Error Estimates for Spaces Arising from Approximation by Translates of a Basic Function

by

M. L. Vail

Department of Mathematics and Computer Science
University of Leicester

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester.

January 2002

Acknowledgements

Many thanks go to Will Light, firstly for introducing me to Approximation Theory as an undergraduate and secondly for all the support and help given over the last three years as my supervisor. My thanks also to EPSRC for providing funding for my PhD.

There are many people who made my years at Leicester both as an undergraduate and as a postgraduate rewarding and enjoyable and I mention only a few here as it is impossible to name them all. To Rob and David, thanks for invaluable informal discussions on this research and for pointing out typos in my thesis; to Richard, Steve and all the postgrads, I'll miss my lunchtime curries; to Rachel, thanks for being an excellent housemate.

To Mum, Dad, Niki and the rest of the family thanks for supporting and encouraging my desire to remain a student for so long.

Finally thanks to Bruce for always being there for me, especially when I needed someone to proof read my thesis.

Abstract

We look at aspects of error analysis for interpolation by translates of a basic function. In particular, we consider ideas of localisation and how they can be used to obtain improved error estimates. We shall consider certain seminorms and associated spaces of functions which arise in the study of such interpolation methods. These seminorms are naturally given in an indirect form, that is in terms of the Fourier Transform of the function rather than the function itself. Thus, they do not lend themselves to localisation. However, work by Levesley and Light [17] rewrites these seminorms in a direct form and thus gives a natural way of defining a local seminorm. Using this form of local seminorm we construct associated local spaces. We develop bounded, linear extension operators for these spaces and demonstrate how such extension operators can be used in developing improved error estimates. Specifically, we obtain improved L_2 estimates for these spaces in terms of the spacing of the interpolation points. Finally, we begin a discussion of how this approach to localisation compares with alternatives.

List Of Standard Notation

$\mathbb{R}$	the real number system
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{C}$	the complex number field
$\mathbb{Z}$	set of all integers
$\mathbb{Z}_+$	set of all non-negative integers
$\mathbb{Z}_+^n$	set of n-tuples of nonnegative numbers
$L^p(\Omega)$	set of all Lebesgue measurable functions $f : \Omega \to \mathbb{C}$
	for which $\int_\Omega \lvert f\rvert^p < \infty$
$\lVert \cdot \rVert_{p,\Omega}$	norm on $L^p(\Omega)$
$L^1_{loc}(\mathbb{R}^n)$	the space of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$
	such that for any compact set K in $\mathbb{R}^n$, $f\vert_K \in L^1(K)$
$C(\Omega)$	the set of continuous functions on Ω
$C^k(\Omega)$	the set of functions on Ω with continuous derivatives up to the k-th order
$C^\infty(\Omega)$	the set of infinitely differentiable continuous functions on Ω
$C_0(\mathbb{R}^n)$	space of all compactly supported, continuous functions on $\mathbb{R}^n$
$C_0^\infty(\mathbb{R}^n)$	space of all functions in $C_0(\mathbb{R}^n)$ which are infinitely differentiable
$\Pi_k(\mathbb{R}^n)$	the space of polynomials on $\mathbb{R}^n$ of degree at most k

Index Of Special Notation

Contents

Chapter 1

Introduction

Interpolation is an important tool in the physical sciences and has a variety of applications. In general we are trying to solve the following type of problem. Information in the form of real or complex values is known only on a finite set of sampling points or nodes in $\mathbb{R}^n$. Our aim is to find a function $s : \mathbb{R}^n \to \mathbb{R}$ whose value at each of the nodes agrees with the given data. The specifics of the interpolation scheme are, of course, dependent on the application in hand. For example, we may require the interpolant s to have a certain degree of smoothness or behaviour at infinity. Some considerations, however, must be taken into account in any interpolation scheme. In order to avoid numerical instability, we must ensure the problem is posed in such a way as to guarantee a unique solution. In addition, we would like small changes in the data to imply small changes in the interpolant. The question of error is also of importance; we would hope to have some means of evaluating how well our scheme works.

The work in this thesis is motivated by interpolation by translates of a basic function. This is now a well-established and relatively simple method of interpolation which can be applied to scattered data. We shall discuss some of the relevant theory in this chapter and introduce certain seminorms and spaces of functions which arise naturally in the study of this type of interpolation problem. The aim of this work is to produce improved error estimates for these spaces by applying localisation methods.

1.1 The Interpolation Problem

We shall be concerned with the following interpolation problem. Interpolation data is given in the form of $x_1, \ldots, x_m \in \mathbb{R}^n$ and corresponding values $d_1, \ldots, d_m \in \mathbb{R}$. Let X be a space of real-valued functions on $\mathbb{R}^n$. We wish to find a function $s \in X$ such that

$$s(x_i) = d_i \qquad \text{for all } i = 1, \ldots, m. \tag{1.1}$$

It is useful to assume that X is a linear space of functions for the following reasons. Suppose $s_1 \in X$ interpolates data $d_1, \ldots, d_m \in \mathbb{R}$ at the points $x_1, \ldots, x_m \in \mathbb{R}^n$, in the sense of (1.1). Then, if X is a linear space, given any $\alpha \in \mathbb{R}$ the function $\alpha s \in X$ interpolates the data $\alpha d_1, \ldots, \alpha d_m \in \mathbb{R}$ at $x_1, \ldots, x_m$. Suppose also that $s_2 \in X$ interpolates the data $a_1, \ldots, a_m \in \mathbb{R}$ at $x_1, \ldots, x_m \in \mathbb{R}^n$. Then, again provided X is a linear space, the function $s_1 + s_2 \in X$ interpolates the data $d_1 + a_1, \ldots, d_m + a_m$ at the points $x_1, \ldots, x_m$.

In order to provide stability for the system it is important that the interpolation conditions (1.1) ensure that s is uniquely defined. Thus, we require that the only function in X which vanishes on the interpolation points is the zero function. However, work dating back to Mairhuber [24] shows that difficulties occur when considering a multidimensional situation. Specifically, suppose X is an m-dimensional space of continuous real-valued functions on a domain $\Omega \subset \mathbb{R}^n$. Suppose also that given any set of points $\{x_1, \ldots, x_m\} \subset \Omega$, the only function $f \in X$ satisfying $f(x_i) = 0$ for all $i = 1, \ldots, m$, is the zero function. Then, either $m = 1$ or the domain Ω is just one dimensional. This result forces us to make X dependent on the given interpolation data.

A solution to this problem presents itself in the form of radial basic function interpolation. We begin with a basic function Φ which is real-valued on $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Letting $A = \{x_1, \ldots, x_m\}$ we define the interpolation space $X = X_A$ to consist of all functions from $\mathbb{R}^n$ to $\mathbb{R}$ of the form

$$x \mapsto \sum_{j=1}^{m} \alpha_j \Phi(|x - x_j|), \qquad \text{with } \alpha_j \in \mathbb{R} \text{ for all } j = 1, \ldots, m.$$

Here $|\cdot|$ denotes the Euclidean norm. Having specified our interpolation space X to be data dependent, we now have a chance of being able to solve the interpolation problem uniquely for s. However, certain restrictions must be placed on the function Φ in order to ensure this.

1.2 Conditionally Positive Definite Functions

We have the following radial basic function interpolation problem. Let Φ be a real-valued function on $\mathbb{R}_+$. Given values $d_1, \ldots, d_m \in \mathbb{R}$ and points $x_1, \ldots, x_m \in \mathbb{R}^n$, we wish to determine

$$s(x) = \sum_{j=1}^{m} \alpha_j \Phi(|x - x_j|),$$

where the α_j are constants to be found subject to the interpolation conditions

$$s(x_i) = d_i, \qquad \text{for all } i = 1, \ldots, m.$$

This is equivalent to solving the linear system

$$\sum_{j=1}^{n} \alpha_j \Phi(|x_i - x_j|) = d_i \qquad \text{for all } i = 1, \dots, m.$$

Writing this in matrix form we have

$$A\alpha = d,$$

where A is the $m \times m$ matrix whose components are given by $A_{ij} = \Phi(|x_i - x_j|)$, $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and $d = (d_1, \dots, d_m) \in \mathbb{R}^m$. Clearly we can solve this system uniquely for α provided A is invertible. With minimal effort one can prove this is the case if Φ is strictly positive definite in the following sense.

Definition 1.2.1 *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$. If, for any set of distinct points $y_1, \dots, y_r \in \mathbb{R}^n$ and constants $c_1, \dots, c_r \in \mathbb{R}$, the quadratic form*

$$\sum_{i=1}^{r} \sum_{j=1}^{r} c_i c_j \Phi(|y_i - y_j|) \geq 0$$

then we say Φ is positive definite. If, in addition,

$$\sum_{i=1}^{r} \sum_{j=1}^{r} c_i c_j \Phi(|y_i - y_j|) = 0$$

if and only if $c_i = 0$ for all $i = 1, \dots, r$, then we say Φ is strictly positive definite.

The interpolation problem is uniquely solvable provided the function Φ is strictly positive definite. The Gaussian, defined by $\Phi(r) = e^{-r^2}$, is an example of a strictly positive definite function, see Powell [28, p.118]; so is the inverse multiquadric defined by $\Phi(r) = (c^2 + |r|^2)^{-1/2}$, where $c > 0$. However, some of the first interpolants of this kind to be used successfully in applications involved basic functions Φ which do not possess this property, for example, Duchon's thin plate splines [6], in which $\Phi(r) = r^2 \ln r$ and Hardy's multiquadric surfaces [14], in which $\Phi(r) = (r^2 + c^2)^{1/2}$, for $c > 0$. However, these functions do possess the following property.

Definition 1.2.2 *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$. If, for any set of distinct points $y_1, \ldots, y_r \in \mathbb{R}^n$ and constants $c_1, \ldots, c_r \in \mathbb{R}$, the quadratic form*

$$\sum_{i=1}^{r} \sum_{j=1}^{r} c_i c_j \Phi(|y_i - y_j|) \geq 0$$

whenever

$$\sum_{i=1}^{r} c_i p(y_i) = 0 \qquad \text{for all } p \in \Pi_{k-1}(\mathbb{R}^n), \tag{1.2}$$

then we say Φ is conditionally positive definite of order k. If, in addition, $c_1, \ldots, c_r$ satisfy (1.2) and

$$\sum_{i=1}^{r} \sum_{j=1}^{r} c_i c_j \Phi(|y_i - y_j|) = 0$$

if and only if $c_i = 0$ for all $i = 1, \ldots, r$, then we say Φ is conditionally strictly positive definite of order k.

We note that the above definition can be extended to functions $\Psi : \mathbb{R}^n \to \mathbb{R}$ by considering $\Psi(y_i - y_j)$ instead of $\Psi(|y_i - y_j|)$.

When considering functions Φ which are conditionally positive definite of order k we add a polynomial of degree $k - 1$ to the interpolant. Assume $\Pi_{k-1}(\mathbb{R}^n)$ has dimension l and suppose polynomials $p_1, \ldots, p_l$ form a basis for $\Pi_{k-1}(\mathbb{R}^n)$. We then let

$$s(x) = \sum_{j=1}^{m} \alpha_j \Phi(|x - x_j|) + \sum_{k=1}^{l} \beta_k p_k(x) \qquad \text{for } x \in \mathbb{R}^n,$$

and hope to solve for constants $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_l$ subject to the conditions

(1) $s(x_i) = d_i$, for all $i = 1, \ldots, m$,

(2) $\sum_{j=1}^{m} \alpha_j p_k(x_j) = 0$, for all $k = 1, \ldots, l$.

The additional degrees of freedom introduced by the polynomial in the interpolant compensate for the addition of condition (2) to the interpolation constraints. We can write the above system of equations in matrix form as follows,

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$$

where A is the $m \times m$ matrix given by $A_{ij} = \Phi(|x_i - x_j|)$, P is the $m \times l$ matrix given by $p_{ij} = p_j(x_i)$, $d = (d_1, \ldots, d_m) \in \mathbb{R}^m$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $\beta = (\beta_1, \ldots, \beta_l) \in \mathbb{R}^l$. This system can be solved uniquely provided the $(m+l) \times (m+l)$ matrix

$$M = \begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix}$$

is invertible. We shall see that this is true if Φ is conditionally positive definite of order k and if in addition the interpolation points $x_1, \ldots, x_m$ have the following property.

Definition 1.2.3 *We say that $y_1, \cdots, y_m \in \mathbb{R}^n$ are unisolvent with respect to $\Pi_k(\mathbb{R}^n)$ if, whenever $p \in \Pi_k(\mathbb{R}^n)$ and $p(y_i) = 0$, for all $i = 1, \cdots, m$, p is the zero polynomial.*

Lemma 1.2.4 *Let Φ be conditionally strictly positive definite of order k, and let $x_1, \ldots, x_m$ be unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^n)$. Assume $\Pi_{k-1}(\mathbb{R}^n)$ has dimension l and suppose polynomials $p_1, \ldots, p_l$ form a basis for $\Pi_{k-1}(\mathbb{R}^n)$. Let A be the $m \times m$ matrix given by $A_{ij} = \Phi(|x_i - x_j|)$ and P be the $m \times l$ matrix given by $P_{ij} = p_j(x_i)$. Then the matrix*

$$M = \begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix}$$

is invertible.

Proof. Suppose $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ and $v = (v_1, \ldots, v_l) \in \mathbb{R}^l$ are such that $(u, v) \in \mathbb{R}^{m+l}$ is in the kernel of M. Then we have

$$Au + Pv = 0 \tag{1.3}$$

and

$$P^T u = 0. \tag{1.4}$$

Equation (1.4) implies $u^T P = 0$. Pre-multiplying Equation (1.3) by u^T and making the substitution $u^T P = 0$ gives

$$u^T A u = 0.$$

Expanding gives

$$\sum_{i=1}^{m} \sum_{j=1}^{m} u_i u_j \Phi(|x_i - x_j|) = 0.$$

Also from Equation (1.4) we have

$$\sum_{i=1}^{m} u_i p_j(x_i) = 0 \qquad \text{for all } j = 1, \ldots, l.$$

Hence, since Φ is conditionally strictly positive definite of order k we must have $u = 0$. From Equation (1.3) we now have $Pv = 0$. Expanding gives

$$q(x_i) = p_1(x_i)v_1 + p_2(x_i)v_2 + \ldots + p_l(x_i)v_l = 0, \qquad \text{for all } i = 1, \ldots, m.$$

Thus q is a polynomial in $\Pi_{k-1}(\mathbb{R}^n)$ with $q(x_i) = 0$, for all $i = 1, \ldots, m$. Hence, by the unisolvency of $x_1, \ldots, x_m$, $q = 0$. Thus, $v = 0$. Therefore, the kernel of M is trivial and hence M is invertible. ∎

The seminal paper by Micchelli [26] shows that the multiquadric is conditionally strictly positive definite of order one. Micchelli also proves that it is possible to solve the multiquadric interpolation problem without a polynomial term being added to the interpolant. This is an interesting feature of conditionally strictly positive definite functions of order one. In the multiquadric case this result backed up Hardy's experimental evidence. In addition, Micchelli gives the following characterisation of conditionally positive definite functions of order k, which extended the work of Schoenberg [35] who had considered the case $k = 0$.

Definition 1.2.5 *A function f is said to be completely monotone on $(0, \infty)$ if $f \in C^\infty(0, \infty)$ and $(-1)^m f^{(m)}(t) \geq 0$ for all $t > 0$ and $m = 0, 1, 2, \ldots$.*

Theorem 1.2.6 *Let $\Phi \in C[0, \infty) \cap C^\infty(0, \infty)$ and suppose that $(-1)^k \Phi^{(k)}$ is completely monotone but not constant on $(0, \infty)$. Let function Ψ be defined by $\Psi(t) = \Phi(t^2)$, for $t \in (0, \infty)$. Then Ψ is conditionally strictly positive definite of order k.*

We have seen that in order to guarantee a solution to the interpolation problem when the basic function Φ is conditionally strictly positive definite of order k, we must ensure that the interpolation points satisfy a unisolvency condition. However, for many common choices of Φ, k is at most two. Thus, the unisolvency condition is not as problematic as it seems at first glance. For example, in two dimensions with $k = 2$, we would simply require three non-collinear points in order to be able to solve the system. If $k = 0$ then no requirement is made on the points at all.

Some common examples of Φ are listed below.

Bare norm	$\Phi(r) = r$	$k = 1$
Thin plate spline	$\Phi(r) = r^2 \ln r$	$k = 2$
Multiquadric	$\Phi(r) = \sqrt{r^2 + c^2}$	$k = 1$
Gaussian	$\Phi(r) = e^{-r^2}$	$k = 0$

1.3 Variational Theory

The first radial basic function interpolants to be actively researched were the thin plate splines. The original motivation came from the aeronautical industry in the early seventies [13]. The problems focussed on finding splines which minimised the bending energy of infinite thin plates subject to interpolation constraints. Duchon, building on earlier work by Atteia [3], took these ideas of minimisation and extended them to higher dimensions. His papers [6, 7] were seminal in the development of a variational theory for these problems. The interpolant is shown to be a *minimal norm interpolant* in the following sense. Let Hilbert space X, with seminorm $|\cdot|$ defined on X, be such that point evaluations are bounded linear functionals on X. Suppose that $f \in X$ is known only on the points $x_1, \ldots, x_m \in \mathbb{R}^n$. We wish to find $s \in X$ such that

i) $s(x_j) = f(x_j)$, for all $j = 1, \ldots, m$,

ii) $|s| \leq |v|$, for all $v \in X$ satisfying $v(x_j) = f(x_j)$ for all $j = 1, \ldots, m$.

Duchon used spaces of distributions which were generalisations of Beppo-Levi spaces (see [5]); the related seminorm taking the form

$$|f| = \left(\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 \, dx \right)^{1/2},$$

for fixed integer $k \geq 0$ and known constants c_α. In this case the solution to the above variational problem takes the form

$$s(x) = \sum_{i=1}^{m} \alpha_i \Phi(|x - x_i|) + p(x),$$

where p is a polynomial of degree $k-1$ and the function Φ has the form

$$\Phi(r) = \begin{cases} r^{2k-n} \ln r, & \text{for } n \text{ even },\\ r^{2k-n}, & \text{otherwise }. \end{cases}$$

Such interpolants are known as surface splines and in one dimension they coincide with natural splines of order $2k$.

Recent work of Light and Wayne [20] generalised that of Duchon, and provides the motivation for much of the work contained in this thesis. A measurable weight function $v : \mathbb{R}^n \to \mathbb{R}$ is introduced and the seminorm is defined, for fixed integer k, by

$$|f|_k = \left(\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 v(x)\, dx \right)^{1/2}.$$

The Fourier transform here is taken in the distributional sense (see Section 1.6) and the constants c_α are defined by the algebraic identity

$$\sum_{|\alpha|=k} c_\alpha x^{2\alpha} = |x|^{2k}, \qquad \text{for all } x \in \mathbb{R}^n.$$

The associated space of functions is given by

$$Z_k(\mathbb{R}^n) = \{ f \in S' : \widehat{D^\alpha f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 v(x)\, dx < \infty,$$
$$\text{for all } \alpha \in \mathbb{Z}^n_+ \text{ with } |\alpha| = k \}.$$

Here we use S' to denote the space of tempered distributions, which we discuss further in Section 1.6.

In order to consider an interpolation problem on these spaces we need to ensure point evaluations make sense. Light and Wayne [20] demonstrated this by proving that $Z_k(\mathbb{R}^n)$ was embedded in $C(\mathbb{R}^n)$. However this is dependent on the weight function v satisfying the following conditions,

(1) $v \in C(\mathbb{R}^n \backslash \{0\})$,

(2) $v(x) > 0$ if $x \neq 0$,

(3) $1/v \in L^1_{loc}(\mathbb{R}^n)$,

(4) there is a $\mu \in \mathbb{R}$ such that $(v(x))^{-1} = \mathcal{O}(|x|^{-2\mu})$ as $|x| \to \infty$.

These conditions also ensure that $Z_k(\mathbb{R}^n)$ is complete with respect to $|\cdot|_k$.

Using the fact that $v(x) > 0$ if $x \neq 0$, one can show that the kernel of the seminorm $|\cdot|_k$ is $\Pi_{k-1}(\mathbb{R}^n)$. Hence, we can define a norm on $Z_k(\mathbb{R}^n)$ as follows. Let $a_1, \ldots, a_l \in \mathbb{R}^n$ be unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^n)$. Then define

$$\|f\|_{Z_k} = \left(\sum_{s=1}^{l} |f(a_s)|^2 + |f|_k^2 \right)^{1/2} \qquad \text{for all } f \in Z(\mathbb{R}^n).$$

Equipped with this norm $Z_k(\mathbb{R}^n)$ is a Hilbert function space. Using reproducing kernel techniques, Light and Wayne [20] were able to show that the minimal norm interpolant to $f \in Z_k(\mathbb{R}^n)$, on points $x_1, \ldots, x_m \in \mathbb{R}^n$ unisolvent with respect to $\Pi_{k-1}(\mathbb{R}^n)$, takes the form

$$s(x) = \sum_{j=1}^{m} \alpha_j \Phi(x - x_j) + p(x).$$

The function $p(x)$ is a polynomial of degree $k-1$, and Φ is a tempered distribution which satisfies

$$\widehat{\Phi}v| \cdot |^{2k} = 1.$$

In addition, Φ is a continuous function and, in a link with the previous section, is also conditionally positive definite of order k. We remark, however, that the basic function Φ is not radial.

1.4 Error Estimates

We turn our attention now to the subject of error estimates in order to provide motivation for the work contained in Chapter 2.

Let X be a space of real valued functions on $\mathbb{R}^n$ with seminorm $|\cdot|$. Suppose we interpolate a function $f \in X$ at points $x_1, \ldots, x_m \in \mathbb{R}^n$ by $s \in X$. Thus,

$$s(x_i) = f(x_i), \qquad \text{for all } i = 1, \ldots m.$$

A typical error estimate has the form

$$|f(x) - s(x)| \leq \mathcal{P}(x, x_1, \ldots, x_m)|f-s|, \qquad \text{for all } x \in \mathbb{R}^n.$$

Here $\mathcal{P}$ is the so-called power function and we shall give more details on how it is derived in Chapter 3. In order to be able to use this estimate we need to know $f - s$ everywhere on $\mathbb{R}^n$. Duchon [7] described for his spaces of distributions how one could use localisation to obtain improved error estimates. Instead of the above global estimate he obtains for

$\Omega \subset \mathbb{R}^n$ a 'local' estimate of the form

$$|f(x) - s(x)| \leq \mathcal{P}(x, x_1, \ldots, x_m)|f - s|_\Omega, \qquad \text{for all } x \in \Omega. \tag{1.5}$$

Note that a localised version of the seminorm appears on the right-hand side and the error estimate is now only true for $x \in \Omega$. Using this local estimate Duchon is able to obtain improved estimates in terms of the spacing of the interpolation points $x_1, \ldots, x_m$. Let $A = \{x_1, \ldots, x_m\}$ and

$$h = \sup_{y \in \Omega} \inf_{x \in A} |y - x|.$$

Suppose using the original estimate one can obtain a constant C independent of f and h such that $\|f - s\|_{p,\Omega} \leq Ch^\beta |f|$ for some β. Making use of the localised error estimate it is possible to improve this to $\|f - s\|_{p,\Omega} \leq Ch^{\beta + n/p}|f|_\Omega$. Exact details of how Duchon obtains these results can be found in [7] or the later work of Light and Wayne [19].

The aim of this work is to apply Duchon's localisation techniques to spaces having a seminorm of the form

$$|f|_k = \left(\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 v(x)\, dx \right)^{1/2},$$

as introduced in the previous section. The first step is to indicate what we mean by a local version of the seminorm $|\cdot|_k$. We notice that this is defined in terms of the Fourier transform of the function. Thus, there is currently no natural way of defining the local seminorm. What is needed is a direct version of the seminorm, defined in terms of the function itself, and not its Fourier transform. The recent paper of Levesley and Light [18] concerned itself with this task. We assume the weight function v satisfies the following conditions,

(1) $v \in S' \bigcap C(\mathbb{R}^n)$ and $v(x) > 0$ for almost all $x \in \mathbb{R}^n$,

(2) $\widehat{v}$ is a measurable function and for any neighbourhood N of the origin, $\widehat{v} \in L^1(\mathbb{R}^n \backslash N)$,

(3) $v(y) = v(-y)$ for all $y \in \mathbb{R}^n$,

(4) $|\widehat{v}(y)| = \mathcal{O}(|y|^s)$ as $y \to 0$, where $n + s + 2 > 0$,

(5) $v(0) = 0$ and $\widehat{v}(x) \leq 0$ for almost all $x \in \mathbb{R}^n$.

Under these conditions, Levesley and Light proved that for all $f \in Z_k(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 v(x) dx = -\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{v}(x-y) |(D^\alpha f)(x) - (D^\alpha f)(y)|^2 \, dx dy.$$

Now we can simply define our local seminorm by

$$|f|_{k,\Omega} = \left(\sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y) |(D^\alpha f)(x) - (D^\alpha f)(y)|^2 \, dx dy \right)^{1/2},$$

where $w = -\frac{1}{2}\hat{v}$.

The second, and less obvious, requirement for the development of the localised version of the error estimate is having to hand certain extension operators. Duchon worked in a Sobolev space setting where the relevant extension theorems were already well known. The development of the required extension operators for the above seminorm is the aim of Chapter 2.

We begin by working with some spaces of continuous functions. Let $C_0^k(\mathbb{R}^n)$ denote the set of all compactly supported functions on $\mathbb{R}^n$ which have continuous derivatives up to the k-th order. For Ω, an open subset of $\mathbb{R}^n$, we define $X(\Omega) = \{g|_\Omega \ : g \in C_0^k(\mathbb{R}^n) \text{ and } |g|_{k,\Omega} < \infty\}$. Similarly we define $X(\mathbb{R}^n) = \{f \in C_0^k(\mathbb{R}^n) : |f|_k < \infty\}$. Now, under appropriate hypotheses on w, $|\cdot|_{k,\Omega}$ defines a seminorm on $X(\Omega)$. We develop a linear extension operator from $X(\Omega)$ to $X(\mathbb{R}^n)$, subject to Ω and the weight function w satisfying certain properties which are detailed in Chapter 2. Using this result we deduce the existence of extensions for functions in $\mathcal{Y}(\Omega)$, the completion of $X(\Omega)$ with respect to the seminorm $|\cdot|_{k,\Omega}$. Outlined below is our principal extension result, the proof of which can be found in Section 2.3 and is again dependent on a suitable choice of Ω and w.

Theorem 1.4.1 *Given $f \in \mathcal{Y}(\Omega)$, there exists a function $f_e \in \mathcal{Y}(\mathbb{R}^n)$ such that*

1) $f_e|_\Omega = f$

2) $|f_e|_{\mathbb{R}^n} \leq M|f|_\Omega$ for some constant M independent of f.

In Chapter 3, using an adaptation of Duchon's methods, we shall demonstrate how these extension theorems can be used to derive improved error estimates for the spaces $\mathcal{Y}(\mathbb{R}^n)$.

1.5 Native Spaces

We discussed in Section 1.3 how the surface splines of Duchon [6, 7] arise naturally from a variational problem described on certain spaces of functions. The work of Madych and Nelson [21, 22, 23] extends this to other radial basic function interpolants, for example multiquadric surfaces. The approach is different to that of Duchon, where the focus is on the seminorm and associated space of functions. Here one begins with a conditionally positive definite function and uses it to construct a *native* space of functions in which one can carry out the appropriate variational arguments. Other papers in this area include those of Wu and Schaback [39], those of Dyn [8, 9] and several papers by Schaback which are accessible through the survey [31].

Many authors considering native spaces work on the whole of $\mathbb{R}^n$. However, definitions of local native spaces, and corresponding extension theorems, can be found for example in the work of Schaback [32] and Iske [16]. In Chapter 4 we shall look at how this approach to localisation compares to that described in the previous section, which makes use of the Levesley-Light direct form seminorm [18]. Of particular interest will be the following convolutional characterisation of the local native space given by Iske.

Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be continuous and strictly positive definite. We define $C_0^\infty(\Omega)$ to be the set of all functions in $C_0^\infty(\mathbb{R}^n)$ whose support is contained in Ω. A bilinear form $(\cdot,\cdot)_\Phi$ is defined on $C_0^\infty(\Omega)$ via

$$(v,w)_\Phi = \int_\Omega \int_\Omega \Phi(y-x)v(x)\overline{w(y)}\,dxdy, \qquad \text{for } v,w \in C_0^\infty(\Omega).$$

Assume that there exists a continuous and positive function $\Psi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ such that

$$(v,w)_\Phi = \int_{\mathbb{R}^n} \Psi(x)\widehat{v}(x)\overline{\widehat{w}(x)}\,dx \qquad \text{for all } v,w \in C_0^\infty(\Omega).$$

Then the quantity $\|\cdot\|_\Phi = \sqrt{(\cdot,\cdot)_\Phi}$ defines a norm on $C_0^\infty(\Omega)$. Since $\Phi \in C(\mathbb{R}^n)$, the convolution $\Phi * w$ is well defined for all $w \in C_0^\infty(\Omega)$ and is given by

$$(\Phi * w)(x) = \int_{\mathbb{R}^n} \Phi(x-y)w(y)\,dy.$$

Let

$$I(\Omega) = \{f\,|_\Omega : f = \Phi * w \text{ for some } w \in C_0^\infty(\Omega)\}.$$

Iske's local native space of Φ is defined as the closure with respect to $\|\cdot\|_\Phi$ of $I(\Omega)$. We remark that this is the simplest case of Iske's results. One could also work with a conditionally positive definite function.

1.6 Distribution Theory and Notation

We end this chapter with some notation and results that will be of use in later chapters. In particular, we introduce some of the theory of distributions which will be needed in Chapter 4. The definitions are taken from Rudin [30] in which further details and proofs of all the results given here can be found.

The space of test functions D consists of the vector space $C_0^\infty(\mathbb{R}^n)$, whose topology is described in [30, Definition 6.3]. The space of distributions, denoted D', consists of all linear functionals on D which are continuous with respect to this topology. If f is a distribution and ϕ is a test function, then we shall use the notation $[f, \phi]$ to denote the action of the distribution on the test function.

For multi-index $\alpha = (\alpha_1 \ldots, \alpha_n) \in \mathbb{Z}_+^n$ the differential operator D^α is given by

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

and has order $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

The space of rapidly decreasing functions consists of those functions for which

$$\sup_{|\alpha|<N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |D^\alpha f(x)| < \infty \qquad N = 0, 1, 2, \ldots. \tag{1.6}$$

These functions form a vector space S whose topology is given by the countable collection of norms (1.6). We denote by S' the space of tempered distributions, that is the set of all continuous linear functionals on S.

We quote now some useful properties of tempered distributions which will be needed in subsequent chapters. If f is a tempered distribution and $\phi \in S$ then ϕf is a tempered distribution whose action is defined by

$$[\phi f, \psi] = [f, \phi\psi] \qquad \psi \in S.$$

For $f \in S$ the Fourier transform of f is defined by

$$\widehat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-ixy}\, dy, \qquad x \in \mathbb{R}^n.$$

Furthermore, the Fourier transform is a continuous, linear, one-to-one mapping of S onto S. We can extend the Fourier transform to tempered distributions by defining for $f \in S'$,

$$[\widehat{f}, \phi] = [f, \widehat{\phi}] \qquad \phi \in S.$$

This distributional Fourier transform is also a continuous, linear, one-to-one mapping of S' onto S'. For $f, g \in L^2(\mathbb{R}^n)$, the Parseval formula states that

$$\int_{\mathbb{R}^n} f\overline{g} = \int_{\mathbb{R}^n} \widehat{f}\overline{\widehat{g}}. \tag{1.7}$$

Taking $f = g$ in Equation (1.7) yields the Plancherel Theorem,

$$\|f\|_2 = \|\widehat{f}\|_2, \qquad \text{for all } f \in L^2(\mathbb{R}^n).$$

The operator B is defined for $\phi \in S$ by $(B\phi)(x) = \phi(-x)$ for all $x \in \mathbb{R}^n$. A useful result is that $(\widehat{\phi})\widehat{\ } = B\phi$, for $\phi \in S$. Given $x \in \mathbb{R}^n$, the shift operator $T_x : S \to S$ is defined by $(T_x\phi)(y) = \phi(y - x)$, $y \in \mathbb{R}^n$. The operators T_x and B are extended to tempered distributions by defining $[T_x f, \phi] = [f, T_{-x}\phi]$ and $[Bf, \phi] = [f, B\phi]$ for all $\phi \in S$.

The convolution of two functions f and g on $\mathbb{R}^n$ is given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy$$

whenever the integral exists. For ϕ and ψ in S the convolution $\phi * \psi$ is well-defined and is itself an element of S. This convolution also satisfies the properties $(\phi * \psi)\widehat{\ } = \widehat{\phi}\widehat{\psi}$ and $(\phi\psi)\widehat{\ } = \widehat{\phi} * \widehat{\psi}$. The convolution of a tempered distribution f with $\phi \in S$ is given by

$$[f * \phi, \psi] = [f, T_x B\psi] \qquad\qquad \psi \in S.$$

Furthermore, it can be shown that $f * \phi \in C^\infty \cap S'$, $(f * \phi)\widehat{\ } = \widehat{f}\widehat{\phi}$ and $(f\phi)\widehat{\ } = \widehat{f} * \widehat{\phi}$.

Finally we end with a remark on notation. We have already in this chapter used the symbol $|\cdot|$ to denote both the Euclidean norm and a generic seminorm. However, as we believe the intended meaning is clear in all cases we have chosen not to develop an alternative piece of notation.

Chapter 2

Extension Theorems

In Section 1.4 we introduced the idea of a direct form seminorm, with particular reference to the seminorm of Levesley and Light [18]. The advantage of a direct form seminorm is that it delivers a very natural way of defining a local seminorm, and hence corresponding local spaces. In this chapter, we shall be interested in spaces of functions which arise from the Levesley-Light direct form seminorms. In particular, we wish to develop certain extension theorems for these spaces, the motivation being that such extension theorems can then be employed in the development of improved error estimates. We shall discuss this application further in Chapter 3.

A special case of the Levesley-Light seminorm is that used in non-integer valued Sobolev space seminorms. Extension theorems for these spaces, both integer and non-integer valued, are well known, see for example Adams [1] or Showalter [36]. We shall make use of this in the sense that our overall *strategy* of proof follows that used in the Sobolev extension theorems. However, at the level of generality we are considering, new techniques must be developed.

2.1 An Extension on $\mathbb{R}^n_+$

We begin with some results on the half plane $\mathbb{R}^n_+$; that is, the set of all points in $\mathbb{R}^n$ whose last coordinate is positive. This focus on the last coordinate means we shall often write a point $x \in \mathbb{R}^n$ in the form $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Using this notation, we can write $\mathbb{R}^n_+ = \{(x', x_n) : x_n > 0\}$.

We define now a linear operator E_α, which extends functions defined on $\mathbb{R}^n_+$, to functions defined on the whole of $\mathbb{R}^n$. This operator will prove essential, not only to the current development, but to subsequent work in this chapter. Indeed, it provides the backbone of the extension operators we shall construct in Section 2.3 for domains more general than $\mathbb{R}^n_+$.

Definition 2.1.1 *Let $k \in \mathbb{Z}_+$ and define $\lambda_1, \ldots, \lambda_{k+1}$ to be the unique solution of the system*

$$\sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^l = 1, \quad l = 0, 1, \ldots, k.$$

For each $f : \mathbb{R}^n_+ \to \mathbb{R}$ and each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, define $E_\alpha f : \mathbb{R}^n \to \mathbb{R}$ by

$$E_\alpha f(x', x_n) = \begin{cases} f(x', x_n), & \text{if } x_n > 0 \\ \\ \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} f(x', -x_n/j), & \text{otherwise.} \end{cases}$$

Many of the results in this section will concern E_θ, where $\theta = (0, \ldots, 0)$. We begin by describing how this operator behaves with regards to continuous functions. We shall initially be interested in the space $Y^k(\mathbb{R}^n_+) = \{g|_{\mathbb{R}^n_+} : g \in C_0^k(\mathbb{R}^n)\}$, where k is some positive integer. The corresponding space $Y^k(\mathbb{R}^n)$ is simply $C_0^k(\mathbb{R}^n)$.

Theorem 2.1.2 *Let $\theta = (0, \ldots, 0)$ and let $f \in Y^k(\mathbb{R}^n_+)$, for some $k \in \mathbb{Z}_+$. Then $E_\theta f \in C_0^k(\mathbb{R}^n)$ and $D^\alpha E_\theta f = E_\alpha D^\alpha f$, for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq k$.*

Proof. Suppose $x = (x', x_n) \in \mathbb{R}^n$, with $x_n \leq 0$, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ with $|\alpha| \leq k$. Then,

$$\begin{aligned} D^\alpha E_\theta f(x', x_n) &= D^\alpha \left[\sum_{j=1}^{k+1} \lambda_j f(x', -x_n/j) \right] \\ &= \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j} \right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \\ &= E_\alpha D^\alpha f(x', x_n). \end{aligned}$$

The relation $D^\alpha E_\theta f(x) = E_\alpha D^\alpha f(x)$ for $x = (x', x_n)$ with $x_n > 0$ is clear and so the formula $D^\alpha E_\theta f = E_\alpha D^\alpha f$ is established for all $f \in Y^k(\mathbb{R}^n_+)$.

It remains to show that $E_\theta f \in C_0^k(\mathbb{R}^n)$. We know from above that $D^\alpha E_\theta f = E_\alpha D^\alpha f$. Since $f \in Y^k(\mathbb{R}^n_+)$, $D^\alpha f \in Y^0(\mathbb{R}^n_+)$. Thus it is clear from the definition of E_α that $E_\alpha D^\alpha f \in C_0(\mathbb{R}^n)$. Hence $D^\alpha E_\theta f \in C_0(\mathbb{R}^n)$ and the result follows. ∎

We now introduce a measurable weight function $w : \mathbb{R}^n \to \mathbb{R}$. We shall assume throughout this section that w satisfies the following two properties,

($\mathcal{W}1$) $w(x) \geq 0$ for almost all $x \in \mathbb{R}^n$;

($\mathcal{W}2$) there exists a constant $M > 0$ such that if $x = (x', x_n) \in \mathbb{R}^n$ and $y = (x', y_n) \in \mathbb{R}^n$ with $|x_n| \geq |y_n|$ then $w(x) \leq M w(y)$.

Now take $\alpha \in \mathbb{Z}^n_+$, and define, for $f \in Y^{|\alpha|}(\mathbb{R}^n_+)$,

$$|f|_{\alpha, \mathbb{R}^n_+} = \left(\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y) |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx dy \right)^{1/2}.$$

The form of the integral is exactly that found in the Levesley-Light seminorm [18]. We shall be interested in functions for which this integral is finite. Thus, we define $X^{\alpha}(\mathbb{R}^n_+)$ to be the set of all $f \in Y^{|\alpha|}(\mathbb{R}^n_+)$ which satisfy $|f|_{\alpha,\mathbb{R}^n_+} < \infty$. The quantity $|f|_{\alpha,\mathbb{R}^n}$ and space $X^{\alpha}(\mathbb{R}^n)$ are similarly defined. We now prove an extension theorem for functions in $X^{\alpha}(\mathbb{R}^n_+)$.

Theorem 2.1.3 *There exists a linear operator $E : X^{\alpha}(\mathbb{R}^n_+) \to X^{\alpha}(\mathbb{R}^n)$ such that for all $f \in X^{\alpha}(\mathbb{R}^n_+)$,*

(i) $Ef(x) = f(x)$, for all $x \in \mathbb{R}^n_+$,

(ii) $|Ef|_{\alpha,\mathbb{R}^n} \leq A|f|_{\alpha,\mathbb{R}^n_+}$, for some positive constant A independent of f.

Proof. Our claim is that a suitable choice for E is the one we have already defined prior to this theorem, $E = E_\theta$, providing $|\alpha| \leq k$. Take $f \in X^{\alpha}(\mathbb{R}^n_+)$. It follows immediately from the construction of E that $Ef(x) = f(x)$ for all $x \in \mathbb{R}^n_+$. We consider

$$|Ef|^2_{\alpha,\mathbb{R}^n} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(x-y)|D^{\alpha}Ef(x) - D^{\alpha}Ef(y)|^2\, dxdy.$$

For convenience we define a measurable function z by

$$z(x,y) = w(x-y)|D^{\alpha}Ef(x) - D^{\alpha}Ef(y)|^2, \qquad \text{for almost all } x,y \in \mathbb{R}^n.$$

Let χ_{++} be the characteristic function of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, χ_{+-} be the characteristic function of $\mathbb{R}^n_+ \times (\mathbb{R}^n \backslash \mathbb{R}^n_+)$, and similarly for χ_{-+} and χ_{--}. Then

$$|Ef|^2_{\alpha,\mathbb{R}^n} = I_{++} + I_{+-} + I_{-+} + I_{--}$$

where, for example,

$$I_{-+} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{-+}(x,y)z(x,y)\;dxdy.$$

Now,

$$\begin{aligned} I_{-+} &= \int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0 w(x'-y',x_n-y_n) \\ &\qquad \left|\left(\sum_{j=1}^{k+1}\lambda_j\left(-\frac{1}{j}\right)^{|\alpha_n|}D^\alpha f(x',-x_n/j)\right)-D^\alpha f(y',y_n)\right|^2 dx_n dx' dy_n dy'. \end{aligned}$$

Recall that since $|\alpha_n|\le k$ we have $\sum_{j=1}^{k+1}\lambda_j\left(-\frac{1}{j}\right)^{|\alpha_n|}=1$. Using this fact and an application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} &\left|\left(\sum_{j=1}^{k+1}\lambda_j\left(-\frac{1}{j}\right)^{|\alpha_n|}D^\alpha f(x',-x_n/j)\right)-D^\alpha f(y',y_n)\right|^2 \\ &= \left|\sum_{j=1}^{k+1}\lambda_j\left(-\frac{1}{j}\right)^{|\alpha_n|}\left(D^\alpha f(x',-x_n/j)-D^\alpha f(y',y_n)\right)\right|^2 \\ &\le \left(\sum_{j=1}^{k+1}\left|\lambda_j\left(-\frac{1}{j}\right)^{|\alpha_n|}\right|^2\right)\left(\sum_{j=1}^{k+1}\left|D^\alpha f(x',-x_n/j)-D^\alpha f(y',y_n)\right|^2\right). \end{aligned}$$

Let $A_1=\sum_{j=1}^{k+1}|\lambda_j|^2\left(-\frac{1}{j}\right)^{2|\alpha_n|}$. Then,

$$\begin{aligned} I_{-+} &\le A_1\sum_{j=1}^{k+1}\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0 w(x'-y',x_n-y_n) \\ &\qquad \left|D^\alpha f(x',-x_n/j)-D^\alpha f(y',y_n)\right|^2\;dx_n dx' dy_n dy'. \end{aligned}$$

Making the substitution $x_n=-js_n$ in the appropriate integral gives

$$\begin{aligned} I_{-+} &\le A_1\sum_{j=1}^{k+1}j\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_0^\infty w(x'-y',-js_n-y_n) \\ &\qquad \left|D^\alpha f(x',s_n)-D^\alpha f(y',y_n)\right|^2\;ds_n dx' dy_n dy'. \end{aligned}$$

Since j, s_n, and y_n only take positive values,

$$|-js_n - y_n| = |js_n + y_n| = js_n + y_n \geq s_n + y_n \geq |s_n - y_n|.$$

Hence, by $(\mathcal{W}2)$, we can find a number $A_2 > 0$ such that

$$\begin{aligned} I_{-+} \ &\leq \ A_2 \sum_{j=1}^{k+1} j \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_0^\infty w(x'-y', s_n - y_n) \\ &\qquad\qquad |D^\alpha f(x', s_n) - D^\alpha f(y', y_n)|^2 \ ds_n dx' dy_n dy'. \end{aligned}$$

Letting $A_3 = A_2 \sum_{j=1}^{k+1} j$ we obtain,

$$I_{-+} \leq A_3 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y) \left| D^\alpha f(x) - D^\alpha f(y) \right|^2 \ dxdy.$$

An almost identical argument furnishes the existence of a constant A_4 such that

$$I_{+-} \leq A_4 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y) \left| D^\alpha f(x) - D^\alpha f(y) \right|^2 \ dxdy.$$

Now, by reasoning very similar to above, we deduce the existence of $A_5 > 0$ such that

$$\begin{aligned} I_{--} \ &= \ \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 w(x'-y', x_n - y_n) \\ &\qquad \left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} \Big(D^\alpha f(x', -x_n/j) - D^\alpha f(y', -y_n/j) \Big) \right|^2 \ dx_n dx' dy_n dy' \\ &\leq \ A_5 \sum_{j=1}^{k+1} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 w(x'-y', x_n - y_n) \\ &\qquad\qquad |D^\alpha f(x', -x_n/j) - D^\alpha f(y', -y_n/j)|^2 \ dx_n dx' dy_n dy'. \end{aligned}$$

The change of variables $x_n = -js_n$ and $y_n = -jt_n$ gives,

$$\begin{aligned} I_{--} \ &\leq \ A_5 \sum_{j=1}^{k+1} j^2 \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_0^\infty w(x'-y', jt_n - js_n) \\ &\qquad\qquad |D^\alpha f(x', s_n) - D^\alpha f(y', t_n)|^2 \ ds_n dx' dt_n dy'. \end{aligned}$$

Again, since s_n, t_n and j take only positive values, we have

$$|jt_n - js_n| = j|s_n - t_n| \geq |s_n - t_n|,$$

and so an application of $(\mathcal{W}2)$ furnishes a constant A_6 such that

$$I_{--} \leq A_6 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x' - y', s_n - t_n) |D^\alpha f(x', s_n) - D^\alpha f(y', t_n)|^2 \, ds_n dx' dt_n dy'.$$

Finally, using

$$\begin{aligned} |Ef|^2_{\alpha,\mathbb{R}^n} &= I_{++} + I_{+-} + I_{-+} + I_{--} \\ &\leq (1 + A_3 + A_4 + A_6) \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x - y) |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx dy, \end{aligned}$$

we obtain $|Ef|_{\alpha,\mathbb{R}^n} \leq \sqrt{1 + A_3 + A_4 + A_6} \; |f|_{\alpha,\mathbb{R}^n_+}$. ∎

We end this section with some results that will be of use later.

Lemma 2.1.4 *Let $\theta = (0, \ldots, 0) \in \mathbb{Z}^n_+$. Let $k \in \mathbb{Z}_+$, and suppose $\alpha \in \mathbb{Z}^n_+$ satisfies $|\alpha| \leq k$. For $f \in X^\alpha(\mathbb{R}^n_+)$ define $Ef = E_\theta f$ as in Definition 2.1.1. Then there exists a constant C such that*

$$\int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2 \, dx \leq C \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx \qquad \textit{for all } f \in X^\alpha(\mathbb{R}^n_+).$$

Proof. We can write,

$$\begin{aligned} \int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2 \, dx &= \int_{\mathbb{R}^n_+} |D^\alpha Ef(x)|^2 \, dx + \int_{\mathbb{R}^n \setminus \mathbb{R}^n_+} |D^\alpha Ef(x)|^2 \, dx \\ &= \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx \\ &\quad + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \right|^2 \, dx_n \, dx'. \end{aligned}$$

We consider the second integral. Let

$$I = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} \left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \right|^2 dx_n\, dx'.$$

An application of the Cauchy-Schwarz inequality gives

$$I \leq \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} \left(\sum_{j=1}^{k+1} \left| \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} \right|^2 \right) \left(\sum_{j=1}^{k+1} |D^\alpha f(x', -x_n/j)|^2 \right) dx_n\, dx'.$$

Letting $c_1 = \sum_{j=1}^{k+1} \left| \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} \right|^2$, and making the change of variables $x_n = -s_n j$ we have

$$I \leq c_1 \sum_{j=1}^{k+1} j \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} |D^\alpha f(x', s_n)|^2\, ds_n dx'.$$

Letting $c_2 = c_1 \sum_{j=1}^{k+1} j$ we have

$$I \leq c_2 \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2\, dx.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2\, dx &\leq \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2\, dx + c_2 \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2\, dx \\ &= (1 + c_2) \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2\, dx. \quad \blacksquare \end{aligned}$$

Definition 2.1.5 *Take $k \in \mathbb{Z}_+$ and let $f \in Y^k(\mathbb{R}^n_+)$. We define*

$$\|f\|^2_{\mathbb{R}^n_+} = \sum_{\substack{\alpha \in \mathbb{Z}^n_+ \\ |\alpha| = k}} c_\alpha |f|^2_{\alpha, \mathbb{R}^n_+} + \sum_{\substack{\alpha \in \mathbb{Z}^n_+ \\ |\alpha| \leq k}} \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2\, dx,$$

where the c_α are constants. Let $\|\cdot\|_{\mathbb{R}^n}$ be similarly defined.

Theorem 2.1.6 *Let $\theta = (0, \ldots, 0) \in \mathbb{Z}^n_+$. For $f \in \bigcap_{|\alpha|=k} X^\alpha(\mathbb{R}^n_+)$, let $Ef = E_\theta f$ be as defined in Theorem 2.1.2. Then, for all $f \in \bigcap_{|\alpha|=k} X^\alpha(\mathbb{R}^n_+)$*

$$\|Ef\|_{\mathbb{R}^n} \leq M \|f\|_{\mathbb{R}^n_+},$$

for some constant M independent of f.

Proof. Take $f \in \bigcap_{|\alpha|=k} X^{\alpha}(\mathbb{R}^n_+)$. Then, by Theorem 2.1.3 and Lemma 2.1.4, there exist a constants M_1 and M_2, independent of f such that

$$\begin{aligned} \|Ef\|^2_{\mathbb{R}^n} &= \sum_{|\alpha|=k} c_\alpha |Ef|^2_{\alpha,\mathbb{R}^n} + \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2 \, dx, \\ &\leq M_1 \sum_{|\alpha|=k} c_\alpha |f|^2_{\alpha,\mathbb{R}^n_+} + M_2 \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx. \end{aligned}$$

Taking $M^2 = \max\{M_1, M_2\}$ gives the result. ■

2.2 Some Preparatory Results

In this section we begin to move towards an extension for more general domains. The actual construction of the extension operator is deferred until Section 2.3; here we shall discuss some technical results that will be essential to our later development. We also introduce some definitions and notation which we will adhere to throughout the rest of the chapter. We begin by recalling the form of the Levesley-Light direct seminorm.

Take $k \in \mathbb{Z}_+$ and suppose Ω is an open subset of $\mathbb{R}^n$. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable weight function, which will be required to satisfy certain properties as we proceed. For $f \in C^k(\Omega)$ we define,

$$|f|_\Omega = \left(\sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y) |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx dy \right)^{1/2}, \tag{2.1}$$

where the constants c_α are defined by the algebraic identity

$$\sum_{|\alpha|=k} c_\alpha x^{2\alpha} = |x|^{2k}, \qquad \text{for all } x \in \mathbb{R}^n.$$

As in Section 2.1, we shall be interested in functions for which $|\cdot|_\Omega$ is finite. We denote by $X(\Omega)$ the set of all f restricted to Ω such that $f \in C_0^k(\mathbb{R}^n)$ and $|f|_\Omega < \infty$. On $X(\Omega)$, $|\cdot|_\Omega$ defines a seminorm with kernel consisting of polynomials restricted to Ω of degree at most k. We define, for $f \in C^k(\mathbb{R}^n)$,

$$\|f\|_\Omega = \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha f(x)|^2 \, dx + |f|_\Omega^2 \right)^{1/2} \qquad \text{for } f \in X(\Omega). \tag{2.2}$$

Then, if Ω is bounded, $\|\cdot\|_\Omega$ defines a norm on $X(\Omega)$.

In the construction of our extension operators we will make use of domain transformations which are k-smooth, by which we mean the following.

Definition 2.2.1 *Let Ω_1 and Ω_2 be domains in $\mathbb{R}^n$, and Φ a bijection from Ω_1 to Ω_2. We say that Φ is k-smooth if, writing $\Phi(x) = (\phi_1(x_1, \dots, x_n), \dots, \phi_n(x_1, \dots, x_n))$ and $\Phi^{-1}(x) = \Psi(x) = (\psi_1(x_1, \dots, x_n), \dots, \psi_n(x_1, \dots, x_n))$, then the functions $\phi_1, \dots, \phi_n$ belong to $C^k(\overline{\Omega}_1)$ and $\psi_1, \dots, \psi_n$ belong to $C^k(\overline{\Omega}_2)$. If $k = 0$ then we will refer to Φ as smooth.*

Definition 2.2.2 *Let Φ be a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$. We say Φ is locally k-smooth if Φ is k-smooth on every bounded domain in $\mathbb{R}^n$.*

As we have already indicated, the results in this section will often need w to satisfy certain conditions. We gather together all the required hypothesis here:

(W1) $w \in L^1(\mathbb{R}^n \backslash N)$ for any neighbourhood N of the origin;

(W2) $w(y) = \mathcal{O}(|y|^s)$ as $y \to 0$, where $n + s + 2 > 0$;

(W3) $\int_A w > 0$ whenever A has positive measure;

(W4) $w(y) = w(-y)$ for all $y \in \mathbb{R}^n$;

(W5) for every locally $(k+1)$-smooth map ϕ on $\mathbb{R}^n$, and every bounded subset Ω of $\mathbb{R}^n$, there is a $K > 0$ such that $w(\phi(x) - \phi(y)) \leq Kw(x-y)$, for all $x, y \in \Omega$;

(W6) there exists a constant $M > 0$ such that if $x = (x', x_n) \in \mathbb{R}^n$ and $y = (x', y_n) \in \mathbb{R}^n$ with $|x_n| \geq |y_n|$, then $w(x) \leq Mw(y)$.

We remark that in the previous section we assumed that w satisfied $w(x) \geq 0$ for almost all $x \in \mathbb{R}^n$. We again assume this here, albeit in the stronger form of (W3). We are now ready to begin developing the technical results needed in Section 2.3. We begin by quoting two standard analysis results which will be of use later (see, for example, Rudin [29] and Apostol [2]).

Theorem 2.2.3 (Fubini's Theorem) *Let f be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m$ and suppose at least one of the integrals*

$$I_1 = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)| \, dx \right) \, dy$$
$$I_2 = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)| \, dy \right) \, dx$$

exists and is finite. Then $I_1 = I_2$.

Theorem 2.2.4 (Lebesgue's Monotone Convergence Theorem) *Let $A \subset \mathbb{R}^n$ be measurable. Let $\{f_n\}$ be a sequence of measurable functions satisfying*

(i) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ for almost every $x \in A$,

(ii) $f_n(x) \to f(x)$ as $n \to \infty$, for almost every $x \in A$.

Then f is measurable, and

$$\int_A f_n(x)\,dx \to \int_A f(x)\,dx \qquad \textit{as } n \to \infty.$$

Lemma 2.2.5 *Let $w : I\!R^n \to I\!R$ be a measurable function satisfying (W1)-(W3). Then the mapping $y \mapsto |y|^2 w(y)$ for $y \in I\!R^n$ is in $L^1_{loc}(I\!R^n)$.*

Proof. Choose $\delta > 0$ and set $N = \{y \in \mathbb{R}^n : |y| < \delta\}$. Then there exists $A > 0$ such that $|w(y)| \leq A|y|^s$ for all $y \in N$. Since $w \in L^1(\mathbb{R}^n \backslash N)$, it is clear that the mapping $y \mapsto |y|^2 w(y)$ for $y \in \mathbb{R}^n$ is in $L^1_{loc}(\mathbb{R}^n \backslash N)$. It suffices to show that this same mapping is in $L^1(N)$. For some appropriate constant B,

$$\int_N |y|^2 w(y)\,dy \leq A \int_N |y|^{s+2}\,dy \leq AB \int_0^\delta r^{n+s+1}\,dr < \infty,$$

by property (W2). ■

Lemma 2.2.6 *Let Ω be an open, convex, bounded subset of $I\!R^n$. Let $w : I\!R^n \to I\!R$ be a measurable function satisfying (W1)-(W4). There exists $A > 0$ such that for each $f \in C^1(\Omega)$,*

$$\int_\Omega \int_\Omega w(x-y)|f(x)-f(y)|^2\,dxdy \leq A \sum_{|\alpha|=1} \int_\Omega |D^\alpha f(x)|^2\,dx.$$

Proof. Since $f \in C^1(\Omega)$, Taylor's formula with integral remainder [15, pg.13] allows us to write

$$|f(x)-f(y)|^2 = \left| \int_0^1 \sum_{|\alpha|=1} (y-x)^\alpha D^\alpha f(x+t(y-x))\,dt \right|^2$$

$$\leq \left(\int_0^1 1\,dt\right)\left(\int_0^1 \Big|\sum_{|\alpha|=1}(y-x)^\alpha D^\alpha f(x+t(y-x))\Big|^2 dt\right)$$

$$\leq \int_0^1 \left(\sum_{|\alpha|=1} 1\right)\left(\sum_{|\alpha|=1}\Big|(y-x)^\alpha D^\alpha f(x+t(y-x))\Big|^2\right) dt.$$

Now, let χ_Ω be the characteristic function of the set Ω. Extend each $D^\alpha f$ to a function on $\mathbb{R}^n$ by setting it to be zero outside Ω. Two applications of Fubini's theorem plus the change of variables $y = z + x$ gives

$$\int_\Omega\int_\Omega w(x-y)|f(x)-f(y)|^2\,dxdy$$

$$\leq n\sum_{|\alpha|=1}\int_\Omega\int_\Omega w(x-y)\int_0^1 |(y-x)^\alpha D^\alpha f(x+t(y-x))|^2\,dtdydx$$

$$= n\sum_{|\alpha|=1}\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(x-y)\chi_\Omega(x)\chi_\Omega(y)\,|(y-x)^\alpha D^\alpha f(x+t(y-x))|^2\,dydxdt$$

$$= n\sum_{|\alpha|=1}\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(z)\chi_\Omega(x)\chi_\Omega(x+z)|z^\alpha|^2\,|D^\alpha f(x+tz)|^2\,dzdxdt$$

$$= n\sum_{|\alpha|=1}\int_0^1\int_{\mathbb{R}^n} w(z)|z^\alpha|^2\int_{\Omega\cap(\Omega-z)}|D^\alpha f(x+tz)|^2\,\chi_\Omega(x)\chi_\Omega(x+z)\,dxdzdt.$$

Since Ω is bounded, we can find $\delta > 0$ such that if $|z| > \delta$ then $\Omega\cap(\Omega-z)$ is empty. Let $B_\delta = \{y\in\mathbb{R}^n : |y|\leq\delta\}$. Then the change of variables $x + tz = v$ gives

$$\int_\Omega\int_\Omega w(x-y)|f(x)-f(y)|^2\,dxdy$$

$$\leq n\sum_{|\alpha|=1}\int_0^1\int_{B_\delta} w(z)|z^\alpha|^2\int_{\Omega\cap(\Omega-z)}|D^\alpha f(x+tz)|^2\,\chi_\Omega(x)\chi_\Omega(x+z)\,dxdzdt$$

$$= n \sum_{|\alpha|=1} \int_0^1 \int_{B_\delta} w(z)|z^\alpha|^2 \int_{\mathbb{R}^n} \chi_\Omega(v - tz)\chi_\Omega(v + (1-t)z) \, |D^\alpha f(v)|^2 \; dvdzdt$$

$$\leq n \sum_{|\alpha|=1} \int_0^1 \int_{B_\delta} w(z)|z^\alpha|^2 \int_{\mathbb{R}^n} |D^\alpha f(v)|^2 \; dvdzdt$$

$$\leq n \sum_{|\alpha|=1} \int_{B_\delta} w(z)|z|^2 \int_\Omega |D^\alpha f(v)|^2 \; dvdz,$$

since $(D^\alpha f)(v) = 0$ for $v \notin \Omega$. Now by Lemma 2.2.5, there is a constant $A > 0$ independent of f such that

$$\int_\Omega \int_\Omega w(x-y)|f(x) - f(y)|^2 \; dxdy \leq An \sum_{|\alpha|=1} \int_\Omega |D^\alpha f(v)|^2 \; dv. \quad \blacksquare$$

Lemma 2.2.7 *Let U, H, G be measurable subsets of $\mathbb{R}^n$ satisfying the following properties*

1) H is a bounded set and $U \subset H \subset G$;

2) there exists a $\delta > 0$ such that for all $x \in G \backslash H$ and $y \in U$, $|x - y| > \delta$.

Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1). Then there exists a constant K such that for all $y \in U$,

$$\left| \int_{G\backslash H} w(x-y) \; dx \right| \leq K.$$

Proof. Define $f : U \to \mathbb{R}$ by $f(y) = \int_{G\backslash H} w(x-y) \; dx$ for $y \in U$. Making the change of variables $x = s + y$ gives

$$f(y) = \int_{T_y} w(s) \; ds,$$

where $T_y = G\backslash H - y$. Take $s \in T_y$. Then $s = x - y$ for some $x \in G\backslash H$ and so by Condition

(2), $|x - y| > \delta$. Now take $N = \{s \in \mathbb{R}^n : |s| < \delta\}$. Then $T_y \subset \mathbb{R}^n \backslash N$ and

$$|f(y)| = \left| \int_{T_y} w(s)\ ds \right| \leq \int_{T_y} |w(s)|\ ds \leq \int_{\mathbb{R}^n \backslash N} |w(s)|\ ds.$$

Setting $K = \int_{\mathbb{R}^n \backslash N} |w(s)|\ ds$ gives the result. ■

In the proof of the next lemma we use of the notion of lower semicontinuity, which we define below. In particular we exploit the fact that a lower semicontinuous function attains its infimum on compact sets. A proof of this can be found in [25].

Definition 2.2.8 *Let f be a function from a topological space to the extended reals. We say f is lower semicontinuous if the set $\{x : f(x) > \alpha\}$ is open for every real α.*

Lemma 2.2.9 *Let H be a bounded subset of $\mathbb{R}^n$. Let U be a subset of H such that $H \backslash U$ has positive measure. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1) and (W3). Then there is a number $K > 0$ such that,*

$$\int_{H \backslash U} w(x - y)\ dx \geq K, \qquad \text{for all } y \in U.$$

Proof. Define f from $\mathbb{R}^n$ to the extended reals by

$$f(y) = \int_{H \backslash U} w(x - y)\ dx = \int_{T_y} w(s)\ ds,$$

where $T_y = H \backslash U - y$ and $y \in \mathbb{R}^n$. Because T_y has positive measure, $f(y) > 0$ for all $y \in \mathbb{R}^n$. We claim f is a lower semicontinuous function on $\mathbb{R}^n$. That is, the set $Y_\alpha = \{y \in \mathbb{R}^n : f(y) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$. Clearly if $\alpha \leq 0$ then Y_α is the whole of $\mathbb{R}^n$ and so is open. Thus we fix $\alpha > 0$. We will show that the set $Y_\alpha^c = \{y \in \mathbb{R} : f(y) \leq \alpha\}$ is closed. Let

$\{v_j\}_{j=0}^{\infty}$ be a sequence in Y_α^c. Then,

$$f(v_j) = \int_{T_{v_j}} w(x)\, dx \leq \alpha, \qquad\qquad \text{for all } j = 0, 1, \ldots.$$

For convenience we shall write T_j for T_{v_j}. Suppose that $\lim_{j\to\infty} v_j = v$. We wish to show that $v \in Y_\alpha^c$. Let N be any neighbourhood of the origin. We define $A = T_v \cap N$ and $A_j = T_j \cap N$. Since $w \in L^1(\mathbb{R}^n \backslash N)$ we have

$$\int_{T_v \backslash A} w(x)\, dx = \lim_{j\to\infty} \int_{T_j \backslash A_j} w(x)\, dx \leq \alpha. \tag{2.3}$$

Let $B(0, 1/m) = \{x \in \mathbb{R}^n : |x| < 1/m\}$ and define $L_m = T_v \cap B(0, 1/m)$. Let χ_m be the characteristic function of L_m. Consider the sequence $\{w_k\}_{k=1}^{\infty}$ defined by $w_k = (1 - \chi_k)w$. Now, for all $x \in \mathbb{R}^n$,

i) $0 \leq w_0(x) \leq w_1(x) \leq \ldots$

ii) $\lim_{k\to\infty} w_k(x) = w(x)$.

Note that in order to ensure Condition (ii) for $x = 0$, we need to define $w(0) = 0$. Now, the Lebesgue Monotone Convergence Theorem and Equation (2.3) give,

$$\int_{T_v} w(x)\, dx = \lim_{k\to\infty} \int_{T_v} (1 - \chi_k)(x) w(x)\, dx = \lim_{k\to\infty} \int_{T_v \backslash L_k} w(x)\, dx \leq \alpha.$$

Therefore, $v \in Y_\alpha^c$ and Y_α^c is closed. Hence, f is lower semicontinuous. Since $U \subset H$ and H is bounded, U lies in some closed ball, centred on the origin. Now f attains its (positive) infimum on this ball, and so the required conclusion follows. ∎

The following result seems to be absolutely crucial in all extensions theorems of this nature. It examines the integral

$$\int_G \int_G w(x - y)|f(x) - f(y)|^2\, dxdy,$$

in the case where f is compactly supported on $U \subset G$. Now if $U \subset H \subset G$ we find that, under certain circumstances, we can in some sense disregard contributions of integrals over $H \times (G \backslash H)$.

Lemma 2.2.10 *Let $U \subset H \subset G$ be measurable subsets of $\mathbb{R}^n$, with H bounded. Suppose that for some $\delta > 0$, $|x-y| > \delta$ for all $x \in G \backslash H$ and $y \in U$. Suppose $w : \mathbb{R}^n \to \mathbb{R}$ is a measurable function satisfying (W1), (W3) and (W4). Let X consist of all functions $f \in C(G)$ for which the mapping $F : G \times G \to \mathbb{R}$ given by $F(x,y) = w(x-y)|f(x)-f(y)|^2$ for $x, y \in \mathbb{R}^n$ is in $L^1(G \times G)$. There is a number K such that*

$$\int_G \int_G F(x,y)\, dxdy \leq K \int_H \int_H F(x,y)\, dxdy,$$

for all $f \in X$ with support in U.

Proof. Let $f \in X$, then f is supported on U and $F(x,y) = w(x-y)|f(x)-f(y)|^2 \in L^1(G \times G)$. Furthermore, since w satisfies $(W4)$, F is symmetrical. Thus we can write

$$\begin{aligned}
\int_G \int_G F(x,y)\, dxdy &= \int_{G\backslash U} \int_{G\backslash U} F(x,y)\, dxdy + 2 \int_U \int_{G\backslash U} F(x,y)\, dxdy \\
&\quad + \int_U \int_U F(x,y)\, dxdy \\
&= 2 \int_U \int_{G\backslash U} F(x,y)\, dxdy + \int_U \int_U F(x,y)\, dxdy \\
&= 2 \int_U \int_{G\backslash H} F(x,y)\, dxdy + 2 \int_U \int_{H\backslash U} F(x,y)\, dxdy
\end{aligned}$$

$$+\int_U\int_U F(x,y)\,dxdy$$

$$= 2\int_U\int_{G\setminus H} F(x,y)\,dxdy + \int_H\int_H F(x,y)\,dxdy.$$

Now, again using the facts that $F \in L^1(G\times G)$, and f is supported in U,

$$\int_U\int_{G\setminus H} F(x,y)\,dxdy = \int_U |f(y)|^2 \int_{G\setminus H} w(x-y)\,dxdy.$$

Lemmas 2.2.7 and 2.2.9 show that there exists constants $K_1, K_2 > 0$ such that

$$\int_{G\setminus H} w(x-y)\,dx \le K_1 \le \frac{K_1}{K_2}\int_{H\setminus U} w(x-y)\,dx.$$

Since f is supported on U, we conclude that

$$\begin{aligned}\int_U\int_{G\setminus H} F(x,y)\,dxdy &\le \frac{K_1}{K_2}\int_U |f(y)|^2\int_{H\setminus U} w(x-y)\,dxdy \\ &= \frac{K_1}{K_2}\int_U\int_{H\setminus U} F(x,y)\,dxdy.\end{aligned}$$

Finally,

$$\begin{aligned}\int_G\int_G F(x,y)\,dxdy &\le \frac{2K_1}{K_2}\int_U\int_{H\setminus U} F(x,y)\,dxdy + \int_H\int_H F(x,y)\,dxdy \\ &\le \left(\frac{K_1}{K_2}+1\right)\int_H\int_H F(x,y)\,dxdy. \quad \blacksquare\end{aligned}$$

We turn our thoughts now to k-smooth mappings as defined at the beginning of this Section. It is important that we have an understanding of them as they will be essential to the development of our extension operator. The necessary results in this area are given in the next four lemmas and in Theorem 2.2.15 which is one of the central results of this Section.

Lemma 2.2.11 *Let Ω_1, Ω_2 be domains in $\mathbb{R}^n$, and ϕ a k-smooth bijection from Ω_1 to Ω_2. For each $f \in C^k(\Omega_2)$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le k$,*

$$D^\alpha(f \circ \phi) = \sum_{0 \le |\beta| \le |\alpha|} P_{\alpha\beta}[(D^\beta f) \circ \phi], \tag{2.4}$$

where each $P_{\alpha\beta}$ is a polynomial of degree at most $|\beta|$ in derivatives of the components of ϕ of orders at most $|\alpha|$.

Proof. The proof is by induction on $|\alpha|$. If $\alpha = 0$, then the result holds with $P_{00} = 1$. Now assume Equation (2.4) holds for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| < m \le k$. Take $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = m$. Then $\alpha = \beta + \gamma$ where $|\beta| < m$ and $|\gamma| = 1$. Now employing the induction hypothesis,

$$\begin{aligned} D^\alpha(f \circ \phi) &= D^\gamma D^\beta(f \circ \phi) \\ &= D^\gamma\left(\sum_{0 \le |\nu| \le m-1} P_{\beta\nu}[(D^\nu f) \circ \phi]\right) \\ &= \sum_{0 \le |\nu| \le m-1} \Big((D^\gamma P_{\beta\nu})[(D^\nu f) \circ \phi] + P_{\beta\nu} D^\gamma[(D^\nu f) \circ \phi]\Big). \end{aligned}$$

The induction hypothesis can now be employed again on part of the second term in the parentheses above giving

$$D^\gamma[(D^\nu f) \circ \phi] = \sum_{0 \le |\mu| \le 1} P_{\gamma\mu}[(D^{\mu+\nu} f) \circ \phi]$$

$$= P_{\gamma 0}[(D^\nu f)\circ\phi] + \sum_{|\mu|=1} P_{\gamma\mu}[(D^{\mu+\nu} f)\circ\phi].$$

Thus,

$$\begin{aligned}
D^\alpha(f\circ\phi) &= \sum_{0\le|\nu|\le m-1} \left(D^\gamma P_{\beta\nu} + P_{\beta\nu}P_{\gamma 0}\right)[(D^\nu f)\circ\phi] \\
&\qquad + \sum_{0\le|\nu|\le m-1} P_{\beta\nu} \sum_{|\mu|=1} P_{\gamma\mu}[(D^{\mu+\nu} f)\circ\phi] \\
&= \sum_{0\le|\nu|\le m-1} \left(D^\gamma P_{\beta\nu} + P_{\beta\nu}P_{\gamma 0}\right)[(D^\nu f)\circ\phi] \\
&\qquad + \sum_{1\le|\nu|\le m} \left(\sum_{\substack{\mu+\delta=\nu \\ |\mu|=1 \\ \delta\ge 0}} P_{\beta\delta}P_{\gamma\mu} \right)[(D^\nu f)\circ\phi].
\end{aligned}$$

We can therefore write

$$D^\alpha(f\circ\phi) = \sum_{0\le|\nu|\le m} P_{\alpha\nu}[(D^\nu f)\circ\phi],$$

where

$$P_{\alpha\nu} = \begin{cases} D^\gamma P_{\beta 0} + P_{\beta 0}P_{\gamma 0}, & \nu = 0 \\ D^\gamma P_{\beta\nu} + P_{\beta\nu}P_{\gamma 0} + \sum_{\substack{\mu+\delta=\nu \\ |\mu|=1 \\ \delta\ge 0}} P_{\beta\delta}P_{\gamma\mu}, & 1\le|\nu|\le m-1 \\ \sum_{\substack{\mu+\delta=\nu \\ |\mu|=1 \\ \delta\ge 0}} P_{\beta\delta}P_{\gamma\mu}, & |\nu| = m. \end{cases}$$

The result now follows by induction. ■

Lemma 2.2.12 *Let ϕ be a k-smooth bijection between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$. There exists a constant K such that for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq k$ and for all $f \in C^k(\Omega_2)$,*

$$\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx \leq K \max_{|\beta| \leq |\alpha|} \int_{\Omega_2} |(D^\beta f)(x)|^2 \, dx.$$

Proof. Take $f \in C^k(\Omega_2)$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq k$. Then, using Lemma 2.2.11

$$D^\alpha(f \circ \phi)(x) = \sum_{|\beta| \leq |\alpha|} P_{\alpha\beta}(x)[(D^\beta f) \circ \phi](x),$$

where each $P_{\alpha\beta}$ is a polynomial of degree at most $|\beta|$ in derivatives of the components of ϕ of orders at most $|\alpha|$.

Thus, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx &= \int_{\Omega_1} \Big| \sum_{|\beta| \leq |\alpha|} P_{\alpha\beta}(x)[(D^\beta f) \circ \phi](x) \Big|^2 \, dx \\
&\leq \int_{\Omega_1} \Big(\sum_{|\beta| \leq |\alpha|} 1 \Big) \Big(\sum_{|\beta| \leq |\alpha|} |P_{\alpha\beta}(x)|^2 \left| [(D^\beta f) \circ \phi](x) \right|^2 \Big) \, dx \\
&\leq \Big(\sum_{|\beta| \leq |\alpha|} 1 \Big)^2 \max_{|\beta| \leq |\alpha|} \int_{\Omega_1} |P_{\alpha\beta}(x)|^2 \left| [(D^\beta f) \circ \phi](x) \right|^2 \, dx \\
&\leq \Big(\sum_{|\beta| \leq |\alpha|} 1 \Big)^2 \max_{|\beta| \leq |\alpha|} \left(\max_{x \in \Omega_1} |P_{\alpha\beta}(x)|^2 \int_{\Omega_1} \left| [(D^\beta f) \circ \phi](x) \right|^2 \, dx \right).
\end{aligned}$$

Now suppose the maximum above over $|\beta| \leq |\alpha|$ occurs at $\beta = \beta_0$. Since Ω_1 is a bounded domain, we can assume that there is a number K_1 such that

$$\left(\sum_{|\beta| \leq |\alpha|} 1 \right)^2 \max_{x \in \Omega_1} |P_{\alpha\beta_0}(x)|^2 \leq K_1.$$

Then,

$$\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx \le K_1 \int_{\Omega_1} \left|[(D_0^\beta f) \circ \phi](x)\right|^2 \, dx.$$

Making the change of variables $x = \phi^{-1}(y)$, we obtain

$$\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx \le K_1 \int_{\Omega_2} \left|(D^{\beta_0} f)(y)\right|^2 \left|J_{\phi^{-1}}(y)\right| \, dy,$$

where $J_{\phi^{-1}}$ is the corresponding Jacobian. Since Ω_2 is bounded, this Jacobian is bounded on Ω_2, and so there is a number K_2, such that

$$\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx \le K_1 K_2 \int_{\Omega_2} |(D^{\beta_0} f)(x)|^2 \, dx$$

as required. ∎

Lemma 2.2.13 *Let ϕ be a (k+1)-smooth bijection between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$. Let $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha|, |\beta| \le k$. Let $P_{\alpha\beta}$ be as in Lemma 2.2.11. Let w be a measurable function satisfying (W1)-(W3). Then there exists a constant K such that*

$$\int_{\Omega_1} w(x-y) \left|P_{\alpha\beta}(x) - P_{\alpha\beta}(y)\right|^2 \, dx \le K,$$

for all $y \in \Omega_1$.

Proof. Recall from Lemma 2.2.11, that $P_{\alpha\beta}$ is a polynomial of degree at most $|\beta|$ in derivatives of the components of ϕ of orders at most $|\alpha|$. Let

$$\phi(x) = (\phi_1(x_1, \ldots, x_n), \ldots, \phi_n(x_1, \ldots, x_n)).$$

Because ϕ is (k+1)-smooth, the functions $\phi_1, \ldots, \phi_n$ are in $C^{k+1}(\overline{\Omega}_1)$. Hence, we can find a constant K_1 such that for all $1 \le i \le n$,

$$|(D^\gamma \phi_i)(x) - (D^\gamma \phi_i)(y)| \le K_1 |x - y|,$$

for all $x, y \in \Omega_1$ and for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq k$. Consequently, we can find a constant K_2 such that $|P_{\alpha\beta}(x) - P_{\alpha\beta}(y)| \leq K_2|x-y|$ for all $x, y \in \Omega_1$ and for all $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha|, |\beta| \leq k$. Hence,

$$\int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\ dx \leq K_2^2 \int_{\Omega_1} |x-y|^2 w(x-y)\ dx.$$

Using the change of variables $x - y = s$ we have

$$\int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\ dx \leq K_2^2 \int_{\Omega_1 - y} |s|^2 w(s)\ ds.$$

Lemma 2.2.5 establishes the existence of a constant $K_3(y) > 0$ such that

$$\int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\ dx \leq K_2^2 K_3(y).$$

Again by Lemma 2.2.5, the map $s \mapsto |s|^2 w(s)$ is in $L^1_{loc}(\mathbb{R}^n)$. Therefore, the function $y \mapsto \int_{\Omega_1 - y} |s|^2 w(s)\ ds$ is continuous. Since Ω_1 is bounded, it follows that $\sup_{y \in \Omega_1} K_3(y) < \infty$. Thus the required result is obtained by taking

$$K = K_2^2 \sup_{y \in \Omega_1} K_3(y). \quad \blacksquare$$

In the following result we will make use of the following simple inequality. For all $a, b \in \mathbb{R}^l$,

$$|a+b|^2 \leq |a|^2 + 2|a||b| + |b|^2 \leq 3(|a|^2 + |b|^2). \tag{2.5}$$

Lemma 2.2.14 *Let ϕ be a (k+1)-smooth bijection between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$. Let w be a measurable function satisfying (W1)-(W3) and (W5). There exists a constant K such that for all $f \in C^k(\Omega_2)$ and all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq k$,*

$$\int_{\Omega_1} \int_{\Omega_1} w(x-y) |D^\alpha (f \circ \phi)(x) - D^\alpha (f \circ \phi)(y)|^2\ dxdy$$

$$\leq K \max_{|\beta|\leq|\alpha|} \int_{\Omega_2}\int_{\Omega_2} w(x-y)|(D^\beta f)(x) - (D^\beta f)(y)|^2\, dxdy + K \max_{|\beta|\leq|\alpha|} \int_{\Omega_2} |(D^\beta f)(x)|^2\, dx.$$

Proof. Take $f \in C^k(\Omega_2)$ and $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leq k$. Observe first that by Lemma 2.2.11,

$$D^\alpha(f\circ\phi)(x) = \sum_{|\beta|\leq|\alpha|} \Big(P_{\alpha\beta}(x)(D^\beta f\circ\phi)(x) \qquad\qquad \text{for } x \in \Omega_1,$$

where each $P_{\alpha\beta}$ is a polynomial of degree at most $|\beta|$ in derivatives of the components of ϕ of orders at most $|\alpha|$. Therefore, using the Cauchy-Schwarz inequality and the remark preceding this Lemma,

$$\begin{aligned}
&|D^\alpha(f\circ\phi)(x) - D^\alpha(f\circ\phi)(y)|^2 \\
&= \left|\sum_{|\beta|\leq|\alpha|} \Big(P_{\alpha\beta}(x)(D^\beta f\circ\phi)(x) - P_{\alpha\beta}(y)(D^\beta f\circ\phi)(y)\Big)\right|^2 \\
&\leq \left(\sum_{|\beta|\leq|\alpha|} 1\right)\left(\sum_{|\beta|\leq|\alpha|} \left|P_{\alpha\beta}(x)(D^\beta f\circ\phi)(x) - P_{\alpha\beta}(y)(D^\beta f\circ\phi)(y)\right|^2\right) \\
&\leq 3\left(\sum_{|\beta|\leq|\alpha|} 1\right)\left(\sum_{|\beta|\leq|\alpha|} |P_{\alpha\beta}(x)|^2 \left|(D^\beta f\circ\phi)(x) - (D^\beta f\circ\phi)(y)\right|^2\right. \\
&\qquad\qquad \left. + \sum_{|\beta|\leq|\alpha|} \left|(D^\beta f\circ\phi)(y)\right|^2 |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\right).
\end{aligned}$$

Put $K_1 = 3\sum_{|\beta|\leq|\alpha|} 1$. Then,

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_1} w(x-y)|D^\alpha(f\circ\phi)(x) - D^\alpha(f\circ\phi)(y)|^2\, dxdy \\
&\quad \leq K_1 \sum_{|\beta|\leq|\alpha|} \int_{\Omega_1}\int_{\Omega_1} w(x-y)|P_{\alpha\beta}(x)|^2 \left|(D^\beta f\circ\phi)(x) - (D^\beta f\circ\phi)(y)\right|^2\, dxdy
\end{aligned}$$

$$+K_1 \sum_{|\beta|\leq|\alpha|} \int_{\Omega_1} \left|(D^\beta f \circ \phi)(y)\right|^2 \int_{\Omega_1} w(x-y)|P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \, dxdy.$$

We examine each of the above integrals in turn. Firstly, since Ω_1 is bounded we can assume that $|P_{\alpha\beta}(x)|^2 \leq K_2$ for all $|\beta| \leq |\alpha|$ and for all $x \in \Omega_1$. Thus, making the changes of variables $x = \phi^{-1}(s)$ and $y = \phi^{-1}(t)$,

$$\int_{\Omega_1}\int_{\Omega_1} w(x-y)|P_{\alpha\beta}(x)|^2 \left|(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)\right|^2 \, dxdy$$

$$\leq \; K_2 \int_{\Omega_1}\int_{\Omega_1} w(x-y) \left|(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)\right|^2 \, dxdy$$

$$\leq \; K_2 \int_{\Omega_2}\int_{\Omega_2} w(\phi^{-1}(s) - \phi^{-1}(t)) \left|(D^\beta f)(s) - (D^\beta f)(t)\right|^2 |J_{\phi^{-1}}(s) J_{\phi^{-1}}(t)| \, dsdt.$$

Using hypothesis (W5) and the fact that $|J_{\phi^{-1}}|$ is bounded on the domain Ω_2, we infer the existence of a constant K_3 such that

$$\int_{\Omega_1}\int_{\Omega_1} w(x-y)|P_{\alpha\beta}(x)|^2 \left|(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)\right|^2 \, dxdy$$

$$\leq K_3 \int_{\Omega_2}\int_{\Omega_2} w(s-t) \left|(D^\beta f)(s) - (D^\beta f)(t)\right|^2 \, dsdt.$$

Considering now the second integral, by Lemma 2.2.13 there is a constant K_4 such that

$$\int_{\Omega_1} \left|(D^\beta f \circ \phi)(y)\right|^2 \int_{\Omega_1} w(x-y)|P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \, dxdy \leq K_4 \int_{\Omega_1} \left|[D^\beta f \circ \phi](x)\right|^2 \, dx.$$

Applying Lemma 2.2.12 there is a constant K_5 such that

$$\int_{\Omega_1} \left|(D^\beta f \circ \phi)(y)\right|^2 \int_{\Omega_1} w(x-y)|P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \, dxdy \leq K_5 \max_{|\beta|\leq|\alpha|} \int_{\Omega_2} \left|(D^\beta f)(x)\right|^2 \, dx.$$

Thus, assuming (with no loss of generality) that $K_5 \geq K_3$,

$$\int_{\Omega_1}\int_{\Omega_1} w(x-y)|D^\alpha(f \circ \phi)(x) - D^\alpha(f \circ \phi)(y)|^2 \, dxdy$$

$$
\begin{aligned}
&\leq K_1K_3 \sum_{|\beta|\leq|\alpha|} \int_{\Omega_2}\int_{\Omega_2} w(s-t)\left|(D^\beta f)(s)-(D^\beta f)(t)\right|^2 dsdt \\
&\qquad +K_1K_5 \sum_{|\beta|\leq|\alpha|} \max_{|\beta|\leq|\alpha|} \int_{\Omega_2} \left|(D^\beta f)(x)\right|^2 dx \\
&\leq K_1K_5 \left(\sum_{|\beta|\leq|\alpha|} 1\right)\left(\max_{|\beta|\leq|\alpha|} \int_{\Omega_2}\int_{\Omega_2} w(s-t)\left|(D^\beta f)(s)-(D^\beta f)(t)\right|^2 dsdt\right. \\
&\qquad \left. + \max_{|\beta|\leq|\alpha|} \int_{\Omega_2} \left|(D^\beta f)(x)\right|^2 dx\right).
\end{aligned}
$$

Taking $K = K_1K_5 \sum_{|\beta|\leq|\alpha|} 1$ completes the proof. ■

Our final result concerning k-smooth mappings and the remaining results in this Section will concern the quantity $\|\cdot\|_\Omega$ as defined at the beginning of this Section. Recall that for bounded domains Ω, $\|\cdot\|_\Omega$ defines a norm on $X(\Omega)$.

Theorem 2.2.15 *Let ϕ be a (k+1)-smooth bijection from a convex, bounded domain Ω_1 into $I\!R^n$. Let $w : I\!R^n \to I\!R$ be a measurable function satisfying (W1)–(W6). Then there is a number K such that*

$$\|f\circ\phi\|_{\Omega_1} \leq K\|f\|_{\phi(\Omega_1)}, \qquad \text{for all } f\in X(\phi(\Omega_1)).$$

Proof. Set $\Omega_2 = \phi(\Omega_1)$. From Lemmas 2.2.12 and 2.2.14 we infer the existence of a constant $K_1 \geq 0$ such that

$$\|f\circ\phi\|^2_{\Omega_1} = \sum_{|\alpha|\leq k} \int_{\Omega_1} |D^\alpha(f\circ\phi)(x)|^2\, dx$$

$$+\sum_{|\alpha|=k} c_\alpha \int_{\Omega_1}\int_{\Omega_1} w(x-y)|D^\alpha(f\circ\phi)(x) - D^\alpha(f\circ\phi)(y)|^2\,dxdy$$

$$\leq K_1 \max_{|\beta|\leq k}\int_{\Omega_2}\int_{\Omega_2} w(x-y)|D^\beta f(x) - D^\beta f(y)|^2\,dxdy$$

$$+K_1 \max_{|\beta|\leq k}\int_{\Omega_2} |D^\beta f(x)|^2\,dx.$$

From Lemma 2.2.6 we infer the existence of a constant $K_2 > 0$ such that

$$\begin{aligned}\|f\circ\phi\|^2_{\Omega_1} &\leq K_1\Bigg(K_2\sum_{|\gamma|\leq k}\int_{\Omega_2}|D^\gamma f(x)|^2\,dx \\ &\qquad + \sum_{|\gamma|=k} c_\alpha \int_{\Omega_2}\int_{\Omega_2} |w(x-y)|D^\gamma f(x) - D^\gamma f(y)|^2\,dxdy\Bigg) \\ &\qquad +K_1\max_{|\beta|\leq k}\int_{\Omega_2}|D^\beta f(x)|^2 dx \\ &\leq K_1(K_2+2)\|f\|_{\Omega_2},\end{aligned}$$

as required. ∎

Lemma 2.2.16 *Let $u \in C_0^\infty(I\!R^n)$ and let Ω be a convex, bounded domain. Let $w : I\!R^n \to I\!R$ satisfy (W1)-(W4). There exists a constant C such that for all $\gamma \in Z\!\!Z^n_+$ with $|\gamma| = k$,*

$$\int_\Omega\int_\Omega w(x-y)|D^\gamma(uf)(x) - D^\gamma(uf)(y)|^2\,dxdy \leq C\|f\|^2_\Omega$$

for all $f \in X(\Omega)$.

Proof. Let

$$I_1 = \int_\Omega \int_\Omega w(x-y)|D^\gamma(uf)(x) - D^\gamma(uf)(y)|^2 \, dxdy.$$

The Leibniz formula (see [30, Section 6.15]) allows us to write

$$D^\gamma(uf) = \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| \le |\gamma|}} C_{\gamma\beta}(D^{\gamma-\beta}u)(D^\beta f),$$

where the $C_{\gamma\beta}$ are suitable numbers. Using this and the Cauchy-Schwarz inequality gives

$$\begin{aligned} I_1 &= \int_\Omega \int_\Omega w(x-y)\Big| \sum_{|\beta| \le |\gamma|} C_{\gamma\beta} \Big\{ (D^{\gamma-\beta}u)(x)(D^\beta f)(x) - (D^{\gamma-\beta}u)(y)(D^\beta f)(y) \Big\} \Big|^2 \, dxdy \\ &\le \left(\sum_{|\beta| \le |\gamma|} |C_{\gamma\beta}|^2 \right) \int_\Omega \int_\Omega w(x-y) \Big(\sum_{|\beta| \le |\gamma|} \Big| (D^{\gamma-\beta}u)(x)(D^\beta f)(x) \\ &\qquad -(D^{\gamma-\beta}u)(y)(D^\beta f)(y) \Big|^2 \Big) \, dxdy. \end{aligned}$$

Now set $c_1 = \sum_{|\beta| \le |\gamma|} |C_{\gamma\beta}|^2$. Then using inequality (2.5) we obtain

$$\begin{aligned} I_1 &\le 3c_1 \sum_{|\beta| \le |\gamma|} \int_\Omega \int_\Omega w(x-y)|D^{\gamma-\beta}u(x)|^2 |D^\beta f(x) - D^\beta f(y)|^2 \, dxdy \\ &\qquad + 3c_1 \sum_{|\beta| \le |\gamma|} \int_\Omega \int_\Omega w(x-y)|D^\beta f(y)|^2 |D^{\gamma-\beta}u(x) - D^{\gamma-\beta}u(y)|^2 \, dxdy. \end{aligned}$$

Now set

$$c_2 = \max_{|\beta| \le |\gamma|} \sup_{x \in \Omega} |D^{\gamma-\beta}u(x)|^2.$$

Lemma 2.2.6 shows that there is a constant c_3 such that

$$I_1 \le 3c_1c_2 \sum_{|\beta| \le |\gamma|} \int_\Omega \int_\Omega w(x-y)|D^\beta f(x) - D^\beta f(y)|^2 \, dxdy$$

$$+3c_1 \sum_{|\beta|\leq|\gamma|} \int_\Omega \int_\Omega w(x-y)|D^\beta f(y)|^2 |D^{\gamma-\beta}u(x) - D^{\gamma-\beta}u(y)|^2 \, dxdy$$

$$\leq \; 3c_1c_2 \sum_{|\beta|=k} \int_\Omega \int_\Omega w(x-y)|D^\beta f(x) - D^\beta f(y)|^2 \, dxdy$$

$$+3c_1c_2c_3 \sum_{1\leq|\beta|\leq k} \int_\Omega |D^\beta f(y)|^2 \, dy$$

$$+3c_1 \sum_{|\beta|\leq|\gamma|} \int_\Omega |D^\beta f(y)|^2 \int_\Omega w(x-y)|D^{\gamma-\beta}u(x) - D^{\gamma-\beta}u(y)|^2 \, dxdy.$$

If we can now show that for each $y \in \Omega$ and every $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq k$,

$$I_2(y) := \int_\Omega w(x-y)|D^\alpha u(x) - D^\alpha u(y)|^2 \, dx$$

is bounded by a constant c_4 dependent only on u and α, then we will obtain

$$I_1 \;\leq\; 3c_1c_2 \sum_{|\beta|=k} \int_\Omega \int_\Omega w(x-y)|D^\beta f(x) - D^\beta f(y)|^2 \, dxdy$$

$$+3c_1c_2c_3 \sum_{1\leq|\beta|\leq k} \int_\Omega |D^\beta f(y)|^2 \, dy$$

$$+3c_1c_4 \sum_{|\beta|\leq|\gamma|} \int_\Omega |D^\beta f(y)|^2 dy.$$

This completes the proof. For the boundedness of I_2 we note that, since $u \in C_0^\infty(\mathbb{R}^n)$, there exists a constant $c_5(\alpha)$ dependent on α, such that

$$|D^\alpha u(x) - D^\alpha u(y)| \leq c_5(\alpha)|x-y|,$$

for all $x, y \in \mathbb{R}^n$. Using the change of variables $x - y = s$, we obtain

$$\begin{aligned} I_2(y) &\leq c_5(\alpha) \int_\Omega w(x-y)|x-y|^2 \, dx \\ &= c_5(\alpha) \int_{\Omega - y} w(s)|s|^2 \, ds. \end{aligned}$$

Lemma 2.2.5 now establishes the boundedness of I_2 on Ω. ■

Lemma 2.2.17 *Let Ω be a bounded, convex, open subset of $\mathbb{R}^n$. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W4). Let $u \in C_0^\infty(\mathbb{R}^n)$. Then there is a number $C > 0$ such that $\|uf\|_\Omega \leq C\|f\|_\Omega$ for all $f \in X(\Omega)$.*

Proof. Let $f \in X(\Omega)$. An application of Lemma 2.2.16 shows that

$$\begin{aligned} \|uf\|_\Omega^2 &= \sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y)|D^\alpha(uf)(x) - D^\alpha(uf)(y)|^2 \, dxdy \qquad (2.6) \\ &\quad + \sum_{|\alpha|\leq k} \int_\Omega |D^\alpha(uf)(x)|^2 \, dx \\ &\leq \sum_{|\alpha|=k} c_\alpha c_1 \|f\|_\Omega^2 + \sum_{|\alpha|\leq k} \int_\Omega |D^\alpha(uf)(x)|^2 \, dx, \qquad (2.7) \end{aligned}$$

for some c_1 independent of f. The Leibniz formula guarantees the existence of constants $c_{\alpha\beta}$ such that

$$D^\alpha(uf) = \sum_{|\beta|\leq|\alpha|} c_{\alpha\beta}(D^{\alpha-\beta}u)(D^\beta f).$$

Hence, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k$, an application of the Cauchy-Schwarz inequality gives

$$\int_\Omega |D^\alpha(uf)(x)|^2 \, dx = \int_\Omega \Big| \sum_{|\beta|\leq|\alpha|} c_{\alpha\beta} D^{\alpha-\beta}u(x) D^\beta f(x) \Big|^2 \, dx$$

$$\leq \left(\sum_{|\beta| \leq |\alpha|} |c_{\alpha\beta}|^2 \right) \int_\Omega \sum_{|\beta| \leq |\alpha|} |D^{\alpha-\beta} u(x) D^\beta f(x)|^2 \, dx.$$

Setting

$$c_2 = \sum_{|\beta| \leq |\alpha|} |c_{\alpha\beta}|^2 \max_{|\beta| \leq |\alpha|} \sup_{x \in \Omega} |D^{\alpha-\beta} u(x)|^2$$

gives,

$$\int_\Omega |D^\alpha (uf)(x)|^2 \, dx \leq c_2 \sum_{|\beta| \leq |\alpha|} \int_\Omega |D^\beta f(x)|^2 \, dx \leq c_2 \|f\|_\Omega^2.$$

Substituting this result back in (2.7) gives

$$\|uf\|_\Omega^2 \leq \sum_{|\alpha|=k} c_\alpha c_1 \|f\|_\Omega^2 + \sum_{|\alpha| \leq k} c_2 \|f\|_\Omega^2,$$

which is the required result providing we take

$$C \geq \sqrt{\sum_{|\alpha|=k} c_\alpha c_1 + \sum_{|\alpha| \leq k} c_2}. \quad \blacksquare$$

2.3 Extension Theorems for More General Domains

We are ready now to construct an extension operator for domains Ω which are considerably more general than $\mathbb{R}^n_+$. Some restrictions on the domain Ω are nevertheless needed. For example, we shall always assume that Ω is bounded. We also require a certain level of smoothness of the boundary of Ω which we now detail.

Let $B = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : |y_j| < 1, \; 1 \leq j \leq n\}$, and set $B_+ = \{y \in B : y = (y', y_n) \text{ and } y_n > 0\}$ and $B_0 = \{y \in B : y = (y', y_n) \text{ and } y_n = 0\}$. We shall assume k is a fixed natural number throughout this section.

Definition 2.3.1 *A bounded, open, convex set Ω in $I\!R^n$ with boundary $\partial\Omega$ will be called a V-domain if the following hold,*

(V1) there exist open sets $G_1, \ldots, G_N \subset I\!R^n$ such that $\partial\Omega \subset \bigcup_{j=1}^{N} G_j$;

(V2) there exist locally (k+1)-smooth maps $\phi_j : I\!R^n \to I\!R^n$ such that $\phi_j(B) = G_j$, $\phi_j(B_+) = G_j \cap \Omega$ and $\phi_j(B_0) = G_j \cap \partial\Omega$, $j = 1, \ldots, N$;

(V3) let Ω_δ be the set of all points in Ω whose distance from $\partial\Omega$ is less than δ. Then for some $\delta > 0$,

$$\Omega_\delta \subset \bigcup_{j=1}^{N} \phi_j \left(\left\{ (y_1, y_2, \ldots, y_n) \in I\!R^n : |y_j| < \frac{1}{k+1},\ 1 \leq j \leq n \right\} \right).$$

We continue to use the notations $|\cdot|_\Omega$ and $\|\cdot\|_\Omega$ as defined in Equations (2.1) and (2.2), as well as the space $X(\Omega)$.

We now embark on the construction which will define our extension. We presume Ω is a V-domain and develop a linear extension operator $L : X(\Omega) \to X(\mathbb{R}^n)$. We note that the notation we are developing here will be used throughout this section in the various results we shall establish. Let

$$Q = \left\{ (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : |y_j| < \frac{1}{k+1},\ 1 \leq j \leq n \right\}.$$

Now set $V_i = \phi_i(Q)$, $i = 1, \ldots, N$. By virtue of (V3) for some $\delta > 0$, $V_1, \ldots, V_N$ form an open cover of Ω_δ. Consequently, we can find an open set V_0 such that $\mathrm{dist}(x, \partial\Omega) \geq \delta$ for all $x \in V_0$, and $\Omega \subset \bigcup_{j=0}^{N} V_j$. Now construct a partition of unity $u_0, \ldots, u_N \in C_0^\infty(\mathbb{R}^n)$ such that, (A1) each u_j is supported in V_j,

(A2) $u_j(x) \geq 0$ for all $x \in \mathbb{R}^n$,

(A3) $\sum_{j=0}^{N} u_j(x) = 1$ for all $x \in \Omega$.

Now take $f \in X(\Omega)$. Then $f = g\mid_\Omega$ for some $g \in C_0^k(\mathbb{R}^n)$ with $|g|_\Omega = |f|_\Omega < \infty$. Thus we can think of f as being in $C_0^k(\mathbb{R}^n)$. We can write,

$$f(x) = \sum_{j=0}^{N} u_j(x) f(x) \qquad\qquad \text{for } x \in \Omega.$$

Now define $\psi_j : \mathbb{R}^n \to \mathbb{R}$ by $\psi_j = (u_j f) \circ \phi_j$, $j = 1, \ldots, N$. Note that $(u_j f)(\phi_j(x)) = 0$ if $\phi_j(x) \notin V_j = \phi_j(Q)$. Hence ψ_j is supported on Q.

Lemma 2.3.2 *Let $s \in C_0^k(\mathbb{R}^n)$ be supported on Q. Define $t = s\mid_{\mathbb{R}^n_+}$, and the extension operator E as in Definition 2.1.1. Then $Et \in C_0^k(\mathbb{R}^n)$ and is supported in B.*

Proof. The fact that $Et \in C_0^k(\mathbb{R}^n)$ is the substance of Theorem 2.1.2. To see that Et is supported in B, suppose $x \notin B$. If $x_n > 0$ then $(Et)(x) = t(x) = s(x) = 0$, since s is supported on Q and $Q \subset B$. If $x_n \leq 0$, then

$$Et(x) = \sum_{i=1}^{k+1} \lambda_i t(x', -x_n/i) = \sum_{i=1}^{k+1} \lambda_i s(x', -x_n/i).$$

Suppose $|x_n| \geq 1$. Then for $1 \leq i \leq k+1$,

$$|x_n/i| \geq \frac{1}{(k+1)}|x_n| \geq \frac{1}{(k+1)}.$$

If $|x_n| < 1$, then since $x \notin B$, there is a j with $1 \leq j \leq n-1$ such that

$$|x_j| \geq 1 \geq \frac{1}{k+1}.$$

We conclude from this that if $x \notin B$, then $(x', -x_n/i) \notin Q$ for $1 \leq i \leq k+1$. Hence, $(Et)(x) = 0$. ■

Define $\Psi_j = \psi_j \mid_{\mathbb{R}^n_+}$. Then by Lemma 2.3.2, $E\Psi_j$ is in $C_0^k(\mathbb{R}^n)$ and is supported in B. Define $\theta_j = E\Psi_j \circ \phi_j^{-1}$. If $x \notin G_j$, it follows that $\phi_j^{-1}(x) \notin B$ and so $E\Psi_j(\phi_j^{-1}(x)) = 0$. From this we conclude that the support of θ_j is in G_j, $j = 1, \ldots, N$. We are now finally in a position to define our extension operator L as

$$Lf = u_0 f + \sum_{i=1}^{N} \theta_i. \tag{2.8}$$

Lemma 2.3.3 *Let Ω be a V-domain. We have $Lf(x) = f(x)$ for all $x \in \Omega$.*

Proof. Take $x \in \Omega$. By reordering if necessary, we can assume that x belongs to $G_1, \ldots, G_M$ but not to $G_{M+1}, \ldots, G_N$. Then,

$$\begin{aligned} Lf(x) &= u_0(x)f(x) + \sum_{i=1}^{M} \theta_i(x) \\ &= u_0(x)f(x) + \sum_{i=1}^{M} E\Psi_i(\phi_i^{-1}(x)). \end{aligned}$$

Now for $i = 1, \ldots, M$, $x \in \Omega \cap G_i$ and so $\phi_i^{-1}(x) \in B_+$. Hence,

$$E\Psi_i(\phi_i^{-1}(x)) = (u_i f)(\phi_i(\phi_i^{-1}(x))) = (u_i f)(x).$$

Finally, because $u_i(x) = 0$, $i = M+1, \ldots, N$,

$$Lf(x) = u_0(x)f(x) + \sum_{i=1}^{M} u_i(x)f(x) = u_0(x)f(x) + \sum_{i=1}^{N} u_i(x)f(x) = f(x). \quad ■$$

From Lemma 2.3.3, we see that L certainly has the potential to be the required extension operator. However, we need to address the question of whether L is bounded. To this end we make the simple observation that

$$\|Lf\|_{\mathbb{R}^n} \leq \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} \|\theta_j\|_{\mathbb{R}^n}.$$

The next result examines the quantities $\|\theta_j\|_{\mathbb{R}^n}$. We shall drop the subscript j temporarily and simply work with $\theta = E\Psi \circ \phi^{-1}$ supported on a set G, which typifies G_j.

Lemma 2.3.4 *Let Ω be a V-domain. Let w satisfy (W1)–(W6). There exists a number $C > 0$ such that,*

$$\|\theta\|_{\mathbb{R}^n} \leq C\|uf\|_{\Omega}, \qquad \textit{for all } f \in X(\Omega).$$

Proof. Let $f \in X(\Omega)$. For $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq k$ we consider the integrals

$$I_1 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(x-y)|D^\alpha\theta(x) - D^\alpha\theta(y)|^2 \, dxdy \quad \text{and} \quad I_2 = \int_{\mathbb{R}^n} |D^\alpha\theta(x)|^2 \, dx.$$

Let $\mathcal{G}$ be a bounded subset of $\mathbb{R}^n$ which contains G. Moreover, suppose there exists $\eta > 0$ such that $|x-y| > \eta$ for all $x \in G$ and $y \in \mathbb{R}^n \backslash \mathcal{G}$. Then, because θ is supported on G, Lemma 2.2.10 provides a number c_1 such that

$$I_1 \leq c_1^2 \int_{\mathcal{G}}\int_{\mathcal{G}} w(x-y)|D^\alpha\theta(x) - D^\alpha\theta(y)|^2 \, dxdy.$$

Again, because θ is supported on G,

$$I_2 = \int_{\mathcal{G}} |D^\alpha\theta(x)|^2 \, dx,$$

and so we conclude that $\|\theta\|_{\mathbb{R}^n} \leq c_1\|\theta\|_{\mathcal{G}}$. Since ϕ^{-1} is a locally $(k+1)$-smooth mapping, Theorem 2.2.15 shows there is a number $c_2 > 0$ such that $\|\theta\|_{\mathcal{G}} = \|E\Psi \circ \phi^{-1}\|_{\mathcal{G}} \leq$

$c_2\|E\Psi\|_{\phi^{-1}(\mathcal{G})}$. Now, by Theorem 2.1.6, we can find a constant $c_3 > 0$ such that

$$\|\theta\|_{\mathbb{R}^n} \leq c_1c_2\|E\Psi\|_{\phi^{-1}(\mathcal{G})} \leq c_1c_2\|E\Psi\|_{\mathbb{R}^n} \leq c_3\|\Psi\|_{\mathbb{R}^n_+}.$$

Since Ψ is supported on $Q_+ \subset B \cap \mathbb{R}^n_+$, we can again apply Lemma 2.2.10 to obtain a constant c_4 such that

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} w(x-y)|D^\alpha\Psi(x)-D^\alpha\Psi(y)|^2\,dxdy \leq c_4\int_{B_+}\int_{B_+} w(x-y)|D^\alpha\Psi(x)-D^\alpha\Psi(y)|^2\,dxdy,$$

for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq k$. Therefore, there exists a constant c_5 such that

$$\|\theta\|_{\mathbb{R}^n} \leq c_3\|\Psi\|_{\mathbb{R}^n_+} \leq c_5\|\Psi\|_{B_+} = c_5\|\psi\|_{B_+}.$$

Moreover, since $\psi = (uf)\circ\phi$, an application of Theorem 2.2.15 shows that there is a constant c_6 such that

$$\begin{aligned}\|\theta\|_{\mathbb{R}^n} \leq c_5\|uf\circ\phi\|_{B_+} &\leq c_6\|uf\|_{\phi(B_+)}\\ &= c_6\|uf\|_{\Omega\cap G}\\ &\leq c_6\|uf\|_\Omega. \quad \blacksquare\end{aligned}$$

Theorem 2.3.5 *Let $\Omega \subset \mathbb{R}^n$ be a V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W6). Let $f \in X(\Omega)$. Then there exists a continuous, linear mapping $L : X(\Omega) \to X(\mathbb{R}^n)$ such that for all $f \in X(\Omega)$,*

1) $Lf\,|_\Omega = f$

2) $\|Lf\|_{\mathbb{R}^n} \leq M\|f\|_\Omega$ for some constant M independent of f.

Proof. Let $f \in X(\Omega)$ and define Lf as in Equation (2.8). By Lemma 2.3.3, $(Lf)(x) = f(x)$ for all $x \in \Omega$. Furthermore,

$$\|Lf\|_{\mathbb{R}^n} \leq \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} \|\theta_j\|_{\mathbb{R}^n}.$$

An application of Lemma 2.3.4 shows that $\|\theta_j\|_{\mathbb{R}^n} \leq c_1 \|u_j f\|_\Omega$ for some suitable constant $c_1 > 0$. Thus,

$$\|Lf\|_{\mathbb{R}^n} \leq \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} c_1 \|u_j f\|_\Omega.$$

An application of Lemma 2.2.17 gives

$$\|Lf\|_{\mathbb{R}^n} \leq \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} c_1 c_2 \|f\|_\Omega,$$

for some number c_2 independent of f. Furthermore, since u_0 is supported on $V_0 \subset \Omega$ we can use Lemma 2.2.10 and a further application of Lemma 2.2.17 to obtain constants $c_3, c_4 > 0$, independent of f, such that

$$\begin{aligned}
\|Lf\|_{\mathbb{R}^n} &\leq c_3 \|u_0 f\|_\Omega + N c_1 c_2 \|f\|_\Omega \\
&\leq c_3 c_4 \|f\|_\Omega + N c_1 c_2 \|f\|_\Omega \\
&\leq (c_3 c_4 + N c_1 c_2) \|f\|_\Omega.
\end{aligned}$$

Using this result and the fact that $f \in X(\Omega)$ we have,

$$\begin{aligned}
|Lf|_{\mathbb{R}^n} &\leq \|Lf\|_{\mathbb{R}^n} \\
&\leq (c_3 c_4 + N c_1 c_2) \|f\|_\Omega \\
&= (c_3 c_4 + N c_1 c_2) \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha f(x)|^2 \, dx + |f|_\Omega^2 \right)^{1/2} \\
&< \infty.
\end{aligned}$$

Thus $Lf \in X(\mathbb{R}^n)$. ■

Let $\mathcal{X}(\Omega)$ be the completion of $X(\Omega)$ with respect to $\|\cdot\|_\Omega$. Let $\mathcal{X}(\mathbb{R}^n)$ be likewise defined. We shall make use of the following standard abstract analysis result, a proof of which can be found in [17].

Lemma 2.3.6 *Let P be a normed space and Q a complete normed space. Let V be a dense linear subspace of P and let T_0 be a continuous mapping of V into Q. Then there is a unique continuous mapping T from P to Q that extends T_0. Further T is linear and $\|T\| = \|T_0\|$.*

We note that the above result also holds if one is considering seminormed rather than normed spaces, the only exception being that the extension T is no longer unique. A straightforward application of Lemma 2.3.6 allows us to deduce the existence of extension results for functions in $\mathcal{X}(\Omega)$ as follows.

Theorem 2.3.7 *Let $\Omega \subset \mathbb{R}^n$ be a V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W6). There exists a continuous linear operator $\mathcal{L} : \mathcal{X}(\Omega) \to \mathcal{X}(\mathbb{R}^n)$ such that for all $f \in \mathcal{X}(\Omega)$,*

1) $\mathcal{L}f\,|_\Omega = f$

2) $\|\mathcal{L}f\|_{\mathbb{R}^n} \leq M\|f\|_\Omega$, for some constant M independent of f.

Now, let $\mathcal{Y}(\mathbb{R}^n)$ be the completion of $X(\mathbb{R}^n)$ with respect to $|\cdot|$. Since $|f|_{\mathbb{R}^n} \leq \|f\|_{\mathbb{R}^n}$ for all $f \in X(\mathbb{R}^n)$, the following result follows immediately from Theorem 2.3.5 .

Theorem 2.3.8 *Let $\Omega \subset \mathbb{R}^n$ be a V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)–(W6). Let $f \in X(\Omega)$. Then there exists a continuous, linear mapping $L : X(\Omega) \to \mathcal{Y}(\mathbb{R}^n)$ such that for all $f \in X(\Omega)$,*

1) $Lf\,|_\Omega = f$

2) $|Lf|_{\mathbb{R}^n} \leq M\|f\|_\Omega$ for some constant M independent of f.

In a similar way to Lemma 2.3.7 we can obtain a second extension result for functions in $\mathcal{X}(\Omega)$.

Theorem 2.3.9 *Let $\Omega \subset \mathbb{R}^n$ be a V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)–(W6). There exists a continuous linear operator $\mathcal{L} : \mathcal{X}(\Omega) \to \mathcal{Y}(\mathbb{R}^n)$ such that for all $f \in \mathcal{X}(\Omega)$,*

1) $\mathcal{L}f\,|_\Omega = f$

2) $|\mathcal{L}f|_{\mathbb{R}^n} \leq M\|f\|_\Omega$, for some constant M independent of f.

Proof. Again this result is derived from Theorem 2.3.8, using Lemma 2.3.6 . ■

Before proving our final extension theorems we quote two more results from abstract analysis which we shall require.

Theorem 2.3.10 *Let X and Y be normed spaces. If there is an open, continuous mapping from X onto Y and X is complete, then so is Y.*

Theorem 2.3.11 *Suppose that $\|.\|_1$ and $\|.\|_2$ are two norms on X both making the space complete. If there exists an $\alpha > 0$ such that*

$$\|x\|_2 \leq \alpha\|x\|_1 \qquad \textit{for all } x \in X,$$

then the norms are equivalent.

Proofs of these results can be found in [17], pages 179 and 218 respectively.

Theorem 2.3.12 *Let $\Omega \subset I\!R^n$ be a V-domain. Let $w : I\!R^n \to I\!R$ be a measurable function satisfying (W1)-(W6). Given $f \in \mathcal{X}(\Omega)$, there exists a function $f_e \in \mathcal{Y}(I\!R^n)$ such that*

1) $f_e |_\Omega = f$

2) $|f_e|_{I\!R^n} \le M|f|_\Omega$ for some constant M independent of f.

Proof. Let $\Pi_{k,\Omega} = \{p |_\Omega : p \in \Pi_k(\mathbb{R}^n)\}$. We shall work with the quotient space

$$\mathcal{X}(\Omega)/\Pi_{k,\Omega} = \{f + \Pi_{k,\Omega} : f \in \mathcal{X}(\Omega)\}.$$

For $f \in \mathcal{X}(\Omega)$ define

$$\|f + \Pi_{k,\Omega}\|_1 = |f|_\Omega,$$

$$\|f + \Pi_{k,\Omega}\|_2 = \inf\{|u|_{\mathbb{R}^n} : u \in \mathcal{Y}(\mathbb{R}^n) \text{ and } u |_\Omega = f\}.$$

We claim that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathcal{X}(\Omega)/\Pi_{k,\Omega}$. Now, $|f|_\Omega = 0$ if and only if $f \in \Pi_{k,\Omega}$, and so $\|\cdot\|_1$ is clearly a norm on $\mathcal{X}(\Omega)/\Pi_{k,\Omega}$. Given $f \in \mathcal{X}(\Omega)$, Theorem 2.3.7 allows us to find an $\mathcal{L}f \in \mathcal{Y}(\mathbb{R}^n)$ which satisfies $\mathcal{L}f |_\Omega = f$ and $|\mathcal{L}f|_{\mathbb{R}^n} < \infty$. Hence, $\|f + \Pi_k\|_2$ exists. Let $f_e \in \mathcal{Y}(\mathbb{R}^n)$ satisfy $|f_e|_{\mathbb{R}^n} = \inf\{|u|_{\mathbb{R}^n} : u \in \mathcal{Y}(\mathbb{R}^n) \text{ and } u |_\Omega = f\}$. Suppose $\|f + \Pi_k\|_2 = 0$, then $|f_e|_{\mathbb{R}^n} = 0$ and $f_e \in \Pi_k$. Since $f_e |_\Omega = f$ this implies $f \in \Pi_{k,\Omega}$. Conversely, suppose $f \in \Pi_{k,\Omega}$. Then f_e is just the polynomial in Π_k for which $f_e |_\Omega = f$, since then $|f_e|_{\mathbb{R}^n} = 0$. Hence $\|\cdot\|_2$ is a norm on $\mathcal{X}(\Omega)/\Pi_{k,\Omega}$.

The quotient map $Q : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)/\Pi_{k,\Omega}$ is defined by $Q(f) = f + \Pi_{k,\Omega}$, for $f \in \mathcal{X}(\Omega)$. This is a linear, continuous, open map from $\mathcal{X}(\Omega)$ to $\mathcal{X}(\Omega)/\Pi_{k,\Omega}$, (see for example [30, p.31]). Since $\mathcal{X}(\Omega)$ is complete we can thus deduce from Lemma 2.3.10 that the normed spaces $(\mathcal{X}(\Omega)/\Pi_k, \|\cdot\|_1)$ and $(\mathcal{X}(\Omega)/\Pi_k, \|\cdot\|_2)$ are also complete. For all $f \in \mathcal{X}(\Omega)$, we have the simple inequality

$$\|f + \Pi_{k,\Omega}\|_1 = |f|_\Omega = |f_e|_\Omega \leq |f_e|_{\mathbb{R}^n} = \|f + \Pi_{k,\Omega}\|_2.$$

Hence, using Lemma 2.3.11, there exits a $\beta > 0$ such that

$$|f_e|_{\mathbb{R}^n} = \|f + \Pi_{k,\Omega}\|_2 \leq \beta\|f + \Pi_{k,\Omega}\|_1 = \beta|f|_\Omega, \qquad \text{for all } f \in \mathcal{X}(\Omega). \quad \blacksquare$$

Our final extension theorem involves the spaces $\mathcal{Y}(\Omega)$ which we define as the completion of $X(\Omega)$ with respect to $|\cdot|$. Before we prove this final result we remark that the following Corollary can be deduced trivially from the previous theorem.

Corollary 2.3.13 *Let $\Omega \subset \mathbb{R}^n$ be a V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W6). Given $f \in X(\Omega)$, there exists a function $f_e \in \mathcal{Y}(\mathbb{R}^n)$ such that*

1) $f_e\,|_\Omega = f$

2) $|f_e|_{\mathbb{R}^n} \leq M|f|_\Omega$ for some constant M independent of f.

Theorem 2.3.14 *Let $\Omega \subset \mathbb{R}^n$ be a V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W6). Given $f \in \mathcal{Y}(\Omega)$, there exists a function $f_e \in \mathcal{Y}(\mathbb{R}^n)$ such that*

1) $f_e\,|_\Omega = f$

2) $|f_e|_{\mathbb{R}^n} \leq M|f|_\Omega$ for some constant M independent of f.

Proof. By Corollary 2.3.12, such extensions exist for functions in $X(\Omega)$. Since $X(\Omega)$ is dense in $\mathcal{Y}(\Omega)$, Lemma 2.3.6 implies the required result. $\blacksquare$

2.4 The Weight Function w and the Domain Ω

The extension results developed in the previous section are dependent on the weight function w satisfying conditions (W1)–(W6), as given in Section 2.2. We give now some examples of weight functions for which these properties hold.

We begin with the familiar non-integer valued Sobolev seminorms. Here the weight function w is defined by $w(x) = |x|^{-n-\lambda}$ for $x \in \mathbb{R}^n$ and $0 < \lambda < 2$. It is clear that w satisfies conditions (W1)–(W4) and (W6). To see that (W5) is satisfied, let ϕ be a locally 1-smooth map on $\mathbb{R}^n$. Then ϕ^{-1} is also locally 1-smooth. Let Ω be a bounded domain. By Taylor's formula, there exists a constant $K > 0$ such that for all $x, y \in \Omega$,

$$|x - y| = |\phi^{-1}(\phi(x)) - \phi^{-1}(\phi(y))| \leq K|\phi(x) - \phi(y)|.$$

Hence, for all $x, y \in \Omega$ with $x \neq y$,

$$w(\phi(x) - \phi(y)) = \frac{1}{|\phi(x) - \phi(y)|^{n+\lambda}} \leq K^{n+\lambda} \frac{1}{|x - y|^{n+\lambda}} = K^{n+\lambda} w(x - y).$$

Since ϕ is a bijection, $x = y$ implies $w(\phi(x) - \phi(y)) = w(x - y) = w(0)$. Hence, $w(\phi(x) - \phi(y)) \leq \max\{K^{n+\lambda}, 1\}\ w(x - y)$ for all $x, y \in \Omega$. Hence, condition (W5) is satisfied.

For our second example let $w(x) = e^{-|x|^2}$ for $x \in \mathbb{R}^n$. Again, it is easily verified that w satisfies conditions (W1)–(W4) and (W6). Let ϕ be a locally smooth map on $\mathbb{R}^n$. Let Ω be a bounded domain. For all $x, y \in \Omega$,

$$|x - y|^2 - |\phi(x) - \phi(y)|^2 \leq |x - y|^2 \leq \sup_{x,y \in \Omega} |x - y|^2.$$

Because Ω is bounded we can find a $K > 0$ such that $\sup_{x,y\in\Omega} |x - y|^2 \leq K$. Then,

$$|\phi(x) - \phi(y)|^2 \geq |x - y|^2 - K \qquad \text{for all } x, y \in \Omega.$$

Thus, for all $x, y \in \Omega$,

$$w(\phi(x) - \phi(y)) = e^{-|\phi(x)-\phi(y)|^2} \leq e^{-|x-y|^2+K} = e^K w(x-y).$$

Consequently, condition (W5) holds.

Our previous example forms part of a family of such examples. Let w be a continuous, positive-valued function in $L^1(\mathbb{R}^n)$ satisfying $w(-x) = w(x)$ for all $x \in \mathbb{R}^n$. We also assume that there exists some ball $B_\delta = \{x \in \mathbb{R}^n : |x| \leq \delta\}$ such that on $\mathbb{R}^n \backslash B_\delta$, $w(x)$ is a decreasing function of $|x|$. It is straightforward to see that w satisfies (W2)–(W4). Furthermore, there exists $A > 0$ such that $w(x) \leq A$ for all $x \in \mathbb{R}^n$. Let ϕ be a locally smooth map on $\mathbb{R}^n$ and let Ω be a bounded domain. Since w is continuous we can find $M > 0$ such that $w(x-y) \geq M$ for all $x, y \in \Omega$. Thus

$$w(\phi(x) - \phi(y)) \leq A \leq \frac{A}{M} w(x-y) \qquad \text{for all } x, y \in \Omega.$$

Hence, (W5) is satisfied. Finally, we examine condition (W6). Take $y \in \mathbb{R}^n$. If $y \in B_\delta$ then a similar argument to that above proves the existence of $C > 0$ such that $w(y) \geq Cw(x)$ for all $x \in \mathbb{R}^n$. If $y \notin B_\delta$, then $w(y) \geq w(x)$ for all $x \in \mathbb{R}^n$ with $|x| \geq |y|$. Thus $w(y) \geq \min\{1, C\}\, w(x)$ whenever $|x| > |y|$, showing (W6) holds.

The condition on the domain is more difficult to exemplify. If Ω is a domain which lies locally on one side of its boundary $\partial\Omega$, then Conditions $(V1)$ and $(V2)$ in Definition 2.3.1 will hold if the boundary $\partial\Omega$ is an $(n-1)$-dimensional, $(k+1)$-smooth manifold in $\mathbb{R}^n$, (see Oden and Reddy [27]). An easy example of a set Ω in $\mathbb{R}^2$, which satisfies $(V3)$, is given by any disc. To construct the open sets G_j for the disc $B(0,r) = \{x \in \mathbb{R}^2 : |x| < r\}$ we can take, for $j = 1, \dots, 8$,

$$G_j = \left\{ x \in \mathbb{R}^2 : x = (\rho\cos\theta, \rho\sin\theta) \text{ and } \tfrac{7r}{8} < \rho < \tfrac{9r}{8},\ \tfrac{(j-1)\pi}{8} < \theta < \tfrac{(j+1)\pi}{8} \right\}.$$

The condition that Ω be a V-domain is a fairly strong requirement on the smoothness of $\partial\Omega$. Of importance to us in Chapter 3 will be the fact that this condition implies the cone condition as defined below, (see Wloka [38, Section 2]).

Definition 2.4.1 *A domain Ω is said to have the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone contained in Ω and congruent to C.*

Chapter 3

Error Estimates

An important question in the study of any interpolation method is that of error. Specifically, we wish to know how well the interpolant reconstructs the original function. Thus, we now turn our attention to the subject of error estimates. As indicated previously, the motivation behind the derivation of the extension results in Chapter 2 was their use in obtaining improved error estimates. In this chapter, we shall provide full details of how this is achieved. However, we shall begin by giving a brief introduction to the subject in a general setting, giving an indication of how one might construct a typical error estimate. We shall introduce the use of the so-called power function in the development of pointwise error estimates, and go on to obtain a simple L_2 estimate.

3.1 Power Functions and Typical Error Estimates

The development we shall use in this section follows that used by Light and Wayne [19], and has its roots in the variational theory of Golomb and Weinberger [11].

Let X be a linear space of continuous real valued functions on $\mathbb{R}^n$, with semi-inner product $(\cdot,\cdot)$. We define a seminorm on X by

$$|f| = \sqrt{(f,f)}, \qquad \text{for } f \in X.$$

We suppose that $|\cdot|$ has a finite dimensional kernel $\mathcal{K}$. Let Ω be a bounded subset of $\mathbb{R}^n$. Let $\mathcal{A}$ be a finite subset of Ω and suppose that $a_1, \ldots, a_\ell \in \mathcal{A}$ are unisolvent with respect to $\mathcal{K}$. We can define an inner product on X by

$$\langle f, g\rangle = \sum_{j=1}^{\ell} f(a_i)g(a_i) + (f,g), \qquad \text{for } f, g \in X.$$

This induces a norm, $\|\cdot\|$, on X via

$$\|g\|^2 = \langle g, g\rangle, \qquad \text{for } g \in X.$$

We shall assume that X is complete with respect to $\|\cdot\|$ and, for each $x \in \mathbb{R}^n$, there exists an $M > 0$ such that

$$|g(x)| \leq M(g,g), \qquad \text{for all } g \in X.$$

Thus X is a Hilbert function space. Given $f \in X$, let $Uf \in X$ be the minimal norm interpolant to f on $\mathcal{A}$. By this we mean $Uf(a) = f(a)$ for all $a \in \mathcal{A}$, and, if $v \in X$ also satisfies $v(a) = f(a)$ for all $a \in \mathcal{A}$, then $\|Uf\| \leq \|v\|$. A useful property of the minimal norm interpolant is that $\|f - Uf\|^2 = \|f\|^2 - \|Uf\|^2$, see Cheney and Light [4, Chapter 30, Theorem 1]. It is straightforward to deduce from this that the minimal norm interpolant also satisfies $|f - Uf|^2 = |f|^2 - |Uf|^2$, a fact we shall make use of later.

We are ready now to begin constructing a pointwise error estimate. We define G, a subspace of X, by

$$G = \{v \in X : v(a_i) = 0 \text{ for all } i = 1, \ldots, \ell\}.$$

Note that G is also a Hilbert space. Let

$$P(x) = \sup_{v \in G}\{|v(x)| : |v| = 1\}.$$

Then, for any $g \in G$, we have

$$P(x) \geq \frac{|g(x)|}{|g|}, \qquad \text{for all } x \in \mathbb{R}^n.$$

Rearranging gives,

$$|g(x)| \leq P(x)|g|, \qquad \text{for all } x \in \mathbb{R}^n. \tag{3.1}$$

Now consider $f - Uf$. Clearly $(f - Uf)(a_i) = 0$ for all $i = 1, \ldots, \ell$, and so $f - Uf \in G$. Thus, using Equation (3.1) and the property of the minimal norm interpolant mentioned above, we have

$$\begin{aligned} |(f - Uf)(x)|^2 &\leq \{P(x)\}^2|f - Uf|^2 \\ &= \{P(x)\}^2(|f|^2 - |Uf|^2) \\ &\leq \{P(x)\}^2|f|^2. \end{aligned}$$

We have derived an error estimate of the form

$$|f(x) - Uf(x)| \leq P(x)|f - Uf| \leq P(x)|f|, \qquad \text{for all } x \in \mathbb{R}^n. \tag{3.2}$$

However, we need to learn more about P for this to be of any practical use. Since G is a Hilbert space, we may make use of the Reisz representation theorem. This establishes, for each $x \in \mathbb{R}^n$, the existence of a unique representer $r_x \in G$, such that $\langle r_x, f\rangle = f(x)$ for all $f \in G$. Fix $x \in \mathbb{R}^n$ and recall that

$$P(x) = \sup_{v \in G}\{|v(x)| : |v| = 1\}.$$

Now, $|v(x)| = |\langle v, r_x\rangle|$ for all $v \in G$ and so we can write

$$P(x) = \sup_{v\in G}\{|\langle v, r_x\rangle| : |v| = 1\}.$$

Now, if $v \in G$, then $|v| = \|v\|$. Thus, using the Cauchy-Schwarz inequality, we have for all $v \in G$ with $|v| = 1$,

$$|\langle v, r_x\rangle| \le \|v\|\ \|r_x\| = |v|\ \|r_x\| = \|r_x\|.$$

Thus, $P(x) \le \|r_x\|$. Also, since $r_x \in G$,

$$P(x) \ge \frac{\langle r_x, r_x\rangle}{|r_x|} = \frac{\|r_x\|^2}{\|r_x\|} = \|r_x\|.$$

Thus, $P(x) = \|r_x\| = \sqrt{\langle r_x, r_x\rangle} = \sqrt{r_x(x)}$ and we have, from Equation (3.2),

$$|f(x) - Uf(x)| \le \sqrt{r_x(x)}\ |f - Uf| \le \sqrt{r_x(x)}\ |f|, \qquad \text{for all } x \in \mathbb{R}^n.$$

The form of the representer r_x can be explicitly calculated. As an example, we recall the spaces of Light and Wayne [20] discussed in Chapter 1. We have a measurable weight function $v : \mathbb{R}^n \to \mathbb{R}$ and, for non-negative integer k, we define

$$Z_k(\mathbb{R}^n) = \{f \in S' : \widehat{D^\alpha f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 v(x)\, dx < \infty,$$
$$\text{for all } \alpha \in \mathbb{Z}^n_+ \text{ with } |\alpha| = k\}.$$

Let $\phi \in S'$ satisfy $\widehat{\phi} = \{|\cdot|^{2k}v\}^{-1}$ and suppose $p_1, \ldots, p_\ell \in \Pi_{k-1}(\mathbb{R}^n)$ are such that $p_s(a_j) = 1$ if $s = j$, and is zero otherwise. Subject to certain conditions on the weight function v, the representer r_x has the form

$$r_x(x) = \phi(0) - \sum_{j=1}^{\ell} p_j(x)\phi(x - a_j) - \sum_{j=1}^{\ell} p_j(x)\phi(a_j - x) + \sum_{i,j=1}^{\ell} p_i(x)p_j(x)\phi(a_i - a_j).$$

Details of how this is obtained can be found in [19].

The function P is often referred to as the power function, the name originating from Schaback [33]. Schaback uses a different approach to the one described above; however, ultimately, he obtains the same function (cf. Light and Wayne [19]).

Having obtained an explicit form for the power function P, we can examine its asymptotic behaviour to establish an error estimate in terms of the spacing of the interpolation points. The usual measure of how densely the points in $\mathcal{A}$ 'fill out' Ω is given by

$$h = \sup_{y\in\Omega} \inf_{x\in A} |y - x|.$$

One hopes to obtain a bound of the form $|P(x)| \leq C_1 h^\beta$, where C_1 is a constant independent of f and h, and $\beta > 0$. Having derived such a bound, we can substitute it into Equation (3.2) to give an error estimate of the form

$$|f(x) - Uf(x)|^2 \leq C_1^2 h^{2\beta} |f|^2, \qquad \text{for all } x \in \mathbb{R}^n.$$

Then, as we would expect, the error on Ω between the function and its interpolant goes to zero as h tends to zero. It is straightforward to move from this inequality to an L_2 error estimate. Integrating both sides over Ω gives

$$\begin{aligned}\int_\Omega |f(x) - Uf(x)|^2 &\leq C_1^2 h^{2\beta} |f|^2 \int_\Omega 1 \\ &\leq C_2 h^{2\beta} |f|^2\end{aligned}$$

for some constant C_2, since Ω is bounded. Thus we have

$$\|f - Uf\|_{2,\Omega} \leq \sqrt{C_2} h^\beta |f|,$$

where C_2 is independent of f and h.

3.2 Improved Error Estimates Using Localisation

We move now from this general setting and return to the spaces considered in Chapter 2. We begin by recalling the relevant definitions. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable weight function which will be required to satisfy certain assumptions as we proceed. Let k be a fixed non-negative integer and define

$$X(\mathbb{R}^n) = \{f \in C_0^k(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(x-y)|D^\alpha f(x) - D^\alpha f(y)|^2 \, dxdy < \infty,$$

$$\text{for all } \alpha \in \mathbb{Z}_+^n \text{ with } |\alpha| = k\}.$$

For $f \in X(\mathbb{R}^n)$ define,

$$|f|_{\mathbb{R}^n} = \left(\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(x-y)|D^\alpha f(x) - D^\alpha f(y)|^2 \, dxdy \right)^{1/2},$$

where the constants c_α are defined by the algebraic identity

$$\sum_{|\alpha|=k} c_\alpha x^{2\alpha} = |x|^{2k}, \qquad \text{for all } x \in \mathbb{R}^n.$$

Let $\mathcal{Y}(\mathbb{R}^n)$ denote the completion of $X(\mathbb{R}^n)$ with respect to $|\cdot|_{\mathbb{R}^n}$. Then $|\cdot|_{\mathbb{R}^n}$ defines a seminorm on $\mathcal{Y}(\mathbb{R}^n)$ with kernel $\Pi_k(\mathbb{R}^n)$. We shall assume that $\Pi_k(\mathbb{R}^n)$ has dimension ℓ. Suppose $a_1, \ldots, a_\ell \in \mathbb{R}^n$ are unisolvent with respect to $\Pi_k(\mathbb{R}^n)$. We can then define a norm on $\mathcal{Y}(\mathbb{R}^n)$ via

$$\|f\|_{\mathcal{Y}}^2 = \sum_{i=1}^{\ell} |f(a_i)|^2 + |f|_{\mathbb{R}^n}^2, \qquad \text{for } f \in \mathcal{Y}(\mathbb{R}^n).$$

We shall see that $\mathcal{Y}(\mathbb{R}^n)$ is complete with respect to this norm, making $(\mathcal{Y}(\mathbb{R}^n), \|\cdot\|_{\mathcal{Y}})$ a Hilbert space.

Lemma 3.2.1 *The space $\mathcal{Y}(I\!R^n)$ is complete with respect to $\|\cdot\|_{\mathcal{Y}}$.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{Y}(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\mathcal{Y}}$. Take $\epsilon > 0$, then there exists a threshold $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\|f_n - f_m\|_{\mathcal{Y}} = \left(\sum_{i=1}^{\ell} |f_n(a_i) - f_m(a_i)|^2 + |f_n - f_m|_{\mathbb{R}^n}^2\right)^{1/2} < \epsilon.$$

It follows that $|f_n - f_m|_{\mathbb{R}^n} < \epsilon$ whenever $m, n > N$. Hence $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to $|\cdot|_{\mathbb{R}^n}$. Since $\mathcal{Y}(\mathbb{R}^n)$ is complete with respect to $|\cdot|$, there is a limit $f \in \mathcal{Y}(\mathbb{R}^n)$ such that $|f - f_n|_{\mathbb{R}^n}$ tends to zero as n tends to infinity. Furthermore, for each $i = 1, \ldots \ell$, $|f_n(a_i) - f_m(a_i)|^2 < \epsilon$ whenever $m, n > N$. Thus, $\{f_n(a_i)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers, and so has a limit $b_i \in \mathbb{R}$ such that $|f_n(a_i) - b_i|$ tends to zero as n tends to infinity.

Now, since $a_1, \ldots, a_\ell \in \mathbb{R}^n$ are unisolvent with respect to $\Pi_k(\mathbb{R}^n)$, there exists a polynomial $q \in \Pi_k(\mathbb{R}^n)$ such that $q(a_i) = b_i + f(a_i)$ for all $i = 1, \ldots, \ell$. Then,

$$\begin{aligned}
\|f_n - (f - q)\|_{\mathcal{Y}}^2 &= \sum_{i=1}^{\ell} |f_n(a_i) - f(a_i) + q(a_i)|^2 + |f_n - f + q|_{\mathbb{R}^n}^2 \\
&\le \sum_{i=1}^{\ell} |f_n(a_i) - f(a_i) + q(a_i)|^2 + (|f_n - f|_{\mathbb{R}^n} + |q|_{\mathbb{R}^n})^2 \\
&= \sum_{i=1}^{\ell} |f_n(a_i) - f(a_i) + q(a_i)|^2 + |f_n - f|_{\mathbb{R}^n}^2 \\
&= \sum_{i=1}^{\ell} |f_n(a_i) - b_i|^2 + |f_n - f|_{\mathbb{R}^n}^2.
\end{aligned}$$

Therefore, $\|f_n - (f - q)\|_{\mathcal{Y}}$ tends to zero as n tends to infinity. Since $f - q \in \mathcal{Y}(\mathbb{R}^n)$, this completes the proof. ∎

For bounded subset Ω of $\mathbb{R}^n$, we define local spaces $X(\Omega)$ by,

$$X(\Omega) = \{f\,|_{\Omega} : f \in C_0^k(\mathbb{R}^n) \text{ and } \int_{\Omega}\int_{\Omega} w(x-y)|D^{\alpha}f(x) - D^{\alpha}f(y)|^2\, dxdy < \infty,$$

for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k\}$.

For $f \in X(\Omega)$, define

$$|f|_\Omega = \left(\sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y) |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx dy \right)^{1/2}.$$

Then $|\cdot|_\Omega$ defines a seminorm on $X(\Omega)$, with kernel $\Pi_{k,\Omega} = \{p|_\Omega : p \in \Pi_k(\mathbb{R}^n)\}$. We denote by $\mathcal{Y}(\Omega)$ the completion of $X(\Omega)$ with respect to $|\cdot|_\Omega$.

We now recall the assumptions (W1)–(W6) made on the weight function w in the previous chapter as we shall refer to these again:

(W1) $w \in L^1(\mathbb{R}^n \backslash N)$ for any neighbourhood N of the origin;

(W2) $w(y) = \mathcal{O}(|y|^s)$ as $y \to 0$, where $n + s + 2 > 0$;

(W3) $\int_A w > 0$ whenever A has positive measure;

(W4) $w(y) = w(-y)$ for all $y \in \mathbb{R}^n$;

(W5) for every locally $(k+1)$-smooth map ϕ on $\mathbb{R}^n$, and every bounded subset Ω of $\mathbb{R}^n$, there is a $K > 0$ such that $w(\phi(x) - \phi(y)) \leq K w(x-y)$, for all $x, y \in \Omega$;

(W6) there exists a constant $M > 0$ such that if $x = (x', x_n) \in \mathbb{R}^n$ and $y = (x', y_n) \in \mathbb{R}^n$ with $|x_n| \geq |y_n|$, then $w(x) \leq M w(y)$.

We shall also require the following additional homogeneity type condition,

(W7) there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $w(\lambda x) = f(\lambda) w(x)$ for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$; furthermore $f(\lambda) \neq 0$ whenever $\lambda \neq 0$.

Now, let Ω be an open subset of $\mathbb{R}^n$. Let $\mathcal{A}$ be a finite subset of Ω such that $a_1, \ldots, a_\ell \in \mathcal{A}$ are unisolvent with respect to $\Pi_k(\mathbb{R}^n)$. Suppose that there is a $\beta > 0$ and a constant C independent of h such that, if $g \in \mathcal{Y}(\mathbb{R}^n)$ satisfies $g(a_i) = 0$ for all $i = 1, \ldots, \ell$, then

$$|g(x)| \leq Ch^\beta |g|_{\mathbb{R}^n} \qquad \text{for all } x \in \Omega.$$

Let $Uf \in \mathcal{Y}(\mathbb{R}^n)$ be the minimal norm interpolant to $f \in \mathcal{Y}(\mathbb{R}^n)$ on $\mathcal{A}$. Now, applying the arguments of the previous section, we could deduce the L_2 error estimate

$$\|f - Uf\|_{2,\Omega} \leq K_1 h^\beta |f|_{\mathbb{R}^n}, \tag{3.3}$$

where K_1 is independent of f and h. However, the arguments found in the remainder of this chapter will demonstrate that, by using a localisation argument, this can be improved to

$$\|f - Uf\|_{2,\Omega} \leq K_2 h^{\beta + n/2} |f|_\Omega,$$

where K_2 is a constant independent of f and h. Comparing this to Equation (3.3) we notice that instead of h^β we now have $h^{\beta+n/2}$ and the local seminorm now appears on the right-hand side of the equation. Furthermore, we need only know f on Ω, that is, we can take $f \in \mathcal{Y}(\Omega)$ rather than in the full space $\mathcal{Y}(\mathbb{R}^n)$. The argument we shall use is based on that found in Light and Wayne [19], which itself uses the work of Duchon [7]. In these two papers, a Sobolev space setting is used. Thus a certain amount of generalisation is needed to ensure this technique works for our spaces. In particular, difficulties occur because our seminorm involves a weight function and a double integral. We shall need some preliminary results before we are able to deduce the improved L_2 error estimate. We begin by recalling the main result of Chapter 2.

Theorem 3.2.2 *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W6). Given $f \in \mathcal{Y}(\Omega)$, there exists a function $f_e \in \mathcal{Y}(\mathbb{R}^n)$ such that*

(1) $f_e\,|_\Omega = f$

(2) $|f_e|_{\mathbb{R}^n} \leq M|f|_\Omega$ for some constant M independent of f.

Our next step is to demonstrate that if Ω is an open ball in $\mathbb{R}^n$, then the constant M in Theorem 3.2.2 can be taken independent of Ω.

Lemma 3.2.3 *Let Ω be a measurable subset of $\mathbb{R}^n$. Let $a, b \in \mathbb{R}^n$ and $h > 0$. Define $\sigma(x) = b + h(x - a)$, for $x \in \mathbb{R}^n$. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W7). Then there exists a constant C_h, dependent on h, such that for all $u \in \mathcal{Y}(\Omega)$,*

$$|u|_\Omega = C_h |u \circ \sigma|_{\sigma^{-1}(\Omega)}.$$

Proof. Take $u \in \mathcal{Y}(\Omega)$, then

$$|u|_\Omega^2 = \sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y)|D^\alpha u(x) - D^\alpha u(y)|^2 \, dxdy.$$

Making the transformation $x = \sigma(s)$ and $y = \sigma(t)$ we have

$$|u|_\Omega^2 = h^{2n} \sum_{|\alpha|=k} c_\alpha \int_{\sigma^{-1}(\Omega)} \int_{\sigma^{-1}(\Omega)} w(\sigma(s) - \sigma(t))|D^\alpha u(\sigma(s)) - D^\alpha u(\sigma(t))|^2 \, dsdt.$$

Now, since w satisfies (W7), there exists a function $f : \mathbb{R} \to \mathbb{R}$ with $f(h) \neq 0$ such that

$$\begin{aligned} w(\sigma(s) - \sigma(t)) &= w(b + h(s-a) - b - h(t-a)) \\ &= w(h(s-t)) \\ &= f(h)w(s-t). \end{aligned}$$

Also, for $|\alpha| = k$,

$$(D^\alpha u)(\xi) = [D^\alpha(u \circ \sigma \circ \sigma^{-1})](\xi) = h^{-k}[D^\alpha(u \circ \sigma)](\sigma^{-1}(\xi)).$$

Thus,

$$[(D^\alpha u) \circ \sigma](\xi) = (D^\alpha u)(\sigma(\xi)) = h^{-k}[D^\alpha(u \circ \sigma)](\xi).$$

Hence,

$$\begin{aligned} |u|_\Omega^2 &= f(h)h^{2n-2k} \sum_{|\alpha|=k} c_\alpha \int_{\sigma^{-1}(\Omega)} \int_{\sigma^{-1}(\Omega)} w(s-t)|D^\alpha(u \circ \sigma)(s) - D^\alpha(u \circ \sigma)(t)|^2 \, dsdt \\ &= f(h)h^{2n-2k}|u \circ \sigma|^2_{\sigma^{-1}(\Omega)}. \end{aligned}$$

Taking $C_h = \sqrt{f(h)h^{2n-2k}}$ gives the result. ■

Lemma 3.2.4 *Let B be any open ball with radius h in $\mathbb{R}^n$. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)-(W7). Given $f \in \mathcal{Y}(B)$, there exists a function $f_B \in \mathcal{Y}(\mathbb{R}^n)$ such that*

(1) $f_B \mid_B = f$

(2) $|f_B|_{\mathbb{R}^n} \leq M|f|_B$ for some constant M independent of f and B.

Proof. Let $f \in \mathcal{Y}(B)$, then, using Lemma 3.2.2, there exists an $f_B \in \mathcal{Y}(\mathbb{R}^n)$ such that $f_B \mid_B = f$ and $|f_B|_{\mathbb{R}^n} \leq M|f|_B$ for some constant M independent of f. By choosing f_B to be the minimal norm extension we can assume that if $v \in \mathcal{Y}(\mathbb{R}^n)$ is such that $v \mid_B = f$ then $|f_B|_{\mathbb{R}^n} \leq |v|_{\mathbb{R}^n}$. We need to show that the constant M can be taken independent of B.

We can write $B = \{x \in \mathbb{R}^n : |x-a| < h\}$ for some $a \in \mathbb{R}^n$. Let $B_0 = \{x \in \mathbb{R}^n : |x| < 1\}$ and define $\sigma(x) = h^{-1}(x-a)$, for $x \in \mathbb{R}^n$. Then $\sigma(B) = B_0$. Let $F = f \circ \sigma^{-1}$, then $F \in \mathcal{Y}(B_0)$. Thus, by Lemma 3.2.2, there exists a function $F_e \in \mathcal{Y}(\mathbb{R}^n)$ such that $F_e\,|_{B_0} = F$ and $|F_e|_{\mathbb{R}^n} \leq K(B_0)|F|_{B_0}$, for some constant $K(B_0)$ independent of F but dependent on B_0. We claim that

(i) $f_B \circ \sigma^{-1}\,|_{B_0} = F$

(ii) $|f_B \circ \sigma^{-1}|_{\mathbb{R}^n} \leq |v|_{\mathbb{R}^n}$ for all $v \in \mathcal{Y}(\mathbb{R}^n)$ such that $v\,|_{B_0} = F$.

Take $x \in B_0$, then $\sigma^{-1}(x) \in B$. Since $f_B\,|_B = f$, it follows that $f_B(\sigma^{-1}(x)) = f(\sigma^{-1}(x))$. Hence, $(f_B \circ \sigma^{-1})(x) = (f \circ \sigma^{-1})(x)$ for all $x \in B_0$ and (i) is satisfied. Now suppose $v \in \mathcal{Y}(\mathbb{R}^n)$ satisfies $v\,|_{B_0} = f \circ \sigma^{-1}$. Then $v(\sigma(x)) = f(x)$ for all $x \in B$. Thus $(v \circ \sigma)\,|_B = f\,|_B$ and the properties of f_B imply that $|f_B|_{\mathbb{R}^n} \leq |v \circ \sigma|_{\mathbb{R}^n}$. By Lemma 3.2.3, there exists a constant $C_h \neq 0$ such that

$$\begin{aligned}
|f_B \circ \sigma^{-1}|_{\mathbb{R}^n} &= C_h |f_B|_{\mathbb{R}^n} \\
&\leq C_h |v \circ \sigma|_{\mathbb{R}^n} \\
&= C_h [C_h]^{-1} |v|_{\mathbb{R}^n} \\
&= |v|_{\mathbb{R}^n}.
\end{aligned}$$

Thus, claim (ii) is satisfied. Now, since $F_e\,|_{B_0} = F$ and $|F_e|_{\mathbb{R}^n} \leq K(B_0)|F|_{B_0}$ we have

$$|f_B \circ \sigma^{-1}|_{\mathbb{R}^n} \leq |F_e|_{\mathbb{R}^n} \leq K(B_0)|F|_{B_0} = K(B_0)|f \circ \sigma^{-1}|_{B_0}.$$

By Lemma 3.2.3, there exists a constant $C_h \neq 0$ such that

$$|f_B|_{\mathbb{R}^n} = |f_B \circ \sigma^{-1} \circ \sigma|_{\mathbb{R}^n}$$

$$
\begin{aligned}
&= [C_h]^{-1}|f_B \circ \sigma^{-1}|_{\mathbb{R}^n} \\
&\leq [C_h]^{-1}K(B_0)|f \circ \sigma^{-1}|_{B_0} \\
&= [C_h]^{-1}K(B_0)C_h|f|_B \\
&= K(B_0)|f|_B,
\end{aligned}
$$

as required. ∎

Throughout the remainder of this chapter we shall make use of the following notation. For $c, r \in \mathbb{R}^n$ we define $B(c, r)$ to be the closed ball in $\mathbb{R}^n$ with centre c and radius r.

Lemma 3.2.5 (Light and Wayne [19]) *Let $\{v_1, \dots v_\ell\}$ be a set of $\Pi_k(\mathbb{R}^n)$-unisolvent points in $\mathbb{R}^n$. Then there exists a $\delta > 0$ such that if $(c_1, \dots, c_\ell) \in B(v_1, \delta) \times B(v_2, \delta) \times \dots \times B(v_\ell, \delta)$, then $\{c_1, \dots, c_\ell\}$ is a set of $\Pi_k(\mathbb{R}^n)$-unisolvent points.*

Definition 3.2.6 *A domain Ω is said to have the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone contained in Ω and congruent to C.*

Lemma 3.2.7 (Duchon [7]) *Let Ω be an open subset of $\mathbb{R}^n$ having the cone property. Then, there exists M, M_1 and $m > 0$ such that to each $0 < h < m$, there corresponds a set $T_h \subset \Omega$ with*

(i) $B(t, h) \subset \Omega$ for all $t \in T_h$,

(ii) $\Omega \subset \bigcup_{t \in T_h} B(t, Mh)$,

(iii) $\sum_{t \in T_h} \chi_{B(t,Mh)} \leq M_1$.

We shall require a slightly modified version of the above lemma, the proof of which is taken from Light and Wayne [19, Lemma 3.6]

Lemma 3.2.8 *Let Ω be an open subset of $I\!R^n$ having the cone property. Let $\mathcal{A}$ be a finite $\Pi_k(I\!R^n)$-unisolvent subset of Ω with*

$$\max_{y\in\Omega}\min_{x\in\mathcal{A}}|y-x|\leq h.$$

Then there exists M_1, M_2, $h_0>0$, and a set $T_h\subset\Omega$, such that

(i) $\Omega\subset\bigcup_{t\in T_h}B(t,M_1h)$,

(ii) $\sum_{t\in T_h}\chi_{B(t,M_1h)}\leq M_2$,

provided $0<h<h_0$.

Furthermore, given $t\in T_h$, there exists $a_1,\ldots,a_\ell\in B(t,M_1h)\cap\mathcal{A}$ such that $a_1,\ldots,a_\ell$ are unisolvent with respect to $\Pi_k(I\!R^n)$.

Proof. Let $\{v_1,\ldots v_\ell\}$ be a set of $\Pi_k(\mathbb{R}^n)$-unisolvent points in $\mathbb{R}^n$. By Lemma 3.2.5 there exists a $\delta>0$ such that if $(c_1,\ldots,c_\ell)\in B(v_1,\delta)\times\ldots\times B(v_\ell,\delta)$, then $\{c_1,\ldots,c_\ell\}$ is a set of $\Pi_k(\mathbb{R}^n)$-unisolvent points. Dilation by a factor of $1/\delta$ creates a new set of points $u_1,\ldots u_\ell$ such that if $(c_1,\ldots,c_\ell)\in B(u_1,1)\times\ldots\times B(u_\ell,1)$, then $\{c_1,\ldots,c_\ell\}$ is a set of $\Pi_k(\mathbb{R}^n)$-unisolvent points.

Choose $R>0$ such that $B(u_i,1)\subset B(0,R)$ for all $i=1,\ldots,\ell$. By Lemma 3.2.7, we can find constants M, M_2 and $m>0$, such that there exists a set $T_h\subset\Omega$ with $B(t,Rh)\subset\Omega$, $\Omega\subset\bigcup_{t\in T_h}B(t,MRh)$ and $\sum_{t\in T_h}\chi_{B(t,MRh)}\leq M_2$, providing $0<Rh<m$. Taking $M_1=MR$ and $h_0=m/R$ delivers the first part of the lemma.

Fix $t \in T_h$ and $h > 0$ such that $0 < h < h_0$. We shall construct $a_1, \ldots, a_\ell \in B(t, M_1 h) \cap \mathcal{A}$ such that $a_1, \ldots, a_\ell$ are unisolvent with respect to $\Pi_k(\mathbb{R}^n)$. Define $\sigma : B(t, MRh) \to B(0, MR)$ by $\sigma(y) = h^{-1}(y - t)$ for $y \in B(t, MRh)$. Due to the spacing of the points in $\mathcal{A}$, each ball $B(u_i, 1)$ must contain at least one image under σ of a point in $\mathcal{A}$. Thus, we can choose $a_1 \ldots, a_\ell \in \mathcal{A}$ such that $\sigma(a_i) \in B(u_i, 1)$, and so $\{\sigma(a_1), \ldots, \sigma(a_\ell)\}$ is a $\Pi_k(\mathbb{R}^n)$-unisolvent set. It follows that $\{a_1, \ldots, a_\ell\}$ is a $\Pi_k(\mathbb{R}^n)$-unisolvent subset of $B(t, MRh) = B(t, M_1 h)$. ■

We are finally ready to compute our improved L_2 error estimate.

Lemma 3.2.9 *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, V-domain. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (W1)–(W7). Let $\mathcal{A}$ be a finite $\Pi_k(\mathbb{R}^n)$-unisolvent subset of Ω with*

$$\max_{y \in \Omega} \min_{x \in \mathcal{A}} |y - x| \leq h.$$

We assume that there exists a $\beta > 0$ and a constant C independent of h such that if A is a $\Pi_k(\mathbb{R}^n)$-unisolvent subset of $\mathcal{A}$, and $g \in \mathcal{Y}(\mathbb{R}^n)$ satisfies $g(a) = 0$ for all $a \in A$, then

$$|g(x)| \leq C h^\beta |g|_{\mathbb{R}^n}, \qquad \text{for all } x \in \mathbb{R}^n.$$

Given $f \in \mathcal{Y}(\Omega)$, let $Uf \in \mathcal{Y}(\mathbb{R}^n)$ be the minimal norm interpolant to f on $\mathcal{A}$. There exists an $h_0 > 0$ and a constant $K > 0$, independent of f and h, such that, provided $h < h_0$,

$$\|f - Uf\|_{2,\Omega} \leq K h^{\beta + n/2} |f|_\Omega, \qquad \text{for all } f \in \mathcal{Y}(\Omega).$$

Proof. By Lemma 3.2.8, there exists constants M_1, M_2, $h_0 > 0$, and a set $T_h \subset \Omega$ such that, $\Omega \subset \bigcup_{t \in T_h} B(t, M_1 h)$ and $\sum_{t \in T_h} \chi_{B(t, M_1 h)} \leq M_2$ provided $0 < h < h_0$. Fix $h < h_0$.

Take $t \in T_h$, then, also by Lemma 3.2.8, we can construct $a_1, \ldots, a_\ell \in B(t, M_1 h) \cap \mathcal{A}$ such that $a_1, \ldots, a_\ell$ are unisolvent with respect to $\Pi_k(\mathbb{R}^n)$. For ease, we shall now write B for $B(t, M_1 h)$.

Take $f \in \mathcal{Y}(\Omega)$ and let $f_\Omega \in \mathcal{Y}(\mathbb{R}^n)$ be the extension of f as described in Lemma 3.2.2. We shall, for the next part of the proof, write f for f_Ω. Let $(f - Uf)_B$ also be as in Lemma 3.2.2. Thus $(f - Uf)_B \mid_B = (f - Uf) \mid_B$, and, due to Lemma 3.2.4, $|(f - Uf)_B|_{\mathbb{R}^n} \leq C_1 |f - Uf|_B$ for some C_1 independent of B.

Now, $(f - Uf)(a_i) = 0$ for all $i = 1, \ldots, \ell$. Since $a_1, \ldots, a_\ell \in B$ it follows that $(f - Uf)_B(a_i) = 0$ for all $i = 1, \ldots, \ell$. Thus, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |(f - Uf)_B(x)|^2 &\leq C^2 h^{2\beta} |(f - Uf)_B|^2_{\mathbb{R}^n} \\ &\leq C^2 C_1^2 h^{2\beta} |f - Uf|^2_B. \end{aligned}$$

For $x \in B$ we have $(f - Uf)_B(x) = (f - Uf)(x)$. Thus, for all $x \in B$,

$$|(f - Uf)(x)|^2 \leq C_2 h^{2\beta} |f - Uf|^2_B,$$

where $C_2 = C^2 C_1^2$ is independent of f and h. Integrating over B gives

$$\begin{aligned} \|f - Uf\|^2_{2,B} &\leq C_2 h^{2\beta} |f - Uf|^2_B \int_B 1 \\ &\leq C_2 C_3 h^{2\beta+n} |f - Uf|^2_B, \end{aligned}$$

for some appropriate constant C_3.

Let $\Omega^* = \bigcup_{t \in T_h} B(t, M_1 h)$. Then,

$$\|f - Uf\|^2_{2,\Omega} \leq \|f - Uf\|^2_{2,\Omega^*}$$

$$
\begin{aligned}
&\leq \sum_{t\in T_h} \|f-Uf\|^2_{2,B(t,M_1h)} \\
&\leq C_2C_3h^{2\beta+n} \sum_{t\in T_h} |f-Uf|^2_{B(t,M_1h)}.
\end{aligned}
$$

Let $B_t = B(t, M_1h)$ and define, for $x, y \in \mathbb{R}^n$,

$$
z(x,y) = \sum_{|\alpha|=k} c_\alpha w(x-y)|D^\alpha(f-Uf)(x) - D^\alpha(f-Uf)(y)|^2.
$$

Then,

$$
\begin{aligned}
\sum_{t\in T_h} |f-Uf|^2_{B(t,M_1h)} &= \sum_{t\in T_h} \int_{B_t}\int_{B_t} z(x,y)\, dxdy \\
&\leq \sum_{t\in T_h} \int_{\mathbb{R}^n}\int_{B_t} z(x,y)\, dxdy \\
&= \sum_{t\in T_h} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{B_t}(x) z(x,y)\, dxdy \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\sum_{t\in T_h} \chi_{B_t}(x)\right) z(x,y)\, dxdy \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} M_2 z(x,y)\, dxdy \\
&= M_2|f-Uf|^2_{\mathbb{R}^n}
\end{aligned}
$$

Thus,

$$
\|f-Uf\|^2_{2,\Omega} \leq C_2C_3M_2h^{2\beta+n}|f-Uf|^2_{\mathbb{R}^n}.
$$

Recall now that we are writing f for f_Ω. Thus, using properties of the minimal norm interpolant, we have

$$
\begin{aligned}
\|f-Uf\|^2_{2,\Omega} &\leq C_2C_3M_2h^{2\beta+n}|f_\Omega - Uf_\Omega|^2_{\mathbb{R}^n} \\
&= C_2C_3M_2h^{2\beta+n}(|f_\Omega|^2_{\mathbb{R}^n} - |Uf_\Omega|^2_{\mathbb{R}^n}) \\
&\leq C_2C_3M_2h^{2\beta+n}|f_\Omega|^2_{\mathbb{R}^n}.
\end{aligned}
$$

Using the properties of the extension f_Ω, there is a constant C_4 independent of f and h such that

$$\|f - Uf\|_{2,\Omega}^2 \leq C_2 C_3 C_4 M_2 h^{2\beta+n} |f|_\Omega^2.$$

Letting $K = \sqrt{C_2 C_3 C_4 M_2}$ we have

$$\|f - Uf\|_{2,\Omega} \leq K h^{\beta+n/2} |f|_\Omega. \quad \blacksquare$$

Chapter 4

Alternative Local Spaces and Seminorms

Many of the ideas discussed in the previous two chapters stem from a variational approach to the interpolation problem. One begins with a Hilbert space of functions and recognises the interpolant as the solution of a variational problem described on the given space. In particular, we have focussed on the spaces of distributions introduced by Light and Wayne [20], and the associated direct form seminorms developed by Light and Levesley [18]. These allowed us to construct the local spaces and seminorms of Chapter 2, for which extension theorems were developed.

An alternative approach to the variational one uses the ideas introduced in Section 1.5. Here one begins with a conditionally positive definite function and constructs around it a *native* Hilbert space in which to study the interpolation problem. This theory leads to its own definitions of local spaces and seminorms. Clearly it is of interest to know whether these two approaches generate the same spaces, particularly with regards to localisation.

This chapter attempts to answer some aspects of this question.

There are many different characterisations of the native space. We shall be interested in that given by Iske in [16]. Beginning with the spaces of Madych and Nelson [23], Iske develops a convolutional representation of the native space. The technical details of this useful reformulation are not trivial. Thus, here, we shall merely quote the relevant definitions.

4.1 Spaces on $\mathbb{R}^n$

We begin our discussion with a comparison of spaces described on the whole of $\mathbb{R}^n$. In particular, we shall focus on the native spaces of Iske [16] and the spaces of distributions of Light and Wayne [20]. We begin with some definitions and assumptions.

Throughout this section we assume $v : \mathbb{R}^n \to \mathbb{R}$ to be a measurable function which satisfies

(v1) $v \in C(\mathbb{R}^n \backslash \{0\})$,

(v2) $v(x) > 0$ for all $x \neq 0$,

(v3) $1/v \in L^1_{loc}(\mathbb{R}^n)$,

(v4) there is a $\mu \in \mathbb{R}$ such that $\mu > n$ and $\{v(x)\}^{-1} = \mathcal{O}(|x|^{-\mu})$ as $|x| \to \infty$.

We define, for non-negative integer k,

$$Z_k(\mathbb{R}^n) = \{f \in S' : \widehat{D^\alpha f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2 v(x)\, dx < \infty$$
$$\text{for all } \alpha \in \mathbb{Z}^n_+ \text{ with } |\alpha| = k\}.$$

This is the space of distributions introduced by Light and Wayne [20], and is discussed in Section 1.3. We shall, in this section, concern ourselves only with the case $k = 0$. Therefore, the relevant space is simply

$$Z_0(\mathbb{R}^n) = \{g \in S' : \widehat{g} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\widehat{g}(x)|^2 v(x)\, dx < \infty\}.$$

For ease of notation we shall, henceforward, write $Z(\mathbb{R}^n)$ for $Z_0(\mathbb{R}^n)$. A norm is defined on $Z(\mathbb{R}^n)$ by,

$$\|g\|_Z = \left(\int_{\mathbb{R}^n} |\widehat{g}(x)|^2 v(x)\, dx\right)^{1/2}, \qquad \text{for } g \in Z(\mathbb{R}^n).$$

We make the assumptions (v1), (v2) and (v4) on v in order to ensure that $Z(\mathbb{R}^n)$ is a subset of the continuous functions, see [20, Theorem 2.18]. The additional assumption, (v3), allows us to derive some useful results concerning the function v.

Lemma 4.1.1 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (v1)–(v4), then $1/v \in L^1(\mathbb{R}^n)$.*

Proof. We can write

$$\int_{\mathbb{R}^n} \frac{1}{|v(x)|}\, dx = \int_{|x|<1} \frac{1}{|v(x)|}\, dx + \int_{|x|\geq 1} \frac{1}{|v(x)|}\, dx.$$

By assumption (v3) on v, $1/v \in L^1_{loc}(\mathbb{R}^n)$, and so the first integral is finite. Now, since $v \in C(\mathbb{R}^n \backslash \{0\})$ and $v(x) > 0$ for all $x \neq 0$, $1/v$ is continuous on the set $\{x \in \mathbb{R}^n : |x| \geq 1\}$. Hence, using this and property (v4) of v, there is a constant C and a $\mu > n$, such that $\{v(x)\}^{-1} \leq C|x|^{-\mu}$ for all $x \in \{x \in \mathbb{R}^n : |x| \geq 1\}$. Hence,

$$\int_{|x|\geq 1} \frac{1}{|v(x)|}\, dx \leq C \int_{|x|\geq 1} |x|^{-\mu}\, dx.$$

Using polar coordinates we can find a $B > 0$ such that

$$\int_{|x|\geq 1} \frac{1}{|v(x)|}\,dx \leq B \int_1^\infty r^{-\mu+n-1}\,dr.$$

This final integral is finite since $-\mu + n - 1 < -n + n - 1 = -1$. ∎

Lemma 4.1.2 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (v1)–(v4). Then there exists a $\psi \in S'$ such that*

$$\widehat{\psi} = 1/v.$$

Furthermore, ψ is a bounded continuous function on $\mathbb{R}^n$, and $\psi(x)$ tends to zero as $|x|$ tends to infinity.

Proof. By Lemma 4.1.1, $1/v \in L^1(\mathbb{R}^n)$. Hence $1/v$ is a tempered distribution, whose action on a test function $\phi \in S$ is given by

$$[1/v, \phi] = \int_{\mathbb{R}^n} \frac{1}{v(x)}\,\phi(x)\,dx.$$

The distributional Fourier transform is a one to one mapping of S' onto S'. Hence, there exists a $\psi \in S'$ such that

$$\widehat{\psi} = 1/v.$$

Then, by properties of Fourier transforms,

$$B\psi = \widehat{\widehat{\psi}} = \widehat{(1/v)}.$$

Since $1/v \in L^1(\mathbb{R}^n)$, it also possesses a Fourier transform in the classical sense, which is continuous and bounded on $\mathbb{R}^n$, and approaches zero at infinity. Since the classical and distributional Fourier transform for $1/v$ must coincide, it follows that ψ possesses these properties. ■

We demonstrate now that the ψ of the previous lemma is also positive definite. In order to do so we make use of the following definition from Gel'fand and Vilenkin [10, Chapter II, Section 3.1].

Definition 4.1.3 *A distribution $\Lambda \in S'$ is called positive definite if for all $\phi \in S$,*

$$[\widehat{\Lambda}, \phi\overline{\phi}] \geq 0.$$

Lemma 4.1.4 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (v1)–(v4). Let $\psi \in S'$ satisfy $\widehat{\psi} = 1/v$. Then ψ is positive definite.*

Proof. Take $\phi \in S$. Then,

$$[\widehat{\psi}, \phi\overline{\phi}] = [1/v, \phi\overline{\phi}] = \int_{\mathbb{R}^n} \frac{1}{v(x)} |\phi(x)|^2 \, dx \geq 0,$$

since, by assumption (v2), $v(x) > 0$ if $x \neq 0$. ■

The space $Z(\mathbb{R}^n)$ has delivered a continuous positive definite function ψ. We shall now take ψ and generate its native space according to Iske [16]. Intuitively we would expect this space to be identical to $Z(\mathbb{R}^n)$. We shall show that, under the right circumstances, this is indeed the case.

Definition 4.1.5 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (v1)–(v4). Let $\psi \in S'$ be such that $\widehat{\psi} = 1/v$. We define*

$$I(\mathbb{R}^n) = \{g : g = \psi * f, \text{ for some } f \in C_0^\infty(\mathbb{R}^n)\}.$$

Some remarks about the definition of $I(\mathbb{R}^n)$ seem to be required here. An element of $I(\mathbb{R}^n)$ is a convolution of the form $\psi * f$, with $f \in C_0^\infty(\mathbb{R}^n)$. This convolution can be taken in two senses. By Lemma 4.1.2, $\psi \in C(\mathbb{R}^n)$. Therefore, the convolution $\psi * f$ is well defined in the classical sense and is given by

$$(\psi * f)(x) = \int_{\mathbb{R}^n} \psi(x-y)f(y)\,dy, \qquad \text{for } x \in \mathbb{R}^n.$$

Also, $\psi \in S'$ and, since $f \in C_0^\infty(\mathbb{R}^n)$, we have $f \in S$. Hence, the convolution $\psi * f$ is well defined in the distributional sense and is given by

$$\psi * f = [\psi, T_x Bf].$$

We would expect these two interpretations of the convolution to coincide. We now demonstrate that this is indeed the case. Since $\psi \in C(\mathbb{R}^n)$, its action on a test function in $C_0^\infty(\mathbb{R}^n)$ is given by integration. Thus, for $f \in C_0^\infty(\mathbb{R}^n)$ we have,

$$\begin{aligned}
[\psi, T_x Bf] &= \int_{\mathbb{R}^n} \psi(y)(T_x Bf)(y)\,dy \\
&= \int_{\mathbb{R}^n} \psi(y)Bf(y-x)\,dy \\
&= \int_{\mathbb{R}^n} \psi(y)f(x-y)\,dy \\
&= \int_{\mathbb{R}^n} \psi(x-y)f(y)\,dy.
\end{aligned}$$

We give now some insights into the elements of $I(\mathbb{R}^n)$.

Lemma 4.1.6 *Let $I(\mathbb{R}^n)$ be as in Definition 4.1.5. Suppose $g \in I(\mathbb{R}^n)$ with $g = \psi * f$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then,*

(i) $g \in C^\infty(\mathbb{R}^n) \cap S'$,

(ii) $\widehat{g} = \widehat{f}\widehat{\psi} \in L^1(\mathbb{R}^n)$.

The Fourier transform here is taken in a distributional sense.

Proof. Take $g \in I(\mathbb{R}^n)$ with $g = \psi * f$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then, since $\psi \in S'$ and $f \in C_0^\infty(\mathbb{R}^n) \subset S$, properties of distributions ensure $g \in S' \cap C^\infty(\mathbb{R}^n)$, and $\widehat{g} = \widehat{f}\widehat{\psi}$. Since $f \in C_0^\infty(\mathbb{R}^n)$, it follows that $\widehat{f} \in S$. By Lemma 4.1.1, $\widehat{\psi} = 1/v \in L^1(\mathbb{R}^n)$. Thus, $\widehat{f}\widehat{\psi} \in L^1(\mathbb{R}^n)$. ■

Definition 4.1.7 *For each $g \in I(\mathbb{R}^n)$, with $g = \psi * f$, and $f \in C_0^\infty(\mathbb{R}^n)$, we define*

$$\|g\|_I^2 = \int_{\mathbb{R}^n} g(x)\overline{f(x)}\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x-y)f(y)\overline{f(x)}\, dydx.$$

The use of the notation $\|\cdot\|_I$ here is not abusive as we shall now demonstrate that this quantity defines a norm on $I(\mathbb{R}^n)$.

Lemma 4.1.8 *Let $I(\mathbb{R}^n)$ and $\|\cdot\|_I$ be as in Definitions 4.1.5 and 4.1.7. Then $\|\cdot\|_I$ defines a norm on $I(\mathbb{R}^n)$.*

Proof. We shall use a distributional argument and make frequent use of the properties of distributions described in Section 1.6. Take $g \in I(\mathbb{R}^n)$. Then $g = \psi * f$ for some $f \in C_0^\infty(\mathbb{R}^n)$. By Lemma 4.1.6, $g \in C^\infty(\mathbb{R}^n)$ and thus g is a distribution whose action on a test function $\phi \in C_0^\infty(\mathbb{R}^n)$ is given by

$$[g, \phi] = \int_{\mathbb{R}^n} g(x)\phi(x)\, dx.$$

Thus, we can write

$$\|g\|_I^2 = \int_{\mathbb{R}^n} g(x)\overline{f(x)}\, dx = [g, \overline{f}].$$

Since $f \in S$, there is a $\theta \in S$ such that $\widehat{\theta} = \overline{f}$. Thus, using simple properties of distributions,

$$\|g\|_I^2 = [g, \overline{f}] = [g, \widehat{\theta}] = [\widehat{g}, \theta].$$

By Lemma 4.1.6, $\widehat{g} = \widehat{f}\widehat{\psi}$. Thus, again using simple properties of distributions,

$$\|g\|_I^2 = [\widehat{f}\widehat{\psi}, \theta] = [\widehat{\psi}, \widehat{f}\theta] = [\widehat{\psi}, \widehat{f}(B\widehat{\overline{f}})] = [\widehat{\psi}, \widehat{f}\,\overline{\widehat{f}}].$$

By Lemma 4.1.1, $\widehat{\psi} = 1/v \in L^1(\mathbb{R}^n)$, and thus, as a tempered distribution, its action on a test function in S is given by integration. Hence,

$$\begin{aligned}
\|g\|_I^2 &= \int_{\mathbb{R}^n} \widehat{\psi}(x)(\widehat{f}\,\overline{\widehat{f}})(x)\, dx \\
&= \int_{\mathbb{R}^n} \widehat{\psi}(x)|\widehat{f}(x)|^2\, dx. \\
&= \int_{\mathbb{R}^n} \frac{1}{v(x)}|\widehat{f}(x)|^2\, dx.
\end{aligned}$$

Since, by assumption (v2), $v(x) > 0$ for all $x \neq 0$, it is clear that $\|g\|_I \geq 0$. Further more, $\|g\|_I = 0$ implies $\widehat{f} = 0$. If $\widehat{f} = 0$, then $f = 0$ and we must have $g = \psi * f = 0$. ■

The native space of ψ is given by the closure of $I(\mathbb{R}^n)$ with respect to $\|\cdot\|_I$. We want to compare this space with our original space $Z(\mathbb{R}^n)$. Our aim is to show that, under suitable circumstances, $I(\mathbb{R}^n)$ is a dense subset of $Z(\mathbb{R}^n)$. We begin by examining the two norms $\|\cdot\|_I$ and $\|\cdot\|_Z$.

Theorem 4.1.9 *If $g \in I(\mathbb{R}^n)$, then $\|g\|_I = \|g\|_Z$.*

Proof. Take $g \in I(\mathbb{R}^n)$, then $g = \psi * f$ for some $f \in C_0^\infty(\mathbb{R}^n)$. An identical argument to that found in the proof of Lemma 4.1.8 gives

$$\|g\|_I^2 = \int_{\mathbb{R}^n} \widehat{\psi}(x)\overline{\widehat{f}}(x)\widehat{f}(x)\, dx.$$

Now, $v\widehat{\psi} = 1$, thus since v is real valued we have

$$v\overline{\widehat{\psi}} = \overline{v\widehat{\psi}} = 1.$$

Using this and the fact that, by Lemma 4.1.6, $\widehat{g} = \widehat{\psi}\widehat{f}$, we have

$$\begin{aligned} \|g\|_I^2 &= \int_{\mathbb{R}^n} (v\overline{\widehat{\psi}})(x)\widehat{\psi}(x)\overline{\widehat{f}}(x)\widehat{f}(x)\, dx \\ &= \int_{\mathbb{R}^n} \overline{\widehat{f}\widehat{\psi}}(x)\widehat{f}\widehat{\psi}(x)v(x)\, dx \\ &= \int_{\mathbb{R}^n} \overline{\widehat{g}}(x)\widehat{g}(x)v(x)\, dx \\ &= \int_{\mathbb{R}^n} |\widehat{g}(x)|^2 v\, dx \\ &= \|g\|_Z^2. \quad \blacksquare \end{aligned}$$

Lemma 4.1.10 *We have $I(\mathbb{R}^n) \subset Z(\mathbb{R}^n)$.*

Proof. Take $g \in I(\mathbb{R}^n)$, then by Lemma 4.1.6, $g \in S'$ and $\widehat{g} \in L^1(\mathbb{R}^n)$. By Lemma 4.1.9, $\|g\|_Z = \|g\|_I < \infty$. Hence $g \in Z(\mathbb{R}^n)$. $\blacksquare$

The following density result concerning $Z(\mathbb{R}^n)$ can be found in [20].

Lemma 4.1.11 *The set $\{g \in Z(\mathbb{R}^n) : \widehat{g} \in C_0^\infty(\mathbb{R}^n)\}$ is dense in $Z(\mathbb{R}^n)$.*

In order to show that $I(\mathbb{R}^n)$ is a dense subset of $Z(\mathbb{R}^n)$ we need to make some further assumptions on the weight function v. Specifically, we strengthen property (v1) and assume $v \in C(\mathbb{R}^n) \cap S'$. We also assume $1/v \in L^2_{loc}(\mathbb{R}^n)$.

Lemma 4.1.12 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying (v2)-(v4), $v \in C(\mathbb{R}^n) \cap S'$ and $1/v \in L^2_{loc}(\mathbb{R}^n)$. Let $\psi \in S'$ be such that $\widehat{\psi} = 1/v$. Let*

$$I(\mathbb{R}^n) = \{g : g = \psi * f, \text{ for some } f \in C_0^\infty(\mathbb{R}^n)\}.$$

Then $I(\mathbb{R}^n)$ is dense in $Z(\mathbb{R}^n)$.

Proof. By Lemma 4.1.10, $I(\mathbb{R}^n) \subset Z(\mathbb{R}^n)$. Take $g \in Z(\mathbb{R}^n)$. Lemma 4.1.11 allows us to assume that $\widehat{g} \in C_0^\infty(\mathbb{R}^n)$. Note that this implies $g \in S \subset L^2(\mathbb{R}^n)$. Since $v \in S'$, $\widehat{Bv} \in S'$. Hence, since $g \in S$, the convolution $f = \widehat{Bv} * g$ is well defined and $f \in S' \cap C^\infty(\mathbb{R}^n)$. By properties of distributions we then have

$$\widehat{f} = (\widehat{Bv} * g)\widehat{\ } = \widehat{g}\widehat{\widehat{Bv}} = \widehat{g}v.$$

Since $\widehat{g} \in C_0^\infty(\mathbb{R}^n)$ and $v \in C(\mathbb{R}^n)$, it follows that $\widehat{f} \in C_0(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Hence, $f \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Since $v = 1/\widehat{\psi}$ we can write $\widehat{g} = (\widehat{g}/\widehat{\psi})\widehat{\psi} = \widehat{f}\widehat{\psi}$.

For $n \in \mathbb{N}$ we can define $\theta_n \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \theta_n(x) \leq 1$ for all $x \in \mathbb{R}^n$, and

$$\theta_n(x) = \begin{cases} 1, & |x| < n \\ 0, & |x| \geq n+1 \end{cases}.$$

Define $f_n = f\theta_n$, then, since $f \in C^\infty(\mathbb{R}^n)$, $f_n \in C_0^\infty(\mathbb{R}^n)$. Let $g_n = \psi * f_n$. Then $g_n \in I(\mathbb{R}^n)$, and, by Lemma 4.1.6, $\widehat{g_n} = \widehat{f_n}\widehat{\psi}$. Since $\widehat{\psi} \in L^2_{loc}(\mathbb{R}^n)$ and $f_n \in C_0^\infty(\mathbb{R}^n)$, we have for some compact set K,

$$\int_{\mathbb{R}^n} |\widehat{g_n}(x)|^2\, dx = \int_{\mathbb{R}^n} |\widehat{f_n}(x)\widehat{\psi}(x)|^2\, dx \leq \sup_{x \in \mathbb{R}^n} |f_n(x)| \int_K |\widehat{\psi}(x)|^2\, dx < \infty.$$

Thus $\widehat{g_n} \in L^2(\mathbb{R}^n)$ and it follows that $g_n \in L^2(\mathbb{R}^n)$.

We now consider the function $g - g_n$. Firstly,

$$(g - g_n)\widehat{\ } = \widehat{g} - \widehat{g_n} = \widehat{f}\widehat{\psi} - \widehat{f_n}\widehat{\psi} = (\widehat{f} - \widehat{f_n})\widehat{\psi} = (f - f_n)\widehat{\ }\widehat{\psi}.$$

Now,

$$\|g - g_n\|_Z = \int_{\mathbb{R}^n} |(g - g_n)\widehat{\ }(x)|^2 v(x)\, dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \overline{\widehat{(g-g_n)}}(x)\widehat{(g-g_n)}(x)v(x)\,dx \\
&= \int_{\mathbb{R}^n} \overline{\widehat{(g-g_n)}}(x)\widehat{(f-f_n)}(x)\widehat{\psi}v(x)\,dx \\
&= \int_{\mathbb{R}^n} \overline{\widehat{(g-g_n)}}(x)\widehat{(f-f_n)}(x)\,dx.
\end{aligned}
$$

The final equality is due to the fact that $\widehat{\psi}v = 1$. Since f, f_n, g and g_n are in $L^2(\mathbb{R}^n)$, it follows that $f - f_n$ and $g - g_n$ are in $L^2(\mathbb{R}^n)$. Thus, we can apply Parseval's Theorem to obtain,

$$
\begin{aligned}
\|g-g_n\|_Z &= \int_{\mathbb{R}^n} \overline{(g-g_n)}(x)(f-f_n)(x)\,dx \\
&= \left|\int_{\mathbb{R}^n} \overline{(g-g_n)}(x)(f-f_n)(x)\,dx\right| \\
&\leq \int_{\mathbb{R}^n} |(g-g_n)(x)||(f-f\theta_n)(x)|\,dx \\
&= \int_{|x|\geq n} |((g-g_n)f)(x)||(1-\theta_n)(x)|\,dx \\
&\leq \int_{|x|\geq n} |((g-g_n)f)(x)|\,dx.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality,

$$
\int_{|x|\geq n} |((g-g_n)f)(x)|\,dx \leq \left(\int_{|x|\geq n} |(g-g_n)(x)|^2\,dx\right)^{1/2} \left(\int_{|x|\geq n} |f(x)|^2\,dx\right)^{1/2} < \infty,
$$

since g, g_n and f are in $L^2(\mathbb{R}^n)$. Thus $(g - g_n)f \in L^1(\mathbb{R}^n)$ and

$$
\int_{|x|\geq n} |((g-g_n)f)(x)|\,dx
$$

tends to zero as n tends to infinity. Hence $\|g - g_n\|_Z$ tends to zero as n tends to infinity and $I(\mathbb{R}^n)$ is dense in $Z(\mathbb{R}^n)$. ■

4.2 Spaces on Domains in $\mathbb{R}^n$

We turn now to spaces defined on a domain $\Omega \subset \mathbb{R}^n$. We begin with a discussion of local native spaces. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be continuous and positive definite. As in the previous section we define

$$I(\mathbb{R}^n) = \{g : g = \psi * f \text{ for some } f \in C_0^\infty(\mathbb{R}^n)\}.$$

A norm on $I(\mathbb{R}^n)$ is defined for $g = \psi * f$, with $f \in C_0^\infty(\mathbb{R}^n)$, by

$$\|g\|_I^2 = \int_{\mathbb{R}^n} g(x)\overline{f(x)}\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x-y)f(y)\overline{f(x)}\, dydx.$$

The form of the above definitions delivers a very natural way of defining a local space $I(\Omega)$ and a local norm $\|\cdot\|_{I(\Omega)}$. We define

$$I(\Omega) = \{g|_\Omega \ : g = f * \psi \text{ for some } f \in C_0^\infty(\mathbb{R}^n) \text{ with support in } \Omega\}.$$

For each $g = f * \psi \in I(\Omega)$, with $f \in C_0^\infty(\mathbb{R}^n)$ having support in Ω, let

$$\|g\|_{I(\Omega)}^2 = \int_\Omega g(x)\overline{f(x)}\, dx = \int_\Omega\int_\Omega \psi(x-y)f(y)\overline{f(x)}\, dydx.$$

The local native space of Iske [16] is obtained by taking the closure of $I(\Omega)$ with respect to $\|\cdot\|_{I(\Omega)}$. Note that here we have chosen to assume ψ to be positive definite in order to simplify the arguments that follow. One could also work with a conditionally positive definite function.

An interesting property of this definition of the native space it that it is straightforward to define an extension operator from $I(\Omega)$ to $I(\mathbb{R}^n)$. Suppose $g \in I(\Omega)$, then $g = (f * \psi)\,|_\Omega$ for some $f \in C_0^\infty(\mathbb{R}^n)$ with support in Ω. We can extend g to a function, $Eg \in I(\mathbb{R}^n)$

simply by taking $Eg = f * \psi$. Two properties of this extension are in stark contrast to our own extension theorems of Chapter 2. Firstly, this extension makes no demands on the domain Ω. Our own extensions required a certain degree of smoothness of the boundary of Ω. Secondly, since the definition of the norm $\|\cdot\|_{I,\Omega}$ exploits the fact that f is compactly supported on Ω, we have

$$\begin{aligned} \|g\|_{I,\Omega}^2 &= \int_\Omega \int_\Omega \psi(x-y) f(x) f(y)\, dxdy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(x-y) f(x) f(y)\, dxdy \\ &= \|Eg\|_I^2. \end{aligned}$$

Thus the extension is isometric. Our own extension operators certainly do not enjoy this property.

Clearly the existence of isometric extensions from the local to the global native space, regardless of the domain Ω, is due the chosen definition of the local native space and local norm. We examine now how this compares with other approaches to localisation, beginning with the particular example of Sobolev spaces. Define

$$W^{m,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n), 0 \le |\alpha| \le m\}.$$

Of particular interest is $W^{m,2}(\mathbb{R}^n)$, $m > n/2$, since it can be shown that this is the native space for the positive definite function $\psi(r) = r^{m-n/2} K_{m-n/2}(r)$ (Schaback [34]). Here K is a Bessel or MacDonald function, defined by

$$K_v(z) = \frac{\pi^{1/2}(z/2)^v}{\Gamma(v+1/2)} \int_1^\infty e^{-zt}(t^2-1)^{v-1/2}\, dt,$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$. Now, classically, one defines the local Sobolev space as

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq |\alpha| \leq m\}.$$

We shall show that, for certain domains Ω, this does not correspond to the localised native space. For, if Ω has a suitably irregular boundary, then there exist no extensions from $W^{m,2}(\Omega)$ to $W^{m,2}(\mathbb{R}^n)$. This is in contrast to the local native space for which extensions exist irrespective of the particular domain Ω.

Definition 4.2.1 *Let Ω be a domain in $\mathbb{R}^n$ and let x_0 be a point on its boundary. Let $B(r, x_0)$ be the open ball of radius r centre x_0 with boundary $\partial B(r, x_0)$. Let $S_r = \partial B(r, x_0) \cap \Omega$ and $A(S_r)$ be the surface area of S_r. We say that Ω has an exponential cusp at x_0 if for every real number k,*

$$\lim_{r \to 0+} \frac{A(S_r)}{r^k} = 0.$$

The following theorem can be found in [1, p. 122].

Theorem 4.2.2 *Let Ω be a domain in $\mathbb{R}^n$ with an exponential cusp. If $q > p$ then there exists $u \in W^{m,p}(\Omega)$ such that $u \notin L^q(\Omega)$.*

Lemma 4.2.3 *Let $m > n/2$ and Ω be a bounded domain in $\mathbb{R}^n$. Then $W^{m,2}(\mathbb{R}^n) |_\Omega \subset L^q(\Omega)$ for all $q \geq 1$.*

Proof. If $m > n/2$ then by the Sobolev Imbedding Theorem (see [1, p. 97]), $W^{m,2}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$. Thus, if $v \in W^{m,2}(\mathbb{R}^n)$ then $v \in L^q_{loc}(\mathbb{R}^n)$ for all $q \geq 1$. Since Ω is bounded it follows that $v |_\Omega \in L^q(\Omega)$. ∎

Lemma 4.2.4 *Let $m > n/2$ and Ω be a bounded domain in $\mathbb{R}^n$ having an exponential cusp. Then there exists $u \in W^{m,2}(\Omega)$ with $u \notin W^{m,2}(\mathbb{R}^n)\mid_\Omega$.*

Proof. From Theorem 4.2.2 there exists a $u \in W^{m,2}(\Omega)$ such that $u \notin L^q(\Omega)$ for $q > 2$. By Lemma 4.2.3 it follows that $u \notin W^{m,2}(\mathbb{R}^n)\mid_\Omega$. ∎

We return now to the spaces of distributions discussed in the previous section. Thus we have a measurable function $v : \mathbb{R}^n \to \mathbb{R}$ and we define

$$Z(\mathbb{R}^n) = \{g \in S' : \widehat{f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\widehat{g}(x)|^2 v(x)\, dx < \infty\}.$$

A norm is defined on $Z(\mathbb{R}^n)$ by,

$$\|g\|_Z = \left(\int_{\mathbb{R}^n} |\widehat{g}(x)|^2 v(x)\, dx\right)^{1/2}, \qquad \text{for } g \in Z(\mathbb{R}^n).$$

Any immediate attempt to define a corresponding local space or norm is hampered by the inclusion of the Fourier transform in the definition of $\|\cdot\|_Z$ and $Z(\mathbb{R}^n)$; if a function is only defined on a domain Ω then we are unable to compute its Fourier transform. As discussed in Section 1.5, a solution to this problem can be found in Levesley and Light [18]. We suppose that v satisfies the following conditions:

(A1) $v \in S' \cap C(\mathbb{R}^n)$ and $v(x) > 0$ for almost all $x \in \mathbb{R}^n$,

(A2) $\widehat{v}$ is a measurable function and, for any neighbourhood N of the origin, $\widehat{v} \in L^1(\mathbb{R}^n \backslash N)$,

(A3) $v(y) = v(-y)$ for all $y \in \mathbb{R}^n$,

(A4) $|\widehat{v}(y)| = \mathcal{O}(|y|^s)$ as $y \to 0$, where $n + s + 2 > 0$, (A5) $v(0) = 0$ and $\widehat{v}(x) \leq 0$ for almost all $x \in \mathbb{R}^n$.

The main result of Section 3 of [18] is that, provided v satisfies the above conditions, then

$$\|g\|_Z = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w(x-y)|g(x) - g(y)|^2 \, dxdy \qquad \text{for all } g \in Z(\mathbb{R}^n),$$

where $w = -\widehat{v}/2$. This gives us a natural way of defining a local norm. Given $\Omega \subset \mathbb{R}^n$ we define for $g : \Omega \to \mathbb{R}^n$,

$$\|g\|_\Omega^2 = \int_\Omega \int_\Omega w(x-y)|g(x) - g(y)|^2 \, dxdy.$$

There are many ways of constructing a local space $Z(\Omega)$, although we clearly wish to consider functions g for which $\|g\|_\Omega < \infty$. In Chapter 2 we chose to mirror the construction of local Sobolev spaces. Specifically, we defined $Z(\Omega)$ to be the completion of the set $\{g\,|_\Omega \colon g \in C_0(\mathbb{R}^n) \text{ and } \|g\|_\Omega < \infty\}$ with respect to $\|\cdot\|_\Omega$. This made the construction of extension theorems for $Z(\Omega)$ simpler, as we effectively were working with a subset of the continuous functions.

An important feature of the Iske local native spaces and norms was that they lead to isometric extension theorems. We demonstrate now that however one chooses to construct the local space $Z(\Omega)$, the form of the local norm $\|\cdot\|_\Omega$ makes it difficult to construct an isometric extension theorem. Suppose that $f : \Omega \to \mathbb{R}^n$ is such that $\|f\|_\Omega < \infty$. Let Ef be any extension of f to the whole of $\mathbb{R}^n$. Now, clearly

$$\int_\Omega \int_\Omega w(x-y)|g(x) - g(y)|^2 \, dxdy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w(x-y)|Ef(x) - Ef(y)|^2 \, dxdy. \tag{4.1}$$

Let $z(x,y) = w(x-y)|Ef(x) - Ef(y)|^2$ for $x, y \in \mathbb{R}^n$. Then we can write,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} z(x,y) \, dxdy = \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} z(x,y) \, dxdy + 2 \int_{\mathbb{R}^n \setminus \Omega} \int_\Omega z(x,y) \, dxdy$$

$$+\int_{\Omega}\int_{\Omega} z(x,y)\ dxdy.$$

In order to get equality in Equation (4.1), we would require the contributions of integrals over $(\mathbb{R}^n \backslash \Omega) \times (\mathbb{R}^n \backslash \Omega)$ and $(\mathbb{R}^n \backslash \Omega) \times \Omega$ to be zero. This is not necessarily true even in the simple case where Ef is obtained by setting f to be zero outside of Ω.

Chapter 5

Conclusions and Further Work

Many of the ideas discussed here have centred around questions of localisation. We began with the seminorms and associated spaces of distributions introduced by Light and Wayne [20] which arise naturally in the study of interpolation by translates of a basic function. These seminorms are defined in an indirect form, that is in terms of the Fourier transform of the function rather than the function itself, and thus do not lend themselves to localisation. To overcome this we use the direct form of such seminorms given by Light and Levesley in [18] since this delivers a very natural way of defining a local seminorm. For $\Omega \subset \mathbb{R}^n$ and fixed $k \in \mathbb{Z}_+$ we define

$$|f|_\Omega = \left(\sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y) |(D^\alpha f)(x) - (D^\alpha f)(y)|^2 \, dxdy \right)^{1/2} .$$

Recall that here w is a measurable weight function and the c_α are known constants.

Using this form of local seminorm we can construct associated local spaces. We chose to consider the spaces $X(\Omega) = \{g|_\Omega : g \in C_0^k(\mathbb{R}^n) \text{ and } |g|_\Omega < \infty\}$ and $\mathcal{Y}(\Omega)$, the completion of $X(\Omega)$ with respect to the seminorm $|\cdot|_\Omega$. In Chapter 2 we proved certain extension theorems for these spaces. In particular, we constructed a bounded, linear extension operator from $\mathcal{Y}(\Omega)$ to $\mathcal{Y}(\mathbb{R}^n)$, subject to restrictions on the domain Ω and the weight function w. The motivation for the development of such extensions was their use in obtaining improved error estimates, and in Chapter 3 we gave a demonstration of this application. Specifically, by adapting work by Duchon [7], we obtained improved L_2 estimates for the spaces $\mathcal{Y}(\mathbb{R}^n)$ in terms of the spacing of the interpolation points.

Finally, in Chapter 5 we began a discussion of how this approach to localisation compares with alternatives. We looked at native spaces and, in particular, the convolutional characterisation given by Iske [16]. Here, the local space was simply the restriction of the global space to the domain in question. On the whole of $\mathbb{R}^n$ we saw that, under the right circumstances, Iske's spaces coincide with the spaces of distributions introduced by Light and Wayne [20]. However, when looking at the localisation of these spaces things are not so clear. In the particular case of Sobolev spaces we saw that, for certain domains with sharp cusps, the Iske local space was smaller than the classical definition of a local Sobolev space. More generally, we might ask how the Levesley-Light local space, which adopts a classical approach to localisation, compares with that of Iske. As in the special case of Sobolev spaces, is the Iske space smaller for unfriendly domains? On the other hand, what properties must a domain possess in order for the native space approach to deliver the same space as a more classical approach? The definition of the local native space as a restriction space implies the existence of extension operators from the local to the global native space irrespective of the chosen domain. This is in contrast to the extension theorems we have developed for the Levesley-Light local spaces which rely on the domain having a certain degree of smoothness. Furthermore, Iske's extensions are norm-preserving whilst the form of the Levesley-Light local seminorm makes it difficult for any associated extension operator to be isometric. Clearly there are some questions on the relationships between these two approaches to localisation which remain a subject for further discussion.

Other questions arising from this work are those of optimal local approximation orders in L_p. That is, can the error estimates achieved in Chapter 3 be improved? Existing work in this area, including that of Wendland [37], suggests that in the case of thin plate splines, for example, the answer is no. However, work by Gutzmer and Iske [12], suggests improvements may be obtainable if, say, one considers functions with some added degree of smoothness. Thus, this is another area which would benefit from further study.

Bibliography

[1] R. A. Adams, *Sobolev spaces*, Academic Press (New York), 1978.

[2] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley (London), 1974.

[3] M. Atteia, *Fonctions spline et noyaux reproduisants d'Aronszayn-Bergman* , R.I.R.O. 4, no. R-3, 1970, pp. 31–43.

[4] W. Cheney and W. Light, *A course in Approximation Theory*, Brooks/Cole Publishing Company (Pacific Grove), 2000.

[5] J. Deny and J. L. Lions, *Les espaces du type de Beppo Levi* , Ann. Inst. Fourier 5, 1954, pp. 305–370.

[6] J. Duchon, *Splines minimizing rotation-invariant seminorms in Sobolov spaces*, in *Constructive Theory of Functions of Several Variables,* Lecture Notes in Mathematics, Vol. 571, eds. W. Schempp and K. Zeller, Springer-Verlag (Berlin), 1977, pp. 85–100.

[7] J. Duchon, *Sur l'erreur d'interpolation des fonctions de plusieurs variables par les* D^m*-splines,* RAIRO Analyse Numerique 12 (4), 1978, pp. 325–334.

[8] N. Dyn, *Interpolation of Scattered Data by Radial Functions*, in *Topics in Multivariate Approximation*, eds. C. K. Chui, L. L. Schumaker and F. L. Utreras, Academic Press (New York), 1987, pp. 47–61.

[9] N. Dyn, *Interpolation and approximation by radial and related functions*, in *Approximation Theory VI: Volume 1*, eds. C. K. Chui, L. L. Schumaker, J. D. Ward, Academic Press (New York), 1989, pp.211–134.

[10] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized functions*, Vol. 4, Academic Press (New York), 1964.

[11] M. Golomb and H. F. Weinberger, *Optimal approximation and error bounds,* in *On Numerical Approximation*, ed. R. E. Langer, University of Wisconsin Press (Madison), 1959, pp. 117–190.

[12] T. Gutzmer and A. Iske, *Detection of Discontinuities in Scattered Data Approximation,* Numerical Algoritms 16:2, 1997, pp 155–170.

[13] R. L. Harder and R. N. Desmarais, *Interpolation using surface splines,* J. Aircraft 9, 1972, pp. 189–191.

[14] R. L. Hardy, *Multiquadric equations of topography and other irregular surfaces*, J. Geophys. Res. C, 1971, pp. 1905–1915.

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer Verlag (Berlin), 1983.

[16] A. Iske, *Characterisation of function spaces associated with conditionally positive definite functions*, in *Mathematical Methods for Curves and Surfaces*, eds. M. Daehlen, T. Lyche, L.L. Schumaker, Vanderbilt University Press (Nashville), 1995, pp. 265-270.

[17] G. J. O. Jameson, *Topology and Normed Spaces*, Chapman and Hall (London), 1974.

[18] J. Levesley and W. Light, *Direct forms for seminorms arising in the theory of interpolation by translates of a basis function*, Advances in Comp. Math. 11, 1999, pp. 161–182.

[19] W. Light and H. Wayne, *On power functions and error estimates for radial basis function interpolation*, J. Approx. Th. 92 (2), 1998, pp. 245–266.

[20] W. Light and H. Wayne, *Spaces of distributions, interpolation by translates of a basis function and error estimates*, Numer. Math. 81, 1999, pp. 414–450.

[21] W. Madych and S. Nelson, *Multivariate interpolation: a variational theory*, preprint, 1983.

[22] W. Madych and S. Nelson, *Multivariate interpolation and conditionally positive definite functions*, Approx. Theory Appl. 4, 1988, pp. 77–89.

[23] W. Madych and S. Nelson, *Multivariate interpolation and conditionally positive definite functions II*, Math. Comp. 54, 1990, pp. 210–230.

[24] J. C. Mairhuber *On Haar's theorem concerning Chebychev approximation problems having unique solutions*, PAMS 7, 1956, pp.609–615.

[25] E. McShane and T. Botts, *Real Analysis*, D. Van Nostrand Company Inc. (Princeton), 1959.

[26] C. A. Micchelli, *Interpolation of scattered data: distance matrices and conditionally positive definite functions*, Constr. Approx. 2, 1986, pp. 11–22.

[27] J. T. Oden and J. N. Reddy, *An introduction to the mathematical theory of finite elements,* Wiley (New York), 1976.

[28] M. J. D. Powell, *Radial Basis Functions in 1990*, in *Advances in Numerical Analysis Volume 2 - Wavelets, subdivision algorithms and radial basis functions,* ed. W.A. Light, Oxford University Press (Oxford), 1992, pp.105–210.

[29] W. Rudin, *Real and Complex Analysis,* McGraw-Hill (New York), 1987.

[30] W. Rudin, *Functional Analysis,* McGraw-Hill (New York), 1991.

[31] R. Schaback, *Multivariate interpolation and approximation by translates of a basis function*, in *Approximation Theory VIII - Vol. 1 : Approximation and Interpolation*, eds. C. K. Chui, L. L. Schumaker, World Scientific Publishing Co. (Singapore), 1995, pp. 491–514.

[32] R. Schaback, *Native Hilbert spaces for radial basis functions I*, in *New developments in Approximation Theory*, Internat. Ser. Numer. Math. **132**, Birkhäuser, Basel, 1999, pp. 255–282.

[33] R. Schaback, *Comparison of Radial Basis Function Interpolants* , in *Multivariate approximation - from CAGD to wavelets*, eds. K. Jetter, F. Utreras, World Scientific (London), 1993, pp. 293–305.

[34] R. Schaback, *Reconstruction of Multivariate Functions from Scattered Data*, preprint.

[35] I. J. Schoenberg, *Metric spaces and completely monotone functions*, Ann. of Math. 39, 1938, pp. 811–841.

[36] R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman (London), 1977.

[37] H. Wendland, *Optimal Approximation Orders in L_p for Radial Basis Functions*, East Journal on Approximations, 6:1, 2000, pp. 1–6.

[38] J. Wloka, *Partial Differential Equations,* Cambridge University Press (Cambridge), 1987.

[39] Z. Wu and R. Schaback, *Local error estimates for radial basis function interpolation to scattered data*, IMA J. Numer. Anal 13, 1993, pp. 13–27.