INTERMEDIATE PROPOSITIONAL LOGICS

BY

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PREFACE.

Intermediate (propositional) logics are, roughly speaking, propositional logics which lie between the intuitionist and classical systems.

Their interest is threefold: they provide an area of study for generalising and systematising results previously restricted to classical and intuitionist logic, in an informative manner; secondly, many problems, such as decidability, which are classically simple, raise formal difficulties in the case of arbitrary intermediate systems; and lastly, problems relating to intermediate logics can often be given an elegant algebraic formulation and solution.

Intermediate logics have been studied by, among others, DUMMETT, JANKOV, TROELSTRA and UMEZAWA.

In this thesis, we present 3 main results on them, already announced in McKAY [12], [13] and [14]:

1) every intermediate logic characterised by a finite pseudocomplemented lattice, is finitely axiomatisable

2) every intermediate logic obtained by augmenting the intuitionist system with a finite set of classically valid, disjunction-free words, is decidable

3) certain intermediate logics possess no axiomatisations

for which the Separation Theorem can be proved.

In addition, we present a number of ancillary results of varying importance.

My thanks are due to my superviser, Professor R.L.GOODSTEIN, for his constant kindness and help throughout my research career. Also I must thank my friend Dr.A.S.TROELSTRA for a stimulating and fertile correspondence over the past 2 years.

Lastly, I salute the memory of STANISLAW JASKOWSKI, whose work on intuitionist logic, first took my fancy.

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1.

Throughout this thesis,"logic" will abbreviate "propositional logic" unless otherwise stated.

Let $\mathcal{A}$ be the alphabet $\rightarrow$,&,v,¬,(,),A_1,A_2,... where $\rightarrow$ (I),&(C),v(D) and ¬ (N) are called the connectives. The letters appearing in parentheses will prove useful at a later stage,and may be considered as the initials of the connectives. A_1,A_2,...are the atoms.

Definition. A string S is a word iff there is a finite sequence of strings $S_1, S_2 \ldots S_n = S$ such that for $i, 1 \leq i \leq n$ either S_i is an atom or S_i is one of the forms $(S_j \rightarrow S_k)$, $(S_j \& S_k)$, $(S_j \vee S_k)$ or $\neg S_j$, $j,k < i$.

Letters P,Q,R (with indices where necessary) will denote arbitrary words. Let H denote the set of words which are theses of intuitionist logic and let K denote the set of words which are theses of classical logic. Both H and K are finitely axiomatisable . As axioms of H we will take the following:

1) $(A_1 \rightarrow (A_2 \rightarrow A_1))$

2) $((A_1 \rightarrow (A_2 \rightarrow A_3)) \rightarrow ((A_1 \rightarrow A_2) \rightarrow (A_1 \rightarrow A_3)))$

3) $((A_1 \& A_2) \to A_1)$

4) $((A_1 \& A_2) \to A_2)$

5) $(A_1 \to (A_2 \to (A_1 \& A_2)))$

) $(A_1 \to (A_1 \vee A_2))$

7) $(A_2 \to (A_1 \vee A_2))$

8) $((A_1 \to A_3) \to ((A_2 \to A_3) \to ((A_1 \vee A_2) \to A_3)))$

9) $((A_1 \to A_2) \to ((A_1 \to \neg A_2) \to \neg A_1))$

10) $(\neg A_1 \to (A_1 \to A_2))$

along with the usual two rules of inference ,modus ponens and substitution. A finite axiomatisation of K can be obtained from this axiomatisation of H in a variety of ways. We will take as our axiomatisation ,that which results by adjoining PEERCE'S rule

11) $(((A_1 \to A_2) \to A_1) \to A_1)$

We will denote these axiomatisations of H and K by $\mathfrak{U}_H$ and $\mathfrak{U}_K$.

<u>Definition</u>. Let X be a set of words such that

1) $H \subseteq X \subseteq K$

2) X is closed with respect to (wrt.) modus ponens and substitution.

Then X is an intermediate logic.

We will denote arbitrary intermediate logics by the letters X,Y,Z (with indices where necessary).

Examples of intermediate logics.

1) The set of words R which are realisable in the sense of KLEENE-ROSE [20], is an intermediate logic distinct from both H and K.

2) DUMMETT'S [4] logic LC, obtained by adding the new axiom $((A_1 \to A_2) \vee (A_2 \to A_1))$ to $\mathcal{U}_H$ is an intermediate logic distinct from H, K and R.

It will be convenient to have a notation to denote the intermediate logic Y obtained from a logic X by adding the words $P_1 P_2 \ldots$ as additional axioms. We write Y as $X(P_1 P_2 \ldots)$ following the notation of TROELSTRA [21].

The following terminology will also prove useful:

Definition. A successor of an intermediate logic X is a logic Y such that $X \subset Y$. Conversely X is a predecessor of Y.

Definition. An immediate successor of an intermediate logic X is a logic Y such that $X \subseteq Y$ and if for a logic Z, $X \subseteq Z \subseteq Y$ holds, then X=Z or Z=Y. Conversely X is an immediate predecessor of Y.

The above definitions are also due to TROELSTRA.

Just as classical logic can be given the structure of a Boolean algebra, so arbitrary intermediate logics can be given the structure of a pseudocomplemented lattice (henceforth abbreviated PL) alternatively known as a pseudo-Boolean algebra, by passing to consideration of their Lindenbaum algebras. Elementary properties of PLs are presented in RASIOWA and SIKORSKI [19]. We now present some notation and definitions relating to these, for the most part following TROELSTRA [21]. We note that eventually we will require more general abstract structures than PLs. However all the structures that we will be concerned with, can be taken to be partially ordered sets with a unit element. Any such structure we will call an algebra. Some of the following notation and definitions are formulated for algebras thus defined, rather than just for PLs.

Greek letters β, γ, δ, (with indices where necessary) will be used to denote arbitrary algebras. The letter λ, will be reserved for Lindenbaum algebras. λ_X will denote the Lindenbaum algebra of a logic X, and λ^n_X the subalgebra of λ_X containing only elements corresponding to words with at most n distinct atoms, which we may, without restriction, assume to be taken from the set $A_1 A_2 \ldots\ldots A_n$. The letter α will always denote the 2-element lattice. The unit element of an algebra β is denoted by 1_β, and the zero element (should it exist) is denoted by 0_β. In general, we take the elements of the algebras β, γ, to be $b_1, b_2, \ldots\ldots c_1, c_2, \ldots\ldots$

The direct product of algebras β and γ, will be denoted by $\beta \times \gamma$, and more generally if $\{\beta_i\}_{i\in I}$ is a set of algebras, then we denote the direct product of all these algebras by $\underset{i\in I}{\times} \beta_i$. If all the β_i are isomorphic to β, then we may write this as β^I. Ordered I-tuples will be denoted by $\langle b_i \rangle_{i\in I}$ etc.

In addition to the operation of direct product, we introduce a summ operation for certain algebras.

<u>Definition.</u> By $\beta+\gamma$, we denote a type of algebra which is obtained as follows. Let β and γ be algebras, γ with a zero element, and let β be isomorphic to β', γ be isomorphic to γ' and let $\beta' \cap \gamma' = 1_{\beta'} = 0_{\gamma'}$. Then $\beta' \cup \gamma'$ is an algebra of type $\beta+\gamma$. Further we set $\beta+\beta = 2\beta$, $n\beta + \beta = (n+1)\beta$.

The definition above can be extended to the denumerable case in a natural manner. $\omega\cdot\beta$ will denote $\beta(1) \cup \beta(2) \quad \ldots \cup \{1\}$ where β is an algebra with a zero element and $\beta(i) \cap \beta(i+1) = 1_{\beta(i)} = 0_{\beta(i+1)}$, $\beta(i)$ isomorphic to β for all i, and $b_i(j) < 1$ for all i, j.

<u>Example</u> . Let β be the PL of Fig1. and γ the PL of Fig.2. Then $\beta+\gamma$ is the PL of Fig. 3.

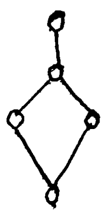

Fig.1

Fig.2

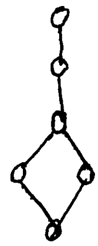

Fig.3

Remark. The sum of algebras thus defined,is a specialisation of the notion of the ordinal sum of posets. The "absorption" of elements ,which occurs in our definition of sum ,is technically convenient,since it allows us to dispense with algebras containing only one element,which would not characterise,in a sense to be defined,any intermediate logic.

The notions of isomorphism,homomorphism,subalgebra etc. will be used in their accustomed way. The letters ϕ, ψ, σ, π, g,h will be used for mappings. ~~Thm~~ If $\psi: \gamma \xrightarrow{\text{onto}} \beta$ and B is a subset of β ,then $\psi^{-1}[B] = \{c : \psi(c) \in B \ \& \ c \in \gamma\}$

The letter ρ will be used for congruence relations. Congruence classes (or cosets) mod ρ will be denoted by $|b|_\rho$ etc. Where there is no ambiguity,we write simply $|b|$.

In the next definition,we treat a PL β as an ordered system $\langle \underline{\beta}, \cup, \cap, \Rightarrow, * \rangle$ where $\underline{\beta}$ is a set and $\cup$, $\cap$, $\Rightarrow$, and $*$ are well defined operations on $\underline{\beta}$.

Definition. Let $\beta = \langle \underline{\beta}, \cup, \cap, \Rightarrow, * \rangle$ be a PL. Then a (PL-) valuation on β ,is a mapping ϕ from the set of words into β such that

$$\phi(P\&Q) = \phi(P)\cap\phi(Q)$$
$$\phi(P\vee Q) = \phi(P)\cup\phi(Q)$$
$$\phi(P\rightarrow Q) = \phi(P)\Rightarrow\phi(Q)$$
$$\phi(\neg P) = \phi(P)^{*}$$

for all words P,Q.

Definition. Let β be a PL. If for every PL-valuation ϕ on β $\phi(P)= 1_{\beta}$,then we say that P is valid on β.

Definition. A PL β is said to characterise an intermediate logic X if $P\in X$ iff P is valid on β.In this case we may write X as $X(\beta)$.

Definition. An intermediate logic X is said to be finite iff it can be characterised by a finite PL.

Remark. In connection with the last definition,it should be noted that the intermediate logic $X(\alpha^{\omega})$ is finite although the PL α^{ω} is clearly not finite . This is the case ,since it can be shown that $X(\alpha^{\omega}) = X(\alpha) = K$.

2.

In this section we prove an important representation theorem for PLs which allows us often to replace an arbitrary PL by an "equivalent " set of PLs of a certain restricted type.

The following definitions and lemmas are well known for PLs, (see e.g. RASIOWA& SIKORSKI[19] Cap 1,13), but we include them for convenience .

Definition. Let β be a PL, and F a subset of β such that 1) $1_\beta \in F$ and 2) if $b_i \in F$, $b_i \Rightarrow b_j \in F$ then $b_j \in F$. Then we say that F is a filter of β . If $F \neq \beta$, then we say that F is proper. A filter F is principal iff there is an element $b_i \in F$ such that for all elements $b_j \in F$ we have that $b_i \leq b_j$.

2.1 Lemma. Let β be a PL, F a filter of β. Define the relation ρ on β by setting $b_i \,\rho\, b_j$ iff $b_i \Rightarrow b_j \in F$ and $b_j \Rightarrow b_i \in F$. Then ρ is a congruence relation on β.

Remark. ρ is called the congruence relation determined by F and we write β/F for the quotient algebra β/ρ to indicate this.

Definition. Let β_1, β_2 be PLs and $h: \beta_1 \xrightarrow{\text{onto}} \beta_2$ be a homomorphism. Then the subset $\ker(h) = \{ b_i \in \beta_1 : h(b_i) = 1_{\beta_2} \}$ is called the kernel of the homomorphism h.

2.2 Lemma. Let β_1, β_2 be PLs and $h: \beta_1 \xrightarrow{onto} \beta_2$ be a homomorphism. Then the kernel of h is a filter.

2.3 Lemma. Let β_1, β_2 be PLs and $h: \beta_1 \xrightarrow{onto} \beta_2$ be a homomorphism. Then h is an isomorphism iff $\ker(h) = \{1_{\beta_1}\}$.

2.4 Lemma. Let β_1, β_2 be PLs and $h: \beta_1 \xrightarrow{onto} \beta_2$ be a homomorphism. Then the quotient algebra $\beta_1/\ker(h)$ is isomorphic to β_2 under the mapping ϕ, where $\phi(|b_i|) = h(b_i)$, $b_i \in \beta_1$.

We now pass to consideration of subdirect products. The general method involved is due to BIRKHOFF [1].

Definition. A PL β is said to have a representation as a subdirect product of a set of PLs $\{\beta_i\}_{i\in I}$ iff there is an embedding $h: \beta \to \bigtimes_{i\in I} \beta_i$ such that the product homomorphisms $g_i = h\circ\pi_i$, $h\circ\pi_i : \beta \to \beta_i$ are onto where π_i denotes the projection homomorphism from $\bigtimes_{i\in I} \beta_i$ into β_i.

Definition. A PL β is subdirectly reducible iff there exists a representation of β as a subdirect product of algebras $\{\beta_i\}_{i\in I}$ such that each of the homomorphisms g_i is proper. In this case we say that β is subdirectly reducible to the set of PLs $\{\beta_i\}_{i\in I}$.

β is said to be subdirectly irreducible iff it is not reducible.

<u>Examples.</u> The PL $\beta=\alpha+\alpha^2$ is subdirectly reducible to the set of PLs containing just 2 copies of the PL, 2α .Each copy is a proper homomorphic image of β ,and β is also embeddable in the PL $(2\alpha)^2$ under the displayed mapping(Fig. 4). On the other hand the PL 2α is subdirectly irreducible,since the only proper homomorphic image of 2α is α . But 2α cannot be embedded in any direct power of α , α^K, since $b_i^{**} \Rightarrow b_i = 1_{\alpha^K}$ for all b_i and K , but this is false for 2α .

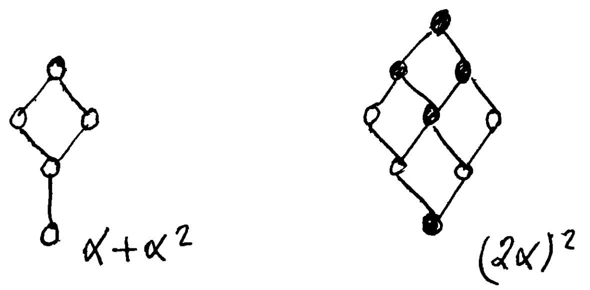

Fig. 4.

We now prove some lemmas on subdirect products.

2.5 <u>Lemma.</u> A PL β is subdirectly reducible to PLs $\{\beta_i\}_{i\in I}$ iff there exist proper homomorphisms $h_i; \beta \to \beta_i$ such that for every element $b\in\beta$,$b\neq 1_\beta$,$h_i(b)\neq 1_{\beta_i}$ for at least one i.

<u>Proof.</u> Suppose that β is subdirectly reducible to the set $\{\beta_i\}_{i\in I}$. By definition,there are proper homomorphisms $g_i=h\circ\pi_i; \beta\to\beta_i$ where h embeds β in $\times_{i\in I}\beta_i$ and the π_i are the projection homomorphisms. Suppose that for all i,there is an element $b\in\beta$ such that $g_i(b)=\pi_i(h(b))=1_{\beta_i}$.Then $h(b)=\langle 1_{\beta_i}\rangle$ $i\in I$,and thus $h(b)=h(1_\beta)$. But h is an isomorphism and hence by a previous lemma

$b=1_\beta$.

Conversely suppose that there exist proper homomorphisms $h_i : \beta \xrightarrow{onto} \beta_i$ such that for every element $b \in \beta$, such that $b \neq 1_\beta$ $h_i(b) = 1_{\beta_i}$ for at least one i. Consider the mapping $h : \beta \to \times_{i \in I} \beta_i$ given by $h(b) = \langle h_i(b) \rangle_{i \in I}$. It is clear that h is a homomorphism of β into $\times_{i \in I} \beta_i$. Also by definition of h, we have that $\ker(h) = \{1_\beta\}$. Hence by lemma 2.3 ,h is an embedding ,which completes the proof.

2.5.1. <u>Corollary</u>. Let $\{F_i\}_{i \in I}$ be a set of proper filters in a PL β such that $\bigcap_{i \in I} F_i = \{1_\beta\}$. Then β is subdirectly reducible to the set of quotient algebras $\{\beta/F_i\}_{i \in I}$.

<u>Proof</u>. Each β/F_i is a proper homomorphic image of β. Suppose $b \in \beta$, and $b \neq 1_\beta$. Then for some F_i , say F_j , $b \notin F_j$ .Hence $h_j(b) \neq 1_{\beta/F_j}$ where h_j is the homomorphism from β onto β/F_j .Hence by the previous lemma, the corollary follows.

2.6. <u>Lemma</u>. A PL β is subdirectly reducible iff there exists a set of proper filters $\{F_i\}_{i \in I}$ in β ,such that $\bigcap_{i \in I} F_i = \{1_\beta\}$.

<u>Proof</u>. By the above corollary ,if there exists such a set, then β is subdirectly reducible. Conversely suppose that β is subdirectly reducible to a set of PLs $\{\beta_i\}_{i \in I}$ Then each β_i is a proper homomorphic image of β. Hence for each i, there is a proper filter F_i of β such that β/F_i is isomorphic to β_i . Let g_i be the isomorphism $g_i : \beta_i \xrightarrow{onto} \beta/F_i$ and h_i the

homomorphism $h_i: \beta \to \beta_i$. For each element $b \in \beta$, $b \neq 1_\beta$ we have that $h_i(b) \neq 1_{\beta_i}$ for some i. Hence $g_i(h_i(b)) \neq 1_{\beta/F_i}$ and thus $b \notin F_i$. Hence $b \notin \bigcap_{i \in I} F_i$ and thus $\bigcap_{i \in I} F_i = 1_\beta$, QED.

2.7. Theorem. A PL β is subdirectly irreducible iff β contains an element $\omega \neq 1_\beta$ such that for all elements $b_i \in \beta$ such that $b_i \neq 1_\beta$, $\omega \geqslant b_i$.

Proof. If β contains an ω-element, then for every set of proper filters $\{F_i\}_{i \in I}$ of β, $\bigcap_{i \in I} F_i \neq 1_\beta$, and hence by the foregoing lemma, β is subdirectly irreducible.

Conversely, suppose that $\beta = \{1_\beta, 0_\beta\} \cup \{b_i\}_{i \in I}$ where no b_i is greater than all the others. Consider the set of corresponding principal filters, F_{b_i}. We have that $\bigcap_{i \in I} F_{b_i} = 1_\beta$. For suppose not. Then there is an element ω such that $\omega \in \bigcap_{i \in I} F_{b_i}$ and hence $\omega \in F_{b_i}$ for all i. Hence $b_i \leqq \omega$ for all i. Contradiction. Since $\bigcap_{i \in I} F_{b_i} = 1_\beta$, β is subdirectly reducible. to the PLs $\{\beta / F_{b_i}\}_{i \in I}$.

On the basis of this theorem, we introduce the next definition.

Definition. A PL β is strongly compact iff it is of type $\gamma + \alpha$, γ arbitrary. Alternatively, β is strongly compact iff it contains an ω-element in the sense of the theorem.

We now present our fundamental representation theorem :

2.8. Theorem. Every PL β is subdirectly reducible to a set of strongly compact PLs.

Proof. It will suffice to show that every PL β is subdirectly reducible to a set of subdirectly irreducible PLs.

Let b be an element of β, $b \neq 1_\beta$. Let $\mathfrak{F}$ be the set of filters of β which do not contain b, partially ordered by inclusion. The unit filter $\{1_\beta\} \in \mathfrak{F}$. Further the sup of any chain in $\mathfrak{F}$ is a filter of β which does not contain b. Hence by ZORN'S lemma, $\mathfrak{F}$ contains a maximal element which we denote by F_b. Let $\{F_{b_i}\}_{i\in I}$ be the set of all such filters, $b_i \neq 1_\beta$. $\bigcap_{i\in I} F_{b_i} = \{1_\beta\}$. Hence by lemma 2.6, β is subdirectly reducible to the set of quotient algebras $\{\beta/F_{b_i}\}_{i\in I}$.

It remains to show that each quotient algebra β/F_{b_j} is subdirectly irreducible. Let h be the homomorphism $h: \beta \longrightarrow \beta/F_{b_j}$, and let f be a non-unit filter of the quotient algebra β/F_{b_j}. Consider the subset of β, $h^{-1}[f]$. $h^{-1}[f]$ is a filter which contains F_{b_j}. Further since f is not the unit filter, there is an element $h(b_i) \in \beta/F_{b_j}$ such that $h(b_i) \in f$ and $h(b_i) \neq h(1_\beta)$. Hence $b \in h^{-1}[f]$, but $b \notin F_{b_j}$. Thus $h^{-1}[f] \supset F_{b_j}$. But since F_{b_j} is maximal, we have that $b_j \in h^{-1}[f]$ and hence $h(b_j) \in f$. We have shown therefore that every filter f in the quotient algebra β/F_{b_j} contains the element $h(b_j)$ and thus the intersection of any set of these filters must contain it as well. Hence by lemma 2.6, β/F_{b_j} is subdirectly irreducible, and our theorem is established.

Remark. The above representation theorem has been proved for PLs .In fact,it carries over to all the algebras we will be concerned with ,in this thesis . The definition of "strongly compact " in terms of a structure with an ω -element ,remains unchanged.

3. We now apply our fundamental representation theorem to the study of intermediate logics.
First of alll ,we prove a theorem which gives a syntactic characterisation of finite logics.

3.1 Lemma. Every intermediate logic $X(\beta)$ can be characterised by a set of strongly compact PLs.

Proof. By the fundamental representation theorem , β is subdirectly reducible to a set of strongly compact PLs $\{\beta_i\}_{i\in I}$. Suppose a word P is valid on β. Then since each β_i is a homomorphic image of β, P must be valid on each β_i and hence on their direct product. Conversely,if P is valid on each β_i ,then it is valid on the product algebra $\bigtimes_{i\in I}\beta_i$, and thus on the subalgebra $h(\beta)$,where h is the embedding of β in $\bigtimes_{i\in I}\beta_i$ .Hence P is valid on β.

Definition. If M is a set of strongly compact PLs which characterise an intermediate logic $X(\beta)$,we call M a characteristic set of $X(\beta)$.

Remark. In general M is not unique,for a given $X(\beta)$.

3.2 Theorem. An intermediate logic $X(\beta)$ is finite iff the word $E_n\in X(\beta)$ for some n,n>2 ,where

$$\begin{aligned} E_n = \; & (A_1\leftrightarrow A_2)\vee(A_1\leftrightarrow A_3)\vee\dots\dots\dots\dots(A_1\leftrightarrow A_n) \\ & \vee(A_2\leftrightarrow A_3)\vee\dots\dots\dots\dots(A_2\leftrightarrow A_n) \\ & \dots\dots\dots\dots\dots\dots \\ & \vee\ (A_{n-1}\leftrightarrow A_n) \end{aligned}$$

and $(A_i \leftrightarrow A_j)$ abbreviates $((A_i \to A_j)\&(A_j \to A_i))$.

Proof. It is well known that if $X(\beta)$ is finite, then $E_n \in X(\beta)$ for some $n > 2$. Suppose conversely that $E_n \in X(\beta)$ for some $n > 2$. Let M be a charcteristic set for $X(\beta)$. E_n is valid on each member of M. But then there can be no member of M with $k > n-1$ elements. There are only a finite number of distinct strongly compact PLs with $k \leq n-1$ elements, and so M has only a finite number of members, each of which is finite. Let γ be the direct product of all the members of M. We have that $X(\gamma) = X(\beta)$, and so $X(\beta)$ is finite. QED.

The next theorem allows us to give a useful characterisation of the intuitionist logic H, in terms of a sequence of explicitly axiomatised, finite logics.

3.3. Theorem. The sequence of finite logics $H(E_n)$, $n = 3,4\ldots$ is characteristic for the intuitionist logic, H.

PRoof. It is clear that if $P \in H$, then $P \in H(E_n)$ for each $n = 3,4..$ Suppose conversely that $P \notin H$. Then since H has the finite model property (FMP) (see HARROP [7], for a discussion of this property, and its relevance for the study of propositional logics), there exists a finite intermediate logic $X(\beta)$ such that $P \notin X(\beta)$. Because $X(\beta)$ is finite, we have that $E_n \in X(\beta)$ for some $n > 2$. Hence $H(E_n) \subseteq X(\beta)$. But then $P \notin H(E_n)$.

Remark. For $n>2$, $H(E_n)=X(\gamma)$, where γ is the direct product of all strongly compact PLs with less than n elements.

3.4 Theorem. Every finite logic $X(\beta)$ has a finite number of successors.

Proof. Since $X(\beta)$ is finite, $E_n \in X(\beta)$ for some $n>2$. Choose the least n such that $E_n \in X(\beta)$. ~~Thx~~ Let M be the characteristic set of $X(\beta)$ which contains all strongly compact PLs which satisfy the axioms of $X(\beta)$, and which have $k<n$ elements. If $X(\beta_1)$ is an arbitrary successor of $X(\beta)$, then every strongly compact PL which satisfies the axioms of $X(\beta_1)$ must also satisfy the axioms of $X(\beta)$. Hence if N is the characteristic set of $X(\beta_1)$ which contains all strongly compact PLs which satisfy the axioms of $X(\beta_1)$, we have that $N \subseteq M$. But there are at most a finite number of distinct subsets of M, and hence there can be at most a finite number of successors of $X(\beta)$.

3.5 Theorem. Every finite intermediate logic $X(\beta)$ is finitely axiomatisable.

Proof. $H(E_n) \subseteq X(\beta)$ for some $n>2$. If $H(E_n) \neq X(\beta)$, then there is some word P_1 such that $P_1 \in X(\beta)$ and $P_1 \notin H(E_n)$. Consider the intermediate logic $H(E_n, P_1)$. If $H(E_n, P_1) \neq X(\beta)$ then there is a word P_2 such that $P_2 \in X(\beta)$ and $P_2 \notin H(E_n, P_1)$. But this approximation procedure must terminate after a finite number of stages, otherwise we determine a strictly increasing infinite sequence of successors of $H(E_n)$ thus contradicting the previous theorem.

Remark. This theorem was first announced by D.H.de JONGH. His proof has not yet appeared. However, DR.TROELSTRA has informed me (personal communication) that it is quite different from the above method.

4.

The main object of the present section is to prove a generalisation of a theorem of DIEGO [3], which will allow us subsequently to establish the decidability of a wide class of intermediate logics. To this end, we introduce more restricted classes of words and certain restricted types of intermediate logics based upon them.

Definition. A word is an I (resp. IC,IN,ICN) word iff it contains only the connective(s) I(resp. I and C,I and N,I,C and N.)

Definition. Let X be an intermediate logic and X_I the set of I words belonging to X. Then X_I is said to be the I fragment of X.

IN,IC and ICN fragments are defined and denoted analogously.

Definition. A set of I words X is an intermediate I logic iff

1) $H_I \subseteq X \subseteq K_I$
2) X is closed wrt. modus ponens and substitution.

Intermediate IC,IN and ICN logics are defined analogously. As indicated in section1, we require more general structures than PLs to provide algebras which will characterise these new restricted intermediate logics. These will be called I algebras, IC algebras and so on. This terminology is due to ALFRED HORN [8] where an account of the properties of these algebras is set out. For the convenience of the reader, we summarise some of his definitions and results here.

Definition. An I-algebra β is an ordered triple $\beta = \langle \underline{\beta}, 1_\beta, \Rightarrow \rangle$ where $\underline{\beta}$ is a set, 1_β is an element of $\underline{\beta}$ and $\Rightarrow$ is a binary operation defined on $\underline{\beta}$ such that for all elements b_1, b_2, b_3 of $\underline{\beta}$

1) if $1 \Rightarrow b_1 = 1_\beta$, then $b_1 = 1_\beta$
2) if $b_1 \Rightarrow b_2 = b_2 \Rightarrow b_1 = 1_\beta$, then $b_1 = b_2$
3) $(b_1 \Rightarrow (b_2 \Rightarrow b_1)) = 1_\beta$
4) $((b_1 \Rightarrow (b_2 \Rightarrow b_3)) \Rightarrow ((b_1 \Rightarrow b_2) \Rightarrow (b_1 \Rightarrow b_3))) = 1_\beta$

Remark. An I algebra β can be given the structure of a partially ordered set by defining $b_i \leq b_j$ iff $b_i \Rightarrow b_j = 1$ for all elements b_i, b_j of β. However in general the structure of β as a poset does not determine the operation $\Rightarrow$ uniquely.

Example. Consider the poset $\alpha^2 + \alpha$. We may take $\Rightarrow$ as the operation of relative pseudo-complementation on $\alpha^2 + \alpha$; or we take it as the operation defined by the conditions $b_i \Rightarrow b_j = 1$ if $b_i \leq b_j$, and is equal to b_j otherwise. On both these interpretations, conditions 1-4 in the definition of an I algebra are satisfied, but the resulting I algebras are not equivalent.

Definition. An IN algebra $\beta = \langle \underline{\beta}, 1_\beta, \Rightarrow, * \rangle$ is an I algebra with an additional unary operation $*$ such that $((b_i \Rightarrow b_j) \Rightarrow ((b_i \Rightarrow b_j^*) \Rightarrow b_i^*)) = 1_\beta$ and $(b_i \Rightarrow (b_i^* \Rightarrow b_j)) = 1_\beta$, for all b_i, b_j of β. IC, ICN algebras are defined in an analogous manner.

4.1 Theorem. An IN algebra $\beta = \langle \underline{\beta}, 1_\beta, \Rightarrow, * \rangle$ is an I algebra with a smallest element 0_β, and an operation $*$ defined by $b_i^* = b_i \Rightarrow 0_\beta$. Conversely any such I algebra is an IN algebra.

Definition. A semi-lattice is a poset such that any 2 elements b_i, b_j have an inf, $b_i \cap b_j$. A semi-lattice is relatively pseudo-complemented iff for any 2 elements b_i, b_j, there is an element $(b_i \Rightarrow b_j)$ such that $b_i \cap b_k \leq b_j$ iff $b_i \leq (b_k \Rightarrow b_j)$.

4.2 Theorem. An IC algebra is a relatively pseudo-complemented semi-lattice and conversely.

Remark. It follows from the above theorems that an ICN algebra β is a relatively pseudo-complemented semi-lattice with a least element 0_β, and a unary operation $*$ defined by $b_i^* = b_i \Rightarrow 0_\beta$.

Before proving our principal theorem of this section, we introduce a useful definition, and some lemmas stemming from it.

Definition. Let $\beta = \langle \underline{\beta}, \Rightarrow, \cap, * \rangle$ be an ICN algebra. Then we denote by $\beta^+ = \langle \underline{\beta}^+, \Rightarrow^+, \cap^+, *^+ \rangle$ the strongly compact ICN algebra obtained from $\underline{\beta}$, by interpolating a new ω-element between 1_β and the other elements of β.

Similarly if $\beta=\langle\beta,\Rightarrow,\cap,*\rangle$ is a strongly compact ICN algebra, We denote by $\beta^{-}=\langle\beta^{-},\Rightarrow^{-},\cap^{-},*^{-}\rangle$ the ICN algebra got by deleting the ω- element. We remark that β^{-} is a sub- ICN algebra of β.

The following lemmas are obvious.

4.3 Lemma. $(\beta^{+})^{-}=\beta$

4.4 Lemma. $(\beta^{-})^{+}=\beta$ where β is a strongly compact ICN algebra.

In addition we have

4.5 Lemma. If β is a strongly compact ICN algebra, and $Gn(\beta)$ is a set of generators for β, then $\omega_{\beta}\in Gn(\beta)$.

PRoof. Suppose contrary to the lemma, $\omega_{\beta}\notin Gn(\beta)$. We then have that $Gn(\beta)\subseteq\beta-\{\omega_{\beta}\}$. But $\beta-\{\omega_{\beta}\}$ is a proper subalgebra of β, under the operations $\Rightarrow$, $\cap$, and $*$ of β. Hence $Gn(\beta)$ does not generate β. Contradiction.

We are now in a position to prove our generalisation of DIEGO'S theorem, which he proved for I algebras in [3].

4.6 Theorem. Every ICN algebra β with a finite number of generators is finite.

Proof. By induction on the number of generators of β.
Suppose that β has one generator b_1. The the free ICN algebra γ

with one generator, has 6 elements, namely b_1, b_1^*, b_1^{**}, $b_1^{**} \Rightarrow b_1$, $b_1^* \wedge b_1$, $b_1 \Rightarrow b_1$.

β is a homomorphic image of γ, and hence contains $k \leq 6$ elements. Suppose then that every ICN algebra with n generators is finite, and consider the ICN algebra β with $(n+1)$ generators $b_1, b_2, \ldots b_n, b_{n+1}$. Analogously to our fundamental representation theorem for PLs, we can show that β is subdirectly reducible to a set of strongly compact ICN algebras $\{\beta_i\}_{i\in I}$. Since by definition of subdirect reducibility β can be embedded in $\bigtimes_{i\in I} \beta_i$, in order to show that β is finite, it will suffice to show that 1) each β_i is finite and 2) that I can be taken to be finite.

As to 1) if $\beta_j \in \{\beta_i\}_{i\in I}$ then β_j is a homomorphic image of β and hence is generated by the set $Gn(\beta_j) = \{h_j(b_1), h_j(b_2) \ldots h_j(b_{n+1})$ where h_j is the homomorphism from β onto β_j. Further since β_j is strongly compact, we have by lemma 4.5 that $\omega_{\beta_j} \in Gn(\beta_j)$. Consider the algebra β_j^-. Since $\omega_{\beta_j} \notin \beta_j^-$, β_j^- is generated by a set of n generators, and hence by the induction hypothesis, is finite. Obviously $(\beta_j^-)^+$ is finite if β_j^- is. But by lemma 4.4 $(\beta_j^-)^+ = \beta_j$, which completes the proof of 1).

We observe that the above proof provides us with a simple primitive recursive bound on the size of the quotient algebras $\{\beta_i\}_{i\in I}$, in the representation of β as a subdirect product of the $\{\beta_i\}_{i\in I}$ where β is an ICN algebra with n generators. Let f(n) be the number of elements in the free ICN algebra with n free generators. Then each $\beta_j \in \{\beta_i\}_{i\in I}$ has at most f(n)+ 1 elements.

From this observation it follows that I can be taken to be finite, since there can be at most a finite number of distinct (i.e. non-isomorphic) ICN algebras with $k \leq f(n)+1$ elements. In fact, as we shall prove later, every finite ICN algebra is a distributive lattice. It can be shown that there are at most 2^{2^n} non-isomorphic distributive lattices with at most n elements and a fortiori there can be at most 2^{2^n} non-isomorphic ICN algebras with at most n elements. This gives us a primitive recursive bound for the size of an ICN algebra β with n generators. β will have $k \leq g(n)$ elements, where $g(n)$ is defined primitively recursively by the conditions $g(1) = 6$
$g(n+1) = \exp(2^{2^{g(n)+1}}, g(n)+1)$, $\exp(a,b) = b^a$

4.6.| Corollary. Every I, IC and IN algebra with a finite number of generators is finite, with primitive recursive bound.

5.

We now apply these results to the study of intermediate logics.

5.1 Theorem. If X is an intermediate ICN logic, then X has the FMP and hence if X is finitely axiomatisable, it is decidable.

Proof. Suppose that a word $P \notin X$. Then if P contains at most n distinct atoms, $|P| \neq 1_{\lambda^n_X}$. By our generalisation of DIEGO'S result, λ^n_X is finite, since it is generated by n elements. Hence X has the FMP, and hence by the result in HARROP [6] if X is finitely axiomatisable, it is decidable. Indeed X is primitively recursively decidable, for we need test P only on all those ICN algebras with $k \leq g(n)$ elements, which satisfy the axioms of X, where g(n) is the primitive recursive function defined in the theorem.

5.1.1 Corollary. If X is an intermediate I, Ic or IN logic, then X has the FMP and hence if X is finitely axiomatisable then it is decidable.

Remark. The above theorem and corollary generalise and complete finally the work of BULL [2].

Remark 2. The above method of proving the decidability of finitely axiomatised intermediate I, IC, IN and ICN logics cannot be extended in general to the case of intermediate ID logics (where we define these in an obvious way) or therefore to intermediate logics in the unrestricted sense. Let $P_1(A_1, A_2) = A_1$

$P_2(A_1, A_2) = A_2$, $P_{2n+1}(A_1, A_2) = (P_{2n-1} \vee P_{2n})$ and $P_{2n+2}(A_1 A_2) = (P_{2n} \to P_{2n-1})$ for n 1,2....These words are all non equivalent,as is seen from NISHIMURA'S paper [18] ,if we substitute $\neg A_1$ for A_2 because we obtain in this way all words with one atom minus $\neg(A_1 \to A_1)$. Hence $\lambda^2_{H_{ID}}$ is infinite. I owe this reference to NISHIMURA'S work to DR.TROELSTRA.

Remark 3. In [1] McKINSEY and TARSKI introduced the concept of the reducibility of a propositional logic. It follows from theorem 4.6,that an intermediate ICN(also I,IC and IN) logic is reducibleiff it is finite. This contrasts with the situation for unrestricted intermediate logics,e.g. $X(\lambda^1_H)$ is not finite but it is 1-reducible,and thus reducible.

In order to prove the main theorem of this section we require the following preliminary lemma:

5.2 Lemma. Let β be a PL with operations $\cup, \cap, \Rightarrow, *$, and let γ be a finite sub-ICN algebra of β. Then 1) γ is aPL with operations $\cup', \cap, \Rightarrow, *$, and 2) if b_1, b_2, $b_1 \cup b_2 \in \gamma$,then $b_1 \cup b_2 = b_1 \cup' b_2$.

Proof. γ contains a least element ,namely $(b_1^* \cap b_1)$. It remains to show that any 2 elements b_1 ,b_2 of γ have a sup with respect to the partial ordering in γ . Let B be the set of upper bounds of b_1, b_2 wrt this ordering. Since β contains the unit element of β ,B is non-empty and obviously it is finite.Hence $\bigcap$B exists and is the required sup. γ will not normally be a sub-PL of β ,but from the definition of union in γ,2) will be satisfied.

5.3 Theorem. Let $X(\beta) = H(P_1, P_2, \ldots P_k)$ be an intermediate logic where all the P_i are ICN words.Then $X(\beta)$ has the FMP and hence is decidable.

PRoof. Suppose that a word $Q \notin X(\beta)$. Then $|Q| \neq 1_{\lambda_{X(\beta)}}$ under the natural mapping ϕ from the set of words onto $\lambda_{X(\beta)}$. Let $Q_1, Q_2, \ldots Q_n = Q$ be the subwords of Q and let F(n) be the sub-ICN algebra of $\lambda_{X(\beta)}$ generated by $|Q_1| |Q_2| \ldots |Q_n|$. By theorem 4.6 F(n) is finite,and by the previous lemma ,it is a PL. All the axioms of H are valid on F(n). In addition all the P_i are valid on F(n) since F(n) is a sub-ICN algebra of $\lambda_{X(\beta)}$. But $|Q| \neq 1_{F(n)}$ by the lemma and the fact that all the subwords of Q are represented in F(n). Hence the mapping ϕ induces a refuting valuation of Q on the finite PL,F(n). $X(\beta)$ therefore has the FMP,and since it is finitely axiomatisable,it must be decidable.

5.3.1 Corollary. Let $X(\beta) = H(P_1, P_2, \ldots P_k)$ be an intermediate logic where all the P_i are of the form

$$(Q_{i_1} \vee Q_{i_2} \vee \ldots \vee Q_{i_m}) \rightarrow (R_{i_1} \vee R_{i_2} \vee \ldots \vee R_{i_n})$$

and where the Q_i ,R_i are ICN words .Then $X(\beta)$ has the FMP and hence it is decidable.

Proof. Each P_i is interdeducible in H with the ICN word P_i' where

$$P_i' = \left[Q_{i_1} \to (((R_{i_1} \to A_j) \& \ldots \& (R_{i_n} \to A_j)) \to A_j)\right] \& \ldots$$
$$\ldots \& \left[Q_{i_m} \to (((R_{i_1} \to A_j) \& \ldots \& (R_{i_n} \to A_j)) \to A_j)\right]$$

where A_j is an atom new to P_i.

Remark. Theorem 5.3 and its corollary allow us to establish the decidability of many of the intermediate logics treated in the literature, in an integrated fashion. We mention some of these: the intermediate logics of JANKOV [10], $\mathcal{L}$ and $\mathfrak{M}$; the intermediate logic LC of DUMMETT [4]; the logics K and H (in the case of the latter we add zero extra ICN axioms!). Of course, we already know that K and H are decidable, but it is perhaps more interesting to see this as part of a more general theory.

A natural question which arises, is to ask whether every intermediate logic can be axiomatised with ICN words, thereby giving decidability for all finitely axiomatisable ones. We shall see presently that the answer to this, is "no".

We conclude this section with a variety of results on intermediate ICN logics. Similar results hold for intermediate I, IC and IN logics.

5.4 Theorem. Let C_n be the IC word

$$\Big[((A_1 \leftrightarrow A_2) \to A_{n+1}) \& \ldots \& ((A_1 \leftrightarrow A_n) \to A_{n+1})$$
$$\& ((A_2 \leftrightarrow A_3) \to A_{n+1}) \& \ldots \& ((A_2 \leftrightarrow A_n) \to A_{n+1})$$
$$(\& \ldots \& (A_{n-1} \leftrightarrow A_n) \to A_{n+1})\Big] \to A_{n+1}$$

Then an ICN logic $X(\beta)$ is finite iff $C_n \in X(\beta)$ for some $n, n > 2$.

Proof. It need only be observed that C_n is interdeducible in H with the word E_n. The proof of Theorem 3.2 then carries over.

Remark. Following from this theorem, we have the analogues of theorems 3.3, 3.4, and 3.5 for intermediate ICN logics.

5.5 Theorem. The immediate predecessors of a finite intermediate ICN logic $X(\beta) = H(P_1, P_2, \ldots P_k)$ are finite, and there are at most finitely many of them.

Proof. Let $P = (P_1 \& P_2 \& \ldots \& P_k)$. Then $X(\beta) = H(P)$. Suppose that $X(\beta_1)$ is an immediate predecessor of $X(\beta)$. Then $P \notin X(\beta_1)$. Suppose that P has n atoms $A_1, A_2, \ldots A_n$, and let $b_1, b_2, \ldots b_n$ be the elements of β_1, assigned to the A_i in a valuation which makes P false on β_1. Let γ be the ICN algebra generated by the b_i. Then γ is a homomorphic image of $\lambda^n_{H_{ICN}}$. Hence the characteristic algebra of any immediate predecessor of $X(\beta)$ must contain a homomorphic image of $\lambda^n_{H_{ICN}}$ for which P fails. But since $\lambda^n_{H_{ICN}}$ is finite, there are at most a finite number of distinct homomorphic images of $\lambda^n_{H_{ICN}}$, and hence only finitely many for which P is refutable, call them $\gamma_1 \gamma_2 \cdots \gamma_p$. Any immediate predecessor of $X(\beta)$ must be one of the logics $X(\beta \times \gamma_1), X(\beta \times \gamma_2) \ldots X(\beta \times \gamma_p)$. For as we have noted, for any immediate predecessor $X(\beta_1)$ of $X(\beta)$, β_1 must contain a homomorphic image of $\lambda^n_{H_{ICN}}$ for which P fails, say γ_i, $1 \leq i \leq p$. Hence $X(\beta_1) \subseteq X(\gamma_i)$. Further $X(\beta_1) \subset X(\beta)$ and thus by the properties of direct product

$X(\beta_1) \subseteq X(\beta \times \gamma_i)$. But $X(\beta \times \gamma_i) \subseteq X(\beta)$. Since $X(\beta_1)$ is an immediate predecessor of $X(\beta)$, we must have that $X(\beta_1) = X(\beta \times \gamma_i)$.

6.

The purpose of this section is to introduce the new concept of strong undefinability of connectives, and to investigate the scope of the Separation theorem in the domain of intermediate logics.

Definition. Let X be an intermediate logic and $\{O_i\}_{i\in I}$ a set of connectives defined for X. Then a connective O_i is said to be weakly definable in X with respect to $\{O_j\}_{j\in I, j\neq i}$ iff for every word $P \in X$ which contains the connective O_i ,there exists a finite set of words P_1 , P_2 ,....P_n in which the connective O_i does not occur ,and such that $\vdash_X P$ iff $\vdash_X P_i$ for all i,$1\leq i \leq n$.

A connective O_i is said to be strongly undefinable in X wrt $\{O_j\}_{j\in I, j\neq i}$ iff it is not weakly definable.

Remark. The above definition is a modification of one formulated by TROELSTRA and mentioned in McKAY [14] .

Remark 2. It will be observed that in the above definition we are interested in the interdeducibility of P with a set of words. One might ask if one might make do with single word ,rather than sets of words. However this is not possible. Counterexample: consider the word $(A_1 \& A_2)$. It is not (in H) interdeducible with a single word P which does not contain the connective & . However it is interdeducible with the set of atoms $\{A_1, A_2\}$.

One may ask what the relation is between the above concept of strong undefinability and the notion of undefinability due to McKINSEY [15] . The following remarks clarify this.

If a connective O_i is strongly undefinable in X wrt. $\{O_j\}_{j\in I,\, j\neq i}$ then it is also undefinable in the sense of McKINSEY. The converse is not true. Thus the connective & is weakly definable ~~inHx~~ in H wrt $\to$, v and $\neg$. We can show this by making use of the equivalences $(A_1\to(A_2\&A_3))\leftrightarrow((A_1\to A_2)\&(A_1\to A_3))$, $((A_1\&A_2)\to A_3)\leftrightarrow(A_1\to(A_2\to A_3))$, $\neg(A_1\&A_2)\leftrightarrow(A_1\to\neg A_2)$ and $(A_1\vee(A_2\&A_3))\leftrightarrow((A_1\vee A_2)\&(A_1\vee A_3))$, and finally the fact that $\vdash_H(P_1\&P_2\ldots\&P_n)$ iff $\vdash_H P_i$ for all $i, 1\le i\le n$. However & is not McKINSEY definable in H.

As a further example, we note that the connective $\neg$ is McKINSEY undefinable wrt $\to$, & and v , in K. However it is weakly definable wrt $\to$, & and v in K. For let P be a word with the atoms $A_1, A_2 \ldots A_n$. Replace each subword P_i' of P of the form $\neg Q$ by $(Q\to A_{n+1})$, and let P^0 be the resulting expression. Then the words P and $((P^0\to A_{n+1})\to A_{n+1})$ are interdeducible in K. Suppose that $\vdash_K((P^0\to A_{n+1})\to A_{n+1})$. Replace A_{n+1} by $(A_1\&\neg A_1)$ and we have $\vdash_K\neg\neg P$ and thus $\vdash_K P$. Conversely, suppose that $\nvdash_K((P^0\to A_{n+1})\to A_{n+1})$. In the refuting assigment of truth values to $((P^0\to A_{n+1})\to A_{n+1})$, A_{n+1} is assigned the value F, and hence under the same assignment we have a refutation of $\neg\neg P$ and hence of P. It may be noted that P^{o} and P are not in general interdeducible in K, as one might be tempted to think. Counterexample: take $P = (\neg\neg A_1 \rightarrow A_1)$.

As we have seen then, the connective & is weakly definable in H wrt $\rightarrow$, v, and $\neg$. What about the other 3 connectives?

6.1 Theorem. The connective $\neg$ is strongly undefinable in H wrt $\rightarrow$, & and v.

Proof. It will suffice to show that the intermediate logic $H(\neg A \vee \neg\neg A)$ has the same ICD fragment as H.
Let $\lambda^{*}_{H_{ICD}}$ be the PL obtained by adding a zero-element to the RPL, $\lambda_{H_{ICD}}$. The word $(\neg A \vee \neg\neg A)$ is valid on $\lambda^{*}_{H_{ICD}}$, and thus $H(\neg A \vee \neg\neg A) \subseteq X(\lambda^{*}_{H_{ICD}})$. But if P is an ICD word such that $P \notin H$, then $P \notin X(\lambda^{*}_{H_{ICD}})$. Hence $H(\neg A \vee \neg\neg A)$ has the same ICD fragment as H. The word $(\neg A \vee \neg\neg A)$ is therefore not interdeducible with a set of ICD words in H.

6.2 Theorem. The connective $\rightarrow$ is strongly undefinable in H wrt. &, v and $\neg$.

Proof. It will suffice to show that the intermediate logic $H((\neg A_1 \rightarrow (A_2 \vee A_3)) \rightarrow (\neg A_1 \rightarrow A_2) \vee (\neg A_1 \rightarrow A_3))$ has the same CDN fragment as H.
It is shown in KREISEL and PUTNAM [11], that the above intermediate logic is prime, i.e. it has the "disjunction property"

$\vdash P \vee Q$ iff $\vdash P$ or $\vdash Q$. Now it can be shown by a simple inductive proof that any CDN word P is equivalent to a disjunction of CN words, $P_1, P_2, \ldots P_n$. Hence P is provable in the KREISEL-PUTNAM logic KPC, iff $\vdash_{KPC} P_i$ for some i , $1 \leq i \leq n$, and thus by the theorem of GODEL-GLIVENKO [5] iff $\vdash_H P_i$ for some i. But this is the case iff $\vdash_H P$ since H also is prime. Hence KPC and H have the same CDN fragments. Our theorem then follows as before.

The only remaining case to consider is the connective v . We shall show that it is also strongly undefinable in H, but in order to do this ,we require to make an interesting detour.

Definition. Let $X(\beta)$ be an intermediate logic. An axiomatisation $H(P_1 P_2 \ldots P_{\ldots}) = X(\beta)$ is said to be normal iff each P_i contains only occurrences of $\rightarrow$ and at most one other connective.

Definition. An intermediate logic $X(\beta)$ is normalisable iff it possesses a normal axiomatisation.

Definition. $X(\beta)$ is said to be separable iff it possesses an axiomatisation for which the Separation theorem (see [8]) can be proved.

Remark. The significance of normalisability lies in the fact that an intermediate logic is separable only if it is normalisable.

The following question naturally arises: is every intermediate logic normalisable? We show that the answer is negative. This contrasts with the fact that the intermediate logics H,K,X($\omega\alpha$) and the finite limitations of the latter X($n\alpha$) $n=1,2,3....$ are all known to be separable (HOSOI [9]).

6.3 Lemma. Let X(β) be an intermediate logic such that $(\neg\neg A \vee (\neg\neg A \rightarrow A)) \notin X(\beta)$. Then β contains a sub-ICN algebra isomorphic to $(\alpha \times 2\alpha)+\alpha$. (Fig. 5)

Proof. β contains an element b such that $(b^{**} \cup (b^{**} \Rightarrow b)) = c < 1_\beta$. It may be checked that $(0_\beta, b, b^*, b^{**}, (b^{**} \Rightarrow b), c, 1_\beta)$ is isomorphic to the ICN algebra $(\alpha \times 2\alpha)+\alpha$.

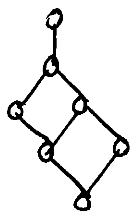

Fig. 5.

Remark. It need not be the case that β contains a sub-PL isomorphic to $(\alpha \times 2\alpha)+\alpha$. For example, $(\neg\neg A \vee (\neg\neg A \rightarrow A))$ is not valid on the PL γ of Fig. 6 ,but γ does not contain a sub-PL isomorphic to $(\alpha \times 2\alpha)+\alpha$.

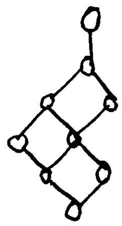

Fig. 6.

Let X(β_S) be SCOTT'S intermediate logic obtained by adding to $\mathcal{U}_H$ the new axiom $P_S = (((\neg\neg A \to A) \to (\neg A \vee A)) \to (\neg\neg A \vee \neg A))$. (See [11])

6.4 <u>Theorem.</u> X(β_S) is not normalisable.

<u>Proof.</u> X(β_S) $\subset$ H($\neg A \vee \neg\neg A$), and hence by theorem 6.1, X(β_S) has the same ICD fragment as H. X(β_S) is therefore normalisable iff it can be obtained by adding a finite set of IN words to $\mathcal{U}_H$. Suppose then that X(β_S) = H(P_1, P_2, ...) where the P_i are IN words. Since P_S is valid on the PL of Fig. , $(\neg\neg A \vee (\neg\neg A \to A)) \notin$ X(β_S). Hence β_S contains a sub-ICN algebra isomorphic to $(\alpha\times 2\alpha)+\alpha$. But P_S is not valid on the PL $(\alpha\times 2\alpha)+\alpha$ Thus some P_i, say P_j, is not valid on the PL $(\alpha\times 2\alpha)+\alpha$, otherwise we have that H(P_S) $\neq$ H(P_1 P_2...). Since P_j is an IN word, P_j must fail on $(\alpha\times 2\alpha)+\alpha$, considered as an ICN algebra. Hence P_j fails on β_S. But then H(P_1 P_2...) $\neq$ X(β_S). Contradiction. Thus X(β_S) is not normalisable.

6.4.1 <u>Corollary.</u> The intermediate logic X(β_S) is not separable.

6.4.2 <u>Corollary.</u> The connective v is strongly undefinable in H wrt $\to$, & and $\neg$

Remark. A similar method can be used to show that KPC is also not separable. It follows that the method of proving decidability set forth in theorem 5.3, is not applicable to these logics.

Remark 2. The above results can be obtained in a totally different manner from the work of MEDVEDEV [17] on recursive realisability and a related notion. In fact MEDVEDEV has established the stronger result that the logics $X(\beta_S)$ and KPC have both the same ICN fragment as H.

We conclude this section with a theorem which establishes the separability of a large number of intermediate logics.

6.5 Theorem. Let $X(\beta)$ be a finite normalisable intermediate logic. Then $X(\beta)$ is finitely separable.

Proof. Let $\mathcal{U}_{X(\beta)}$ be a normal axiomatisation of $X(\beta)$. Then by theorem 3.5, we can take $\mathcal{U}_{X(\beta)}$ to be finite. Let therefore $\mathcal{U}_{X(\beta)}$ be the set of axioms which results from adding the words $P_1, P_2 \ldots. P_n$ to $\mathcal{U}_H$. As in the paper of HOSOI [9] we will define an axiomatisation $\mathcal{U}_X$ of an intermediate logic X to be I complete iff every I word which is a thesis of X is derivable from the I axioms of $\mathcal{U}_X$. IC, IN etc completeness may be defined analogously.

First of all, we extend $\mathcal{U}_{X(\beta)}$ to a finite set of axioms $\mathcal{U}^{+}_{X(\beta)}$ which is I, IN and ID complete, as follows. Since the fragments

$X(\beta)_I$, $X(\beta)_{IN}$ and $X(\beta)_{ID}$ can be viewed respectively as finite intermediate I,IN and ID logics,they are finitely axiomatisable. Let the respective sets of axioms be $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$. Then the set of axioms $\mathcal{U}^+_{X(\beta)} = \mathcal{U}_{X(\beta)} \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$ satisfies our stipulation. In addition $\mathcal{U}^+_{X(\beta)}$ is IC,ICN and ICD complete. Here we make use of the fact that the connective & is weakly definable wrt the other 3 connectives. It remains to show that $\mathcal{U}^+_{X(\beta)}$ is IDN complete. For this we require the lemma

Lemma.Let γ be a finite IDN algebra. Then γ is a finite PL.

This lemma may be proved in a manner similar to that of lemma 5.2

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Suppose now that an IDN word Q is not derivable from the I,ID and IN axioms,of $\mathcal{U}^+_{X(\beta)}$. Let $X(\delta)$ be the finite IDN logic defined by these axioms. Such a finite logic can be found since C'_n is derivable from the I axioms of $\mathcal{U}^+_{X(\beta)}$,where C'_n is the I word obtained from the word C_n ,in a natural manner. Q fails on δ . By the lemma , δ is a PL.Further all the axioms of $\mathcal{U}^+_{X(\beta)}$ satisfy δ . This is obviously the case for the I,ID and IN axioms. Once more using the fact that & is weakly definable in H, we have that all the remaining axioms of $\mathcal{U}^+_{X(\beta)}$ are valid on δ as well. Hence $Q \notin X(\beta)$ and so the axiomatisation $\mathcal{U}^+_{X(\beta)}$ is IDN complete.

6.5.| Corollary. The sequence of logics $H(C'_n)$ n 3,4... are all finitely separable.

<u>Remark.</u> In connection with a conjecture in HOSOI [9] we note that for the logics $H(C_n')$, $n \geqslant 6$, the connective v is not McKINSEY definable in terms of the other connectives.

On the basis of the present work, the following open problems arise.

1) Do all intermediate logics have the FMP ?
2) Are all finitely axiomatisable intermediate logics decidable? If the answer to 1) is yes, then of course the answer to the present question is positive. As to the converse, there exist logics which are decidable but which do not possess the FMP.
3) Are all intermediate logics finitely axiomatisable? A.A.MARKOV informed me that his student JANKOV has answered this question in the negative, but no proofs have as yet appeared.
4) Does every prime intermediate logic have the same ICN and ICD fragments as H ? Is the converse true ? Both are true of the prime intermediate logics known.

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Abstract

The main object of the thesis is to investigate a variety of questions relating to the set of intermediate propositional logics. Let H denote the set of words which are intuitionist theses and let K denote the set of words which are classical theses. Then a set of words X is an intermediate (propositional) logic iff 1) $H \subseteq X \subseteq K$ and 2) X is closed wrt modus ponens and substitution.

Of special interest among intermediate logics, are those which are characterised by a finite pseudocomplemented lattice. We prove the important result that every such finite logic is finitely axiomatisable. This result is one of the many consequences of the fundamental representation theorem for pseudocomplemented lattices (PLs) whereby every PL is subdirectly reducible to a set of so-called strongly compact PLs. In addition we provide a neat syntactic characterisation of finite logics, and show that H is the limit of a certain sequence of explicitly axiomatised finite logics.

In addition we consider more restricted types of intermediate logics, in particular intermediate ICN logics. By generalising a result of DIEGO, to show that every ICN algebra with a finite number of generators, is finite, we manage to prove that every finitely axiomatised intermediate ICN logic is decidable with primitive recursive bound. This generalises

and completes earlier work of BULL. The same methods are then applied to obtain a proof of the decidability of all those intermediate logics, obtained by adding a finite set of disjunction-free words, as additional axioms to H. Many older results in the literature are then seen to be special cases of this general result.

We introduce the new concept of strong undefinability of a propositional connective, and examine its relation to McKINSEY'S related notion. It is shown that the connectives of implication, disjunction and negation, are all strongly undefinable in H, whereas conjunction is weakly definably.

Lastly we investigate the scope of the so-called Separation theorem in the field of intermediate logics. It is shown that certain intermediate logics treated in the literature do not possess any axiomatisation for which the Separation theorem can be proved.