot \frac{\partial^2}{\partial y^2}(\overline{\delta\phi}) + \frac{\partial}{\partial x}(F\delta x) + \frac{\partial}{\partial y}(F\delta y) \Big] dx\,dy \tag{5.42}$$

It is easy to verify that

$$
\begin{aligned}
\frac{\partial F}{\partial p}\cdot\frac{\partial}{\partial x}(\overline{\delta\phi}) &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\cdot\overline{\delta\phi}\right) - \overline{\delta\phi}\cdot\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right),\\
\frac{\partial F}{\partial q}\cdot\frac{\partial}{\partial y}(\overline{\delta\phi}) &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\cdot\overline{\delta\phi}\right) - \overline{\delta\phi}\cdot\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right),\\
\frac{\partial F}{\partial r}\cdot\frac{\partial^2}{\partial x^2}(\overline{\delta\phi}) &= \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial r}\cdot\frac{\partial}{\partial x}(\overline{\delta\phi})\right\} - \frac{\partial}{\partial x}\left\{\overline{\delta\phi}\cdot\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r}\right)\right\} + \overline{\delta\phi}\,\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial r}\right),\\
\frac{\partial F}{\partial t}\cdot\frac{\partial^2}{\partial y^2}(\overline{\delta\phi}) &= \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial t}\cdot\frac{\partial}{\partial y}(\overline{\delta\phi})\right\} - \frac{\partial}{\partial y}\left\{\overline{\delta\phi}\cdot\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right)\right\} + \overline{\delta\phi}\,\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial t}\right),\\
\frac{\partial F}{\partial s}\cdot\frac{\partial^2}{\partial x\partial y}(\overline{\delta\phi}) &= \frac{1}{2}\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial s}\cdot\frac{\partial}{\partial y}(\overline{\delta\phi}) - \overline{\delta\phi}\cdot\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial s}\right)\right\} +\\
&\quad + \frac{1}{2}\frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial s}\cdot\frac{\partial}{\partial x}(\overline{\delta\phi}) - \overline{\delta\phi}\,\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial s}\right)\right\} + \overline{\delta\phi}\cdot\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial s}\right).
\end{aligned}
\tag{5.43}
$$

Using (5.43), we then can write δJ in (5.42) in the form

$$
\begin{aligned}
\delta J = \iint_R \Big[&\overline{\delta\phi}\left\{\frac{\partial F}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial r}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial s}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial t}\right)\right\} +\\
&+ \frac{\partial}{\partial x}\left\{F\delta x + \frac{\partial F}{\partial p}\overline{\delta\phi} + \frac{\partial F}{\partial r}\cdot\frac{\partial}{\partial x}(\overline{\delta\phi}) - \overline{\delta\phi}\cdot\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r}\right) + \frac{1}{2}\left(\frac{\partial F}{\partial s}\cdot\frac{\partial}{\partial y}(\overline{\delta\phi}) - \overline{\delta\phi}\,\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial s}\right)\right)\right\} +\\
&+ \frac{\partial}{\partial y}\left\{F\delta y + \frac{\partial F}{\partial q}\overline{\delta\phi} + \frac{\partial F}{\partial t}\cdot\frac{\partial}{\partial y}(\overline{\delta\phi}) - \overline{\delta\phi}\cdot\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial F}{\partial s}\cdot\frac{\partial}{\partial x}(\overline{\delta\phi}) - \overline{\delta\phi}\,\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial s}\right)\right)\right\}\Big]\,dx\,dy
\end{aligned}
\tag{5.44}
$$

Applying the Green's theorem in 2 dimensional xy-plane to (5.44) and then using (5.13), (5.23) and (5.24) in the line integrals, we finally obtain

$$
\begin{aligned}
\delta J = &\iint_R \overline{\delta\phi}\left[\frac{\partial F}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial r}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial s}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial t}\right)\right] dx\,dy +\\
&+ \oint_C \left[P\,dy - Q\,dx\right],
\end{aligned}
\tag{5.45}
$$

where c is a closed curve bounding region R , and the functions P and Q are defined as follows:

$$P = \left\{ \frac{\partial F}{\partial r}\,\delta\phi_x + \frac{1}{2}\,\frac{\partial F}{\partial s}\,\delta\phi_y \right\} + \delta\phi \left\{ \frac{\partial F}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r}\right) - \frac{1}{2}\,\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial s}\right) \right\} +$$

$$+ \delta x \left[F - p\left\{ \frac{\partial F}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r}\right) - \frac{1}{2}\,\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial s}\right) \right\} - \left(r\,\frac{\partial F}{\partial r} + \frac{1}{2}\,s\,\frac{\partial F}{\partial s} \right) \right] -$$

$$- \delta y \left[q\left\{ \frac{\partial F}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r}\right) - \frac{1}{2}\,\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial s}\right) \right\} + \left(s\,\frac{\partial F}{\partial r} + \frac{1}{2}\,t\,\frac{\partial F}{\partial s} \right) \right],$$

(5.46)

and

$$Q = \left\{ \frac{1}{2}\,\frac{\partial F}{\partial s}\,\delta\phi_x + \frac{\partial F}{\partial t}\,\delta\phi_y \right\} + \delta\phi \left\{ \frac{\partial F}{\partial q} - \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial s}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right) \right\} -$$

$$- \delta x \left[p\left\{ \frac{\partial F}{\partial q} - \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial s}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right) \right\} + \left(\frac{1}{2}\,r\,\frac{\partial F}{\partial s} + s\,\frac{\partial F}{\partial t} \right) \right] +$$

$$+ \delta y \left[F - q\left\{ \frac{\partial F}{\partial q} - \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial s}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right) \right\} - \left(\frac{1}{2}\,s\,\frac{\partial F}{\partial s} + t\,\frac{\partial F}{\partial t} \right) \right].$$

(5.47)

This result is the same as in section 336 of Forsyth's " Calculus of Variations " text book, [17] .

If the functional contains the partial derivatives of order not higher than one, i.e.,

$$J = \iint_R F(x, y, \phi, p, q)\,dx\,dy$$

then we can deduce from (5.45) - (5.47) that

$$\delta J = \iint_R \overline{\delta\phi} \left[\frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) \right] dx\,dy +$$

$$+ \oint_c \left[\delta\phi \left\{ \frac{\partial F}{\partial p}\,dy - \frac{\partial F}{\partial q}\,dx \right\} + \delta x \left\{ F\,dy - p\left(\frac{\partial F}{\partial p}\,dy - \frac{\partial F}{\partial q}\,dx \right) \right\} - \right.$$

$$\left. - \delta y \left\{ F\,dx + q\left(\frac{\partial F}{\partial p}\,dy - \frac{\partial F}{\partial q}\,dx \right) \right\} \right],$$

(5.48)

where $p \equiv \frac{\partial \phi}{\partial x}$, $q \equiv \frac{\partial \phi}{\partial y}$ and C is a closed curve bounding the region R .

By referring to the remark 3 in page 175 of Gelfand and Fomin's text book [18] , we shall state without proof that if

$$J = \iint_R F(x, y, \phi_1, \ldots, \phi_n, \phi_{1x}, \ldots, \phi_{nx}, \phi_{1y}, \ldots, \phi_{ny})\, dx\,dy$$

then the first variation δJ can be written in the form

$$\begin{aligned}
\delta J = & \iint_R \sum_{i=1}^{n} \left[\overline{\delta\phi_i} \left\{ \frac{\partial F}{\partial \phi_i} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial y}\right)}\right) \right\} \right] dx\,dy + \\
& + \oint_C \left[\sum_{i=1}^{n} \delta\phi_i \left\{ \frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} dy - \frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial y}\right)} dx \right\} + \right. \\
& \quad + \delta x \left\{ F dy - \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x} \left(\frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} dy - \frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial y}\right)} dx \right) \right\} - \\
& \quad \left. - \delta y \left\{ F dx + \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial y} \left(\frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial x}\right)} dy - \frac{\partial F}{\partial\left(\frac{\partial \phi_i}{\partial y}\right)} dx \right) \right\} \right],
\end{aligned} \tag{5.49}$$

where C is a closed curve bounding the region R .

CHAPTER 6

THE GENERAL STATEMENT OF THE TWO DIMENSIONAL HARMONIC CONTROL PROBLEM WITH THE SHAPE OF THE DOMAIN AS THE CONTROL : COMPARISON OF METHODS OF FORSYTH AND GELFAND/FOMIN.

The problem we discuss in this chapter can be stated as follows:

Let $\phi(x,y)$ be a harmonic function in the two dimensional xy -plane, i.e., $\phi(x,y)$ satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 , \qquad (x,y) \in S \tag{6.1}$$

where S is a domain in xy-plane bounded by a closed curve $\Gamma = \Gamma_1 \cup \Gamma_2$. The part Γ_1 of the curve Γ is assumed to be fixed and the part Γ_2 can be varied (Fig. 29).

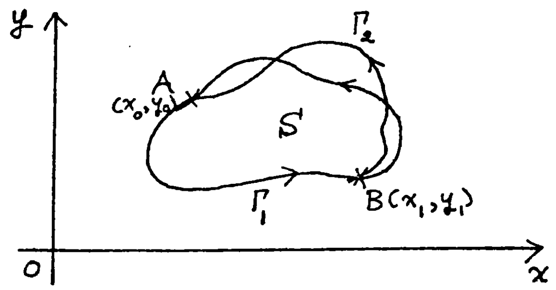


Fig. 29

The boundary conditions are given in the form

$$M(x,y,\phi,\phi_x,\phi_y) = 0 , \qquad (x,y) \in \Gamma_1 \tag{6.2}$$

$$N(x,y,\phi,\phi_x,\phi_y) = 0 , \qquad (x,y) \in \Gamma_2 \tag{6.3}$$

When we choose different curves Γ_2 we obtain different functions ϕ which satisfy (6.1) - (6.3). We wish to find the shape of Γ_2, in other words the equation of the curve Γ_2, for which the functional

$$I = \iint_S F(x,y,\phi,\phi_x,\phi_y)\,dx\,dy + \int_{\Gamma_2} [V(x,y,\phi)\,dx + W(x,y,\phi)\,dy], \qquad (6.4)$$

has an extremum subject to the constraint

$$\iint_S G(x,y,\phi,\phi_x,\phi_y)\,dx\,dy = K, \qquad (6.5)$$

where K is a constant. For example, when $G \equiv 1$, (6.5) could be an area constraint.

Let $x = \xi(t)$; $y = \eta(t)$ be the equations of the curve Γ_2 in a parametric form with parameter t. Suppose that $\xi(t)$ and $\eta(t)$ are continuous functions and have continuous derivatives $\dot{\xi}(t)$ and $\dot{\eta}(t)$ and do not vanish simultaneously, when t varies from t_0 which corresponds to the fixed point $A(x_0,y_0)$ and t_1 corresponding to the fixed point $B(x_1,y_1)$, (see Fig. 29).

We shall investigate the problem by two methods depending on using the result in (5.45) or (5.49) in Chapter 5 and we establish that the governing equations are the same by each method. Let us call the first method which uses (5.45) the Forsyth method and the second one which uses (5.49) the Gelfand and Fomin method.

Method 1. Forsyth method

We introduce the Lagrange multipliers $\gamma(x,y)$ and ν where ν is a constant and write a new functional J in the form

$$J = \iint_S \left[H(x,y,\phi,\phi_x,\phi_y) + \gamma(x,y)\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\}\right] dx\,dy -$$

$$- \int_{t_0}^{t_1} \left[V(\xi(t),\eta(t),\phi(\xi,\eta))\dot{\xi} + W(\xi(t),\eta(t),\phi(\xi,\eta))\dot{\eta}\right] dt,$$

$$J = I_1 - I_2 \tag{6.6}$$

where $$H(x,y,\phi,\phi_x,\phi_y) = F(x,y,\phi,\phi_x,\phi_y) + \nu G(x,y,\phi,\phi_x,\phi_y) \tag{6.7}$$

$$I_1 = \iint_S \left[H(x,y,\phi,\phi_x,\phi_y) + \gamma(x,y)\{\phi_{xx} + \phi_{yy}\}\right] dx\,dy \tag{6.8}$$

and $$I_2 = \int_{t_0}^{t_1} \left[V(\xi,\eta,\phi(\xi,\eta))\dot{\xi} + W(\xi,\eta,\phi(\xi,\eta))\dot{\eta}\right] dt \tag{6.9}$$

Since the curve Γ_2 which is a part of the boundary of the domain S varies then S is a variable domain, and since the functional I_1 in (6.8) contains the second order partial derivatives of ϕ thus by using (5.45) in Chapter 5 we can write δI_1 in the form

$$\delta I_1 = \iint_S \overline{\delta\phi}\left[\frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \phi_y}\right) + \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2}\right] dx\,dy +$$

$$+ \oint_{\Gamma_1 \cup \Gamma_2} \left[P\,dy - Q\,dx \right] , \tag{6.10}$$

where P and Q are obtained by using (5.46) and (5.47) in Chapter 5, as follows:

$$P = \gamma\,\delta\phi_x + (\delta\phi)\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x}\right) + (\delta x)\left[H - \phi_x\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x}\right) - \gamma\phi_{xx}\right] -$$

$$- (\delta y)\left[\phi_y\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x}\right) + \gamma\phi_{xy}\right] , \tag{6.11}$$

$$Q = \gamma\,\delta\phi_y + (\delta\phi)\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y}\right) - (\delta x)\left[\phi_x\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y}\right) + \gamma\phi_{xy}\right] +$$

$$+ (\delta y)\left[H - \phi_y\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y}\right) - \gamma\phi_{yy}\right] . \tag{6.12}$$

Since Γ_1 is assumed to be fixed, hence $\delta x = 0$, $\delta y = 0$ on Γ_1 and since on Γ_2 we have $x = \xi(t)$; $y = \eta(t)$, $t_0 \le t \le t_1$, hence $\delta x = \delta\xi(t)$; $\delta y = \delta\eta(t)$; $dx = \dot{\xi}(t)\,dt$ and $dy = \dot{\eta}(t)\,dt$. Therefore δI_1 in (6.10) can be written in the form

$$\delta I_1 = \iint_S \overline{\delta\phi}\left[\frac{\partial H}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial\phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial\phi_y}\right) + \frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\gamma}{\partial y^2}\right] dx\,dy +$$

$$+ \int_{\Gamma_1}\left[\left\{\gamma\delta\phi_x + (\delta\phi)\left(\frac{\partial H}{\partial\phi_x} - \frac{\partial\gamma}{\partial x}\right)\right\}dy - \left\{\gamma\delta\phi_y + (\delta\phi)\left(\frac{\partial H}{\partial\phi_y} - \frac{\partial\gamma}{\partial y}\right)\right\}dx\right] -$$

$$- \int_{t_0}^{t_1}\left[\left\{\gamma\delta\phi_x + (\delta\phi)\left(\frac{\partial H}{\partial\phi_x} - \frac{\partial\gamma}{\partial x}\right) + (\delta\xi(t))\left(H - \phi_x\left(\frac{\partial H}{\partial\phi_x} - \frac{\partial\gamma}{\partial x}\right) - \gamma\phi_{xx}\right) -\right.\right.$$

$$-(\delta\eta(t))\left(\phi_y\left(\frac{\partial H}{\partial\phi_x} - \frac{\partial\gamma}{\partial x}\right) + \gamma\phi_{xy}\right)\Big\}\dot{\eta}(t) - \Big\{\gamma\delta\phi_y + (\delta\phi)\left(\frac{\partial H}{\partial\phi_y} - \frac{\partial\gamma}{\partial y}\right) -$$

$$\left.-(\delta\xi(t))\left(\phi_x\left(\frac{\partial H}{\partial\phi_y} - \frac{\partial\gamma}{\partial y}\right) + \gamma\phi_{xy}\right) + (\delta\eta(t))\left(H - \phi_y\left(\frac{\partial H}{\partial\phi_y} - \frac{\partial\gamma}{\partial y}\right) - \gamma\phi_{yy}\right)\Big\}\dot{\xi}(t)\right]_{x=\xi(t),\,y=\eta(t)} dt\,. \tag{6.13}$$

The functional I_2 in (6.9) depends on three functions $\xi(t)$, $\eta(t)$ and $\phi(\xi(t),\eta(t))$. The limits of integration t_0 and t_1 are fixed, and the values of $\xi(t)$ and $\eta(t)$ when $t = t_0$ and $t = t_1$, i.e., at the fixed points $A(x_0, y_0)$ and $B(x_1, y_1)$, are known. The variation of this kind of functional can be found by standard methods and we can write δI_2 in the form

$$\delta I_2 = \int_{t_0}^{t_1}\left[\left\{\frac{\partial V}{\partial\xi}\dot{\xi} + \frac{\partial W}{\partial\xi}\dot{\eta} - \frac{dV}{dt}\right\}\delta\xi(t) + \left\{\frac{\partial V}{\partial\eta}\dot{\xi} + \frac{\partial W}{\partial\eta}\dot{\eta} - \frac{dW}{dt}\right\}\delta\eta(t) +\right.$$

$$\left. + \left\{\frac{\partial V}{\partial\phi}\dot{\xi} + \frac{\partial W}{\partial\phi}\dot{\eta}\right\}\delta\phi\left(\xi(t),\eta(t)\right)\right] dt$$

$$= \int_{t_0}^{t_1} \left[\left\{ \left(\frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \right) \dot{\eta} - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \cdot \dot{\xi} \right\} \delta \xi(t) + \right.$$

$$+ \left\{ \left(\frac{\partial V}{\partial \eta} - \frac{\partial W}{\partial \xi} - \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \right) \dot{\xi} - \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \cdot \dot{\eta} \right\} \delta \eta(t) +$$

$$\left. + \left\{ \frac{\partial V}{\partial \phi} \dot{\xi} + \frac{\partial W}{\partial \phi} \dot{\eta} \right\} \delta \phi(\xi, \eta) \right] dt. \tag{6.14}$$

Substituting (6.13) and (6.14) into the variation of (6.6), we obtain

$$\delta J = \iint_S \overline{\delta\phi} \left[\frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \phi_y} \right) + \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} \right] dx\, dy +$$

$$+ \oint_{\Gamma_1} \left[\gamma (\delta\phi_x\, dy - \delta\phi_y\, dx) + (\delta\phi) \left\{ \left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x} \right) dy - \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y} \right) dx \right\} \right] -$$

$$- \int_{t_0}^{t_1} \left[\gamma (\delta\phi_x \dot{\eta} - \delta\phi_y \dot{\xi}) + \{\delta\phi(\xi,\eta)\} \left\{ \left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x} + \frac{\partial W}{\partial \phi} \right) \dot{\eta} - \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y} - \frac{\partial V}{\partial \phi} \right) \dot{\xi} \right\} + \right.$$

$$+ \{\delta \xi(t)\} \left\{ \left(H - \gamma \phi_{xx} - \phi_x \left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x} \right) + \frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \right) \dot{\eta} + \right.$$

$$\left. + \left(\gamma \phi_{xy} + \phi_x \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y} \right) - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \right) \dot{\xi} \right\} -$$

$$- \{\delta \eta(t)\} \left\{ \left(\gamma \phi_{xy} + \phi_y \left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x} \right) + \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \right) \dot{\eta} + \right.$$

$$\left. \left. + \left(H - \gamma \phi_{yy} - \phi_y \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y} \right) - \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \right) \dot{\xi} \right\} \right]_{\substack{x = \xi(t) \\ y = \eta(t)}} dt \tag{6.15}$$

A necessary condition for the functional I to have an extremum subject to (6.1) and (6.5) is that $\delta I = 0$ for all non-zero arbitrary variations $\overline{\delta\phi}$, $\delta\phi$, $\delta\phi_x$, $\delta\phi_y$, δx and δy. We assume that the variations for each separate part of the boundary are independent of the variations for every other part. Hence we shall have the following equations to be satisfied.

$$\frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \phi_y}\right) + \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} = 0 \,, \quad (x,y) \in S \,, \tag{6.16}$$

$$\left[\gamma(\delta\phi_x dy - \delta\phi_y dx) + (\delta\phi)\left\{\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x}\right)dy - \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y}\right)dx\right\}\right] = 0, \quad (x,y) \in \Gamma_1 \,, \tag{6.17}$$

and

$$\begin{aligned}
&\gamma(\delta\phi_x \dot{\eta} - \delta\phi_y \dot{\xi}) + (\delta\phi)\left\{\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x} + \frac{\partial W}{\partial \phi}\right)\dot{\eta} - \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y} - \frac{\partial V}{\partial \phi}\right)\dot{\xi}\right\} + \\
&+ \{\delta\xi(t)\}\left[\left\{H - \gamma\phi_{xx} - \phi_x\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x}\right) + \frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta} + \right.\\
&\qquad \left. + \left\{\gamma\phi_{xy} + \phi_x\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y}\right) - \frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}\right] - \\
&- \{\delta\eta(t)\}\left[\left\{\gamma\phi_{xy} + \phi_y\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \gamma}{\partial x}\right) + \frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta} + \right.\\
&\qquad \left. + \left\{H - \gamma\phi_{yy} - \phi_y\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \gamma}{\partial y}\right) - \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}\right] = 0, \quad \text{on } \Gamma_2
\end{aligned} \tag{6.18}$$

on Γ_2 where $x = \xi(t)$; $y = \eta(t)$.

The conditions (6.17) and (6.18) must be consistent with the variation of the given conditions in (6.2) and (6.3) respectively.

By taking the variation of (6.2) we have

$$\frac{\partial M}{\partial \phi}\delta\phi + \frac{\partial M}{\partial \phi_x}\delta\phi_x + \frac{\partial M}{\partial \phi_y}\delta\phi_y = 0 \,, \quad (x,y) \in \Gamma_1 \,; \tag{6.19}$$

since we assume $x = \xi(t)$; $y = \eta(t)$, $t_0 \le t \le t_1$ on Γ_2 , from (6.3) we have

$$\frac{\partial N}{\partial \xi}\delta\xi + \frac{\partial N}{\partial \eta}\delta\eta + \frac{\partial N}{\partial \phi}\delta\phi(\xi,\eta) + \frac{\partial N}{\partial \phi_x}\delta\phi_x + \frac{\partial N}{\partial \phi_y}\delta\phi_y = 0 \;, \qquad x=\xi(t);\, y=\eta(t) \tag{6.20}$$

Hence, for consistency between (6.17) and (6.19), also (6.18) and (6.20) we obtain the transversality conditions as follows:

$$\frac{\left(\frac{\partial H}{\partial \phi_x}-\frac{\partial \gamma}{\partial x}\right)dy-\left(\frac{\partial H}{\partial \phi_y}-\frac{\partial \gamma}{\partial y}\right)dx}{\frac{\partial M}{\partial \phi}} = \frac{\gamma\, dy}{\frac{\partial M}{\partial \phi_x}} = \frac{-\gamma\, dx}{\frac{\partial M}{\partial \phi_y}} \;, \qquad (x,y)\in\Gamma_1 \tag{6.21}$$

and

$$\frac{\left\{H-\gamma\phi_{xx}-\phi_x\left(\frac{\partial H}{\partial \phi_x}-\frac{\partial \gamma}{\partial x}\right)+\frac{\partial W}{\partial \xi}-\frac{\partial V}{\partial \eta}-\frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta}+\left\{\gamma\phi_{xy}+\phi_x\left(\frac{\partial H}{\partial \phi_y}-\frac{\partial \gamma}{\partial y}\right)-\frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}}{\frac{\partial N}{\partial \xi}} =$$

$$= \frac{-\left\{\gamma\phi_{xy}+\phi_y\left(\frac{\partial H}{\partial \phi_x}-\frac{\partial \gamma}{\partial x}\right)+\frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta}-\left\{H-\gamma\phi_{yy}-\phi_y\left(\frac{\partial H}{\partial \phi_y}-\frac{\partial \gamma}{\partial y}\right)-\frac{\partial V}{\partial \eta}+\frac{\partial W}{\partial \xi}+\frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}}{\frac{\partial N}{\partial \eta}} =$$

$$= \frac{\left\{\frac{\partial H}{\partial \phi_x}-\frac{\partial \gamma}{\partial x}+\frac{\partial W}{\partial \phi}\right\}\dot{\eta}-\left\{\frac{\partial H}{\partial \phi_y}-\frac{\partial \gamma}{\partial y}-\frac{\partial V}{\partial \phi}\right\}\dot{\xi}}{\frac{\partial N}{\partial \phi}} = \frac{\gamma\dot{\eta}}{\frac{\partial N}{\partial \phi_x}} = \frac{-\gamma\dot{\xi}}{\frac{\partial N}{\partial \phi_y}} \;, \tag{6.22}$$

on Γ_2 where $x=\xi(t)$; $y=\eta(t)$; $t_0 \le t \le t_1$.

<u>Method 2. Gelfand and Fomin method</u>

Let us introduce two new functions $f(x,y)$ and $g(x,y)$ defined as

$$f(x,y) = \frac{\partial \phi}{\partial x} \quad ; \quad g(x,y) = \frac{\partial \phi}{\partial y} \;, \tag{6.23}$$

hence we can write (6.1) - (6.3) in the form

$$f_x + g_y = 0 , \qquad (x,y) \in S \tag{6.24}$$

$$\left.\begin{array}{ll} M(x,y,\phi,f,g) = 0, & (x,y) \in \Gamma_1 \\ N(x,y,\phi,f,g) = 0, & (x,y) \in \Gamma_2 , \; x = \xi(t); \; y = \eta(t) \end{array}\right\} \tag{6.25}$$

and

By using (6.4), (6.5) and (6.23) - (6.25) we write a new functional J^* of the form

$$J^* = \iint_S \Phi(x,y,\phi,\phi_x,\phi_y, f, f_x, f_y, g, g_x, g_y)\,dx\,dy + \oint_{\Gamma_1} \mu_1(s)\, M(x,y,\phi,f,g)\,ds - \int_{t_0}^{t_1} \left[V(\xi,\eta,\phi(\xi,\eta))\dot{\xi} + W(\xi,\eta,\phi(\xi,\eta))\dot{\eta} + \mu_2(t) N(\xi,\eta,\phi,f,g) \right] dt .$$

$$\therefore \; J^* = I_1^* + I_2^* - I_3^* \tag{6.26}$$

where

$$\Phi = H(x,y,\phi,\phi_x,\phi_y) + \lambda_1(x,y)(\phi_x - f) + \lambda_2(x,y)(\phi_y - g) + \lambda_3(x,y)(f_x + g_y), \tag{6.27}$$

$H = F(x,y,\phi,\phi_x,\phi_y) + \nu\, G(x,y,\phi,\phi_x,\phi_y)$ as defined in (6.7),

the functions $\lambda_1(x,y)$, $\lambda_2(x,y)$, $\lambda_3(x,y)$, $\mu_1(s)$, $\mu_2(t)$ and the constant ν are Lagrange multipliers. The functionals I_1^* , I_2^* and I_3^* are defined as follows:

$$I_1^* = \iint_S \Phi(x,y;\phi,\phi_x,\phi_y, f, f_x, f_y, g, g_x, g_y)\,dx\,dy , \tag{6.28}$$

$$I_2^* = \oint_{\Gamma_1} \mu_1(s)\, M(x,y,\phi,f,g)\,ds , \tag{6.29}$$

and

$$I_3^* = \int_{t_0}^{t_1} \left[V(\xi,\eta,\phi(\xi,\eta))\,\dot{\xi} + W(\xi,\eta,\phi(\xi,\eta))\,\dot{\eta} + \mu_2(t)\, N(\xi,\eta,\phi,f,g) \right] dt \tag{6.30}$$

The functional I_1^* in (6.28) involves three dependent variables ϕ , f , g and their first order derivatives, defined on the variable domain S . Applying the Gelfand/Fomin theorem (5.49) from Chapter 5 to this problem and by using (6.27) and (6.28) we can write the first variation of I_1^* in the form

$$\begin{aligned}
\delta I_1^* = & \iint_S \left[\left\{ \frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x}\right) - \frac{\partial \lambda_1}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \phi_y}\right) - \frac{\partial \lambda_2}{\partial y} \right\} \overline{\delta\phi} + \left\{ -\lambda_1 - \frac{\partial \lambda_3}{\partial x} \right\} \overline{\delta f} + \left\{ -\lambda_2 - \frac{\partial \lambda_3}{\partial y} \right\} \overline{\delta g} \right] dx\,dy \\
& + \oint_{\Gamma_1} \left[\left\{ \left(\frac{\partial H}{\partial \phi_x} + \lambda_1\right)\frac{dy}{ds} - \left(\frac{\partial H}{\partial \phi_y} + \lambda_2\right)\frac{dx}{ds} \right\} \delta\phi + \left\{ \lambda_3 \frac{dy}{ds} \right\} \delta f + \left\{ -\lambda_3 \frac{dx}{ds} \right\} \delta g \right] ds - \\
& - \int_{t_0}^{t_1} \Bigg[\left\{ \left(\frac{\partial H}{\partial \phi_x} + \lambda_1\right)\dot{\eta} - \left(\frac{\partial H}{\partial \phi_y} + \lambda_2\right)\dot{\xi} \right\} \delta\phi + \left\{ \lambda_3 \dot{\eta} \right\} \delta f + \left\{ -\lambda_3 \dot{\xi} \right\} \delta g + \\
& + \left\{ \left(H - \phi_x\left(\frac{\partial H}{\partial \phi_x} + \lambda_1\right) - f_x \lambda_3 \right)\dot{\eta} + \left(\phi_x\left(\frac{\partial H}{\partial \phi_y} + \lambda_2\right) + g_x \lambda_3 \right)\dot{\xi} \right\} \delta\xi(t) - \\
& - \left\{ \left(\phi_y\left(\frac{\partial H}{\partial \phi_x} + \lambda_1\right) + f_y \lambda_3 \right)\dot{\eta} + \left(H - \phi_y\left(\frac{\partial H}{\partial \phi_y} + \lambda_2\right) - g_y \lambda_3 \right)\dot{\xi} \right\} \delta\eta(t) \Bigg]_{\substack{x=\xi(t) \\ y=\eta(t)}} dt
\end{aligned} \tag{6.31}$$

where $\delta x = 0$; $\delta y = 0$ on Γ_1 , and $\delta x = \delta\xi(t)$; $\delta y = \delta\eta(t)$; $dx = \dot{\xi}\,dt$ and $dy = \dot{\eta}\,dt$ on Γ_2 .

As already mentioned in method 1, the first variation of the functionals I_2^* and I_3^* , defined in (6.29) and (6.30), can be written in the form

$$\delta I_2^* = \oint_{\Gamma_1} \left[\mu_1(s) \left\{ \frac{\partial M}{\partial \phi}\delta\phi + \frac{\partial M}{\partial f}\delta f + \frac{\partial M}{\partial g}\delta g \right\} \right] ds \tag{6.32}$$

and

$$\delta I_3^* = \int_{t_0}^{t_1} \Big[\Big\{ \Big(\frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \Big) \dot{\eta} - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \cdot \dot{\xi} + \mu_2(t) \frac{\partial N}{\partial \xi} \Big\} \delta \xi(t) +$$

$$+ \Big\{ \Big(\frac{\partial V}{\partial \eta} - \frac{\partial W}{\partial \xi} - \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \Big) \dot{\xi} - \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \cdot \dot{\eta} + \mu_2(t) \frac{\partial N}{\partial \eta} \Big\} \delta \eta(t) +$$

$$+ \Big\{ \frac{\partial V}{\partial \phi} \dot{\xi} + \frac{\partial W}{\partial \phi} \dot{\eta} \Big\} \delta \phi(\xi, \eta) + \mu_2(t) \Big\{ \frac{\partial N}{\partial \phi} \delta \phi + \frac{\partial N}{\partial f} \delta f + \frac{\partial N}{\partial g} \delta g \Big\} \Big] \, dt \tag{6.33}$$

It then follows from (6.26), (6.31), (6.32) and (6.33) that

$$\delta J^* = \iint_S \Big[\Big\{ \frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\Big(\frac{\partial H}{\partial \phi_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial H}{\partial \phi_y}\Big) - \frac{\partial \lambda_1}{\partial x} - \frac{\partial \lambda_2}{\partial y} \Big\} \delta \phi + \Big\{ -\lambda_1 - \frac{\partial \lambda_3}{\partial x} \Big\} \delta f + \Big\{ -\lambda_2 - \frac{\partial \lambda_3}{\partial y} \Big\} \delta g \Big] \, dx\, dy$$

$$+ \oint_{\Gamma_1} \Big[\Big\{ \Big(\frac{\partial H}{\partial \phi_x} + \lambda_1 \Big) \frac{dy}{ds} - \Big(\frac{\partial H}{\partial \phi_y} + \lambda_2 \Big) \frac{dx}{ds} + \mu_1(s) \frac{\partial M}{\partial \phi} \Big\} \delta \phi + \Big\{ \lambda_3 \frac{dy}{ds} + \mu_1(s) \frac{\partial M}{\partial f} \Big\} \delta f +$$

$$+ \Big\{ -\lambda_3 \frac{dx}{ds} + \mu_1(s) \frac{\partial M}{\partial g} \Big\} \delta g \Big] \, ds \; -$$

$$- \int_{t_0}^{t_1} \Big[\Big\{ \Big(\frac{\partial H}{\partial \phi_x} + \lambda_1 + \frac{\partial W}{\partial \phi} \Big) \dot{\eta} - \Big(\frac{\partial H}{\partial \phi_y} + \lambda_2 - \frac{\partial V}{\partial \phi} \Big) \dot{\xi} + \mu_2(t) \frac{\partial N}{\partial \phi} \Big\} \delta \phi + \Big\{ \lambda_3 \dot{\eta} + \mu_2(t) \frac{\partial N}{\partial f} \Big\} \delta f +$$

$$+ \Big\{ -\lambda_3 \dot{\xi} + \mu_2(t) \frac{\partial N}{\partial g} \Big\} \delta g + \Big\{ \Big(H - f_x \lambda_3 - \phi_x \Big(\frac{\partial H}{\partial \phi_x} + \lambda_1 \Big) + \frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \Big) \dot{\eta} +$$

$$+ \Big(g_x \lambda_3 + \phi_x \Big(\frac{\partial H}{\partial \phi_y} + \lambda_2 \Big) - \frac{\partial V}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \Big) \dot{\xi} + \mu_2(t) \frac{\partial N}{\partial \xi} \Big\} \delta \xi(t) - \Big\{ \Big(f_y \lambda_3 + \phi_y \Big(\frac{\partial H}{\partial \phi_x} + \lambda_1 \Big) +$$

$$+ \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \Big) \dot{\eta} - \mu_2(t) \frac{\partial N}{\partial \eta} + \Big(H - g_y \lambda_3 - \phi_y \Big(\frac{\partial H}{\partial \phi_y} + \lambda_2 \Big) - \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \Big) \dot{\xi} \Big\} \delta \eta(t) \Big]_{\substack{x = \xi(t) \\ y = \eta(t)}} \, dt \, . \tag{6.34}$$

As before, the necessary condition for I to have an extremum subject to (6.5) and (6.23) - (6.25) is $\delta J^* = 0$ for all arbitrary variations $\overline{\delta\phi}$, $\delta\phi$, $\overline{\delta f}$, δf , $\overline{\delta g}$, δg , $\delta\xi$ and $\delta\eta$. Hence we have the following equations to be satisfied.

$$\frac{\partial H}{\partial \phi}-\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \phi_y}\right)-\frac{\partial \lambda_1}{\partial x}-\frac{\partial \lambda_2}{\partial y}=0, \quad (x,y)\in S, \tag{6.35}$$

$$\left.\begin{aligned} -\lambda_1-\frac{\partial \lambda_3}{\partial x}&=0, \quad (x,y)\in S,\\ -\lambda_2-\frac{\partial \lambda_3}{\partial y}&=0, \quad (x,y)\in S, \end{aligned}\right\} \tag{6.36}$$

$$\left.\begin{aligned} \left(\frac{\partial H}{\partial \phi_x}+\lambda_1\right)\frac{dy}{ds}-\left(\frac{\partial H}{\partial \phi_y}+\lambda_2\right)\frac{dx}{ds}+\mu_1(s)\frac{\partial M}{\partial \phi}&=0, \quad (x,y)\in \Gamma_1\\ \lambda_3\frac{dy}{ds}+\mu_1(s)\frac{\partial M}{\partial f}&=0, \quad (x,y)\in \Gamma_1\\ -\lambda_3\frac{dx}{ds}+\mu_1(s)\frac{\partial M}{\partial g}&=0, \quad (x,y)\in \Gamma_1 \end{aligned}\right\} \tag{6.37}$$

and on Γ_2 we have

$$\begin{aligned}
&\left[\left\{H-f_x\lambda_3-\phi_x\left(\frac{\partial H}{\partial \phi_x}+\lambda_1\right)+\frac{\partial W}{\partial \xi}-\frac{\partial V}{\partial \eta}-\frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta}+\left\{g_x\lambda_3+\phi_x\left(\frac{\partial H}{\partial \phi_y}+\lambda_2\right)-\frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}+\right.\\
&\left.+\mu_2(t)\frac{\partial N}{\partial \xi}\right]=0,\\
&\left[\left\{f_y\lambda_3+\phi_y\left(\frac{\partial H}{\partial \phi_x}+\lambda_1\right)+\frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta}+\left\{H-g_y\lambda_3-\phi_y\left(\frac{\partial H}{\partial \phi_y}+\lambda_2\right)-\frac{\partial V}{\partial \eta}+\frac{\partial W}{\partial \xi}+\frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}-\right.\\
&\left.-\mu_2(t)\frac{\partial N}{\partial \eta}\right]=0,\\
&\left[\left(\frac{\partial H}{\partial \phi_x}+\lambda_1+\frac{\partial W}{\partial \phi}\right)\dot{\eta}-\left(\frac{\partial H}{\partial \phi_y}+\lambda_2-\frac{\partial V}{\partial \phi}\right)\dot{\xi}+\mu_2(t)\frac{\partial N}{\partial \phi}\right]=0,\\
&\left[\lambda_3\dot{\eta}+\mu_2(t)\frac{\partial N}{\partial f}\right]=0,\\
&\left[-\lambda_3\dot{\xi}+\mu_2(t)\frac{\partial N}{\partial g}\right]=0.
\end{aligned} \tag{6.38}$$

By using (6.23) and (6.36), we can write (6.35), (6.37) and (6.38) in the form

$$\frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \phi_y}\right) + \frac{\partial^2 \lambda_3}{\partial x^2} + \frac{\partial^2 \lambda_3}{\partial y^2} = 0 \ , \quad (x,y) \in S \ . \tag{6.39}$$

$$\frac{\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x}\right)\frac{dy}{ds} - \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y}\right)\frac{dx}{ds}}{\frac{\partial M}{\partial \phi}} = \frac{\lambda_3 \frac{dy}{ds}}{\frac{\partial M}{\partial \phi_x}} = \frac{-\lambda_3 \frac{dx}{ds}}{\frac{\partial M}{\partial \phi_y}} = -\mu_1(s) \ ,$$

$$(x,y) \in \Gamma_1 \tag{6.40}$$

and

$$\frac{\left\{H - \lambda_3 \phi_{xx} - \phi_x\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x}\right) + \frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta} + \left\{\lambda_3 \phi_{xy} + \phi_x\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y}\right) - \frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}}{\frac{\partial N}{\partial \xi}} =$$

$$= \frac{-\left\{\lambda_3 \phi_{xy} + \phi_y\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x}\right) + \frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}\dot{\eta} - \left\{H - \lambda_3 \phi_{yy} - \phi_y\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y}\right) - \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}\dot{\xi}}{\frac{\partial N}{\partial \eta}} =$$

$$= \frac{\left\{\frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x} + \frac{\partial W}{\partial \phi}\right\}\dot{\eta} - \left\{\frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y} - \frac{\partial V}{\partial \phi}\right\}\dot{\xi}}{\frac{\partial N}{\partial \phi}} = \frac{\lambda_3 \dot{\eta}}{\frac{\partial N}{\partial \phi_x}} = \frac{-\lambda_3 \dot{\xi}}{\frac{\partial N}{\partial \phi_y}} = -\mu_2(t) \ , \tag{6.41}$$

where $x = \xi(t)$; $y = \eta(t)$ on Γ_2 .

It will now be observed that the condition (6.39) with the natural boundary condition (6.40) and the transversality condition (6.41) of method 2 are the same as in (6.16), (6.21) and (6.22) of method 1 where $\lambda_3(x,y) \equiv \gamma(x,y)$.

Not all the equations in (6.22) or (6.41) are independent. In fact it can be verified as follows:

On Γ_2 : $x = \xi(t)$; $y = \eta(t)$ and from the boundary condition as defined in (6.3), i.e.,

$$N(x, y, \phi, \phi_x, \phi_y) = 0 ,$$

we have

$$\frac{\partial N}{\partial \xi}\dot{\xi} + \frac{\partial N}{\partial \eta}\dot{\eta} + \frac{\partial N}{\partial \phi}\cdot\frac{d\phi}{dt} + \frac{\partial N}{\partial \phi_x}\cdot\frac{d\phi_x}{dt} + \frac{\partial N}{\partial \phi_y}\cdot\frac{d\phi_y}{dt} = 0 \tag{6.42}$$

where

$$\frac{d\phi}{dt} = \phi_x\dot{\xi} + \phi_y\dot{\eta} \;;\; \frac{d\phi_x}{dt} = \phi_{xx}\dot{\xi} + \phi_{xy}\dot{\eta} \quad\text{and}\quad \frac{d\phi_y}{dt} = \phi_{xy}\dot{\xi} + \phi_{yy}\dot{\eta} \tag{6.43}$$

It is clear that (6.41) can be written in the form

$$A(x,y)\dot{\eta} + B(x,y)\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \xi} = 0 , \tag{6.44}$$

$$C(x,y)\dot{\eta} + D(x,y)\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \eta} = 0 , \tag{6.45}$$

$$E(x,y)\dot{\eta} + G(x,y)\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \phi} = 0 , \tag{6.46}$$

$$\lambda_3\dot{\eta} + \mu_2(t)\frac{\partial N}{\partial \phi_x} = 0 , \tag{6.47}$$

and

$$-\lambda_3\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \phi_y} = 0 . \tag{6.48}$$

where

$$A(x,y) \equiv \left\{H - \lambda_3\phi_{xx} - \phi_x\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x}\right) + \frac{\partial W}{\partial \xi} - \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}$$

$$B(x,y) \equiv \left\{\lambda_3\phi_{xy} + \phi_x\left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y}\right) - \frac{\partial V}{\partial \phi}\cdot\frac{\partial \phi}{\partial \xi}\right\}$$

$$C(x,y) \equiv -\left\{\lambda_3\phi_{xy} + \phi_y\left(\frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x}\right) + \frac{\partial W}{\partial \phi}\cdot\frac{\partial \phi}{\partial \eta}\right\}$$

$$D(x,y) \equiv -\left\{ H - \lambda_3 \phi_{yy} - \phi_y \left(\frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y} \right) - \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \phi} \cdot \frac{\partial \phi}{\partial \xi} \right\}$$

$$E(x,y) \equiv \left\{ \frac{\partial H}{\partial \phi_x} - \frac{\partial \lambda_3}{\partial x} + \frac{\partial W}{\partial \phi} \right\}$$

$$G(x,y) \equiv -\left\{ \frac{\partial H}{\partial \phi_y} - \frac{\partial \lambda_3}{\partial y} - \frac{\partial V}{\partial \phi} \right\}$$

Using (6.42) and (6.43) it may be verified that equations (6.44) - (6.48) are connected by the identity

$$\left\{ A(x,y)\dot{\eta} + B(x,y)\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \xi} \right\}\dot{\xi} + \left\{ C(x,y)\dot{\eta} + D(x,y)\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \eta} \right\}\dot{\eta} +$$

$$+ \left\{ E(x,y)\dot{\eta} + G(x,y)\dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \phi} \right\}\left\{ \phi_x \dot{\xi} + \phi_y \dot{\eta} \right\} + \left\{ \lambda_3 \dot{\eta} + \mu_2(t)\frac{\partial N}{\partial \phi_x} \right\}\left\{ \phi_{xx}\dot{\xi} + \phi_{xy}\dot{\eta} \right\} +$$

$$+ \left\{ -\lambda_3 \dot{\xi} + \mu_2(t)\frac{\partial N}{\partial \phi_y} \right\}\left\{ \phi_{xy}\dot{\xi} + \phi_{yy}\dot{\eta} \right\} \equiv 0 .$$

The same is true in the case of equation (6.22).

CHAPTER 7

A PARTICULAR TWO DIMENSIONAL HARMONIC CONTROL PROBLEM IN AN ANNULAR REGION WITH THE SHAPE OF THE DOMAIN AS THE CONTROL.

In this chapter we consider a particular two dimensional problem using plane polar coordinates (r, θ). The problem can be stated as follows:

Let $\phi(r,\theta)$ be a harmonic function so that

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0 , \qquad (r,\theta) \in S, \tag{7.1}$$

or

$$\nabla^2\phi(r,\theta) = 0 , \qquad (r,\theta) \in S,$$

where ∇^2 is a Laplacian operator and S is a doubly connected domain bounded by two closed curves C_1 and C_2 as shown in Fig. 30 . We assume that the closed curve C_2 is a given fixed circle $r = R$; $0 \leq \theta \leq 2\pi$ and the closed curve C_1 is a smooth curve of the form $r = g(\theta)$; $0 \leq \theta \leq 2\pi$ and $0 < g(\theta) < R$ (see Fig. 30). The function $\phi(r,\theta)$ is also assumed to be continuous on the boundaries.

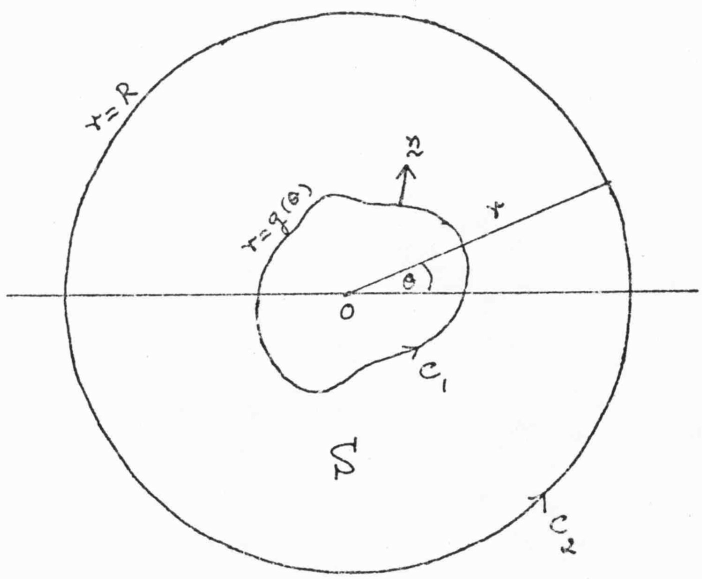


Fig. 30

The boundary conditions on the boundaries C_1 and C_2 are given and defined as follows:

$$A(\theta,g,g')\frac{\partial\phi}{\partial r} + B(\theta,g,g')\frac{\partial\phi}{\partial\theta} + C(\theta,g,g')\phi = F_2(\theta,g,g') \qquad \text{on } C_1 \text{ ,} \tag{7.2}$$

where $r = g(\theta)$; $0 \leq \theta \leq 2\pi$ on C_1 ; $g' \equiv \dfrac{dg(\theta)}{d\theta}$ and $A - Bg' \neq 0$;

and $\phi(R,\theta) = \beta$ on C_2 : $r = R$; $0 \leq \theta \leq 2\pi$, (7.3)

where β is a constant.

If the curve C_1 : $r = g(\theta)$; $0 \leq \theta \leq 2\pi$ is known then the corresponding $\phi(r,\theta)$ can be calculated from the boundary value problem (7.1) - (7.3), and a different curve C_1 gives a different value of $\phi(r,\theta)$.

Here we wish to find the curve C_1 : $r = g(\theta)$; $0 \leq \theta \leq 2\pi$ so that the functional I , defined as

$$I = \iint_S \left[\phi_r^2 + \frac{1}{r^2}\phi_\theta^2 \right] r\,dr\,d\theta \text{ ,} \tag{7.4}$$

has an extremum subject to the given area constraint

$$\iint_S r\,dr\,d\theta = K \text{ ,} \tag{7.5}$$

where K is a constant.

In this problem the optimum shape of C_1 is closely linked with the nature of the boundary condition (7.2) and, as an elementary example, we may note that when condition (7.2) is replaced by the simple Neumann condition

$$\frac{\partial\phi}{\partial r} = a_0 \quad \text{, on } C_1 \qquad (a_0 \text{ is a constant}),$$

and we disregard the constraint (7.5), the problem has the following simple solution. We can in this case look for a solution in which

C_1 is a circle of radius R_0 , where R_0 is unknown, so that

$$\frac{\partial\phi}{\partial r} = a_0 \quad , \qquad r = R_0$$

Then the solution for ϕ in S is a function of r only and is given by

$$\phi = a_0 R_0 \log \frac{r}{R} + \beta$$

We can now find the value of I , as defined in (7.4), and if we now regard I as a function of R_0 it follows that I attains an extremum when $\frac{dI}{dR_0} = 0$, that is

$$2\pi a_0^2 R_0 \left(2 \log \frac{R}{R_0} - 1\right) = 0$$

if $R_0 \neq 0$, thus $\qquad R_0 = R e^{-1/2}$

and it is easily shown that I attains a maximum in this case at $R_0 = \frac{R}{\sqrt{e}}$.

We can look upon the above problem as a steady flow problem in hydrodynamics in which the liquid supply to the annulus is a_0 per unit length across C_1 . The total flow of liquid across C_1 is $Q = 2\pi R_0 a_0$ and the integral I is a measure of the total kinetic energy of the liquid in the annulus. We see that as $R_0 \to R$, I must tend to zero since the annulus area tends to zero; furthermore as $R_0 \to 0$ the liquid supply $Q \to 0$ and thus we have $I = 0$ when $R_0 = 0$ and when $R_0 = R$, hence I being a continuous function of R_0 it follows from the mean value theorem that I must have a maximum at some point between $R_0 = 0$ and $R_0 = R$.

Next let us turn attention to the problem stated earlier in the chapter. We shall use the Forsyth method which has been mentioned in Chapter 6 to solve this problem.

Let us set the new functional J as follows:

$$J = \iint_S \left[H(r,\theta,\phi,\phi_r,\phi_\theta) + \gamma(r,\theta)\left\{ r\phi_{rr} + \phi_r + \frac{1}{r}\phi_{\theta\theta} \right\} \right] dr\,d\theta , \tag{7.6}$$

where

$$H(r,\theta,\phi,\phi_r,\phi_\theta) = r\phi_r^2 + \frac{1}{r}\phi_\theta^2 + \nu r \tag{7.7}$$

the function $\gamma(r,\theta)$ and a constant ν are Lagrange multipliers.

Here J is a functional involving two independent variables r and θ, one dependent variable ϕ and its partial derivatives up to second order, and since we also can show that

$$\iint_S \left[\frac{\partial P}{\partial r} + \frac{\partial Q}{\partial \theta} \right] dr\,d\theta = \oint_\Gamma \left[P\,d\theta - Q\,dr \right] ,$$

where Γ is a curve bounding a domain S. Hence, by using (5.45) - (5.47) in which we write r and θ instead of x and y, also p, q, r, s and t denote $\frac{\partial\phi}{\partial r}$, $\frac{\partial\phi}{\partial\theta}$, $\frac{\partial^2\phi}{\partial r^2}$, $\frac{\partial^2\phi}{\partial r\partial\theta}$ and $\frac{\partial^2\phi}{\partial\theta^2}$ respectively, the first variation δJ of (7.6) can be written in the form

$$\delta J = \iint_S (\overline{\delta\phi}) \left[\frac{\partial H}{\partial \phi} - \frac{\partial}{\partial r}\left(\frac{\partial H}{\partial \phi_r} + \gamma \right) - \frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial \phi_\theta} \right) + \frac{\partial^2}{\partial r^2}(r\gamma) + \frac{1}{r}\frac{\partial^2 \gamma}{\partial \theta^2} \right] dr\,d\theta +$$

$$+ \oint_{C_2} \left[P\,d\theta - Q\,dr \right] - \oint_{C_1} \left[P\,d\theta - Q\,dr \right] , \tag{7.8}$$

where H is defined in (7.7) and the functions P and Q are obtained from (5.46) and (5.47) as follows:

On C_2 : $r = R$; $0 \le \theta \le 2\pi$, a fixed given circle, we shall have

$\delta r = 0$, $\delta\theta = 0$, $dr = 0$ and since from the condition (7.3), we also have $\delta\phi = 0$, hence

$$\left.\begin{aligned} P &= r\gamma\,\delta\phi_r \\ Q &= \frac{\gamma}{r}\,\delta\phi_\theta \end{aligned}\right\} \text{ on } C_2 \tag{7.9}$$

On C_1 : $r = g(\theta)$; $0 \le \theta \le 2\pi$, we shall have $\delta r = \delta g(\theta)$, $\delta\theta = 0$ and $dr = g'd\theta$, hence

$$\left.\begin{aligned} P &= \{r\gamma\delta\phi_r\} + (\delta\phi)\left\{\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r}\right\} + \{\delta g(\theta)\}\left[H - \phi_r\left\{\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r}\right\} - r\gamma\phi_{rr}\right] \\ Q &= \left\{\frac{\gamma}{r}\delta\phi_\theta\right\} + (\delta\phi)\left\{\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right\} - \{\delta g(\theta)\}\left[\phi_r\left\{\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right\} + \frac{\gamma}{r}\phi_{r\theta}\right] \end{aligned}\right\} \tag{7.10}$$

By substituting (7.9) and (7.10) into (7.8), we obtain

$$\begin{aligned} \delta J = &\iint_S (\overline{\delta\phi})\left[\frac{\partial H}{\partial\phi} - \frac{\partial}{\partial r}\left(\frac{\partial H}{\partial\phi_r} + \gamma\right) - \frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial\phi_\theta}\right) + \frac{\partial^2}{\partial r^2}(r\gamma) + \frac{1}{r}\frac{\partial^2\gamma}{\partial\theta^2}\right] dr\,d\theta + \int_0^{2\pi}\left[r\gamma\delta\phi_r\right]_{r=R} d\theta - \\ &- \int_0^{2\pi}\Big[r\gamma\left\{\delta\phi_r - \frac{g'}{r^2}\delta\phi_\theta\right\} + (\delta\phi)\left\{\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right)\right\} + \\ &+ \{\delta g(\theta)\}\left\{H - r\gamma\phi_{rr} + \frac{\gamma g'}{r}\phi_{r\theta} - \phi_r\left(\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right)\right)\right\}\Big]_{r=g(\theta)} d\theta . \end{aligned} \tag{7.11}$$

A necessary condition for I in (7.4) to have an extremum subject to the constraints (7.1) and (7.5) is that $\delta J = 0$ which leads to the following equations.

$$\frac{\partial H}{\partial\phi} - \frac{\partial}{\partial r}\left(\frac{\partial H}{\partial\phi_r}\right) - \frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial\phi_\theta}\right) + r\frac{\partial^2\gamma}{\partial r^2} + \frac{\partial\gamma}{\partial r} + \frac{1}{r}\frac{\partial^2\gamma}{\partial\theta^2} = 0 , \quad (r,\theta) \in S \tag{7.12}$$

Since $\delta\phi_r$ is arbitrary on C_2 $\therefore$ $\delta\phi_r \neq 0$ and then its coefficient is zero, i.e.,

$$\gamma(R,\theta) = 0, \text{ on } C_2 : \; r = R \;, \; 0 \le \theta \le 2\pi. \qquad (7.13)$$

and

$$\int_0^{2\pi} \Big[r\gamma\Big\{\delta\phi_r - \frac{g'}{r^2}\delta\phi_\theta\Big\} + (\delta\phi)\Big\{\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\Big(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\frac{\partial \gamma}{\partial \theta}\Big)\Big\} +$$

$$+ \{\delta g(\theta)\}\Big\{ H - r\gamma\phi_{rr} + \frac{\gamma g'}{r}\phi_{r\theta} - \phi_r\Big(\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\Big(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\frac{\partial \gamma}{\partial \theta}\Big)\Big)\Big\}\Big]_{r=g(\theta)} d\theta = 0.$$

(7.14)

On the boundary C_1 , since we have $r = g(\theta)$, thus

$$\frac{d}{d\theta}\{\delta\phi(r,\theta)\} = \Big[g'\cdot\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}\Big]\delta\phi(r,\theta) = g'\cdot\delta\phi_r + \delta\phi_\theta \qquad (7.15)$$

By using (7.15), we can write (7.14) in the form

$$\int_0^{2\pi} \Big[r\gamma\Big\{1 + g'^2/g^2\Big\}\delta\phi_r - \frac{r\gamma g'}{g^2}\cdot\frac{d}{d\theta}(\delta\phi) + \Big\{\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\Big(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\frac{\partial \gamma}{\partial \theta}\Big)\Big\}\delta\phi +$$

$$+ \Big\{ H - r\gamma\phi_{rr} + \frac{\gamma g'}{r}\phi_{r\theta} - \phi_r\Big(\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\Big(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\frac{\partial \gamma}{\partial \theta}\Big)\Big)\Big\}\delta g(\theta)\Big]_{r=g(\theta)} d\theta = 0.$$

(7.16)

Taking the variation of the given boundary condition on C_1 which is defined in (7.2), we have

$$A\,\delta\phi_r + B\,\delta\phi_\theta + C\,\delta\phi = \Big[\frac{\partial F_2}{\partial g} - \frac{\partial A}{\partial g}\phi_r - \frac{\partial B}{\partial g}\phi_\theta - \frac{\partial C}{\partial g}\phi\Big]\delta g(\theta) +$$

$$+ \Big[\frac{\partial F_2}{\partial g'} - \frac{\partial A}{\partial g'}\phi_r - \frac{\partial B}{\partial g'}\phi_\theta - \frac{\partial C}{\partial g'}\phi\Big]\delta g' ,$$

on C_1 : $r = g(\theta)$; $0 \le \theta \le 2\pi$, and by using (7.15), we obtain

$$\left\{A-Bg'\right\}\delta\phi_r + B\frac{d}{d\theta}(\delta\phi) + C\delta\phi = \left[\frac{\partial F_2}{\partial g} - \frac{\partial A}{\partial g}\phi_r - \frac{\partial B}{\partial g}\phi_\theta - \frac{\partial C}{\partial g}\phi\right]\delta g(\theta) +$$
$$+\left[\frac{\partial F_2}{\partial g'} - \frac{\partial A}{\partial g'}\phi_r - \frac{\partial B}{\partial g'}\phi_\theta - \frac{\partial C}{\partial g'}\phi\right]\delta g', \quad \text{on } C_1 \tag{7.17}$$

We are assuming that $A - Bg' \neq 0$, and then by substituting $\delta\phi_r$ from (7.17) into (7.16), we have

$$\int_0^{2\pi}\Bigg[\left\{-g\left(B+\frac{Ag'}{g^2}\right)\cdot\frac{\gamma}{(A-Bg')}\right\}\frac{d}{d\theta}(\delta\phi) + \left\{\frac{\partial H}{\partial \phi_r} - r\frac{\partial\gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right) - \frac{g(1+g'^2/g^2)\gamma C}{(A-Bg')}\right\}\delta\phi +$$
$$+\left\{H - r\gamma\phi_{rr} + \frac{\gamma g'}{r}\phi_{r\theta} - \phi_r\left(\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right)\right) + \frac{g(1+g'^2/g^2)\gamma}{(A-Bg')}\cdot\left(\frac{\partial F_2}{\partial g} - \phi_r\frac{\partial A}{\partial g} - \phi_\theta\frac{\partial B}{\partial g} -\right.\right.$$
$$\left.\left. - \phi\frac{\partial C}{\partial g}\right)\right\}\delta g(\theta) + \left\{\frac{g(1+g'^2/g^2)}{(A-Bg')}\cdot\gamma\cdot\left(\frac{\partial F_2}{\partial g'} - \phi_r\frac{\partial A}{\partial g'} - \phi_\theta\frac{\partial B}{\partial g'} - \phi\frac{\partial C}{\partial g'}\right)\right\}\delta g'\Bigg]_{r=g(\theta)} d\theta = 0.$$

By performing an integration by parts of the first and the last integrals and using the assumption that these functions are single-valued, we obtain

$$\int_0^{2\pi}\Bigg[\left\{\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right) - \frac{g(1+g'^2/g^2)\gamma C}{(A-Bg')} + \frac{d}{d\theta}\left(g\frac{\gamma\cdot(B+Ag'/g^2)}{(A-Bg')}\right)\right\}_{r=g(\theta)}\delta\phi +$$
$$+\left\{H - r\gamma\phi_{rr} + \frac{\gamma g'}{r}\phi_{r\theta} - \phi_r\left(\frac{\partial H}{\partial\phi_r} - r\frac{\partial\gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial\phi_\theta} - \frac{1}{r}\frac{\partial\gamma}{\partial\theta}\right)\right) + \frac{g(1+g'^2/g^2)\gamma}{(A-Bg')}\cdot\left(\frac{\partial F_2}{\partial g} - \phi_r\frac{\partial A}{\partial g} - \phi_\theta\frac{\partial B}{\partial g} - \phi\frac{\partial C}{\partial g}\right) -\right.$$
$$\left.-\frac{d}{d\theta}\left(\frac{g(1+g'^2/g^2)\gamma}{(A-Bg')}\cdot\left(\frac{\partial F_2}{\partial g'} - \phi_r\frac{\partial A}{\partial g'} - \phi_\theta\frac{\partial B}{\partial g'} - \phi\cdot\frac{\partial C}{\partial g'}\right)\right)\right\}_{r=g(\theta)}\delta g(\theta)\Bigg]_{r=g(\theta)} d\theta = 0, \quad \text{on } C_1.$$

Since on C_1, $\delta\phi \neq 0$, $\delta g \neq 0$, the following conditions must be satisfied on $\mathcal{C}_1$.

$$\left[\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\frac{\partial \gamma}{\partial \theta}\right) - \frac{\gamma g(1+g'^2/g^2)c}{(A-Bg')} + \frac{d}{d\theta}\left\{\frac{g.(B+Ag'/g^2)\gamma}{(A-Bg')}\right\}_{r=g(\theta)}\right]_{r=g(\theta)} = 0 ,$$

(7.18)

and

$$\left[\frac{d}{d\theta}\left\{\frac{g(1+g'^2/g^2)\gamma}{(A-Bg')}\cdot\left(\frac{\partial F_2}{\partial g'} - \phi_r\frac{\partial A}{\partial g'} - \phi_\theta\frac{\partial B}{\partial g'} - \phi\frac{\partial C}{\partial g'}\right)\right\}_{r=g(\theta)} - H + g\gamma\phi_{rr} - \frac{\gamma g'}{g}\phi_{r\theta} + \right.$$

$$\left. + \phi_r\left\{\frac{g(1+g'^2/g^2)\gamma c}{(A-Bg')} - \frac{d}{d\theta}\left(\frac{g(B+Ag'/g^2)\gamma}{(A-Bg')}\right)\right\}_{r=g(\theta)} - \frac{g(1+g'^2/g^2)\gamma}{(A-Bg')}\cdot\left(\frac{\partial F_2}{\partial g} - \phi_r\frac{\partial A}{\partial g} - \phi_\theta\frac{\partial B}{\partial g} - \phi\frac{\partial C}{\partial g}\right)\right]_{r=g(\theta)} = 0$$

(7.19)

where H is defined in (7.7).

Therefore, we conclude that if the functional I in (7.4) has an extremum subject to the constraints (7.1) and (7.5) and the boundary conditions (7.2) and (7.3) then the conditions (7.12), (7.13), (7.18) and (7.19) must be satisfied. The optimum curve $\mathcal{C}_1$: $r = g(\theta)$, $0 \leq \theta \leq 2\pi$ will be found from these necessary conditions.

Let us consider when the boundary condition on $\mathcal{C}_1$ in (7.2) is replaced by

$$a\frac{\partial \phi}{\partial n} + b\frac{\partial \phi}{\partial s} + k\phi = F_1(r,\theta) \qquad \text{on } \mathcal{C}_1 : r = g(\theta) \ ; 0 \leq \theta \leq 2\pi,$$

(7.20)

where a , b and k are constants, $\frac{\partial}{\partial n}$ is the partial derivative operator, along the inward normal $\underset{\sim}{n}$ to the curve $\mathcal{C}_1$

(see Fig. 30), and $\frac{\partial}{\partial s}$ is the partial derivative along the curve C_1. The function $F_1(r,\theta)$ is prescribed and assumed to have the continuous derivatives up to the second order.

Let us find the relations between $\frac{\partial\phi}{\partial n}$ and $\frac{\partial\phi}{\partial r}$, $\frac{\partial\phi}{\partial\theta}$ also between $\frac{\partial\phi}{\partial s}$ and $\frac{\partial\phi}{\partial r}$, $\frac{\partial\phi}{\partial\theta}$.

Since we know that

$$\left.\begin{aligned}\frac{\partial\phi}{\partial n} &= \frac{\partial\phi}{\partial x}\cdot\frac{dy}{ds} - \frac{\partial\phi}{\partial y}\cdot\frac{dx}{ds}\\ \frac{\partial\phi}{\partial s} &= \frac{\partial\phi}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{ds}\end{aligned}\right\} \qquad (7.21)$$

and in polar coordinates, it is easy to find the following relations

$$\left.\begin{aligned}\phi_x &= (\cos\theta)\,\phi_r - \left(\frac{\sin\theta}{r}\right)\phi_\theta\\ \phi_y &= (\sin\theta)\,\phi_r + \left(\frac{\cos\theta}{r}\right)\phi_\theta\end{aligned}\right\} \qquad (7.22)$$

On C_1 : $r = g(\theta)$; $0 \le \theta \le 2\pi$, we can find that $ds = g\sqrt{1+g'^2/g^2}\,d\theta$ thus

$$\left.\begin{aligned}\frac{dx}{ds} &\approx \frac{dx}{g\sqrt{1+g'^2/g^2}\,d\theta} = \left[-\sin\theta + \frac{g'}{g}\cos\theta\right]\cdot\frac{1}{\sqrt{1+g'^2/g^2}}\\ \frac{dy}{ds} &\approx \frac{dy}{g\sqrt{1+g'^2/g^2}\,d\theta} = \left[\cos\theta + \frac{g'}{g}\sin\theta\right]\cdot\frac{1}{\sqrt{1+g'^2/g^2}}\end{aligned}\right\} \qquad (7.23)$$

Substituting (7.22) and (7.23) into (7.21), we obtain

$$\frac{\partial\phi}{\partial n} = \left[\phi_r - \frac{g'}{g^2}\phi_\theta\right]\cdot\frac{1}{\sqrt{1+g'^2/g^2}} \quad , \text{ on } C_1 : r = g(\theta) \;, 0 \le \theta \le 2\pi, \qquad (7.24)$$

and

$$\frac{\partial \phi}{\partial \lambda} = \left[\frac{g'}{g}\phi_r + \frac{1}{g}\phi_\theta\right] \cdot \frac{1}{\sqrt{1+g'^2/g^2}} \quad , \text{ on } C_1 : r = g(\theta) \quad , 0 \leq \theta \leq 2\pi,$$

(7.25)

By using (7.24) and (7.25), the boundary condition (7.20) can be written in the form

$$\left(a + \frac{bg'}{g}\right)\phi_r + \left(\frac{b}{g} - \frac{ag'}{g^2}\right)\phi_\theta + k\sqrt{1+g'^2/g^2}\,\phi = F_1(r,\theta).\sqrt{1+g'^2/g^2} \quad , \quad \text{on } C_1$$

(7.26)

Comparing (7.26) with (7.2), we have

$$A \equiv \left(a + \frac{bg'}{g}\right) \;;\; B \equiv \left(\frac{b}{g} - \frac{ag'}{g^2}\right);\; C \equiv k\sqrt{1+g'^2/g^2} \;;\; F \equiv F_1(g,\theta)\sqrt{1+g'^2/g^2}$$

$$\therefore \; A - Bg' \equiv a(1 + g'^2/g^2) \quad \text{and} \quad \left(B + \frac{g'A}{g^2}\right) \equiv \frac{b}{g}\left(1 + \frac{g'^2}{g^2}\right).$$

Hence the transversality conditions on C_1 in (7.18) and (7.19) corresponding to the boundary condition (7.20) will be

$$\left[\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\cdot\frac{\partial \gamma}{\partial \theta}\right) - \frac{g\gamma k\sqrt{1+g'^2/g^2}}{a} + \frac{b}{a}\left\{\frac{\partial \gamma}{\partial r}\cdot g' + \frac{\partial \gamma}{\partial \theta}\right\}\right]_{r=g(\theta)} = 0,$$

(7.27)

and

$$\left[\frac{d}{d\theta}\left\{\frac{\gamma}{a}\cdot\left(b\phi_r - \frac{a}{g}\phi_\theta - \frac{(F_1 - k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{g'}{g}\right)\right\}_{r=g(\theta)} + H - g\gamma\phi_{rr} + \frac{\gamma g'}{g}\phi_{r\theta} - \frac{g\gamma k\phi_r\sqrt{1+g'^2/g^2}}{a} +\right.$$

$$+ \frac{b}{a}\phi_r\left(\frac{\partial \gamma}{\partial r}g' + \frac{\partial \gamma}{\partial \theta}\right) + \frac{\gamma}{a}\left\{\frac{bg'}{g}\phi_r + \frac{b}{g}\phi_\theta - \frac{2ag'}{g^2}\phi_\theta + g\frac{\partial F_1}{\partial g}\sqrt{1+g'^2/g^2} - \right.$$

$$\left.\left. - \frac{(F_1 - k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{g'^2}{g^2}\right\}\right]_{r=g(\theta)} = 0. \qquad (7.28)$$

For the case when $a = 1$, $b = 0$ i.e., when (7.20) is replaced by the boundary condition

$$\frac{\partial\phi}{\partial n} + k\phi = F_1(r,\theta), \quad \text{on } C_1 : r = g(\theta) \; ; 0 \le \theta \le 2\pi, \tag{7.29}$$

the transversality conditions on C_1 for this case will follow from (7.27) and (7.28) as follows:

$$\left[\frac{\partial H}{\partial \phi_r} - r\frac{\partial \gamma}{\partial r} - g'\cdot\left(\frac{\partial H}{\partial \phi_\theta} - \frac{1}{r}\frac{\partial \gamma}{\partial \theta}\right) - g\gamma k\sqrt{1+g'^2/g^2} \right]_{r=g(\theta)} = 0 \quad , \text{ on } C_1 \tag{7.30}$$

and

$$\Bigg[\frac{d}{d\theta}\left\{ \gamma\left(\frac{\phi_\theta}{g} + \frac{(F_1 - k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{g'}{g} \right)\right\}_{r=g(\theta)} - H + g\gamma\phi_{rr} - \frac{\gamma g'}{g}\phi_{r\theta} + \phi_r g\gamma k\sqrt{1+g'^2/g^2} +$$

$$+ \gamma\left\{ \frac{2g'}{g^2}\phi_\theta - g\frac{\partial F_1}{\partial g}\sqrt{1+g'^2/g^2} + \frac{(F_1 - k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{g'^2}{g^2} \right\}\Bigg]_{r=g(\theta)} = 0 \quad , \text{ on } C_1 \tag{7.31}$$

where H is defined in (7.7).

We shall now discuss further the problem of finding the curve C_1 : $r = g(\theta)$; $0 \le \theta \le 2\pi$ which provides an extremum of the functional I defined in (7.4) subject to the constraints (7.5) and the function $\phi(r,\theta)$ satisfies the following boundary value problem.

$$\left.\begin{aligned} \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} &= 0, && (r,\theta) \in S, \\ \frac{\partial\phi}{\partial n} + k\phi &= F_1(r,\theta), && \text{on } C_1 : r = g(\theta) ; 0 \le \theta \le 2\pi, \\ \phi(R,\theta) &= \beta, && \text{on } C_2 : r = R ; 0 \le \theta \le 2\pi, \end{aligned}\right\} \tag{7.32}$$

where S is a domain $0 < g(\theta) \le r \le R$; $0 \le \theta \le 2\pi$ as shown in Fig. 30 .

The necessary conditions for this problem have been derived and defined as in (7.12), (7.13), (7.30) and (7.31). After substituting H from (7.7) these conditions can be written in the form

$$\left.\begin{aligned} \gamma_{rr} + \frac{1}{r}\gamma_r + \frac{1}{r^2}\gamma_{\theta\theta} &= 0, && (r,\theta) \in S, \\ \gamma(R,\theta) &= 0, && \text{on } C_2 : r=R;\ 0 \le \theta \le 2\pi, \\ \frac{\partial \gamma}{\partial n} + k\gamma &= 2[F_1(r,\theta) - k\phi], && \text{on } C_1 : r = g(\theta);\ 0 \le \theta \le 2\pi, \end{aligned}\right\} \quad (7.33)$$

where $\frac{\partial}{\partial n}$ is defined in (7.24). The other transversality condition follows from (7.31) that

$$\Bigg[\frac{d}{d\theta}\left\{\frac{\gamma\phi_\theta}{g} + \frac{(F_1 - k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{\gamma g'}{g}\right\}_{r=g(\theta)} - g\phi_r^2 - \frac{1}{g}\phi_\theta^2 - g\nu + g\gamma\phi_{rr} - \frac{\gamma g'}{g}\phi_{r\theta} + \frac{2\gamma g'\phi_\theta}{g^2} -$$

$$- g\gamma\cdot\left(\frac{\partial F_1}{\partial g} - k\phi_r\right)\cdot\sqrt{1+g'^2/g^2} + \gamma\cdot\frac{(F_1 - k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{g'^2}{g^2}\Bigg]_{r=g(\theta)} = 0 . \quad (7.34)$$

The boundary value problems (7.32) and (7.33) can be solved by using the single layer potential theory and will be discussed in the next chapter.

In order to simplify the problem, let us first consider the problem when the curve C_1 is a circle $r=r_0$ and $F_1(r,\theta)=\alpha$ where α $(\neq 0)$ is a constant. Let the corresponding $\phi(r,\theta)$ be ϕ_0, then the boundary value problem (7.32) becomes

$$\left.\begin{aligned} \nabla^2\phi_0 &= 0, && \text{in } S, \\ \frac{\partial\phi_0}{\partial r} + k\phi_0 &= \alpha, && \text{on a circle } C_0 : r = r_0, \\ \phi_0 &= \beta, && \text{on } C_2 : r = R . \end{aligned}\right\} \quad (7.35)$$

The solution ϕ_o will be a function of r only, hence the general solution of $\nabla^2\phi_o = 0$ will be of the form

$$\phi_o = A \log r + B$$

where A and B are arbitrary constants which can be found by using the boundary conditions on $r = r_o$ and $r = R$, that is

$$A = \frac{r_o(\alpha - k\beta)}{1 + r_o k \log \frac{r_o}{R}} \quad \text{and} \quad B = \beta - \frac{r_o(\alpha - k\beta)\log R}{1 + r_o k \log \frac{r_o}{R}}$$

hence

$$\phi_o = r_o M \log \frac{r}{R} + \beta \tag{7.36}$$

where

$$M = \frac{(\alpha - k\beta)}{1 + r_o k \log \frac{r_o}{R}} \tag{7.37}$$

Consider next the problem when the curve C_1 is only a small departure from a circle $r = r_o$ and expressed in the form

$$r = r_o + \varepsilon a_1(\theta) + O(\varepsilon^2) \quad , \quad r_o \neq 0 \; ; \tag{7.38}$$

and the function $F_1(r,\theta)$ is prescribed in the form

$$F_1(r,\theta) = \alpha + \varepsilon f_1(r,\theta) + O(\varepsilon^2) \; ,$$

where ε is a small quantity parameter. Let the corresponding solution of the boundary value problem (7.32) be

$$\phi = \phi_o + \varepsilon \phi_1 + O(\varepsilon^2) \tag{7.39}$$

where ϕ_o is defined in (7.36). Since $\nabla^2\phi_o = 0$ and $\nabla^2\phi = 0$, it then follows from (7.39) that

$$\nabla^2 \phi_1 = 0 . \tag{7.40}$$

Let us find the boundary conditions on C_1 and C_2 for the function $\phi_1(r,\theta)$. By using (7.24), (7.36) and (7.39) in (7.32) we obtain

$$\left[\frac{r_0 M}{r} + k r_0 M \log \frac{r}{R}\right]_{r=r_0+\varepsilon a_1(\theta)} + k\beta + \varepsilon\left[\frac{\partial \phi_1}{\partial r} + k\phi\right]_{r=r_0} = \alpha + \varepsilon f_1(r_0,\theta) + O(\varepsilon^2)$$

or

$$\left[\frac{\partial \phi_1}{\partial r} + k\phi_1\right]_{r=r_0} = f_1(r_0,\theta) - kM a_1(\theta) + \frac{M a_1(\theta)}{r_0} + O(\varepsilon) \; ; \tag{7.41}$$

and on C_2 : $r = R$, we have

$$\phi_1(R,\theta) = 0 , \qquad r = R ; \; 0 \le \theta \le 2\pi \tag{7.42}$$

Next we shall solve for ϕ_1 from the boundary value problem (7.40) - (7.42).

The general solution for a single-valued $\phi_1(r,\theta)$ of Laplace's equation $\nabla^2\phi_1 = 0$ can be written in the form of series as follows:

$$\phi_1(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty}\left(C_n \cos n\theta + D_n \sin n\theta\right) r^n + \sum_{n=1}^{\infty}\left(C_n' \cos n\theta + D_n' \sin n\theta\right) r^{-n}, \tag{7.43}$$

where A_0 , B_0 , C_n , C_n' , D_n , D_n' ; $(n = 1, 2, 3, \ldots\ldots)$ are arbitrary constants which can be found by using the boundary conditions (7.41) and (7.42).

Using the boundary condition (7.41), and (7.43), we have

$$f_1(r_0,\theta) - kMa_1(\theta) + \frac{Ma_1(\theta)}{r_0} = kA_0 + B_0\left(\frac{1}{r_0} + k\log r_0\right) +$$
$$+ \sum_{n=1}^{\infty} \left(C_n \cos n\theta + D_n \sin n\theta\right)\left(kr_0^n + nr_0^{n-1}\right) +$$
$$+ \sum_{n=1}^{\infty} \left(C_n' \cos n\theta + D_n' \sin n\theta\right)\left(kr_0^{-n} - nr_0^{-n-1}\right) + O(\varepsilon),$$

in which by using the property of the periodic functions $\sin n\theta$ and $\cos n\theta$; $(n=1, 2, 3, \ldots\ldots)$, we obtain the following relations

$$r_0 k A_0 + B_0(1 + r_0 k \log r_0) = \frac{1}{2\pi}\int_0^{2\pi} \left\{ r_0 f_1(r_0,\theta) + Ma_1(\theta)(1 - kr_0)\right\} d\theta \,; \tag{7.44}$$

$$(kr_0^{n+1} + nr_0^n)C_n + (kr_0^{-n+1} - nr_0^{-n})C_n' = \frac{1}{\pi}\int_0^{2\pi} \left\{ r_0 f_1(r_0,\theta) + Ma_1(\theta)(1 - kr_0)\right\} \cos n\theta \, d\theta \,;$$

$$n = 1, 2, 3, \ldots\ldots \tag{7.45}$$

and

$$(kr_0^{n+1} + nr_0^n)D_n + (kr_0^{-n+1} - nr_0^{-n})D_n' = \frac{1}{\pi}\int_0^{2\pi} \left\{ r_0 f_1(r_0,\theta) + Ma_1(\theta)(1 - kr_0)\right\} \sin n\theta \, d\theta \,;$$

$$n = 1, 2, 3, \ldots\ldots \tag{7.46}$$

Similarly, by using the condition (7.42), and (7.43), it will lead to the following relations

$$\left.\begin{aligned} A_0 &= -B_0 \log R \,, \\ C_n' &= -C_n R^{2n} \,, \quad n = 1, 2, \ldots \\ D_n' &= -D_n R^{2n} \,, \quad n = 1, 2, \ldots \end{aligned}\right\} \tag{7.47}$$

Solving the equations (7.44) - (7.47) for A_0 , B_0 , C_n , D_n , C_n' and D_n' , we have

$$B_0 = -\frac{A_0}{\log R} = \frac{1}{2\pi(1+r_0 k \log\frac{r_0}{R})}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta) + (1-kr_0)Ma_1(\theta)\right\}d\theta ,$$

$$C_n = -C_n' R^{-2n} = \frac{\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta)+(1-kr_0)Ma_1(\theta)\right\}\cos n\theta\, d\theta}{\pi\left\{kr_0^{n+1}(1-r_0^{-2n}R^{2n})+nr_0^n(1+r_0^{-2n}R^{2n})\right\}} , \quad n=1,2,\ldots$$

$$D_n = -D_n' R^{-2n} = \frac{\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta)+(1-kr_0)Ma_1(\theta)\right\}\sin n\theta\, d\theta}{\pi\left\{kr_0^{n+1}(1-r_0^{-2n}R^{2n})+nr_0^n(1+r_0^{-2n}R^{2n})\right\}} , \quad n=1,2,\ldots$$

Hence $\phi_1(r,\theta)$ in (7.43) can be written in the form

$$\phi_1(r,\theta) = \frac{\log\frac{r}{R}}{2\pi(1+r_0 k\log\frac{r_0}{R})}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta)+(1-kr_0)Ma_1(\theta)\right\}d\theta +$$

$$+\sum_{n=1}^{\infty}\frac{r^n(1-r^{-2n}R^{2n})\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta')+(1-kr_0)Ma_1(\theta')\right\}\cos n(\theta'-\theta)\,d\theta'}{\pi\left\{kr_0^{n+1}(1-r_0^{-2n}R^{2n})+nr_0^n(1+r_0^{-2n}R^{2n})\right\}} . \tag{7.48}$$

Substituting (7.36) and (7.48) into (7.39), we obtain

$$\phi(r,\theta) = \beta + r_0 M\log\frac{r}{R} + \frac{\varepsilon}{\pi}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta')+(1-kr_0)Ma_1(\theta')\right\}\left\{\frac{\log\frac{r}{R}}{2(1+r_0 k\log\frac{r_0}{R})} +\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{r^n(1-r^{-2n}R^{2n})\cos n(\theta'-\theta)}{kr_0^{n+1}(1-r_0^{-2n}R^{2n})+nr_0^n(1+r_0^{-2n}R^{2n})}\right\}d\theta' + O(\varepsilon^2) , \tag{7.49}$$

where $M = \dfrac{(\alpha - k\beta)}{(1+r_0 k\log\frac{r_0}{R})}$.

We use the same method to solve the boundary value problem (7.33) for $\gamma(r,\theta)$. When $r=g(\theta)=r_0+\varepsilon a_1(\theta)+O(\varepsilon^2)$ and $F_1(r,\theta) = \alpha+\varepsilon f_1(r,\theta)+O(\varepsilon^2)$ the system (7.33) can be written in the form

$$\nabla^2 \gamma(r,\theta) = 0, \qquad (r,\theta) \in S$$

$$\frac{\partial \gamma}{\partial n} + k\gamma = 2\alpha + 2\varepsilon f_1(r,\theta) - 2k\phi + O(\varepsilon^2) \quad \text{on } C_1 : r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2),$$

$$\gamma(R,\theta) = 0, \qquad \text{on } C_2 : r = R;\ 0 \le \theta \le 2\pi,$$

where $\frac{\partial}{\partial n}$ is defined in (7.24) and for this case will be

$$\frac{\partial}{\partial n} \equiv \frac{\partial}{\partial r} - \frac{\varepsilon a_1'(\theta)}{r_0^2} \cdot \frac{\partial}{\partial \theta} + O(\varepsilon^2)$$

Let us consider the boundary condition on C_1, i.e.,

$$\frac{\partial \gamma}{\partial n} + k\gamma = 2\alpha + 2\varepsilon f_1(r,\theta) - 2k\phi + O(\varepsilon^2), \quad r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2) \tag{7.50}$$

By using (7.49), we have

$$[\phi]_{r=r_0+\varepsilon a_1(\theta)} = r_0 M \log\frac{r_0}{R} + \beta + \varepsilon M a_1(\theta) + \frac{\varepsilon}{\pi}\int_0^{2\pi} \left\{ r_0 f_1(r_0,\theta') + (1-kr_0) M a_1(\theta') \right\} \cdot \left\{ \right.$$

$$\cdot \left\{ \frac{\log\frac{r_0}{R}}{2(1+r_0 k \log\frac{r_0}{R})} + \sum_{n=1}^{\infty} \frac{r_0^n (1 - r_0^{-2n} R^{2n}) \cos n(\theta'-\theta)}{k r_0^{n+1}(1 - r_0^{-2n} R^{2n}) + n r_0^n (1 + r_0^{-2n} R^{2n})} \right\} d\theta' + O(\varepsilon^2). \tag{7.51}$$

Thus, by substituting (7.51) into (7.50), we obtain the boundary condition on C_1 of the form

$$\frac{\partial \gamma}{\partial n} + k\gamma = m + \varepsilon N(r,\theta) + O(\varepsilon^2), \qquad \text{on } C_1 : r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2). \tag{7.52}$$

where

$$m \equiv 2\alpha - 2kr_0 M \log\frac{r_0}{R} - 2k\beta = 2M, \tag{7.53}$$

and M is defined in (7.37); and

$$N(r,\theta) \equiv 2f_1(r,\theta) - 2kMa_1(\theta) - \frac{2k}{\pi}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta') + (1-kr_0)Ma_1(\theta')\right\}\left\{\frac{\log\frac{r_0}{R}}{2(1+r_0 k\log\frac{r_0}{R})} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{r_0^n(1-r_0^{-2n}R^{2n})\cos n(\theta'-\theta)}{kr_0^{n+1}(1-r_0^{-2n}R^{2n}) + nr_0^n(1+r_0^{-2n}R^{2n})}\right\} d\theta' .$$

(7.54)

Now we have the following boundary value problem to be solved.

$$\nabla^2\gamma(r,\theta) = 0 , \qquad (r,\theta)\in S' ,$$

$$\frac{\partial\gamma}{\partial n} + k\gamma = m + \varepsilon N(r,\theta) + o(\varepsilon^2), \text{ on } C_1 : r = r_0 + \varepsilon a_1(\theta) + o(\varepsilon^2),$$

$$\gamma(R,\theta) = 0 , \qquad \text{on } C_2 : r = R ;\ 0 \le \theta \le 2\pi.$$

where m and $N(r,\theta)$ are defined in (7.53) and (7.54) respectively, and $\frac{\partial}{\partial n} \equiv \frac{\partial}{\partial r} - \frac{\varepsilon a_1'(\theta)}{r_0^2}\cdot\frac{\partial}{\partial\theta} + o(\varepsilon^2)$.

It is clear that this problem is of the same pattern as the boundary value problem for $\phi(r,\theta)$. Hence the solution for $\gamma(r,\theta)$ will be similar to (7.49) with α , β and $f_1(r,\theta)$ are replaced by m , zero and $N(r,\theta)$ respectively, that is

$$\gamma(r,\theta) = \frac{r_0 m\log\frac{r}{R}}{(1+r_0 k\log\frac{r_0}{R})} + \frac{\varepsilon}{\pi}\int_0^{2\pi}\left\{r_0 N(r_0,\theta') + \frac{(1-kr_0)ma_1(\theta')}{(1+kr_0\log\frac{r_0}{R})}\right\}\left\{\frac{\log\frac{r}{R}}{2(1+r_0 k\log\frac{r_0}{R})} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{r^n(1-r^{-2n}R^{2n})\cos n(\theta'-\theta)}{kr_0^{n+1}(1-r_0^{-2n}R^{2n}) + nr_0^n(1+r_0^{-2n}R^{2n})}\right\} d\theta' + o(\varepsilon^2) ,$$

(7.55)

where $m = \dfrac{2(\alpha - k\beta)}{(1 + r_0 k\log\frac{r_0}{R})}$ as defined in (7.53).

Let us set

$$P = \frac{m}{(1+kr_0\log\frac{r_0}{R})} = \frac{2(\alpha-k\beta)}{(1+r_0k\log\frac{r_0}{R})^2} \tag{7.56}$$

and

$$S_n(r,r_0) = \frac{r^n(1-r^{-2n}R^{2n})}{kr_0^{n+1}(1-r_0^{-2n}R^{2n})+nr_0^n(1+r_0^{-2n}R^{2n})}, \quad n=1,2,\dots \tag{7.57}$$

By using (7.54), (7.56) and (7.57), we can write

$$\begin{aligned} r_0N(r_0,\theta')+\frac{(1-kr_0)ma_1(\theta')}{(1+kr_0\log\frac{r_0}{R})} &= 2r_0f_1(r_0,\theta')+(1-2kr_0-k^2r_0^2\log\tfrac{r_0}{R}).P.\,a_1(\theta')- \\ &-\frac{2kr_0}{\pi}\int_0^{2\pi}\Big\{r_0f_1(r_0,\theta'')+(1-kr_0)(1+kr_0\log\tfrac{r_0}{R}).\tfrac{P}{2}.a_1(\theta'')\Big\}.\Big\{ \\ &\cdot\Big\{\frac{\log\frac{r_0}{R}}{2(1+kr_0\log\frac{r_0}{R})}+\sum_{n=1}^{\infty}S_n(r_0,r_0)\cos n(\theta''-\theta')\Big\}d\theta''. \end{aligned}$$

Hence $\gamma(r,\theta)$ in (7.55) becomes

$$\begin{aligned} \gamma(r,\theta) &= r_0P\log\frac{r}{R}+\frac{\varepsilon}{\pi}\int_0^{2\pi}\Big[\Big\{2r_0f_1(r_0,\theta')+(1-2kr_0+k^2r_0^2\log\tfrac{r_0}{R}).P.a_1(\theta')\Big\}.\Big\{\frac{\log\frac{r}{R}}{2(1+kr_0\log\frac{r_0}{R})}+ \\ &+\sum_{n=1}^{\infty}S_n(r,r_0)\cos n(\theta'-\theta)\Big\}-kr_0\Big\{2r_0f_1(r_0,\theta')+(1-kr_0)(1+kr_0\log\tfrac{r_0}{R}).P.a_1(\theta')\Big\}.\Big\{ \\ &\cdot\Big\{\frac{(\log\frac{r_0}{R})(\log\frac{r}{R})}{2(1+kr_0\log\frac{r_0}{R})^2}+\sum_{n=1}^{\infty}S_n(r,r_0)\,S_n(r_0,r_0)\cos n(\theta'-\theta)\Big\}\Big]d\theta'+O(\varepsilon^2), \end{aligned}$$

(7.58)

where P and $S_n(r,r_0)$ are defined in (7.56) and (7.57) respectively.

Now we have

$$\left.\begin{aligned} r = g(\theta) &= r_0+\varepsilon a_1(\theta)+O(\varepsilon^2), \\ F_1(r,\theta) &= \alpha+\varepsilon f_1(r,\theta)+O(\varepsilon^2), \\ \phi(r,\theta) &= \phi_0+\varepsilon\phi_1(r,\theta)+O(\varepsilon^2), \\ \gamma(r,\theta) &= \gamma_0+\varepsilon\gamma_1(r,\theta)+O(\varepsilon^2), \end{aligned}\right\} \tag{7.59}$$

where r_0 and $a_1(\theta)$ are unknown; α and $f_1(r,\theta)$ are prescribed, and, $\phi(r,\theta)$ and $\gamma(r,\theta)$ have been evaluated and defined in (7.49) and (7.58) respectively in which ϕ_0 and γ_0 are functions of r only.

The transversality condition on C_1 in (7.34) can be rewritten here as follows:

$$\frac{d}{d\theta}\left\{\frac{\gamma\phi_\theta}{g}+\frac{(F_1-k\phi)}{\sqrt{1+g'^2/g^2}}\cdot\frac{\gamma g'}{g}\right\}_{r=g(\theta)}-\left\{g\phi_r^2+\frac{1}{g}\phi_\theta^2+g\nu-g\gamma\phi_{rr}+\frac{\gamma g'}{g}\phi_{r\theta}-\right.$$

$$\left.-\frac{2\gamma\phi_\theta g'}{g^2}+g\gamma.\left(\frac{\partial F_1}{\partial r}-k\phi_r\right).\sqrt{1+g'^2/g^2}-\frac{(F_1-k\phi)}{\sqrt{1+g'^2/g^2}}.\gamma.\frac{g'^2}{g^2}\right\}_{r=g(\theta)}=0\,. \tag{7.34}$$

Substituting (7.59) into (7.34) and neglecting the terms of degree higher than one relative to ε , we obtain

$$\left[\frac{\varepsilon\gamma_0}{r_0}\frac{d}{d\theta}\left\{\frac{\partial\phi_1}{\partial\theta}+a_1'(\theta)(\alpha-k\phi_0)\right\}_{r=r_0}-\varepsilon\left\{a_1(\theta)\left(\nu+\left(\frac{\partial\phi_0}{\partial r}\right)^2-\gamma_0\frac{\partial^2\phi_0}{\partial r^2}-\gamma_0 k\frac{\partial\phi_0}{\partial r}\right)+\right.\right.$$

$$\left.\left.+r_0\left(2\frac{\partial\phi_0}{\partial r}\cdot\frac{\partial\phi_1}{\partial r}+\gamma_0\frac{\partial f_1}{\partial r}-\gamma_0\frac{\partial^2\phi_1}{\partial r^2}-\gamma_1\frac{\partial^2\phi_0}{\partial r^2}\right)-kr_0\left(\gamma_0\frac{\partial\phi_1}{\partial r}+\gamma_1\frac{\partial\phi_0}{\partial r}\right)\right\}\right]_{r=r_0}-$$

$$-\left\{r_0\left(\nu+\left(\frac{\partial\phi_0}{\partial r}\right)^2-\gamma_0\frac{\partial^2\phi_0}{\partial r^2}-\gamma_0 k\frac{\partial\phi_0}{\partial r}\right)\right\}_{r=r_0+\varepsilon a_1(\theta)}+O(\varepsilon^2)=0\,, \tag{7.60}$$

where ϕ_0 , γ_0 , ϕ_1 and γ_1 can be obtained from (7.49) and (7.58).

We shall discuss the problem in more detail only for the case when $k=0$.

Case $k=0$.

The transversality condition (7.60) becomes

$$\left[\frac{\varepsilon\gamma_0}{r_0}\cdot\frac{d}{d\theta}\left\{\frac{\partial\phi_1}{\partial\theta}+\alpha\, a_1'(\theta)\right\}_{r=r_0}-\varepsilon\left\{a_1(\theta)\left(\nu+\left(\frac{\partial\phi_0}{\partial r}\right)^2-\gamma_0\frac{\partial^2\phi_0}{\partial r^2}\right)+\right.\right.$$

$$\left.\left.+r_0\left(2\frac{\partial\phi_0}{\partial r}\cdot\frac{\partial\phi_1}{\partial r}+\gamma_0\frac{\partial f_1}{\partial r}-\gamma_0\frac{\partial^2\phi_1}{\partial r^2}-\gamma_1\frac{\partial^2\phi_0}{\partial r^2}\right)\right\}\right]_{r=r_0}-r_0\left\{\nu+\left(\frac{\partial\phi_0}{\partial r}\right)^2-\gamma_0\frac{\partial^2\phi_0}{\partial r^2}\right\}_{r=r_0+\varepsilon a_1(\theta)}+o(\varepsilon^2)=0\,;$$

(7.61)

The functions $\phi(r,\theta)$ and $\gamma(r,\theta)$ in (7.49) and (7.58) will be

$$\phi(r,\theta)=\phi_0+\varepsilon\,\phi_1(r,\theta)+o(\varepsilon^2)$$

$$=\beta+\alpha r_0\log\frac{r}{R}+\frac{\varepsilon}{\pi}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right\}\cdot\left\{\frac{\log\frac{r}{R}}{2}+\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{r^n(1-r^{-2n}R^{2n})}{n r_0^n(1+r_0^{-2n}R^{2n})}\cos n(\theta'-\theta)\right\}d\theta'+o(\varepsilon^2)\,,$$

(7.62)

$$\gamma(r,\theta)=\gamma_0+\varepsilon\,\gamma_1(r,\theta)+o(\varepsilon^2)$$

$$=2r_0\alpha\log\frac{r}{R}+\frac{2\varepsilon}{\pi}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right\}\cdot\left\{\frac{\log\frac{r}{R}}{2}+\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{r^n(1-r^{-2n}R^{2n})}{n r_0^n(1+r_0^{-2n}R^{2n})}\cos n(\theta'-\theta)\right\}d\theta'+o(\varepsilon^2)\,,$$

(7.63)

hence

$$\gamma_0=2\phi_0-2\beta\quad;\quad\gamma_1=2\phi_1\,,$$

where

$$\phi_0=\beta+\alpha r_0\log\frac{r}{R}\,,$$

$$\phi_1=\frac{1}{\pi}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right\}\left\{\frac{\log\frac{r}{R}}{2}+\sum_{n=1}^{\infty}\frac{r^n(1-r^{-2n}R^{2n})}{n r_0^n(1+r_0^{-2n}R^{2n})}\cos n(\theta'-\theta)\right\}d\theta'.$$

Calculating the first, second and third bracket of (7.61) we obtain respectively, as follows:

$$\left[\varepsilon\frac{\gamma_0}{r_0}\cdot\frac{d}{d\theta}\left\{\frac{\partial\phi_1}{\partial\theta}+\alpha a_1'(\theta)\right\}\right]_{r=r_0}$$

$$=2\varepsilon\alpha^2\left(\log\frac{r_0}{R}\right)a_1''(\theta)-\frac{2\varepsilon\alpha\log\frac{r_0}{R}}{\pi}\int_0^{2\pi}\left\{r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right\}\left\{\sum_{n=1}^{\infty}\frac{n(1-r_0^{-2n}R^{2n})}{(1+r_0^{-2n}R^{2n})}\cos n(\theta'-\theta)\right\}d\theta'+O(\varepsilon^2);$$

(7.64)

$$\varepsilon\left[a_1(\theta)\left\{\nu+\left(\frac{\partial\phi_0}{\partial r}\right)^2-\gamma_0\frac{\partial^2\phi_0}{\partial r^2}\right\}+r_0\left(2\frac{\partial\phi_0}{\partial r}\cdot\frac{\partial\phi_1}{\partial r}+\gamma_0\frac{\partial f_1}{\partial r}-\gamma_0\frac{\partial^2\phi_1}{\partial r^2}-\gamma_1\frac{\partial^2\phi_0}{\partial r^2}\right)\right]_{r=r_0}$$

$$=\varepsilon\left[a_1(\theta)\left(\nu+\alpha^2+2\alpha^2\log\frac{r_0}{R}\right)+2\alpha r_0^2\left\{\frac{\partial f_1}{\partial r}\right\}_{r=r_0}\cdot\log\frac{r_0}{R}\right]+$$

$$+\frac{2\varepsilon\alpha}{\pi}\int_0^{2\pi}\left[r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right]\left[\frac{1}{2}+\log\frac{r_0}{R}+\sum_{n=1}^{\infty}\left\{1+\log\frac{r_0}{R}+\frac{(1-r_0^{-2n}R^{2n})}{n(1+r_0^{-2n}R^{2n})}-\right.\right.$$

$$\left.\left.-\frac{n(1-r_0^{-2n}R^{2n})}{(1+r_0^{-2n}R^{2n})}\log\frac{r_0}{R}\right\}\cos n(\theta'-\theta)\right]d\theta'+O(\varepsilon^2);$$

(7.65)

and

$$\left[r_0\left\{\nu+\left(\frac{\partial\phi_0}{\partial r}\right)^2-\gamma_0\frac{\partial^2\phi_0}{\partial r^2}\right\}\right]_{r=r_0+\varepsilon a_1(\theta)}$$

$$=r_0\left(\alpha^2+\nu+2\alpha^2\log\frac{r_0}{R}\right)-4\alpha^2\varepsilon a_1(\theta)\log\frac{r_0}{R}+O(\varepsilon^2).$$

(7.66)

By substituting (7.64) - (7.66) into (7.61), we shall have the equation satisfied by the unknown r_0 $(\neq 0)$ and $a_1(\theta)$ as follows:

$$r_0\left(\alpha^2+\nu+2\alpha^2\log\frac{r_0}{R}\right)+\varepsilon\left[a_1(\theta)\left(\alpha^2+\nu+2\alpha^2\log\frac{r_0}{R}\right)+2\alpha\log\frac{r_0}{R}\left\{r_0^2\left.\frac{\partial f_1}{\partial r}\right|_{r=r_0}-\alpha a_1''(\theta)-2\alpha a_1(\theta)\right\}\right]+$$

$$+\frac{2\varepsilon\alpha}{\pi}\int_0^{2\pi}\left[r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right]\left[\frac{1}{2}+\log\frac{r_0}{R}+\sum_{n=1}^{\infty}\left\{1+\log\frac{r_0}{R}+\frac{(1-r_0^{-2n}R^{2n})}{n(1+r_0^{-2n}R^{2n})}\right\}\cos n(\theta'-\theta)\right]d\theta'+O(\varepsilon^2)=$$

$$=0. \tag{7.67}$$

Hence $\quad \alpha^2+\nu+2\alpha^2\log\frac{r_0}{R} = 0 ,$

or
$$r_0 = Re^{-\lambda/2}, \tag{7.68}$$

where
$$\lambda = 1+\frac{\nu}{\alpha^2} ; \tag{7.69}$$

and

$$a_1(\theta)\left(\alpha^2+\nu+2\alpha^2\log\frac{r_0}{R}\right)+2\alpha\log\frac{r_0}{R}\left[r_0^2\left\{\frac{\partial f_1}{\partial r}\right\}_{r=r_0}-\alpha a_1''(\theta)-2\alpha a_1(\theta)\right]+$$

$$+\frac{2\alpha}{\pi}\int_0^{2\pi}\left[r_0 f_1(r_0,\theta')+\alpha a_1(\theta')\right]\left[\frac{1}{2}+\log\frac{r_0}{R}+\sum_{n=1}^{\infty}\left\{1+\log\frac{r_0}{R}+\frac{(1-r_0^{-2n}R^{2n})}{n(1+r_0^{-2n}R^{2n})}\right\}\cos n(\theta'-\theta)\right]d\theta'=0. \tag{7.70}$$

By using r_0 defined in (7.68) we then can write (7.70) in the form

$$a_1''(\theta)+2a_1(\theta)+\frac{2}{\lambda\pi}\int_0^{2\pi}a_1(\theta')\left[b_0+\sum_{n=1}^{\infty}b_n\cos n(\theta'-\theta)\right]d\theta' = W(\theta), \tag{7.71}$$

where

$$\left.\begin{aligned} b_0 &= \frac{1-\lambda}{2} \\ b_n &= 1-\frac{\lambda}{2}+\frac{(1-e^{\lambda n})}{n(1+e^{\lambda n})}, \quad n=1,2,\dots \end{aligned}\right\} \tag{7.72}$$

and

$$W(\theta)=\frac{R^2e^{-\lambda}}{\alpha}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda/2}}-\frac{2Re^{-\lambda/2}}{\alpha\lambda\pi}\int_0^{2\pi}f_1(Re^{-\lambda/2},\theta')\left[b_0+\sum_{n=1}^{\infty}b_n\cos n(\theta'-\theta)\right]d\theta', \tag{7.73}$$

the constant λ is defined in (7.69).

Hence the unknown function $a_1(\theta)$ must satisfy the integro-differential equation (7.71). Since $a_1(\theta)$ and $f_1(r,\theta)$ should be periodic functions with period 2π and also continuous, thus we shall look for a solution of (7.71) of the form

$$a_1(\theta) = A_0 + \sum_{m=1}^{\infty}\left(A_m \cos m\theta + B_m \sin m\theta\right), \tag{7.74}$$

where

$$\left.\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} a_1(\theta)\, d\theta \,, \\ A_m &= \frac{1}{\pi}\int_0^{2\pi} a_1(\theta)\cos m\theta\, d\theta\,, \quad m=1,2,\ldots\ldots \\ B_m &= \frac{1}{\pi}\int_0^{2\pi} a_1(\theta)\sin m\theta\, d\theta\,, \quad m=1,2,\ldots\ldots \end{aligned}\right\} \tag{7.75}$$

Calculating $a_1''(\theta)$ from (7.74) and substituting it into (7.71) we then can write (7.71) in the form

$$2a_1(\theta) = W(\theta) + \sum_{m=1}^{\infty} m^2\left(A_m\cos m\theta + B_m \sin m\theta\right) - \frac{2}{\lambda\pi}\int_0^{2\pi} a_1(\theta')\left[b_0 + \sum_{n=1}^{\infty} b_n \cos n(\theta'-\theta)\right]d\theta'$$

using (7.75), we have

$$2a_1(\theta) = W(\theta) + \sum_{m=1}^{\infty}\left[\left(m^2 - \frac{2b_m}{\lambda}\right)\left(A_m\cos m\theta + B_m\sin m\theta\right)\right] - \frac{4}{\lambda}A_0 b_0 \tag{7.76}$$

Hence, by using (7.75), (7.76) and the property of the periodic functions $\cos m\theta$, $\sin m\theta$; ($m=1, 2, 3, \ldots\ldots$), we can find A_0 , A_m and B_m as follows:

$$\left.\begin{aligned} A_0 &= \frac{1}{4\pi\left(1+\frac{2}{\lambda}b_0\right)}\int_0^{2\pi} W(\theta)\, d\theta \\ A_m &= \frac{1}{\pi\left(2-m^2+\frac{2b_m}{\lambda}\right)}\int_0^{2\pi} W(\theta)\cos m\theta\, d\theta\,, \quad m=1,2,\ldots\ldots \\ B_m &= \frac{1}{\pi\left(2-m^2+\frac{2b_m}{\lambda}\right)}\int_0^{2\pi} W(\theta)\sin m\theta\, d\theta\,, \quad m=1,2,\ldots\ldots \end{aligned}\right\} \tag{7.77}$$

Substituting (7.77) into (7.74) and using (7.72), we obtain

$$a_1(\theta) = \frac{\lambda}{4\pi}\int_0^{2\pi} W(\theta')\,d\theta' + \int_0^{2\pi} W(\theta')\sum_{m=1}^{\infty}\frac{\cos m(\theta'-\theta)\,d\theta'}{\pi\left\{1-m^2+\frac{2}{\lambda}\left(1-\frac{1}{m}\tanh\frac{\lambda m}{2}\right)\right\}}\ , \tag{7.78}$$

where $W(\theta)$ and λ are defined in (7.73) and (7.69) respectively.

After substituting $W(\theta)$, defined in (7.73) in which b_0 and b_n are in terms of λ as in (7.72), we obtain the expression of $a_1(\theta)$ in terms of unknown λ as follows:

$$\begin{aligned} a_1(\theta) = {} & \frac{\lambda R^2 e^{-\lambda}}{4\pi\alpha}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda/2}} d\theta' + \frac{R e^{2-\lambda}}{\alpha\pi}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda/2}}\cdot\sum_{m=1}^{\infty}\frac{\cos m(\theta'-\theta)\,d\theta'}{\left\{1-m^2+\frac{2}{\lambda}\left(1-\frac{1}{m}\tanh\frac{\lambda m}{2}\right)\right\}} - \\ & - \frac{2Re^{-\lambda/2}}{\alpha\lambda\pi}\int_0^{2\pi} f_1(Re^{-\lambda/2},\theta')\cdot\sum_{m=1}^{\infty}\frac{\left\{1-\frac{\lambda}{2}-\frac{1}{m}\tanh\frac{\lambda m}{2}\right\}\cos m(\theta'-\theta)\,d\theta'}{\left\{1-m^2+\frac{2}{\lambda}\left(1-\frac{1}{m}\tanh\frac{\lambda m}{2}\right)\right\}} - \\ & - \frac{Re^{-\lambda/2}(1-\lambda)}{2\pi\alpha}\int_0^{2\pi} f_1(Re^{-\lambda/2},\theta')\,d\theta'\ , \end{aligned} \tag{7.79}$$

where λ is defined in terms of the unknown Lagrange multiplier ν as in (7.69).

This unknown λ can be found from the constraint (7.5), as follows:

Since $$\int_0^{2\pi}\int_{r_0+\varepsilon a_1(\theta)}^{R} r\,dr\,d\theta = K$$

or $$\frac{1}{2}\int_0^{2\pi}\left[R^2-r_0^2-2\varepsilon r_0 a_1(\theta)\right]d\theta = K + O(\varepsilon^2)$$

using (7.75), we have

$$\pi(R^2 - r_0^2) - 2\pi\varepsilon r_0 A_0 = K + O(\varepsilon^2) \tag{7.80}$$

The constant A_0 is defined in (7.77) and expressed in terms of λ as follows:

$$A_0 = \frac{\lambda}{4\pi}\left[\frac{R^2 e^{-\lambda}}{\alpha}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda/2}} d\theta - \frac{2Re^{-\lambda/2}(1-\lambda)}{\alpha\lambda}\int_0^{2\pi} f_1(Re^{-\lambda/2},\theta)\, d\theta\right]. \tag{7.81}$$

By using (7.68), i.e., $r_0 = Re^{-\lambda/2}$ and (7.81), we can write (7.80) in the form

$$e^{-\lambda} + \frac{\varepsilon e^{-\lambda}(\lambda-1)}{\alpha\pi}\int_0^{2\pi} f_1(Re^{-\lambda/2},\theta)\, d\theta + \frac{\varepsilon R\lambda e^{-3\lambda/2}}{2\alpha\pi}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda/2}} d\theta = 1 - \frac{K}{\pi R^2} + O(\varepsilon^2). \tag{7.82}$$

To solve for λ , we assume λ is written in the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2). \tag{7.83}$$

Substituting (7.83) into (7.82), we obtain

$$e^{-\lambda_0}\left[1 - \varepsilon\lambda_1 + \frac{\varepsilon(\lambda_0 - 1)}{\alpha\pi}\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta)\, d\theta + \frac{\varepsilon R\lambda_0 e^{-\lambda_0/2}}{2\alpha\pi}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} d\theta\right] = 1 - \frac{K}{\pi R^2} + O(\varepsilon^2).$$

Hence

$$e^{-\lambda_0} = 1 - \frac{K}{\pi R^2}$$

or

$$\lambda_0 = -\log\left(1 - \frac{K}{\pi R^2}\right) \tag{7.84}$$

and

$$\lambda_1 = \frac{(\lambda_0-1)}{\alpha\pi}\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta)\,d\theta + \frac{R\lambda_0 e^{-\lambda_0/2}}{2\alpha\pi}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} d\theta\,, \tag{7.85}$$

where λ_0 is defined in (7.84).

Therefore

$$\begin{aligned}\lambda &= \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2)\\ &= \lambda_0 + \frac{\varepsilon}{\alpha\pi}\left[(\lambda_0-1)\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta)\,d\theta + \frac{R\lambda_0 e^{-\lambda_0/2}}{2}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} d\theta\right] + O(\varepsilon^2)\end{aligned} \tag{7.86}$$

where λ_0 is defined in (7.84).

It then follows from (7.68) that

$$\begin{aligned}r_0 &= Re^{-(\lambda_0+\varepsilon\lambda_1)/2}\\ &= R\sqrt{1-\frac{K}{\pi R^2}}\left[1-\frac{\varepsilon}{2\alpha\pi}\left\{(\lambda_0-1)\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta)\,d\theta + \frac{R\lambda_0 e^{-\lambda_0/2}}{2}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} d\theta\right\}\right] + O(\varepsilon^2)\end{aligned} \tag{7.87}$$

and from (7.79), we shall have

$$\begin{aligned}a_1(\theta) &= \frac{\lambda_0 R^2 e^{-\lambda_0}}{4\pi\alpha}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} d\theta' + \frac{Re^{-\lambda_0/2}(\lambda_0-1)}{2\pi\alpha}\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta')\,d\theta' +\\ &+ \frac{R^2 e^{-\lambda_0}}{\pi\alpha}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}}\cdot\sum_{m=1}^{\infty}\frac{\cos m(\theta'-\theta)\,d\theta'}{\left\{1-m^2+\frac{2}{\lambda_0}\left(1-\frac{1}{m}\tanh\frac{\lambda_0 m}{2}\right)\right\}} -\\ &- \frac{2Re^{-\lambda_0/2}}{\pi\alpha\lambda_0}\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta')\cdot\sum_{m=1}^{\infty}\frac{\left\{1-\frac{\lambda_0}{2}-\frac{1}{m}\tanh\frac{\lambda_0 m}{2}\right\}\cos m(\theta'-\theta)\,d\theta'}{\left\{1-m^2+\frac{2}{\lambda_0}\left(1-\frac{1}{m}\tanh\frac{\lambda_0 m}{2}\right)\right\}} + O(\varepsilon)\,,\end{aligned} \tag{7.88}$$

where λ_0 is defined in (7.84).

Hence the optimum curve $C_1 : r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2)$; $0 \le \theta \le 2\pi$ will be defined in the following form; by using (7.87) and (7.88),

$$r = R\sqrt{1-\frac{K}{\pi R^2}} + \frac{\varepsilon}{\pi\alpha}\Bigg[R^2 e^{-\lambda_0}\int_0^{2\pi}\left\{\frac{\partial f_1}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} \cdot \sum_{m=1}^{\infty}\frac{\cos m(\theta'-\theta)\,d\theta'}{\left\{1-m^2+\frac{2}{\lambda_0}\left(1-\frac{1}{m}\tanh\frac{\lambda_0 m}{2}\right)\right\}} -$$

$$-\frac{2Re^{-\lambda_0/2}}{\lambda_0}\int_0^{2\pi} f_1(Re^{-\lambda_0/2},\theta')\sum_{m=1}^{\infty}\frac{\left\{1-\frac{\lambda_0}{2}-\frac{1}{m}\tanh\frac{\lambda_0 m}{2}\right\}\cos m(\theta'-\theta)\,d\theta'}{\left\{1-m^2+\frac{2}{\lambda_0}\left(1-\frac{1}{m}\tanh\frac{\lambda_0 m}{2}\right)\right\}}\Bigg] + O(\varepsilon^2), \tag{7.89}$$

where $0 \le \theta \le 2\pi$, and λ_0 is defined in (7.84) as

$$\lambda_0 = -\log\left(1-\frac{K}{\pi R^2}\right).$$

We note here that the unknown curve C_1 is depending on the behaviour of the given function $f_1(r,\theta)$ on C_1 .

It is clear from (7.89) that when $\varepsilon = 0$, the optimum curve C_1 will be a circle of radius $R\sqrt{1-\frac{K}{\pi R^2}}$. This result can be checked by using elementary calculus.

Let us give an example to illustrate the optimum curve C_1 : $r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2)$ in (7.89).

Find a curve $C_1 : r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2)$; $0 \le \theta \le 2\pi$ which gives an extremum of the functional

$$I = \int_0^{2\pi}\int_{r=r_0+\varepsilon a_1(\theta)}^{R}\left\{\phi_r^2 + \frac{1}{r^2}\phi_\theta^2\right\} r\,dr\,d\theta ,$$

subject to the constraint

$$\int_0^{2\pi}\int_{r_0+\varepsilon a_1(\theta)}^{R} r\,dr\,d\theta = K ,$$

where $\phi(r,\theta)$ satisfies the boundary value problem

$$\nabla^2\phi(r,\theta) = 0, \quad \text{in } S : 0 < r_0 + \varepsilon a_1(\theta) \le r \le R \; ; \; 0 \le \theta \le 2\pi ,$$

$$\frac{\partial\phi}{\partial n} = \alpha + \varepsilon \frac{r}{2}\cos^2\theta , \text{ on } C_1 : r = r_0 + \varepsilon a_1(\theta) \; ; \; 0 \le \theta \le 2\pi ,$$

$$\phi = \beta , \qquad \text{on } C_2 : r = R ; \; 0 \le \theta \le 2\pi ,$$

the constants R , K , β and α $(\neq 0)$ are given and ε is assumed to be so small that all powers greater than the first can be neglected.

In this example $f_1(r,\theta)$ is given to be

$$f_1(r,\theta) = \frac{r}{2}\cos^2\theta$$

$$\therefore \quad \frac{\partial f_1}{\partial r} = \frac{1}{2}\cos^2\theta$$

$$\int_0^{2\pi} \left\{\frac{\partial f}{\partial r}\right\}_{r=Re^{-\lambda_0/2}} \cos m(\theta'-\theta)\, d\theta' = \frac{\pi}{4}\cos m\theta , \qquad \text{when } m = 2$$

$$= 0 , \qquad \text{when } m \neq 2$$

and

$$\int_0^{2\pi} f_1(Re^{-\lambda_0/2}, \theta')\cos m(\theta'-\theta)\, d\theta' = \frac{\pi}{4} . Re^{-\lambda_0/2}\cos m\theta , \qquad \text{when } m = 2$$

$$= 0 , \qquad \text{when } m \neq 2$$

Hence, (7.89) becomes

$$r = R\sqrt{1 - \frac{K}{\pi R^2}} + \frac{\varepsilon R^2 (1 - \frac{K}{\pi R^2})}{4\alpha}\left[\frac{(2 - \frac{2}{\lambda_0} + \frac{1}{\lambda_0}\tanh\lambda_0)}{(\frac{2}{\lambda_0} - \frac{1}{\lambda_0}\tanh\lambda_0 - 3)} \cos 2\theta \right] + O(\varepsilon^2)$$

or

$$r = R\sqrt{1 - \frac{K}{\pi R^2}} + \frac{\varepsilon R^2 (1 - \frac{K}{\pi R^2})}{4\alpha}\left[\frac{\lambda_0}{3\lambda_0 + \tanh\lambda_0 - 2} - 1 \right] \cos 2\theta + O(\varepsilon^2) ,$$

where

$$\lambda_0 = -\log\left(1 - \frac{K}{\pi R^2}\right) .$$

By substituting the expression of λ_0 , we obtain the curve C_1 of the form

$$r = R\sqrt{1-\frac{K}{\pi R^2}} + \frac{\varepsilon R^2\left(1-\frac{K}{\pi R^2}\right)}{4\alpha}\left[\frac{(2\pi^2R^4-2K\pi R^2+K^2)\log\left(1-\frac{K}{\pi R^2}\right)\cos 2\theta}{3(2\pi^2R^4-2K\pi R^2+K^2)\log\left(1-\frac{K}{\pi R^2}\right)+4\pi^2R^4-6K\pi R^2+3K^2} - \right.$$

$$\left. - \cos 2\theta\right] + O(\varepsilon^2), \qquad 0 \leq \theta \leq 2\pi .$$

CHAPTER 8

THE SOLUTION OF LAPLACE'S EQUATION IN AN ANNULAS USING SINGLE LAYER POTENTIAL THEORY.

We shall investigate in this chapter the solution of the boundary value problems which have arisen in Chapter 7. Those problems can be rewritten here as follows:

$$\left.\begin{array}{lll} \nabla^2\phi(r,\theta) = 0\,, & (r,\theta) \in S\,, & \\ \frac{\partial\phi}{\partial n} + k\phi = F_1(r,\theta), & \text{on } C_1 : r = g(\theta)\,;\ 0 \le \theta \le 2\pi, & \\ \phi(R,\theta) = \beta\,, & \text{on } C_2 : r = R\,;\ 0 \le \theta \le 2\pi\,, & \end{array}\right\} \quad (8.1)$$

and

$$\left.\begin{array}{lll} \nabla^2\gamma(r,\theta) = 0\,, & (r,\theta) \in S, & \\ \frac{\partial\gamma}{\partial n} + k\gamma = 2\left[F_1(r,\theta) - k\phi\right] & \text{on } C_1 : r = g(\theta)\,;\ 0 \le \theta \le 2\pi, & \\ \gamma(R,\theta) = 0 & \text{on } C_2 : r = R\,;\ 0 \le \theta \le 2\pi, & \end{array}\right\} \quad (8.2)$$

where S is a doubly connected domain bounded by two closed curves C_1 and C_2 , and $\frac{\partial}{\partial n}$ is a partial derivative along the inward normal to C_1 , (see Fig. 30 page 127).

Both boundary value problems are of the same pattern and we shall consider first the problem (8.1).

Two methods are discussed and based on the theory of potential: in the first method we use the idea of image theory since the curve C_2 is a circle, and in the second method we use the general theory of logarithmic potential of a single layer.

Method 1.

Using the notations as shown in Fig. 31, we suppose that a line charge of density e located on c_1 at a point $P(\xi,\eta)$ a distance ρ from the centre O of c_2, and an image line charge of density e' is at a point $P'(\xi',\eta')$ a distance ρ' from the centre O along OP produced.

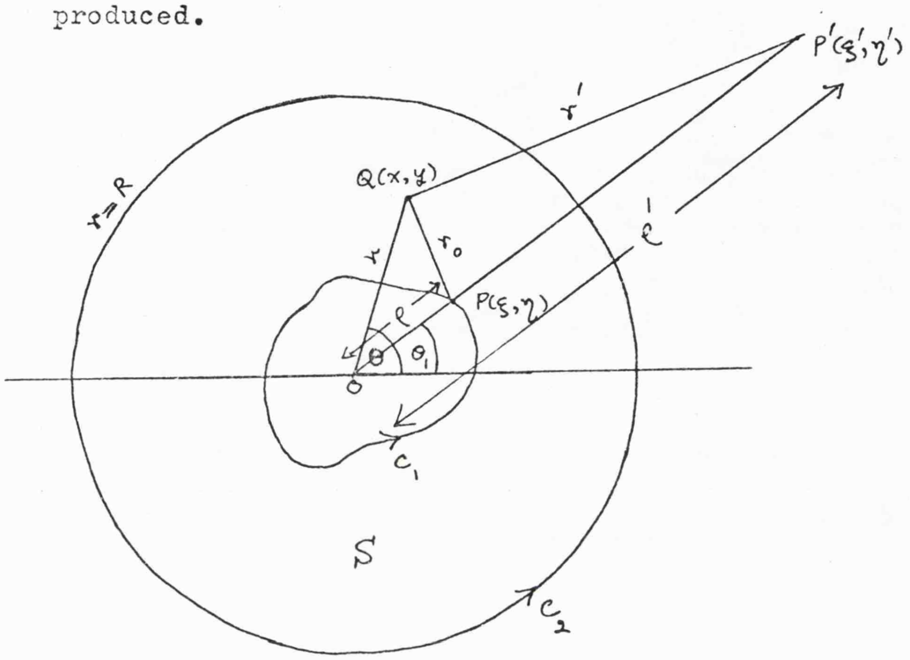


Fig. 31

$Q(x,y)$ is any point in a domain S with a distance r from the centre O.

The potential at any point $Q(x,y)$ due to the line charge at $P(\xi,\eta)$ and its image at $P'(\xi',\eta')$ can be verified to be

$$\Phi(Q) = \beta + e\left[\log\frac{1}{r_0} - \log\frac{1}{r'} - \log\left(\frac{R}{\rho}\right)\right]. \tag{8.3}$$

Let the point Q be (r,θ) in plane polar coordinates, we then have the relations

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad 0 \le \theta \le 2\pi ;$$

and the point $P(\xi,\eta)$ correspond to $P(\rho,\theta_1)$ in which $\rho = g(\theta_1)$, hence

$$\left.\begin{aligned} \xi &= g(\theta_1)\cos\theta_1 \\ \eta &= g(\theta_1)\sin\theta_1 \end{aligned}\right\} \quad 0 \le \theta_1 \le 2\pi ;$$

similarly, for the image point $P'(\xi',\eta')$ a distance $\rho' = \frac{R^2}{\rho}$ from the centre O , we have

$$\left.\begin{aligned} \xi' &= \frac{R^2}{g(\theta_1)}\cos\theta_1 \\ \eta' &= \frac{R^2}{g(\theta_1)}\sin\theta_1 \end{aligned}\right\} \quad 0 \le \theta_1 \le 2\pi .$$

Thus the distances r_0 and r' can be evaluated and defined as follows:

$$\begin{aligned} r_0^2 &= (x-\xi)^2+(y-\eta)^2 \\ &= r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta) \end{aligned}$$

and

$$\begin{aligned} (r')^2 &= (x-\xi')^2+(y-\eta')^2 \\ &= r^2+\frac{R^4}{g^2(\theta_1)}-\frac{2rR^2}{g(\theta_1)}\cos(\theta_1-\theta) \end{aligned}$$

hence

$$\log\frac{1}{r_0}-\log\frac{1}{r'}-\log\left(\frac{R}{\rho}\right) = \frac{1}{2}\log\left[\frac{\frac{r^2g^2(\theta_1)}{R^2}+R^2-2rg(\theta_1)\cos(\theta_1-\theta)}{r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta)}\right]. \tag{8.4}$$

Since $\Phi(Q)$ in (8.3) is a potential at any point $Q(x,y)$ in a domain S due to a line charge of density e placed at point $P(\xi,\eta)$ on the curve C_1 , we also can write the potential at Q due to a distribution of sources around C_1 with strength $\mu(s)$ in the form

$$\phi(x,y) = \oint_{C_1} \mu(s)\left[\log\frac{1}{r_0} - \log\frac{1}{r'} - \log\left(\frac{R}{\rho}\right)\right] ds + \beta . \tag{8.5}$$

On C_1 : $r = g(\theta_1)$; $0 \le \theta_1 \le 2\pi$, we can find $ds = g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\, d\theta_1$

and by using (8.4), $\phi(x,y)$ in (8.5)becomes

$$\phi(r,\theta) = \beta + \frac{1}{2}\int_0^{2\pi} \mu(\theta_1) g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\, \log\left[\frac{\frac{r^2 g^2(\theta_1)}{R^2} + R^2 - 2rg(\theta_1)\cos(\theta_1-\theta)}{r^2 + g^2(\theta_1) - 2rg(\theta_1)\cos(\theta_1-\theta)}\right] d\theta_1$$

(8.6)

It can be verified that $\phi(x,y)$ or $\phi(r,\theta)$ defined in (8.6) satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 ,$$

and the boundary condition

$$\phi(R,\theta) = \beta , \qquad \text{on } C_2 : r = R ; 0 \le \theta \le 2\pi .$$

Next we shall calculate $\frac{\partial\phi}{\partial n_i}$ on the boundary C_1 where $\underset{\sim}{n_i}$ is a unit normal vector to C_1 and directed outward the domain S (see Fig. 32).

Let $N(\xi_0,\eta_0)$ be a point on C_1 with plane polar - coordinates $\{g(\theta_0), \theta_0\}$, i.e.,

$$\left.\begin{aligned} \xi_0 &= g(\theta_0)\cos\theta_0 \\ \eta_0 &= g(\theta_0)\sin\theta_0 \end{aligned}\right\} \quad 0 \le \theta_0 \le 2\pi ;$$

through which the point $Q(x,y)$ passes when it moves to the boundary C_1 along $\underset{\sim}{n_i}$.

By partial differentiating (8.5) with respect to n_i , we obtain

$$\frac{\partial \phi}{\partial n_i} = -\int_0^{2\pi} \mu(\theta_1)\left[\frac{1}{r_0}\frac{\partial r_0}{\partial n_i} - \frac{1}{r'}\frac{\partial r'}{\partial n_i}\right] \cdot g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\, d\theta_1 , \qquad (8.7)$$

where as before we have

$$\left.\begin{aligned} r_0^2 &= r^2 + g^2(\theta_1) - 2rg(\theta_1)\cos(\theta_1-\theta) \\ (r')^2 &= r^2 + \frac{R^4}{g^2(\theta_1)} - \frac{2rR^2}{g(\theta_1)}\cos(\theta_1-\theta) \end{aligned}\right\} \qquad (8.8)$$

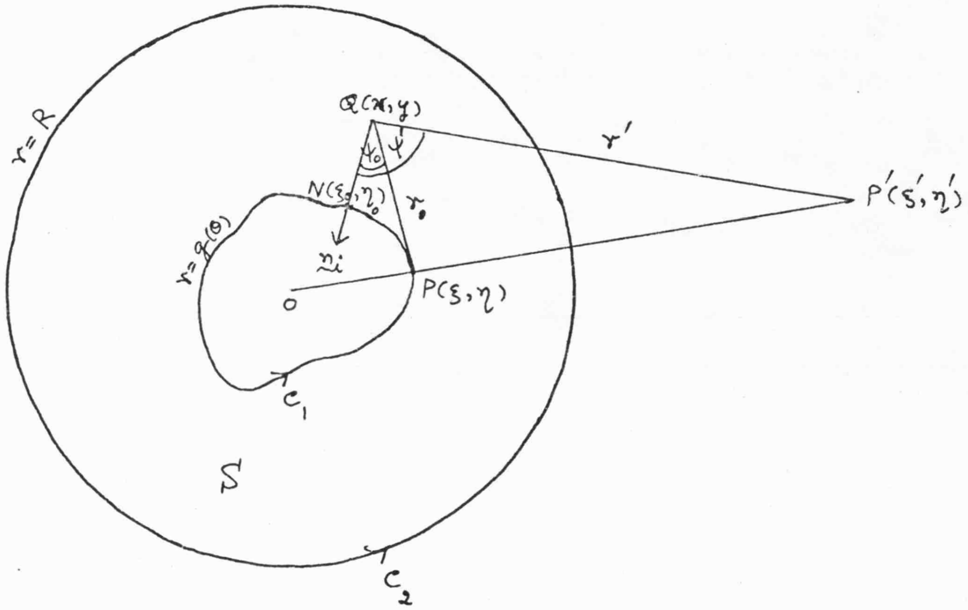


Fig. 32

Since $\quad -\delta r_0 = \delta n_i \cos\psi_0 ,$ (see Fig. 32)

$$\therefore \quad \frac{\partial r_0}{\partial n_i} \simeq -\cos\psi_0$$

Similarly, $\qquad \dfrac{\partial r'}{\partial n_i} \simeq -\cos\psi'$

where ψ_0 and ψ' are the angles between r_0 and n_i , and, r' and n_i respectively. Hence (8.7) becomes

$$\frac{\partial \phi}{\partial n_i} = \int_0^{2\pi} \mu(\theta_1)\left[\frac{\cos\psi_0}{r_0} - \frac{\cos\psi'}{r'}\right] \cdot g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}} \cdot d\theta_1$$

The function $\frac{\partial \phi}{\partial n_i}$ is not continuous at the boundary C_1 and it can be shown that [51],

$$\lim_{Q \to N_+} \frac{\partial \phi}{\partial n_i} = \pi \mu(N) + \int_0^{2\pi} \mu(\theta_1) \left[\frac{\cos \psi_0}{r_0} - \frac{\cos \psi'}{r'} \right]_{\text{at } N(\xi_0, \eta_0)} \cdot g(\theta_1) \cdot \sqrt{1 + \frac{g'^2(\theta_1)}{g^2(\theta_1)}}\, d\theta_1 \,, \tag{8.9}$$

where the integral has to be interpreted as a Cauchy principal value.

Consider Fig. 33, we can express the term $\left[\frac{\cos \psi_0}{r_0}\right]_{\text{at } N(\xi_0, \eta_0)}$ as follows:

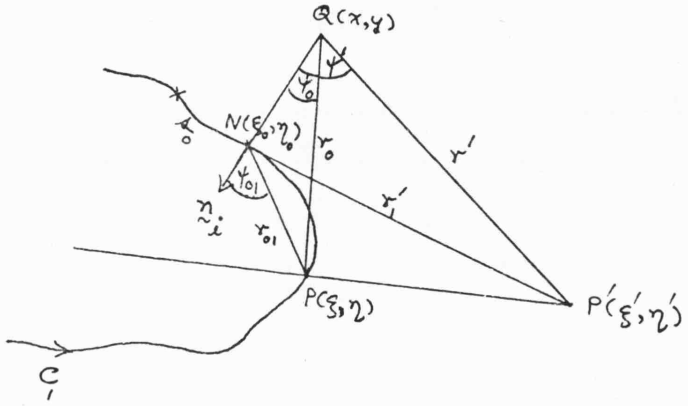


Fig. 33

$$\left[\frac{\cos \psi_0}{r_0}\right]_{\text{at } N(\xi_0, \eta_0)} = \frac{\cos \psi_{01}}{r_{01}}$$

$$= \frac{\cos (r_{01}, n_i)}{r_{01}}$$

$$= \frac{\cos (n_i, \xi_0 - r_{01}, \xi_0)}{r_{01}}$$

$$= \frac{1}{r_{01}} \left[\cos (n_i, \xi_0) \cos (r_{01}, \xi_0) + \sin (n_i, \xi_0) \sin (r_{01}, \xi_0) \right]$$

or

$$\left[\frac{\cos\psi_0}{r_0}\right]_{\text{at } N(\xi_0,\eta_0)} = \frac{1}{r_{01}}\left[\cos(r_{01},\xi_0)\cos(n_i,\xi_0) + \cos(r_{01},\eta_0)\cos(n_i,\eta_0)\right] \tag{8.10}$$

Since

$$\cos(n_i,\xi_0) = -\frac{d\eta_0}{ds_0} = -\left[\cos\theta_0 + \frac{g'(\theta_0)}{g(\theta_0)}\sin\theta_0\right]\cdot\frac{1}{\sqrt{1+\frac{g'^2(\theta_0)}{g^2(\theta_0)}}}$$

$$\cos(n_i,\eta_0) = \frac{d\xi_0}{ds_0} = \left[-\sin\theta_0 + \frac{g'(\theta_0)}{g(\theta_0)}\cos\theta_0\right]\cdot\frac{1}{\sqrt{1+\frac{g'^2(\theta_0)}{g^2(\theta_0)}}}$$

$$\cos(r_{01},\xi_0) = \frac{\xi-\xi_0}{r_{01}} = \frac{g(\theta_1)\cos\theta_1 - g(\theta_0)\cos\theta_0}{r_{01}}$$

$$\cos(r_{01},\eta_0) = \frac{\eta-\eta_0}{r_{01}} = \frac{g(\theta_1)\sin\theta_1 - g(\theta_0)\sin\theta_0}{r_{01}}$$

where

$$\begin{aligned} r_{01}^2 &= (\xi-\xi_0)^2 + (\eta-\eta_0)^2 \\ &= g^2(\theta_1) + g^2(\theta_0) - 2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0) \end{aligned}$$

Hence, (8.10) becomes

$$\left[\frac{\cos\psi_0}{r_0}\right]_{\text{at } N(\xi_0,\eta_0)} = \frac{g(\theta_0) - g(\theta_1)\cos(\theta_1-\theta_0) + \frac{g(\theta_1)g'(\theta_0)}{g(\theta_0)}\sin(\theta_1-\theta_0)}{\sqrt{1+\frac{g'^2(\theta_0)}{g^2(\theta_0)}}\left[g^2(\theta_1)+g^2(\theta_0)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)\right]} \tag{8.11}$$

Similarly, by using the same method, $\left[\frac{\cos\psi'}{r'}\right]_{\text{at } N(\xi_0,\eta_0)}$ can be expressed in the form

$$\begin{aligned} \left[\frac{\cos\psi'}{r'}\right]_{\text{at } N(\xi_0,\eta_0)} &= \frac{\cos(r_1',n_i)}{r_1'} \\ &= \frac{g(\theta_0) - \frac{R^2}{g(\theta_1)}\cos(\theta_1-\theta_0) + \frac{R^2 g'(\theta_0)}{g(\theta_1)g(\theta_0)}\sin(\theta_1-\theta_0)}{\sqrt{1+\frac{g'^2(\theta_0)}{g^2(\theta_0)}}\left[\frac{R^4}{g^2(\theta_1)} + g^2(\theta_0) - \frac{2R^2 g(\theta_0)}{g(\theta_1)}\cos(\theta_1-\theta_0)\right]}. \end{aligned} \tag{8.12}$$

To satisfy the boundary condition on C_1 ; i.e., $\frac{\partial\phi}{\partial n} + k\phi = F_1(r,\theta)$ where $\underset{\sim}{n}$ is in the opposite direction to $\underset{\sim}{n}_i$, it follows from (8.6) and (8.9) that $\mu(\theta_1)$ must satisfy the integral equation

$$k\beta - F_1(g(\theta_0),\theta_0) = \pi\mu(\theta_0) + \int_0^{2\pi}\mu(\theta_1)g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\Bigg[\left\{\frac{\cos\psi_0}{r_0} - \frac{\cos\psi'}{r'}\right\}_{\text{at } N(\xi_0,\eta_0)} -$$

$$-\frac{k}{2}\log\left\{\frac{\frac{g^2(\theta_0)g^2(\theta_1)}{R^2} + R^2 - 2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)}{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)}\right\}\Bigg]d\theta_1 , \tag{8.13}$$

where $\left[\frac{\cos\psi_0}{r_0}\right]_{\text{at } N(\xi_0,\eta_0)}$ and $\left[\frac{\cos\psi'}{r'}\right]_{\text{at } N(\xi_0,\eta_0)}$ are defined in (8.11) and (8.12) respectively.

Therefore, the solution of the boundary value problem (8.1) is defined in (8.6) where the unknown $\mu(\theta_1)$ satisfies the Fredholm integral equation (8.13).

Some kernels in (8.13) can be changed into Fourier series which are degenerate. In fact we have

$$\log\left\{1+r^2-2r\cos(\theta_1-\theta)\right\}^{\frac{1}{2}} = -\sum_{n=1}^{\infty}\frac{r^n\cos n(\theta_1-\theta)}{n} , \qquad \text{when } |r|<1; \tag{8.14}$$

and since $\frac{rg(\theta_1)}{R^2} < 1$, then we can write

$$\log\left\{\frac{r^2g^2(\theta_1)}{R^2} + R^2 - 2rg(\theta_1)\cos(\theta_1-\theta)\right\}^{\frac{1}{2}} = \log R - \sum_{n=1}^{\infty}\frac{r^n g^n(\theta_1)\cos n(\theta_1-\theta)}{nR^{2n}} , \tag{8.15}$$

We note that (8.15) is also valid when $r = g(\theta_0)$.

Also from (8.14), we have

$$\frac{r - \cos(\theta_1-\theta)}{1+r^2-2r\cos(\theta_1-\theta)} = -\sum_{n=1}^{\infty} r^{n-1}\cos n(\theta_1-\theta) \,, \qquad \text{when } |r|<1 \text{ ; all } \theta \,,$$

and

$$\frac{\sin(\theta_1-\theta)}{1+r^2-2r\cos(\theta_1-\theta)} = \sum_{n=1}^{\infty} r^{n-1}\sin n(\theta_1-\theta) \,, \qquad \text{when } |r|<1 \text{ ; all } \theta \,.$$

By using these two relations and since $\frac{g(\theta_0)\,g(\theta_1)}{R^2} < 1$, we can write

$$\frac{g(\theta_0) - \frac{R^2}{g(\theta_1)}\cos(\theta_1-\theta_0) + \frac{R^2 g'(\theta_0)}{g(\theta_0)\,g(\theta_1)}\sin(\theta_1-\theta_0)}{g^2(\theta_0) + \frac{R^4}{g^2(\theta_1)} - 2\frac{R^2 g(\theta_0)}{g(\theta_1)}\cos(\theta_1-\theta_0)}$$

$$= -\sum_{n=1}^{\infty} \frac{g^n(\theta_1)\,g^{n-1}(\theta_0)\cos n(\theta_1-\theta_0)}{R^{2n}} + \sum_{n=1}^{\infty} \frac{g'(\theta_0)\,g^n(\theta_1)\,g^{n-2}(\theta_0)}{R^{2n}}\sin n(\theta_1-\theta_0) \,. \tag{8.16}$$

By using (8.15) and (8.16), the solution of the boundary value problem (8.1), i.e., defined in (8.6) and (8.13), can be written in the form

$$\phi(r,\theta) = \beta + \int_0^{2\pi} \mu(\theta_1)\,g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\left[\log R - \sum_{n=1}^{\infty}\frac{r^n g^n(\theta_1)\cos n(\theta_1-\theta)}{n R^{2n}} - \right.$$
$$\left. -\ \frac{1}{2}\log\left\{r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta)\right\}\right] d\theta_1 \,, \tag{8.17}$$

where $\mu(\theta_1)$ satisfies the Fredholm integral equation of the form

$$k\beta - F_1(g(\theta_0),\theta_0) = \pi\mu(\theta_0) + \int_0^{2\pi} \mu(\theta_1)\, K(\theta_1,\theta_0)\, d\theta_1 \quad , \tag{8.18}$$

where the kernel $K(\theta_1,\theta_0)$ is defined as follows:

$$\begin{aligned} K(\theta_1,\theta_0) = {} & g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\Bigg[\frac{k}{2}\log\left\{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)\right\} - \\ & - k\log R + k\sum_{n=1}^{\infty}\frac{g^n(\theta_0)\, g^n(\theta_1)\cos n(\theta_1-\theta_0)}{n R^{2n}} + \\ & + \frac{1}{\sqrt{1+\frac{g'^2(\theta_0)}{g^2(\theta_0)}}}\left\{\frac{g(\theta_0)-g(\theta_1)\cos(\theta_1-\theta_0)+\frac{g'(\theta_0)\, g(\theta_1)}{g(\theta_0)}\sin(\theta_1-\theta_0)}{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)} + \right. \\ & \left. + \sum_{n=1}^{\infty}\left(\frac{g^n(\theta_1)\, g^{n-1}(\theta_0)}{R^{2n}}\cos n(\theta_1-\theta_0) - \frac{g'(\theta_0)\, g^n(\theta_1)\, g^{n-2}(\theta_0)}{R^{2n}}\sin n(\theta_1-\theta_0)\right)\right\}\Bigg] . \end{aligned} \tag{8.19}$$

In a similar way using (8.2) and (8.17) we can derive an integral formula for $\gamma(r,\theta)$. Finally we substitute for ϕ and γ in (7.34) which provides an integro-differential equation for $g(\theta)$.

<u>Method 2.</u>

Let $Q(x,y)$ be any point in domain S corresponding to $Q(r,\theta)$ in polar coordinates, i.e.,

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad 0 \le \theta \le 2\pi \; ;$$

$P(\xi,\eta)$ be a point on C_1 where

$$\left.\begin{aligned} \xi &= g(\theta_1)\cos\theta_1 \\ \eta &= g(\theta_1)\sin\theta_1 \end{aligned}\right\} \quad 0 \le \theta_1 \le 2\pi\,;$$

$N(\xi_0, \eta_0)$ be another point on C_1 where we have

$$\left.\begin{aligned} \xi_0 &= g(\theta_0)\cos\theta_0 \\ \eta_0 &= g(\theta_0)\sin\theta_0 \end{aligned}\right\} \quad 0 \le \theta_0 \le 2\pi\,;$$

and $P_2(\xi_2, \eta_2)$ be a point on C_2 where

$$\left.\begin{aligned} \xi_2 &= R\cos\theta_2 \\ \eta_2 &= R\sin\theta_2 \end{aligned}\right\} \quad 0 \le \theta_2 \le 2\pi$$

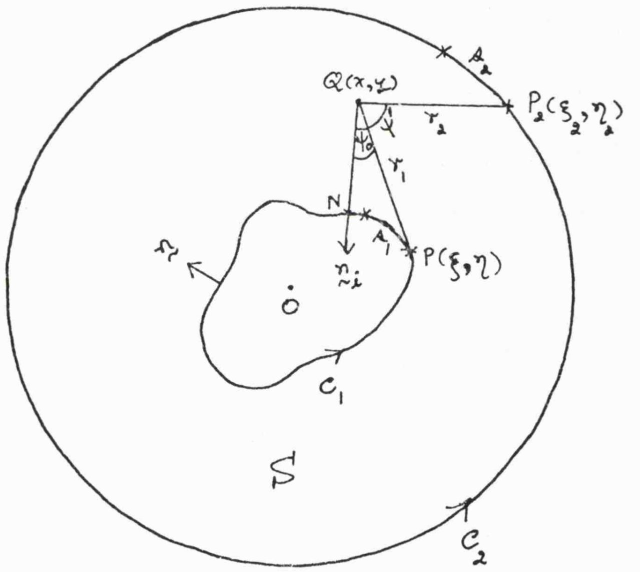


Fig. 34

Consider the logarithmic potential at $Q(x,y)$ of a single layer of densities $\mu_1(s_1)$ and $\mu_2(s_2)$ distributed over the curves C_1 and C_2 respectively, i.e.,

$$\phi(x,y) \equiv \phi(r,\theta) = \oint_{C_1} \mu_1(s_1) \log\frac{1}{r_1}\, ds_1 + \oint_{C_2} \mu_2(s_2) \log\frac{1}{r_2}\, ds_2 \,, \tag{8.20}$$

where r_1 and r_2 are distances between Q and P, and, Q and P_2 respectively (see Fig. 34).

$$\therefore\ r_1^2 = (x-\xi)^2+(y-\eta)^2 = r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta)$$

$$r_2^2 = (x-\xi_2)^2+(y-\eta_2)^2 = r^2+R^2-2rR\cos(\theta_2-\theta)$$

On C_1 : $r=g(\theta_1)$; $0\le\theta_1\le 2\pi$, we have $ds_1 = \sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\cdot g(\theta_1)d\theta_1$

On C_2 : $r=R$; $0\le\theta_2\le 2\pi$, we have $ds_2 = R\,d\theta_2$

Hence (8.20) can be written in the form

$$\phi(r,\theta) = -\frac{1}{2}\int_0^{2\pi}\mu_1(\theta_1)g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\cdot\log\left[r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta)\right]d\theta_1 -$$

$$-\frac{R}{2}\int_0^{2\pi}\mu_2(\theta_2)\log\left[r^2+R^2-2rR\cos(\theta_2-\theta)\right]d\theta_2 ,$$

(8.21)

where $\mu_1(\theta_1)$ and $\mu_2(\theta_2)$ are unknown functions which have to be determined. It can be verified that $\phi(r,\theta)$ in (8.21) is a harmonic function, i.e., $\nabla^2\phi(r,\theta) = 0$.

The unknown $\mu_1(\theta_1)$ and $\mu_2(\theta_2)$ can be found by using the boundary conditions on C_1 and C_2 . Since the logarithmic potential of a single layer is continuous on the boundary, it then follows from (8.21) and the condition $\phi=\beta$ on C_2 that

$$\beta = -\frac{1}{2}\int_0^{2\pi}\mu_1(\theta_1)g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\log\left[R^2+g^2(\theta_1)-2Rg(\theta_1)\cos(\theta_1-\theta)\right]d\theta_1 -$$

$$-\frac{R}{2}\int_0^{2\pi}\mu_2(\theta_2)\log\left[2R^2\left\{1-\cos(\theta_2-\theta)\right\}\right]d\theta_2 , \qquad 0\le\theta\le 2\pi ;$$

(8.22)

and since its normal derivative is discontinuous on the boundary, as in Method 1 , we have

$$\lim_{Q \to N_+} \frac{\partial \phi}{\partial n_i} = \pi \mu_1(N) + \oint_{C_1} \mu_1(s_1) \left[\frac{\cos(r_1, n_i)}{r_1} \right]_{at\, N(\xi_0, \eta_0)} ds_1 + \oint_{C_2} \mu_2(s_2) \left[\frac{\cos(r_2, n_i)}{r_2} \right]_{at\, N(\xi_0, \eta_0)} ds_2 \tag{8.23}$$

where the integral around C_1 is interpreted as a Cauchy principal value.

The expressions of $\left[\frac{\cos(r_1, n_i)}{r_1} \right]_{at\, N(\xi_0, \eta_0)}$ and $\left[\frac{\cos(r_2, n_i)}{r_2} \right]_{at\, N(\xi_0, \eta_0)}$ can be found in the same way as in Method 1 and given as follows:

$$\left[\frac{\cos(r_1, n_i)}{r_1} \right]_{at\, N(\xi_0, \eta_0)} = \frac{g(\theta_0) - g(\theta_1)\cos(\theta_1 - \theta_0) + \frac{g(\theta_1)\, g'(\theta_0)}{g(\theta_0)} \sin(\theta_1 - \theta_0)}{\sqrt{1 + \frac{g'^2(\theta_0)}{g^2(\theta_0)}} \left[g^2(\theta_1) + g^2(\theta_0) - 2g(\theta_0)\, g(\theta_1) \cos(\theta_1 - \theta_0) \right]} \tag{8.24}$$

and

$$\left[\frac{\cos(r_2, n_i)}{r_2} \right]_{at\, N(\xi_0, \eta_0)} = \frac{g(\theta_0) - R\cos(\theta_2 - \theta_0) + \frac{R\, g'(\theta_0)}{g(\theta_0)} \sin(\theta_2 - \theta_0)}{\sqrt{1 + \frac{g'^2(\theta_0)}{g^2(\theta_0)}} \left[R^2 + g^2(\theta_0) - 2R\, g(\theta_0) \cos(\theta_2 - \theta_0) \right]} \tag{8.25}$$

By using (8.21), (8.23) and the condition $\frac{\partial \phi}{\partial n} + k\phi = F_1(r, \theta)$ on C_1 : $r = g(\theta_1)$; $0 \le \theta_1 \le 2\pi$, we obtain

$$-F_1(g(\theta_0),\theta_0) = \pi\mu_1(\theta_0) + \int_0^{2\pi}\mu_1(\theta_1)\,g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\left[\left\{\frac{\cos(r_1,n_i)}{r_1}\right\}_{at\,N(\xi_0,\eta_0)} + \right.$$

$$\left. + \frac{k}{2}\log\left\{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)\right\}\right]d\theta_1 +$$

$$+ R\int_0^{2\pi}\mu_2(\theta_2)\left[\left\{\frac{\cos(r_2,n_i)}{r_2}\right\}_{at\,N(\xi_0,\eta_0)} + \frac{k}{2}\log\left\{g^2(\theta_0)+R^2-2Rg(\theta_0)\cos(\theta_2-\theta_0)\right\}\right]d\theta_2 ,$$

(8.26)

where $\left[\frac{\cos(r_1,n_i)}{r_1}\right]_{at\,N(\xi_0,\eta_0)}$ and $\left[\frac{\cos(r_2,n_i)}{r_2}\right]_{at\,N(\xi_0,\eta_0)}$ are defined in (8.24) and (8.25).

Hence the solution of the boundary value problem (8.1) is defined in (8.21) in which the unknown functions $\mu_1(\theta_1)$ and $\mu_2(\theta_2)$ can be found from a system of integral equations (8.22) and (8.26).

As in Method 1 , some terms can be written in the form of Fourier series as follows:

$$\log\left\{r^2+R^2-2rR\cos(\theta_2-\theta)\right\}^{\frac{1}{2}} = \log R - \sum_{n=1}^{\infty}\frac{r^n\cos n(\theta_2-\theta)}{nR^n} , \quad \left|\frac{r}{R}\right|<1 \text{ all } \theta ;$$

(8.27)

also

$$\log\left[2R^2\left\{1-\cos(\theta_2-\theta)\right\}\right]^{\frac{1}{2}} = \log R + \log\left\{2\sin\left(\frac{\theta_2-\theta}{2}\right)\right\}$$

$$= \log R - \sum_{n=1}^{\infty}\frac{\cos n(\theta_2-\theta)}{n} ,$$

(8.28)

and

$$\left[\frac{\cos(r_2,n_i)}{r_2}\right]_{\text{at } N(\xi_0,\eta_0)} = \frac{g(\theta_0) - R\cos(\theta_2-\theta_0) + \dfrac{R g'(\theta_0)}{g(\theta_0)}\sin(\theta_2-\theta_0)}{\sqrt{1+\dfrac{g'^2(\theta_0)}{g^2(\theta_0)}}\left[R^2+g^2(\theta_0) - 2Rg(\theta_0)\cos(\theta_2-\theta_0)\right]}$$

$$= \left[-\sum_{n=1}^{\infty}\frac{g^{n-1}(\theta_0)\cos n(\theta_2-\theta_0)}{R^n} + \sum_{n=1}^{\infty}\frac{g'(\theta_0)\, g^{n-2}(\theta_0)\sin n(\theta_2-\theta_0)}{R^n}\right]\Bigg/\sqrt{1+\frac{g'^2(\theta_0)}{g^2(\theta_0)}} \quad , \quad \left|\frac{g(\theta_0)}{R}\right|<1$$

(8.29)

By using (8.27) - (8.29), we can write the solution of the problem (8.1), defined in (8.21), (8.22) and (8.26), in the form

$$\phi(r,\theta) = -\frac{1}{2}\int_0^{2\pi}\mu_1(\theta_1)\, g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\,\log\left[r^2+g^2(\theta_1) - 2rg(\theta_1)\cos(\theta_1-\theta)\right]d\theta_1 -$$

$$- R\int_0^{2\pi}\mu_2(\theta_2)\left[\log R - \sum_{n=1}^{\infty}\frac{r^n\cos n(\theta_2-\theta)}{nR^n}\right]d\theta_2 \, ,$$

(8.30)

where $\mu_1(\theta_1)$ and $\mu_2(\theta_2)$ satisfy the Fredholm integral equations

$$\beta = -\int_0^{2\pi}\mu_1(\theta_1)\, g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\left[\log R - \sum_{n=1}^{\infty}\frac{g^n(\theta_1)\cos n(\theta_1-\theta)}{nR^n}\right]d\theta_1 -$$

$$- R\int_0^{2\pi}\mu_2(\theta_2)\left[\log R - \sum_{n=1}^{\infty}\frac{\cos n(\theta_2-\theta)}{n}\right]d\theta_2 \; ;$$

(8.31)

and

$$-F_1(g(\theta_0),\theta_0) = \pi\mu_1(\theta_0) + \int_0^{2\pi}\mu_1(\theta_1)K_1(\theta_1,\theta_0)d\theta_1 + \int_0^{2\pi}\mu_2(\theta_2)K_2(\theta_2,\theta_0)d\theta_2 \, ,$$

(8.32)

where

$$K_1(\theta_1,\theta_0) = g(\theta_1)\sqrt{1+\frac{g'^2(\theta_1)}{g^2(\theta_1)}}\left[\frac{k}{2}\log\left\{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)\right\}+\right.$$

$$\left.+\frac{g(\theta_0)-g(\theta_1)\cos(\theta_1-\theta_0)+\dfrac{g'(\theta_0)g(\theta_1)}{g(\theta_0)}\sin(\theta_1-\theta_0)}{\sqrt{1+\dfrac{g'^2(\theta_0)}{g^2(\theta_0)}}\left\{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)\right\}}\right],$$

and

$$K_2(\theta_2,\theta_0) = R\left[k\left\{\log R-\sum_{n=1}^{\infty}\frac{g^n(\theta_0)\cos n(\theta_2-\theta_0)}{R^n}\right\}-\right.$$

$$\left.-\frac{1}{\sqrt{1+\dfrac{g'^2(\theta_0)}{g^2(\theta_0)}}}\sum_{n=1}^{\infty}\left\{\frac{g^{n-1}(\theta_0)\cos n(\theta_2-\theta_0)}{R^n}-\frac{g'(\theta_0)g^{n-2}(\theta_0)\sin n(\theta_2-\theta_0)}{R^n}\right\}\right].$$

It is not difficult to show that these two methods are equivalent. The second method is useful when the outer boundary C_2 is not a circle, otherwise the first method is preferable since there is only one unknown $\mu(\theta)$ to be evaluated from the integral equation.

We now consider two boundary value problems which have already been discussed in Chapter 7 but here we apply Method 1 to find the solution. Referring to the problem (8.1),

Case 1: $k \neq 0$; $g(\theta) = r_0$; $F_1(r,\theta) = \alpha$, where α and r_0 are constants and $0 < r_0 < R$.

Since $g(\theta) = r_0$; $0 \leq \theta \leq 2\pi$, we have $g'(\theta) = 0$ and it then follows from (8.17) - (8.19) that

$$\phi(r,\theta) = \beta - r_0\int_0^{2\pi}\mu(\theta_1)\left[\log\frac{r}{R}+\sum_{n=1}^{\infty}\frac{r_0^n r^n(1-r^{-2n}R^{2n})}{nR^{2n}}\cos n(\theta_1-\theta)\right]d\theta_1 \, ;$$

where $0 < r_0 < r < R$; $0 \leq \theta \leq 2\pi$, (8.33)

$\mu(\theta_1)$ satisfies the Fredholm integral equation

$$k\beta - \alpha = \pi\mu(\theta_0) + r_0\int_0^{2\pi}\mu(\theta_1)\Big[k\log r_0 + k\log\Big\{2\sin\Big(\frac{\theta_1-\theta_0}{2}\Big)\Big\} - k\log R +$$

$$+ k\sum_{n=1}^{\infty}\frac{r_0^{2n}}{R^{2n}}\cdot\frac{\cos n(\theta_1-\theta_0)}{n} + \frac{1}{2r_0} + \sum_{n=1}^{\infty}\frac{r_0^{2n-1}}{R^{2n}}\cos n(\theta_1-\theta_0)\Big]d\theta_1$$

(8.34)

We can easily show that

$$\log\Big\{2\sin\Big(\frac{\theta_1-\theta_0}{2}\Big)\Big\} = -\sum_{n=1}^{\infty}\frac{\cos n(\theta_1-\theta_0)}{n} ,$$

hence (8.34) becomes

$$k\beta - \alpha = \pi\mu(\theta_0) + \int_0^{2\pi}\mu(\theta_1)\Big[\frac{1}{2} + r_0 k\log\frac{r_0}{R} + \sum_{n=1}^{\infty}\Big\{\frac{r_0^{2n}}{R^{2n}} + \frac{k r_0^{2n+1}}{nR^{2n}} - \frac{r_0 k}{n}\Big\}\cos n(\theta_1-\theta_0)\Big]d\theta_1$$

(8.35)

We set

$$A_0 = \int_0^{2\pi}\mu(\theta_1)\,d\theta_1 ,$$

$$A_n = \int_0^{2\pi}\mu(\theta_1)\cos n\theta_1\,d\theta_1 , \quad n = 1, 2, \ldots\ldots\ldots$$

$$B_n = \int_0^{2\pi}\mu(\theta_1)\sin n\theta_1\,d\theta_1 , \quad n = 1, 2, \ldots\ldots\ldots$$

it then follows from (8.35) that

$$\mu(\theta_0) = \frac{1}{\pi}\Big[(k\beta-\alpha) - \Big(\frac{1}{2} + r_0 k\log\frac{r_0}{R}\Big)A_0 - \sum_{n=1}^{\infty}\Big\{\frac{r_0^{2n}}{R^{2n}} + \frac{k r_0^{2n+1}}{nR^{2n}} - \frac{r_0 k}{n}\Big\}\cdot\Big\{A_n\cos n\theta_0 + B_n\sin n\theta_0\Big\}\Big]$$

(8.36)

By integrating (8.36) from 0 to 2π , we have

$$A_0 = \frac{k\beta - \alpha}{1 + r_0 k\log\frac{r_0}{R}}$$

Multiply both sides of (8.36) by $\cos n\theta_0$ and integrate from $\theta_0 = 0$ to $\theta_0 = 2\pi$, we have

$$A_n = 0 \quad , \quad n = 1, 2, \ldots\ldots$$

Similarly, multiply (8.36) by $\sin n\theta_0$ and integrate from 0 to 2π we also have

$$B_n = 0 \quad , \quad n = 1, 2, \ldots$$

Hence (8.36) becomes

$$\mu(\theta_0) = -\frac{1}{2\pi}\left[\frac{\alpha - k\beta}{1 + r_0 k \log\frac{r_0}{R}}\right]$$

and then it follows from (8.33) that

$$\phi(r,\theta) = \beta + \frac{r_0(\alpha - k\beta)}{(1 + r_0 k \log\frac{r_0}{R})} \cdot \log\frac{r}{R}$$

which is the same as (7.36) in Chapter 7.

<u>Case 2: $k = 0$; $g(\theta) = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2)$; $F_1(r,\theta) = \alpha + \varepsilon f_1(r,\theta) + O(\varepsilon^2)$</u>,

<u>where α and r_0 are constants and $0 < r_0 < R$.</u>

When $k = 0$ and since $\{g'(\theta)\}^2 = O(\varepsilon^2)$, it follows from (8.17) - (8.19) that

$$\phi(r,\theta) = \beta + \int_0^{2\pi} \mu(\theta_1) g(\theta_1)\Big[\log R - \sum_{n=1}^{\infty} \frac{r^n g^n(\theta_1)\cos n(\theta_1 - \theta)}{n R^{2n}} -$$

$$- \frac{1}{2}\log\left\{r^2 + g^2(\theta_1) - 2rg(\theta_1)\cos(\theta_1 - \theta)\right\}\Big] d\theta_1 \;,$$

(8.37)

where $\mu(\theta_1)$ satisfies the integral equation

$$-F_1(g(\theta_0), \theta_0) = \pi\mu(\theta_0) + \int_0^{2\pi} \mu(\theta_1) K(\theta_1, \theta_0)\, d\theta_1 \qquad (8.38)$$

where

$$K(\theta_1,\theta_0) = g(\theta_1)\left[\frac{g(\theta_0)-g(\theta_1)\cos(\theta_1-\theta_0)+\frac{g'(\theta_0)g(\theta_1)}{g(\theta_0)}\sin(\theta_1-\theta_0)}{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)}\right. +$$

$$\left. +\sum_{n=1}^{\infty}\left\{\frac{g^n(\theta_1)g^{n-1}(\theta_0)}{R^{2n}}\cos n(\theta_1-\theta_0) - \frac{g'(\theta_0)g^n(\theta_1)g^{n-2}(\theta_0)}{R^{2n}}\sin n(\theta_1-\theta_0)\right\}\right]$$

(8.39)

By using $g(\theta) = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2)$ and neglecting the terms of degree greater than one relative to ε , we can easily find the following expressions.

$$\log\{r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta)\}^{\frac{1}{2}} = \log r - \sum_{n=1}^{\infty}\frac{r_0^n\cos n(\theta_1-\theta)}{nr^n} - \varepsilon a_1(\theta_1)\sum_{n=1}^{\infty}\frac{r_0^{n-1}\cos n(\theta_1-\theta)}{r^n} ,$$

and

$$\frac{g(\theta_0)-g(\theta_1)\cos(\theta_1-\theta_0)+\frac{g'(\theta_0)g(\theta_1)}{g(\theta_0)}\sin(\theta_1-\theta_0)}{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)}$$

$$= \frac{1}{2r_0} + \frac{\varepsilon}{2r_0^2\{1-\cos(\theta_1-\theta_0)\}}\left[a_1(\theta_0)\cos(\theta_1-\theta_0) - a_1(\theta_1) + a_1'(\theta_0)\sin(\theta_1-\theta_0)\right]$$

thus we have

$$g(\theta_1)\log\{r^2+g^2(\theta_1)-2rg(\theta_1)\cos(\theta_1-\theta)\}^{\frac{1}{2}}$$

$$= \{r_0+\varepsilon a_1(\theta_1)\}\log r - \sum_{n=1}^{\infty}\frac{r_0^{n+1}\cos n(\theta_1-\theta)}{nr^n} - \sum_{n=1}^{\infty}\varepsilon a_1(\theta_1)\frac{r_0^n(n+1)\cos n(\theta_1-\theta)}{nr^n} + O(\varepsilon^2) ,$$

(8.40)

and

$$g(\theta_1)\left[\frac{g(\theta_0)-g(\theta_1)\cos(\theta_1-\theta_0)+\dfrac{g'(\theta_0)g(\theta_1)}{g(\theta_0)}\sin(\theta_1-\theta_0)}{g^2(\theta_0)+g^2(\theta_1)-2g(\theta_0)g(\theta_1)\cos(\theta_1-\theta_0)}\right]$$

$$= \frac{1}{2}+\frac{\varepsilon}{2r_0\{1-\cos(\theta_1-\theta_0)\}}\left[\{a_1(\theta_0)-a_1(\theta_1)\}\cos(\theta_1-\theta_0)+a_1'(\theta_0)\sin(\theta_1-\theta_0)\right]+O(\varepsilon^2). \tag{8.41}$$

Substituting (8.40) and (8.41) into (8.37) and (8.39) respectively, we obtain

$$\phi(r,\theta) = \beta-\int_0^{2\pi}\mu(\theta_1)\Big[\{r_0+\varepsilon a_1(\theta_1)\}\log\frac{r}{R}+\sum_{n=1}^{\infty}\frac{r^n r_0^{n+1}(1-r^{-2n}R^{2n})}{nR^{2n}}\cos n(\theta_1-\theta)+$$

$$+\varepsilon a_1(\theta_1)\sum_{n=1}^{\infty}\frac{(n+1)r_0^n r^n(1-r^{-2n}R^{2n})}{nR^{2n}}\cos n(\theta_1-\theta)\Big]\,d\theta_1+O(\varepsilon^2), \tag{8.42}$$

where $\mu(\theta_1)$ satisfies the Fredholm integral equation of the second kind as follows:

$$-\alpha-\varepsilon f_1(g(\theta_0),\theta_0)=\pi\mu(\theta_0)+\int_0^{2\pi}\mu(\theta_1)\Big[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{r_0^{2n}}{R^{2n}}\cos n(\theta_1-\theta_0)+$$

$$+\frac{\varepsilon}{2r_0}\left\{\frac{(a_1(\theta_0)-a_1(\theta_1))\cos(\theta_1-\theta)+a_1'(\theta_0)\sin(\theta_1-\theta_0)}{1-\cos(\theta_1-\theta_0)}\right\}+$$

$$+\varepsilon\sum_{n=1}^{\infty}\frac{r_0^{2n-1}}{R^{2n}}\Big\{\big((n-1)a_1(\theta_0)+(n+1)a_1(\theta_1)\big)\cos n(\theta_1-\theta_0)-a_1'(\theta_0)\sin n(\theta_1-\theta_0)\Big\}\Big]\,d\theta_1$$

$$+\ O(\varepsilon^2). \tag{8.43}$$

We suppose that

$$\mu(\theta) = \mu_0(\theta)+\varepsilon\mu_1(\theta)+O(\varepsilon^2) \tag{8.44}$$

it then follows from (8.43) two integral equations

$$-\alpha = \pi\mu_0(\theta_0) + \int_0^{2\pi} \mu_0(\theta_1)\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{r_0^{2n}}{R^{2n}} \cos n(\theta_1-\theta_0)\right] d\theta_1 \tag{8.45}$$

and

$$-f_1(g(\theta_0),\theta_0) = \pi\mu_1(\theta_0) + \int_0^{2\pi} \mu_1(\theta_1)\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{r_0^{2n}}{R^{2n}} \cos n(\theta_1-\theta_0)\right] d\theta_1 +$$

$$+\int_0^{2\pi} \mu_0(\theta_1)\left[\frac{1}{2r_0}\left\{\frac{\left(a_1(\theta_0)-a_1(\theta_1)\right)\cos(\theta_1-\theta_0)+a_1'(\theta_0)\sin(\theta_1-\theta_0)}{1-\cos(\theta_1-\theta_0)}\right\}+\right.$$

$$+\sum_{n=1}^{\infty} \frac{r_0^{2n-1}}{R^{2n}}\left\{\left((n-1)a_1(\theta_0)+(n+1)a_1(\theta_1)\right)\cos n(\theta_1-\theta_0)-\right.$$

$$\left.\left. - a_1'(\theta)\sin n(\theta_1-\theta_0)\right\}\right] d\theta_1 . \tag{8.46}$$

The integral equation (8.45) is the same as (8.35) when $k=0$, thus its solution $\mu_0(\theta_0)$ will be

$$\mu_0(\theta_0) = -\frac{\alpha}{2\pi} \tag{8.47}$$

We set

$$\left.\begin{aligned} B_0 &= \int_0^{2\pi} \mu_1(\theta_1)\, d\theta_1 , \\ B_n &= \int_0^{2\pi} \mu_1(\theta_1)\cos n\theta_1\, d\theta_1 , \qquad n=1,2,3,\ldots\ldots \\ D_n &= \int_0^{2\pi} \mu_1(\theta_1)\sin n\theta_1\, d\theta_1 , \qquad n=1,2,3,\ldots\ldots \end{aligned}\right\} \tag{8.48}$$

By using (8.47), (8.48) and since the Cauchy principal value of

$$\int_0^{2\pi} \frac{\sin(\theta_1-\theta_0)\, d\theta_1}{1-\cos(\theta_1-\theta_0)} = 0$$

thus the integral equation (8.46) has the solution as follows:

$$\mu_1(\theta_0) = -\frac{1}{\pi}\Big[f_1(g(\theta_0),\theta_0) + \frac{B_0}{2} + \sum_{n=1}^{\infty}\frac{r_0^{2n}}{R^{2n}}\left(B_n\cos n\theta_0 + D_n\sin n\theta_0\right) - \frac{\alpha}{4\pi r_0}\int_0^{2\pi}\frac{a_1(\theta_0)\cos(\theta_1-\theta_0)}{1-\cos(\theta_1-\theta_0)}d\theta_1 +$$

$$+\frac{\alpha}{4\pi r_0}\int_0^{2\pi}\frac{a_1(\theta_1)\cos(\theta_1-\theta_0)}{1-\cos(\theta_1-\theta_0)}d\theta_1 - \frac{\alpha}{2\pi}\sum_{n=1}^{\infty}\frac{(n+1)r_0^{2n-1}}{R^{2n}}\int_0^{2\pi}a_1(\theta_1)\cos n(\theta_1-\theta_0)\,d\theta_1\Big]. \tag{8.49}$$

By integrating (8.49) from $\theta_0 = 0$ to $\theta_0 = 2\pi$, we have

$$B_0 = -\frac{1}{2\pi}\int_0^{2\pi} f_1(g(\theta_0),\theta_0)\,d\theta_0 \tag{8.50}$$

The coefficients B_n and D_n ($n=1, 2, 3, \ldots\ldots$) can be found by multiplying (8.49) by $\cos n\theta_0$ and $\sin n\theta_0$ respectively and integrating from $\theta_0 = 0$ to $\theta_0 = 2\pi$. Hence we can find that

$$\frac{\pi r_0^{2n}}{R^{2n}}\left(1 + r_0^{-2n}R^{2n}\right)\left\{B_n\cos n\theta_0 + D_n\sin n\theta_0\right\}$$

$$= -\int_0^{2\pi} f_1(g(\theta_1),\theta_1)\cos n(\theta_1-\theta_0)d\theta_1 + \frac{\alpha}{4\pi r_0}\int_0^{2\pi}\left[a_1(\theta_1)\cos n(\theta_1-\theta_0)\left\{\int_0^{2\pi}\frac{\cos(\xi-\theta_1)\,d\xi}{1-\cos(\xi-\theta_1)}\right\}\right]d\theta_1 -$$

$$-\frac{\alpha}{4\pi r_0}\int_0^{2\pi}\left[a_1(\theta_1)\left\{\int_0^{2\pi}\frac{\cos n(\xi-\theta_0)}{1-\cos(\theta_1-\xi)}d\xi\right\}\right]d\theta_1 + \frac{\alpha r_0^{2n-1}(n+1)}{2R^{2n}}\int_0^{2\pi}a_1(\theta_1)\cos n(\theta_1-\theta_0)\,d\theta_1 ,$$

$$n = 1, 2, 3, \ldots\ldots \tag{8.51}$$

It is not difficult to verify that

$$\int_0^{2\pi}\frac{\cos n(\xi-\theta_0)}{1-\cos(\theta_1-\xi)}d\xi = \cos n(\theta_1-\theta_0)\left[-2\pi n + \int_0^{2\pi}\frac{d\eta}{1-\cos(\eta-\theta_1)}\right],$$

hence, (8.51) becomes

$$\frac{r_0^{2n}}{R^{2n}}\left\{B_n \cos n\theta_0 + D_n \sin n\theta_0\right\} = -\frac{1}{\pi\left(1+r_0^{-2n}R^{2n}\right)}\int_0^{2\pi} f_1(g(\theta_1),\theta_1)\cos n(\theta_1-\theta_0)\,d\theta_1 +$$

$$+\frac{\alpha r_0^{2n-1}}{2\pi R^{2n}}\left\{n+\frac{(1-r_0^{-2n}R^{2n})}{(1+r_0^{-2n}R^{2n})}\right\}\int_0^{2\pi} a_1(\theta_1)\cos n(\theta_1-\theta_0)\,d\theta_1 .$$

(8.52)

By substitutuing (8.50) and (8.52) into (8.49), we obtain

$$\mu_1(\theta_0) = -\frac{1}{\pi}\left[f_1(g(\theta_0),\theta_0) - \frac{1}{4\pi}\int_0^{2\pi} f_1(g(\theta_1),\theta_1)\,d\theta_1 - \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{f_1(g(\theta_1),\theta_1)\cos n(\theta_1-\theta_0)\,d\theta_1}{(1+r_0^{-2n}R^{2n})} +\right.$$

$$-\frac{\alpha}{\pi r_0}\sum_{n=1}^{\infty}\frac{1}{(1+r_0^{-2n}R^{2n})}\int_0^{2\pi} a_1(\theta_1)\cos n(\theta_1-\theta_0)\,d\theta_1 - \frac{\alpha}{4\pi r_0}\int_0^{2\pi}\frac{a_1(\theta_0)\cos(\theta_1-\theta_0)\,d\theta_1}{1-\cos(\theta_1-\theta_0)}$$

$$+\frac{\alpha}{4\pi r_0}\int_0^{2\pi}\frac{a_1(\theta_1)\cos(\theta_1-\theta_0)}{1-\cos(\theta_1-\theta_0)}\,d\theta_1 .$$

(8.53)

By using (8.44), the solution $\phi(r,\theta)$ in (8.42) can be written in terms of $\mu_0(\theta)$ and $\mu_1(\theta)$ as follows:

$$\phi(r,\theta) = \beta - \int_0^{2\pi}\mu_0\left[\left\{r_0+\varepsilon a_1(\theta_1)\right\}\log\frac{r}{R} + \sum_{n=1}^{\infty}\frac{r^n r_0^{n+1}}{R^{2n}}(1-r^{-2n}R^{2n})\,\frac{\cos n(\theta_1-\theta)}{n} +\right.$$

$$\left.+\varepsilon a_1(\theta_1)\sum_{n=1}^{\infty}\frac{(n+1)r_0^n r^n(1-r^{-2n}R^{2n})}{nR^{2n}}\cos n(\theta_1-\theta)\right]d\theta_1 -$$

$$-\varepsilon\int_0^{2\pi}\mu_1(\theta_1)\left[r_0\log\frac{r}{R} + \sum_{n=1}^{\infty}\frac{r^n r_0^{n+1}}{nR^{2n}}(1-r^{-2n}R^{2n})\cos n(\theta_1-\theta)\right]d\theta_1 + O(\varepsilon^2).$$

(8.54)

Substituting (8.47) and (8.53) into (8.54) we then get the result in the form

$$\phi(r,\theta) = \beta + \alpha r_0 \log\frac{r}{R} + \frac{\varepsilon}{\pi}\int_0^{2\pi}\left\{r_0 f_1(g(\theta_1),\theta_1) + \alpha a_1(\theta_1)\right\}\left\{\frac{\log\frac{r}{R}}{2} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{r^n(1 - r^{-2n}R^{2n})}{n r_0^n(1 + r_0^{-2n}R^{2n})} \cos n(\theta_1 - \theta)\right\} d\theta_1 + O(\varepsilon^2) , \tag{8.55}$$

which is the same as (7.62) in Chapter 7.

CONCLUSION OF PART II

The particular problem of the annulus which we discussed in Chapter 7 is an example which illustrates the application of the theory of the variation of the functional defined on a variable domain. Since the boundary value problems for the state function ϕ and the Lagrange multiplier γ are of the same pattern both can be solved by using the method of separation of variables for the case when the unknown curve C_1 is in the form $r = r_0 + \varepsilon a_1(\theta) + O(\varepsilon^2)$ and the given boundary conditions are

$$\frac{\partial\phi}{\partial n} = \alpha + \varepsilon f_1(r,\theta) + O(\varepsilon^2), \quad \text{on } C_1 ;$$

$$\phi = \beta , \qquad \text{on } C_2 : r = R .$$

The problem of the annulus is completely solved for this case by substituting the expressions for ϕ and γ , which are in terms of unknown functions r_0 and $a_1(\theta)$, into the transversality condition on the unknown boundary and solve for the unknown boundary shape. The Fredholm integral equations which occur in this case have the kernels of degenerate type which are not difficult to solve.

For the case when the unknown curve C_1 is in the general form $r = g(\theta)$; $0 \leq \theta \leq 2\pi$ and the given condition on C_1 is $\frac{\partial\phi}{\partial n} + k\phi = F_1(r,\theta)$, the boundary value problems for ϕ and γ may be solved by using the method of logarithmic potential of a single layer which has been discussed in Chapter 8. Numerical work is needed here in order to solve for the optimum curve $r = g(\theta)$. The existence and uniqueness for the solution of the optimum shape problem have not been studied in this thesis and much work needs to be done in this area.

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SUMMARY

In Part I, the problem of heating a thin plate or material travelling through a furnace, in which the system is described by first order linear partial differential equations, is introduced as an example of optimal control theory in distributed parameter systems. The variational technique in a fixed domain is used to obtain the necessary conditions for optimality. Many cases of the problem with the state equation described by first order linear partial differential equations are discussed, in which the control function enters into the state equation in different positions. The problems are analysed and solved by making use of characteristic curves.

In Part II, we have studied the variation of a functional defined on a variable domain, and we apply it to the problem of finding the optimum shape of the domain in which some performance criterion has an extremum. The problem in which the state equation is Laplace's equation defined on the variable domain of an annular shape with given boundary conditions is discussed and completely solved for the case when the inner boundary of the domain is only a small departure from a circle. We also introduce the method of logarithmic potential of a single layer to solve the boundary value problem of Laplace's equation with mixed boundary conditions and two simple examples are solved by using this method which leads to coupled integral equations.