A REVIEW AND THE DEVELOPMENT OF BOUNDING METHODS IN CONTINUUM MECHANICS

by

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SUMMARY

Energy theorems and kindred inequalities have long been a basis for the analysis of redundant structures and the material continuum. In the first section of this thesis we trace the development of the principal results of elasticity, time-independent inelasticity and creep, from the principle of virtual work and the well-known theorems of linear elasticity to recent results which describe the deformation of general inelastic materials under time-varying loads. In certain instances where incompleteness is apparent in the theory an attempt is made to remedy this; in particular we present a new view of the upper bound shakedown theorem - an area which remains relatively unexplored in comparison with the lower bound theorem and the limit theorems. A discussion of the fundamental material requirements which permit the establishment of many of the inequalities is included.

In the following section we obtain new bounding results for a class of constitutive relations using a thermodynamic formalism as the basis of the discussion. The bounds turn out to be both simple in form and insensitive to the detailed aspects of the material behaviour. Cyclic work bounds are derived in which the cyclic stress history known as the "rapid cycle solution" gives a simple physical meaning to the bounding results. Examples are given for linear viscoelastic models, the non-linear viscous model and the Bailey-Orowan recovery model. A displacement bound is derived which is expressed in terms of two plasticity solutions and the result of a simple creep test. Examples are given and the results we obtain for the Bree problem are compared with O'Donnell's solutions which are in use in current design.

In the third section, new results are obtained for the behaviour of a general viscoelastic material subjected to cyclic loading. The existence and uniqueness of a stationary cyclic state of stress is proved and a lower

work bound for the general non-linear material is derived. An upper work bound is obtained for the general linear material in terms of the rapid cycle solution and we describe a simple method for obtaining this solution without the need for a full analysis. The role of the constitutive equation in the bounding theory is investigated when the method based on a state variable description is compared with the results obtained from the use of a history-dependent constitutive relation. We go on to show how a knowledge of the response of a viscoelastic body to constant loading is sufficient to determine its general long-term cyclic strain behaviour.

In the final section we bring together the existing theorems concerning small deformations of time-dependent materials and large deformations of time-independent materials. The problem posed has dual complexity as a result of the dependence of the deformation on the stress history and the dependence of the stress on the changing geometry. We obtain a general displacement bound in terms of suitably defined conjugate variables referred to the undeformed configuration. In an example which follows it is shown that the employment of such variables may in some cases reduce the difficulty of bounding non-linear deformations to a level that is comparable with the linear case.

CONTENTS

SECTION 1

A REVIEW OF DEVELOPMENTS IN THE BOUNDING METHODS OF CONTINUUM MECHANICS

INTRODUCTION

In this first section we draw together the work of many authors in an attempt to describe a continuous history of the development of the methods of bounding exact material and structural behaviour in terms of displacement, deformation and energy expressions. The motivation to generate such bounds lies in the inherent intractability of many structural problems with regard to analytical solution, even when apparently simple material models are employed. It might be argued further, that exact solutions of realistic design problems are impossible to obtain, and that the finite element method of analysis comes closest to providing acceptably accurate solutions. We hope to demonstrate that bounding methods exist with the capacity to generate acceptably accurate solutions to a wide variety of continuum and structural problems, and that these methods may offer both a simpler and a cheaper approach to the designer.

In pursuit of completeness, the well-known theorems of linear elasticity are included; it may be seen that their format is a basis for many of the bounding methods applicable to inelastic materials. There follows a description of the inequalities that apply primarily to incremental plasticity; mention is also made of some results of the deformation theory that appear to belong to the same class. The limit theorems are discussed, and in view of continuing interest, the shakedown theorems are considered in some detail, including the presentation of a new proof of the upper bound theorem.

Mention is made of the skeletal point and reference stress methods of structural analysis; selection of an appropriate reference stress permits the simplification of the dissipation terms appearing in some upper bound expressions.

Purely creeping and creeping elastic bodies are discussed in terms of properties of appropriate convex functions, and the extension, due to

Ponter and Leckie, to include plastic behaviour up to specific factors of the collapse or shakedown loads is included, together with Ainsworth's development of the bounding theorems for cases above the shakedown limit.

A separate approach lay open as a result of Martin's intuitive notion of the existence of paths in stress space which maximise the complementary work between two prescribed states of stress. Consequent results are given, along with Ponter's extensions into strain space, and strain-time and stress-time spaces. These extremum quantities may be utilised to define an associated elastic problem under conditions related to the given problem for an inelastic body; this gives rise to extended bounds and to results which give added meaning to deformation theories of plasticity. In a later section we demonstrate a further use of the extremum-path quantities in bounding the time-dependent non-linear deformation of a body.

Ponter has defined a functional which may be exploited in cases where it can be shown to possess an upper bound; we discuss how this enables more general energy theorems to be derived, and in a subsequent section we employ the functional to obtain new bounds for cyclically loaded viscoelastic bodies. It is shown that under certain circumstances this functional is related to Martin's maximum complementary work expression.

Finally, mention is made of some work on the fundamental precepts that give rise to the various bounding methods. Foremost in the early development were the contributions of Hill, Drucker and Iliushin, and an attempt is made to provide the basis for a general framework, encompassing some of the work of these authors.

In view of its central role in the majority of the theorems discussed in this section, we begin with a statement of the principle of virtual work.

1. The Principle of Virtual Work

In its broadest form, known as d'Alembert's principle, the principle of virtual work is one of the most general summarising statements in the mechanics of material systems. All related statements of principle, including Hamilton's, are derivable from it.*

In the main, we will confine discussion to quasi-static changes, and inertia terms may be ignored. Two distinct versions of the principle appear in the literature:

a) the increment of work is zero for a virtual displacement, compatible with any constraints, from a position of equilibrium;

b) the increment of complementary work is zero for an infinitesimal virtual change in force from a geometrically compatible state.

Finite displacements are permitted in a) if the form of the equilibrium equations is undisturbed, or if they are defined in the deformed state (Malvern 1969).

Writing the equilibrium equations in the form:

$$\left.\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} + X_i &= 0 \\ \sigma_{ij} n_j - P_i &= 0 \end{aligned}\right\} \quad \dagger \qquad (1)$$

for external loads X_i per unit volume and P_i per unit surface area with unit outward normal n_j , and the compatibility conditions:

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*See, e.g., Washizu (1975), or Leech (1958); full references are given in an appendix to this section.

†A list of the notation used in this section is contained in an appendix.

$$\left.\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ u_i &= u_i^{\,o} \end{aligned}\right\} \tag{2}$$

for prescribed displacement $u_i^{\,o}$ on part of the surface Su, we may state the two principles as follows:

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}dv - \int_V X_i\delta u_i dv - \int_{Sp} P_i\delta u_i ds = 0 \tag{3}$$

$$\int_V \varepsilon_{ij}\delta\sigma_{ij}dv - \int_{Su} u_i^{\,o}\,\delta\sigma_{ij}n_j ds = 0 \tag{4}$$

Sp is the part of the surface upon which P_i is applied. It may be observed in (4) that $\delta\sigma_{ij}$ is a "residual stress field", in equilibrium with zero applied forces.

If both the static variables (σ_{ij}, X_i, P_i) and the kinematic variables $(\varepsilon_{ij}, u_i, u_i^{\,o})$ are continuous, we may incorporate (3) and (4) into one statement:

$$\int_V \sigma_{ij}^{\,*}\,\varepsilon_{ij}^{\,c}\,dv = \int_V X_i^{\,*}\,u_i^{\,c}\,dv + \int_{Sp} P_i^{\,*}\,u_i^{\,c}\,ds + \int_{Su} \sigma_{ij}^{\,*}\,n_j u_i^{\,o^c}\,ds \tag{5}$$

where the starred quantities are an equilibrium set and the superscript 'c' indicates a compatible set.

Equation (5) is a corner-stone in much of the subsequent theory described in this section.

2. Time independent material behaviour

Summary

Here we include energy theorems for elastic materials, plastic materials and for bodies subjected to constant imposed displacements. In view of the continuing interest in the phenomenon of shakedown, the upper bound shakedown theorem is discussed in some detail.

An overall view of the behaviour of an elastic, perfectly-plastic body (subjected to boundary conditions described in Chapter 1) emerges from the theorems of elasticity and shakedown - the latter including the limit theorems as special cases. It is summarised in the following statement.

If, after the application of finite deforming agencies, an equilibrium stress field arises and yield has not been reached, the body deforms only elastically and does not collapse, and the actual static and kinematic variables minimise the total potential energy and total complementary energy functions. If yield is reached, but the body deforms so as to be capable of storing and dissipating energy internally at a rate not lower than that with which the external agencies do work, then some steady state (possibly cyclic) is eventually reached, in which the static and kinematic variables are determined by equality of internal and external energy quantities. In such circumstances there are three possibilities: shakedown, incremental collapse and reverse plasticity, the first being distinct in that the rate of energy dissipation is zero, and the second in that there is a non-zero accumulation of deformation between points in time at which the deforming agencies have identical values. On the other hand, if work is done externally at a greater rate than can be stored and dissipated internally, then the body acquires kinetic energy and collapses.

Small-displacement elasticity

The elastic strain tensor is related to stress by

$$e_{ij} = C_{ijk\ell}\,\sigma_{k\ell} \tag{6}$$

where, by virtue of the symmetry of e_{ij} and $\sigma_{k\ell}$,

$$C_{ijk\ell} = C_{jik\ell} = C_{ij\ell k} \tag{7}$$

An increment of strain energy density is defined as

$$\delta E = \sigma_{ij}\,\delta e_{ij} \tag{8}$$

and so
$$E(e_{ij}) = \frac{1}{2}\,C_{ijk\ell}^{-1}\,e_{k\ell}\,e_{ij} \tag{9}$$

The existence of $E(e_{ij})$ such that $E(o) = 0$ is sufficient to ensure that

$$C_{ijk\ell} = C_{k\ell ij}{}^{*} \tag{10}$$

The principle of virtual work now yields the theorem of minimum potential energy:

$$U_p(e_{ij},u_i) \leqslant U_p\,(e_{ij}^{\ c},u_i^{\ c}) \tag{11}$$

where
$$U_p(e_{ij},u_i) = \int_V E(e_{ij})dv - \int_V X_i u_i dv - \int_{Sp} P_i u_i ds \lesssim 0 \tag{12}$$

and (e_{ij},u_i) represent actual quantities, while $(e_{ij}^{\ c},u_i^{\ c})$ represents any compatible set. The proof of (11) requires $E(e_{ij})$ to be positive-definite; this may be established by accepting the proposition of Gibbs concerning stable thermodynamic equilibrium on the basis of the second law of thermodynamics: for an isolated system in equilibrium, the internal energy is a minimum amongst states with equal entropy. The unstrained state is such a state and a variation from it thus gives a positive increment of internal energy,

* Hill (1956), Jeffreys (1931)

$$\delta E > 0 \tag{12A}$$

Noting (10) it follows that E is positive definite (Malvern 1969). The complementary energy theorem follows in a similar way; the complementary energy density is defined through

$$\delta E_c = e_{ij} \delta \sigma_{ij} \tag{13}$$

and

$$E_c(\sigma_{ij}) = \frac{1}{2} C_{ijk\ell} \sigma_{k\ell} \sigma_{ij} \tag{14}$$

and (10) and (12A) ensure that

$$U_c(\sigma_{ij}) \leq U_c(\sigma_{ij}^*) \tag{15}$$

where

$$U_c(\sigma_{ij}) = \int_V E_c(\sigma_{ij}) dv - \int_{Su} u_i^o \sigma_{ij} n_j ds \tag{16}$$

and σ_{ij} represents the actual stress, while σ_{ij}^* represents any statically admissible stress.

Finite displacements

Equality (12) implies the minimum potential energy theorem only if the second variation, $\delta^2 U_p(e_{ij}^c)$, is positive for all kinematically admissible displacements. The state is then said to be stable.

A dual result for complementary energy is complicated by the coupling of stress and displacement terms but some progress has been made by Koiter (1973), by redefining the total complementary energy in terms of the two Piola stress tensors, referred to the undeformed configuration. Although uncoupling of stresses and displacements is then achieved, Koiter admits to the difficulty of application of the resulting stationary principle. (Examples are given for the special case of semi-linear isotropic materials.)

Imposed strains

A generalisation of the minimum complementary energy theorem (15), to include constant imposed strains p_{ij} (for example, plastic strains) is due to Colonetti and independently, Reissner. $U_c(\sigma_{ij}^*)$ is redefined as

$$U_c(\sigma_{ij}^*) = \int_V E_c(\sigma_{ij}^*)dv + \int_V \sigma_{ij}^* p_{ij} dv - \int_{Su} u_i^o \sigma_{ij}^* n_j ds \qquad (17)$$

and U_c is minimised by the actual stress field.

The generalisation of the minimum potential energy theorem was given by Greenberg in 1949. $U_p(\varepsilon_{ij}^c)$ is redefined as

$$U_p(\varepsilon_{ij}^c) = \int_V E(e_{ij}^c)dv - \int_V X_i u_i^c dv - \int_{Sp} P_i u_i^c ds \qquad (18)$$

where $\varepsilon_{ij}^c = p_{ij} + e_{ij}^c$ is compatible. U_p is minimised by the actual strain field.

Both theorems are discussed in Koiter's general review (1960).

Increments

Analogous minimum principles for stress and strain increments were derived substantially by Prager and Hodge, Greenberg and Bauer. The energy and complementary energy terms are respectively defined to be

$$U_p(d\varepsilon_{ij}^c) = \frac{1}{2}\int_V d\sigma_{ij}^c d\varepsilon_{ij}^c dv - \int_V dX_i du_i^c dv - \int_{Sp} dP_i du_i^c ds \qquad (19)$$

$$U_c(d\sigma_{ij}^*) = \frac{1}{2}\int_V d\sigma_{ij}^* d\varepsilon_{ij}^* dv - \int_{Su} d\sigma_{ij}^* n_j du_i^o ds \qquad (20)$$

and each is minimised by the actual values in the body.

Proofs of the theorems may be found in Koiter (1960) and Hill (1950).

Plastic materials

The historical development of minimum and stationary principles in incremental plasticity are comprehensively described in Koiter's summarising paper (1960). An outline of the early results which led to the limit theorems is given here, followed by results for the deformation theory of plasticity. The limit and shakedown theorems conclude this chapter on time-independent material behaviour.

Hill's principle of maximum plastic work

Hill (1947) proved that the rate of working of the external forces is a maximum for the actual stress field amongst all admissible stresses. That is

$$U_c(\sigma_{ij}^*) \equiv - \int_{Su} \sigma_{ij}^* n_j du_i^o \, ds \tag{21}$$

is a minimum for the actual stress field. For the proof, Hill assumed the Mises yield criterion, and took the whole body to be deforming plastically. Equation (21) is a special case of the upper bound limit theorem.

In 1950, he proved that the principle was equivalent to the following material inequality, which has subsequently become known as the maximum work principle:

$$(\sigma_{ij}^* - \sigma_{ij})dp_{ij} \leq 0 \tag{22}$$

which applies to general convex yield surfaces, and remains valid when not all of the body is at yield. It may be remarked that (22) is distinct from other extremum principles in the discussion above, in that the maximum is non-analytic.

Hill's principle plays a central role in the derivations of the limit and shakedown theorems.

Markov's principle

This is expressed in terms of admissible increments. Assuming the entire body is yielding,

$$U_p(dp_{ij}^{\,c}, du_i^{\,c}) = \sqrt{2}\,k \int_V \sqrt{dp_{ij}^{\,c}\, dp_{ij}^{\,c}}\, dv - \int_{Sp} P_i du_i^{\,c}\, ds \tag{23}$$

is a minimum for the actual kinematic quantities, (dp_{ij}, du_i). The Mises yield criterion is again assumed:

$$S_{ij}S_{ij} \leq 2\,k^2 \tag{24}$$

where S_{ij} is the stress deviator,

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{ii}\delta_{ij} \quad . \tag{25}$$

In the case where not all of the body is at yield, the principle holds, but the minimum is non-analytic. (Hill 1950).

Deformation theory plasticity

The minimum principles of deformation theory are closely related to those of small displacement elasticity. Provided the stress-strain relations do not alter during loading - that is, unloading from the yield surface is forbidden - extremum principles follow in an analogous way to (11) and (15).

A strain-hardening material may be described by

$$S_{ij} = \mu p'_{ij} \tag{26}$$

where $\mu = \mu(p'_{ij})$ and p'_{ij} is the plastic strain deviator. Kachanov, in 1942, derived minimum principles based on (26). A perfectly plastic (Mises) material is known as a Hencky material and the complementary energy minimum principle

$$U_c(\sigma_{ij}) \leq U_c(\sigma_{ij}^*) = \frac{1}{2}\int_V C_{ijk\ell}\sigma_{ij}^*\sigma_{k\ell}^*\,dv - \int_{Su}\sigma_{ij}^* n_j u_i^o\,ds \qquad (27)$$

is known as the Haar-Karman principle (1909). A proof was given by Greenberg in 1949.

In the case of the rigid-plastic Hencky material, results exist of an analogous nature to those of Hill and Markov:

$$U_c = -\int_{Su} \sigma_{ij}^* n_j u_i^o\,ds \qquad (28)$$

is a minimum for the actual stress field; this was known as Sadowsky's principle of maximum plastic resistance (1943); and

$$U_p = \sqrt{2}\,k \int_V \sqrt{p_{ij}^c\, p_{ij}^c}\,dv - \int_{Sp} P_i u_i^c\,ds \qquad (29)$$

is a minimum for the actual kinematic quantities.

It is generally agreed that the deformation theory is an unsatisfactory description of plastic behaviour except in the case of proportional loading, and the above results are included for completeness and historical interest.

The limit theorems of incremental plasticity

It is proved in Koiter's review (1960) that during the collapse of an elastic, perfectly-plastic body, the elastic strain rates and the stress rates are zero and the body behaves in a rigid-plastic fashion, with the collapse mechanism being a purely plastic kinematically admissible strain increment, dp_{ij}. This observation enables Hill's principle to be invoked in the proof of the limit theorems.

The lower bound collapse theorem states that if there exists any admissible stress distribution in the body for which collapse would not occur, then collapse does not occur. A simple proof using Hill's principle is given in Washizu (1975), and another, using Drucker's postulates (which are discussed in a later section herein) is given by Koiter (1960). As the name suggests, the theorem may be used to generate a lower bound on the safety factor for a given problem.

The upper bound collapse theorem concerns increments of work - externally, $dW_e = \int_V X_i du_i^c dv + \int_{Sp} P_i du_i^c ds$

and internally, $dW_i = \int_V \sigma_{ij} dp_{ij}^c dv$

which result from an admissible collapse mechanism. The theorem states that collapse will occur if there exists any admissible kinematic set (du_i^c, dp_{ij}^c) for which $dW_e > dW_i$, but if the inequality is reversed for all admissible mechanisms, the body will not collapse. Once again, the proof may be accomplished through virtual work and Hill's principle (Washizu) or Drucker's postulates (Koiter). An upper bound to the safety factor may be determined on application of this theorem.

The design technique of achieving close bounds on either side of the safety factor is illustrated through examples by Neal (1964).

It may be noted that the limit theorems do not provide information about the magnitude of displacements before collapse occurs; nevertheless Koiter demonstrated that for a safety factor exceeding unity throughout a loading programme, the total plastic work in the structure may be bounded from above. This, of course, does not rule out severe local deformation but it may be regarded as a bound on the mean degree of plastic deformation.

The shakedown theorems

It is well known that a body may collapse through the application of loads whose maximum values, if kept constant, would have been safe on the basis of the limit theorems. A simple example of a portal frame is cited by Neal (1964). Collapse occurs by accumulated increments of plastic deformation. On the other hand, the body is said to shake down if, after an initial period of plastic straining, it develops constant residual stresses that allow it to respond entirely elastically to subsequent load variations. The theorems describing shakedown are generalisations of the preceding collapse theorems.

The lower bound theorem was first given in a restricted form by Bleich in 1932 and then, more generally, by Melan in 1936. Following Koiter's account, it is assumed for simplicity that the body has rigid supports: $u_i^o = 0$ on Su. The following quantities are required in the discussion of the theorems:

$\hat{\sigma}_{ij}$ is the stress occurring if the body were purely elastic

$\hat{\varepsilon}_{ij}$ corresponds to $\hat{\sigma}_{ij}$

$\rho_{ij}(t)$ is the residual stress in the body if, at time t, the loads were (slowly) removed without causing additional plastic strain

e_{ij}^r corresponds to ρ_{ij}

Δp_{ijo} is defined to be a kinematically admissible cyclic accumulation of purely plastic strain: $\Delta p_{ijo} = \int_{cycle} \dot{p}_{ijo} dt$

$\dot{\rho}_{ijo}$ is the unique residual stress rate corresponding to $\dot{p}_{ijo}$

$\dot{e}_{ijo}$ corresponds to $\dot{\rho}_{ijo}$

$\bar{\rho}_{ij}$ is an arbitrary constant residual stress field.

The lower bound shakedown theorem states that if any $\bar{\rho}_{ij}$ exists such that $(\hat{\sigma}_{ij} + \bar{\rho}_{ij})$ does not exceed yield in the body during a loading programme between prescribed limits, then the body will shake down. The proof given in Koiter is a simplified version of Melan's, due to Symonds (1950). It is shown that as Δp_{ijo} is taken to be admissible, then $\rho_{ijo}(o) = \rho_{ijo}(T)$, where $0 \leqslant t \leqslant T$ denotes a cycle of loading. Consequently

$$\int_o^T \dot{e}_{ijo} dt = 0 \qquad . \qquad (30)$$

We now assume the existence of an admissible stress σ_{ij}^* which does not exceed yield throughout the loading, and define the positive-definite quantity

$$E = \int_V \frac{1}{2} C_{ijk\ell} (\sigma_{ij} - \sigma_{ij}^*)(\sigma_{k\ell} - \sigma_{k\ell}^*) dv \tag{31}$$

As
$$\sigma_{ij} = \hat{\sigma}_{ij} + \rho_{ij} \tag{32}$$

and we may set

$$\sigma_{ij}^* = \hat{\sigma}_{ij} + \bar{\rho}_{ij} \quad , \tag{33}$$

then on differentiating (31) we obtain

$$\begin{aligned} \frac{\partial E}{\partial t} &= \int_V C_{ijk\ell} (\sigma_{ij} - \sigma_{ij}^*) \frac{\partial \rho_{k\ell}}{\partial t} dv \\ &= \int_V (\sigma_{ij} - \sigma_{ij}^*) \frac{\partial e_{ij}^r}{\partial t} dv \end{aligned} \tag{34}$$

From (32), the actual elastic strain is given by

$$e_{ij} = \hat{\varepsilon}_{ij} + e_{ij}^r$$

and so the total strain is

$$\varepsilon_{ij} = \hat{\varepsilon}_{ij} + e_{ij}^r + p_{ij} \tag{35}$$

Substituting (35) into (34) and applying virtual work to the self-equilibrating stress $(\sigma_{ij} - \sigma_{ij}^*)$ and the admissible strain rates $\frac{\partial \varepsilon_{ij}}{\partial t}$ and $\frac{\partial \hat{\varepsilon}_{ij}}{\partial t}$ we obtain

$$\frac{\partial E}{\partial t} = - \int_V (\sigma_{ij} - \sigma_{ij}^*) \frac{\partial p_{ij}}{\partial t} dv . \tag{36}$$

and from Hill's maximum work principle it follows that $\frac{\partial E}{\partial t} < 0$ unless either $\sigma_{ij} = \sigma_{ij}^*$, in which case $p_{ij} \equiv 0$ by hypothesis, or $\sigma_{ij} = \hat{\sigma}_{ij} + \rho_{ij}$ where ρ_{ij} is time-constant, in which case the body has shaken down to a state distinct from σ_{ij}^*.

As E is positive definite, E cannot decrease indefinitely and so the body shakes down.

Note that an alternative form of (36) is

$$\frac{\partial E}{\partial t} = - \int_V (\rho_{ij}(t) - \bar{\rho}_{ij}) \frac{\partial p_{ij}}{\partial t} dv$$

and the maximum work principle requires that additional plastic straining changes the residual stress $\rho_{ij}(t)$ in such a way as to approach a shakedown state.

Once again, no deformation limits result, but Koiter has shown that for a shakedown safety factor above unity, the total plastic work may be bounded above.

The first shakedown theorem was subsequently demonstrated to be a special case of the theorem due to Frederick and Armstrong (1966), in which it is proved that two bodies differing only in their initial patterns of internal stress will develop identical patterns of internal stresses in regions of creep and plasticity if they are subjected to the same variations of temperature and loading.

The upper bound shakedown theorem

The upper bound theorem was first given by Koiter in 1956. It states that if there exists any $\dot{p}_{ijo}$ (defined above) for which the accumulation of external work exceeds the accumulated internal dissipation, then shakedown will not occur. On the other hand, if the inequality is reversed for all $\dot{p}_{ijo}$, then the body will shake down.

The accumulations of energy are, respectively,

$$\Delta W_e = \int_{\text{cycle}} \left\{ \int_V X_i \dot{u}_{io} dv + \int_{Sp} P_i \dot{u}_{io} ds \right\} dt$$

$$\Delta W_i = \int_{\text{cycle}} \int_V \sigma_{ijo} \dot{p}_{ijo} dv dt$$

where $\int_{\text{cycle}} \dot{u}_{io} dt$ is compatible with Δp_{ijo}, and σ_{ijo} is the stress on the yield surface corresponding to $\dot{p}_{ijo}$.

The proof is given by Koiter, but as he observed, difficulties lie in the application of the theorem. They appear to centre on the fact that while Δp_{ijo} may be defined to be admissible, in general $\dot{p}_{ijo}$ is not admissible, and during a cycle the loads do not remain on the yield surface, so - distinct from the circumstance of the upper bound limit theorem - energy rates cannot be equated through virtual work.

An alternative statement of both shakedown theorems has been given by D.A. Gokhfeld (1977); for the second theorem he still uses the quantity $\dot{p}_{ijo}$ as defined here, but he re-expresses the theorem in terms of the <u>constant</u> set $(\sigma_{ij}^{o}, X_i^{o}, P_i^{o})$ as follows: shakedown will occur if for any $\dot{p}_{ijo}$,

$$\int_{cycle} \left\{ \int_V X_i^{o} \dot{u}_{io} dv + \int_{Sp} P_i^{o} \dot{u}_{io} ds \right\} dt < \int_{cycle} \int_V \sigma_{ij}^{o} \dot{p}_{ijo} dv\, dt \quad . \tag{37}$$

In terms of the constant component of stress, σ_{ij}^{o}, a "fictitious yield surface" is defined by

$$\sigma_{ij}^{o} \leqslant \sigma_{ijx}^{o} \tag{38}$$

and the second shakedown theorem then takes this form: shakedown is impossible if there exists a $\dot{p}_{ijo}$ such that

$$\int_V X_i^{o} \Delta u_{io} dv + \int_{Sp} P_i^{o} \Delta u_{io} ds \geqslant \int_V \sigma_{ijx}^{o} \Delta p_{ijo} dv \quad . \tag{39}$$

A new view of the second theorem is now given. We suppose for simplicity that the body is subjected only to surface traction variations between prescribed limits. The cycle is imagined as a sequence of stages in which subsets of the complete set of loads are first applied and then removed. During a typical stage it follows that

$$\int P_i u_i^{a} ds = E + \int \int \sigma_{ij}^{y} \dot{p}_{ij} dt\, dv \tag{40}$$

and

$$\int - P_i u_i^{b} ds = - E \tag{41}$$

where the superscripts a and b refer to the application and removal of the loading subset; E is a quantity of recoverable elastic energy and σ_{ij}^{y} is the stress on the yield surface corresponding to $\dot{p}_{ij}$. Combining (40) and (41) and noting that σ_{ij}^{y} is constant during this stage we obtain

$$\int P_i \, u_i \, ds = \int \sigma_{ij}^{y} p_{ij} \, dv \tag{42}$$

for the completed stage. We now choose a set of such stages so that the assumed accumulations of deformation u_i constitute a collapse mechanism. Substitution of each such u_i into (42), together with a compatible strain p_{ij} and corresponding yield stress σ_{ij}^{y} gives a value for the load P_i . Together, in any sequence, the set of loads so obtained is sufficient to cause incremental collapse. That they are an upper bound in this sense is evident from Hill's principle: in (42), σ_{ij}^{y} corresponds to the assumed p_{ij} ; suppose the actual stress is σ_{ij}' ; then by Hill's principle, $(\sigma_{ij}^{y} - \sigma_{ij}') \, p_{ij} \geq 0$, and on application of virtual work,

$$\int (P_i - P_i')u_i ds \geq 0$$

where P_i is the calculated value of load and P_i' is the actual value.

In effect, the theorem provides an upper bound to the energy required to bring about incremental collapse.

3. Time dependent behaviour

Summary

We begin this chapter with inequalities that fall within the class of theorems known as the elastic analogue. We see that the property of convexity of certain functionals is central to the derivation of energy principles for time dependent materials.* There follows a brief discussion of the skeletal point and reference stress methods of structural analysis and then we diverge from the "classical" energy methods with the notion of extremal paths, due initially to Martin (1965), whereby a comparison quasi-elastic body may be used to bound the actual material behaviour. The notion was first discussed in the time-independent cases and was extended by Ponter; the methods recover earlier results due to Leckie and Martin and also enable the formulation of a generalisation of the Haar Karman principle of deformation theory plasticity.

Energy and displacement bounds for creeping plastic materials were obtained by Ponter and Leckie as extensions of the Leckie-Martin results, and Ponter extended the bounds to cases of variable loading. The bounds apply to loading below a certain factor of the shakedown limit; Ainsworth's discussion of loading above this factor is also included. These bounding theorems are strongly dependent on the material constitutive relations; Ponter made further progress by devising displacement and work bounds in terms of a functional whose determination constitutes a separate problem from the energy theorems themselves; this is discussed in relation to perfectly-plastic and non-linear viscous materials, and in a separate section, viscoelastic materials are included.

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*The more general significance of convexity is discussed in a later chapter.

The elastic analogue

The inclusion of time-dependent behaviour in the family to which potential and complementary energy theorems may be applied was noted by Hoff (1954). At about this time, Odqvist, Hoff and others had used the name "elastic analogue" in describing certain time-dependent problems (Odqvist, 1974). Hill (1956) developed the notion along the lines of general variational principles: if there exists a one-one relation between σ_{ij} and $\dot{\varepsilon}_{ij}$ such that

$$\frac{\partial \sigma_{ij}}{\partial \dot{\varepsilon}_{k\ell}} = \frac{\partial \sigma_{k\ell}}{\partial \dot{\varepsilon}_{ij}} \geq 0 \tag{43}$$

then the existence of a state function

$$E = \int \sigma_{ij} d\dot{\varepsilon}_{ij} \tag{44}$$

is assured, and furthermore E is convex, in the sense that

$$E(\dot{\varepsilon}_{ij}{}^{2}) - E(\dot{\varepsilon}_{ij}{}^{1}) \geq (\dot{\varepsilon}_{ij}{}^{2} - \dot{\varepsilon}_{ij}{}^{1}) \frac{\partial E}{\partial \dot{\varepsilon}_{ij}} \quad * \tag{45}$$

A minimum principle follows, similar in format to the potential energy theorem of elasticity, (11), with strain rates and velocities taking the places of strains and displacements.

Similarly, Hill quotes a minimum complementary principle analogous to (15), involving the function

$$E_c = \int \dot{\varepsilon}_{ij} \, d\sigma_{ij} \quad . \tag{46}$$

With various substitutions for E and E_c , Hill recovered principles for rigid-perfect-plasticity, Newtonian viscosity, viscoplasticity, work-

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* Mention is made in a subsequent section of the connection between equation (43) and Onsager's reciprocal relations

hardening elastic-plasticity and non-linear elasticity and he obtained new results for a viscoelastic Maxwell solid. [Hill observed that the minima were non-analytic for perfectly plastic materials.]

Martin (1965) defined the functionals $E_c(\sigma_{ij})$ and $E(\dot{\varepsilon}_{ij})$ for a material obeying Norton's law:

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_o} = \left(\frac{\sigma}{\sigma_o}\right)^n \tag{47}$$

in the uniaxial case, and

$$\frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_o} = \phi^n \left(\frac{\sigma_{ij}}{\sigma_o}\right) \frac{\partial\phi}{\partial\left(\frac{\sigma_{ij}}{\sigma_o}\right)} \tag{48}$$

in general. Assuming ϕ^{n+1} is convex, it follows that E_c and E are also convex and Martin derived the following inequality for a problem in which P_i^* is the load on Sp :

$$\frac{1}{n+1} \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* dv + \frac{n}{n+1} \int_V \sigma_{ij}^c \dot{\varepsilon}_{ij}^c dv \geq \int_{Sp} P_i^* \dot{u}_i^c ds \tag{49}$$

where $\dot{\varepsilon}_{ij}^*$ is associated with σ_{ij}^* and may not be admissible, likewise σ_{ij}^c, associated with $\dot{\varepsilon}_{ij}^c$.

Martin went on to use (49) to produce a point displacement-rate bound for the creeping body.

An extension to the elastic analogue was made by Hult (1962) who demonstrated that a constitutive equation of the form

$$\dot{\varepsilon}_{ij}(t) = \frac{3}{2} k(t) \frac{\{\sigma_e(t)\}^{n-1}}{\{\varepsilon_e(t)\}^m} S_{ij} \quad , \tag{50}$$

which includes primary creep, results in a stress distribution which is constant and identical to the "steady-state" case, (48), though the strain rates now vary in time.

[Here, σ_e and ε_e are effective stress and strain $\sqrt{\frac{3}{2} S_{ij}S_{ij}}$ and $\sqrt{\frac{2}{3} \varepsilon_{ij}\varepsilon_{ij}}$ respectively.]

The solution to this problem is known as the "stationary state" solution. For proportional loading the stress variation is also proportional and follows the time variation of the loading.

Leckie and Martin (1967) pointed out that a range of laws describing primary and secondary creep may be incorporated into the general form of (50); moreover they all predict the same stationary stress distribution. The following results for elastic-creeping bodies were obtained:

1) The stress following step loading eventually becomes equal to the stationary solution, σ_{ij}^{s}, which is identical to the stress in a purely creeping body.

2) Upper and lower bounds on the total energy dissipated by external loads at large times:

$$\int_V (E(\varepsilon_{ij}^{s}) - E(\varepsilon_{ij}^{o^+}))dv + \int_V \int_{o^+}^{T} \sigma_{ij}^{s}\,\dot{\varepsilon}_{ij}^{s}\,dtdv \leq D^T \leq (n+2)\int_V \{E(\varepsilon_{ij}^{s}) - E(\varepsilon_{ij}^{o^+})\}dv + \int_V \int_{o^+}^{T} \sigma_{ij}^{s}\,\dot{\varepsilon}_{ij}^{s}\,dtdv \qquad (51)$$

where $D^T = \int_{Sp} P_i(u_i(T) - u_i(o^+))ds$ and $\dot{\varepsilon}_{ij}^{s}$ corresponds to σ_{ij}^{s} in (50).

3) A general intermediate-time upper bound is formed by adding the term $\frac{1}{n}\int_V E(\varepsilon_{ij}^{*})dv$ to the large-time upper bound in (51) and replacing σ_{ij}^{s} with an arbitrary equilibrium stress, σ_{ij}^{*}.

4) Point displacement bounds, extending Martin's earlier result (1965), to intermediate and large times for the elastic-creeping material.

5) Improved upper and lower bounds on external energy dissipation at large times, based on the assumption that the stress redistribution is small:

$$2\int_V \{E(\varepsilon_{ij}^{s}) - E(\varepsilon_{ij}^{o^+})\}dv + \int_V \int_{o^+}^{T} \sigma_{ij}^{s}\,\dot{\varepsilon}_{ij}^{s}\,dtdv \leq D^T \leq (n+1)\int_V \{E(\varepsilon_{ij}^{s}) - E(\varepsilon_{ij}^{o^+})\}dv + \int_V \int_{o^+}^{T} \sigma_{ij}^{s}\,\dot{\varepsilon}_{ij}^{s}\,dtdv \qquad (52)$$

The significance of stress redistribution may be estimated from the quantity

$$E^* = \frac{\int E(\varepsilon_{ij}^{s})dv - \int E(\varepsilon_{ij}^{o+})dv}{\int E(\varepsilon_{ij}^{o+})dv} \quad . \tag{53}$$

Leckie and Martin showed that in several practical structures, $E^* < 25\%$, indicating that the effects of redistribution might be neglected, in which case sufficient accuracy is obtained from addition of elastic and stationary terms. They confirmed this by considering a two-bar structure in which redistribution is severe: $E^* = 50\%$ - and they obtained bounds from which a reasonably close estimate of displacement could be made.

Skeletal point and reference stress methods

A method of calculating the deformation of certain bodies made of elastic-creeping materials was given by Marriott and Leckie (1964). They observed that in such bodies as beam sections, thick tubes and transversely loaded flat circular plates, there exist locations where the elastic and stationary stress values are identical. It was argued that at such points, the stress during redistribution cannot deviate significantly from this "skeletal value" and so the total creep strain at such "skeletal points" is accurately estimated by the stationary solution. This, in turn, may be obtained by testing the material at an arbitrary stress, resulting in a creep rate $\dot{v}_{ij}^{a}$, and solving the non-linear elastic analogue problem at the same stress to obtain a steady-state creep rate $\dot{v}_{ij}^{*a}$. Then the actual displacement rate in the body, $\dot{u}_i$ is given in terms of the elastic analogue solution, $\dot{u}_i^*$ for the actual stress, as follows:

$$\dot{u}_i = \frac{\dot{u}_i^* \, \dot{v}_{ij}^{a}}{\dot{v}_{ij}^{*a}} \quad . \tag{54}$$

In essence, the skeletal point observation is used to justify the omission of the effects of redistribution in deformation calculations.

Anderson et al (1965), and later MacKenzie (1968), obtained deformation rate estimates by testing the material at a specific stress, called the "reference stress", chosen in such a way as to minimise the effect on the solution of unknown material parameters. The displacement rate was expressed in the form

$$\dot{u}_i = \dot{\varepsilon}_o \, F(\text{geometry}) . \, G(t) . J(x^n, n) \tag{55}$$

where the uniaxial stationary creep law is assumed to have the form

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_o} = \left(\frac{\sigma}{\sigma_o}\right)^n \tag{56}$$

and x is the ratio of applied traction to σ_o .

The function J is the only one explicitly containing n . For choices of n on either side of the anticipated value, the two corresponding values of $\dot{u}_i$ are equated, making x determinate, and so σ_o becomes a known function, σ_R , of the applied loads. The quantity σ_R is known as the "reference stress", and assuming J is a fairly smooth function of n in the chosen range, the choice $\sigma_o = \sigma_R$ makes $\dot{u}_i$ fairly insensitive to the actual value of n . A material test at stress $\sigma_o = \sigma_R$ gives $\dot{\varepsilon}_o$ and hence $\dot{u}_i$ is obtained.

It transpires that the stresses at the skeletal points of Leckie and Marriott are identical to the reference stresses for those structures. Marriott (1970) pointed out that this is not a general result.

Sim (1968) provided a simple method for selecting the reference stress σ_R , using the solution for $n \to \infty$, corresponding to the rigid perfectly-plastic solution:

$$\sigma_R = \sigma_Y \frac{P_i}{P_L} \tag{57}$$

where P_i is the actual load, P_L the limit load and σ_Y the yield stress ($\frac{\sigma_Y}{P_L}$ is a function of geometry only) .

The stationary solution, (54), may be calculated in a simple way by using the reference stress in (57) to obtain $\dot{v}_{ij}^{a}$, and taking $n = 1$ as the lower range for n, in which case $\dot{u}_i^*$ and $\dot{v}_{ij}^{*a}$ correspond to a linear elastic analogue solution:

$$\dot{u}_i = \frac{\hat{u}_i \cdot \dot{v}_{ij}^{R}}{\sigma_R/E} \tag{58}$$

where $\hat{u}_i$ is the linear elastic solution for load P_i, $\dot{v}_{ij}^{R}$ is determined from testing at σ_R and E is Young's modulus. (Goodall, Leckie, Ponter and Townley 1979).

Despite the elegance of these displacement methods, it may be observed that for the purpose of describing the overall response of structures to general loading, it is also desirable to obtain bounds on the total energy dissipation.

4. Extremal paths

Many of the bounds for elastic, creeping materials were based on the convexity equation in the form of (49). Martin derived this from the more general inequality

$$\int_V E(\dot{\varepsilon}_{ij}^{c})dv + \int_V E_c(\sigma_{ij}^{*})dv \geq \int_S P_i^{*}\dot{u}_i^{c} ds \tag{59}$$

in which $E = \int \sigma_{ij} d\dot{\varepsilon}_{ij}$ and $E_c = \int \dot{\varepsilon}_{ij} d\sigma_{ij}$. Martin (1966) generalised (59) to a class of the time independent materials by adding to the assumed stability postulates the statement that there exist paths between specified points in stress space for which the net complementary work has a maximum value. Such paths are called "extremal paths".

In order to apply the resulting bounds, these paths must be obtained for the materials in question; this was investigated by Martin, Ponter (1968, 1969 a and b) and Ponter and Martin (1972):

1. if an entirely elastic path exists, this is extremal and the maximum complementary work, $\bar{\omega}$, is zero;

2. for an elastic, work-hardening body with only one non-zero principal stress, a monotonic change in stress is an extremal path;

3. for a rigid, perfectly-plastic material with a convex yield surface and associated flow rule, any path within the yield surface (but not touching it) is extremal, and $\bar{\omega} = 0$;

4. for an isotropically strain-hardening material, initially unstressed, extremal paths are radial stress paths. For such paths, a deformation theory is an appropriate material description;

5. for a linear kinematic hardening material with a Mises yield criterion, extremal paths are radial and a deformation theory is again appropriate.

Martin's extremal paths led to a fertile source of energy theorems. Ponter (1968) pointed out the physically more identifiable notion of minimum work paths in strain space and he also showed that in the absence of rigid behaviour, an extremal strain path defines a unique terminal stress - a situation more reminiscent of elasticity than plasticity. Indeed, Ponter defined an associated elastic material in terms of the minimum work function, ω , as follows:

$$\sigma_{ij}^{1} = \left.\frac{\partial\omega}{\partial\varepsilon_{ij}}\right|_{\varepsilon_{ij}=\varepsilon_{ij}^{2}} \tag{60}$$

and
$$\omega(0,1) - \omega(0,2) - (\varepsilon_{ij}^{1} - \varepsilon_{ij}^{2}) \left.\frac{\partial\omega}{\partial\varepsilon_{ij}}\right|_{\varepsilon_{ij}=\varepsilon_{ij}^{2}} \gtrsim 0 \tag{61}$$

where 0, 1 and 2 refer respectively to the initial point and two terminal points in stress and strain space. Clearly ω may be interpreted as a convex elastic strain energy density.

Dual results follow for $\bar{\omega}$, the maximum complementary energy, and in terms of $\bar{\omega}$ Martin's generalised version of (59) takes the form

$$\bar{\omega}(0,2) + W(0,1) - (\sigma_{ij}^{2}\ \varepsilon_{ij}^{1} - \sigma_{ij}^{o}\ \varepsilon_{ij}^{o}) \geqslant 0 \tag{62}$$

where $W(0,1)$ is the actual work done in the body.

From (62), Ponter and Martin (1972) obtained the following inequalities:

$$\hat{U}_c(\sigma_{ij}^{2}) \geqslant \hat{U}_c(\hat{\sigma}_{ij}) \geqslant U_c(\sigma_{ij}^{1}) \tag{63}$$

in which $U_c(\sigma_{ij}^{1})$ is the total complementary energy in the body, $\hat{U}_c(\sigma_{ij}^{2})$ is the total complementary energy associated with the arbitrary admissible σ_{ij}^{2} in the associated elastic material, and $\hat{\sigma}_{ij}$ is the elastic solution. Since equation (60) and its dual define a state of strain corresponding to a given stress and vice versa, it may be interpreted as the constitutive equation of a deformation theory material, and (63) is a generalisation of the Haar Karman principle.

A complete dual of (63), in terms of U_p , is not obtainable because of the lack of duality in the directions of the inequalities:

$$U_p(\varepsilon_{ij}^{c}) \geqslant \hat{U}_p(\varepsilon_{ij}^{c}) \geqslant \hat{U}_p(\hat{\varepsilon}_{ij}) \tag{64}$$

where ε_{ij}^{c} may be taken to be the actual strain

As Ponter and Martin point out, (63) bounds the material in terms of a less stiff elastic body with an arbitrary admissible stress; (64) provides a bound only in terms of an elastic solution.

Both theorems are based on the hypothesis that $\bar{\omega}$ exists. Ogden (1976) extended the theory by assuming that ω exists simultaneously with $\bar{\omega}$ and he permitted non-linear deformations by defining a non-linear associated elastic problem. He obtained the extended result:

$$U_p(\varepsilon_{ij}^{1}) \equiv -U_c(\sigma_{ij}^{1}) \geqslant \hat{U}_p(\varepsilon_{ij}^{1}) \geqslant \hat{U}_p(\hat{\varepsilon}_{ij}) \equiv -\hat{U}_c(\hat{\sigma}_{ij}) \geqslant -\hat{U}_c(\sigma_{ij}^{2}) \tag{65}$$

Martin and Ponter used (64) to formulate bounds for impulsively loaded plastic structures. Martin had previously bounded the response of an impulsively loaded elastic body and Wierzbicki used a similar method with rigid, perfectly-plastic structures, justifying the use of the virtual work equation with changing geometry in a "large-small" problem by showing that a correction term takes the sign that cannot weaken the inequality. The result (64) was used to show that an elastic analysis could be rigorously applied to a rigid, perfectly-plastic body. The deformation theory problem thus defined was solved for a fixed-end beam subjected to a transverse impulse. The bounds obtained from the deformation theory and an incremental theory analysis were close, differences arising from the non-optimality of paths in the latter case in the early stages of deformation.

Extremal paths for time-dependent materials

Ponter (1968) broadened the results above by redefining the complementary energy density and work density as follows:

$$\bar{\omega}(0,1) \geqslant \bar{W}(0,1) = \int_{t_o}^{t_1} \varepsilon_{ij}\dot{\sigma}_{ij}dt \qquad (66)$$

$$\omega(0,2) \leqslant W(0,2) = \int_{t_o}^{t_2} \sigma_{ij}\dot{\varepsilon}_{ij}dt \qquad (67)$$

The associated elastic material may now be regarded as possessing a time-varying strain-energy density. Particular extremal paths were obtained:

1. linear viscoelasticity, characterised by the constitutive relation

   $\sigma(t) = \int_o^t G(t-\tau)\dot{\varepsilon}(\tau)d\tau$: the extremal path is given by

   $\int_o^T G|t-\tau|\dot{\varepsilon}(\tau)d\tau = \sigma(T)$, giving $\omega(=\bar{\omega}) = \frac{1}{2}\,\sigma(T)\varepsilon(T)$;

2. non-linear viscosity and strain-hardening creep:

   $$\bar{\omega}(0,\sigma_{ij}^{*}(T)) = E_c(\sigma_{ij}^{*}) + f(n)\,.\,\frac{1}{n}\,D^c\,\left(\frac{n}{n+1}\,\sigma_{ij}^{*}\right)$$

   where $f(n) = 1$ for non-linear viscosity, and $1 \leqslant f(n) \leqslant 1.36$ for $0 \leqslant n \leqslant \infty$, for strain-hardening.

Particular results were obtained for the non-linear Maxwell model:

$$\dot{\varepsilon}_{ij} = C_{ijk\ell}\dot{\sigma}_{ij} + \frac{\partial}{\partial\sigma_{ij}} \left\{\frac{\phi(\sigma_{ij})}{n+1}\right\} .$$

The extremal path is a radial path:

$$\sigma_{ij}(0) = 0 \ , \ \sigma_{ij}(t) = \frac{n}{n+1}\,\sigma_{ij}(T) \ , \ 0 < t < T \ . \qquad (68)$$

Ponter derived a displacement bound for this material; it reproduced an earlier result of Leckie and Martin (1967), derived from convex functionals, since these authors had adopted the optimal stress history ((68)) as their choice of arbitrary admissible stress.

Despite this link between the two distinct bounding methods, much of the energy subsequently expended by workers in this field was directed towards developing the "classical" convexity-type theorems.

5. Creep and plasticity

Summary

Leckie and Ponter (1969) extended the theorems of Leckie and Martin (1967) for the elastic analogue to include plastic strains in an elastic, creeping material. They defined a "stationary plastic creep solution" and obtained bounds for the body in this state, and they went on to show that provided the external load did not exceed a factor of $\frac{n}{n+1}$ times the limit load, the plastic strain was insignificant. Ponter (1970a) extended this to variable loading, the limit being $\frac{n}{n+1}$ times the shake-down load. He considered cyclic loading in particular, and generated upper and lower work bounds, the optimum choices of which were shown to have a physically-meaningful identity. Ponter provided examples of several structures to support his results.

Ainsworth (1977) extended the range of variable loading above $\frac{n}{n+1}$ times the shakedown load by expressing general bounds in terms of a "cyclic plasticity solution"; he also lent support to the theorems with structural examples.

Constant loading

Just as the stationary solution σ_{ij}^{s} is important in bounding elastic, creeping materials, the "stationary plastic creep solution" is useful here: following step loading the stress approaches this solution, denoted by σ_{ij}^{*}, and the total strain rate becomes the sum of creep and plastic components only. Leckie and Ponter (1969) showed that σ_{ij}^{*} is history independent and that the creep dissipation rate is minimised by $\sigma_{ij} = \sigma_{ij}^{*}$, subject to the yield condition, $f(\sigma_{ij}) \leqslant 0$. (The absolute minimum is given by the unrestricted σ_{ij}^{s}.) The authors drew analogy with the minimum complementary energy theorem of elasticity, and the Haar Karman principle.

The convexity-based results, (49) and (59) again provide the foundation for bounding statements.

1. Upper and lower bounds on the external work in the stationary state:

$$\int_V \dot{D}(\sigma_{ij}^{s})dv \leqslant \int_{Sp} P_i \dot{u}_i^{*} ds \leqslant \int_V \dot{D}(\sigma_{ij}^{g})dv \tag{69}$$

where $\dot{u}_i^{*}$ is the stationary plastic-creep displacement rate, σ_{ij}^{g} is an arbitrary admissible stress subject to the restricted yield criterion

$$f\left(\frac{n+1}{n}\sigma_{ij}^{g}\right) \leqslant 0 \tag{70}$$

and $\dot{D}(\sigma_{ij})$ is the creep dissipation rate.

2. Point displacement bound:

$$R_i \dot{u}_i^{R} \leqslant \frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1} \int_V \dot{D}(\sigma_{ij}^{D})dv$$

where R_i is a dummy load and σ_{ij}^{D} is in equilibrium with $P_i + R_i$, and $f(\sigma_{ij}^{D}) \leqslant 0$.

3. An intermediate-time work bound was given, with additional elastic energy terms in the upper bound in (69).

An interpretation of the bounds can be obtained by imagining the body to be loaded proportionally: $P_i = \lambda P_i^L$ where P_i^L is the limit load, independent of creep strains. When $\lambda = 1$ there exists a unique equilibrium set (P_i^L, σ_{ij}^L), the plastic limit solution; thus when $\lambda = \frac{n}{n+1}$, the stress $\sigma_{ij} = \frac{n\sigma_{ij}^L}{n+1}$ is in equilibrium with load $\frac{n}{n+1} P_i^L$ and is the only distribution satisfying the restricted yield condition, (70). For this reason, (69) is not applicable for loading above $\frac{n}{n+1} P_i^L$ and so it may appear less surprising that the bounds in (69) contain no plastic term - the contribution of plastic straining below this level of loading is small.

The effect of stress concentrations may also be considered: suppose we take the maximum effective stress, $\bar{\sigma}_m$, to have just reached yield when $\lambda = \frac{n}{n+1}$. Calladine (1963) has shown that a linear interpolation of maximum effective stress as a function of n between $n = 1$ and $n = \infty$ provides a reasonable approximation for a range of common structures. His formula is $F_n = \frac{n+1}{2n}$ where $F_n = \frac{\bar{\sigma}_m(n)}{\bar{\sigma}_m(n=1)}$. We have $\bar{\sigma}_m(n) = \sigma_y$ = the yield stress, so $\bar{\sigma}_m(n=1) = \frac{2n}{n+1}\sigma_y$; this is the instantaneous response of the body in the absence of plasticity; we may infer that plastic deformation occurs only during redistribution. [Calladine's maximum concentration factor is 2 ; if there are higher levels than this in the body, Ponter (1970) considers them to be "severe but local".]

The upper bound in (69) may be simplified by an appropriate choice of reference stress. Writing the creep dissipation function in the form

$$\dot{D}(\sigma_{ij}^g) = \sigma_o \dot{v}_o \, \phi^{n+1}(\sigma_{ij}^g/\sigma_o) \tag{71}$$

where $\phi(\sigma_{ij}) = \bar{\sigma}$ = effective stress, and noting that as σ_{ij}^g is in equilibrium with P_i, we may set

$$\sigma_{ij}^g = \sigma_{ij}^L \frac{P_i}{P_i^L} \quad ; \tag{72}$$

then

$$\phi\left(\frac{\sigma_{ij}^g}{\sigma_o}\right) = \frac{P_i}{P_i^L}\, \phi\left(\frac{\sigma_{ij}^L}{\sigma_o}\right) \quad . \tag{73}$$

Now $\phi(\sigma_{ij}^{L}) = \bar{\sigma}^{L} \leq \sigma_y$, so (73) becomes $\phi\left(\frac{\sigma_{ij}^{g}}{\sigma_o}\right) \leq \frac{P_i}{P_i^L}\frac{\sigma_y}{\sigma_o}$. The choice of σ_o is open: setting $\sigma_o = \sigma_R = \frac{P_i\sigma_y}{P_i^L}$ the bound, (69), takes the simple form

$$\int_{Sp} P_i\dot{u}_i^{*}\,ds \leq \sigma_o\dot{v}_o \int_V \left(\frac{P_i\sigma_y}{P_i^L\sigma_o}\right)^{n+1} dv = \sigma_o\dot{v}_o V \tag{74}$$

[In a minimum-weight design structure, $\phi(\sigma_{ij}^{L}) = \sigma_y$ and $\dot{D}(\sigma_{ij})$ is constant throughout V so equality holds in (74) for $\lambda \leq 1$.]

Ponter and Leckie (1969) used examples of a beam in flexure, a thick cylinder under internal pressure and a transversely loaded flat circular plate to demonstrate that for loading below $\frac{n}{n+1}P_i^L$, the effect of plastic deformations was indeed small.

<u>Variable loading</u>

The role of the shakedown theorems in elastic-plastic design has been mentioned in an earlier chapter. In many design situations, however, operating temperatures may result in the occurrence of non-negligible creep straining and any residual stress tending to build up in the material will relax away in time. In such circumstances the deformations may become excessive for loading below the shakedown limit.

An extension to the constant-load bounds above was achieved by Ponter (1970), who derived an upper bound on the total inelastic work:

$$\int_V\int_o^T \sigma_{ij}\dot{\varepsilon}_{ij}'\,dtdv \leq \frac{\mu}{\mu-1}\,[A(o)-A(T)] + \frac{1}{n(\mu-1)}\left(\frac{n\mu}{n+1}\right)^{n+1}\int_V\int_o^T \phi(\sigma_{ij}^{*})g(t)\,dtdv \tag{75}$$

where $\dot{\varepsilon}_{ij}$ is the inelastic strain rate, μ is a positive constant, $A(t)$ is the positive definite elastic energy term:

$$A(t) = \int \frac{1}{2}\,C_{ijk\ell}(\sigma_{ij}(t) - \sigma_{ij}^{*}(t))(\sigma_{k\ell}(t) - \sigma_{k\ell}^{*}(t))\,dv \tag{76}$$

and σ_{ij}^{*} is an arbitrary admissible stress, subject to the constraint that

$\mu P_i(t)$ is within the plastic shakedown limit.

With $\mu = \frac{n+1}{n}$ the bound becomes

$$\int_V \int_o^T \sigma_{ij}\dot{\varepsilon}'_{ij}\,dtdv \leq (n+1)[A(o)-A(T)] + \int_V \int_o^T \phi(\sigma_{ij}^{\ *})g(t)\,dtdv \qquad (77)$$

and for cyclic loading the average work per cycle at large times is bounded also:

$$W_{(ave)} \leq \frac{1}{N}\int_V \int_o^T \phi(\sigma_{ij}^{\ *})g(t)\,dtdv \qquad (78)$$

where N is the number of cycles.

The bounds are dominated by creep terms: it is concluded that for loading below $\frac{n}{n+1}$ of the shakedown limit, plastic straining may be regarded as insignificant.

Interest concentrated on the particular case of cyclic loading. Ponter showed that the stress approaches a cyclic stationary state, irrespective of its initial value. For a body in the cyclic stationary state, with $0 \leq t \leq T$ defining a cycle, the total work done on the body, W_o^T, may be bounded from above and below by creep terms:

$$\int_V \int_{cycle} \sigma_{ij}^{\ s}\,\dot{v}_{ij}^{\ s}\,dtdv \leq W_o^T \leq \int_V \int_{cycle} \sigma_{ij}^{\ g}\,\dot{v}_{ij}^{\ g}\,dtdv \qquad (79)$$

where $\sigma_{ij}^{\ s}$ is the purely viscous stationary solution, and $\sigma_{ij}^{\ g}$ is arbitrary, subject to the restricted yield condition $f(\frac{n+1}{n}\sigma_{ij}^{\ g}) \leq 0$.

An example of step-cycles of loading applied to a two-bar body, in which the bounds might be expected to be fairly severely tested, resulted in a maximum difference between them of 4.5%. [Ponter 1972] It may be noted that bounds on the total inelastic work from the commencement of loading would differ by a greater amount, due to the additional work corresponding to the redistribution of stress to the cyclic stationary state.

A very general displacement bound, encompassing those of Leckie and Martin (1967), and Leckie and Ponter (1969), was derived by Ponter (1972):

$$\int_{S_T} T_i(u_i(T)-u_i(o))ds \leqslant \int_{S_T} T_i(\hat{u}_i(T)-\hat{u}_i(o))ds + A(o)-A(T) + \int_V\int_o^T \frac{1}{n}\phi\left(\frac{n}{n+1}\sigma_{ij}^*\right)g(t)dtdv \tag{80}$$

in which T_i is a constant dummy load acting on S_T, $\hat{u}_i$ is the elastic solution due to the real load P_i, σ_{ij}^* is arbitrary, subject to $f(\sigma_{ij}^*) \leqslant 0$ and $\{\phi(\sigma_{ij})g(t)\}$ is the creep dissipation rate. Displacement or deformation bounds may be obtained with a suitable choice for T_i.

The cyclic work and displacement bounds were tested for beam sections and portal frames [Leckie and Ponter (1972)] using shakedown solutions. It was confirmed that the effects of plasticity may be regarded as insignificant for loads lower than $\frac{n}{n+1}$ times the shakedown limit.

Physical interpretations of the bounds

The work bounds in (79) refer to the cyclic stationary state; Ponter (1973a) sought the optimum choice of stress σ_{ij}^g in this state and succeeded in showing that the two bounds represent extreme modes of the body's cyclic behaviour. As might be expected, the lower bound corresponds to very slow cycling (compared with a characteristic material time). The upper bound can be written in terms of stress

$$\sigma_{ij}^g = \hat{\sigma}_{ij} + \bar{\rho}_{ij} \tag{81}$$

where $\hat{\sigma}_{ij}$ is the elastic solution and $\bar{\rho}_{ij}$ is an arbitrary constant residual stress, and Ponter showed that the optimum upper bound is provided by the value of $\bar{\rho}_{ij}$ that gives a unique, compatible accumulation of creep strain, Δv_{ij}^g over a cycle. Furthermore, this choice makes σ_{ij}^g identical to the actual stress in the body when the cycling is fast.

An example of a two-bar structure indicated that for cycles in which the applied load did not reverse in sign, the bounds were very close, in which case a knowledge of the stationary solution may well be acceptable. On the other hand, characteristic times for common structural materials indicated that for practical cases of cyclic loading, the body's response may, in general, be nearer to the upper bound. Similar close bounds were found using optimum stress histories for a beam under variable bending moment and a thick tube under variable internal pressure.

The same interpretation of the optimum stress was made in the case of the displacement bound, (80). Once again, provided the applied load does not change sign during the cycle, the upper and lower displacement bounds are approximately equal; this was confirmed for a laterally-loaded cantilever.

Loading above $\frac{n}{n+1}$ of the shakedown limit

The results of the previous section require the load variations to be within $\frac{n}{n+1}$ of the shakedown load. Ainsworth, (1977), noted the necessity to examine structural behaviour beyond this limit in circumstances of severe thermal straining, as opposed to purely mechanical loading which may be accommodated by suitable design.

By definition, it is now impossible to find an admissible stress of the form $\sigma_{ij}^{g} = \hat{\sigma}_{ij} + \bar{\rho}_{ij}$ which does not violate yield during a cycle, and so Ainsworth defined and discussed a "cyclic plasticity solution", by virtue of which he was able to extend Ponter's bounds. The cyclic plasticity solution is defined for a body without creep; two models were considered - perfect plasticity and linear kinematic strain-hardening plasticity. In the perfectly plastic model, the plastic strain rates are periodic and so either reverse plasticity or ratchetting occurs; for a hardening body the plastic strains are periodic and ratchetting is excluded. Stress rates in both materials are periodic, and in view of these properties, Ainsworth formulated the following bound:

$$\int_0^T \int_S R_i \dot{u}_i \, ds dt \leq \int_0^T \int_S R_i \dot{u}_i^* \, ds dt + \frac{1}{n} \left(\frac{n}{n+1}\right)^{n+1} \int_0^T \int_V \dot{D} \left(\frac{\sigma_{ij}^*}{\sigma_o}\right) dv dt \qquad (82)$$

in which $R_i(t)$ is a cyclic dummy load, and starred terms refer to the cyclic plasticity solution for loading $P_i(t) + R_i(t)$. [$P_i(t)$ is the real load on the actual creeping body.]

For loading below the restricted limit, (82) reduces to (80).

Result (82) yields a displacement bound, a work bound or a creep dissipation bound depending on the choice of R_i. In the latter, the unique, optimum solution again provides an admissible accumulation of inelastic strain over a cycle, and the bound again becomes exact for rapid cycling.

More recently, Ainsworth has included the transient response leading to the cyclic stationary state, by adding the elastic term [A(o) - A(T)], where $A(t) = \int_V E_c(\sigma_{ij}^* - \sigma_{ij})dv$, to the bound in (82).

The theorems were applied to three structures: a beam, a tube and a two-bar body, and all were considered to be subjected to constant mechanical loads and cyclic thermal strains. Good estimates were obtained for the actual behaviour for realistic cycle times and Ainsworth deduced that the optimum stress in the upper bound is approximated in many practical situations, as Ponter observed for cases of loading below the shakedown limit.

6. Bounding theorems for general materials

Thus far, the most general bounding theorems have covered cases of constant and variable loading with imposed strains, in which the total mechanical strain has been expressed as

$$\varepsilon_{ij} = e_{ij} + p_{ij} + v_{ij} \quad . \tag{83}$$

Specific properties of elasticity, plasticity and creep, such as positive definite strain energy, maximum work and convexity of the dissipation function have been employed in formulating the bounds, of which (75), (80) and (82) are primary examples.

Ponter (1974) put forward a more general approach in which the material description remains unspecified in the bounds. Earlier results were obtained as special cases, and in a later section herein, new results are given for a general linear viscoelastic material.

The total strain is rewritten as

$$\varepsilon_{ij} = e_{ij} + \varepsilon'_{ij} + \delta_{ij} \tag{84}$$

in which ε'_{ij} are stress-history dependent inelastic strains and δ_{ij} are externally induced. In formulating the theorems, Ponter required that there exists an energy-type functional which possesses an upper bound amongst all possible stress histories $\sigma_{ij}(t)$:

$$W(s_{ij}(t),\sigma_{ij}(t)) = \int_o^T (s_{ij}(t) - \sigma_{ij}(t))\, \dot{\varepsilon}'_{ij}(t)dt \leq w(s_{ij}(t),T) \tag{85}$$

where $s_{ij}(t)$ is a prescribed history of stress.

A work bound and a displacement bound were obtained; it may be noted that no specific constitutive relations are assumed:

a) an upper bound on the total inelastic work: setting

$$s_{ij}(t) = \mu\sigma_{ij}^{*}(t) \quad \text{where} \quad \mu > o \quad \text{and} \quad \sigma_{ij}^{*} = \hat{\sigma}_{ij}(t) + \bar{\rho}_{ij}(x_i) \ ,$$

it was shown that

$$\int_V \int_o^T \sigma_{ij}\dot{\varepsilon}'_{ij}dtdv \leq \frac{\mu}{\mu - 1}\{A(o) - A(T)\} + \frac{1}{\mu - 1}\int_V w(\mu\sigma_{ij}^{*}(t),T)dv \tag{86}$$

b) an upper displacement bound: setting

$$s_{ij}(t) = \sigma_{ij}^{**}(t) = \hat{\sigma}_{ij}(t) + \hat{\sigma}_{ij}^{T_i} + \bar{\rho}_{ij}(x_i) \quad ,$$

where $\hat{\sigma}_{ij}^{T_i}$ is the elastic solution for constant dummy load T_i, it follows that

$$\int_{Sp} T_i(u_i(T) - u_i(o))ds \leq A(o) - A(T) + \int_{Sp} T_i(\hat{u}_i(T) - \hat{u}_i(o))ds + \int_{Sp} w(\sigma_{ij}^{**}(t),T)dv \tag{87}$$

It may be observed that (86) and (87) provide generalisations of (75) and (80) respectively.

In the application of these theorems to particular materials, w must be evaluated. Ponter (1975) discussed the perfectly plastic model with an associated flow rule: provided $f(s_{ij}) \leqslant 0$, w is zero, otherwise w is unbounded. The bounds recover previous results established by Koiter (1960) and Ponter (1972). Ponter went on to use w in the description of the dynamic loading of an elastic, perfectly-plastic body and he recovered the results of Martin where the static elastic solution to an associated problem had been employed. The method applied to dynamic loading also permitted the formulation of general bounds in terms of dynamic elastic solutions; in particular, an example of an impulsively loaded elastic, perfectly-plastic beam was given, for which bounds were calculated for the maximum and time-averaged deflections.

<u>The nature of the functional $w(s_{ij}(t),T)$</u>

It was observed previously that the maximum complementary work, $\bar{\omega}(s_{ij})$ is also zero for a perfectly-plastic material, provided $f(s_{ij}) \leqslant 0$. Ponter (1975) examined the connection between the two functionals, w and $\bar{\omega}$.

It transpires that for materials that are stable in the Drucker sense, that is:

$$d\sigma_{ij}\, d\varepsilon_{ij} \geqslant 0 \qquad , \tag{88}$$

then

$$w(s_{ij},T) = \bar{\omega}(s_{ij}) \tag{89}$$

provided s_{ij} is constant in time and if $\sigma_{ij}(o) = 0$, $\sigma_{ij}(T) = s_{ij}$ and $\varepsilon_{ij}(o) = 0$.

On the other hand, if W is defined in (85) in terms of elastic strains, W is in general unbounded, despite (88) being satisfied, and so a distinction exists in Ponter's formulation between elastic and inelastic stable materials.

For a perfectly-plastic material with a non-associated flow rule, W is bounded provided $s_{ij}(t)$ lies within an effective yield surface formed by the envelope of hyperplanes passing through the actual yield surface which are orthogonal to the plastic strain increments at points of contact - thus W exists for a material which may be unstable in the Drucker sense.

For isotropic hardening plasticity, W takes the value of the maximum complementary work to the stress at which the yield surface $f(s_{ij}(t))$ achieves its maximum value.

In the case of linear kinematic hardening plasticity, the result is that W is bounded above: $W \leq \frac{1}{2} C'_{ijk\ell}\varepsilon_{ij}^{*} \varepsilon_{k\ell}^{*}$ provided that there exists a strain $\varepsilon_{k\ell}^{*}$ such that

$$f(s_{ij}(t) - C'_{ijk\ell}\varepsilon_{k\ell}) \leq 0 , \quad 0 \leq t, \leq T .$$

The condition requires s_{ij} to satisfy the yield condition corresponding to $\varepsilon_{k\ell}^{*}$; by suitable definition of $\varepsilon_{k\ell}^{*}$ it again becomes possible to identify W with a particular value of the maximum complementary work functional.

Stationary state creep, described by the flow rule

$$\dot{v}_{ij} = \frac{\partial}{\partial \sigma_{ij}} G(\sigma_{ij}) ,$$

where

$$G = G(\phi) = k \frac{\phi^{n+1}}{n+1} ,$$

results in W taking the form

$$W = \int_0^T \frac{k}{n} \left(\frac{n}{n+1}\right)^{n+1} \{\phi(s_{ij})\}^{n+1} dt$$

which is again identical to the maximum complementary work if s_{ij} is constant.

For viscoplasticity described by the constitutive relation

$$\dot{\varepsilon}_{ij} = k \frac{\partial}{\partial \sigma_{ij}} \left[\frac{\{\phi(\sigma_{ij}) - \sigma_o\}^{n+1}}{n+1} \right] \quad , \quad \phi > \sigma_o$$

$$= 0 \quad , \quad \phi \leqslant \sigma_o$$

then w is given by

$$w = \int_o^T \frac{k}{n} \left(\frac{n}{n+1}\right)^{n+1} \{\phi(s_{ij}) - \sigma_o\}^{n+1} dt \quad , \quad \phi(s_{ij}) > \sigma_o$$

$$= 0 \quad , \quad \phi(s_{ij}) \leqslant \sigma_o$$

The nature and application of $w(s_{ij}(t),T)$ in the case of linear viscoelasticity is discussed in detail in a separate section.

7. Fundamental material inequalities

All of the bounding theorems discussed in this section have depended implicitly upon some postulate concerning fundamental material behaviour. Principal amongst such postulates have been the contributions of Hill (1950) and Drucker (1951), and these and others are discussed in a general way in terms of properties of functions of stress and strain or strain-rate.

Several variational principles and bounding theorems have been presented for a variety of boundary value problems and material types. It is of interest to extract from the derivations of these results what appear to be the common fundamental properties that are necessary for the generation of such results, and examine them in search of an underlying theoretical core. Existing attempts include the work of Hill (1956, 1968 a and b), Drucker (1967) and Iliushin (1961), but no clear agreement has been reached on the nature or identity of such a core.

There appear to be four distinct bases for the derivations discussed in this section; these are

i) the convexity of a function

ii) the existence of a positive definite function

iii) a geometrical property of a surface, distinct from convexity

iv) the Schwartz inequality.

The efforts of the authors mentioned above have been to provide physically reasonable and mathematically consistent grounds for these bases. Drucker's work (1951) concerns the existence of a non-negative work quantity arising from a cycle of loading. This may be shown to lead to an associated flow rule and to a convex yield function as necessary consequences.

Iliushin's postulate (1961) concerns cycles of deformations and, unlike Drucker's, distinguishes between elastic and perfectly-plastic behaviour. Normality and convexity also follow as consequences, although the latter holds only for isotropic media.

Hill (1968) defines conjugate stress and strain increment variables in terms of their product, which represents an increment of work done in the material. In terms of these variables he discusses the postulates of Drucker and Iliushin, and Martin's notion (1966) of maximum complementary work. Hill points out that the validity of a postulate hinges on how such variables are measured; that is, with respect to what configurations their values are determined. He concludes, however, that normality is assured independently of the chosen postulate, and in most practical circumstances convexity also holds. He also demonstrates that logarithmic strain is generally the most suitable measure.

More recently, in a paper by Ponter, Kestin and Bataille (1979), the postulates of Drucker and Iliushin are shown to be essentially non-thermodynamic in nature; in particular, the property of normality of the plastic strain increment is shown to arise from particular properties of dislocations which produce an average, macroscopic behaviour, whereas the nature of the yield surface is determined by local (small-scale) material properties.

For linear viscous and viscoelastic materials, Kestin (1966) and Biot (1958) have considered the consequences of Onsager's reciprocal relations concerning irreversible thermodynamic processes.* The following points emerge:

a) Onsager's relations are applicable to these materials but not to plastic materials;

b) the coefficients in the stress-strain, or stress-strain-rate relations are symmetric in the sense that if $\varepsilon_{ij} = C_{ijk\ell}\, \sigma_{k\ell}$, then $C_{ijk\ell} = C_{k\ell ij}$ by virtue of Onsager's relations.

Ziegler (1966) shows that a further consequence of Onsager's relations is that the strain-rate, $\dot{\varepsilon}_{ij}$, in a purely dissipative material, is normal to the dissipation surface in stress space: $\dot{D}(\sigma_{ij}) = \sigma_{ij}\, \dot{\varepsilon}_{ij}$. The distinction between plasticity and viscosity appears as a difference in character between plastic yield surfaces and surfaces of constant dissipation rate for viscous materials: Onsager's relations require the dissipation rate surface to be definable both in terms of σ_{ij} and $\dot{\varepsilon}_{ij}$: this is not possible in plasticity.

Stability of an elastic material is demonstrated thermodynamically (Washizu, 1975) by consideration of the state function A, where $dA = \sigma_{ij} de_{ij}$. "Stability" is used in the sense that in straining, energy is absorbed by the material and is not extractable simultaneously from the straining agency and the material.

* Onsager's relations are not defined in the non-linear case, although Ziegler (1966) shows that certain consequences of the relations which may be derived in the linear case also have physical meaning in the general case.

Another view of material stability, following from the properties of scalar functions of stress and strain or strain-rate is now discussed.

We suppose that there exist functions $f = f(\sigma_i)$ and $g = g(\varepsilon_i)$ of nine-component stress and strain or strain-rate tensors respectively, which are differentiable with continuous third partial derivatives.*

The Taylor series for $f(\sigma_i)$ may be written as follows:

$$f(\overset{2}{\sigma}_i) = f(\overset{1}{\sigma}_i) + (\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)\frac{\partial f}{\partial \overset{1}{\sigma}_i} + \frac{1}{2}(\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)\frac{\partial^2 f}{\partial \overset{1}{\sigma}_i \partial \overset{1}{\sigma}_j}(\overset{2}{\sigma}_j - \overset{1}{\sigma}_j) + A(\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)^3 \tag{90}$$

where $\frac{\partial}{\partial \overset{1}{\sigma}_i}$ denotes the value of the partial derivative at $\sigma_i = \overset{1}{\sigma}_i$, and where $A(\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)^3$ approaches zero at least as fast as $(\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)^3$ when $\overset{2}{\sigma}_i \to \overset{1}{\sigma}_i$.

The symmetric matrix $\frac{\partial^2 f}{\partial \overset{1}{\sigma}_i \partial \overset{1}{\sigma}_j}$ is called the Hessian matrix of $f(\sigma_i)$ at $\sigma_i = \overset{1}{\sigma}_i$; the term

$$(\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)\frac{\partial^2 f}{\partial \overset{2}{\sigma}_i \partial \overset{1}{\sigma}_j}(\overset{2}{\sigma}_j - \overset{1}{\sigma}_j)$$

is the quadratic form associated with this Hessian matrix.

The function $f(\sigma_i)$ is said to be convex if, for any pair $\overset{2}{\sigma}_i$ and $\overset{1}{\sigma}_i$,

$$f(\overset{2}{\sigma}_i) - f(\overset{1}{\sigma}_i) \geq (\overset{2}{\sigma}_i - \overset{1}{\sigma}_i)\frac{\partial f}{\partial \overset{1}{\sigma}_i} \tag{91}$$

Assuming $f(\sigma_i)$ to be sufficiently smooth, it is evident from (90) that a necessary and sufficient condition for convexity is that the quadratic form associated with $\frac{\partial^2 f}{\partial \overset{1}{\sigma}_i \partial \overset{2}{\sigma}_i}$ is positive semi-definite. [Hill (1956)]. If

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*For notational brevity we write ε_i to represent both strain and strain-rate; the selection is made clear in context.

the quadratic form is positive definite, strict inequality holds in (91) for all $\sigma_i^2 \neq \sigma_i^1$ and $f(\sigma_i)$ is strictly convex.

We now choose various forms for the functions $f(\sigma_i)$ and $g(\varepsilon_i)$ and examine the consequences of the expressions (90) and (91).

i) Suppose $f(\sigma_i)$ is a convex potential function, in the sense that

$$\frac{\partial f}{\partial \sigma_i} = \varepsilon_i \quad . \tag{92}$$

If we choose $\sigma_i^2 = \sigma_i^1 + \delta\sigma_i$, we obtain from (90) and (92)

$$f(\sigma_i^2) - f(\sigma_i^1) = (\sigma_i^2 - \sigma_i^1)\frac{f}{\partial\sigma_i^1} + \frac{1}{2}\delta\sigma_i \frac{\partial\varepsilon_i}{\partial\sigma_j}\delta\sigma_j + A(\delta\sigma_i)^3 \tag{93}$$

and the convexity expression, (91), implies that

$$\delta\sigma_i \frac{\partial\varepsilon_i}{\partial\sigma_j}\delta\sigma_j = \delta\sigma_i\delta\varepsilon_i \geq 0 \tag{94}$$

ii) If $g(\varepsilon_i)$ is a convex potential function in the sense that

$$\frac{\partial g}{\partial \varepsilon_i} = \sigma_i \quad , \tag{95}$$

the same result follows. This may be expressed as

$$\delta g(\varepsilon_i) = \sigma_i\delta\varepsilon_i + \frac{1}{2}\delta\sigma_i\delta\varepsilon_i + 0(\delta)^3 \quad . \tag{96}$$

A critical point is defined by $\frac{\partial g}{\partial \varepsilon_i} = 0$, that is $\sigma_i = 0$. If $g(\varepsilon_i)$ is regarded as the total-potential function, and the set ε_i is taken to represent both the internal and external kinematic variables, this condition is identical to the equations of equilibrium. Furthermore, from (96), the equilibrium is stable if

$$\delta\sigma_i\delta\varepsilon_i \geq 0 \; ; \tag{97}$$

that is, if $g(\varepsilon_i)$ is convex.

iii) Similarly, if $f(\sigma_i)$ is regarded as the total complementary energy, and $\frac{\partial f}{\partial \sigma_i} = 0$, the compatibility equations are obtained.

iv) The notion of a stable material may be obtained from (96): if $g(\varepsilon_i)$ is a convex potential function, then $\delta g \geq \sigma_i \delta\varepsilon_i$, and taking ε_i to represent quantities of strain, δg represents the actual increment of absorbed energy, which exceeds the increment that would occur if the stress remained constant during the incremental strain. It is thus possible to write $\delta g = \sigma_i' \delta\varepsilon_i$, where σ_i' represents a mean stress during the change in strain, and so $\sigma_i'\delta\varepsilon_i \geq \sigma_i\delta\varepsilon_i$. Stability is assured in this sense; that is, when an increment of energy is absorbed there is an overall increase in the magnitudes of the components of current stress in the directions of the strain increments, and the material may be said to resist the change.*

v) The case $f(\sigma_i^2) \leq f(\sigma_i^1)$ results in $(\sigma_i^2 - \sigma_i^1)\varepsilon_i^1 \leq 0$, on application of (93). When $f(\sigma_i)$ represents a convex yield function in deformation theory plasticity with an associated flow rule, the result is a generalisation of that given by Sadowsky, (29).

vi) We may include incremental plasticity if we suppose that $f(\sigma_i)$ is a convex potential-type function in the sense that

$$\delta\varepsilon_i = \delta\lambda \frac{\partial f}{\partial \sigma_i} \quad , \tag{98}$$

where $\delta\lambda > 0$ for $\delta\varepsilon_i \neq 0$.

From (90), we obtain,

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*We may equivalently express this in terms of effective stress and effective strain increment.

$$f(\sigma_i^2) = f(\sigma_i^1) + \frac{(\sigma_i^2 - \sigma_i^1)}{\delta\lambda}\delta\varepsilon_i + \frac{1}{2}(\sigma_i^2 - \sigma_i^1)\frac{\partial^2 f}{\partial\sigma_i^1\partial\sigma_j^1}(\sigma_j^2 - \sigma_j^1) + A(\sigma_i^2 - \sigma_i^1)^3. \tag{99}$$

If we choose

$$\sigma_i^2 = \sigma_i^1 + \delta\sigma_i \quad , \tag{100}$$

which represents loading into the yield surface when $f(\sigma_i^2) \geqslant f(\sigma_i^1)$, then on noting the convexity expression, (91), we obtain

$$\delta\sigma_i\delta\varepsilon_i \geqslant 0 \tag{101}$$

which is discussed in the context of material stability by Drucker.

On the other hand, we may choose σ_i^1 and σ_i^2 such that

$$f(\sigma_i^1) \geqslant f(\sigma_i^2) \quad . \tag{102}$$

On noting (91), equation (99) gives

$$f(\sigma_i^1) - f(\sigma_i^2) = \frac{(\sigma_i^1 - \sigma_i^2)}{\delta\lambda}\ \delta\varepsilon_i - \alpha - A(\sigma_i^2 - \sigma_i^1)^3$$

where α is a positive definite quadratic form, and from (102) we obtain

$$(\sigma_i^1 - \sigma_i^2)\ \delta\varepsilon_i \geqslant 0 \tag{103}$$

Expressions of this type, due to Hill and Drucker, were used extensively in formulating bounds for plastic materials.

In summary, we have included descriptions of elasticity, deformation theory plasticity and, by analogy, stationary creep by defining $f(\sigma_i)$ through equation (92). Incremental plasticity is described by equation (98).

It is evident that a range of material inequalities, appertaining to the notion of material stability, may be derived for those materials for which an appropriate smooth convex potential function may be defined.

APPENDIX 1.1

Notation in general use in Section 1

$A(t) = \int_V \frac{1}{2} C_{ijk\ell}(\sigma_{ij}(t) - \sigma_{ij}^{s}(t))(\sigma_{k\ell}(t) - \sigma_{k\ell}^{s}(t))dv$ = elastic complementary energy expression.

C : constant

$C_{ijk\ell}$: fourth-order elastic-constants tensor

D^t : creep energy dissipation at time t

$E(\varepsilon_{ij})$: positive definite strain-energy density function, defined as $\int \sigma_{ij} d\varepsilon_{ij}$

$E_c(\sigma_{ij})$: complementary energy density function, defined as $\int \varepsilon_{ij} d\sigma_{ij}$

E^* : elastic energy redistribution factor

e_{ij} : elastic strain tensor

e^r_{ij} : elastic strain corresponding to residual stress, $\dot{\rho}_{ij}$

$\dot{e}_{ijo}$: elastic strain rate corresponding to residual stress rate $\dot{\rho}_{ijo}$

$f(\sigma_{ij})$: yield function

G : flow potential

$G(t-\tau)$: relaxation function

h : hardening coefficient

$H(t)$: Heaviside function

k : yield stress in pure shear

$k(t)$: time-weighting function in creep law

n : creep index

n_j : unit outward normal to a surface

P_i : externally applied surface traction

P_L : limit load

p_{ij} : plastic strain tensor

p'_{ij} : plastic strain deviator

Δp_{ijo} : kinematically admissible accumulation of plastic strain over a cycle of deformation

$\dot{p}_{ijo}$: strain rates corresponding to Δp_{ijo}

R_i : dummy load

S : safety factor

S_u : surface with prescribed displacements, u_i^o

S_p : surface with prescribed tractions, P_i

s_{ij} : prescribed stress field, unless defined in context as the stress deviator, $\sigma_{ij} - \frac{1}{3}\sigma_{ii}\delta_{ij}$

t : generic time

T : prescribed final time

$t = 0^+$: completion of elastic response to step loading

t_i : unit dummy load

T_i : dummy load

u_i : displacement vector

U_p : total potential energy

U_c : total complementary energy

$\dot{u}_{io}$: displacement rate compatible with strain rate $(\dot{p}_{ijo} + \dot{e}_{ijo})$

$\hat{U}_p$: total potential energy in associated elastic body

$\hat{U}_c$: total complementary energy in associated elastic body

v_{ij} : creep strain tensor

W : functional of stresses σ_{ij} and s_{ij} (or σ_{ij}^*), unless defined in the context as $\int \sigma_{ij} d\varepsilon_{ij}$

W' : inelastic work quantity

$\bar{W}$: complementary energy density $\int \varepsilon_{ij} d\sigma_{ij}$

w : maximum value of the functional W

W_e : total work done by external forces

W_i : total work done by internal forces

x_i : position coordinate

X_i : externally applied body force (per unit volume)

δ_{ij} : Kronecker delta tensor

ε_{ij} : total strain tensor

$\varepsilon_{ij}^{c}, u_i^{c}$: compatible set

ε_{ij}^{*} : strain corresponding to σ_{ij}^{*}

$\hat{\varepsilon}_{ij}$: strain corresponding to $\hat{\sigma}_{ij}$

$\dot{\varepsilon}_{o}$: material constant

ε_{ij}^{s} : stationary state strain

ε_{ij}^{m} : strain value on an extremal path

$\dot{\varepsilon}'_{ij}$: inelastic strain rate

μ : proportionality factor

ρ_{ij} : residual stress field

$\dot{\rho}_{ijo}$: residual stress rate corresponding to $\dot{p}_{ijo}$

$\bar{\rho}_{ij}$: constant residual stress

σ_{ij} : stress tensor

σ_{ij}^{*}, P_i^{*} : equilibrium set. σ_{ij}^{*} is also used for the stationary plastic creep solution

σ_{ij}^{c} : stress corresponding to ε_{ij}^{c}

σ_{y} : yield stress in pure tension

σ : equal to $\frac{1}{3}\sigma_{ii}$

$\hat{\sigma}_{ij}$: stress in a purely elastic body

σ_{ijo} : stress corresponding to $\dot{p}_{ijo}$

σ_{o} : material constant

$\sigma_{e}, \bar{\sigma}$: effective stress, defined as $\sqrt{\frac{3}{2} s_{ij} s_{ij}}$

σ_{ij}^{s} : stationary state solution

σ_{ij}^{m} : stress value on an extremal path

σ_{ij}^{g} : general equilibrium stress

$\phi\left(\frac{\sigma_{ij}}{\sigma_{o}}\right)$: homogeneous function

ϕ_{v}, ϕ_{s} : potential functions for external forces

$\bar{\omega}$: maximum complementary energy density

ω : minimum work density

SECTION 2

BOUNDS BASED ON A THERMODYNAMICAL APPROACH TO AN INTERNAL STATE DESCRIPTION OF MATERIAL BEHAVIOUR

CHAPTER 1

INTRODUCTION

A considerable amount of work is being carried out in industry to develop creep constitutive equations for use in numerical solutions for structural problems at high temperatures, particularly in the context of nuclear and gas turbine design. Much of the work is not found in the open literature as the cost of generating material data discourages publication. There are, however, three major groups that we know to be working in related fields:

1. Oak Ridge Laboratories of the United States Atomic Energy Authority are developing constitutive equations based upon Rice (1970) and Ponter and Leckie (1976).

2. Constitutive equations for gas turbine design are being developed by Chaboche and others at O.N.E.R.A. in Paris, based upon the doctorate work of Chaboche (e.g. Chaboche 1977).

3. Some development of a constitutive equation due to Besselling* is taking place in the Nuclear Research Laboratories of T.N.O. in Belgium.

A summary of recent developments and an attempt to understand their similarities and dissimilarities has been given by Chaboche and Rousselier (1982a and b). The details of these models are complex because of the need to model material data as precisely as possible. The approach taken by all three groups involves the use of internal state variables which are close to the models discussed in this section of the thesis.

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* A description of Beselling's equation in the context of the work in this thesis is given by Ponter (J.de Mec., 1976).

The purpose of this section is to attempt to give some insight into the behaviour of a class of constitutive equations by using a thermodynamic formalism as the basis of discussion. We find that a number of properties are required for the existence of bounding results, but the bounds themselves appear very insensitive to the details of the constitutive relationship - in other words the same or very similar results exist for a range of relationships, provided they all yield identical results for a class of simple material tests, namely creep tests conducted at constant stress and temperature.

The bounds do not provide particularly detailed information about the creep deformation of a structure. At most, they yield the displacement of a particular point and the best results are usually for the average deflection rate of the applied load. The bounds are, however, very simple in form and they indicate that the overall deformation of the structure may not depend strongly on the details of the material behaviour. This suggests that the considerable expenditure on the development of constitutive equations for industry, which produces superficially dissimilar equations derived from considerable material data bases in different laboratories will produce solutions to structural problems which may well differ only in detail.

In Chapter 2 we give the thermodynamic formalism used as the basis for the theory, in which two scalar functions, the free energy function and the rate potential describe the capacity of the material to internally store and dissipate energy. If the rate potential is convex, this is shown to be sufficient for the stress and internal state histories to converge to values which are independent of their initial values. This is followed in Chapters 3 and 4 by the derivation of general bounds for a class of rate potentials, using the structure of the thermodynamic formalism.

In Chapter 5, properties of the stationary cyclic condition are used to obtain bounds on the cyclic work done on the body. The importance of the rapid cycle solution emerges naturally and gives a simple physical meaning to the bounding results. This is supported by the results of bounds calculated for linear viscoelastic models. Work bounds are also obtained for the Bailey-Orowan material and the non-linear viscous model.

A displacement bound is derived in Chapter 6. This bound has a particularly simple form, being very insensitive to detailed material properties and requiring for its determination two plasticity solutions and the result of a single creep test at constant stress. Examples are given to illustrate the use of the bound and we conclude by comparing our results for the Bree problem with those of O'Donnell which are in current use in industrial design.

CHAPTER 2

A THERMODYNAMIC APPROACH TO CONSTITUTIVE RELATIONS INVOLVING INTERNAL VARIABLES

In this section we construct the description of the current mechanical behaviour of an inelastic material by supposing that we may determine the current values of a sufficient number of internal variables so that a knowledge of the previous history of stress and deformation is not required. The chosen internal variables are deformation - like in character and they are related to corresponding force-like variables or "affinities" through the free energy function ψ . The evolution of the internal variables is determined by the nature of an assumed rate potential, Ω . Properties of this potential result in the second law being satisfied and also permit the formulation of general work and displacement bounds. A circumstance is discussed in which the potential may be expressed in a simplified form in terms of a single characteristic internal variable.

2.1 Initial Assumptions

We suppose that the total strain, $\underline{\varepsilon}$, consists of an instantaneously recoverable component, $\underline{e}$, a thermal expansion component $\underline{\varepsilon}^{\theta}$ and a component $\underline{\varepsilon}'$ such that

$$\underline{\varepsilon} = \underline{e} + \underline{\varepsilon}^{\theta} + \underline{\varepsilon}' \tag{2.1}$$

The stress in the material is linearly related to the strain component $\underline{e}$ through the equations

$$\left.\begin{aligned} \underline{\sigma} &= \underset{\sim}{c}\, \underline{e} \\ \underline{e} &= \underset{\sim}{c}^{-1}\, \underline{\sigma} \end{aligned}\right\} \tag{2.2}$$

where $\underset{\sim}{c}$ is a fourth-order elastic constants tensor and $\underset{\sim}{c}^{-1}$ is its multiplicative inverse.

The thermal strain is given by

$$\underline{\varepsilon}^{\theta} = \frac{1}{3}\gamma\,(\theta-\theta_o)\underline{\delta} \tag{2.3}$$

where γ is the cubic expansivity, θ is the current temperature and θ_o is the initial temperature.

We assume that there exists a stress-free state known as the "ground state" in which the inelastic strain rate $\dot{\underline{\varepsilon}}'$ vanishes. The following argument concerns deviations from this state.

In order to describe the current material behaviour without a knowledge of its previous history, we suppose that in addition to the strain components we can evaluate a set of internal variables β_i (i=1,2,...). The strain components, the internal variables and the temperature θ are assumed to be sufficient to determine the current free energy per unit volume, ψ, which is defined as follows:

$$\begin{aligned}\psi &= \psi(\underline{\varepsilon},\underline{\varepsilon}',\beta_i,\theta)\\ &= \frac{1}{2}\,\underset{\sim}{c}(\underline{\varepsilon} - \underline{\varepsilon}' - \frac{1}{3}\gamma\,\underline{\delta}(\theta-\theta_o))(\underline{\varepsilon} - \underline{\varepsilon}' - \frac{1}{3}\gamma\,\underline{\delta}(\theta-\theta_o))\\ &\qquad + f(\beta_i) + g(\theta)\end{aligned} \tag{2.4}$$

where

$$f = \frac{1}{2}\,B_{ij}\,\beta_i\,\beta_j \tag{2.5}$$

and B_{ij} are constants.* Specifically the β_i terms may represent quantities associated with the displacement discontinuities in the material such as dislocations, and if linear elastic behaviour is assumed then $f(\beta_i)$ takes the form of eqn.(2.5).

A sufficient condition for the transformation of f into canonical form with real non-negative coefficients is $f \geq 0$, equivalent to

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* In this context β_i are conceived as thermodynamical variables so that we assume that there exists a conceptual, if not realisable reversible path from the ground state to the current state. See, e.g., Ponter, Bataille and Kestin 1979.

requiring the ground state to be a state of minimum free energy. We may then write f in the form

$$f = f(\alpha_1) = \frac{1}{2} c_i \alpha_i^2 \tag{2.6}$$

where c_i are the positive eigenvalues of the matrix B_{ij} (Courant and Hilbert 1953). From Eqns.(2.4) and (2.6) the rate of change of ψ is given by

$$\dot{\underline{\psi}} = \frac{\partial \psi}{\partial \underline{\varepsilon}} \dot{\underline{\varepsilon}} + \frac{\partial \psi}{\partial \underline{\varepsilon}'} \dot{\underline{\varepsilon}}' + \sum_i \frac{\partial \psi}{\partial \alpha_i} \dot{\alpha}_i + \frac{\partial \psi}{\partial \theta} \dot{\theta}$$

$$= \underline{\sigma}\, \dot{\underline{\varepsilon}} - \underline{\sigma}\, \dot{\underline{\varepsilon}}' + \sum_i \frac{\partial f}{\partial \alpha_i} \dot{\alpha}_i - \xi \dot{\theta} \tag{2.7}$$

where ξ is the entropy per unit volume.*

If we define $\frac{\partial f}{\partial \alpha_i}$ as follows:

$$A_i = \frac{\partial f}{\partial \alpha_i}$$

then from Eqn.(2.6)

$$A_i = c_i \alpha_i \tag{2.8}$$

and from Eqn.(2.7)

$$A_i = \frac{\partial \psi}{\partial \alpha_i} \tag{2.9}$$

and also

$$\underline{\sigma} = - \frac{\partial \psi}{\partial \underline{\varepsilon}'} \tag{2.10}$$

The force quantities A_i and $\underline{\sigma}$ thus defined are the "affinities" associated with the deformation quantities α_i and $\underline{\varepsilon}'$ through the free energy function.

Eqn.(2.4) may be written in the form

$$\psi = \psi_e(\underline{\varepsilon} - \underline{\varepsilon}' - \underline{\varepsilon}^\theta) + \psi_1 (\alpha_i) + g(\theta) \tag{2.11}$$

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* The entropy is given by $\xi = - \frac{\partial \psi}{\partial \theta}$ and thus from Eqn.(2.4), $\xi = - g'(\theta) + \frac{1}{3} \underline{\sigma} \underline{\delta} \gamma$. This coincides with the treatment by Ponter (1979) given in terms of the Gibbs potential rather than the free energy.

where ψ_e is the component of ψ that is instantaneously recoverable by removal of the applied forces and ψ_1 is the "latent energy"*, an accumulation of elastic strain energy that is locked into the material during inelastic straining but which is recoverable, at least in principle.

The rate of energy dissipation per unit volume in the body is given by the following:

$$D = \underline{\sigma}\ \dot{\underline{\varepsilon}}' - \dot{\psi}_1 \tag{2.12}$$

If D is non-negative then the second law is satisfied: the proof is given in Appendix 2.1, together with a derivation of Eqn.(2.12).

If we regard the components of $\dot{\underline{\varepsilon}}'$ and the set $\dot{\alpha}_i$ as components of a single flux vector, $\dot{\underline{\eta}} = [\dot{\underline{\varepsilon}}', \dot{\alpha}_i]$, and similarly $\underline{\sigma}$ and A_i as components of a single affinity vector $\underline{\Sigma} = [\underline{\sigma}, -A_i]$, then the dissipation rate takes the form of the scalar product

$$D = \underline{\Sigma}\ \dot{\underline{\eta}} \tag{2.13}$$

2.2. The Rate Potential, Ω

To complete the basic material description we require a statement on the evolution of the fluxes $\dot{\underline{\eta}}$. To this end we adopt the following hypothesis: the flux vector $\dot{\underline{\eta}}$ is the gradient of a potential $\Omega(\underline{\Sigma})$:

$$\dot{\underline{\eta}} = \frac{\partial \Omega}{\partial \underline{\Sigma}} \tag{2.14}$$

Eqn.(2.14) may be expressed as

$$\dot{\underline{\varepsilon}}' = \frac{\partial \Omega}{\partial \underline{\sigma}} \tag{2.15}$$

and

$$\dot{\alpha}_i = -\frac{\partial \Omega}{\partial A_i} \tag{2.16}$$

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* A term used by Taylor and Quinney in their experimental work, (1934 and 1937), in which they showed that from 5 to 15% of the input work remained in the body while the rest appeared as heat.

the latter of which becomes, on account of Eqn.(2.8),

$$\dot{\alpha}_i = -\frac{1}{c_i}\frac{\partial\Omega}{\partial\alpha_i} \tag{2.17}$$

where Ω is regarded as a function of $\underline{\sigma}$ and α_i .

Eqn.(2.15) was obtained by Rice [1970] for the inelastic deformation resulting from dislocation motion in polycrystalline materials. Eqn.(2.16) is consistent with Schmid's law* if we take the rates $\dot{\alpha}_i$ to represent the velocities of dislocation motion, and the affinities A_i to represent the conjugate shear stresses.

The set α_i which is defined through the free energy function is exceedingly large. A simplification is achieved by assuming that the potential may be represented by a smaller number of state variables than this complete set. In the simplest case suppose that the chosen potential can be expressed as follows:

$$\Omega = \Omega(\underline{\sigma},s) \tag{2.18}$$

where s is a single state variable. The component of free energy known as the latent energy, ψ_1 , is given by

$$\psi_1 = \sum_{i=1}^{m} \frac{1}{2} c_i\alpha_i^2 \tag{2.19}$$

where m is the number of variables in the set α_i . We assume that the following relationship holds:

$$\psi_1 = \sum \frac{1}{2} c_i\alpha_i^2 = \frac{1}{2}\lambda s^2 \tag{2.20}$$

where λ is a constant. This assumption implies that the state variable s is always sufficient to determine the latent energy, while a large number of differing sets of α_i can correspond to the same value of s .

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* Consider an element of a slip plane in the material, with normal pressure p and shear stress τ acting. Schmid's law states that the velocity of dislocation motion depends only on τ and not on p . See, e.g., Van Vlack, 1970.

From Eqn.(2.20),

$$\dot{\psi}_1 = \sum_i c_i \alpha_i \dot{\alpha}_i = \lambda s \, \dot{s} \tag{2.21}$$

and from Eqns.(2.17) and (2.21) we obtain

$$\dot{\psi}_1 = - \sum_i \alpha_i \frac{\partial \Omega}{\partial \alpha_i} = - \frac{\partial \Omega}{\partial s} \sum_i \alpha_i \frac{\partial s}{\partial \alpha_i} \tag{2.22}$$

as Ω is a function of $\underline{\sigma}$ and s .

Differentiating Eqn.(2.20) with respect to α_i yields

$$c_i \alpha_i = \lambda s \frac{\partial s}{\partial \alpha_i}$$

and substituting into Eqn.(2.22):

$$\dot{\psi}_1 = \frac{1}{\lambda s} \frac{\partial \Omega}{\partial s} \sum_i c_i \alpha_i^2$$

or, on account of Eqn.(2.20):

$$\dot{\psi}_1 = - s \frac{\partial \Omega}{\partial s} \tag{2.23}$$

Finally, from Eqns.(2.21) and (2.23) we obtain

$$\dot{s} = - \frac{1}{\lambda} \frac{\partial \Omega}{\partial s} \tag{2.24}$$

As a result, it is not necessary to know the nature of the individual α_i to define the gross properties of the material. Eqns.(2.15) and (2.24) provide a simplified model of metallic behaviour which, on the micro-scale, involves the motion of very large numbers of dislocations.

2.3 Convexity of Ω

We next consider what functional properties are required of Ω . In the process of describing the inelastic behaviour of solids and fluids, Hill, as long ago as 1956, commented on the 'unifying concept' of a convex function. In the general case $\Omega = \Omega(\underline{\Sigma})$, we suppose that Ω is convex*,

* A sufficient condition for convexity of Ω is $d\underline{\Sigma} d\underline{\dot{\eta}} \geq 0$, as shown in Ponter (1976b). That paper discussed Eqn.(2.24) not from a thermodynamical viewpoint but with the intention of bringing together several types of constitutive relationship.

and thus

$$\Omega_1 \geq \Omega_2 + (\underline{\Sigma}_1 - \underline{\Sigma}_2) \left.\frac{\partial \Omega}{\partial \underline{\Sigma}}\right|_{\underline{\Sigma}_2} \tag{2.25}$$

Setting $\underline{\Sigma}_1 = \underline{\Sigma}$ and $\underline{\Sigma}_2 = 0$ and noting that we are free to assign $\Omega(0) = 0$ we obtain

$$\Omega \geq 0 \tag{2.26}$$

If we now set $\underline{\Sigma}_1 = 0$ and $\underline{\Sigma}_2 = \underline{\Sigma}$ in equality (2.25) we obtain

$$0 \geq \Omega - \underline{\Sigma} \frac{\partial \Omega}{\partial \underline{\Sigma}} \tag{2.27}$$

Combining Eqns.(2.13) and (2.14) we obtain

$$D = \underline{\Sigma} \frac{\partial \Omega}{\partial \underline{\Sigma}} \tag{2.28}$$

and so from (2.27) and (2.28)

$$D \geq \Omega \tag{2.29}$$

It is evident from inequalities (2.26) and (2.29) that the convexity of Ω is a sufficient condition to satisfy the second law.

A related result is obtained if we adopt the initial assumption that the dissipation rate D rather than the potential Ω is convex. In this case,

$$D_1 \geq D_2 + (\underline{\Sigma}_1 - \underline{\Sigma}_2) \left.\frac{\partial D}{\partial \underline{\Sigma}}\right|_{\underline{\Sigma}_2}$$

Setting $\underline{\Sigma}_1 = 0$ and $\underline{\Sigma}_2 = \underline{\Sigma}$ it follows that

$$0 \geq D - \underline{\Sigma} \frac{\partial D}{\partial \underline{\Sigma}}$$

and on substitution from Eqn.(2.28):

$$0 \geq \underline{\Sigma} \frac{\partial \Omega}{\partial \underline{\Sigma}} - \underline{\Sigma} \left[\frac{\partial \Omega}{\partial \underline{\Sigma}} + \underline{\Sigma} \frac{\partial^2 \Omega}{\partial \underline{\Sigma} \partial \underline{\Sigma}} \right]$$

or

$$\underline{\Sigma} \frac{\partial^2 \Omega}{\partial \underline{\Sigma} \partial \underline{\Sigma}} \underline{\Sigma} \geq 0 \tag{2.30}$$

and inequality (2.30) assures the convexity of Ω^*.

We thus have a choice of starting points in formulating Ω. In principle if we can make observations which lead to an understanding of the force-flux behaviour that produces dissipation in a material then we might hope to be able to deduce the form of a suitable potential, Ω. In practice it appears more fruitful to make intuitive assumptions about the form Ω may take after specifying the possible nature of the state variable or variables. At the very least, such assumptions must permit the second law to be satisfied. As observed above, a sufficient condition is that Ω is convex**. We will adopt this as a hypothesis in Chapter 3.

2.4 Convergence of the stress and internal state histories to a stationary cyclic condition

In a later chapter we will place some emphasis on the conditions in a body that is subjected to repeated cycles of loading. As a preliminary we first show that under such loading the stress and state in the body approach a stationary cyclic condition.

Two identical bodies are imagined, differing in initial stress and internal state but subjected to identical histories of loading and temperature from time $t = 0$. From the principle of virtual work,

$$\int_0^T \int_V \underline{\tilde{\sigma}}\, \underline{\tilde{\dot{\varepsilon}}}\, dt dv = 0 \tag{2.31}$$

where $\underline{\tilde{\sigma}} = \underline{\sigma}_1(t) - \underline{\sigma}_2(t)$ represents the difference in stresses between the two bodies and $\underline{\tilde{\dot{\varepsilon}}} = \underline{\dot{\varepsilon}}_1(t) - \underline{\dot{\varepsilon}}_2(t)$ represents the difference in strain-rates. Considering for simplicity the case in which the potential may be

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* Hill 1956. On the other hand it does not appear possible to prove the convexity of D from the convexity of Ω.

** It is not clear what are the <u>necessary</u> conditions on Ω so as to satisy the second law as well as to provide the basis for the derivation of the bounds that follow.

expressed in the form $\Omega = \Omega(\underline{\sigma}, s)$, the convexity condition has the following form:

$$\Omega_1 \geq \Omega_2 + (\underline{\sigma}_1 - \underline{\sigma}_2) \left.\frac{\partial \Omega}{\partial \underline{\sigma}}\right|_{\underline{\sigma}_2} + (s_1 - s_2) \left.\frac{\partial \Omega}{\partial s}\right|_{s_2}$$

Reversing the subscripts and adding we obtain

$$0 \geq [\underline{\sigma}_1 - \sigma_2] \left[\left.\frac{\partial \Omega}{\partial \underline{\sigma}}\right|_{\underline{\sigma}_2} - \left.\frac{\partial \Omega}{\partial \underline{\sigma}}\right|_{\underline{\sigma}_1} \right] + [s_1 - s_2] \left[\left.\frac{\partial \Omega}{\partial s}\right|_{s_2} - \left.\frac{\partial \Omega}{\partial s}\right|_{s_1} \right]$$

or

$$[\underline{\sigma}_1 - \underline{\sigma}_2]\ [\dot{\underline{\varepsilon}}_1' - \dot{\underline{\varepsilon}}_2'] \geq \lambda [s_1 - s_2]\ [\dot{s}_1 - \dot{s}_2]$$

or

$$\underline{\tilde{\sigma}}\ \dot{\underline{\tilde{\varepsilon}}}' \geq \lambda\, \tilde{s}\ \dot{\tilde{s}} \tag{2.32}$$

where $\tilde{s} = s_1 - s_2$.   From Eqns.(2.1),(2.2) and (2.31)

$$\tilde{A}(t) + \int_V \int_0^T \underline{\tilde{\sigma}}\ \dot{\underline{\tilde{\varepsilon}}}'\, dt dv = \tilde{A}(0) \geq 0 \tag{2.33}$$

where $\tilde{A}(t) = \frac{1}{2} \int \underset{\sim}{C}^{-1}\ \underline{\tilde{\sigma}}(t) \underline{\tilde{\sigma}}(t) dv$.   From Eqns.(2.32) and (2.33),

$$\tilde{A}(T) + \tilde{B}(T) \leq \tilde{A}(0) + \tilde{B}(0) \tag{2.34}$$

where

$$\tilde{B}(t) = \frac{1}{2} \lambda \int_V [s_1(t) - s_2(t)]^2 dv$$

The time origin $t = 0$ may be assigned arbitrarily and in consequence the quantity $\tilde{A}(t) + \tilde{B}(t)$ can never increase.   If Ω is strictly convex then equality can only occur in (2.32) when both sides vanish.   As both $\tilde{A}(t)$ and $\tilde{B}(t)$ are positive, strict inequality in (2.34) then requires both $(\underline{\sigma}_1 - \underline{\sigma}_2)$ and $(s_1 - s_2)$ to converge to zero.   As a result the stress and state histories approach a condition that is independent of their initial values and dependent only on the histories of mechanical and thermal loading.   When these histories take the form of repeated cycles of period Δt then the stress and state approach a condition in which their values also repeat in a cycle of period Δt, provided such a condition exists. This condition is known as the stationary cyclic condition*.

* A slightly stronger proof of this result may be obtained when the constitutive relation between $\underline{\sigma}$ and $\underline{\dot{\varepsilon}}'$ has a linear form. This is discussed in Appendix 2.2.

CHAPTER 3

AN ENERGY INEQUALITY

We now show that the constitutive relations discussed in Chapter 2 permit the derivation of an energy inequality. This inequality leads to the formulation of general work and displacement bounds which are given in Chapter 4.

A simple form of the rate potential Ω which involves a single state variable s is the following:

$$\Omega = F[\phi(\underline{\sigma}) - s] + G(s) \tag{3.1}$$

where (i) ϕ is a homogeneous function of degree one in the components of $\underline{\sigma}$;

(ii) F is convex in $(\underline{\sigma}, s)$ space and is homogeneous of degree $p + 1$ in ϕ and s ;

(iii) G is convex and homogeneous of degree $q + 1$ in s .

This particular form of the potential has been chosen because it includes some known material models such as the Bailey Orowan recovery model [Ponter and Leckie 1973] and linear viscoelastic models (Ponter 1976b) and in addition it may provide the basis of a more general result.*

Following from Chapter 2 the rate equations are assumed to take the following form:

$$\dot{\underline{\varepsilon}}' = \frac{\partial \Omega}{\partial \underline{\sigma}} (\underline{\sigma}, s) \tag{3.2}$$

$$\dot{s} = -\frac{1}{\lambda} \frac{\partial \Omega}{\partial s} (\underline{\sigma}, s) \tag{3.3}$$

The convexity condition for F is expressed as follows:

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* Eqn.(3.1) also resembles aspects of the Chaboche model. This is discussed in Appendix 2.3.

$$F_1 \geq F_2 + (\underline{\sigma}_1 - \underline{\sigma}_2) \left.\frac{\partial F}{\partial \sigma}\right|_{\underline{\sigma}_2} + (s_1 - s_2) \left.\frac{\partial F}{\partial s}\right|_{s_2} \tag{3.4}$$

As a result of the homogeneity of F and ϕ we obtain

$$(p + 1)F = \underline{\sigma} \frac{\partial F}{\partial \underline{\sigma}} + s \frac{\partial F}{\partial s} \tag{3.5}$$

Setting $\underline{\sigma}_1 = \frac{p}{p+1} \underline{\sigma}^{**}$, $s_1 = \frac{p}{p+1} s^{**}$, $\underline{\sigma}_2 = \underline{\sigma}$ and $s_2 = s$, and combining (3.4) and (3.5) we obtain

$$F \left[\frac{p}{p+1} \left\{\phi(\underline{\sigma}^{**}) - s^{**}\right\}\right] \geq \frac{p}{p+1}\left[\underline{\sigma}^{**} \frac{\partial F}{\partial \underline{\sigma}} + s^{**} \frac{\partial F}{\partial s} - \underline{\sigma} \frac{\partial F}{\partial \underline{\sigma}} - s \frac{\partial F}{\partial s}\right] \tag{3.6}$$

Following a similar procedure for G we obtain

$$G\left[\frac{q}{q+1} s^{**}\right] \geq \frac{q}{q+1} [s^{**} - s] \frac{\partial G}{\partial s} \tag{3.7}$$

Adding inequalities (3.6) and (3.7):

$$\frac{p+1}{p} F\left[\frac{p}{p+1} \left\{\phi (\underline{\sigma}^{**}) - s^{**}\right\}\right] + \frac{q+1}{q} G\left[\frac{q s^{**}}{q+1}\right] \geq$$

$$[\underline{\sigma}^{**} - \underline{\sigma}] \frac{\partial F}{\partial \underline{\sigma}} + [s^{**} - s] \frac{\partial F}{\partial s} + (s^{**} - s) \frac{\partial G}{\partial s} \tag{3.8}$$

On account of equations (3.2) and (3.3) the last inequality takes the form

$$[\underline{\sigma}^{**} - \underline{\sigma}] \ \underline{\dot{\varepsilon}}' \leq \frac{p+1}{p} F\left[\frac{p}{p+1} \left\{\phi(\underline{\sigma}^{**}) - s^{**}\right\}\right] + \frac{q+1}{q} G\left[\frac{q s^{**}}{q+1}\right]$$

$$+ \lambda \dot{s}[s^{**} - s] \tag{3.9}$$

Integrating (3.9) from $t = 0$ to $t = T$ results in an upper bound on the functional

$$W = \int_0^T [\underline{\sigma}^{**} - \underline{\sigma}]\underline{\dot{\varepsilon}}' \, dt \tag{3.10}$$

as follows

$$W \leq w(\underline{\sigma}^{**}, s^{**}, T) = \int_0^T \left[\frac{p+1}{p} F\left[\frac{p}{p+1} \left\{\phi(\underline{\sigma}^{**}) - s^{**}\right\}\right] + \frac{q+1}{q} G \left[\frac{q s^{**}}{q+1}\right]\right.$$

$$\left. + \lambda \dot{s}\{s^{**} - s\}\right] dt \tag{3.11}$$

If Ω has the more general form:

$$\Omega = \sum_{i=1}^{k} F_i\left\{\phi(\underline{\sigma}) - s\right\} + \sum_{j=1}^{l} G_j(s)$$

with each F_i and G_j convex and homogeneous of degree p_i+1 and q_j+1 respectively $(i=1,2,\ldots k$ and $j=1,2,\ldots l)$ then the above argument may be followed through to the result:

$$w(\underline{\sigma}^{**},s^{**},T) = \int_0^T \left\{ \sum_{i=1}^{k} \frac{p_i+1}{p_i} F_i\left[\frac{p_i}{p_i+1}\left\{\phi(\underline{\sigma}^{**}) - s^{**}\right\}\right]\right.$$

$$\left. + \sum_{j=1}^{l} \frac{q_j+1}{q_j} G_j\left[\frac{q_j s^{**}}{q_j+1}\right] + \lambda\,\dot{s}\left\{s^{**}-s\right\}\right\} dt \qquad (3.12)$$

CHAPTER 4

GENERAL WORK AND DISPLACEMENT BOUNDS

Inequality (3.1) is now used to obtain bounds on the work and displacement in the body. We first specify that the arbitrary state s^{**} is constant in the interval $0 \leq t \leq T$. The problem is now described as follows: for $t \leq 0$ a body of volume V remains in a stress-free condition. At $t = 0$ a history of loading $\underline{P}$ and temperature $\theta(\underline{x},t)$ commences, resulting in a history of stress $\underline{\sigma}$, inelastic strain rate $\dot{\underline{\varepsilon}}'$ and internal state s. We assume that the elastic material constants are independent of temperature. The elastic solutions corresponding to the loading history $\underline{P}$ and the temperature history θ are denoted by $\hat{\underline{\sigma}}_P$ and $\hat{\underline{\sigma}}_\theta$ respectively.

A further elastic solution is defined for a constant dummy load $\underline{T}$ acting on part of the surface of the body, S_T. This solution is denoted by $\hat{\underline{\sigma}}_T$.

The bounds obtained in this section are expressed in terms of $\hat{\underline{\sigma}}_P$, $\hat{\underline{\sigma}}_\theta$, $\hat{\underline{\sigma}}_T$ and a number of time independent quantities defined as follows:

$\bar{\underline{\rho}}(x)$ is a time-constant residual stress field

s^* is a constant state variable

$\mu > 1$ is a constant number

In terms of the above quantities we define the equilibrium distributions of stress $\hat{\underline{\sigma}} = \hat{\underline{\sigma}}_P + \hat{\underline{\sigma}}_\theta$ and $\underline{\sigma}^* = \hat{\underline{\sigma}} + \bar{\underline{\rho}}$, and the time-constant state $s^{**} = \mu s^*$.

4.1 Total inelastic work bound

The following upper bound on the inelastic work done in the body in time T was obtained by Ponter (1974):

$$\int_V \int_0^T \underline{\sigma}\, \underline{\dot{\varepsilon}}' dtdv \leq \frac{\mu}{\mu-1} \left\{A(0) - A(T)\right\} + \frac{1}{\mu-1} \int_V w(\mu\underline{\sigma}^*, T)dv \qquad (4.1)$$

where

$$A(t) = \frac{1}{2} \int_V \underset{\sim}{c}^{-1}[\underline{\rho}(t) - \underline{\bar{\rho}}][\underline{\rho}(t) - \underline{\bar{\rho}}]dv$$

and

$$\underline{\rho}(t) = \underline{\sigma}(t) - \underline{\hat{\sigma}}(t) .$$

Setting $\underline{\sigma}^{**} = \mu\underline{\sigma}^*$ and substituting from inequality (3.11) we obtain the following:

$$\int_V \int_0^T \underline{\sigma}\, \underline{\dot{\varepsilon}}' \, dtdv \leq \frac{\mu}{\mu-1} [A(0) - A(T)] + \frac{1}{\mu-1} \int_V \left[\int_0^T \left[\frac{p+1}{p} F\left\{\frac{\mu p}{p+1} (\phi(\underline{\sigma}^*) - s^*)\right\} \right.\right.$$

$$\left.\left. + \frac{q+1}{q} G \left\{\frac{\mu q s^*}{q+1}\right\} \right] dt + \frac{\lambda}{2} [\mu s^* - s(0)]^2 - \frac{\lambda}{2} [\mu s^* - s(T)]^2 \right] dv \quad (4.2)$$

Clearly the positive time-dependent quantities $A(T)$ and $[\mu s^* - s(T)]^2$ may be omitted from (4.2) to leave a bound expressed only in terms of initial and prescribed quantities.

4.2 Total displacement bound

Here we define $\underline{\sigma}^{**}$ as follows:

$$\underline{\sigma}^{**} = \underline{\sigma}^* + \underline{\hat{\sigma}}_T$$

The following displacement bound was obtained by Ponter (1974):

$$-\int_{S_T} \underline{T}\, \Delta\underline{u}\, ds \leq A(0) - A(T) + \int_{S_T} \underline{T}\, \Delta\underline{\hat{u}}\, ds + \int_V w(\underline{\sigma}^{**}, s^{**})dv \qquad (4.3)$$

where $\Delta\underline{u}$ is the displacement of surface S_T in the interval $0 \leq t \leq T$ and $\Delta\underline{\hat{u}}$ is the elastic solution in the same interval.

Combining inequalities (4.3) and (3.11) and omitting the positive terms $A(T)$ and $[s^{**} - s(T)]^2$ we obtain

$$\int_{S_T} \underline{T}\, \Delta \underline{u}\, ds \leq A(0) + \int_{S_T} \underline{T}\, \Delta \underline{\hat{u}}\, ds + \int_V \left[\int_0^T \left[\frac{p+1}{p} F\left\{ \frac{p}{p+1} (\phi(\underline{\sigma}^{**}) - s^{**}) \right\} + \frac{q+1}{q} G\left\{ \frac{qs^{**}}{q+1} \right\} \right] dt + \frac{\lambda}{2} [s^{**} - s(0)]^2 \right] dv \tag{4.4}$$

In the following chapters the general bounds (4.2) and (4.4) are interpreted and expressed in simplified form.

CHAPTER 5

WORK BOUNDS FOR A BODY IN THE STATIONARY CYCLIC CONDITION

In this chapter it is assumed that the stress and internal state in the body have reached the stationary cyclic condition.* Certain properties of this condition will be exploited in deriving a bound on the cyclic work done on the body. These properties are listed in part one. In part two the work bound is expressed in simplified form in two particular cases. The first, applying to materials possessing a homogeneous form of the rate potential, is expressed in terms of the "rapid cycle solution". We illustrate this in part three with numerical examples for linear viscoelastic materials, and we also show that by employing a specific characteristic material time we may state the conditions under which the body's behaviour is accurately described by the upper bound. The second case applies to circumstances in which it is possible to find a surface in stress space which encloses the rapid cycle stress history. This is illustrated in part four with the derivation of an existing work bound for an example of the Bailey-Orowan recovery model. Another material which may be described using a rate potential is the non-linear viscous model. For completeness, the work bound obtained by Ponter (1974) for this model is derived in Appendix 2.9.

The notation of the previous chapter is retained except for the definition of the cycle time:

$t = 0$ corresponds to the start of the cycle

$t = T = \Delta t$ corresponds to the end of the cycle.

5.1 Properties of the stationary cyclic condition

Three properties are quoted below. These will be used to simplify the general work inequality (4.2).

---

*Implying that the thermal and mechanical loading histories are cyclic.

5.1A

The internal state $s(t)$ in the body is assumed to be cyclic, from which it follows that

$$\Delta s = \int_0^{\Delta t} \dot{s}\, dt = 0$$

In consequence, from Eqn.(3.3) we have

$$\int_0^{\Delta t} \frac{\partial \Omega}{\partial \dot{s}}\, dt = 0 \tag{5.1}$$

5.1B

As the cycle time approaches zero the stationary cyclic state converges to a value s^r which is constant within a cycle. The proof is given in Appendix 2.4.

5.1C

The rapid cycle solution $(\underline{\sigma}^r, s^r)$ makes the quantity $\int_V \int_0^{\Delta t} \Omega(\underline{\sigma}^*, s^*) dt dv$ a minimum amongst the set $(\underline{\sigma}^*, s^*)$:

$$\int_V \int_0^{\Delta t} \Omega(\underline{\sigma}^r, s^r) dt dv \leq \int_V \int_0^{\Delta t} \Omega(\underline{\sigma}^*, s^*) dt dv \tag{5.2}$$

The proof is given in Appendix 2.5.

5.2. Cyclic work bounds

The three properties listed above are now combined with inequality (4.2) to obtain two simplified cyclic work bounds. In the stationary cyclic condition the general work bound (4.2) reduces to the following inequality:

$$\int_V \int_0^{\Delta t} \underline{\sigma}\, \dot{\underline{\varepsilon}}\, dt dv \leq \frac{1}{\mu - 1} \int_v \int_0^{\Delta t} \left[\frac{p+1}{p} F\left[\frac{\mu p}{p+1} \left\{ \phi(\underline{\sigma}^*) - s^* \right\} \right] + \frac{q+1}{q} G\left[\frac{\mu q s^*}{q+1} \right] \right] dt dv$$

(5.3)

If we choose a value of μ such that

$$\frac{\mu^{p+1}}{(\mu-1)p}\left(\frac{p}{p+1}\right)^{p+1} = \frac{\mu^{q+1}}{(\mu-1)q}\left(\frac{q}{q+1}\right)^{q+1} = Z(p,q) \tag{5.4}$$

then on noting the homogeneity of F , G and ϕ , inequality (5.3) takes the form

$$\int_V\int_0^{\Delta t} \underline{\sigma}\,\dot{\underline{\varepsilon}}\,dtdv \leq Z \int_V\int_0^{\Delta t} D(\underline{\sigma}^*,s^*)dtdv \tag{5.5}$$

Details of the derivation of (5.5) and values of $Z(p,q)$ are given in Appendices 2.6 and 2.7 respectively. It may be seen that Z exceeds unity by a small amount.

Two particular cases in which the bound (5.5) may be further simplified are as follows:

Case 1

When F and G are homogeneous of the same degree $(n+1)$ or when either is identically zero, then $D(\underline{\sigma},s) = (n+1)\Omega(\underline{\sigma},s)$ and $Z = 1$. From(5.5) we then obtain

$$\int_V\int_0^{\Delta t} \underline{\sigma}\,\dot{\underline{\varepsilon}}\,dtdv \leq \int_V\int_0^{\Delta t} D(\underline{\sigma}^r,s^r)dtdv \tag{5.6}$$

on account of property 5.1C*. Combining Eqns.(5.6) and (2.29) and noting property 5.1A we obtain the following bound:

$$\int_V\int_0^{\Delta t} \underline{\sigma}\,\dot{\underline{\varepsilon}}\,dtdv \leq \int_V\int_0^{\Delta t} \underline{\sigma}^r\,\dot{\underline{\varepsilon}}^r\,dtdv \tag{5.7}$$

Thus the cyclic work done on the body has an attainable upper bound in the energy dissipated by the rapid cycle stress distribution.

---

* If $p \neq q$ then there remains the possibility that the optimum histories $(\underline{\sigma}^*,s^*)$ in (5.5) are not the rapid cycle solution $(\underline{\sigma}^r,s^r)$.

Case 2

When the stress $\underline{\sigma}^*$ is known to be confined within a certain region of stress space ϕ_o given by $\phi(\underline{\sigma}^*) \lesssim \phi_o$ then it follows that $D(\phi(\underline{\sigma}^*)) \lesssim D(\phi_o)$ and so, from (5.5), we obtain the bound:

$$\int_V \int_0^{\Delta t} \underline{\sigma}\ \underline{\dot{\varepsilon}}\ dtdv \lesssim ZV \int_0^{\Delta t} D(\phi_o)dt \qquad (5.8)$$

where $D(\phi_o)$ is the dissipation rate at the surface of the confined region in stress space.

5.3. Linear viscoelastic materials

The linear viscoelastic model has a rate potential that is homogeneous of degree two. As such, it represents an example of a material to which the work bound (5.6) applies. We calculate and compare the upper bound, the lower bound and the actual increment of work for two simple models, one a solid and one a fluid.

The dissipation rate in an element of the general (non-linear) viscoelastic material shown in Figure 2.1 is given by

$$D(r,s_i) = \frac{r^{n+1}}{k} + \frac{s_1^{n_1+1}}{k_1} + \frac{s_2^{n_2+1}}{k_2} + \ldots \qquad (5.8)$$

where $\sigma = r + s_0 + s_1 + s_2 + \ldots\ldots$ and k_i and n_i are material constants.

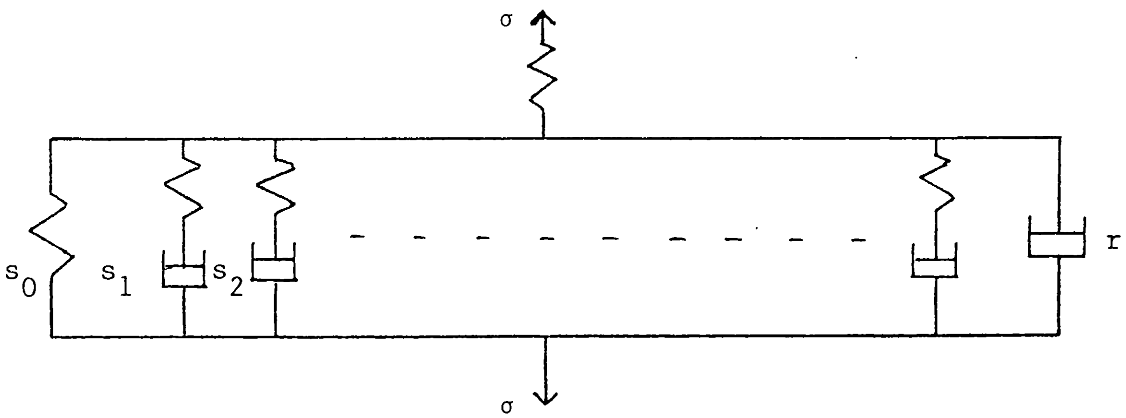

Fig.2.1

From the definition of Ω it follows that

$$\Omega = \frac{r^{n+1}}{(n+1)k} + \frac{s_1^{n_1+1}}{(n_1+1)k_1} + \frac{s_2^{n_2+1}}{(n_2+1)k_2} + \ldots \qquad (5.9)$$

In the linear case $n = n_1 = n_2 \ldots\ldots = 1$ and from Eqn.(5.8):

$$D = \frac{r^2}{k} + \sum_{i=1}^{m} \frac{s_i^2}{k_i} \qquad (5.10)$$

Bound (5.6) applies:

$$\int_V \int_0^{\Delta t} \sigma\, \dot{\varepsilon}\, dtdv \leq \int_V \int_0^{\Delta t} D(\sigma^r, s^r) dtdv \qquad (5.6)$$

Two particular material models are discussed below. A uniaxial description is given: generalisation to the multiaxial case has been discussed by Bland (1960).

1. <u>Three parameter solid, Figure 2.2</u>

Equation (5.10) yields

$$D = \frac{r^2}{k} = \frac{(\sigma - s)^2}{k} \qquad (5.11)$$

and

$$\Omega = \frac{1}{2} D = \frac{(\sigma - s)^2}{2k} \qquad (5.12)$$

The rapid cycle state s^r is obtained from properties 5.1A and 5.1B as follows:

From Eqns.(6.18) and (5.1):

$$\int_0^{\Delta t} (\sigma - s) dt = 0$$

and from property 5.1B:

$$s^r = \frac{1}{\Delta t} \int_0^{\Delta t} \sigma^r\, dt \qquad (5.13)$$

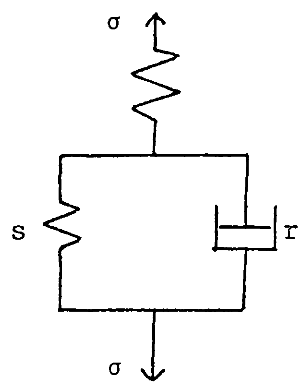

Fig.2.2

For an applied stress in the form of a sequence of simple steps given by

$$\left.\begin{array}{ll} \sigma = \sigma_o & 0 \leq t < \frac{1}{2}\Delta t \\ \quad = 0 & \frac{1}{2}\Delta t \leq t < \Delta t \end{array}\right\} \tag{5.14}$$

it follows that

$$s^r = \frac{1}{2}\sigma_o \tag{5.15}$$

On substituting Eqn.(5.15) into (5.11) we obtain

$$D(\sigma^r, s^r) = \frac{(\sigma^r - \frac{1}{2}\sigma_o)^2}{k} \tag{5.16}$$

Integrating Eqn.(5.16) with respect to time and substituting into inequality (5.6) we obtain the following work bound:

$$\int_V \int_0^{\Delta t} \sigma\,\dot{\varepsilon}\,dtdv \leq \Delta t \int_V \frac{\sigma_o^2}{4k}\,dv \tag{5.17}$$

2. <u>Four parameter fluid, Figure 2.3</u>

Eqn.(5.10) yields:

$$D = \frac{r^2}{k} + \frac{s^2}{k_1} = \frac{(\sigma-s)^2}{k} + \frac{s^2}{k_1} \tag{5.18}$$

and from Eqn.(5.9):

$$\Omega = \frac{r^2}{2k} + \frac{s^2}{2k_1} = \frac{(\sigma-s)^2}{2k} + \frac{s^2}{2k_1} \tag{5.19}$$

As above, s^r is obtained from properties 5.1A and 5.1B:

$$\int_0^{\Delta t} \left[-\frac{(\sigma^r - s^r)}{k} + \frac{s^r}{k_1} \right] dt = 0$$

and therefore

$$s^r = \frac{k_1}{\Delta t(k+k_1)} \int_0^{\Delta t} \sigma^r\,dt$$

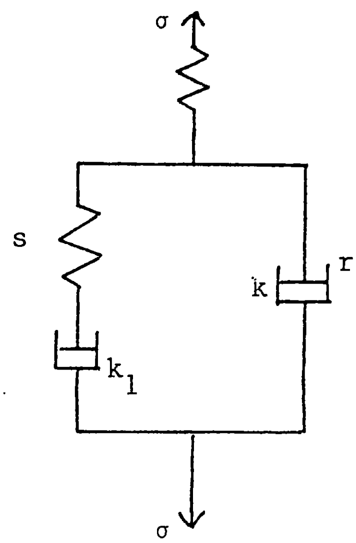

Fig.2.3

For the step-stressing of Eqn.(5.14) we obtain $s^r = \frac{k_1 \sigma_o}{2(k+k_1)}$ and on substituting for s^r in Eqn.(5.18) and integrating with respect to time we obtain the bound

$$\int_V \int_0^{\Delta t} \sigma \dot{\varepsilon} \, dt dv \leq \Delta t \int_V \frac{\sigma_o^2 (2k+k_1)}{4k(k+k_1)} dv \tag{5.20}$$

The cyclic work per unit volume,

$$W = \int_0^{\Delta t} \sigma \dot{\varepsilon} \, dt$$

was calculated exactly for both models. The upper bound, W_u was obtained from Eqns.(5.17) and (5.20) respectively and the lower work bound, W_L was determined from the expression

$$W_L = \int_0^{\Delta t} D(\sigma^{ss}) dt$$

where σ^{ss} is the steady state solution (i.e. the asymptotic solution for very slow cycling). A non-dimensionalised form of the cycle time was defined as

$$\alpha = \frac{\Delta t}{2\tau}$$

where $\tau = \frac{E}{k}$ for both models. The parameter τ determines the transient creep response, given by

$$\varepsilon' = \frac{\sigma_o}{E} (1 - \exp(- \frac{t}{\tau}))$$

as shown in Figure 2.4.

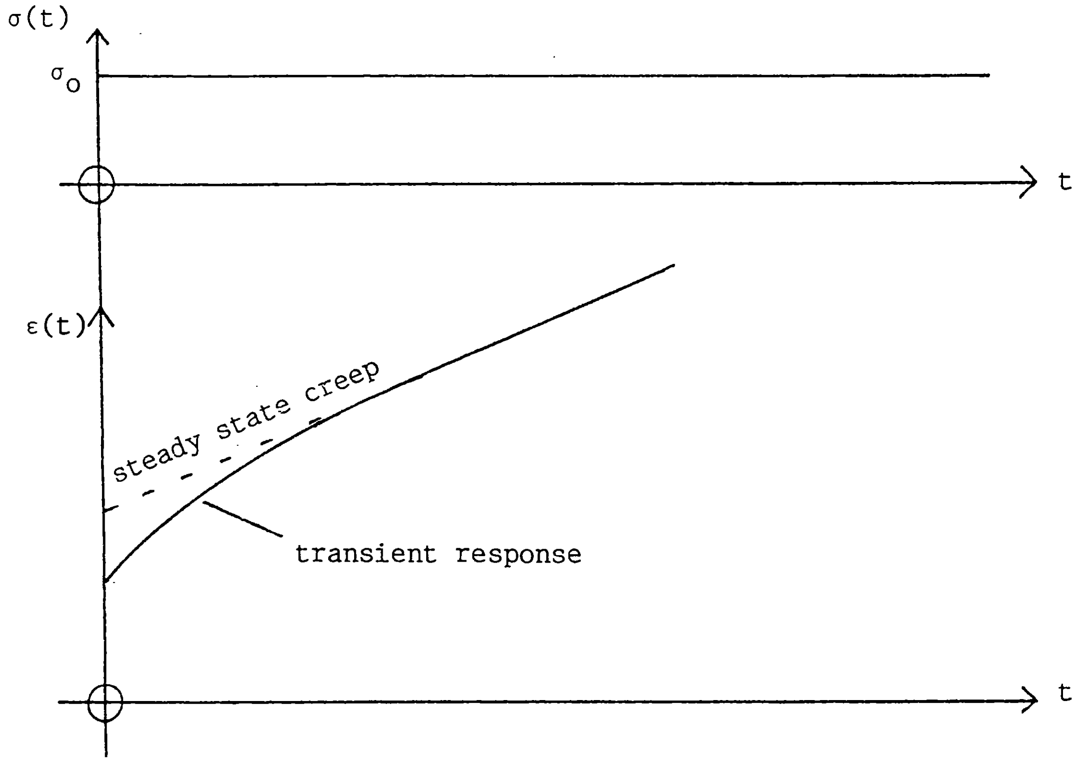

Fig. 2.4

A comparison of the work quantities was obtained by computing the expression

$$X = \frac{W-W_L}{W_u-W_L}$$

for a range of values of α . Details of the calculations are given in Appendix 2.8. The results for the two models are shown in Figure 2.5 in which we see that when the cycle time is smaller than the "transient time" τ , there is little redistribution of the residual stress field during a cycle and the material closely follows upper bound behaviour.

A further point is that as a result of comparing the cycle time with the appropriate material response time τ , the results for the two models, one a solid and one a fluid, coincide.

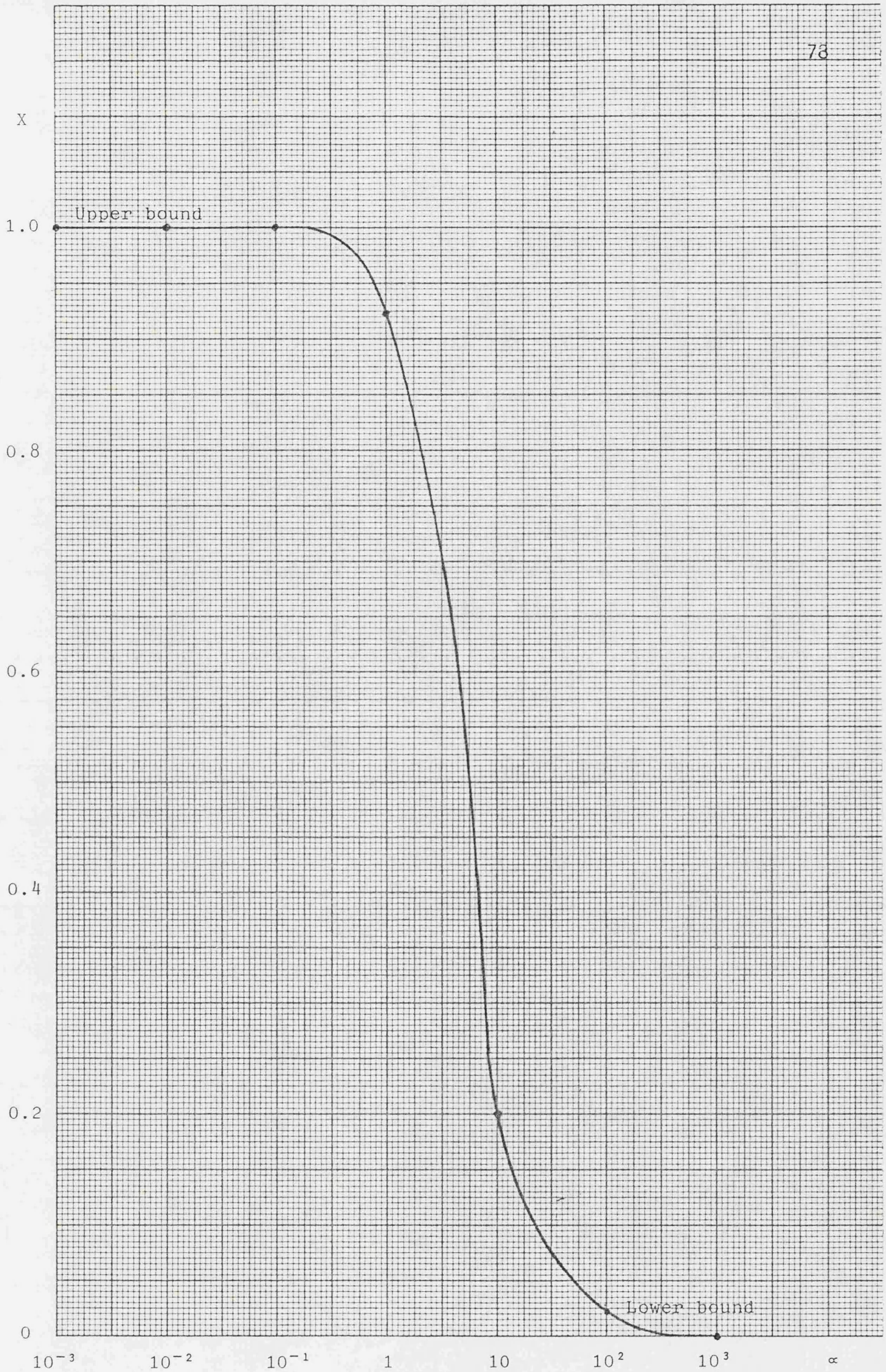

Figure 2.5

Relation between the work done per cycle and the cycle time
for two viscoelastic models

5.4. The Bailey-Orowan recovery model

In this material the deformation is governed by variations of the value of the yield stress which occur on account of strain hardening and thermal softening. The state variable s is a measure of the yield stress. An example of the possible dependence of the strain-rate on s is given by the expression

$$\underline{\dot{\varepsilon}}' = F'\,[\phi(\underline{\sigma}) - s]\,\frac{\partial\phi}{\partial\underline{\sigma}} \tag{5.21}$$

in which

$$\left.\begin{aligned} F' &> 0 \quad \text{when}\ \phi \geq s \\ F' &= 0 \quad \text{when}\ \phi \leq s \end{aligned}\right\} \tag{5.22}$$

When $\phi < s$ no creep occurs and s decreases in time. When ϕ increases rapidly the deformation resembles plastic behaviour. When $\phi = s$ and $\underline{\sigma}$ is constant, then $G(s) = G[\phi(\underline{\sigma})]$. In this case, steady state creep occurs, given by the following:

$$\underline{\dot{\varepsilon}}' = \frac{\partial G(\phi)}{\partial\underline{\sigma}} \tag{5.23}$$

Consider the following convex function:

$$\left.\begin{aligned} F[\phi(\underline{\sigma}) - s] &= \frac{k\bar{\sigma}}{p+1}\left\{\frac{\phi - s}{\bar{\sigma}}\right\}^{p+1} \quad , \quad \phi(\underline{\sigma}) \geq s \\ &= 0 \qquad\qquad\qquad\qquad , \quad \phi(\underline{\sigma}) \leq s \end{aligned}\right\} \tag{5.24}$$

where k and $\bar{\sigma}$ are constants. The derivative of F with respect to its argument is thus

$$\left.\begin{aligned} F' &= k\left\{\frac{\phi - s}{\bar{\sigma}}\right\}^{p} \quad , \quad \phi(\underline{\sigma}) \geq s \\ &= 0 \qquad\qquad\quad , \quad \phi(\underline{\sigma}) \leq s \end{aligned}\right\} \tag{5.25}$$

We now suppose that $\bar{\sigma}$ decreases in size and that $\phi(\underline{\sigma})$ approaches s from above in such a way that F' remains finite and positive. In such circumstances F' tends towards the form given by (5.22).

Suppose that we define a state variable s_o^* as follows:

$$s_o^* = \underset{0 \leq t \leq \Delta t}{\text{maximum}} \; [\phi(\underline{\sigma}^*)] \tag{5.26}$$

which we denote as

$$s_o^* = \phi(\underline{\sigma}_o^*) \tag{5.27}$$

It follows from (5.25) that

$$F\{\phi(\underline{\sigma}^*), s_o^*\} = 0 \tag{5.28}$$

and also that

$$G(s_o^*) = G(\phi(\underline{\sigma}_o^*)) \tag{5.29}$$

With $s^* = s_o^* = \phi(\underline{\sigma}_o^*)$ in inequality (5.3) we obtain the following, in view of Eqns.(5.28) and (5.29):

$$\int_V \int_0^{\Delta t} \underline{\sigma}\ \underline{\dot{\varepsilon}}\ dtdv \leq \frac{1}{\mu-1} \int_V \int_0^{\Delta t} \frac{q+1}{q}\, G\left[\frac{\mu q \phi(\underline{\sigma}_o^*)}{q+1}\right] dtdv$$

The optimum value of μ is $\mu = \frac{q+1}{q}$ and hence

$$\int_V \int_0^{\Delta t} \underline{\sigma}\ \underline{\dot{\varepsilon}}\ dtdv \leq \int_V \int_0^{\Delta t} (q+1)\ G[\phi(\underline{\sigma}_o^*)]dtdv \tag{5.30}$$

As $F[\phi(\underline{\sigma}^*), s_o^*]$ vanishes, the integral on the right of inequality (5.30) is $\Omega[\phi(\underline{\sigma}^*), s_o^*]$. Consequently from property 5.1C we obtain

$$\int_V \int_0^{\Delta t} \underline{\sigma}\ \underline{\dot{\varepsilon}}\ dtdv \leq \int_V \int_0^{\Delta t} (q+1)\ G[\phi(\underline{\sigma}_o)]dtdv \tag{5.31}$$

where $\underline{\sigma}_o$ is the point in the rapid cycle stress history that maximises $\phi(\underline{\sigma})$.

The steady state creep rate corresponding to $\underline{\sigma}_o$ is given by

$$\underline{\dot{\varepsilon}}_o^{ss} = \left.\frac{\partial G(\phi)}{\partial \underline{\sigma}}\right|_{\underline{\sigma}_o}$$

Further, as G is homogeneous,

$$\underset{\sim}{\sigma}_o \, \underline{\dot{\varepsilon}}'^{ss} = \underset{\sim}{\sigma}_o \left. \frac{\partial G}{\partial \underline{\sigma}} \right|_{\underset{\sim}{\sigma}_o} = (q+1)\; G\{\phi(\underset{\sim}{\sigma}_o)\} \tag{5.32}$$

Combining (5.31) and (5.32) we obtain

$$\int_V \int_0^{\Delta t} \underline{\sigma}\, \underline{\dot{\varepsilon}}\, dtdv \lesssim V\, \Delta t\, \underset{\sim}{\sigma}_o\, \underset{\sim}{\dot{\varepsilon}}_o^{ss}$$

This bound is an example of the type described in case 2, Chapter 5.2*.

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* This recovers an earlier result [Ponter, Univ. of Leic., 1973] from a more general basis.

CHAPTER 6

DISPLACEMENT BOUNDS FOR A BODY IN THE STATIONARY CYCLIC CONDITION

We assume in this chapter that the stress and internal state in the body have reached the stationary cyclic condition. The general displacement bound (4.4) is simplified in part one, initially by omitting the terms that repeat cyclically and by using the homogeneity of the potentials F and G . The resulting inequality is expressed in terms of the admissible stress $\underline{\sigma}^{**}$ and the arbitrary state s^{**} , and with a suitable choice of these quantities an upper bound on the displacement increment can then be obtained. We simplify further by assuming that $\underline{\sigma}^{**}$ satisfies certain inequalities that relate to properties of an associated elastic, perfectly plastic body, and a form of the displacement bound is obtained which requires the knowledge of two plasticity solutions together with the result of a single uniaxial creep test under constant reference stress.

In part two we give an example of the displacement bound for a body that is subjected to proportional loading under isothermal conditions.

Thermal loading is included in an example in part three in which we apply the bound in the case of an axially loaded tube with a through-thickness temperature gradient. The upper bound answers are compared with those obtained from the O'Donnell-Porowski diagram.

6.1. A cyclic displacement bound

In the stationary cyclic condition the general displacement bound (4.4) takes the form

$$\int_{S_T} \underline{T}\ \Delta\underline{u}\ ds \leq \int_V \int_0^{\Delta t} \left\{ \frac{p+1}{p} F\left[\frac{p}{p+1} \left(\phi(\underline{\sigma}^{**}) - s^{**}\right) \right] + \frac{q+1}{q} G\left[\frac{qs^{**}}{q+1} \right] \right\} dt dv \qquad (6.1)$$

On account of the homogeneity of the functions F and G we express this as follows:

$$\int_{S_T} T\,\Delta\underline{u}\,ds \leq \int_V \int_0^{\Delta t} \left\{ \frac{k_1(p+1)}{p} \left[\frac{p}{p+1} \left(\phi(\underline{\sigma}^{**}) - s^{**} \right) \right]^{p+1} + \frac{k_2(q+1)}{q} \left[\frac{qs^{**}}{q+1} \right]^{q+1} \right\} dt dv \qquad (6.2)$$

where k_1 and k_2 are material constants. This expression could be used to obtain a displacement bound, given any equilibrium distribution of stress $\underline{\sigma}^{**}$ and any state variable s^{**}.

We will simplify inequality (6.2) by placing certain restrictions on the stress $\underline{\sigma}^{**}$. First we assume that the function $\phi(\underline{\sigma})$, being linear and homogeneous in the components of $\underline{\sigma}$ satisfies the following form of the Schwartz inequality:

$$\phi(\underline{\sigma}_1 + \underline{\sigma}_2) \leq \phi(\underline{\sigma}_1) + \phi(\underline{\sigma}_2) \quad ^\dagger$$

Writing $\underline{\sigma}^{**}$ in the form $\underline{\sigma}^{**} = \hat{\underline{\sigma}}_T + \hat{\underline{\sigma}} + \bar{\underline{\rho}}$ we thus obtain

$$\phi(\underline{\sigma}^{**}) \leq \phi(\hat{\underline{\sigma}}_T) + \phi(\hat{\underline{\sigma}} + \underline{\rho}) \qquad (6.3)$$

We now consider an identical problem in which the body is composed of an elastic, perfectly plastic material. The following assumptions are adopted:

1. We suppose that T^L is the plastic limit load for a yield condition $\phi(\underline{\sigma}) \leq \sigma_Y$, and that

$$\gamma\,\sigma_o = \frac{T\,\sigma_Y}{T^L} \qquad (6.4)$$

where γ and σ_o are constants. Then

$$\phi(\hat{\underline{\sigma}}_T) \leq \gamma\,\sigma_o \qquad (6.5)$$

2. We assume that for the actual loading there exists some value of effective stress σ_o for which the following inequality holds:

$$\phi(\underline{\sigma}^*) \leq \sigma_o \qquad (6.6)$$

for some stress $\underline{\sigma}^*$ given by $\underline{\sigma}^* = \hat{\underline{\sigma}} + \bar{\underline{\rho}}$. The quantity σ_o may be

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†This is satisfied, e.g., by $\phi = \sqrt{\frac{3}{2}\underline{\sigma}'\,\underline{\sigma}'}$

regarded as the lowest yield stress for which the actual loading remains safe. In the examples that follow we illustrate the circumstances in which it is possible to find a value of σ_o satisfying inequality (6.6).

In addition to these assumptions we define the time constant state variable $\bar{s}$ as follows:

$$\bar{s} = \frac{s^{**}}{1+\gamma} \tag{6.7}$$

On substituting inequalities (6.3),(6.5) and (6.6) and Eqn.(6.7) into inequality (6.2) we obtain

$$\int_{S_T} \underline{T}\, \Delta\underline{u}\, ds \leq \int_V \int_0^{\Delta t} \left\{ \frac{k_1(p+1)}{p}\left[\frac{p(\gamma+1)}{p+1}\right]^{p+1} (\sigma_o-\bar{s})^{p+1} + \frac{k_2(q+1)}{q}\left[\frac{q(\gamma+1)}{q+1}\right]^{q+1} \bar{s}^{q+1}\right\} dtdv \tag{6.8}$$

If we set $\mu = \gamma + 1$ and choose μ as in Eqn.(5.4) we obtain the following from (6.8):

$$\int_{S_T} \underline{T}\, \Delta\underline{u}\, ds \leq Z\gamma \int_V \int_0^{\Delta t} \left\{(p+1)k_1(\sigma_o-\bar{s})^{p+1} + (q+1)k_2\bar{s}^{q+1}\right\} dtdv$$

or, since $(p+1)F + (q+1)G = \underline{\sigma}\,\frac{\partial\Omega}{\partial\underline{\sigma}} + s\,\frac{\partial\Omega}{\partial s} = D(\underline{\sigma},s)$,

$$\int_{S_T} \underline{T}\, \Delta\underline{u}\, ds \leq Z\gamma \int_V \int_0^{\Delta t} D(\sigma_o,\bar{s})dtdv \tag{6.9*}$$

If $\underline{T}$ is a dummy point load we obtain from (6.9):

$$\Delta u \leq \frac{Z\gamma}{T} \int_V \int_0^{\Delta t} D(\sigma_o,\bar{s})dtdv$$

or

$$\Delta u \leq \frac{Z\gamma\sigma_o}{T} \int_V \int_0^{\Delta t} \dot{\varepsilon}_o^{ss}\, dtdv \tag{6.10}$$

where $\dot{\varepsilon}_o^{ss}$ is the steady state creep rate resulting from the constant stress σ_o and the actual temperature field $\theta(\underline{x},t)$. From Eqn.(6.4),

---

* Details of the derivation of (6.9) are similar to those which are given in Appendix 2.6 for inequality (5.5).

(6.10) becomes

$$\Delta u \leqslant \frac{Z\sigma_Y}{T^L} \int_V \int_0^{\Delta t} \dot{\varepsilon}_o^{ss} \, dtdv \tag{6.11}$$

Inequality (6.11) is a point displacement bound expressed in terms of two plasticity solutions. It is not strongly sensitive to the values of the material constants p and q since, as shown in Appendix 2.7, for a practical range of values of p and q, Z lies in the range $1 \leqslant Z \leqslant 1.1$.

For some variable-temperature problems it may be possible to determine a reference temperature history $\theta_R(t)$ such that

$$\int_V \dot{\varepsilon}_o^{ss} \, dv = V \, \dot{\varepsilon}_o^{ss} \, (\theta_R) \tag{6.12}$$

In such cases the displacement bound takes the form

$$\Delta u \leqslant \frac{ZV\sigma_Y}{T^L} \int_0^{\Delta t} \dot{\varepsilon}_o^{ss} \, (\theta_R) dt \tag{6.13}$$

In the remaining two parts of this chapter we give examples of the use of the displacement bound, first for cycles of proportional loading under isothermal conditions and then for the Bree-type problem.

6.2 Proportional loading under isothermal conditions

The type of problem we consider here involves loading of the form

$$\underline{P}(t) = \lambda(t) \, \underline{P}^m$$

where $\lambda(t)$ has a period Δt and $0 \leqslant \lambda(t) \leqslant 1$, and $\underline{P}^m$ is the maximum value of $\underline{P}$.

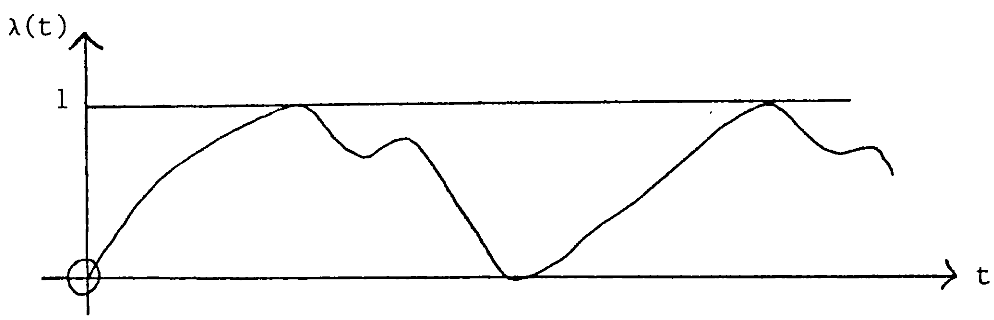


Fig. 2.6

Problems of this type include a pressure vessel under internal pressure and a circular plate under central load.

We suppose initially that the body is composed of an elastic perfectly plastic material with a yield stress σ_Y and a collapse load $\underline{P}^L$. At $\underline{P} = \underline{P}^L$ the stress in the body is given by $\underline{\sigma} = \underline{\sigma}^L$, where

$$\phi\,(\underline{\sigma}^L) \leq \sigma_Y$$

and $\underline{\sigma}^L$ is in equilibrium with $\underline{P}^L$. For the actual loading, when $\underline{P} = \underline{P}^m$ the stress is given by $\underline{\sigma} = \underline{\sigma}^m$, where $\underline{\sigma}^m$ is in equilibrium with $\underline{P}^m$. Suppose that

$$\underline{P}^m = \eta \underline{P}^L$$

where η is a constant number. Then as the equilibrium equations are linear in $\underline{\sigma}$ and $\underline{P}$ it follows that

$$\underline{\sigma}^m = \eta \underline{\sigma}^L \tag{6.14}$$

and also, as ϕ is homogeneous of degree one,

$$\phi(\underline{\sigma}^m) = \eta\phi(\underline{\sigma}^L) \leq \eta\sigma_Y \tag{6.15}$$

We now define a reference stress σ_o as follows:

$$\sigma_o = \eta\sigma_Y \tag{6.16}$$

and so, from (6.15) and (6.16),

$$\phi(\underline{\sigma}^m) \leq \sigma_o \tag{6.17}$$

Thus σ_o is the least value of yield stress that would permit the elastic, perfectly plastic body to remain safe under the actual loading.

For the displacement bound we now need to find an equilibrium stress distribution with the form $\underline{\sigma}^* = \hat{\underline{\sigma}} + \underline{\bar{\rho}}$. We denote the elastic solution for the actual loading by $\lambda(t)\hat{\underline{\sigma}}^m$ where $\hat{\underline{\sigma}}^m$ is in equilibrium with $\underline{P}^m$, and we define a time-constant residual stress field as follows:

$$\underline{\bar{\rho}} = \underline{\sigma}^m - \underline{\hat{\sigma}}^m \tag{6.18}$$

and so $\underline{\sigma}^*$ takes the form

$$\underline{\sigma}^* = \lambda(t)\underline{\hat{\sigma}}^m + (\underline{\sigma}^m - \underline{\hat{\sigma}}^m) \tag{6.19}$$

The stress $\underline{\sigma}^*$ varies along a straight line in stress space between the points $\underline{\sigma}^* = \underline{\sigma}^m$ when $\lambda=1$ and $\underline{\sigma}^* = \underline{\sigma}^m - \underline{\hat{\sigma}}^m$ when $\lambda=0$.

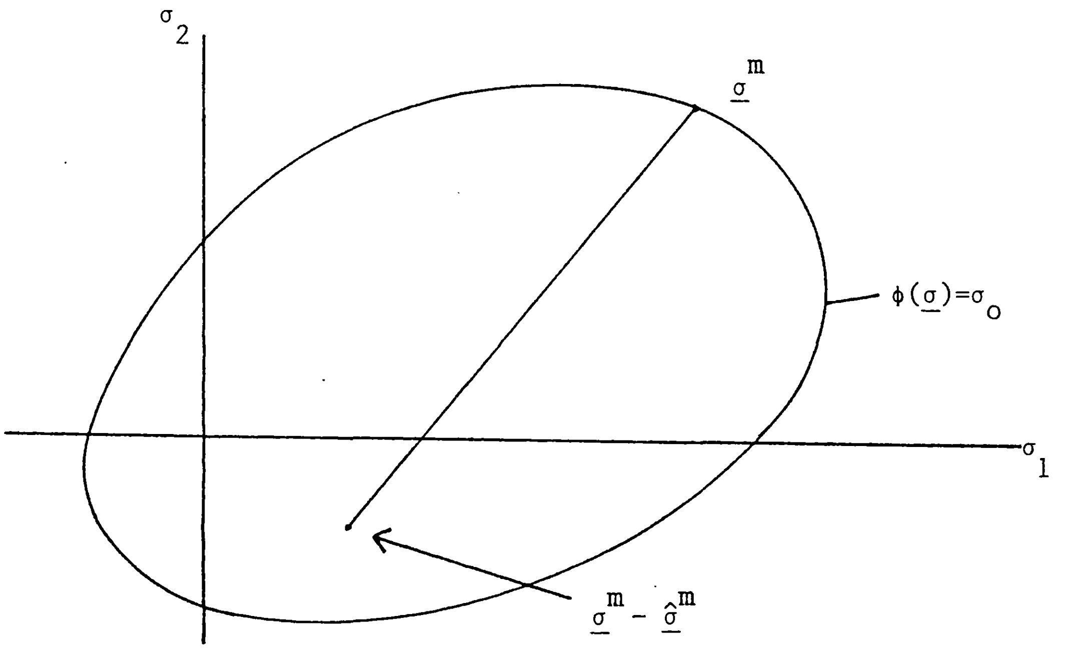


Fig. 2.7

For the bound, $\underline{\sigma}^*$ must satisfy

$$\phi(\underline{\sigma}^*) \leq \sigma_o \tag{6.20}$$

Our definition of σ_o assures that this is satisfied when $\lambda=1$ but we must also ensure that it is satisfied at $\lambda=0$. This is ascertained by finding the limit load and the elastic solution for the actual problem. We illustrate this with the example of a beam under simple bending, shown under limit bending moment in Figure 2.8:

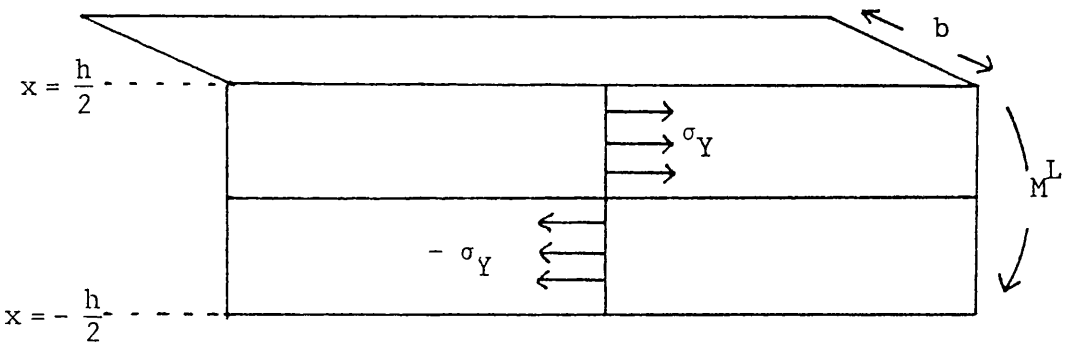


Fig. 2.8

The stress field at the limit load is given by the following:

$$\left.\begin{aligned} M &= M^L \\ \underline{\sigma} = \underline{\sigma}^L &= \sigma_Y \quad , \quad 0 < x \leqslant \frac{h}{2} \\ &= -\sigma_Y \quad , \quad -\frac{h}{2} \leqslant x < 0 \end{aligned}\right\} \qquad (6.21)$$

The equilibrium condition gives

$$M_L = \sigma_Y \frac{bh^2}{4} \qquad (6.22)$$

and, for the actual loading,

$$\left.\begin{aligned} \sigma^m &= \frac{4M^m}{bh^2} \quad , \quad 0 < x \leqslant \frac{h}{2} \\ &= -\frac{4M^m}{bh^2} \quad , \quad -\frac{h}{2} \leqslant x < 0 \end{aligned}\right\} \qquad (6.23)$$

The elastic solution is $\frac{\hat{\sigma}}{x} = \frac{M}{I}$ from which we obtain

$$\hat{\sigma}^m = \frac{12\, x\, M^m}{bh^3} \qquad (6.24)$$

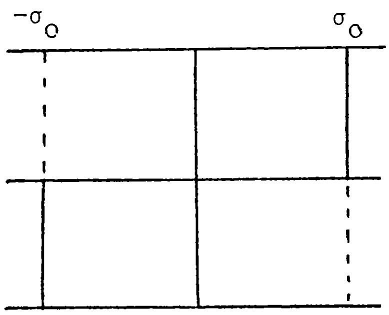
From (6.23) and (6.24) we have, for $\lambda=0$:

$$\left.\begin{aligned} \underline{\sigma}^* = \underline{\sigma}^m - \underline{\hat{\sigma}}^m &= \frac{4M^m}{bh^2} - \frac{12\, x\, M^m}{bh^3} \quad , \quad 0 < x \leqslant \frac{h}{2} \\ &= -\frac{4M^m}{bh^2} - \frac{12\, x\, M^m}{bh^3} \quad , \quad -\frac{h}{2} \leqslant x < 0 \end{aligned}\right\} \qquad (6.25)$$

The reference stress is defined as $\sigma_o = \eta\sigma_Y = \frac{M^m}{M^L}\sigma_Y$ and so from Eqn.(6.22),

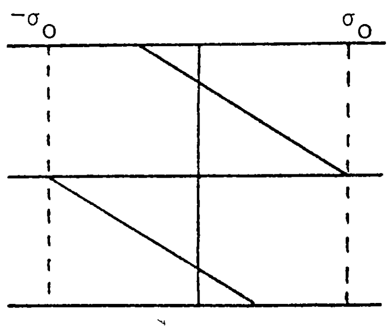
$$\sigma_o = \frac{4M^m}{bh^2} \qquad (6.26)$$

The two extreme values of $\underline{\sigma}^*$ given by Eqns.(6.23) and (6.25) are shown in Figures 2.9 and 2.10 respectively, and it is evident that the inequality $\phi(\underline{\sigma}^*) \leq \sigma_o$ is satisfied.



$\lambda=1$: Eqn.(6.23)

Fig. 2.9



$\lambda=0$: Eqn.(6.25)

Fig. 2.10

For problems of this type we may express the bound in the form

$$\frac{\Delta u}{\Delta t} \leq Z \frac{V\sigma_Y}{T^L} \dot{\varepsilon}^{ss}(\sigma_o, \theta_o) \tag{6.27}$$

where $\sigma_o = \dfrac{P^m \sigma_Y}{P^L}$.

Determination of the bound requires the plasticity solutions $\dfrac{\sigma_Y}{T^L}$ and $\dfrac{\sigma_Y}{P^L}$ and the steady state creep rate at stress σ_o and temperature θ_o .

6.3 Thermal and mechanical loading

We consider a class of problems in which there is mechanical loading $\underline{P}(t)$ given by

$$\underline{P}(t) = \lambda(t) \underline{P}^m \tag{6.28}$$

and thermal loading due to an applied temperature field $\theta(\underline{x},t)$ given by

$$\theta(\underline{x},t) = g(\underline{x},t)\Delta\theta^m \tag{6.29}$$

The functions $\lambda(t)$ and $g(\underline{x},t)$ are periodic in time with period Δt, and they satisfy

$$\left.\begin{aligned} 0 \leq \lambda(t) \leq 1 \\ 0 \leq g(\underline{x},t) \leq 1 \end{aligned}\right\} \tag{6.30}$$

$\underline{P}^m$ is the maximum mechanical load and $\Delta\theta^m$ is the maximum temperature difference occurring in the body during a cycle.

The stress field in equilibrium with $\underline{P}(t)$ is denoted by $\underline{\sigma}_p$ and the thermal stress field is denoted by $\underline{\sigma}_\theta$.

We suppose that the body is composed of an elastic, perfectly plastic material and that for a given yield stress σ_Y we know the criterion for shakedown, expressed in the form

$$h(\underline{\sigma}_p^{\,s}, \underline{\sigma}_\theta^{\,s}) \leq \sigma_Y \tag{6.31}$$

where h is a homogeneous function of degree one in its argument and $\underline{\sigma}_p^{\,s}$ and $\underline{\sigma}_\theta^{\,s}$ are in equilibrium with the mechanical and thermal shakedown limit loads. We now imagine the actual loading histories to be proportionally increased at the start of each cycle as follows:

$$\left.\begin{aligned} \underline{P}(t) &= \alpha\,\lambda(t)\underline{P}^m \\ \theta(\underline{x},t) &= \alpha\, g(\underline{x},t)\Delta\theta^m \end{aligned}\right\} \tag{6.32}$$

where $\alpha \geq 1$. The mechanical and thermal stresses in equilibrium with this loading are $\alpha\underline{\sigma}_p$ and $\alpha\underline{\sigma}_\theta$ respectively. Eventually, at some value $\alpha = \alpha_o$ and at some instant in the cycle at which the stresses are $\alpha_o\underline{\sigma}_p^{\,o}$ and $\alpha_o\underline{\sigma}_\theta^{\,o}$ we obtain

$$h(\alpha_o\, \underline{\sigma}_p^{\,o}, \alpha_o\, \underline{\sigma}_\theta^{\,o}) = \sigma_Y \tag{6.33}$$

indicating that under the scaled-up loading the stress has reached some point on the shakedown boundary, as shown in Figure 2.11.

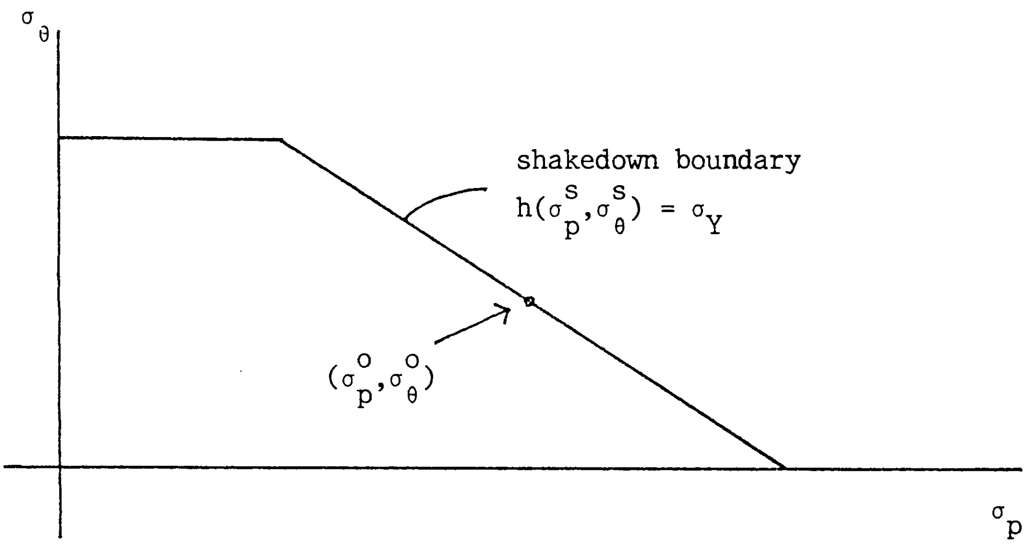


Fig. 2.11

In consequence α_o is the largest value of α for which the body shakes down, and so for the loading given by Eqns.(6.32) it is possible to find some equilibrium stress distribution $\alpha_o\underline{\sigma}^*$ of the form $\alpha_o\underline{\sigma}^* = \alpha_o(\hat{\underline{\sigma}} + \underline{\bar{\rho}})$ such that

$$\phi\,(\alpha_o\underline{\sigma}^*) \leq \sigma_Y \qquad (6.34)$$

where $\underline{\sigma}^*$ is in equilibrium with the actual loading. We now define a reference stress σ_o as follows:

$$\sigma_o = \frac{\sigma_Y}{\alpha_o} \qquad (6.35)$$

Combining (6.34) and (6.35) we obtain

$$\phi(\underline{\sigma}^*) \leq \sigma_o \qquad (6.36)$$

as required for the displacement bound. Combining (6.33) and (6.35) we have

$$h(\underline{\sigma}_p^{\,o},\underline{\sigma}_\theta^{\,o}) = \sigma_o \qquad (6.37)$$

for the point $(\underline{\sigma}_p^{\,o},\underline{\sigma}_\theta^{\,o})$. For the stress field as a whole, from the definition of the point $(\underline{\sigma}_p^{\,o},\underline{\sigma}_\theta^{\,o})$ and Eqn.(6.37) we have

$$h(\underline{\sigma}_p,\underline{\sigma}_\theta) \leq \sigma_o \qquad (6.38)$$

It is apparent from (6.37) and (6.38) that the reference stress σ_o is the smallest value of yield stress for which the body will shake down under the actual loading. Inequality (6.38) defines a "reduced shakedown boundary" corresponding to yield stress σ_o.

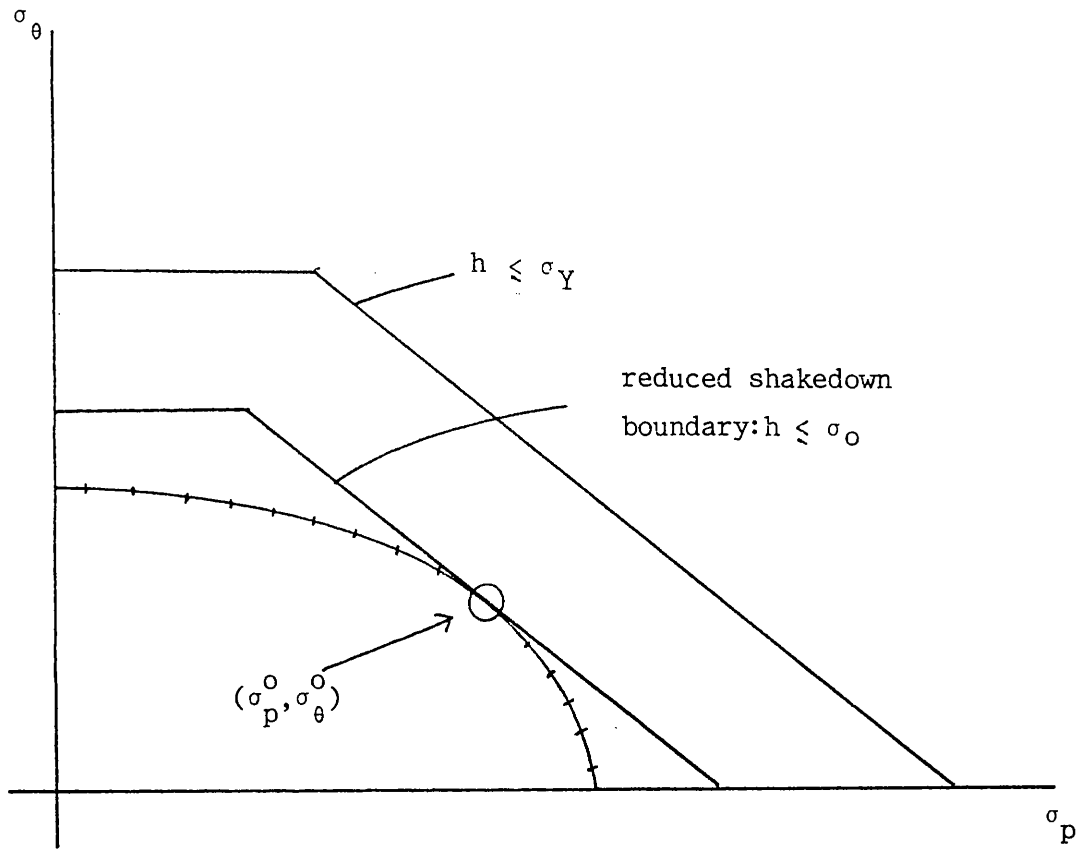


Figure 2.12

[The hatched line is the contour followed by the equilibrium stress histories σ_p and σ_θ during a cycle]

As an illustration, consider the problem of an axially loaded tube subjected to a temperature difference between its inner and outer walls (e.g. a nuclear fuel can supporting its own weight). This may be simplified to the case of the rectangular beam shown in Figure 2.14:

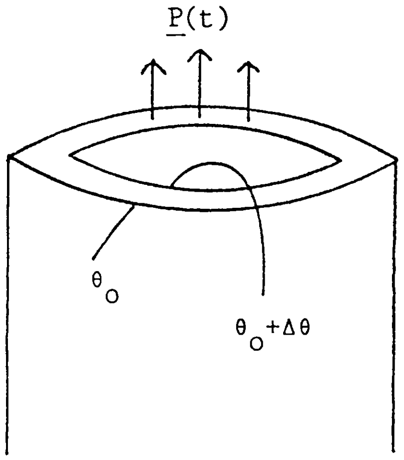


Fig. 2.13

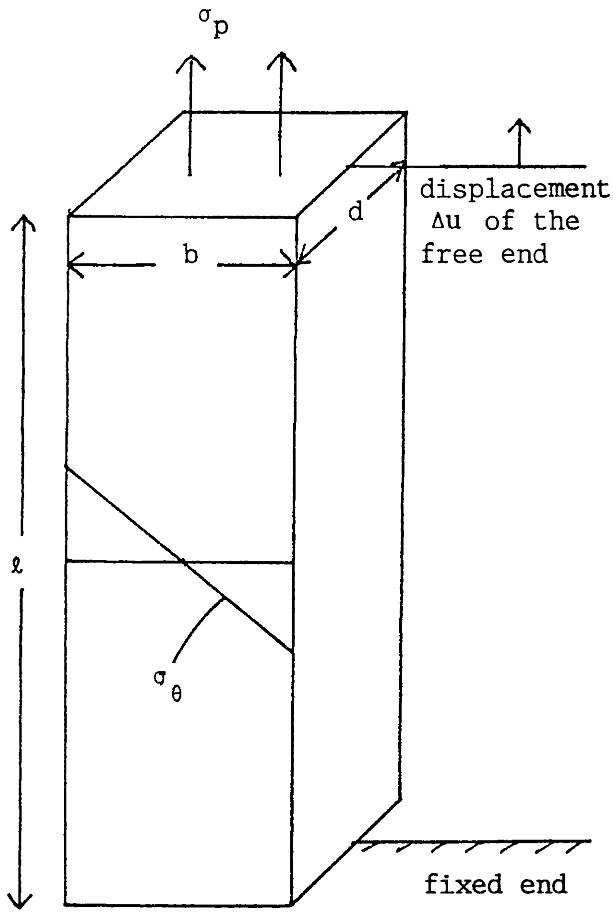


Fig. 2.14

The beam is under axial stress σ_p and thermal stress σ_θ given by $\sigma_\theta = E\alpha \frac{\Delta\theta}{2}$.

This is a Bree-type problem, for which the shakedown boundary is known to have the form

$$\left.\begin{aligned} h(\sigma_p, \sigma_\theta) &= \frac{\sigma_\theta}{2} = \sigma_Y \qquad , \ \sigma_\theta > 4\ \sigma_p \\ &= \sigma_p + \frac{1}{4}\sigma_\theta = \sigma_Y \ , \ \sigma_\theta < 4\ \sigma_p \end{aligned}\right\} \qquad (6.39)$$

as shown in Figure 2.15:

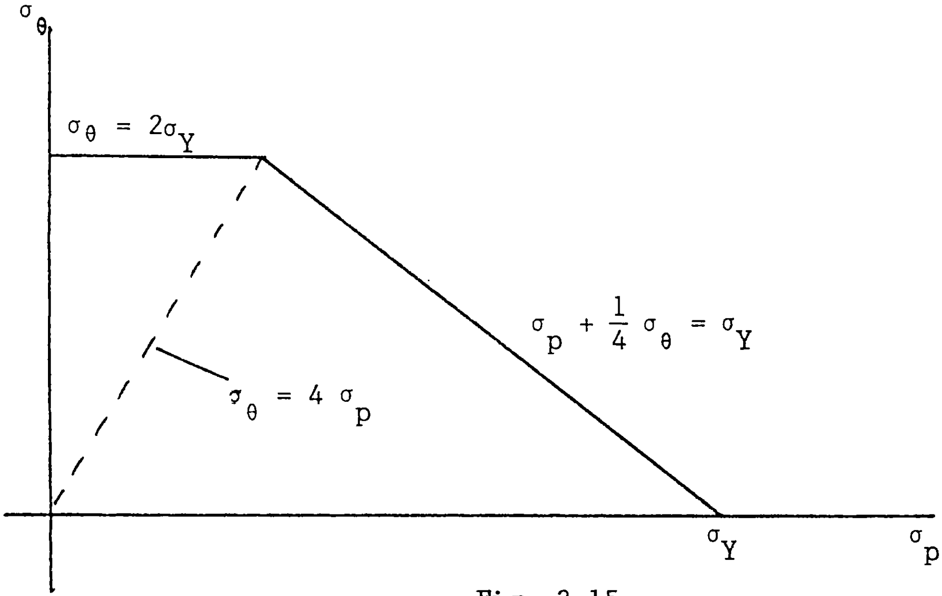


Fig. 2.15

From Eqns.(6.37) and (6.39) the reference stress is given by

$$\left.\begin{aligned} \sigma_o &= \frac{\sigma_\theta^o}{2} && \text{if } \sigma_\theta^o > 4\,\sigma_p^o \\ &= \sigma_p^o + \frac{\sigma_\theta^o}{4} && \text{if } \sigma_\theta^o < 4\,\sigma_p^o \end{aligned}\right\} \tag{6.40}$$

where $(\sigma_p^o\ ,\sigma_\theta^o)$ is the point in the stress history that lies on the reduced shakedown boundary.

The body's volume is

$$V = d\,\ell\,b \tag{6.41}$$

and the dummy limit load is given by

$$T^L = \sigma_Y\,db \tag{6.42}$$

and so

$$\frac{V\sigma_Y}{T^L} = \ell \tag{6.43}$$

and the displacement bound (6.13) becomes

$$\frac{\Delta u}{\ell} \lesssim Z \int_0^{\Delta t} \dot{\varepsilon}_o^{ss}\ (\sigma_o, \theta_R)dt \tag{6.44}$$

or

$$\Delta\varepsilon \lesssim Z\ \Delta\varepsilon_o^{ss} \tag{6.45}$$

where $\Delta\varepsilon_o^{ss}$ is the creep strain in time Δt at constant stress σ_o and reference temperature history $\theta_R(t)$, and Z is close to unity.

It is of interest to compare the results obtained from the method described in this section with those used in current design that are based on the O'Donnell-Porowski diagram*. The latter are derived from the rapid cycle solution for a specific material model in which the total strain is given by

$$\underline{\varepsilon} = \underline{e} + \underline{\varepsilon}^c + \underline{\varepsilon}^p$$

in which the creep strain $\underline{\varepsilon}^c$ is given by an n-power law and $\underline{\varepsilon}^p$ is the plastic strain assuming perfect plasticity; there is no recovery included in the model.

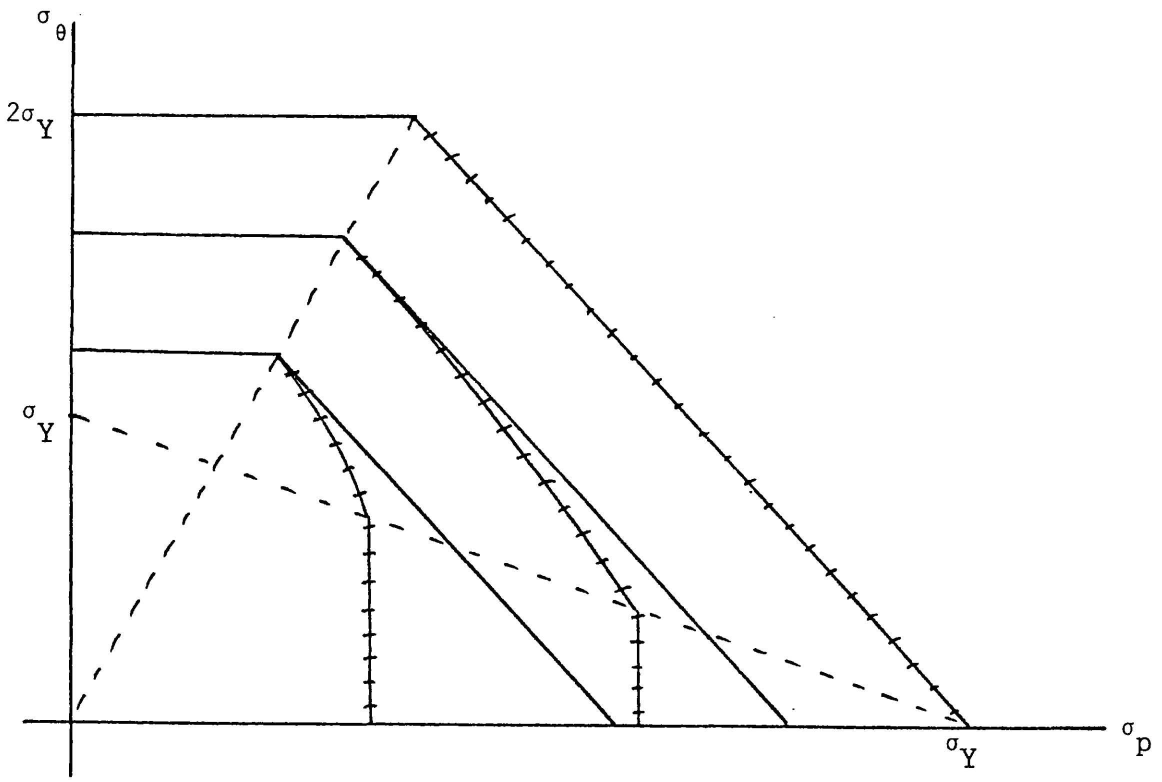


Fig. 2.16
The O'Donnell-Porowski results are shown as hatched lines

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* See, e.g. Ponter and Cocks 1982

The conclusions we reach from the work of this section are that we cannot justify the use of the O'Donnell-Porowski diagram in obtaining displacement bounds, but the nature of our method makes it highly likely that the answers which we obtain are safe.

APPENDIX 2.1

We define $\dot{\xi} = \dot{\xi}^r + \dot{\xi}^i$ as the rate of change of entropy per unit volume, where $\dot{\xi}^r = \frac{\dot{q}}{\theta}$ is the rate of entropy input corresponding to thermal power input $\dot{q}$ into unit volume at temperature θ, and $\dot{\xi}^i = \frac{D}{\theta}$ is the rate of internal entropy production resulting from dissipation rate D per unit volume at temperature θ. We have assumed that no internal entropy production occurs as a result of the flow of energy within the body due to internal temperature gradients.

On account of the second law, $\dot{\xi}^i \geq 0$ and hence $D \geq 0$.

That the dissipation defined above corresponds to the function in Eqn.(2.12) may be seen as follows: the Gibbs relation is

$$\dot{U} = \dot{W} + \dot{q}$$

where $\dot{U}$ is the rate of change of internal energy per unit volume and $\dot{W}$ is the rate of working of the external forces per unit volume. The free energy is defined as follows:

$$\psi = U - \theta\xi$$

and so

$$\begin{aligned} \dot{\psi} &= \dot{U} - \theta\dot{\xi} - \xi\dot{\theta} \\ &= \dot{W} + \dot{q} - \theta\dot{\xi} - \xi\dot{\theta} \\ &= \underline{\sigma}\,\underline{\dot{\varepsilon}} - D - \xi\dot{\theta} \end{aligned} \qquad (2.1.1)$$

as $\dot{W} = \underline{\sigma}\,\underline{\dot{\varepsilon}}$.

From Eqn.(2.7),

$$\dot{\psi} = \underline{\sigma}\,\underline{\dot{\varepsilon}} - \underline{\sigma}\,\underline{\dot{\varepsilon}}' + \dot{\psi}_1 - \xi\dot{\theta} \qquad (2.1.2)$$

and a combination of Eqns. (2.1.1) and (2.1.2) yields:

$$D = \underline{\sigma}\,\underline{\dot{\varepsilon}}' - \dot{\psi}_1 \qquad (2.1.3)$$

APPENDIX 2.2

Stationary cyclic condition for materials with a linear relationship between stress and inelastic strain rate.

For such materials

$$\underline{\sigma} = \underset{\sim}{c}' \, \dot{\underline{\varepsilon}}' \tag{2.2.1}$$

where $\underset{\sim}{c}'$ is a fourth order tensor of constants.

Consequently,

$$\underline{\tilde{\sigma}} = \underline{\sigma}_1 - \underline{\sigma}_2 = \underset{\sim}{c}' \, (\dot{\underline{\varepsilon}}'_1 - \dot{\underline{\varepsilon}}'_2)$$

or

$$\underline{\tilde{\sigma}} = \underset{\sim}{c}' \, \underline{\tilde{\dot{\varepsilon}}}' \tag{2.2.2}$$

From Eqns. (2.2.1) and (2.2.2) we see that the quantities $\underline{\tilde{\sigma}}$ and $\underline{\tilde{\dot{\varepsilon}}}'$ are related in the same way as $\underline{\sigma}$ and $\dot{\underline{\varepsilon}}'$. In consequence the expression $\int_V \int_0^T \underline{\tilde{\sigma}} \, \underline{\tilde{\dot{\varepsilon}}} \, dtdv$ represents the work done in the body by the stress $\underline{\tilde{\sigma}}$. As $\tilde{\theta} = 0$ it follows from Eqns. (2.2.2) and (2.2.3) that

$$\int_V \int_0^T \underline{\tilde{\sigma}} \, \underline{\tilde{\dot{\varepsilon}}} \, dtdv = \int_V \int_0^T [\tilde{\dot{\psi}} + \tilde{D}] \, dtdv = 0 \tag{2.2.3}$$

where

$$\tilde{\dot{\psi}} = \frac{d}{dt}\psi[\underline{\tilde{\varepsilon}}, \underline{\tilde{\varepsilon}}', \tilde{\alpha}_i, \underline{\tilde{\theta}}] \quad \text{and} \quad \tilde{D} = D(\underline{\tilde{\Sigma}})$$

From Eqns.(2.11) we obtain

$$\tilde{\dot{\psi}} = \dot{\psi}_e(\underline{\tilde{\varepsilon}} - \underline{\tilde{\varepsilon}}') + \dot{\psi}_1(\tilde{\alpha}_i) \tag{2.2.4}$$

Noting Eqns.(2.1) and (2.2) we rewrite $\dot{\psi}_e$ in terms of $\underline{\tilde{\sigma}}$:

$$\dot{\psi}_e = \dot{\phi}_e(\underline{\tilde{\sigma}}) \, .$$

Similarly, from Eqn. (2.8),

$$\dot{\psi}_1(\tilde{\alpha}_i) = \dot{\phi}_1(\tilde{A}_i) \, .$$

Eqn. (2.2.4) now becomes:

$$\tilde{\dot{\psi}} = \dot{\phi}_e(\underline{\tilde{\sigma}}) + \dot{\phi}_1(\tilde{A}_i) \tag{2.2.5}$$

Substituting (2.2.5) into (2.2.3) we obtain

$$\int_V \left[\int_0^T D(\tilde{\underline{\Sigma}})dt + \phi_e(\tilde{\underline{\sigma}}(T)) + \phi_1(A_i(T)) \right] dv = \int_V \left[\phi_e(\tilde{\underline{\sigma}}(o)) + \phi_1(A_i(o)) \right] dv \qquad (2.2.6)$$

The right-hand side of (2.2.6) is positive and constant in time. Each term on the left-hand side is positive and as T increases, the term $\int_0^T D(\tilde{\underline{\Sigma}})dt$ increases without limit unless $\tilde{\underline{\Sigma}} \to 0$.

As in Chapter 2.4 we have shown that the stress and state approach a condition that depends only on the history of loading. If this loading is cyclic the stress and state approach the stationary cyclic condition. The proof above does not require the assumption that the potential Ω is convex.

APPENDIX 2.3

A form of the "Chaboche model" [e.g. Chaboche 1977, Chaboche and Rousselier 1982] may be expressed as follows:

$$\Omega = F[J(\underline{\sigma} - \underline{s}) - R] + G(\underline{s}) \qquad (2.3.1)$$

where the state variables $\underline{s}$ and R describe kinematic and isotropic behaviour respectively, and $J = \sqrt{3J_2}$. The rate equations take the form

$$\begin{aligned} \dot{\underline{\varepsilon}}' &= \frac{\partial\Omega}{\partial\underline{\sigma}} \\ \dot{\underline{s}} &= -\frac{1}{\lambda}\frac{\partial\Omega}{\partial\underline{s}} \\ \dot{P} &= -\frac{\partial\Omega}{\partial R} \end{aligned} \qquad (2.3.2)$$

where $\dot{P} = \sqrt{\frac{2}{3}\dot{\underline{\varepsilon}}''\dot{\underline{\varepsilon}}''}$, $\dot{\underline{\varepsilon}}''$ being the deviator of $\dot{\underline{\varepsilon}}'$.

APPENDIX 2.4

Proof of property 5.1B

We first show that the rate of change of state in the body remains finite at all times. From Eqn.(2.24), if $\dot{s}$ is unbounded, so also is $\frac{d\Omega}{ds}$. If this is so when, say, $s = s_2$, then we contravene the convexity inequality

$$\Omega_1 \geq \Omega_2 + (\underline{\sigma}_1 - \underline{\sigma}_2) \left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}_2} + (s_1 - s_2) \left.\frac{\partial\Omega}{\partial s}\right|_{s_2}$$

since by definition, $\frac{\partial\Omega}{\partial\underline{\sigma}} = \dot{\underline{\varepsilon}}'$ is finite and Ω_1 and Ω_2 are finite on account of inequality (2.29).

Property 5.1B follows if we denote the value of $s(t)$ at the start of the cycle by $s(o)$, and at some subsequent time t by $s(t) = s(o) + \Delta s$, since

$$|\Delta s| \leq t|\dot{s}|_{max} \leq \Delta t|\dot{s}|_{max}$$

where $|\dot{s}|_{max}$ is greatest numerical value of $\dot{s}$ that occurs during the cycle. The value of $|\dot{s}|_{max}$ remains bounded and so it follows that as $\Delta t \to o$, $s(t) \to s(o)$. The asymptotic value of $s(o)$ as $\Delta t \to o$ is the "rapid cycle state", s^r.

APPENDIX 2.5

Proof of property 5.1C

From the convexity of Ω we have the following:

$$\Omega(\underline{\sigma}^*, s^*) - \Omega(\underline{\sigma}^r, s^r) - (\underline{\sigma}^* - \underline{\sigma}^r) \left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}^r} - (s^* - s^r) \left.\frac{\partial\Omega}{\partial s}\right|_{s^r} \geq 0$$

Integrating over a cycle and using properties 5.1A and 5.1B we obtain

$$\int_0^{\Delta t} \Omega(\underline{\sigma}^*, s^*)dt \geq \int_0^{\Delta t} \Omega(\underline{\sigma}^r, s^r)dt - \int_0^{\Delta t} (\underline{\sigma}^* - \underline{\sigma}^r) \left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}^r} dt \qquad (2.5.1)$$

The last term in (2.5.1) may be written as $(\bar{\underline{\rho}}-\bar{\underline{\rho}}^r)\Delta\underline{\varepsilon}'^r$. Integrating (2.5.1) over the volume and applying the principle of virtual work results in this term vanishing and the required result is obtained.

For completeness we remark that the admissible accumulation of strain given by

$$\Delta\underline{\varepsilon}'^* = \int_0^{\Delta t} \left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}^*} dt$$

is unique. If the contrary were so, suppose that $\underline{\sigma}_1^* = \hat{\underline{\sigma}} + \bar{\underline{\rho}}_1$ and $\underline{\sigma}_2^* = \hat{\underline{\sigma}} + \bar{\underline{\rho}}_2$ both give rise to admissible accumulations. By the principle of virtual work

$$\int_V \int_0^{\Delta t} (\underline{\sigma}_1^*-\underline{\sigma}_2^*)\ (\dot{\underline{\varepsilon}}_1'^*-\dot{\underline{\varepsilon}}_2'^*)dtdv = 0 \tag{2.5.2}$$

From convexity of ,

$$\Omega_1 \geq \Omega_2 + (\underline{\sigma}_1^*-\underline{\sigma}_2^*)\left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}_2^*} + (s_1^*-s_2^*)\left.\frac{\partial\Omega}{\partial s}\right|_{s_2^*} \tag{2.5.3}$$

with equality only for $\underline{\sigma}_1^* = \underline{\sigma}_2^*$ and $s_1^* = s_2^*$. Reversing the subscripts in (2.5.3) and adding the two inequalities we obtain

$$(\underline{\sigma}_1^*-\underline{\sigma}_2^*)\left\{\left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}_1^*} - \left.\frac{\partial\Omega}{\partial\underline{\sigma}}\right|_{\underline{\sigma}_2^*}\right\} + (s_1^*-s_2^*)\left\{\left.\frac{\partial\Omega}{\partial s}\right|_{s_1^*} - \left.\frac{\partial\Omega}{\partial s}\right|_{s_2^*}\right\} \geq 0$$

Integrating over a cycle and recalling that s_1^* and s_2^* are time-constants we obtain

$$\int_0^{\Delta t} (\underline{\sigma}_1^*-\underline{\sigma}_2^*)(\dot{\underline{\varepsilon}}_1'^*-\dot{\underline{\varepsilon}}_2'^*)dt \geq 0 \tag{2.5.4}$$

Comparison of (2.5.2) and (2.5.4) yields the result that $\Delta\underline{\varepsilon}'^*$ is unique.

APPENDIX 2.6

Derivation of inequality (5.5)

On account of the homogeneity of F and G , Eqn.(5.3) becomes

$$\int_V \int_0^{\Delta t} \underline{\sigma}\ \dot{\underline{\varepsilon}}\ dtdv \leq \int_V \int_0^{\Delta t} \left\{ \frac{\mu^{p+1}}{p(\mu-1)} \left\{\frac{p}{p+1}\right\}^{p+1} (p+1)F(\phi(\underline{\sigma}^*)-s^*) + \frac{\mu^{q+1}}{q(\mu-1)} \left(\frac{q}{q+1}\right)^{q+1} (q+1)G(s^*)\right\} dtdv \qquad (2.6.1)$$

With μ chosen as in Eqn.(5.4) we obtain

$$\int_0^{\Delta t}\int_V \underline{\sigma}\ \dot{\underline{\varepsilon}}\ dtdv \leq Z \int_V \int_0^{\Delta t} \left\{(p+1)\, F\, (\phi(\sigma^*)-s^*) + (q+1)G(s^*)\right\} dtdv \qquad (2.6.2)$$

As F and G are homogeneous,

$$(p+1)F\{\phi(\underline{\sigma}^*)\} + (q+1)G(s^*) = \underline{\sigma}^* \frac{\partial F}{\partial \underline{\sigma}^*} + s^* \left[\frac{\partial F}{\partial s^*} + \frac{\partial G}{\partial s^*}\right] = \underline{\sigma}^* \frac{\partial \Omega}{\partial \underline{\sigma}^*} + s^* \frac{\partial \Omega}{\partial s^*} \qquad (2.6.3)$$

and a combination of (2.6.2) , (2.6.3) and Eqns.(2.13) and (2.14) results in inequality (5.5).

APPENDIX 2.7

Values of Z(p,q)

p,q	Z
equal	1
1,3	1.1
3,5	1.03
3.7	1.08
5,7	1.01
5,9	1.04
7,9	1.01

APPENDIX 2.8

Evaluation and comparison of the terms in bounds (5.17) and (5.20)

Three-parameter model

In the first half of the j-th cycle the inelastic strain is given by

$$\varepsilon'(t) = \frac{\sigma_o}{E}\left[1 - R\, e^{-Et/k}\right] \qquad (2.8.1)$$

where $R = 1 - e^{\alpha} + e^{2\alpha} - \ldots + e^{2(j-1)\alpha}$

$$\alpha = \frac{E\Delta t}{2k} \quad , \qquad (j-1)\,\Delta t \leqslant t \leqslant (j-\tfrac{1}{2})\Delta t$$

The cyclic work per unit volume is given by

$$\Delta W = \sigma_o \int_0^{\frac{1}{2}\Delta t} \dot{\varepsilon}'\, dt \qquad (2.8.2)$$

and so

$$\Delta W = \frac{\sigma_o^2}{E}\left[\left\{1-e^{-\alpha}+e^{-2\alpha}\ldots+ e^{-(2j-2)\alpha}\right\} - \left\{e^{-\alpha}-e^{-2\alpha} + \ldots + e^{-(2j-1)\alpha}\right\}\right]$$

$$= \frac{\sigma_o^2}{E}\,\frac{(1-e^{-\alpha})}{1+e^{-\alpha}}\left\{1 + e^{-(2j-1)\alpha}\right\}$$

In the cyclic stationary condition, $j \to \infty$ and so

$$\Delta W = \frac{\sigma_o^2}{E}\,\frac{(1-e^{-\alpha})}{(1+e^{-\alpha})} \qquad (2.8.3)$$

The upper bound in (5.17) is given by

$$\Delta W_u = \frac{\sigma_o^2\,\Delta t}{4k} \qquad (2.8.4)$$

and the lower bound, ΔW_L , vanishes for the 3-parameter model. The relationship between ΔW and the upper and lower bounds is illustrated by computing the expression

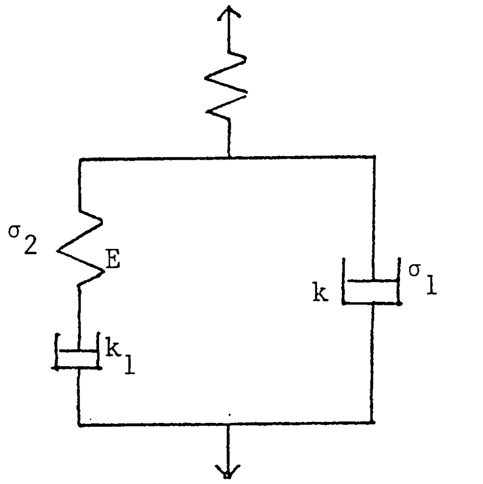
$$X = \frac{\Delta W - \Delta W_L}{\Delta W_u - \Delta W_L}$$

for various values of the non-dimensionalised cycle time, α . From (2.8.3) and (2.8.4), for the 3-parameter model X is given by

$$X = \frac{2}{\alpha}\,\frac{e^{\alpha} - 1}{e^{\alpha} + 1} \qquad (2.8.5)$$

The relation between X and α is shown in Fig.2.5.

<u>Four parameter model</u>

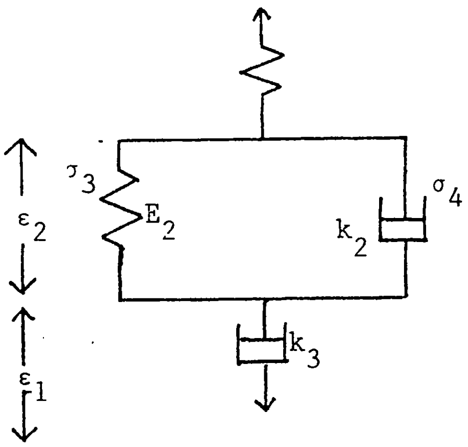


The differential equation describing the response of this model is as follows:

$$\sigma + \frac{k_1}{E}\dot{\sigma} = [k+k_1]\dot{\varepsilon} + \frac{k\,k_1}{E}\ddot{\varepsilon} \qquad (2.8.6)$$

An equivalent model is shown below. The differential equation for this model is

$$\sigma + \frac{k_2+k_3}{E_2}\dot{\sigma} = k_3\dot{\varepsilon} + \frac{k_2 k_3}{E_2}\ddot{\varepsilon} \qquad (2.8.7)$$



The coefficients in (2.8.6) and (2.8.7) may be equated to give

$$\left.\begin{aligned} k_2 &= \frac{k\,(k+k_1)}{k_1} \\ k_3 &= k + k_1 \\ E_2 &= E\left[\frac{k+k_1}{k_1}\right]^2 \end{aligned}\right\} \qquad (2.8.8)$$

We now observe that the increment of work done in the Voight-part of the equivalent model is identical to that in the 3-parameter model above, given in Eqn. (2.8.3). The increment of work done in the remaining dashpot is $\frac{\sigma_o^2\,\Delta t}{2\,k_3}$ and so the total increment is given by

$$\Delta W = \sigma_o^2\left[\frac{1-e^{-\alpha}}{1+e^{-\alpha}}\,\frac{1}{E_2} + \frac{\Delta t}{2k_3}\right] \qquad (2.8.9)$$

where $\alpha = \frac{E_2\,\Delta t}{2\,k_2}$.

The upper bound in (5.20) is given by

$$\Delta W_u = \frac{\Delta t\,\sigma_o^2\,(2k+k_1)}{4k\,(k+k_1)} \qquad (2.8.10)$$

Substituting for k_2, k_3 and E_2 in Eqn. (2.8.9) and combining the result with Eqn. (2.8.10) we obtain

$$\frac{\Delta W}{\Delta W_u} = \frac{2\lambda}{\alpha} \frac{e^{\alpha}-1}{e^{\alpha}+1} + 1 - \lambda \tag{2.8.11}$$

where $\lambda = \dfrac{k_1}{2k+k_1}$ and $0 \leq \lambda \leq 1$.

We note that for rapid cycling $\alpha \to 0$ and $\dfrac{\Delta W}{\Delta W_u} \to 1$ for all values of λ .

The lower bound on the cyclic work is given by

$$\Delta W_L = \int_0^{\Delta t} \frac{\sigma^2}{k_3} dt$$

and so

$$\Delta W_L = \frac{\sigma_o^2 \Delta t}{2(k+k_1)} \tag{2.8.12}$$

As in the case of the 3-parameter model the expression $X = \dfrac{\Delta W - \Delta W_L}{\Delta W_u - \Delta W_L}$ was computed.

From (2.8.9) , (2.8.10) and (2.8.12) it may be shown that

$$X = \frac{2}{\alpha} \frac{e^{\alpha}-1}{e^{\alpha}+1} \tag{2.8.13}$$

Eqn. (2.8.13) is identical to (2.8.5) for the 3-parameter model and consequently Fig.2.5 also represents the variation of X with α for the 4-parameter model.

APPENDIX 2.9.

Cyclic work bound for the non-linear viscous model

For this model we have

$$\frac{\dot{\underline{\varepsilon}}'}{v_o} = k\phi^n \frac{\partial\phi}{\partial\left(\frac{\underline{\sigma}}{\sigma_o}\right)} \qquad (2.9.1)$$

$$D(\underline{\sigma}) = \underline{\sigma}\,\dot{\underline{\varepsilon}}' = \frac{\underline{\sigma}}{\sigma_o}\sigma_o v_o k\phi^n \frac{\partial\phi}{\partial\left(\frac{\underline{\sigma}}{\sigma_o}\right)}$$

i.e.

$$D(\underline{\sigma}) = \sigma_o v_o k\phi^{n+1}\left(\frac{\underline{\sigma}}{\sigma_o}\right) \qquad (2.9.2)$$

From Eqn.(2.28) we thus obtain

$$\Omega(\underline{\sigma}) = \frac{\sigma_o v_o k\ \phi^{n+1}}{n+1} \qquad (2.9.3)$$

This coincides with the form of Eqn.(3.1) if we set $s^{**} = 0$, $p = n$, $G \equiv 0$ and

$$F\{\phi(\underline{\sigma}^{**})\} = \frac{\sigma_o v_o k}{n+1}\ \phi^{n+1}(\underline{\sigma}^{**})$$

In this case we obtain from (3.11)

$$W = \int_0^{\Delta t} \frac{n+1}{n}\,\frac{\sigma_o v_o k}{n+1}\left\{\frac{n\phi(\underline{\sigma}^{**})}{n+1}\right\}^{n+1} dt$$

and the bound (4.2) becomes

$$\int_V\int_0^{\Delta t} \underline{\sigma}\,\dot{\underline{\varepsilon}}\,dtdv \leq \frac{1}{\mu-1}\int_V\int_0^{\Delta t}\frac{\sigma_o v_o k}{n}\left\{\frac{n\phi(\mu\underline{\sigma}^{*})}{n+1}\right\}^{n+1} dtdv \qquad (2.9.4)$$

As ϕ is homogeneous of degree one we may use the fact that $\phi(\mu\underline{\sigma}^{*}) = \mu\phi(\underline{\sigma}^{*})$ to obtain the optimum value of μ, viz. $\mu_{opt} = \frac{n+1}{n}$. On substitution into inequality (2.9.4)

$$\int_V\int_0^{\Delta t} \underline{\sigma}\,\dot{\underline{\varepsilon}}\,dtdv \leq \int_V\int_0^{\Delta t}\sigma_o v_o k\phi^{n+1}(\underline{\sigma}^{*})dtdv = \int_V\int_0^{\Delta t} D(\underline{\sigma}^{*})dtdv$$

Finally, using result 5.1C we obtain

$$\int_V \int_0^{\Delta t} \underline{\sigma}\,\dot{\underline{\varepsilon}}\,dtdv \lesssim \int_V \int_0^{\Delta t} D(\underline{\sigma}^r)dtdv$$

This is the optimum upper cyclic work bound for the non-linear viscous material.

SECTION 3

CYCLICALLY LOADED VISCOELASTIC BODIES

CHAPTER 1

INTRODUCTION

A general method for bounding the deformation of an inelastic body was devised by Ponter (1974, 1976a). The bounds are expressed in terms of a functional of a prescribed stress history and the determination of this functional for the material in question is central to the application of Ponter's method. Among the material models for which results have been previously obtained are perfect plasticity, non-linear viscosity and the Bailey-Orowan recovery model (Ponter 1973b,1974). In Section 2 of this thesis we derived an expression for the functional by using an internal state variable description of material behaviour, thereby including non-linear viscosity, the Bailey-Orowan model, a form of the Chaboche model and the viscoelastic model. In Section 3 we develop the bounding method specifically for the class of viscoelastic materials. This class has the advantage of allowing a reasonably accessible material description to be devised by using linear theory, while still including several physically significant features, namely transient creep, viscous flow and recovery.

The description of the problem is given in Chapter 2 and the general material models are described in Chapter 3. An important class of problems is that in which the applied loading varies cyclically in time, and in Chapter 4 we give a general convergence proof for the stresses, which for cyclic loading establishes the existence and uniqueness of a stationary cyclic state of stress.

In Chapter 5 we derive a lower work bound for the general material. In the following chapter the material model is confined to the linear case and we use a state variable description to obtain an upper bound on the cycle of work done in a general linear material.

The bound is expressed in terms of the rapid cycle stress history and in Chapters 7 and 8 we describe a simple method for obtaining this solution without the need for a full cyclic analysis. The method is illustrated with an example. In the course of this we prove that even when the stress field within the body is very different from the steady state solution, a knowledge of the response to step-loading is sufficient to determine the long-term cyclic strain behaviour.

In Chapter 10 we compare the bounding method based on the state variable description of the material with that obtained using a history-dependent constitutive relation in the form of a hereditary integral for the general linear model. A cyclic work bound and a total inelastic strain bound are obtained using the latter approach and a numerical example is given.

Finally in Chapter 11 two of the main results of the section are presented in the form of a simple geometrical analogy.

CHAPTER 2

A DESCRIPTION OF THE BODY

The body under consideration has a volume V and a surface S of which a part S_T is subjected to tractions $\underline{P}(\underline{x},t)$ while the remainder S_u is subjected to displacements $\underline{u}(\underline{x},t)$. The current stress in the body is denoted by $\underline{\sigma}(\underline{x},t)$.

The strains are assumed to be sufficiently small to remain within the realms of classical continuum mechanics. The total strain $\underline{\varepsilon}(\underline{x},t)$ is taken to consist of an elastic component $\underline{e}$ which is capable of undergoing instantaneous change, and an inelastic component $\underline{\varepsilon}'$, the rate of change of which is finite for finite stress:

$$\underline{\varepsilon} = \underline{e} + \underline{\varepsilon}' \tag{2.1}$$

The elastic strain is given by

$$\left.\begin{aligned} \underline{\sigma} &= \underset{\approx}{c}\, \underline{e} \\ \underline{e} &= \underset{\approx}{c}^{-1}\, \underline{\sigma} \end{aligned}\right\} \tag{2.2}$$

where $\underset{\approx}{c}$ is a fourth-order tensor of elastic constants and $\underset{\approx}{c}^{-1}$ is its multiplicative inverse.

Following the description of three-dimensional viscoelasticity given by Bland (1960) we adopt the following hypotheses:

A. The material is homogeneous and its volume is imagined to be subdivided into infinitesimal close-packed rectangular parallelpipeds, known as "boxes".

B. The behaviour of each box is mechanically equivalent to a network of elastic and viscous elements.

C. The boxes represent the smallest portions of the material which have the same properties as the material in bulk.

Bland has shown that with this material classification the general three-dimensional linear viscoelastic constitutive relations may be simplified so that each deviatoric stress component is related only to the corresponding strain component, and the dilatational stress is related solely to the dilatational strain*, each through a uniaxial-type relationship of the form

$$\sigma(t) = B\dot{\varepsilon}(t) + \int_0^t G(t-\tau)\dot{\varepsilon}(\tau)d\tau + G(t)\varepsilon_o \qquad (2.3)$$

where $\varepsilon_o = \varepsilon(o)$, B is a constant and $G(t)$ is a positive function with the following form:

$$G(t) = A + \sum_i c_i \exp(-\lambda_i t) \qquad (2.4)$$

in which A , c_i and λ_i are constants.

For the purposes of this section we will require a relation involving only the inelastic component of strain, ε' , and so for the material described above, Eqn.(2.3) is rewritten as

$$\sigma(t) = B\dot{\varepsilon}'(t) + \int_0^t G'(t-\tau)\dot{\varepsilon}'(\tau)d\tau + G'(t)\varepsilon' \qquad (2.5)$$

where $G'(t)$ does not contain the modulus of the instantaneous element in the model.

Two of the simplest models to which Eqn.(2.5) applies are the two-parameter "Maxwell element" and the three-parameter model containing the "Voight element". For the former we have $G' \equiv 0$ and $B = k$, the viscosity of the dashpot, and from Eqn.(2.5)

$$\sigma(t) = k\,\dot{\varepsilon}'(t) \qquad (2.6)$$

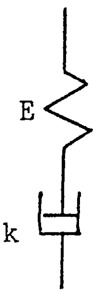


Fig.3.1 Maxwell element

* The dilatational response of a viscoelastic body is frequently taken to be purely elastic.

For the Voight element, $G' = E_1$ and $B = k_1$ and thus from Eqn.(2.5)

$$\sigma(t) = k_1 \dot{\varepsilon}'(t) + E_1 \varepsilon'(t) \qquad (2.7)$$

The Maxwell and Voight elements are the fundamental components of the general models described in the following chapter.

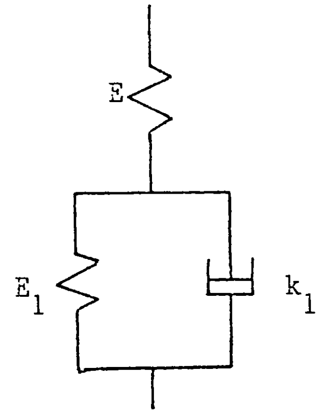


Fig.3.2

Three-parameter model

CHAPTER 3

THE GENERAL LINEAR VISCOELASTIC MATERIAL

The behaviour of a real linear viscoelastic material may be modelled to any required accuracy by combining sufficient numbers of Maxwell or Voight elements. Two equivalent general models* are shown in Figures 3.3 and 3.4.

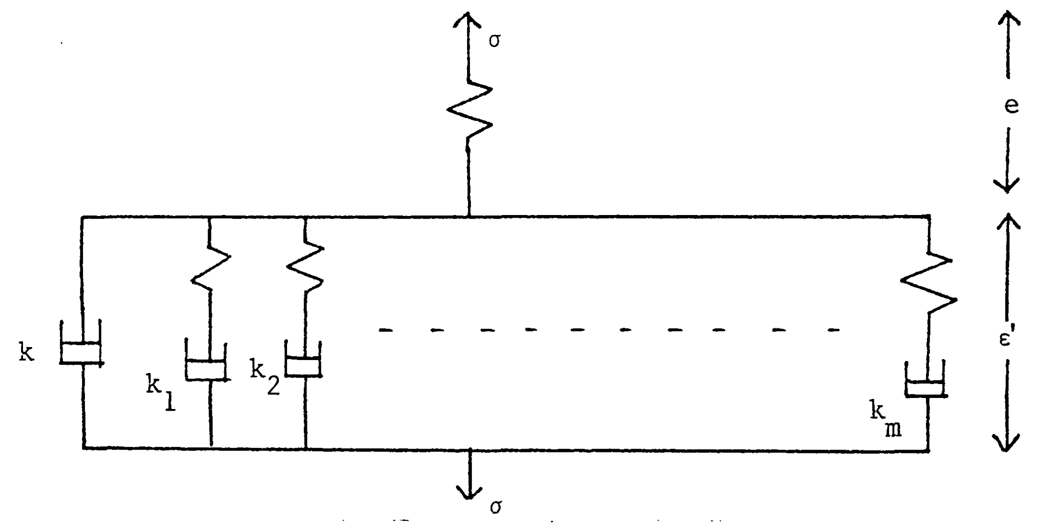


Fig.3.3 The Generalised Maxwell model

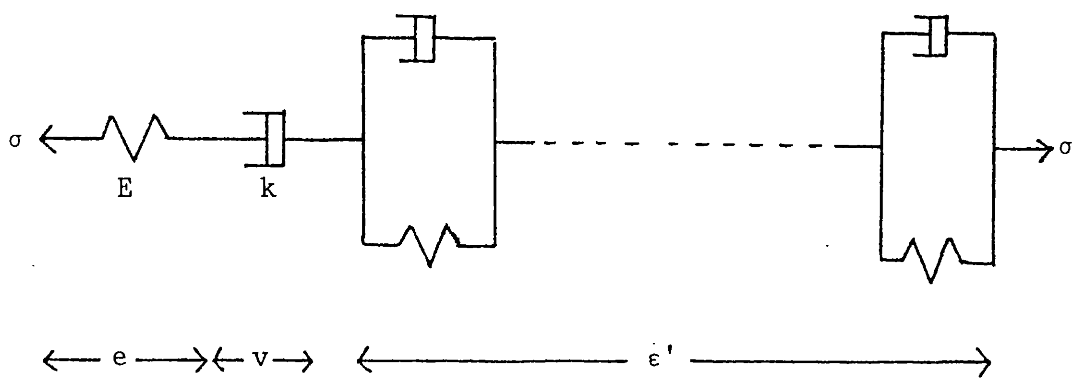


Fig.3.4 The Generalised Voight model

* See, for example, Flugge, "Viscoelasticity" (1967), Chapter 2.

The strain in the generalised Maxwell model is adequately described by Eqn.(2.1) but in the case of the generalised Voight model it is convenient to subdivide the total strain into three components:

$$\varepsilon = e + v + \varepsilon' \qquad (3.1)$$

in which v is the purely viscous or "steady state" component and ε' is the total strain due to the individual Voight elements.

CHAPTER 4

CONVERGENT STRESS HISTORIES

Throughout much of this section we will be concerned with the properties of viscoelastic bodies which have been subjected to repeated cycles of loading. At this point we prove the existence of a corresponding stationary cyclic state of stress and we show that irrespective of its initial value the actual stress history eventually reaches this state.*

Considering the generalised Voight model, Fig.3.4, to which Eqn.(3.1) applies, the components of strain e and v are related to the stress σ as follows:

$$e(t) = E^{-1}\sigma(t) \tag{4.1}$$

$$\dot{v}(t) = k^{-1}\sigma(t) \tag{4.2}$$

The dissipation rate in unit volume due to the purely viscous element is given by the positive function

$$D(\sigma) = \frac{\sigma(t)^2}{k} \tag{4.3}$$ **

The total strain ε' from the m Voight elements is given by

$$\varepsilon' = \sum_{i=1}^{m} \varepsilon_i' \tag{4.4}$$

and the behaviour of a single typical Voight element may be described as follows:

$$\sigma(t) = \sigma_i^e(t) + \sigma_i^v(t) \tag{4.5}$$

---

* The stationary cyclic state of stress for a class of constitutive relations which includes linear viscoelasticity has also been discussed by Ponter (1976b).

** In three dimensions, $\sigma_{ij}\dot{v}_{ij} = D\left(\frac{\sigma_{ij}}{\sigma_o}\right) = \sigma_o\dot{v}_o\phi\left(\frac{\sigma_{ij}}{\sigma_o}\right)^2$ where ϕ is convex and homogeneous of degree one in its argument.

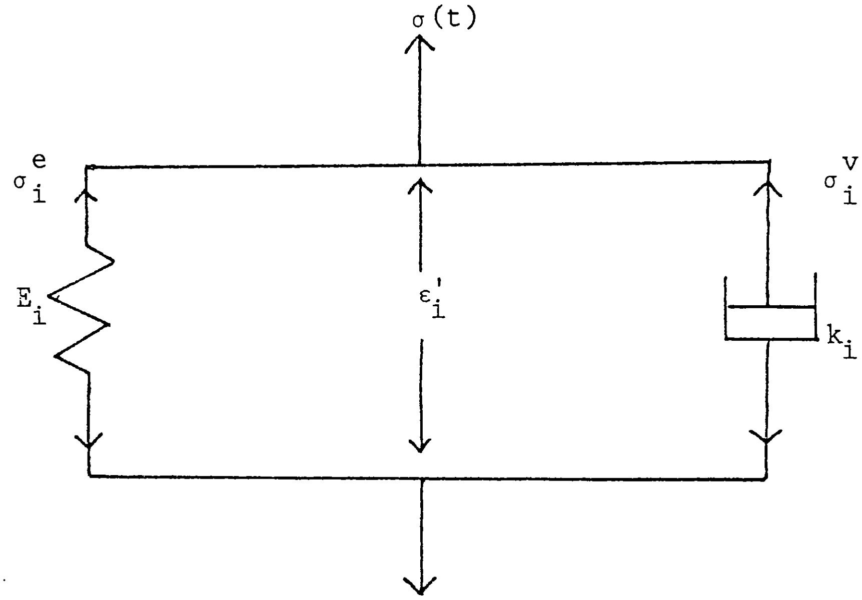


Fig. 3.5

where

$$\sigma_i^e = E_i \ \varepsilon_i' \tag{4.6}$$

$$\sigma_i^v = k_i \ \dot{\varepsilon}_i' \tag{4.7}$$

The dissipation rate in unit volume due to the single Voight element is given by

$$D(\sigma_i^v) = \sigma_i^v \ \dot{\varepsilon}_i' = \frac{(\sigma_i^v)^2}{k_i} \tag{4.8}$$

We now consider two identical bodies, differing in initial stress distributions, subjected to identical loading histories $\underline{P}(t)$. Denoting the stresses and strain rates in the bodies by $\sigma_1(t), \sigma_2(t)$ and $\dot{\varepsilon}_1(t), \dot{\varepsilon}_2(t)$, it follows from the principle of virtual velocities that for all time t :

$$\begin{aligned} &\int_V \left\{\sigma_1(t)-\sigma_2(t)\right\}\left\{\dot{\varepsilon}_1(t)-\dot{\varepsilon}_2(t)\right\}dv \\ &= \int_{S_T} \left\{\underline{P}(t)-\underline{P}(t)\right\}\left\{\dot{u}_1(t)-\dot{u}_2(t)\right\}ds \\ &= 0 \end{aligned} \tag{4.9}$$

assuming, for simplicity, rigid supports except where $\underline{P}$ acts.

Thus:

$$\int_V \int_o \left\{\sigma_1(\tau)-\sigma_2(\tau)\right\}\left\{\dot{\varepsilon}_1(\tau)-\dot{\varepsilon}_2(\tau)\right\} dt dv = 0$$

which we write as

$$\int_V \int_o^t \tilde{\sigma}(\tau)\dot{\tilde{\varepsilon}}(\tau) d\tau dv = 0 \tag{4.10}$$

where the symbol "~" represents the difference between the two bodies. Substituting from Eqns.(3.1),(4.1) and (4.2) into (4.10) we obtain

$$\int_V \left\{ \int_o^{\tilde{e}(t)} \tilde{\sigma}\, d\tilde{e} + \int_o^t \tilde{\sigma}\, \dot{\tilde{v}}\, dt + \int_o^t \tilde{\sigma}\, \dot{\tilde{\varepsilon}}' d\tau \right\} dv = 0 \tag{4.11}$$

The third term in Eqn.(4.11) is the work done by the stress $\tilde{\sigma}$ on the Voight elements. From Eqns.(4.4) to (4.9):

$$\int_o^t \tilde{\sigma}\, \dot{\tilde{\varepsilon}}'\, d\tau = \sum_{i=1}^m \left\{ \int_o^{\tilde{\varepsilon}_i'(t)} E_i \tilde{\varepsilon}_i' d\tilde{\varepsilon}_i' + \int_o^t D(\tilde{\sigma}_i^v) d\tau \right\} \tag{4.12}$$

Substitution of Eqn.(4.12) into (4.11) and integration of the elastic terms yields

$$\int_V \left\{ \frac{\tilde{\sigma}(t)^2}{2E} + \int_o^t D(\tilde{\sigma}(\tau)) d\tau + \sum_{i=1}^m \left[\frac{E_i \tilde{\varepsilon}_i'(t)^2}{2} + \int_o^t D(\tilde{\sigma}_i^v) d\tau \right] \right\} dv$$

$$= \int_V \left\{ \frac{\tilde{\sigma}(o)^2}{2E} + \sum_{i=1}^m \frac{E_i \tilde{\varepsilon}_i'(o)^2}{2} \right\} dv \tag{4.13}$$

The right-hand side of Eqn.(4.13) is the sum of positive time-constant terms; the left-hand side is the sum of positive time dependent terms. As time $t \to \infty$ the left hand side becomes unbounded unless $\tilde{\sigma}(t) \to o$, and consequently equality in (4.13) requires that $\sigma_1(t)$ and $\sigma_2(t)$ converge in time.

Consider now the particular case of repeated cycles of loading with a period of Δt. We imagine that we begin with two identical bodies with identical initial stress fields, but that the loading history on one body starts at an interval Δt before that on the other body and so the second body is one cycle behind the first. According to the proof above, the stress histories in the bodies converge and after sufficient time they are identical. This is the "stationary cyclic state" of stress. Furthermore, if in the proof above the history $\sigma_1(t)$ is chosen to be the stationary cyclic state, it follows that any other admissible history $\sigma_2(t)$ in the body eventually converges with this stationary state.

CHAPTER 5

A LOWER BOUND ON THE CYCLIC WORK

In this chapter it is first shown that the stress distribution following step loading eventually becomes σ^s , the "steady state solution". From this it is shown that σ^s provides a lower bound on the increment of cyclic work done in the body. The proof is given for a general non-linear viscoelastic material.

The material under consideration is the generalised three dimensional non-linear Voight model of Figure 3.4, with the following properties:

$$\underline{\varepsilon} = \underline{e} + \underline{v} + \underline{\varepsilon}' \tag{5.1}$$

$$\underline{\sigma} = \underset{\sim}{C}\,\underline{e} \tag{5.2}$$

$$\frac{\dot{\underline{v}}}{\dot{v}_o} = k\,\phi^n \frac{\partial\phi}{\partial\left(\frac{\underline{\sigma}}{\sigma_o}\right)} \tag{5.3}$$

where ϕ is homogeneous of degree one in $\left(\frac{\underline{\sigma}}{\sigma_o}\right)$, n is odd and $\dot{v}_o, \sigma_o$ and k are constants. The stress in the i-th Voight element consists of two components $\underline{\sigma}_i^e$ and $\underline{\sigma}_i^v$, where

$$\underline{\sigma}_i^e + \underline{\sigma}_i^v = \underline{\sigma} \tag{5.4}$$

The inelastic strain $\underline{\varepsilon}'$ is the sum of the strains in the m Voight elements:

$$\underline{\varepsilon}' = \sum_{i=1}^{m} \underline{\varepsilon}_i' \tag{5.5}$$

Analogous to Eqns.(5.2) and (5.3) we have

$$\underline{\sigma}_i^e = \underset{\sim}{C}_i\,\underline{\varepsilon}_i' \tag{5.6}$$

$$\frac{\dot{\underline{\varepsilon}}_i'}{\dot{v}_o} = k_i\,\phi^{n_i} \frac{\partial\phi}{\partial\left(\frac{\underline{\sigma}_i^v}{\sigma_o}\right)} \tag{5.7}$$

If a step load $\underline{P}(t) = \underline{P}H(t)$ is applied to the surface of the body, we have for $t > 0$

$$\int_S \dot{\underline{P}}\,\dot{\underline{u}}\,ds = \int_V \dot{\underline{\sigma}}\,\dot{\underline{\varepsilon}}\,dv = 0 \tag{5.8}$$

and thus

$$-\int_V \dot{\underline{\sigma}}(\dot{\underline{v}} + \dot{\varepsilon}')dv = \int_V \dot{\underline{\sigma}}\,\dot{\underline{e}}\,dv$$

and so, from Eqn.(5.4):

$$-\int_V \left\{\dot{\underline{\sigma}}\,\dot{\underline{v}} + \sum_{i=1}^{m} \dot{\underline{\sigma}}_i^v\,\dot{\underline{\varepsilon}}_i'\right\}dv = \int_V \left\{\dot{\underline{\sigma}}\,\dot{\underline{e}} + \sum_{i=1}^{m} \dot{\underline{\sigma}}_i^e\,\dot{\underline{\varepsilon}}_i'\right\}dv \tag{5.9}$$

From Eqns.(5.2) and (5.6), the right-hand side of (5.9) is positive as long as $\dot{\sigma} > 0$. Consequently

$$\int_V \left\{\dot{\underline{\sigma}}\,\dot{\underline{v}} + \sum_{i=1}^{m} \dot{\underline{\sigma}}_i^v\,\dot{\underline{\varepsilon}}_i'\right\}dv < 0 \qquad \text{if} \quad \dot{\underline{\sigma}} > 0 \tag{5.10}$$

On account of Eqns.(5.3) and (5.7) we may rewrite (5.10) as

$$\frac{d}{dt}\int_V \left\{ k\,\phi\left(\frac{\underline{\sigma}}{\sigma_o}\right)^{n+1} + \sum_{i=1}^{m} k_i\,\phi\left(\frac{\underline{\sigma}_i^v}{\sigma_o}\right)^{n_i+1}\right\}dv < 0 \tag{5.11}$$

The integral in (5.11) is positive as long as $\underline{\sigma}$ and $\underline{\sigma}_i^v$ are non-zero and as a result of inequality (5.11) this integrand decreases in time until $\underline{\sigma}$ and $\underline{\sigma}_i^v$ become constant. Their values are then denoted respectively by $\underline{\sigma}^s$ and $\underline{\sigma}_i^{vs}$, the "steady state solution". Furthermore, as $\underline{\sigma}^s$ and $\underline{\sigma}_i^{vs}$ are constant, then $\underline{\sigma}_i^{es}$ is constant and so $\dot{\underline{\varepsilon}}_i' = 0$, which results in the steady state stress $\underline{\sigma}_i^{vs}$ vanishing.

The steady state solution is now used to obtain a lower work bound. From inequality (5.11) and the definition of $\underline{\sigma}^s$ it follows that

$$\int_V \sigma_o\,\dot{v}_o\left\{k\,\phi\left(\frac{\underline{\sigma}}{\sigma_o}\right)^{n+1} + \sum_{i=1}^{m} k_i\,\phi\left(\frac{\underline{\sigma}_i^v}{\sigma_o}\right)^{n_i+1}\right\}dv$$
$$\geq \int_V \sigma_o\,\dot{v}_o k\,\phi\left(\frac{\underline{\sigma}^s}{\sigma_o}\right)^{n+1} dv \tag{5.12}$$

Noting Eqns.(5.3) and (5.7), (5.12) becomes

$$\int_V \left\{ \underline{\sigma}\, \dot{\underline{v}} + \sum_{i=1}^{m} \underline{\sigma}_i^{v}\, \dot{\underline{\varepsilon}}_i' \; dv \geq \int_V \underline{\sigma}^s\, \dot{\underline{v}}^s \; dv = \int_V D(\underline{\sigma}^s) dv \right. \tag{5.13}$$

Integrating over a cycle and noting that the increment of elastic energy over a stationary state cycle is zero we obtain

$$\int_V \int_o^{\Delta t} \underline{\sigma}\, \dot{\underline{\varepsilon}} \; dt dv \geq \int_V \int_o^{\Delta t} D(\underline{\sigma}^s) dv \tag{5.14}$$

Inequality (5.14) provides a lower bound on the cyclic work done in the general non-linear viscoelastic body.

CHAPTER 6

AN UPPER BOUND ON THE CYCLIC WORK IN A GENERAL LINEAR VISCOELASTIC MATERIAL

In Section 2 of this thesis a general work bound was obtained for a class of constitutive equations which included linear viscoelasticity. Simple examples* were used to demonstrate that the body's behaviour closely resembled the upper bound for cycle times which were less than a material transient time. In this chapter we obtain the upper bound expression for the general linear viscoelastic material.

The following relations from Section 2 are required here:

1. The upper bound on the cyclic work is given by

$$\int_V \int_0^{\Delta t} \sigma \dot{\varepsilon} \, dtdv \leq \int_V \int_0^{\Delta t} D(\sigma^r, s^r) dtdv \tag{6.1}$$

where σ^r is the rapid cycle stress history and s^r is an internal state variable.†

2. The "rate potential" for the material, Ω is related to the dissipation rate D as follows:

$$\Omega = \tfrac{1}{2} D \tag{6.2}$$

3. The internal state variable in the body, s is cyclic, from which it follows that

$$\int_0^{\Delta t} \frac{\partial \Omega}{\partial s} dt = 0 \tag{6.3}$$

4. For rapid cycling, the state variable $s=s^r$ is constant within a cycle.

---

* The three-parameter solid and the four parameter fluid.

† From Bland's result we may express the bounds for the linear material in terms of the scalars σ^r and s^r. (See page 112).

We begin by observing that the internal state of a generalised linear viscoelastic model such as the Maxwell model contains m independent variables. For example, if the stresses in the m Maxwell elements in Figure 3.6 are known, then the behaviour of the model is a known function of the current stress $\sigma(t)$. We denote these internal variables by $s_1, s_2, \ldots, s_m$ and the stress in the single dashpot by r, where

$$\sigma(t) = r + \sum_{i=1}^{m} s_i \tag{6.4}$$

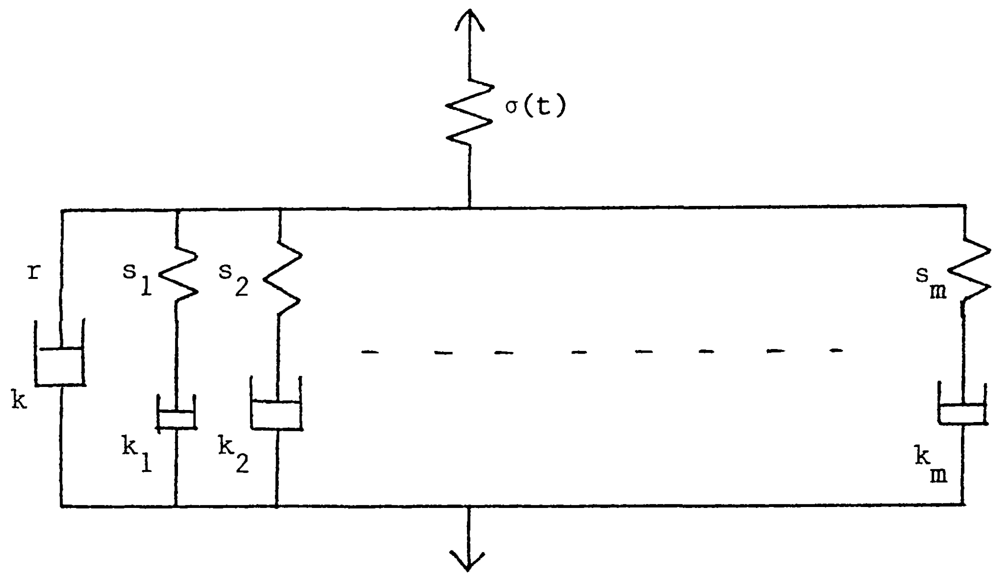


Fig. 3.6

The dissipation rate in the model is given by

$$D = \frac{r^2}{k} + \sum_{i=1}^{m} \frac{s_i^2}{k_i} \tag{6.5}$$

Using Eqns.(6.2) and (6.5) we now have the rate potential Ω, which we write in the form

$$\Omega = F(\sigma, s_i) + \sum_{i=1}^{m} G(s_i) \tag{6.6}$$

where

$$F = \frac{(\sigma - s_1 - s_2 - \ldots - s_m)^2}{2k} \tag{6.7}$$

and

$$G(s_i) = \frac{s_i^2}{2k_i} \tag{6.8}$$

For the bound we require the rapid cycle states s_i^r . The condition for the general state s_i to be cyclic is given by Eqn.(6.3). Combining Eqns.(6.3), (6.6), (6.7) and (6.8) we obtain

$$\int_0^{\Delta t} \left\{ \frac{-\sigma + \Sigma s_i}{k} + \frac{s_i}{k_i} \right\} dt = 0 \quad , \quad i=1,2,\dots,m \tag{6.9}$$

For rapid cycling, $s_i = s_i^r$ = constant during the cycle and so from Eqns.(6.9) we obtain

$$\frac{k\, s_i^r}{k_i} + \sum_{i=1}^{m} s_i^r = \frac{1}{\Delta t} \int_0^{\Delta t} \sigma^r \, dt \tag{6.10}$$

From Eqns.(6.10) it follows that

$$\frac{s_1^r}{k_1} = \frac{s_2^r}{k_2} = \dots\dots = \frac{s_m^r}{k_m} \tag{6.11}$$

and by combining (6.10) and (6.11) we obtain

$$s_i^r \left\{ \frac{k_1}{k_i} + \frac{k_2}{k_i} + \dots\dots + \frac{k_m}{k_i} + \frac{k}{k_i} \right\} = \frac{1}{\Delta t} \int_0^{\Delta t} \sigma^r \, dt \tag{6.12}$$

For simplicity we will suppose that the time-average of the stress σ^r during a cycle is given by $\frac{1}{2}\sigma_o$ where σ_o is its maximum value. Eqns.(6.12) then take the form

$$s_i^r = \frac{k_i \, \sigma_o}{2(k + \Sigma k_i)} \tag{6.13}$$

Substituting Eqns.(6.13) into (6.5), the dissipation for rapid cycling is then given by

$$D(\sigma^r, s_i^r) = \frac{1}{k} \left\{ \sigma^r - \frac{\sigma_o \, \Sigma k_i}{2(k + \Sigma k_i)} \right\}^2 + \left\{ \frac{\sigma_o}{2(k + \Sigma k_i)} \right\}^2 \Sigma k_i \tag{6.14}$$

The upper bound on the cyclic work per unit volume in the general linear model is obtained by integrating Eqn.(6.14) over the cycle.

We note that for cycles of loading in the form of on-off steps, since the rapid cycle solution σ^r is given by

$$\sigma^r = \hat{\sigma}(t) + \bar{\rho} \tag{6.15}$$

where $\hat{\sigma}(t)$ is the elastic solution and $\bar{\rho}$ is time constant, then σ^r also has step form and the integral of Eqn.(6.14) may be simplified to the following:

$$\int_0^{\Delta t} D(\sigma^r, s^r)dt = \frac{\Delta t\ \sigma_o^2\ (2k+\Sigma k_i)}{4k\ (k+\Sigma k_i)} \tag{6.16}$$

and thus the cyclic work bound for the general model is given by

$$\int_V \int_0^{\Delta t} \sigma\,\dot{\varepsilon}\,dtdv \lesssim \Delta t \int_V \frac{\sigma_o^2\ (2k+\Sigma k_i)}{4k\ (k+\Sigma k_i)}\ dv \tag{6.17}$$

In order for this bound to be useful we require simple methods for determining both the material constants and the rapid cycle solution. A material test to find the parameters k and Σk_i is described in Appendix 3.1. A method for determining σ^r is described in detail in Chapter 8 using a property of the cyclic strain which is derived in Chapter 7.

Bounds for specific material models derived from the general material are discussed in Appendix 3.2 but we include an example below in which the bound takes a particularly simple form.

The general linear viscoelastic solid

When one of the k_i is made indefinitely large in the general model, the fluid response vanishes and we obtain the general linear viscoelastic solid shown in Figure 3.7. The bound (6.17) reduces to the following form for this model:

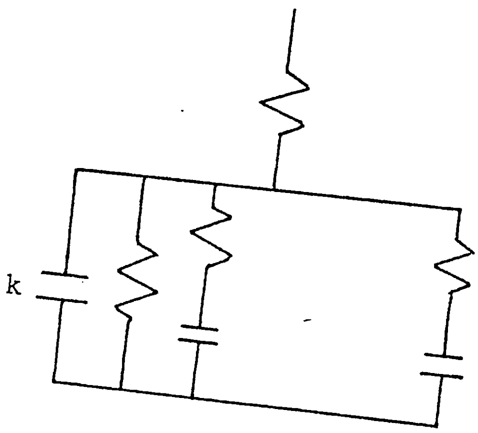


Fig. 3.7

$$\int_{V}\int_{0}^{\Delta t} \sigma \dot{\varepsilon}\, dtdv \leq \Delta t \int \frac{\sigma_o^2}{4k}\, dv \qquad (6.18)$$

Inequality (6.18) is identical to the bound obtained in Section 2 for the three-parameter model, suggesting that the Maxwell elements are redundant in this model when the body undergoes rapid cycles. This may be confirmed by inspecting Eqn.(6.13), in which the only non-zero value of s_i is the stress in the elastic element (denoted by s_o in Fig.3.8). We may conclude therefore that the simple three-parameter model is sufficient for describing the rapid cycle behaviour of the general linear viscoelastic solid.

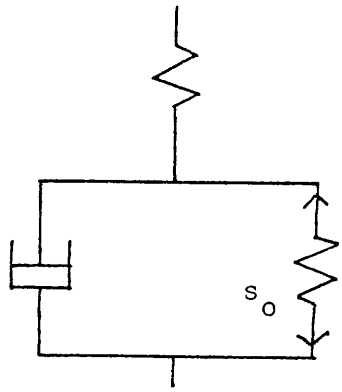


Fig. 3.8
Three parameter model

CHAPTER 7

THE CYCLIC INCREMENT OF STRAIN

In this chapter we investigate the magnitude of the accumulation of strain in the stationary cyclic state. A property of the strain increment is established which permits the determination of the rapid cycle solution σ^r in the following chapter.

We start by observing that in the stationary cyclic state the accumulated cyclic strain, $\Delta\varepsilon$ is purely viscous, irrespective of the nature of the stress history. Thus

$$\Delta\varepsilon = \Delta v = \frac{1}{k} \int_0^{\Delta t} \sigma(t)dt \tag{7.1}$$

In general there is a time-dependent stress redistribution which also depends on the history of strain. Only in the extreme cases of slow and rapid cycling is the redistribution effectively equal to zero, permitting direct integration in Eqn.(7.1). For the general cycle we begin by defining the quantities $\tilde{\sigma}$ and $\tilde{\varepsilon}$ as follows:

$$\tilde{\sigma}(t_1) = \sigma(t_1) - \sigma^s(t_1) \tag{7.2}$$

$$\tilde{\varepsilon}(t_2) = \varepsilon(t_2) - \varepsilon^s(t_2) \tag{7.3}$$

where t_1 and t_2 are arbitrary instants within the cycle, $\varepsilon(t_2)$ corresponds to $\sigma(t_2), \varepsilon^s(t_2)$ corresponds to the steady state stress $\sigma^s(t_2)$ and all terms in (7.2) and (7.3) are admissible. From the principle of virtual work:

$$\int_V \tilde{\sigma}(t_1)\tilde{\varepsilon}(t_2)dv = 0 \tag{7.4}$$

Specifically from Eqn.(7.4)

and

$$\left.\begin{aligned} \int_V \tilde{\sigma}(t)\tilde{\varepsilon}(T)dv &= 0 \\ \int_V \tilde{\sigma}(t)\tilde{\varepsilon}(T+\Delta t)dv &= 0 \end{aligned}\right\} \tag{7.5}$$

from which we obtain

$$\int_V \tilde{\sigma}(t)\, \Delta\tilde{\varepsilon}\, dv = 0 \tag{7.6}$$

As has been observed, the accumulations of strain in the stationary cyclic state are purely viscous and so we may write Eqn.(7.6) as

$$\int_V \tilde{\sigma}(t)\ \Delta\tilde{v}\ dv = 0 \tag{7.7}$$

In view of the fact that Eqn.(7.7) holds for arbitrary time t, and that $\Delta\tilde{v}$ is independent of the value of t, we obtain

$$\int_V \int_o^{\Delta t} \tilde{\sigma}(t)\ \Delta\tilde{v}\ dvdt = 0$$

or

$$\int_V \Delta\tilde{v} \int_o^{\Delta t} \tilde{\sigma}(t) dtdv = 0$$

and so from Eqn.(7.1) it follows that

$$\int_V k(\Delta\tilde{v})^2\ dv = 0 \tag{7.8}$$

Equation (7.8) requires that $\Delta\tilde{v} \equiv 0$, or in other words

$$\Delta v = \Delta v^s \tag{7.9}$$

where Δv is the total cycle of strain in the actual body and Δv^s is the total cycle in the same location in the body, resulting from a stress distribution σ^s. Thus, in the stationary cyclic state, <u>the strain accumulates at a rate which is independent of the cycle time</u>. This may be expressed as follows:

$$\frac{\Delta\varepsilon}{\Delta t} = \frac{\Delta\varepsilon^r}{\Delta t^r} = \frac{\Delta v^s}{\Delta t^s} \tag{7.10}$$

where $\Delta\varepsilon^r$ and Δv^s are the strain increments for rapid and slow cycles of length Δt^r and Δt^s respectively.

CHAPTER 8

RAPID CYCLING

In obtaining the upper bound on the cyclic work, (6.17), we observed that a simple method was required for determining the rapid cycle stress solution σ^r. In this chapter we describe such a method and illustrate its use through the example of a cyclically loaded viscoelastic beam.

8.1 The upper bound stress history from the compatibility condition

We consider a cycle of loading $0 \leq t \leq \Delta t$ in which the following load is applied:

$$\underline{P}(t) = \underline{P}_o \, f(\frac{t}{\Delta t}) \tag{8.1}$$

where $\underline{P}_o$ is the maximum load in the cycle. The corresponding stress histories in the extreme cases are given by

$$\sigma^s(t) = \sigma_o^s \, f(\frac{t}{\Delta t}) \qquad 0 \leq t \leq \Delta t^s \tag{8.2}$$

for slow cycles, and

$$\sigma^r(t) = \hat{\sigma}(t) + \bar{\rho}^r \qquad 0 \leq t \leq \Delta t^r \tag{8.3}$$

for fast cycles. The elastic solution is given by

$$\hat{\sigma}(t) = \hat{\sigma}_o \, f(\frac{t}{\Delta t}) \tag{8.4}$$

The cyclic behaviour of a non-linear Maxwell material was described by Ponter (1972,1973a) in which he showed that $\Delta\varepsilon^r$, the strain increment corresponding to σ^r, is uniquely admissible, and that the constant residual stress $\bar{\rho}^r$ is unique in generating $\Delta\varepsilon^r$. A consequence of this in the linear case is that we may choose an arbitrary constant residual stress $\bar{\rho}$ and use the compatibility condition to determine $\bar{\rho}^r$. A suitable form for $\bar{\rho}$ is this:

$$\bar{\rho} = \mu(\sigma_o^s - \hat{\sigma}_o) \tag{8.5}$$

where μ is an arbitrary constant to be determined. Clearly $\bar{\rho}$ in Eqn.(8.5) is in equilibrium with zero applied loads.

The compatibility condition requires that the strain increment generated by the stress field $(\hat{\sigma}+\rho)$ is to be admissible, and so in view of Eqn.(7.10) it follows that

$$\frac{1}{\Delta t}\int_0^{\Delta t} (\hat{\sigma}+\bar{\rho})dt = \frac{1}{\Delta t^s}\int_0^{\Delta t} \sigma^s \, dt \tag{8.6}$$

Setting $\tau = \frac{t}{\Delta t}$ and substituting from Eqns.(8.2) and (8.4) we obtain from (8.6):

$$\int_0^1 (\hat{\sigma}_o f(\tau) + \bar{\rho})d\tau = \int_0^1 \sigma_o^s \, f(\tau)d\tau$$

or

$$\bar{\rho} = (\sigma_o^s - \hat{\sigma}_o) \int_0^1 f(\tau)d\tau \tag{8.7}$$

Comparing Eqns.(8.5) and (8.7) it follows that if $\mu = \int_0^1 f(\tau)d\tau$ then $\bar{\rho}$ satisfies the compatibility condition, and since $\bar{\rho}=\bar{\rho}^r$ is unique in this respect, we have

$$\bar{\rho}^r = (\sigma_o^s - \hat{\sigma}_o) \int_0^1 f(\tau)d\tau \tag{8.8}$$

Since the compatibility condition, Eqn.(8.6) is in the form of a strain rate integral over a cycle, the condition is applicable to any material for which the mean rate of accumulation of strain over a cycle $\frac{\Delta\varepsilon}{\Delta t}$ is proportional to the mean stress during the cycle $\frac{1}{\Delta t}\int_0^{\Delta t}\sigma(t)dt$. We have observed that in the stationary cyclic state a viscoelastic material accumulates only viscous strain. Consequently Eqn.(8.8) applies to the general linear viscoelastic material. On account of this the rapid cycle stress history required for the upper work bound is given by

$$\sigma^r = \hat{\sigma}_o \, f(\tau) + (\sigma_o^s - \hat{\sigma}_o) \int_0^1 f(\tau)d\tau \tag{8.9}$$

where $f(\tau)$ is prescribed, and we see that the problem of determining σ^r amounts to a determination of the elastic and steady state solutions for step loading.

8.2 An example

To illustrate the use of the compatibility condition in determining the rapid cycle solution, consider the cyclically loaded viscoelastic beam shown in Figure 3.9.

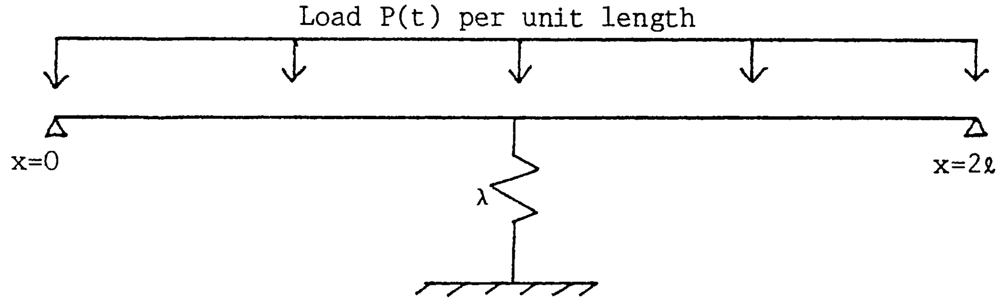


Fig. 3.9

We first imagine the beam to be uniformly step-loaded and we find the central reaction R in the following circumstances:

$R = R_o^s$: the stationary solution

$R = \hat{R}_o$: the elastic solution

Using Eqn.(8.9) we are then able to determine the rapid cycle reaction R^r . Following this we consider the beam under step cycles of loading and we determine the rapid cycle reaction from the full cyclic solution.

Step loading

The elastic solution for the vertical deflection $y(x)$ is given by

$$y(x) = \frac{P_o}{24EI}\left[(8\ell^3-4\ell x^2+x^3)x - \frac{5\lambda\ell^4 x(3\ell^2-x^2)}{2(6EI+\lambda\ell^3)}\right] \qquad (8.10)$$

On account of the "correspondence principle" (Flugge, 1967) the solution for a linear viscoelastic beam may be obtained by replacing E in Eqn. (8.10) by a function of the Laplace transforms of the differential generators for the viscoelastic material. For a Maxwell material, the central deflection becomes

$$y(\ell,t) = \frac{5P_o\ell}{4\lambda}\,[1 - B_1 e^{-k_1 t}] \tag{8.11}$$

where

$$B_1 = \frac{6EI}{6EI + \lambda\ell^3} \tag{8.12}$$

and

$$k_1 = \frac{\lambda\ell^3 E}{k(6EI + \lambda\ell^3)} \tag{8.13}$$

For a three-parameter Voight material the central deflection is given by

$$y(\ell,t) = \frac{5P_o\ell^4[1-B_2 e^{-k_2 t}]}{4[\lambda\ell^3 + \frac{6IEE'}{E+E'}]} \tag{8.14}$$

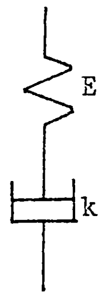


Fig.3.10

where

$$B_2 = \frac{6IE'^2}{(6IE'+\lambda\ell^3)(E+E')} \tag{8.15}$$

and

$$k_2 = \frac{6IEE' + \lambda\ell^3(E+E')}{k(6IE' + \lambda\ell^3)} \tag{8.16}$$

In both materials the central reaction is given by

$$R(t) = \lambda y(\ell,t) \tag{8.17}$$

At $t = 0$ we obtain the elastic solutions:

$$\hat{R}_o = \frac{5P_o\ell}{4}(1-B_1) \quad : \text{Maxwell model} \tag{8.18}$$

$$\hat{R}_o = \frac{5\lambda P_o\ell^4\,(1-B_2)}{4\left[\lambda\ell^3 + \frac{6IEE'}{(E+E')}\right]} \quad : \text{Three-parameter model} \tag{8.19}$$

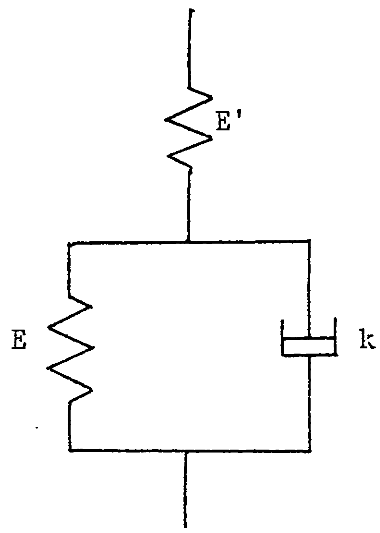


Fig.3.11

As $t \to \infty$ we obtain the stationary solutions:

$$R_o^s = \frac{5P_o\ell}{4} \quad : \quad \text{Maxwell model} \tag{8.20}$$

$$R_o^s = \frac{5\lambda P_o \ell^4}{4\left[\lambda\ell^3 + \frac{6IEE'}{(E+E')}\right]} \quad : \text{Three-parameter model} \tag{8.21}$$

As a result of Eqns.(8.18) to (8.21) we may write

$$\hat{R}_o = R_o^s\,[1 - B_i] \quad , \quad i=1,2 \tag{8.22}$$

to describe both materials. Furthermore, from Eqns.(8.11) to (8.17) we have

$$R(t) = R_o^s\,(1 - B_i\, e^{-k_i t}) \quad , \quad i=1,2 \tag{8.23}$$

for the reaction at time t in both materials.

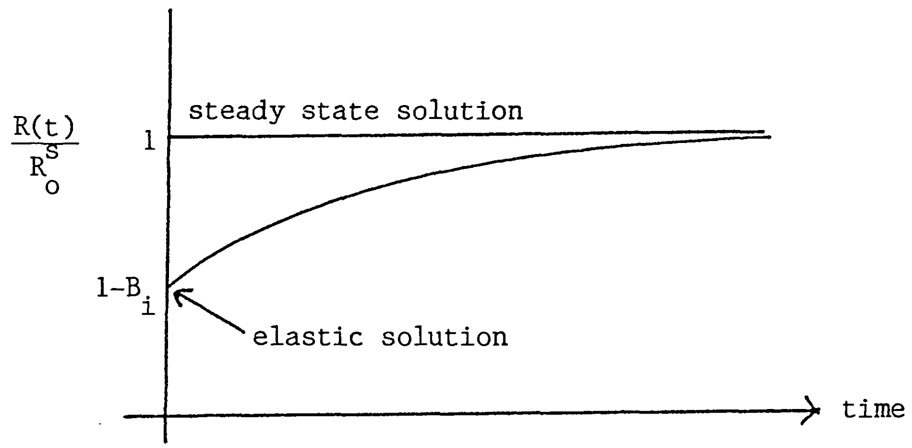


Fig.3.12

The actual loading

Suppose that the actual loading applied to the beam is as follows:

$P(t) = P_o$ during each first half cycle

$P(t) = 0$ for the remainder

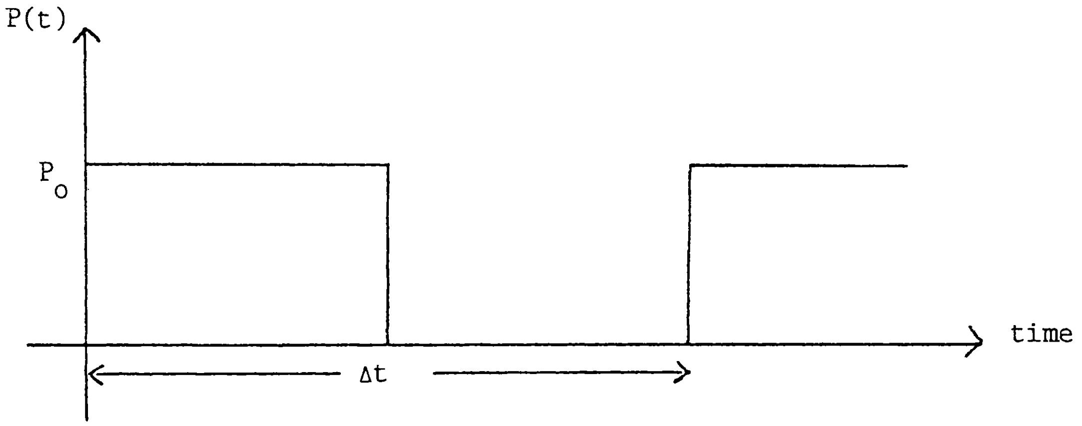


Fig.3.13

From the definition of $f(\frac{t}{\Delta t})$ in Eqn.(8.1) it follows that

$$\int_o^1 f(\tau)d\tau = \tfrac{1}{2} \tag{8.24}$$

If we combine Eqns.(8.24) and (8.9) we obtain

$$R^r = \hat{R}_o + \tfrac{1}{2}\,(R_o^s - \hat{R}_o) \quad : \quad \text{First half cycles}$$

$$= \frac{1}{2}\,(R_o^s - \hat{R}_o) \quad : \quad \text{Second half cycles}$$

and from Eqn.(8.22) this becomes

$$\left.\begin{aligned} \frac{R^r}{R_o^s} &= 1 - \tfrac{1}{2}\, B_i \quad : \quad \text{First half cycles} \\ &= \tfrac{1}{2}\, B_i \quad : \quad \text{Second half cycles} \end{aligned}\right\} \tag{8.25}$$

Eqns.(8.25) are the rapid cycle reaction for both materials, derived from the compatibility condition.

The cyclic solution

To obtain the same answer from the full cyclic solution we proceed as follows: we first sum the series of terms obtained from Eqn.(8.23) in which R_o^s is given in first half-cycles by Eqns.(8.20) and (8.21) respectively for the two materials. After the necessary algebra this yields the following:

$$\frac{R(t)}{R_s^o} = 1 - \frac{B_i}{1 + \exp(-\frac{1}{2} k_i \Delta t)} \quad : \quad \text{start of a cycle}$$

$$= 1 - \frac{B_i}{1 + \exp(\frac{1}{2} k_i \Delta t)} \quad : \quad \text{end of first half cycle}$$

$$= \frac{B_i}{1 + \exp(-\frac{1}{2} k_i \Delta t)} \quad : \quad \text{start of second half of cycle}$$

$$= \frac{B_i}{1 + \exp(\frac{1}{2} k_i \Delta t)} \quad \quad \text{end of cycle}$$

The rapid cycle solution is now obtained by letting Δt approach zero, from which we obtain

$$\left.\begin{aligned} \frac{R^r}{R_o^s} &= 1 - \tfrac{1}{2} B_i \quad : \quad \text{throughout first half cycles} \\ &= \tfrac{1}{2} B_i \quad : \quad \text{throughout the remainder} \end{aligned}\right\} \quad (8.26)$$

Eqn.(8.26) is the rapid cycle solution in the two materials calculated from the full cyclic solution, and it is identical to the result obtained from the compatibility condition using only the elastic and steady state solutions.

CHAPTER 9

AN EXTENDED WORK INEQUALITY

The increment of work done in a stationary state cycle in a linear viscoelastic body has been shown to be bounded above by the dissipation due to the rapid cycle stress distribution and below by that due to the steady state distribution. Combining the two bounds (6.7) and (5.14) we have the following extended work inequality:

$$\int_V \int_0^{\Delta t} D(\sigma^s)\,dt\,dv \leq \int_V \int_0^{\Delta t} \sigma(t)\dot{\varepsilon}(t)\,dt\,dv \leq \int_V \int_0^{\Delta t} D(\sigma^r)\,dt\,dv \qquad (9.1)$$

At this point it might be emphasised that while we have shown the strain to accumulate at equal rates in fast and slow cycling (Eqn.(7.10)) the corresponding work quantities generally differ. For example, for a linear Maxwell material the work bounds are given by the following:

$$\text{Upper bound : } \int_V \int_0^{\Delta t} D(\sigma^r)\,dt\,dv = \int_V \frac{1}{k} \int_0^{\Delta t} (\hat{\sigma} + \bar{\rho}^r)^2\,dt\,dv$$

$$\text{Lower bound : } \int_V \int_0^{\Delta t} D(\sigma^s)\,dt\,dv = \int_V \frac{1}{k} \int_0^{\Delta t} (\sigma^s)^2\,dt\,dv$$

Non-dimensionalising the cycle times, the difference between these bounds becomes

$$\int_V \frac{1}{k} \int_0^1 \left\{ (\hat{\sigma}+\bar{\rho}^r)^2 - (\sigma^s)^2 \right\} dt\,dv$$

$$= \int_V \frac{dv}{k} \int_0^1 \{ (\hat{\sigma}_o f(\tau) + \bar{\rho}^r)^2 - (\sigma_o^s f(\tau))^2 \} d\tau$$

$$= \int_V \frac{dv}{k} (\hat{\sigma}_o^2 - \sigma_o^{s2})\ (\alpha - \beta) \qquad (9.2)$$

on noting Eqn.(8.8), where $\alpha = \int_0^1 f^2 d\tau$ and $\beta = \left\{ \int_0^1 f d\tau \right\}^2$. The factor $(\alpha-\beta)$ is non-negative, for $\int_0^1 \left\{ f(\tau) - \int_0^1 f(\tau) d\tau \right\}^2 d\tau \geq 0$ and so $\int_0^1 \left\{ f^2 - 2f \int_0^1 f d\tau + \beta \right\} d\tau \geq 0$. Moreover unless $P(t) \equiv P_o$ for all t, the factor is non-zero. Also, Ponter (1972) has shown for a Maxwell material that the dissipation corresponding to σ^s is an absolute minimum amongst admissible stress fields. Consequently the integral in Eqn.(9.2) is never negative, although it may be zero, as in plane stress and plane strain problems where $\hat{\sigma}_o = \sigma_o^s$. However, for any visco-elastic material whose rheological model includes a Voight element, the upper bound contains additional positive terms and the two bounds can never be equal.

CHAPTER 10

FURTHER INEQUALITIES

The basis of the upper work bound obtained in Chapter 6 was the determination of an appropriate form of the functional $w(\underline{\sigma}^{**}(t),T)$ defined by Ponter (1974). This functional was used in section two to obtain general work and displacement bounds for a state variable description of material behaviour and the bound in Chapter 6 was a particular example of such a description. In this chapter we investigate the result of using a different description of material behaviour. An alternative form for w is obtained and specific examples are derived in which w depends only upon the arbitrary stress history $\sigma^{**}(t)$ and not on any internal variable. An existing work bound is recovered and a new strain bound is derived. The method is investigated through the calculation of an example for the latter bound.

We begin with the determination of w.

10.1 The functional w by calculus of variations

Following Ponter (1974), the functional W is defined as follows:

$$W(\underline{\sigma}^{**}(t),\underline{\sigma}(t)) = \int_0^T \{\underline{\sigma}^{**}(t)-\underline{\sigma}(t)\}\dot{\underline{\varepsilon}}'(t)dt \qquad (10.1)$$

where T is a prescribed time, the inelastic strain rate $\dot{\underline{\varepsilon}}'$ corresponds to the stress history $\underline{\sigma}(t)$ in the body and $\underline{\sigma}^{**}(t)$ is an arbitrary prescribed stress history. It is assumed that this functional is bounded above and we denote the optimum upper bound (if it exists) by $w(\underline{\sigma}^{**}(t),T)$.

From Eqn.(2.5) and on account of Bland's description of linear viscoelasticity, the functional W in Eqn.(10.1) consists of the sum of nine deviatoric and one dilatational energy expressions of the form

$$W_j = \int_0^T \left\{\sigma^{**}(t) - B\dot{\varepsilon}'(t) - \int_0^t G'(t-\tau)\dot{\varepsilon}'(\tau)d\tau - G'(t)\varepsilon'_o\right\}\dot{\varepsilon}'(t)dt \quad (10.2)$$

which we may rewrite as follows:

$$W_j = \int_0^T \left\{\sigma^{**}(t) - B\dot{\varepsilon}'(t) - G'(t)\varepsilon'_o\right\}\dot{\varepsilon}'(t)dt - \frac{1}{2}\int_0^T\int_0^T G'|t-\tau|\,\dot{\varepsilon}'(\tau)\dot{\varepsilon}'(t)d\tau\,dt \quad (10.3)^*$$

where $j = 1,2,\ldots,10$. For brevity henceforth we shall write W_j as W and w_j as w, understanding that each represents a single typical member of the set of ten energy expressions.

The maximum functional w is now obtained. We consider a variation in $\dot{\varepsilon}'(t)$ from the history that optimises W :

$$\dot{\varepsilon}'(t) = \dot{\varepsilon}'_{opt}(t) + \alpha\dot{\eta}(t) \quad ; \quad 0 \leqslant t \leqslant T \quad (10.4)$$

where $\dot{\eta}(t)$ is an arbitrarily fixed function subject to $\dot{\eta}(0) = \dot{\eta}(T) = 0$. From Eqn.(10.3) we obtain

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* A proof of the equality of the right-hand sides of Eqns.(10.2) and (10.3) is given in Appendix 3.3.

$$W(\dot{\varepsilon}'_{opt} + \alpha\dot{\eta}) = \int_o^T \{\sigma^{**}(t) - B\dot{\varepsilon}'_{opt}(t) - \alpha B\dot{\eta}(t) - G'(t)\varepsilon'_o\}\{\dot{\varepsilon}'_{opt}(t) + \alpha\dot{\eta}(t)\}dt$$
$$- \frac{1}{2}\int_o^T\int_o^T G'|t-\tau|\{\dot{\varepsilon}'_{opt}(\tau) + \alpha\dot{\eta}(\tau)\}\{\dot{\varepsilon}'_{opt}(t) + \alpha\dot{\eta}(t)\}d\tau dt$$
$$= \int_o^T \sigma^{**}(t) - B\dot{\varepsilon}'_{opt}(t) - G'(t)\varepsilon'_o\}\dot{\varepsilon}'_{opt}(t)dt + \alpha\int_o^T \{\dot{\eta}(t)\sigma^{**}(t) - 2B\dot{\eta}(t)\dot{\varepsilon}'_{opt}(t)$$
$$- \dot{\eta}(t)G'(t)\varepsilon'_o\}dt - \alpha^2 B\int_o^T \dot{\eta}^2(t)dt$$
$$- \frac{1}{2}\int_o^T\int_o^T G'|t-\tau|\dot{\varepsilon}'_{opt}(\tau)\dot{\varepsilon}'_{opt}(t)d\tau dt - \frac{\alpha}{2}\int_o^T\int_o^T G'|t-\tau|\{\dot{\varepsilon}'_{opt}(t)\dot{\eta}(\tau)$$
$$+ \dot{\varepsilon}'_{opt}(\tau)\dot{\eta}(t)\}dtd\tau - \frac{\alpha^2}{2}\int_o^T\int_o^T G'|t-\tau|\dot{\eta}(\tau)\dot{\eta}(t)dtd\tau$$
$$= \int_o^T \{\sigma^{**}(t) - B\dot{\varepsilon}'_{opt}(t) - G'(t)\varepsilon'_o - \frac{1}{2}\int_o^T G'|t-\tau|\dot{\varepsilon}'_{opt}(\tau)d\tau\}\dot{\varepsilon}'_{opt}(t)dt$$
$$+ \alpha\Bigg[\int_o^T \sigma^{**}(t)\dot{\eta}(t)dt - 2B\int_o^T \dot{\eta}(t)\dot{\varepsilon}'_{opt}(t)dt - \int_o^T \dot{\eta}(t)G'(t)\varepsilon'_o dt$$
$$- \frac{1}{2}\int_o^T\int_o^T G'|t-\tau|\{\dot{\varepsilon}'_{opt}(t)\dot{\eta}(\tau) + \dot{\varepsilon}'_{opt}(\tau)\dot{\eta}(t)\}d\tau dt\Bigg]$$
$$- \alpha^2\left[B\int_o^T \dot{\eta}^2(t)dt + \frac{1}{2}\int_o^T\int_o^T G'|t-\tau|\dot{\eta}(\tau)\dot{\eta}(t)d\tau dt\right] \qquad (10.5)$$

Equation (10.5) has the form $W(\dot{\varepsilon}') = W(\dot{\varepsilon}'_{opt}) + \alpha X - \alpha^2 Y$ where Y is positive. The conditions for a maximum value of W with respect to $\dot{\varepsilon}'(t)$ is thus $X = 0$, or

$$\sigma^{**}(t) = 2B\dot{\varepsilon}'_{opt}(t) + G'(t)\varepsilon'_o + \int_o^T G'|t-\tau|\dot{\varepsilon}'_{opt}(\tau)d\tau \qquad (10.6)^*$$

---

* For then $\int_o^T \sigma^{**}(t)\dot{\eta}(t)dt = 2B\int_o^T \dot{\varepsilon}'_{opt}(t)\dot{\eta}(t)dt + \int_o^T \dot{\eta}(t)G'(t)\varepsilon'_o dt +$
$\int_o^T\int_o^T \dot{\eta}(t)G'|t-\tau|\dot{\varepsilon}'_{opt}(\tau)d\tau dt$ and the last term is identical to
$\int_o^T\int_o^T \dot{\eta}(\tau)G'|t-\tau|\dot{\varepsilon}'_{opt}(t)dtd\tau$ by interchanging the variables t and τ.
It thus follows that the coefficient of α vanishes, and W attains a maximum. (See, e.g. Elsgolc "Calculus of Variations" 1961).

Equation (10.6) is a Fredholm equation of the second kind with a symmetrical kernel (see, e.g. Tricomi "Integral Equations", 1967). Although in the general case it does not appear possible to invert Eqn.(10.6) there are simple cases where this can be done and we may then obtain expressions for w in terms of $\sigma^{**}(t)$.* The stress history σ_{opt} that corresponds to the optimising strain rate history $\dot{\varepsilon}'_{opt}$ is given by the constitutive equation:

$$\sigma_{opt}(t) = B\dot{\varepsilon}'_{opt}(t) + G'(t)\varepsilon'_{o} + \int_{o}^{t} G'(t-\tau)\varepsilon'_{opt}(\tau)d\tau \qquad (10.7)$$

Substituting eqns.(10.6) and (10.7) into (10.1) we obtain the maximum value of W :

$$w = \int_{o}^{T}\left\{B\dot{\varepsilon}'_{opt}(t) + \frac{1}{2}\int_{o}^{T} G'|t-\tau|\,\dot{\varepsilon}'_{opt}(\tau)d\tau\right\}\dot{\varepsilon}'_{opt}(t)dt$$

or

$$w = \frac{1}{2}\int_{o}^{T} \{\sigma^{**}(t) - G'(t)\varepsilon'_{o}\}\dot{\varepsilon}'_{opt}(t)dt \qquad (10.8)$$

In the following simple cases the functional w can be expressed entirely in terms of $\sigma^{**}(t)$:

The Maxwell material

For this model we have $B = k$ and $G' = 0$ and so from Eqn.(10.6):

$$\dot{\varepsilon}'_{opt} = \frac{1}{2k}\sigma^{**}(t) \qquad (10.9)$$

and from Eqn.(10.8):

$$w(\sigma^{**},T) = \frac{1}{4k}\int_{o}^{T}(\sigma^{**}(t))^{2}\,dt \qquad (10.10)$$

Fig.3.14

* The general case is discussed in Appendix 3.4

The three-parameter Voight solid

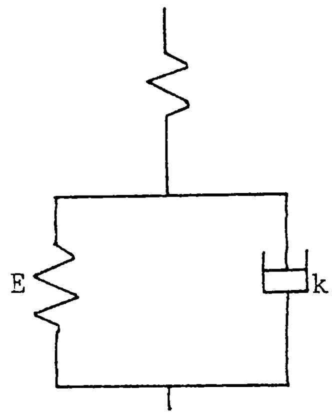

Fig.3.15

For this model, $B = k$ and $G' = E$. From Eqn.(10.6):

$$\sigma^{**}(t) = 2k\dot{\varepsilon}'_{opt}(t) + E\varepsilon'_{opt}(T) \quad (10.11)$$

Equation (10.11) can be inverted by integrating from $t = 0$ to $t = T$:

$$\int_o^T \sigma^{**}dt = \varepsilon'_{opt}(T)\,[2k+ET] - 2k\varepsilon'_o \qquad (10.12)$$

Case 1 If $\varepsilon'_o = 0$ we obtain

$$\dot{\varepsilon}'_{opt}(t) = \frac{1}{2k}\left[\sigma^{**}(t) - \frac{E}{ET+2k}\int_o^T \sigma^{**}dt\right]$$

and from Eqn.(10.8)

$$w(\sigma^{**},T) = \frac{1}{4k}\left[\int_o^T (\sigma^{**})^2 dt - \frac{E}{ET+2k}\left\{\int_o^T \sigma^{**}dt\right\}^2\right] \qquad (10.13)$$

Case 2 If the interval from $t = 0$ to $t = T$ is a cycle of loading in which the body is in a stationary cyclic state of stress we have $\varepsilon'_{opt}(T) = \varepsilon'_o$ and so from Eqns.(10.11) and (10.12):

$$\dot{\varepsilon}'_{opt}(t) = \frac{1}{2k}\left[\sigma^{**}(t) - \frac{1}{T}\int_o^T \sigma^{**}dt\right] \qquad (10.14)$$

and from Eqn.(10.8):

$$w(\sigma^{**},T) = \frac{1}{4k}\left[\int_o^T (\sigma^{**})^2 dt - \frac{1}{T}\left\{\int_o^T \sigma^{**}dt\right\}^2\right] \qquad (10.15)$$

In the remaining parts of this chapter we show how the functional w can be used to formulate work and strain bounds. In the former case we use Eqn.(10.15) to recover a work bound for the three parameter model that was previously obtained using the state variable approach.

In the latter case we calculate the exact accumulation of inelastic strain for a three parameter model and using Eqn.(8.13) we calculate the corresponding upper bound.

10.2 A total inelastic work bound

The stress $\underline{\sigma}^{**}$ in Eqn.(10.8) is arbitrary. By setting $\underline{\sigma}^{**} = \mu\underline{\sigma}^{**} = \mu(\hat{\underline{\sigma}} + \bar{\underline{\rho}})$, where $\hat{\underline{\sigma}}$ is the elastic solution, $\bar{\underline{\rho}}$ is an arbitrary residual stress field and μ is an arbitrary constant, the following bound on the total inelastic work was obtained by Ponter (1974):

$$\int_V \int_o^T \underline{\sigma}\,\dot{\underline{\varepsilon}}'\,dtdv \leq \frac{\mu}{\mu-1}\{A(0)-A(T)\} + \frac{1}{\mu-1}\int_V w(\mu\underline{\sigma}^*, T)dv \qquad (10.16)$$

where the elastic terms $A(0)$ and $A(T)$ are defined from the equations

$$A(t) = \int_V E\{\underline{\sigma}(t) - \underline{\sigma}^*(t)\}dv$$

$$E(\underline{\sigma}) = \tfrac{1}{2}\underline{\sigma}\,\underset{\sim}{C}\,\underline{\sigma}$$

and where $\mu > 1$.

10.3 The cyclic work bound

In the stationary cyclic state with $0 \leq t \leq T$ denoting one cycle, the elastic term $(A(0) - A(T))$ vanishes. For a linear visco-elastic material, $w(\sigma^{**})$ is a quadratic function of σ^{**} and so inequality (10.16) becomes

$$\int_V \int_o^T \sigma\,\dot{\varepsilon}\,dtdv \leq \frac{\mu^2}{\mu-1}\int_V w(\sigma^*)dv$$

in which it is easily shown that $\mu_{opt} = 2$. Inequality (10.16) then reduces to

$$\int_V \int_o^T \sigma\,\dot{\varepsilon}\,dtdv \leq \int_V w(2\sigma^*)dv \qquad (10.17)$$

We can further optimise inequality (10.17) with an appropriate choice of the arbitrary stress σ^* , by using the fact that a small variation in σ^* from its optimum value will result in the first variation of $\int w(2\sigma^*)dv$ vanishing. It is first noted that a variation in $\sigma^*(t)$ gives rise to a corresponding variation in $\dot{\varepsilon}'_{opt}(t)$, obtained from Eqn.(10.6):

$$\delta(2\sigma^*) = 2B\delta\dot{\varepsilon}'_{opt}(t) + \int_0^T G'|t-\tau|\delta\dot{\varepsilon}'_{opt}(\tau)d\tau \tag{10.18}$$

From Eqns.(10.6) and (10.8) we express w entirely in terms of $\dot{\varepsilon}'_{opt}$ as follows:

$$w = \frac{1}{2}\int_0^T\left[2B\dot{\varepsilon}'_{opt}(t) + \int_0^T G'|t-\tau|\dot{\varepsilon}'_{opt}(\tau)d\tau\right]\dot{\varepsilon}'_{opt}(t)dt \tag{10.19}$$

and hence

$$\delta\int_V w\,dv = \int_V\int_0^T\left[2B\dot{\varepsilon}'_{opt}(t)\delta\dot{\varepsilon}'_{opt}(t) + \dot{\varepsilon}'_{opt}(t)\int_0^T G'|t-\tau|\delta\dot{\varepsilon}'_{opt}(\tau)d\tau\right]dtdv$$

$$= 0 \tag{10.20}$$

Substituting Eqn.(10.18) into (10.20):

$$\int_V\int_0^T\delta(2\sigma^*)\dot{\varepsilon}'_{opt}\,dtdv = 0 \tag{10.21}$$

Since $\sigma^* = \hat{\sigma} + \bar{\rho}$ where $\hat{\sigma}$ is the elastic solution for prescribed load, it follows that $\delta\sigma^* = \delta\bar{\rho}$ = constant and so Eqn.(10.21) becomes

$$\int_V \delta\bar{\rho}\,\Delta\,\varepsilon_{opt}\,dv = 0 \tag{10.22}$$

Now $\delta\bar{\rho}$ is in equilibrium with zero applied loads and so Eqn.(10.22) is satisfied if $\bar{\rho}$ is such that $\Delta\varepsilon_{opt}$ is compatible. This condition is satisfied by the residual stress $\bar{\rho}^r$ corresponding to the rapid cycle stress $\sigma^r = \hat{\sigma} + \bar{\rho}^r$ and so the optimum form of inequality (10.17) is

$$\int_V\int_0^T\sigma\,\dot{\varepsilon}\,dtdv \leq \int_V w(2\sigma^r)dv \tag{10.23}$$

This is the optimum upper cyclic work bound, obtained from the variational method.

To illustrate, if we substitute from Eqn.(10.15) for the three-parameter model we obtain

$$\int_V \int_0^{\Delta t} \sigma \dot{\varepsilon}\, dt dv \leq \int_V \frac{1}{k} \left[\int_0^{\Delta t} (\sigma^r)^2 dt - \frac{1}{\Delta t} \left\{ \int_0^{\Delta t} \sigma^r dt \right\} \right] dv \qquad (10.24)$$

In the particular case of the step-cycles of stress described in Chapter 6 we would recover inequality (6.18), which we previously derived with the state variable description of material behaviour.

10.4 Inelastic strain bound

The functional w is next used to obtain a bound on the total inelastic strain. From Eqn.(10.1) and the definition of w we have

$$\int_0^T \{\sigma^{**}(t) - \sigma(t)\}\dot{\varepsilon}'(t)dt \leq w(\sigma^{**},T) \qquad (10.25)$$

We now substitute $\sigma^{**}(t) = \sigma(t) + \bar{\sigma}$, where $\bar{\sigma}$ is constant in time, to obtain

$$\varepsilon'(T) - \varepsilon'_o \leq \frac{w(\sigma+\bar{\sigma})}{\bar{\sigma}} \qquad (10.26)$$

This bound on the total inelastic strain can be optimised with respect to the arbitrary $\bar{\sigma}$ once the particular nature of w is specified for the material model. To illustrate, we consider a three-parameter model subjected to step-stressing given by

$$\sigma(t) = \sum_{j=1}^{\infty} \{H(t-(j-1)\Delta t) - H(t-(j-\tfrac{1}{2})\Delta t)\}$$

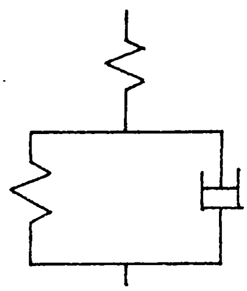

Fig.3.16

The expression for w for this model is given in Eqn.(10.13). Using this we calculate the value of the right-hand side of inequality (10.26), and we compare it with the left-hand side calculated using the full solution.

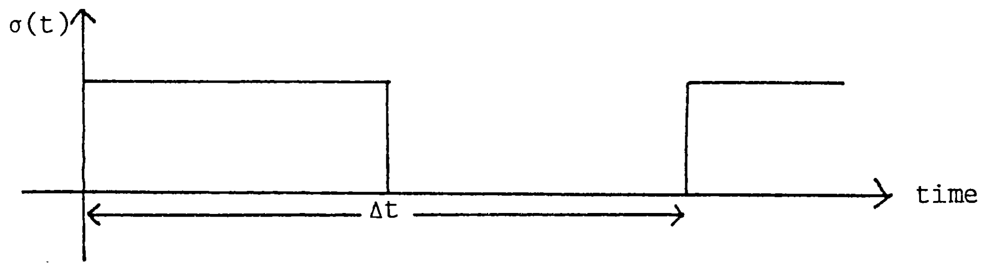

Fig.3.17

Details of the computation are given in Appendix 3.5. The period Δt and the total time T were varied and the percentage by which the bound exceeded the inelastic strain was calculated. The result of the computation indicated that for small total time the bound gave an excess of 21% and that this increased as the total time increased.

This strain bound is an example of a non-attainable upper bound and it therefore contrasts with the cyclic work bound derived from the same basis which was shown by calculation in section two to produce good answers whenever the cycle time was small.

We have shown that the bounding method based on the hereditary integral constitutive equation gives work bound results that equal those based on the state variable method. The result of the strain bound is, on the other hand, less promising, and it appears that the versatility of the state variable approach in encompassing several material descriptions at once may offer more advantages than an approach based on the detailed properties of a single material.

CHAPTER 11

STRESS REDISTRIBUTION

In this final chapter of Section 3 a description is given of the cyclic stress redistribution occurring in a body which is in the stationary cyclic state. We conclude with a simple geometrical analogy in terms of the redistribution for two of the results obtained in this section, namely the extended work bounds, (9.1), and the fact that the average strain rate is independent of the cycle time.

We imagine a generalised Voight model, Figure 3.4, subjected to on-off cycles of loading. The cycle length is given by $0 \leq \tau \leq 1$ where $\tau = \frac{t}{\Delta t}$. We suppose further that some suitable transient response time for the material, t_m, can be defined from a uniaxial creep test along similar lines to those described in Section 2. In terms of t_m, material behaviour may be called "dominantly elastic" if $\Delta t << t_m$ and the behaviour approaches the steady state if $\Delta t >> t_m$. Comparison of the stresses in a single body which is subjected to sets of loading of varying cycle times is equivalent to the comparison of a set of bodies for which t_m varies while Δt is fixed.*

In Figure 3.18 the following histories are shown: the rapid cycle solution $\sigma^r = \hat{\sigma} + \bar{\rho}^r$, the steady state solution σ^s, the history for some intermediate cycle time denoted by $\sigma(\tau)$ and the corresponding strain history $\varepsilon(\tau)$. As may be seen in Figure 3.18, for the extreme histories σ^r and $\sigma^{\bar{s}}$ there is no effective redistribution.

We have established in this section that for arbitrary cycle times the stress history $\sigma(\tau)$ is such that the following two properties hold:

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* Variation of t_m may be accomplished by fixing all the viscous coefficients and varying all the elastic moduli proportionally through the body; the distributions $\hat{\sigma}$ and σ^s are then unchanged.

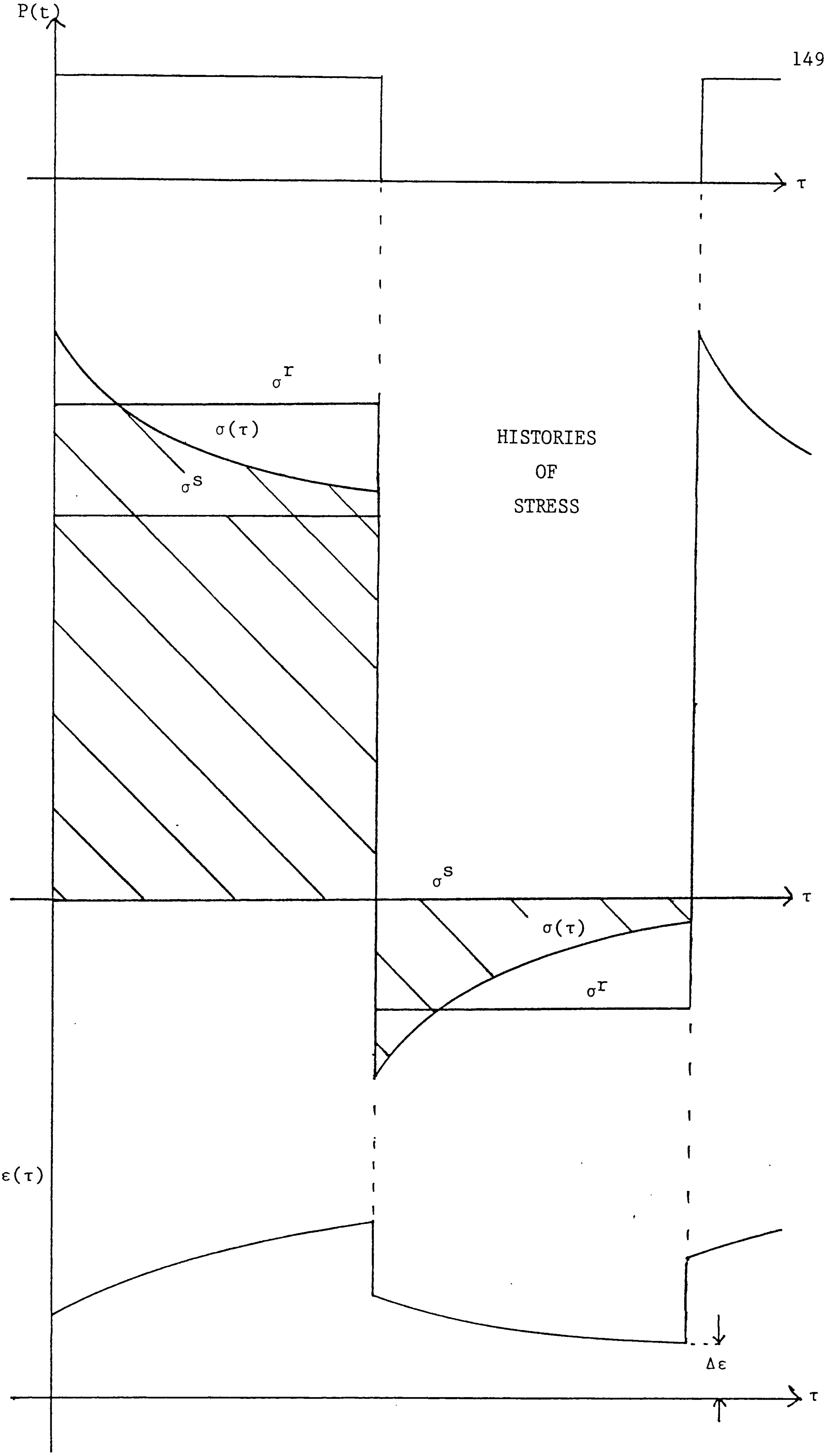

P(t)
τ
σ^r
σ(τ)
σ^s
HISTORIES
OF
STRESS
τ
σ^s
σ(τ)
σ^r
ε(τ)
Δε
τ

Fig.3.18

1. The mean rate of accumulation of strain $\frac{\Delta\varepsilon}{\Delta t}$ is independent of cycle time.

2. The cyclic work is bounded as follows:

$$\int_V \int_0^{\Delta t} D(\sigma^s)dtdv \leq \int_V \int_0^{\Delta t} D(\sigma)dtdv \leq \int_V \int_0^{\Delta t} D(\sigma^r)dtdv \qquad (11.1)$$

In terms of the stress $\sigma(\tau)$, the first property implies that $\int_0^1 \sigma(\tau)d\tau$ is constant which in geometrical terms requires the total areas enclosed in Figure 3.18 by the stress histories σ^r, σ^s and $\sigma(\tau)$ to be identical.

In the case of the simple Maxwell material, inequality (11.1) can also be interpreted in geometrical terms, for we may then write the bounds as follows:

$$\int_V \int_0^1 (\sigma^s)^2 \, d\tau dv \leq \int_V \int_0^1 \sigma^2 \, d\tau dv \leq \int_V \int_0^1 (\sigma^r)^2 d\tau dv \qquad (11.2)$$

If we denote the average value of σ^2 over the volume by $\overline{\sigma^2}$ then (11.2) takes the form

$$\int_0^1 \overline{(\sigma^s)^2} d\tau \leq \int_0^1 \overline{\sigma^2} \, d\tau \leq \int_0^1 \overline{(\sigma^r)^2} \, d\tau \qquad (11.3)$$

which places in order of size the average volumes of revolution obtained in the diagram above from the stress histories σ^s, σ and σ^r.

From further consideration of the stress redistribution it may be possible to determine precisely the way in which the dissipation rate, and hence the rate of internal entropy production vary with time. [The increment of entropy production over a cycle under isothermal conditions is proportional to the volume of revolution mentioned above.] For example, it appears probable that the rate of internal entropy production $\dot{\xi}^i$ decreases monotonically as the stress redistributes towards the steady state σ^s. Certainly we know that $\dot{\xi}^i$ is a minimum when $\sigma = \sigma^s$. Furthermore, in the rapid cycle state the residual stress is "frozen", in the sense that the history $\dot{\sigma}^r$ does not approach the history σ^s, and the mean rate of entropy production over a cycle is a maximum amongst all admissible stress fields.

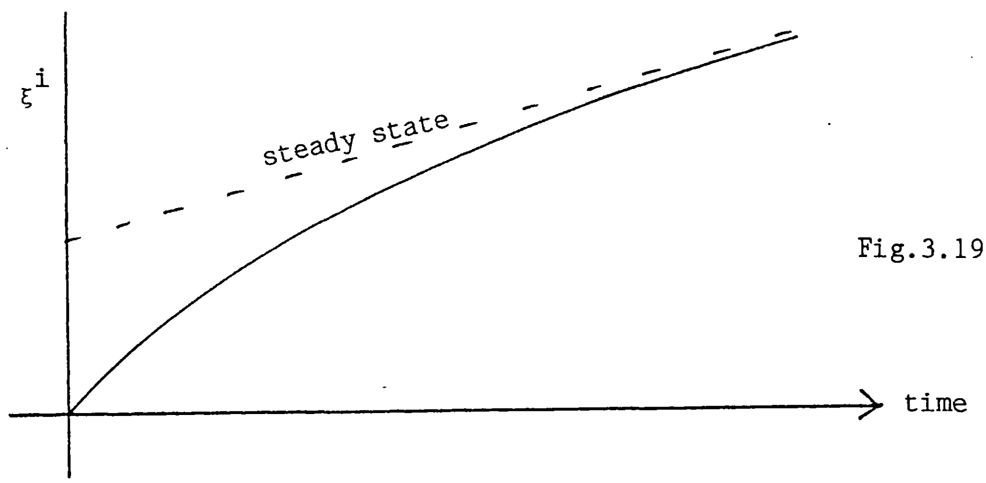


Fig.3.19

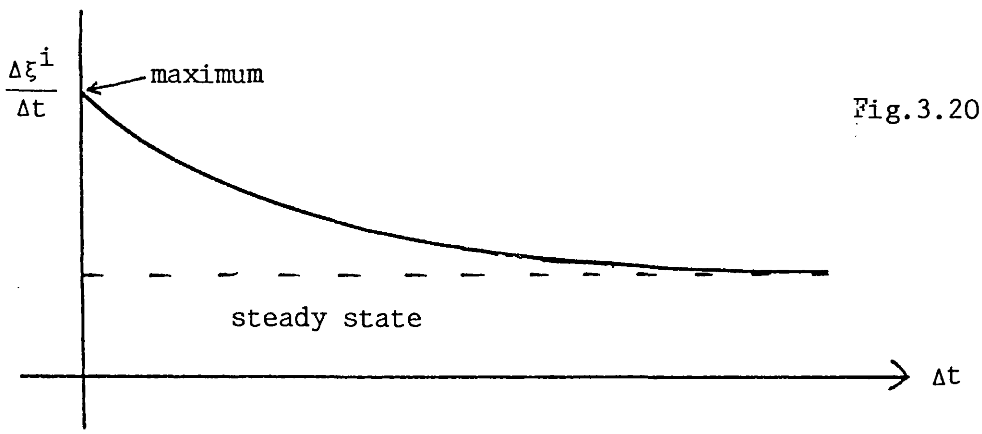


Fig.3.20

It is therefore tempting to suggest a general association between the redistribution within a creeping body and the mean rate of internal entropy production. This is an area that remains open for further study.

APPENDIX 3.1

A method for determining the material parameters required for the upper work bound (6.17)

Consider a body consisting of the general viscoelastic material shown in Fig.3.6, in which there is a uniaxial stress field $\overset{s}{\sigma}$ resulting from a constant load. The steady state of stress in the body satisfies

$$\overset{s}{\sigma} = \overset{s}{r} + \overset{s}{s}_1 + \overset{s}{s}_2 + \dots + \overset{s}{s}_m$$

where
$$\overset{s}{r} = k\,\dot{\overset{s}{v}}$$
$$s_i = k_i\,\dot{\overset{s}{v}} \quad ; \quad i = 1,2,\dots,m$$

and so the steady state creep rate is given by

$$\dot{\overset{s}{v}} = \frac{\overset{s}{\sigma}}{k + \Sigma k_i} \tag{3.1.1}$$

Suppose now that the load is instantaneously removed. The internal state of stress immediately after this instant is then given by

$$r(0) + s_1(0) + s_2(0) + \dots + s_m(0) = 0 \tag{3.1.2}$$

where
$$s_i(0) = k_i\,\dot{\overset{s}{v}} \quad ; \quad i = 1,2,\dots,m \tag{3.1.3}$$

and the removal of the load is assigned as $t = 0$.

From (3.1.2) and (3.1.3) we have

$$r(0) = -\,\dot{\overset{s}{v}}\sum_i k_i \tag{3.1.4}$$

and so the inelastic strain rate at $t = 0$ is given by

$$\dot{\varepsilon}'(0) = -\frac{\dot{\overset{s}{v}}\,\Sigma k_i}{k} \tag{3.1.5}$$

A simple creep-relaxation test conducted at known stress $\overset{s}{\sigma}$ allows $\dot{\overset{s}{v}}$ and $\dot{\varepsilon}'(0)$ to be determined and then by combining Eqns.(3.1.1) and (3.1.5) the parameters k and $\sum_i k_i$ can be found. It is easily shown that the bound (6.17) would then reduce to the following form:

$$\int_V \int_o^{\Delta t} \sigma \dot{\varepsilon} \, dt dv \lesssim \left(\frac{2 \dot{v}^s - \dot{\varepsilon}'(0)}{\sigma^s} \right) \frac{\Delta t}{4} \int_V \sigma_o^2 \, dv \qquad (3.1.6)$$

APPENDIX 3.2

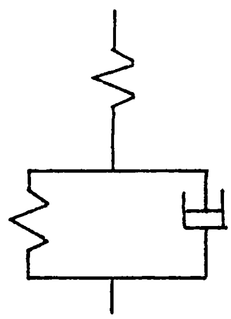
Particular cyclic work bounds are obtained from inequality (6.16) by substitution of the appropriate material constants. Three simple cases are shown below.

1. Two-parameter Maxwell element

$$k_i = 0$$

$$\text{Bound} = \Delta t \int_o \frac{\sigma_o^2}{2k} \, dv \qquad (3.2.1)$$

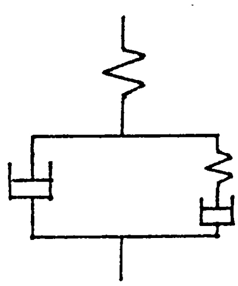
2. Three-parameter Voight solid.



Inequality (6.17) applies:

$$\text{Bound} = \Delta t \int_V \frac{\sigma_o^2}{4k} \, dv \qquad (3.2.2)$$

3. Four-parameter fluid



$$k_i = 0 \;, \; i \gtrsim 2$$

$$\text{Bound} = \Delta t \int_V \frac{\sigma_o^2 \, (2k+k_1)}{4k \, (k+k_1)} \, dv \qquad (3.2.3)$$

Equations (3.2.2) and (3.2.3) recover the examples given in Section 2.

We also note that for any given material, the viscous coefficient k can be determined from the initial inelastic strain rate $\dot{\varepsilon}'(0)$ of a previously unstressed body when a step stress $\sigma_o H(t)$ is applied:

$$k = \frac{\sigma_o}{\dot{\varepsilon}'(0)}$$

Having obtained k, bounds (3.2.1) and (3.2.2) may then be evaluated for given applied stress. It may be seen from inequality (6.16) that for the given stress history these are respectively the greatest and least upper work bounds for any possible linear viscoelastic model.

APPENDIX 3.3

Proof of the equality of $\int_o^T \int_o^t f(t-\tau)x(t)x(\tau)dtd\tau$ and $\frac{1}{2}\int_o^T \int_o^T f|t-\tau|x(t)x(\tau)dtd\tau$

Let $$I = \int_o^T \int_o^t f(t-\tau)x(t)x(\tau)dtd\tau$$

and $$I' = \int_o^T \int_t^T f(\tau-t)x(t)x(\tau)dtd\tau$$

Now $$I = \underset{\delta\to o}{\mathrm{lt}} \sum_{m=o}^{n} \sum_{j=o}^{m} f(m\delta-j\delta)x(m\delta)x(j\delta)\delta^2$$

where $j\delta = \tau$, $m\delta = t$, $n\delta = T$

Denoting the argument of I by $g(m,j)$ we have

$$I = \underset{\delta\to o}{\mathrm{lt}} \sum_{m=o}^{n} \sum_{j=o}^{m} g(m,j)$$

Similarly, $$I' = \underset{\delta\to o}{\mathrm{lt}} \sum_{m=o}^{n} \sum_{j=m}^{n} g(j,m)$$

Consequently $$I - I' = \underset{\delta\to o}{\mathrm{lt}} \left\{ \sum_{m=o}^{n} \sum_{j=o}^{m} g(m,j) - \sum_{m=o}^{n} \sum_{j=m}^{n} g(j,m) \right\}$$

Expanding the summations we obtain the following expression for the terms within the parenthesis:

$$
\begin{aligned}
\left\{\;\right\} = {} & g(0,0) \\
& + g(1,0) + g(1,1) \\
& + \\
& \vdots \\
& + g(k,0) + g(k,1) + \ldots\ldots\ldots\ldots + g(k,k) \\
& + \\
& \vdots \\
& + g(n,0) + g(n,1) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + g(n,n) \\
& - g(0,0) - g(1,0) - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - g(n,0) \\
& \qquad\quad - g(1,1) - g(2,1) \ldots\ldots\ldots\ldots\ldots\ldots - g(n,1) \\
& \qquad\qquad\qquad\qquad - \ldots\ldots\ldots\ldots\ldots\ldots \\
& \qquad\qquad\qquad\qquad\quad -g(k,k) - \ldots\ldots\ldots - g(n,k) \\
& \qquad\qquad\qquad\qquad\qquad\qquad - \ldots\ldots\ldots\ldots \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - g(n,n)
\end{aligned}
$$

By inspection, the columns of $\sum_{o}^{n} \sum_{j=o}^{m} g(m,j)$ are identical to the rows of $\sum_{o}^{n} \sum_{j=m}^{n} g(j,m)$ and so it follows that $\left\{\;\right\} \equiv 0$ for arbitrary δ.

Consequently $I = I'$ and as

$$I + I' = 2I = \int_{o}^{T}\int_{o}^{T} f\,|t-\tau|\, x(t)x(\tau)\,d\tau dt$$

the result is proved.

APPENDIX 3.4

The problem of inverting Eqn.(10.6) and hence finding an explicit formulation for w in terms of arbitrary stress history $\sigma^{**}(t)$ may be illustrated in a geometrical context as follows: we represent the problem of maximising W as a volume-maximising task in three dimensional Cartesian space, the axes representing stress, strain-rate and time as shown in Fig.3.21:

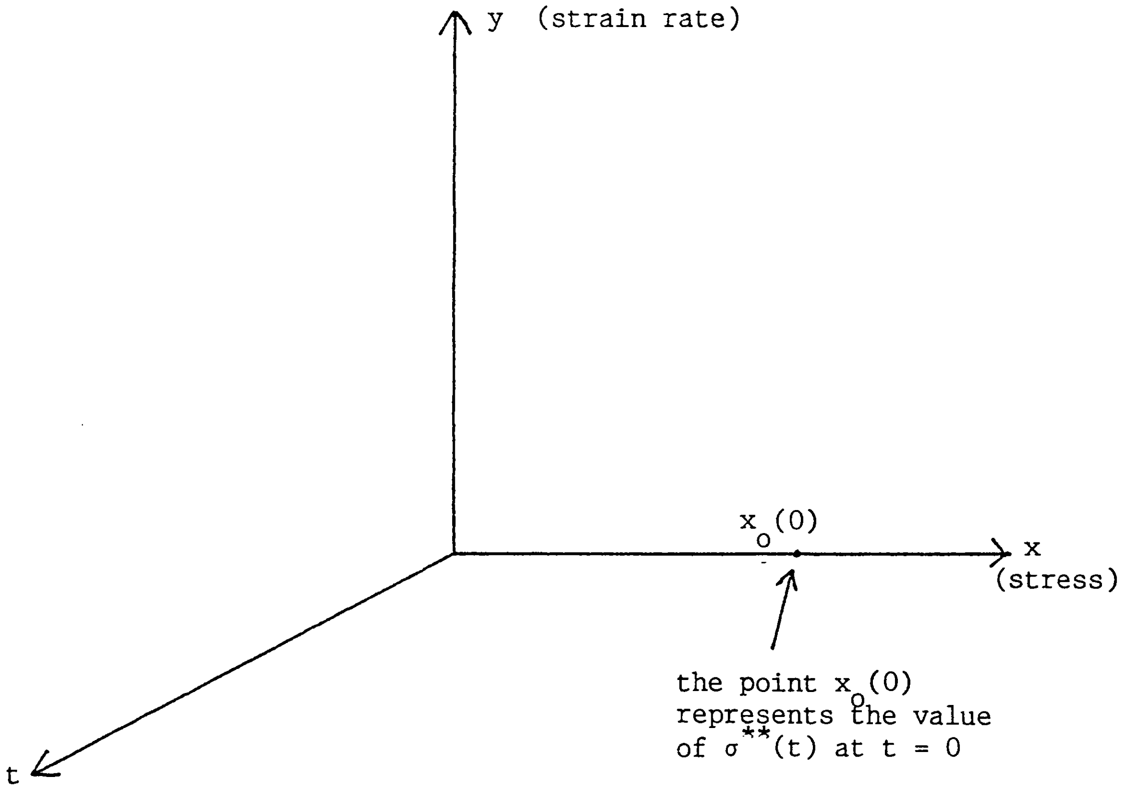


Fig.3.21

W takes the form

$$W = \int_0^T \{x_o(t) - x(t)\} y(x,t)dt \qquad (3.4.1)$$

where $x_o(t)$ represents the arbitrary prescribed function $\sigma^{**}(t)$ and $x(t)$ and $y(x,t)$ represent the actual stress and inelastic strain rate.

Pure viscosity

In this case y is a function of x only, as follows:

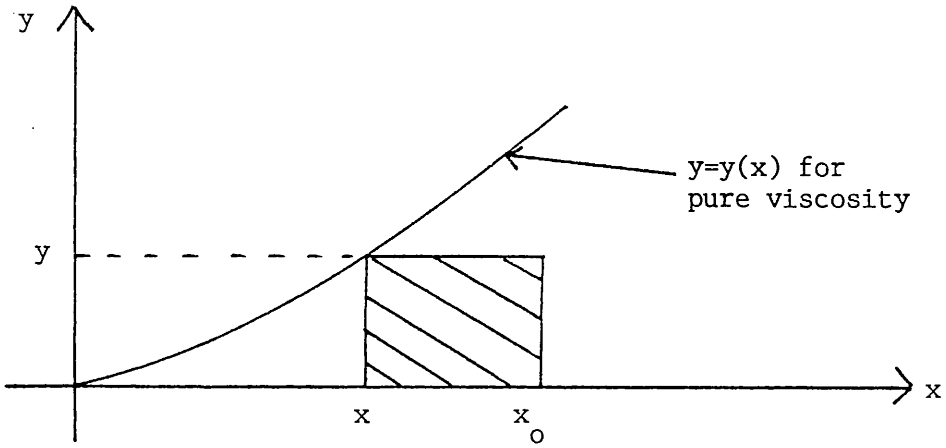
$$y = k\,x^n$$

and t does not appear explicitly in (3.4.1). W is maximised by first maximising the area $(x_o - x)y$ and then by integrating with respect to to time. In the (x,y) plane this presents a simple problem of differential calculus in which it is straightforward to show that the value of x which gives the maximum area is

$$x(t) = \frac{nx_o(t)}{n+1}$$

from which we obtain

$$w = \frac{1}{n}\int_o^T D\left(\frac{n\sigma^{**}(t)}{n+1}\right)dt$$



Perfect plasticity

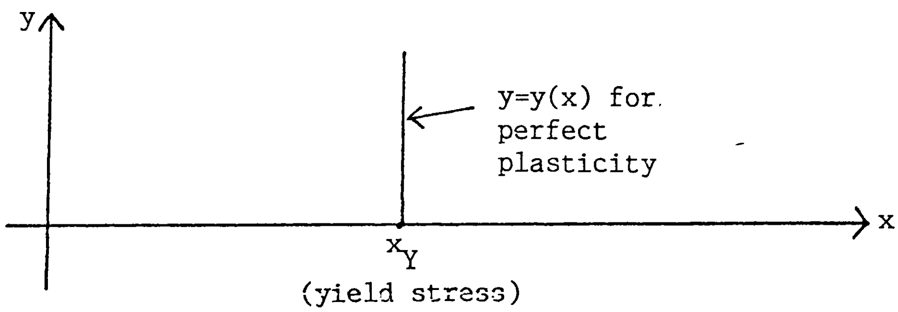
Again, t does not appear explicitly in Eqn.(3.4.1) and the problem reduces to a two dimensional exercise which in this case is trivial:

If $x < x_Y$ then $y = 0$ and so $W = 0$

If $x = x_Y$ and $x_o < x$ then $W < 0$

If $x = x_Y$ and $x_o = x_Y$ then $W = 0$

and so $\max(W) = 0$.



Viscoelasticity

The additional problem arising in viscoelasticity is that the strain rate is in general an explicit function of time as well as of stress and so the maximising condition for the area also varies with time. Consequently we lie outside the essentially two dimensional situations described above.

APPENDIX 3.5

Calculations for the Strain Bound

Combining Eqn.(10.13) for the three parameter model with inequality (10.26) we obtain

$$\varepsilon(T) \lesssim \frac{1}{4k\bar{\sigma}} \int_0^T (\sigma+\bar{\sigma})^2 dt - \frac{E}{4k\bar{\sigma}(2k+ET)} \left\{\int_0^T (\sigma+\bar{\sigma})dt\right\}^2$$

which may be rearranged to the following:

$$\varepsilon(T) \lesssim \frac{T}{16\bar{\sigma}k(2k+ET)} [4k(\bar{\sigma}^2 + (1+\bar{\sigma})^2) + ET] \qquad (3.5.1)$$

The optimum value of $\bar{\sigma}$ is

$$\bar{\sigma} = \sqrt{\frac{ET + 4k}{8k}} \qquad (3.5.2)$$

To evaluate $\varepsilon(T)$ we note that the strain during the two half cycles is given by

$$\varepsilon'(t) = \frac{1}{E}\left\{1 - \exp\left(-\frac{Et}{k}\right)\right\} \; ; \quad 0 \lesssim t \lesssim \frac{\Delta t}{2}$$

$$\varepsilon'(t) = \frac{1}{E}\left\{\exp\left(\frac{E\Delta t}{2k}\right) - 1\right\}\exp\left(-\frac{Et}{k}\right) \; ; \quad \frac{\Delta t}{2} \lesssim t \lesssim \Delta t$$

during the first cycle, and for the j-th cycle:

$$\varepsilon'(t) = \frac{1}{E}\left\{ 1 - R_1 \exp\left(-\frac{Et}{k}\right)\right\} \quad ; \quad (j-1)\Delta t \lesssim t \lesssim (j-\tfrac{1}{2})\Delta t$$

$$\varepsilon'(t) = \frac{1}{E}\left\{ - R_2 \exp\left(-\frac{Et}{k}\right)\right\} \quad ; \quad (j-\tfrac{1}{2})\Delta t \lesssim t \lesssim j\Delta t$$

where

$$R_1 = 1 - \exp\left(\frac{E\Delta t}{2k}\right) + \exp\left(\frac{2E\Delta t}{2k}\right) - \ldots\ldots + \exp\left(\frac{2(j-1)E\Delta t}{2k}\right)$$

and

$$R_2 = 1 - \exp\left(\frac{E\Delta t}{2k}\right) + \exp\left(\frac{2E\Delta t}{2k}\right) - \ldots\ldots - \exp\left(\frac{(2j-1)E\Delta t}{2k}\right)$$

A series evaluation of $\varepsilon(T)$ and a combination of Eqns.(3.5.1) and (3.5.2) gives the strain bound in the form:

$$\frac{1 - \exp\left(-\frac{ET}{k}\right)}{1 + \exp\left(\frac{E\Delta t}{2k}\right)} \lesssim \frac{ET}{8k(2k+ET)}\sqrt{\frac{8k}{ET+4k}}\left[ET + 4k\left\{1 + \sqrt{\frac{ET+4k}{8k}}\right\}\right] \qquad (3.5.3)$$

This expression was written in non-dimensional form in terms of the total time $\left(\frac{ET}{k}\right)$ and the cycle time $\left(\frac{E\Delta t}{2k}\right)$ in the form $\varepsilon \lesssim \varepsilon_u$ and the quantity

$$X = \frac{\varepsilon_u' - \varepsilon}{\varepsilon} \times 100$$

was evaluated for various values of total time and cycle time. The best bound was obtained for small total time, for which $X \approx 21\%$.

SECTION 4

A DISPLACEMENT BOUND FOR TIME-DEPENDENT MATERIALS UNDERGOING NON-LINEAR DEFORMATIONS

INTRODUCTION

The principal difficulty involved in the solution of large-deformation time-dependent problems arises from the fact that the deformation at the current time depends upon the entire previous history of stress, which itself depends on the changing geometry of the structure. To obtain explicit information about the deformation, it is necessary to make statements about both the history dependence of the material and the permissible histories of displacement. We achieve this through separate bounding properties: the material behaviour is described through the concepts of minimum work and maximum complementary work, and the displacements are subject to a potential energy inequality.

A general displacement bound is obtained for a body whose consitutive relation takes a general form, and an example follows in which the bound is compared with the actual displacement of a simple structure composed of a non-linear Maxwell material. In the example, it is shown that the employment of appropriate conjugate variables reduces the complexity of determining the bound to a level that is comparable with the linear case, and in both cases the bound provides an accurate estimation of displacement in the actual structure.

CHAPTER 1

EXTREMAL PROPERTIES OF TIME-DEPENDENT MATERIALS

Summary

The small deformation behaviour of time-dependent materials has been described by the energy principles of Ponter (1969a).

In the time-independent case, Ogden (1975) has extended the theorems of Ponter and Martin (1971) to permit non-linear strain-deformation relations.

Here we combine these principles by deriving an energy theorem which gives rise to an upper bound on the displacement of a general inelastic body undergoing non-linear deformations.

Introduction

The notion of a path in stress-space which gives rise to an extreme value of the complementary work associated with the end points of the path was first postulated by Martin (1966). The dual concept of a path in strain-space giving rise to an extreme value of a work quantity was discussed by Ponter (1968). Together, these authors derived energy principles (1971) based on the extremal paths, which relate the small-strain behaviour of time-independent inelastic materials to that of associated elastic comparison materials.

Ogden (1975) extended these principles, firstly with the assumption of the simultaneous existence of extremal paths in stress and strain space, and secondly with the allowance of non-linear deformations. The latter was accomplished by incorporating non-linear elastic comparison bodies*

* In this context, Ogden replaces the variables "stress and strain" with the conjugate variables "nominal stress and displacement gradient relative to the undeformed configuration".

A description of time-dependent materials was given by Ponter (1969). He defined an associated elastic material whose strain energy density is a function of time. Here, we extend Ponter's results to include non-linear deformations.

Material and structural stability

The complete class of "large deformation" problems contains many members that are inherently unstable in a structural, or possibly material, sense. Here, we will show that it is possible to bound the large deformation behaviour of stable inelastic structures by comparison with associated elastic structures. As a preliminary to the formulation of the associated elastic material, we define appropriate restrictions to ensure the stability of the inelastic body.

We consider a body in equilibrium under quasi-statically applied loads. The traction is currently $\underline{F}$ and the stress is $\underline{\sigma}$. Now the traction is quasi-statically increased to $\underline{F} + \delta\underline{F}$; we ask - what effect has this increase on the equilibrium configuration?

The result of additional $\delta\underline{F}$ is to cause the material to accelerate in the direction of $\delta\underline{F}$. This acceleration produces a deformation that results in additional strain in the material. In general, as a result of the changed strain, the previous stress value changes. If this change occurs in such a way that the resultant of $\underline{F} + \delta\underline{F}$ and $\underline{\sigma}$ approaches zero, then the acceleration decreases and a new equilibrium state is approached. On the other hand, if the resultant of $\underline{F} + \delta\underline{F}$ and $\underline{\sigma}$ does not approach zero, the body acquires kinetic energy and the state is regarded as unstable.

Such instability, arising from a non-zero resultant of the internal and external forces, may arise in two distinct ways.

The material may have entered a region of strain space in which the stress-strain relation is such that $d\underline{\sigma}\,d\underline{\varepsilon} < 0$. Possible causes of this include the formation of cracks and voids on grain boundaries in a metal that is repeatedly loaded and unloaded, or is subjected to a large neutron flux in a reactor. The instability in such cases is inherently a material property.

Alternatively, the body may have entered a regime of deformation where the applied load reduces for an increase in deformation: $d\underline{F}\,d\underline{x} < 0$. An example of this situation occurs in the simple arch, in which the pin-jointed bars are linear elastic.

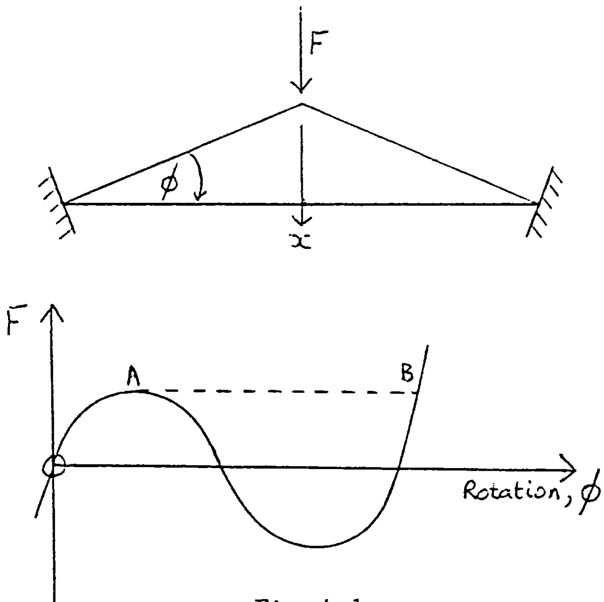


Fig.4.1

The load-rotation relation shows that as F increases, ϕ also increases up to the point A, whereupon the changing geometry permits ϕ to increase while F decreases. In practice, if an increase in F occurs just before A is reached, the structure snaps through and oscillates about B, with appreciable kinetic energy. Here, the material has not behaved in an unstable fashion; clearly $d\underline{\sigma}\,d\underline{\varepsilon}$ is positive throughout. The instability has occurred because the resultant force on the body has diverged from zero once A has been reached. This situation arises because the deformation occurring in the particular geometry of the structure gives rise to strains which in turn generate stresses that would satisfy equilibrium with a <u>reduced</u> load. The geometry, rather than the material, prevents a new equilibrium stress distribution from arising when F increases to $F + \delta F$. This is an example of structural instability. In such problems it is possible to find two configurations which are not accessible to each other via quasi-static variations of load through intermediate states of stable equilibrium.[†]

In the large deformation problems under consideration in this section, we exclude material instability by requiring that $d\underline{\sigma} d\underline{\varepsilon} \geq 0$ throughout. We further exclude structural instability by requiring all possible configurations to be accessible from the unloaded state via quasi-static variations of load through intermediate states of stable equilibrium. The first requirement will permit us to formulate a convex "comparison-elastic" strain energy density function, and the second allows us to employ the divergence theorem in establishing a potential energy inequality.

Energy and complementary energy inequalities

We first define conjugate stress and strain variables according to Hill's formulation (1968, 1956). For a particular choice of strain measure $\underline{\varepsilon}$, the conjugate stress $\underline{\sigma}$ is defined by the requirement that the increment of work per unit reference volume, dW, corresponding to a strain increment $d\underline{\varepsilon}$, is given by

$$dW = \underline{\sigma} d\underline{\varepsilon} \tag{1}$$

This definition includes the interpretations of $\underline{\sigma}$ and $\underline{\varepsilon}$ as true stress and infinitesimal strain, generalised stress and strain - such as moment and curvature of a beam, and nominal stress and displacement gradient.*

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† viz. A and B in the example

* Hill (1968) considered two reference configurations: a spatially fixed configuration, taken to be the initial state, and the currently deformed configuration. He investigated the invariance of certain inequalities for various strain measures and corresponding conjugate stresses. Ogden's choice of variables in the non-linear case assures the validity of certain energy inequalities (see also Hill 1956) that we extend here to time-dependent materials. In this section, the terms "stress" and "strain" will be used in the general sense, to include all suitable conjugate variables.

We consider a history-dependent material undergoing a history of strain specified only by $\underset{\sim}{\varepsilon} = 0$ at time $t = 0$ and $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}(T)$, prescribed, at $t = T$. From (1), the work per unit reference volume corresponding to this history is given by

$$W(\underset{\sim}{\varepsilon}(t),T) = \int_0^T \underset{\sim}{\sigma}(t)\dot{\underset{\sim}{\varepsilon}}(t)dt \tag{2}$$

which, in general depends upon the strain path as well as the end points.

Following Ogden, we now define a domain in strain space known as the "primary domain", in which the following inequality holds:

$$\dot{\underset{\sim}{\sigma}}(t)\dot{\underset{\sim}{\varepsilon}}(t) > 0 \quad . \tag{3}$$

We are concerned with the part of the primary domain that includes the origin; this is the domain of local material stability under dead loading, and the ensuing discussion presumes that all the strains mentioned are from within this domain.

We now assume that there exists a strain history for which $W(\underset{\sim}{\varepsilon}(t),T)$ has a minimum value; such a history is called a minimum work path in strain-time space, and we define the minimum work $\omega(\underset{\sim}{\varepsilon}(T))$ by

$$W(\underset{\sim}{\varepsilon}(t),T) \geqslant \omega(\underset{\sim}{\varepsilon}(T)) \tag{4}$$

The following results were proved by Ponter (1969a) for small strains:

i) the terminal strain and the minimum work function define a unique terminal stress:

$$\underset{\sim}{\sigma}(T) = \frac{\partial\omega(\underset{\sim}{\varepsilon}(T))}{\partial\underset{\sim}{\varepsilon}} \tag{5}$$

ii) the minimum work function is convex in $\underline{\varepsilon}(T)$:

$$\omega(\underline{\varepsilon}_1(T)) - \omega(\underline{\varepsilon}_2(T)) - \frac{\partial\omega(\underline{\varepsilon}_2(T))}{\partial\underline{\varepsilon}} (\underline{\varepsilon}_1(T) - \underline{\varepsilon}_2(T)) \geq 0 \quad (6)$$

These results define an "associated elastic material" whose strain energy density at time T is $\omega(\underline{\varepsilon}(T))$.

As Ogden observed, the basis for such a definition is an inequality which is identical to our assumption (3), and consequently the results (5) and (6) hold for finite strains that lie within the primary domain, although the associated elastic problem is non-linear.

Dual results exist for the extremal path in stress space from $\underline{\sigma}(0) = 0$ to $\underline{\sigma}(T)$, although Ogden points out that the complementary work

$$\bar{W}(\underline{\sigma}(t),T) = \int_0^T \underline{\varepsilon}(t)\dot{\underline{\sigma}}(t)dt \quad (7)$$

is not invariant with respect to the choice of conjugate variables. The maximum complementary work $\bar{\omega}(\underline{\sigma}(T))$ is defined by

$$\bar{W}(\underline{\sigma}(t),T) \leq \bar{\omega}(\underline{\sigma}(T)) \quad (8)$$

and the terminal strain is given by

$$\underline{\varepsilon}(T) = \frac{\partial\bar{\omega}(\underline{\sigma}(T))}{\partial\underline{\sigma}(T)} \quad (9)$$

Furthermore, $\bar{\omega}$ is convex and

$$\omega(\underline{\varepsilon}(T)) + \bar{\omega}(\underline{\sigma}(T)) = \underline{\varepsilon}(T)\underline{\sigma}(T) \quad (10)$$

CHAPTER 2

THE BOUNDARY VALUE PROBLEM

1. A potential energy inequality for the associated elastic body

We now investigate the extremal properties of the associated elastic material. The material is assumed to occupy a volume V and a surface S in its initial configuration; this is taken to be the reference configuration for deformation measurement. On a part of the surface, Sp, external forces $\hat{\underline{P}}(t)$ act, and the remainder of the surface is assumed to have rigid supports.

The potential energy functional for the body is defined as follows:

$$\hat{U}^P(\hat{\underline{\varepsilon}}(T)) = \int_V \omega(\hat{\underline{\varepsilon}}(T))dv - \int_{Sp} \hat{\underline{P}}\hat{\underline{u}}(T)ds \qquad (11)$$

where $\hat{\underline{\varepsilon}}(T)$ is the strain in the comparison elastic body at time T.

Now, if $(\hat{\underline{\varepsilon}}_1(T), \hat{\underline{u}}_1(T))$ is an arbitrary kinematically admissible set, and if $\hat{\underline{\varepsilon}}_1(t)$ is in the primary domain for $0 \leqslant t \leqslant T$, it follows from the convexity of $\omega(\hat{\underline{\varepsilon}}(T))$ and the divergence theorem that

$$\hat{U}^P(\hat{\underline{\varepsilon}}_1(T)) \geqslant \hat{U}^P(\hat{\underline{\varepsilon}}(T)) \qquad (12)$$

A form of this potential energy inequality was given for non-linear elastic materials by Hill (1956).

Clearly we may select the strains and displacements that occur in an otherwise identical inelastic body, denoted by $(\underline{\varepsilon}(T), \underline{u}(T))$, as values for the arbitrary admissible set. With this substitution we obtain the inequality

$$\int_V \omega(\underline{\varepsilon}(T))dv - \int_{Sp} \hat{\underline{P}}\underline{u}(T)ds \geqslant U^P(\hat{\underline{\varepsilon}}(T)) \qquad (13)$$

---

* In the non-linear case, corresponding to the defined nominal stress $\hat{\underline{\sigma}}$, the vector $\hat{\underline{P}}$ is correctly defined as the nominal traction - not the actual applied force (see also Hill, 1956). The nominal traction satisfies equilibrium with $\hat{\sigma}$ on Sp.

In the example which follows, we show that $\hat{\underline{P}}$ is identical to $\mu\underline{F}$, where $\underline{F}$ is the actual applied force and μ is a determinate scalar constant.

On account of the definition of minimum work, (4), we now obtain the result

$$\int_V W(\underline{\varepsilon}(t),T)dv - \int_{Sp} \hat{\underline{P}}\underline{u}(T)ds \geq \hat{U}^P (\hat{\underline{\varepsilon}}(T)) \qquad (14)^*$$

We observe that the choice of tractions on the inelastic body is unspecified in (14).

2. The inelastic body

The tractions on the inelastic body are now assumed to act on Sp and to take the form

$$\begin{aligned} \underline{P}(t) &= f\left(\frac{t}{\Delta t}\right) \underline{P}_o \; : \; 0 \leq t \leq \Delta t \\ &= \underline{P}_o \; : \; t \geq \Delta t \end{aligned} \qquad (15)$$

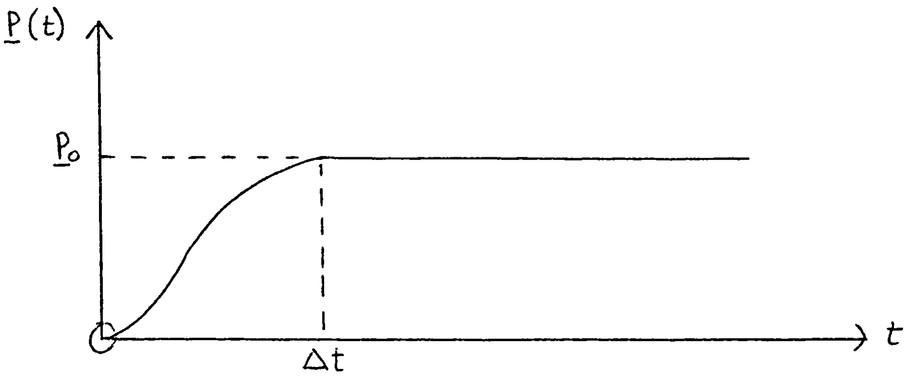

Fig.4.2

where $f(0) = 0$, $f(1) = 1$ and $f(x)$ is a monotonically increasing, continuous function in $0 \leq x \leq 1$.

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*In the small deformation case, this corresponds with the result of Ponter (1968, eqn.(38)). In the time-independent non-linear case it corresponds with Ogden's result (1975, eqn.(28)).

The degree of inequality in (14) may be seen as arising from inequalities (4) and (12). In (4), the extent to which the terms differ depends upon how far from an extremal path is the actual strain path in the body. In (12) the inequality depends on the difference between the values at time T of the quantities $\underline{\varepsilon}$ and $\underline{u}$ in the inelastic body and the comparison elastic body.

The first term in (14) is defined as

$$\int_V W(\underline{\varepsilon}(t),T)dv = \int_V \int_o^T \underline{\sigma}(t)\dot{\underline{\varepsilon}}(t)dtdv \tag{16}$$

where $\underline{\sigma}(t)$ is the conjugate stress in the inelastic body, in equilibrium with $\underline{P}(t)$ on Sp . We will now assume that Δt is large enough to avoid inertia effects during the application of $\underline{P}$, but that it is sufficiently small compared with T that the dissipation of creep energy in $0 \leqslant t \leqslant \Delta t$ is negligible. Consequently, from eqn.(16) we obtain

$$\int_V W(\underline{\varepsilon}(t),T)dv = \int_V \int_o^{\underline{\varepsilon}(\Delta t)} \bar{\underline{\sigma}}d\bar{\underline{\varepsilon}}dv + \int_V \int_{\Delta t}^T \underline{\sigma}(t)\dot{\underline{\varepsilon}}(t)dtdv \tag{17}$$

where $\bar{\underline{\sigma}}$ and $\bar{\underline{\varepsilon}}$ are instantaneous elastic terms. During the time $t \geqslant \Delta t$, the stress $\underline{\sigma}(t)$ is in equilibrium with the constant traction $\underline{P}_o$ and the strain rate $\dot{\underline{\varepsilon}}$ is compatible with displacement rate $\dot{\underline{u}}$ on Sp and zero displacement rate on the remainder of the surface. On application of the divergence theorem* we obtain from eqn.(17):

$$\int_V W(\underline{\varepsilon}(t),T)dv = \int_V \int_o^{\underline{\varepsilon}(\Delta t)} \bar{\underline{\sigma}}d\bar{\underline{\varepsilon}}dv + \int_{Sp} \underline{P}_o\{\underline{u}(T) - \underline{u}(\Delta t)\}ds$$

$$= \int_{Sp} \underline{P}_o\underline{u}(T)ds - \int_V \int_o^{\underline{\sigma}(\Delta t)} \bar{\underline{\varepsilon}}d\bar{\underline{\sigma}}dv \tag{18}$$

---

* From the divergence theorem,

$$\int_V (\sigma_{ij}u_j),_i dv = \int_S \sigma_{ij}u_jn_i ds$$

where V and S denote the reference configuration. Internal equilibrium requires that $\sigma_{ij,i} = 0$, and thus

$$\int_V \sigma_{ij}u_{j,i}dv = \int_S u_j\sigma_{ij}n_ids \ , \quad \text{or} \int_V \sigma_{ij}\varepsilon_{ij}dv = \int_S u_jP_jds \ ,$$

according to the above definitions of ε_{ij} and P_j .

CHAPTER 3

A POINT DISPLACEMENT BOUND

The inequality (14) may be combined with (18) as follows;

$$\int_{Sp} \{\underline{P}_o - \hat{\underline{P}}\}\underline{u}(T)ds - \int_V \int_o^{\underline{\sigma}(\Delta t)} \bar{\underline{\varepsilon}}d\bar{\underline{\sigma}}dv \geq \hat{U}_p(\hat{\underline{\varepsilon}}(T)) \tag{19}$$

Taking $\underline{P}$ and $\hat{\underline{P}}$ to be point loads we obtain

$$(\hat{P} - P_o)u(T) \leq - \hat{U}_p(\hat{\underline{\varepsilon}}(T)) - \int_V \int_o^{\underline{\sigma}(\Delta t)} \bar{\underline{\varepsilon}}d\bar{\underline{\sigma}}dv$$

and on noting (10) this becomes

$$(\hat{P} - P_o)u(T) \leq \int_V \left\{\bar{\omega}(\hat{\underline{\sigma}}(T))dv - \int_o^{\underline{\sigma}(\Delta t)} \bar{\underline{\varepsilon}}d\bar{\underline{\sigma}}\right\} dv \tag{20}$$

If we now write $\hat{P} = \mu P_o$ where $\mu > 1$ is a constant, we obtain the following displacement bound:

$$u(T) \leq \frac{1}{(\mu - 1)P_o} \int_V \left\{ \bar{\omega}(\hat{\underline{\sigma}}(T)) - \int_o^{\underline{\sigma}(\Delta t)} \bar{\underline{\varepsilon}}d\bar{\underline{\sigma}}\right\} dv \tag{21}$$

in which the displacement, $u(T)$, in the inelastic body is bounded by terms which are elastic in nature.

<u>"Large-small" problems</u>

Examples where a body is subjected to significant changes in geometry, but where the strain remains limited, include beams, shells, plates and trusses under lateral loading, where membrane stiffness is of interest. In the section that follows, the two sides of the displacement bound (21) are calculated for an elastic creeping truss in order to assess the circumstances under which an accurate estimate of displacement may be obtained from an elastic-type analysis.

CHAPTER 4

AN EXAMPLE: THE NON-LINEAR DEFLECTIONS OF AN ELASTIC CREEPING TRUSS

The body is composed of a non-linear Maxwell material whose total strain rate consists of a linear elastic component, $\dot{\underline{e}}$, and a non-linear viscous component, $\dot{\underline{v}}$:

$$\dot{\underline{\varepsilon}} = \dot{\underline{e}} + \dot{\underline{v}}$$

$$\underline{e} = \underline{c}\,\underline{\sigma} \tag{22}$$

$$\dot{\underline{v}} = k \frac{\partial}{\partial \underline{\sigma}} \left\{\frac{\phi^{n+1}(\underline{\sigma})}{n+1}\right\}$$

where $\underline{c}$ is a fourth-order tensor, k and n are numerical constants and ϕ is a homogeneous function of degree one.

Ponter (1969a) has derived the extremal properties of the non-linear Maxwell material. The maximum complementary work path to a stress $\hat{\underline{\sigma}}(T)$ is given by

$$\hat{\underline{\sigma}}(t) = \frac{n}{n+1}\,\hat{\underline{\sigma}}(T) , \quad 0 < t < T \tag{23}$$

with instantaneous changes at $t = 0$ and $t = T$ to satisfy $\hat{\underline{\sigma}}(0) = 0$ and $\hat{\underline{\sigma}}(t) = \hat{\underline{\sigma}}(T)$ at $t = T$. The maximum complementary work is then given by

$$\bar{\omega}(\hat{\underline{\sigma}}(T)) = \frac{1}{2}\,\hat{\underline{\sigma}}(T)\underline{c}\hat{\underline{\sigma}}(T) + \frac{kT}{n}\,\phi^{n+1}\left(\frac{n\hat{\underline{\sigma}}(T)}{n+1}\right) \quad . \tag{24}$$

From (9), the associated elastic strain, $\hat{\underline{\varepsilon}}(T)$, is given by

$$\hat{\underline{\varepsilon}}(T) = \underline{c}\hat{\underline{\sigma}}(T) + \frac{kT}{n}\left(\frac{n}{n+1}\right)^{n+1} \frac{\partial}{\partial \underline{\sigma}} \left\{\phi^{n+1}(\hat{\underline{\sigma}}(T))\right\} \tag{25}$$

which is not generally invertible.*

---

*Consequently it is not possible to obtain a formulation for $\omega(\hat{\underline{\varepsilon}}(T))$ that is explicit in $\hat{\underline{\varepsilon}}(T)$ in the general case.

In the unixial case, the constitutive relations for the actual material reduce to expressions of the form

$$\left.\begin{aligned} e &= \frac{\sigma}{E} \\ \dot{\varepsilon} &= \frac{\dot{\sigma}}{E} + k\sigma^n \end{aligned}\right\} \qquad (26)$$

and the associated elastic material is defined by

$$\hat{\varepsilon} = \frac{\hat{\sigma}}{E} + kT\left(\frac{n\hat{\sigma}}{n+1}\right)^n \qquad (27)$$

The two-bar truss

This structure is symmetrical, with the bars initially at an angle α below the horizontal. Under load $P(t)$ the bars deflect to an angle $\phi(t)$, with a central displacement of $u(t)$ and a bar extension of $x(t)$.

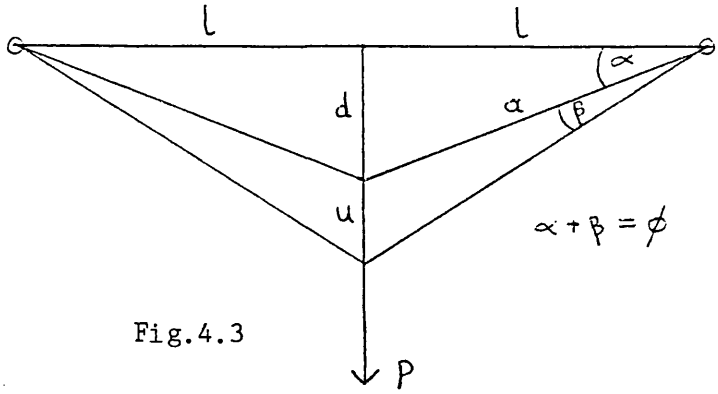

Fig.4.3

The strain-displacement relation for the body is obtained as follows:

$$a^2 = d^2 + \ell^2 \text{ : initial configuration}$$

$$(a+x)^2 = (d+u)^2 + \ell^2 \text{ : at time } t .$$

Thus $2ax + x^2 = 2du + u^2$

or $$\frac{u}{d} = \sqrt{\left(\frac{x}{d}\right)^2 + \frac{2ax}{d^2} + 1} - 1$$

or $$\frac{u}{d} = \sqrt{\varepsilon^2\mathrm{cosec}^2\alpha + 2\varepsilon\,\mathrm{cosec}^2\alpha + 1} - 1 \qquad (28)$$

where $\varepsilon = \frac{x}{a}$.

Consequently, though ε is limited*, the strain-displacement relation may be highly non-linear, as seen in Figure 4.4.

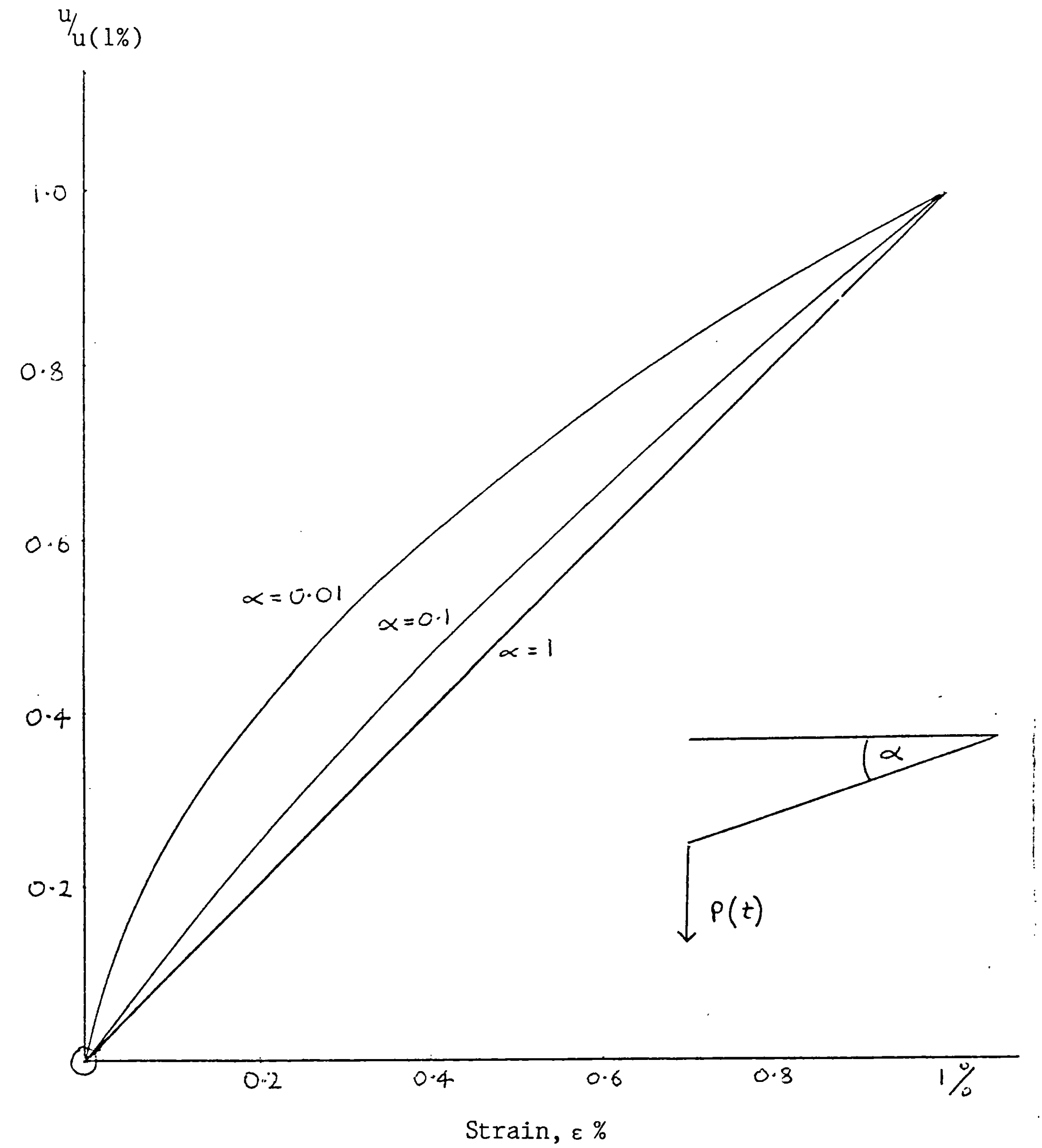

Fig.4.4. Effect of initial sag, α , on the linearity of the strain-displacement relation for the two-bar truss.

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*For a strain of 1%, when $\alpha = 10^{-1}$rad, $\frac{u}{d} \simeq 0.7$, and when $\alpha = 10^{-2}$rad, $\frac{u}{d} \simeq 13$. On the other hand, it may be shown that

$$\varepsilon = \sqrt{\left\{\frac{u^2}{d^2} + \frac{2u}{d}\right\} \sin^2\alpha + 1} - 1$$

and it is evident that whenever $\left(\frac{u}{d}\right)$ is small, $\varepsilon \simeq \frac{u}{d} \sin^2\alpha$ is also small.

The loading is assumed to result in homogeneous uniaxial distributions of stress and strain in each bar, so that an integral over the initial volume may be replaced by a product of the integrand and the term v = 2aA, where A is the initial cross-section of the bar. Consequently the displacement bound (21) becomes

$$u(T) = \frac{2Aa}{(\mu-1)P_o}\left[\bar{\omega}(\hat{\underline{\sigma}}(T)) - \int_o^{\underline{\sigma}(\Delta t)} \bar{\underline{\varepsilon}}\, d\bar{\underline{\sigma}}\right] \tag{29}$$

<u>Conjugate variables</u>

As we have observed, our choice of deformation measurement referred to the original configuration of a body necessitates that either the deformations be small throughout, or that we employ conjugate variables when there is significant geometry change. After Hill (1956 , 1968) and Ogden (1975) we adopt the following choice:

u_i is the actual displacement in the i-direction;

ε_{ij} is the displacement gradient $\frac{\partial u_j}{\partial x_i}$, where x_i is the initial position;

σ_{ij} is the j-component of force, divided by the component of area upon which the force acts that was originally perpendicular to the i-direction (σ_{ij} is known as the "nominal stress");

P_i is the nominal traction in equilibrium with σ_{ij} on Sp.

With this choice of variables it follows from the divergence theorem that

$$\int_V \sigma_{ij}\varepsilon_{ij}dv = \int_{Sp} P_i u_i ds \tag{30}$$

In the two-dimensional truss we refer deformations to the vertical (X_1) and horizontal (X_2) directions, so that $u_1 = u$ and $u_2 = 0$, and $\varepsilon_{11} = \frac{\partial u_1}{\partial X_1} = \frac{u}{d}$, with $\varepsilon_{ij} = 0$ otherwise.

According to the definition of conjugate stress σ_{ij}, the work density corresponding to a change $\delta\varepsilon_{ij}$ is given by $\delta W = \sigma_{ij}\,\delta\varepsilon_{ij}$.

Consequently $SW = \sigma_{11}\frac{\delta u}{d} = \frac{F\delta u}{2Aa}$ where F is the actual force (rather than the nominal force) acting on the structure. Thus

$$\sigma_{11} = \frac{Fd}{2Aa} = \frac{F\sin\alpha}{2A}$$

Finally, from the divergence theorem we may obtain the nominal load, P :

$$\int_V \sigma_{ij}\varepsilon_{ij}dv = \int_S P_j u_j ds$$

$$\therefore \qquad 2Aa \,.\, \frac{F\sin\alpha}{2A} \,.\, \frac{u}{d} = Pu$$

or $P = F$; that is, the nominal force is identical to the actual applied force in this problem.

For brevity we denote the terms σ_{11} and ε_{11} by σ and ε and so our set of conjugate variables is u , the actual displacement, and

$$\left.\begin{aligned} \varepsilon &= \frac{u}{d} \\ \sigma &= \frac{F\sin\alpha}{2A} \\ P &= F^* \end{aligned}\right\} \tag{31}$$ †

*The time dependence of F is thus given in (15).

†In the small strain-finite displacement case where Eulerian strain and Cauchy stress are used to formulate the problem, the corresponding equations are $\varepsilon = \left\{\frac{u}{d} + \frac{1}{2}\frac{u^2}{d^2}\right\}\sin^2\alpha$, $\sigma = \frac{F}{2A\sin\phi(t)}$ and integrations refer to the currently deformed volume and surface.

The large-displacement bound

The non-linear problem is now posed in terms of the actual material (26), the comparison elastic material (27), the conjugate variables (31) and the bound (29). We next obtain the solutions to the actual material problem and the comparison problem in order to assess the accuracy of the displacement bound.

From (26) it follows that

$$\int_0^{\bar{\sigma}(\Delta t)} \bar{\varepsilon}\, d\bar{\sigma} = \frac{\bar{\sigma}(\Delta t)\,\bar{\varepsilon}(\Delta t)}{2}$$

and so from (31) ,

$$\frac{2Aa}{P_o}\int_0^{\bar{\sigma}(\Delta t)} \bar{\varepsilon}\, d\bar{\sigma} = \frac{2Aa}{P_o} \cdot \frac{P_o \sin\alpha}{2.2A} \cdot \frac{\bar{u}}{d} = \frac{1}{2}\,\bar{u} \qquad (32)$$

where $\bar{u}$ is the instantaneous elastic displacement.

From (27),

$$\bar{\omega}(\hat{\sigma}(T)) = \frac{\hat{\sigma}^2}{2E} + \frac{kT}{n}\left(\frac{n\hat{\sigma}}{n+1}\right)^{n+1}$$

$$= \frac{\hat{\sigma}^2}{2E} + \frac{\hat{\varepsilon}\,\hat{\sigma}}{n+1} - \frac{\hat{\sigma}^2}{(n+1)E}$$

$$= \frac{\bar{\varepsilon}\hat{\sigma}^2(n-1)}{2\bar{\sigma}(n+1)} + \frac{\hat{\varepsilon}\,\hat{\sigma}}{n+1}$$

On substituting for $\bar{\varepsilon}$, $\hat{\varepsilon}$, $\bar{\sigma}$ and $\hat{\sigma}$ from (31), and noting that $\hat{P} = \mu P_o$, we obtain

$$\frac{2Aa}{P_o}\,\bar{\omega}(\hat{\sigma}(T)) = \frac{\mu^2\bar{u}(n-1)}{2(n+1)} + \frac{\mu\hat{u}}{n+1} \qquad (33)$$

The bound, (29), now becomes

$$u(T) \leqslant \frac{1}{\mu-1}\left[\frac{\mu^2\bar{u}(n-1)}{2(n+1)} + \frac{\mu\hat{u}}{n+1} - \frac{1}{2}\,\bar{u}\right] \qquad (34)$$

From (31) and (26) we obtain

$$\frac{\bar{u}}{d} = \frac{P_o \sin\alpha}{2AE} = P_1 \sin\alpha \qquad (35)$$

where $P_1 = \frac{P_o}{2AE}$ = non-dimensional applied load

and $$\frac{\dot{u}}{d} = k\left(\frac{P_o \sin\alpha}{2A}\right)^n \quad \text{for} \quad t \geqslant \Delta t \tag{36}$$

Integrating (36) yields

$$\frac{u(T) - \bar{u}}{d} = kTE^n \, (P_1 \sin\alpha)^n$$

or $$\frac{u(T) - \bar{u}}{d} = \tau (P_1 \sin\alpha)^n \tag{37}$$

where $\tau = kTE^n$ = non-dimensional total time.

From (31) and (27),

$$\frac{\hat{u}}{d} = \frac{\hat{P}\sin\alpha}{2AE} + kT\left(\frac{n\hat{P}\sin\alpha}{(n+1).2A}\right)^n$$

$$= \mu P_1 \sin\alpha + \tau\left(\frac{n\mu P_1 \sin\alpha}{n+1}\right)^n \tag{38}$$

Combining (35), (37) and (38) we eliminate τ and P_1 to obtain a relation between the actual and comparison displacement terms:

$$\frac{\hat{u}}{d} = \frac{\mu\bar{u}}{d} + \frac{u - \bar{u}}{d} \left(\frac{n\mu}{n+1}\right)^n \tag{39}$$

We may now substitute (39) into the bound to obtain

$$u \leqslant (u - \bar{u}) \left(\frac{\mu n}{n+1}\right)^n . \frac{\mu}{n+1} - u(\mu - 2) + \frac{\bar{u}}{2}(\mu^2 - 1) \tag{40}$$

The specification of the actual material parameters, together with the value of the total time, is equivalent to the determination of the values of n and $\frac{\bar{u}}{u(T)}$, the latter being the proportion of the total displacement made up by the instantaneous elastic displacement. Consequently we may evaluate the displacement bound - and optimise it with respect to μ - directly from (40). Denoting the right-hand side of (40) by u_B , we define the quantity

$$E = \frac{u_B - u}{u} \times 100\% \qquad (41)$$

as a measure of the accuracy of the bound. On rearranging (40),

$$E = 100 \left[\left(1 - \frac{\bar{u}}{u}\right) \left(\frac{\mu}{n+1}\right) \left(\frac{\mu n}{n+1}\right)^n - (\mu - 1) + \frac{\bar{u}}{2u} (\mu^2 - 1) \right] \qquad (42)$$

and an accurate bound is indicated by a small value of E.

Results

Values of E are given in Appendix 4.1 for a range of specifications for the material and total time. Certain observations are apparent from the results:

1. Provided the instantaneous elastic deformation contributes no more than 10% of the total deformation, the optimum bound is not worse than 4% in excess of the actual displacement. In a typical situation, such as 1% elastic deformation with a creep index of 3, the bound is less than 0.1% in excess.

2. If μ is taken to be $\left(\frac{n+1}{n}\right)$, then provided the instantaneous elastic deformation contributes no more than 10% to the total, the non-optimum bound is not worse than 5% in excess.

3. The optimum value of μ rapidly approaches $\frac{n+1}{n}$, and E becomes small, both as n increases and as the proportional contribution of instantaneous elastic deformation decreases.

It is evident that a highly accurate estimate of deformation is obtainable in this time-dependent non-linear problem by conducting an elastic-type analysis.

The bounding method for the linear problem

We include this simpler case as an indication that in this problem at least, the use of conjugate variables in the non-linear situation reduces the degree of difficulty of analysis to a level that is comparable with the linear situation.

The formulation of the bound is unchanged but the relations between the variables differ from above since σ and ε now denote true stress and infinitesimal strain:

$$\sigma = \frac{P}{2A\sin\alpha} \tag{43}$$

$$\varepsilon = \frac{u}{d}\sin\alpha \tag{44}$$

The terms in the bound are $\frac{2Aa}{P_o} \cdot \frac{1}{2}\bar{\sigma}\bar{\varepsilon}$ and $\frac{2Aa}{P_o}\left[\frac{\bar{\varepsilon}\sigma^2(n-1)}{2\bar{\sigma}(n+1)} + \frac{\hat{\varepsilon}\hat{\sigma}}{n+1}\right]$ as in the non-linear problem. Upon substitution from (43) and (44) we obtain

$$u(T) \leqslant \frac{1}{\mu-1}\left[\frac{\mu\hat{u}}{n+1} + \frac{\bar{u}}{2(n+1)}\{\mu^2(n-1) - (n+1)\}\right] \tag{45}$$

which is identical to the non-linear expression, (34).

We next obtain the relation between $\hat{u}$, $\bar{u}$ and $u(T)$. From (43), (44) and (26)

$$u(T) - \bar{u} = \frac{a\tau}{\sin\alpha}\left(\frac{P_1}{\sin\alpha}\right)^n \tag{46}$$

and
$$P_1 = \frac{\bar{u}}{d}\sin^3\alpha \tag{47}$$

and from (43), (44) and (27),

$$\frac{\hat{u}}{d}\sin^2\alpha = \frac{\mu P_1}{\sin\alpha} + \tau\left(\frac{\mu n P_1}{(n+1)\sin\alpha}\right)^n . \tag{48}$$

Combining (46), (47) and (48) we obtain

$$\hat{u} = \mu\bar{u} + (u - \bar{u}) \left(\frac{\mu n}{n + 1}\right)^n \qquad (49)$$

which is identical to (39) for the non-linear situation.

The bound resulting from (49) and (45) in the linear case is therefore identical to the non-linear bound, (42). Moreover it is apparent that once the nature of the conjugate variables is determined, the degree of complexity of the two problems is similar.

The use of conjugate variables in this problem effectively uncouples the result of changing geometry from the strain-displacement and equilibrium equations, reducing the situation to a quasi-linear one. Given the restrictions that we have imposed to limit unstable behaviour, it is evident that when formulated in terms of appropriate variables, the existing energy theorems of Ponter, Martin and Ogden can be extended into the regime of time-dependent non-linear problems.

APPENDIX 4.1

RESULTS OF COMPUTATION OF THE BOUNDING INEQUALITY (42)

$u(T) : u(0+) = 10 : 1$			
E %	n	μ_{opt}	$\frac{n.\mu_{opt}}{n+1}$
4.09	1	1.818	0.909
0.53	3	1.317	0.988
0.2	5	1.195	0.996
0.1	7	1.140	0.997
0.06	9	1.109	0.998

Table 4.1

$u(T) : u(0+) = 100 : 1$			
E %	n	μ_{opt}	$\frac{n.\mu_{opt}}{n+1}$
0.49	1	1.98	0.99
0.06	3	1.332	0.999
0.02	5	1.2	1.0
0.01	7	1.143	1.0
0.006	9	1.111	1.0

Table 4.2

Tables 4.1 and 4.2 show the magnitude of the excess, E , of the bound over the actual solution. The bound has been optimised with respect to the loading factor, μ.

$\frac{u(T)}{u(0+)}$	E %		
	n=1	n=5	n=9
10	5.0	0.2	0.06
20	2.5	0.1	0.03
40	1.25	0.05	0.015
60	0.83	0.03	0.01
80	0.62	0.025	0.008
100	0.5	0.02	0.006

Table 4.3

Table 4.3 shows values of the bound excess, E , when the loading factor, μ, is fixed at $\mu = \frac{n+1}{n}$. Increasing $\frac{u(T)}{u(0+)}$ represents increasing time. T.

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A REVIEW AND THE DEVELOPMENT OF BOUNDING METHODS IN CONTINUUM MECHANICS

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ABSTRACT

Energy theorems and kindred inequalities have long been a basis for the analysis of redundant structures and the material continuum. We trace the development of the principal results of elasticity, time-independent inelasticity and creep, from the principle of virtual work and the well-known theorems of linear elasticity to recent results which describe the deformation of general inelastic materials under time-varying loads. A new view of the upper bound shakedown theorem is given and a discussion of the fundamental material requirements which permit the establishment of many of the inequalities is included.

In Section 2 we derive new bounding results for a general class of constitutive relations using a thermodynamic formalism as the basis of the discussion. Simple work and displacement bounds are derived which are insensitive to detailed aspects of the material behaviour. Several examples are included and a comparison is offered between our solutions and those which are in current design use.

In the third section new results are obtained for the behaviour of a general viscoelastic material subjected to cyclic loading. The existence and uniqueness of a stationary cyclic state of stress is proved and a lower work bound for the general non-linear material is derived. An upper bound for the linear material is obtained and we describe methods for determining this bound from the results of simple creep and relaxation tests. The bounding theory based on an internal state material description is compared with that using a history-dependent description. We show how a knowledge of the response of a viscoelastic body to constant loading is sufficient to determine the general long-term cyclic strain behaviour.

In the final section we unite the theorems concerning small deformations of time-dependent materials and large deformations of time-independent materials. A general displacement bound is derived which is expressed in terms of conjugate variables defined in the undeformed state. An example is given in which it is shown that the use of such variables can reduce the difficulty of bounding non-linear deformations to a level that is comparable with the linear case.