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On Finite Dimensional Jacobian Algebras

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Abstract We show that Jacobian algebras arising from every tagged triangulation of a sphere with n-punctures, with $n \geq 5$, are finite dimensional algebras. We consider also a family of cyclically oriented quivers and we prove that, for any primitive potential, the associated Jacobian algebra is finite dimensional.

Keywords Jacobian algebras · closed surfaces · cyclically oriented quivers

Mathematics Subject Classification (2000) MSC 16G20 · MSC 13F60 · MSC 57M20 · MSC 16T30

1 Introduction

Let k be an algebraically closed field. A potential W for a quiver Q is a possibly infinite linear combination of cyclic paths in the complete path algebra $k\langle\langle Q\rangle\rangle$. The Jacobian algebra $\mathcal{P}(Q,W)$ associated to a quiver with a potential (Q,W) is the quotient of the complete path algebra $k\langle\langle Q\rangle\rangle$ modulo the Jacobian ideal $J(W)$. Here, $J(W)$ is the topological closure of the ideal of $k\langle\langle Q\rangle\rangle$ which is generated by the cyclic derivatives of W with respect to the arrows of Q.

Quivers with potential were introduced in[DWZ08] in order to construct additive categorifications of cluster algebras with skew-symmetric exchange matrix. For the just mentioned categorification it is crucial that the potential for Q be non-degenerate, i.e. that it can be mutated along with the quiver arbitrarily, see [DWZ08] for more details on quivers with potentials.

In the same year, Fomin, Shapiro and Thurston gave in [FST08] a class of cluster algebras arising from tagged triangulations of surfaces with marked points. More precisely, each triangulation $\mathbb{T}$ of a surface with marked points (S,M) by tagged arcs corresponds to a cluster and the corresponding exchange matrix is conveniently coded into a quiver $Q(\mathbb{T})$. Later, a link between these papers was established by Labardini-Fragoso in [LF09] where he associated a potential $W(\mathbb{T})$ to every ideal

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triangulation $\mathbb{T}$ of a surface with marked points, and proved that ideal triangulations related by a flip have QPs related by QP-mutation and if the surface has non-empty boundary, then the corresponding Jacobian algebra is finite dimensional. Moreover, it was defined a non-degenerate potential $W(\mathbb{T})$ to every *tagged* triangulation and it is also compatible with *flips of tagged arcs* (see [LF][Theorem 7.1] and [GLFS][Corollary 8.15]).

In the first part of this work, we study Jacobian algebras associated with tagged triangulations of a sphere with $n \geq 5$ punctures. Our main result is the following:

Theorem 1 *Let (S, M) be a sphere with n-punctures, where $n \geq 5$. For every tagged triangulation $\mathbb{T}$ of (S, M), the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is finite dimensional.*

The case of a sphere with 4 punctures was studied by Barot and Geiss (in [BG12], Section 5), the algebra associated to this surface is a tubular cluster algebra.

In the theory of cluster algebras, primitive potentials, which are a linear combination of all the oriented chordless cycles in a quiver Q, appear in many contexts, for example in cluster tilted algebras of Dynkin type ([DWZ08], Section 9). Also, it follows from [BT13] that cluster tilted algebras with cyclically oriented quivers have a primitive potential (see definition of a cluster tilted algebra in Section 4).

In the proof of our main Theorem 1, we use a particular ideal triangulation $\mathbb{T}$ of a sphere with n-punctures such that the quiver associated to $\mathbb{T}$ is cyclically oriented but its associated potential is not primitive.

In the second part of this work, we give a class of cyclically oriented quivers such that any primitive potential induces a finite dimensional Jacobian algebra.

The paper is organized as follows: In Section 2, we recall some definitions of quivers with potentials, path algebras, Jacobian algebras and ideal (tagged) triangulations of surfaces. In Section 3, we prove that every Jacobian algebra associated with a tagged triangulation of a sphere with $n \geq 5$ punctures are finite dimensional. Finally, in Section 4, we give a combinatorial description of a quiver Q such that any of its primitive potentials induce a finite dimensional Jacobian algebra.

Remark 1 While we were finishing this manuscript, we became aware of the recent paper [Lad], where Ladkani showed that Jacobian algebras of surfaces with an empty boundary and arbitrary particular genus are finite dimensional algebras.

2 Preliminaries

2.1 Quivers and potentials

In this subsection, we fix notations for path algebras and complete path algebras, and recall basic definitions of quivers with potential (cf. [DWZ08]).

Let Q be a finite quiver and k be a field. We denote by R the k-vector space k^{Q_0}, by A the k-vector space k^{Q_1} and, for each nonnegative integer d by A^d the R-bimodule $\underbrace{A \otimes_R \cdots \otimes_R A}_{d}$.

With this notation, the *path algebra* of Q is the k-algebra defined as the (graded) tensor algebra

$$k\langle Q\rangle = \bigoplus_{d=0}^{\infty} A^d$$

and the *complete path algebra* of Q is the k-vector space defined by

$$k\langle\langle Q\rangle\rangle = \prod_{d=0}^{\infty} A^d.$$

Also, $k\langle\langle Q\rangle\rangle$ is a topological k-algebra with the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the ideal $\prod_{d=1}^{\infty} A^d$.

Remark 2 An important topological property of $k\langle\langle Q\rangle\rangle$ with the $\mathfrak{m}$-adic topology, for this work, is the following:

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $k\langle\langle Q\rangle\rangle$ converges if and only if for every $d \geq 0$, the sequence $(x_n^{(d)})_{n\in\mathbb{N}}$ does, and

$$\lim_{n\to\infty} x_n = \sum_{d\geq 0} \lim_{n\to\infty} x_n^{(d)},$$

where $x_n^{(d)}$ denotes the degree-d component of x_n.

Remark 3 Throughout this work, we will use the following notation for paths. Let Q be a finite quiver. For an arrow α of Q, we denote by $s(\alpha)$ its source and by $t(\alpha)$ its target. A path of length n, with $n > 0$, from a source a to a target c is a sequence of arrows $(a \mid \alpha_1, \alpha_2, \ldots, \alpha_n \mid c)$ with α_i an arrow of Q for all i, where $1 \leq i \leq n$, such that $s(\alpha_1) = a$, $t(\alpha_n) = c$ and $t(\alpha_i) = s(\alpha_{i+1})$ for all i such that $i \leq i < n$. Such a path is briefly denoted by $\alpha_1\alpha_2\ldots\alpha_n$.

Notice that the elements of $k\langle\langle Q\rangle\rangle$ are (possibly infinite) k-linear combinations of paths in Q.

Denote by $k\langle\langle Q\rangle\rangle_{\text{cyc}}$ the k-subspace of $k\langle\langle Q\rangle\rangle$ whose element are k-linear combinations of cycles in Q.

Definition 1 [DWZ08, Definition 3.1]

- A potential W is any element of the k-subspace $k\langle\langle Q\rangle\rangle_{\text{cyc}}$.
- For every arrow a in Q_1, we define the *cyclic derivative* ∂_a as the continuous k-linear map

  $$k\langle\langle Q\rangle\rangle_{\text{cyc}} \to k\langle\langle Q\rangle\rangle$$

  acting on paths by

  $$\partial_a(a_1\cdots a_d) = \sum_{k=1}^{d} \delta_{aa_k} a_{k+1}\cdots a_d a_1\cdots a_{k-1}$$

- The Jacobian ideal $J(W)$ of a potential W is the closure of the ideal

  $$\mathrm{I}(W) = \langle \partial_a(W) \mid a \in Q_1\rangle$$

  in $k\langle\langle Q\rangle\rangle$.
- The Jacobian algebra $\mathcal{P}(Q, W)$ is the quotient $k\langle\langle Q\rangle\rangle/J(W)$.

2.2 Triangulations of surfaces

In this subsection, we review some facts concerning triangulations of surfaces (cf. [FST08]).

Definition 2 [FST08, Definition 2.1] A *bordered surface with marked points* is a pair (S, M), where S is a connected oriented 2-dimensional Riemann surface with a (possibly empty) boundary and M is a finite and non-empty set of points in S, called *marked points*, such that there is at least one marked point on each connected component of the boundary of S.

The set P of marked points in the interior of S are called *punctures*.

In this paper, we study spheres with n punctures, $n \geq 5$, due to the following definitions, in order to avoid surfaces that cannot be triangulated or there is only one triangulation, we need to exclude:

– spheres with one or two punctures;
– unpunctured or once-punctured monogons;
– unpunctured digons; and
– unpunctured triangles

Definition 3 [FST08, Definition 2.2 and 2.4] A (simple) *arc* γ in (S,M) is a curve in S such that:

– the endpoints of γ are marked points in M;
– γ does not intersect itself, except that its endpoints may coincide;
– γ is not contractible into M or into the boundary of S;
– γ does not cut out an unpunctured monogon or an unpunctured digon.

Two arcs are *compatible* if there are arcs in their respective isotopy classes whose relative interiors do not intersect.

An arc whose endpoints coincide is called a *loop*.

Definition 4 [FST08, Definition 2.6] An *ideal triangulation* of (S,M) is any maximal collection of pairwise compatible arcs whose relative interiors do not intersect each other.

The arcs of the triangulation cut the surface S into *ideal triangles*. The three sides of an ideal triangle do not have to be distinct, i.e., we allow *self-folded* triangles.

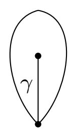

Fig. 1: Self-folded ideal triangle

An easy calculation shows that any ideal triangulation of a sphere with n punctured consists of $3n$ arcs.

3 Jacobian algebras arising from a sphere with n-punctures

The algebra arising from a sphere with punctures was studied for first time by Barot and Geiss in [BG12]. They prove that the tubular cluster algebra of type (2,2,2,2) corresponds to a sphere with 4-punctures (see definition of a cluster tilted algebra in Section 4). In this section, we study the Jacobian algebras arising from a sphere with n-punctures, where $n \geq 5$.

It is well known that finite-dimensionality of Jacobian algebra, with non-degenerate potential, is preserved by mutations (see [DWZ08][Proposition 6.4]), if (S,M) is not a closed surface with exactly one puncture, then any two tagged triangulations of (S,M) are related by a sequence of flips (see [Mos88], [FST08][Proposition 7.10]) and that any two tagged triangulations related by a flip have quivers with potentials related by QP-mutation (see [LF], [GLFS]). Thus, in order to prove the Theorem 1, it is enough to prove that there exists a tagged triangulation $\mathbb{T}$ such that the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}),W(\mathbb{T}))$ is finite dimensional. For that reason, we give a particular ideal triangulation with that property. Recall that ideal triangulations correspond to tagged triangulations with non-negative signature.

Consider the ideal triangulation $\mathbb{T}$ and the quiver $Q(\mathbb{T})$ depicted in Figure 2. For notational convenience we label the punctures on the north and south poles with p_{n+1} and p_{n+2}, so we will consider the sphere with $(n+2)$-punctures, where $n \geq 3$. The labels we have assigned to the arrows in Figure 2 will be kept throughout the paper.

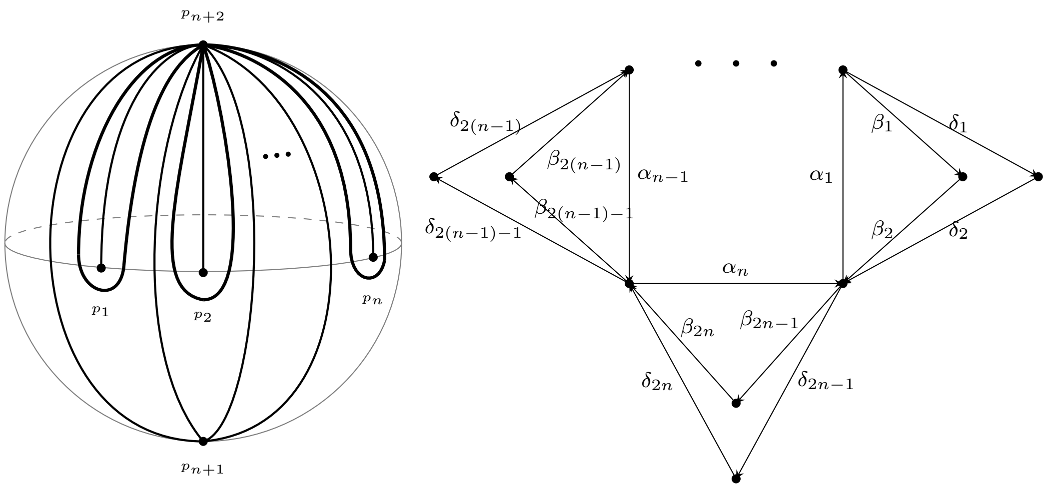

Fig. 2: Quiver associated with the ideal triangulation $\mathbb{T}$

For each puncture $p_i \in P$ in the sphere, we choose a non-zero scalar $x_i \in k$. By [LF09, Definition 23], the potential $W(\mathbb{T})$ associated with the ideal triangulation $\mathbb{T}$ and according to the label in the arrows of the quiver in Figure 2 is:

$$\begin{aligned} W(\mathbb{T}) &= x_{n+1}\alpha_1 \dots \alpha_n + x_{n+2}\delta_{2n-1}\delta_{2n} \dots \delta_1\delta_2 \\ &+ \textstyle\sum_{i=1}^{n} \alpha_i\beta_{2i-1}\beta_{2i} + \sum_{i=1}^{n}(-x_i{}^{-1})\alpha_i\delta_{2i-1}\delta_{2i} \end{aligned}$$

Remark 4 Observe that the quiver $Q(\mathbb{T})$ is the opposite quiver of the one Labardini-Fragoso works with.

Before we prove Theorem 1, we establish some useful identities in the Jacobian algebra $\mathcal{P}P(Q(\mathbb{T}), W(\mathbb{T}))$.

Lemma 1 *The following identities hold in the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$:*

$$\begin{aligned} \beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} &= x_{i-1}\beta_{2i-1}\beta_{2(i-1)}\beta_{2(i-1)-1}\beta_{2(i-1)} && (1) \\ &= (x_{i-1}/x_i)\delta_{2i-1}\delta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)} && (2) \\ &= x_i^{-1}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} && (3) \end{aligned}$$

for every $i = 1, \dots, n$

Proof Let Λ be the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$. Since

$$\partial_{\alpha_{i-1}}(W(\mathbb{T})) = \beta_{2(i-1)-1}\beta_{2(i-1)} + x_{n+1}\alpha_i \dots \alpha_{i-2} - x_{i-1}^{-1}\delta_{2(i-1)-1}\delta_{2(i-1)},$$

then in Λ we have the identity

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = x_{n+1}x_{i-1}\beta_{2i-1}\beta_{2i}\alpha_i \dots \alpha_{i-2}$$

$$+x_{i-1}\beta_{2i-1}\beta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}.$$

Observe that $\partial_{\beta_{2i-1}}(W(\mathbb{T})) = \beta_{2i}\alpha_i$, then the first term on the right hand is in the Jacobian ideal, therefore

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = x_{i-1}\beta_{2i-1}\beta_{2i}\beta_{2(i-1)}\beta_{2(i-1)-1}\beta_{2(i-1)}$$

This establishes the first identity. The second identity can be proved in a similar fashion and is left to the reader. Let us show the third identity. By the relations induced by $\partial_{\alpha_{i-1}}(W(\mathbb{T}))$, we have:

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = -x_{n+1}\alpha_{i+1}\ldots\alpha_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)} \\ +x_{i-1}^{-1}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$$

Denote by ρ the term $\alpha_{i+1}\ldots\alpha_{i-2}\alpha_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)}$. Notice that ρ is a path of length $n+1$. We claim that ρ is in the Jacobian ideal.

Using $\partial_{\delta_{2(i-1)}}(W(\mathbb{T}))$, we have the following identity

$$\alpha_{i-1}\delta_{2(i-1)-1} = x_{i-1}x_{n+2}\delta_{2(i-2)-1}\delta_{2(i-2)}\ldots\delta_1\delta_2\ldots\delta_{2(i-1)-1}.$$

Then, replacing $\alpha_{i-1}\delta_{2(i-1)-1}$ in ρ, we have

$$\rho = x_{n+2}x_{i-1}\alpha_{i+1}\ldots\alpha_{i-2}\delta_{2(i-2)-1}\delta_{2(i-2)}\ldots\delta_1\delta_2\ldots\delta_{2(i-1)-1}\delta_{2(i-1)}.$$

Observe that ρ is a path of length $3n$ in Λ.

Replacing $\delta_{2(i-2)-1}\delta_{2(i-2)}$ by the relation induced by $\partial_{\alpha_{i-2}}(W(\mathbb{T}))$, the path ρ is a path of length $4n-3$. Iterating this process and using the topology of the Jacobian algebra (see Remark 2), we have that ρ is in the Jacobian ideal.

Then, $\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = x_{i-1}^{-1}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$. □

Lemma 2 *The following identities hold in the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$.*

$$\alpha_i\delta_{2i-1}\delta_{2i} = x_ix_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)}\alpha_{i-1} \tag{4}$$
$$\alpha_i\alpha_{i+1}\delta_{2(i+1)-1} = \delta_{2(i-1)}\alpha_{i-1}\alpha_i = 0 \tag{5}$$

for every $i = 1,\ldots,n$

Proof The identity (4) follows as the first two identities in Lemma 1. We prove the second one.

Notice that

$$\partial_{\delta_{2(i+1)}}(W(\mathbb{T})) = -x_{i+1}^{-1}\alpha_{(i+1)}\delta_{2(i+1)-1} + x_{n+2}\delta_{2(i)-1}\delta_{2i}\ldots\delta_{2(i+1)-1},$$

then we have the identity

$$\alpha_i\alpha_{i+1}\delta_{2(i+1)-1} = x_{i+1}x_{n+2}\alpha_i\delta_{2i-1}\delta_{2i}\ldots\delta_{2(i+1)-1}\delta_{2(i+1)}.$$

Let ρ be the path $\alpha_i\delta_{2i-1}\delta_{2i}\ldots\delta_{2(i+1)-1}\delta_{2(i+1)}$ By the identity (3) in Lemma 1 we have that ρ is equal to

$$\alpha_i\beta_{2(i)-1}\beta_{2i}\delta_{2(i-1)-1}\ldots\delta_{2(i+1)},$$

which is in the Jacobian ideal because it contains a factor $\alpha_i\beta_{2i-1} = \partial_{\beta_{2i}}(W(\mathbb{T}))$.

Then $\alpha_i\alpha_{i+1}\delta_{2(i+1)-1} = 0$ in $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$. □

Lemma 3 *Let ρ be a non zero path of length 5 that starts in α_i or δ_{2i-1}. If ρ does not involve any arrow β_i for $i = 1,\ldots,2n$, then ρ is either the path $\alpha_i\ldots\alpha_{i+4}$ or $\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$ in $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$.*

Proof First suppose ρ starts with δ_{2i-1}. Then ρ is one of the following:

- $\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}\delta_{2(i-2)-1}$ or
- $\delta_{2i-1}\delta_{2i}\alpha_i\delta_{2i-1}\delta_{2i} = x_ix_{i-1}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}\alpha_{i-1}$ or
- $\delta_{2i-1}\delta_{2i}\alpha_i\alpha_{i+1}x$, where $x = \alpha_{i+2}$ or $\delta_{2(i+1)-1}$

By Lemma 1, the second option is zero, and by Lemma 2, the third option is zero. Then $\rho = \delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}\delta_{2(i-2)-1}$

Now suppose ρ starts with α_i, then ρ is one of the following:

– $\alpha_i\delta_{2i-1}\delta_{2i}\alpha_i\delta_{2i-1}$ or
– $\alpha_i\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$ or
– $\alpha_i \dots \alpha_{i+4}$

By the first part of Lemma 2 the first one is equal to

$$x_i x_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)}\alpha_{i-1}\alpha_i\delta_{2i-1},$$

and by the second part of the same Lemma, that path is zero. Finally, the path $\alpha_i\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$ by Lemma 1 is equal to

$$x_i\alpha_i\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)},$$

there force this path is also zero. Then $\rho = \alpha_i \dots \alpha_{i+4}$ □

Remark 5 Notice that, since $\alpha_i\beta_{2i-1} = \partial_{\beta_{2i}}(W(\mathbb{T}))$ and $\beta_{2i}\alpha_i = \partial_{\beta_{2i-1}}(W(\mathbb{T}))$, every path in which a path $\alpha_i\beta_{2i-1}$ and $\beta_{2i}\alpha_i$ appear as a factor is zero in $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$. Moreover, every path in which at least an arrow α_i and at least an arrow β_i appear as factors is zero. The last assertion follows from applying repeatedly Lemmas 1 and 2 until we obtain a factor $\alpha_i\beta_{2i-1}$ or $\beta_{2i}\alpha_i$. For example, the path $\delta_{2i-1}\delta_{2i}\alpha_i\delta_{2i-1}\delta_{2i}$ is equivalent to $x_i x_{i-1}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}\alpha_{i-1}$ by Lemma 2, and it is equivalent to $x_i^2 x_{i-1}^2\beta_{2i-1}\beta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}\alpha_{i-1}$ by Lemma 1, which is a zero path.

Now, we can prove our main result.

Proof (Proof of Theorem 1) Since finite-dimensionality of Jacobian algebras is invariant under mutations (cf. [DWZ08, Corollary 6.6]), flips of tagged arcs are compatible with mutations of quivers with potentials (cf. [LF, Theorem 7.1] and [GLFS][Corollary 8.15]) and any two tagged triangulations of (S, M) are related by a sequence of flips (cf. [FST08][Proposition 7.10]), it is enough to show that $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is finite dimensional, where $\mathbb{T}$ is the triangulation in Figure 2. We shall prove that every path of length at least $2n+2$ belongs to the Jacobian ideal $J(W(\mathbb{T}))$.

Let ρ be a path of length at least $2n+2$. Without loss of generality we can assume that ρ starts with β_{2n-1} or δ_{2n-1}. Denote by Q_ρ the set of arrows of the path ρ. By Remark 5, it is enough to analyze when $Q_\rho \subset Q(\mathbb{T})_1 \setminus \{\alpha_1, \dots, \alpha_n\}$ or $Q_\rho \subset Q(\mathbb{T})_1 \setminus \{\beta_1, \dots, \beta_{2n}\}$.

Consider the first case. Without loss of generality we can assume that ρ starts with β_{2n-1} or δ_{2n-1}. Then by Lemma 1,

$$\rho = x\delta_{2n-1}\delta_{2n}\dots\delta_1\delta_2\delta_{2n-1}\delta_{2n}\rho',$$

where ρ' is the rest of the path ρ and $x \in k$ is the product certain scalars x_j, because we can always change a factor $\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$ or $\beta_{2i-1}\beta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}$ or $\delta_{2i-1}\delta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}$ by $\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$.

Hence, by the relation induced of the partial derivative

$$\partial_{\delta_2}(W(\mathbb{T})) = x_{n+2}\delta_{2n-1}\delta_{2n}\dots\delta_1 + x_1\alpha_1\delta_1,$$

we have that

$$\rho = -xx_1\alpha_1\delta_1\delta_2\delta_{2n-1}\delta_{2n}\rho'.$$

Then by Lemma 1 the path ρ is zero in $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$.

Now suppose $Q_\rho \subset Q(\mathbb{T})_1 \setminus \{\beta_1, \dots, \beta_{2n}\}$. By Lemma 3, the only non zero factor of length 5 is $\alpha_i \dots \alpha_{i+4}$ or $\delta_{2i-1}\dots\delta_{2(i-2)-1}$. Then we can assume that $\rho = \delta_{2n-1}\delta_{2n}\dots\delta_1\delta_2\delta_{2n-1}\delta_{2n}\rho'$ or $\rho = \alpha_1\dots\alpha_n\alpha_1\dots\alpha_n\rho'$. But we have already proof that the first option is a zero path. The second one using the relation induced by $\partial_{\alpha_1}(W(\mathbb{T}))$, we have:

$$\rho = x_{n+1}^2 x_1^2\alpha_1\delta_1\delta_2\alpha_1\delta_1\delta_2\rho'$$

that it is zero by Lemma 2. □

4 Potentials in a class of cyclically oriented Quivers

In this section, we construct finite dimensional Jacobian algebras from quivers with certain combinatorial characteristics.

First we recall the definition of primitive potential (c.f [DWZ08], Section 9) and cyclically oriented quivers.

Definition 5 ([BT13], Definition 3.1) A *walk of length p* in a quiver Q is a $(2p+1)$-tuple

$$w = (x_0, \alpha_1, x_1, \alpha_2, \ldots, x_{p-1}, \alpha_p, x_p)$$

such that for all i we have $x_i \in Q_0$, $\alpha \in Q_1$ and $\{s(\alpha_i), t(\alpha_i)\} = \{x_i, x_{i-1}\}$. The walk w is *oriented* if either $s(\alpha_i) = x_{i-1}$ and $t(\alpha_i) = x_i$ for all i or $s(\alpha_i) = x_i$ and $t(\alpha_i) = x_{i-1}$ for all i. Furthermore, w is called a *cycle* if $x_0 = x_p$. A cycle of length 1 is called a *loop*. We often omit the vertices and abbreviate w by $\alpha_1 \cdots \alpha_p$. An oriented walk is a *path*.

A cycle $c = (x_p, \alpha_1, x_1, \alpha_2, \ldots, x_{p-1}, \alpha_p, x_p)$ is called *non-intersecting* if its vertices $x_1, \ldots, x_p$ are pairwise distinct. A non-intersecting cycle of length 2 is called 2-cycle. If c is a non-intersecting cycle then any arrow $\beta \in Q \setminus \{\alpha_1, \ldots, \alpha_p\}$ with $\{s(\beta), t(\beta)\} \subseteq \{x_1, \ldots, x_p\}$ is called a *chord* of c. A cycle c is called *chordless* if it is non-intersecting and there is no chord of c.

A quiver Q without loops and 2-cycles is called *cyclically oriented* if each chordless cycle is oriented. Note that this implies that there are no multiple arrows in Q. A quiver without oriented cycles is called *acyclic* and an algebra whose quiver is acyclic is called *triangular*.

Definition 6 Let Q be a quiver. A *primitive potential S* is a lineal combination of every oriented chordless cycle in Q with non-zero scalars.

Buan, Marsh, Reineke, Rieten and Todorov introduced in [BMR$^+$06] a cluster category $\mathcal{C}_A$ associated to a hereditary algebra A and proved that $\mathcal{C}_A$ is endowed with a cluster-tilting object. The endomorphism algebra of a cluster-tilting object is called *cluster tilted algebra* and it was proven in [Kel11] by Keller that any cluster tilted algebra is a Jacobian algebra.

Barot and Trepode gave in [BT13] an explicit description of the minimal relations in cluster tilted algebras with cyclically oriented quivers, and it follows from this result that the potential associated with this kind of algebras is primitive.

In [Ami09] Amiot introduced a cluster category $\mathcal{C}_{(Q,W)}$ associated to a quiver with potential (Q, W), and she proved that when the Jacobian algebra $\mathcal{P}(Q, W)$ is finite dimensional, the category $\mathcal{C}_{(Q,W)}$ is endowed with a cluster-tilting object whose endomorphism algebra is isomorphic to $\mathcal{P}(Q, W)$. In these context, the endomorphism algebra of a cluster-tilting object is called *2-Calabi-Yau tilted algebra*.

Observe that the quiver $Q(\mathbb{T})$ in Figure 2 is cyclically oriented, however the potential $W(\mathbb{T})$ is not primitive, showing that the previous result does not extend to any Jacobian algebras. Another example of similar behavior are Jacobian algebras arising from the usual tetrahedron triangulation $\mathbb{T}_1$ of the sphere (S, M), with $\mid M \mid= 4$. Consider the quiver with potential $(Q(\mathbb{T}_1), W(\mathbb{T}_1)_t)$ of $\mathbb{T}_1$, where

$$W(\mathbb{T})_t = \sum_{1}^{3} \alpha_i\beta_i\gamma_i + \alpha_1\alpha_2\alpha_3 + \delta_1\delta_2\delta_3 + \gamma_1\beta_5\delta_2 + \beta_6\beta_3\delta_1 + t\delta_3\beta_4\beta_1,$$

and $t \in \mathbb{C} \cup (0, 1]$. Observer that $Q(\mathbb{T}_1)$ is a cyclically oriented quiver (see Figure 3) and $W(\mathbb{T}_1)_t$ is a primitive potential, but $\Lambda_1 = \mathcal{P}(Q(\mathbb{T}_1), W(\mathbb{T}_1)_1)$ is an infinite dimensional algebra and the corresponding potential W_1 is degenerate [GLFS][Section 9.9]. If $t \neq 0$, then the Jacobian algebras Λ_t are finite-dimensional and tame [GLFS] of tubular type [BG12].

Definition 7 ([BT13], Definition 3.3) A path γ which is anti-parallel to an arrow η in a quiver Q is a *shortest path* if the full subquiver generated by the induced oriented cycle $\eta\gamma$ is chordless.

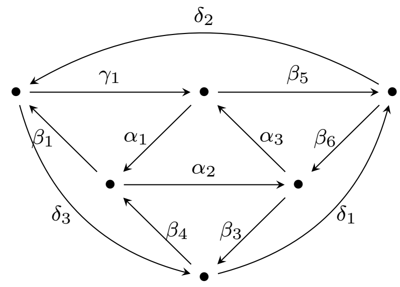

Fig. 3: Quiver of the usual tetrahedron triangulation $\mathbb{T}_1$ triangulation of an sphere with 4 marked points

A path $\gamma = (x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{x}_2 \to \cdots \to x_L)$ is called *shortest directed path* if there exists no arrow $x_i \to x_j$ in Q with $1 \leq i+1 < j \leq L$. A walk $\gamma = (x_0 — x_1 — x_2 — \cdots — x_L)$ is called a *shortest walk* if there is no edge joining x_i with x_j with $1 \leq i+1 < j \leq L$ and $(i,j) \neq (0,L)$ (we write a horizontal line to indicate an arrow oriented in one of the two possible ways).

Definition 8 Let Q be a cyclically oriented quiver such that for any arrow α there are at most 2 shortest path anti-parallel to α and $c = \beta_0\beta_1 \ldots \beta_L$ an oriented chordless cycle. We construct a sequence of triples $(\alpha_n, \rho_n, \rho'_n) \in Q_1 \times k\langle Q\rangle \times k\langle Q\rangle$, $n \in \mathbb{N} \cup \{0\}$, in the following way:

Step 0 We denote by α_0 the arrow β_0 and by ρ'_0 the path $\beta_1 \ldots \beta_L$. If there exists a shortest path ρ_0 anti-parallel to α_0 different to ρ'_o, then the first element of the sequence is $(\alpha_0, \rho_0, \rho'_0)$. Otherwise, the sequence is constant to the element $(\alpha_0, 0, \rho'_0)$.

Step 1 We denote by α_1 the arrow in the path ρ_0 such that $t(\alpha_0) = s(\alpha_1)$ and by ρ'_1 the shortest path anti-parallel to α_1 in $\alpha_0\rho_0$. If there exists a shortest path ρ_1 anti-parallel to α_1 different to ρ'_1, then the second element of the sequence is $(\alpha_1, \rho_1, \rho'_1)$. Otherwise, $(\alpha_n, \rho_n, \rho'_n) = (\alpha_1, 0, \rho'_1)$ for each $n \geq 1$.

Step 2 We denote by α_2 the arrow in the path ρ_1 such that $s(\alpha_1) = t(\alpha_2)$ and by ρ'_2 the shortest path anti-parallel to α_2 in $\alpha_1\rho_1$. If there exists a shortest path ρ_2 anti-parallel to α_2 different to ρ'_2, then the third element of the sequence is $(\alpha_2, \rho_2, \rho'_2)$. Otherwise, $(\alpha_n, \rho_n, \rho'_n) = (\alpha_2, 0, \rho'_2)$ for each $n \geq 2$.

$$\vdots$$

Step i We denote by α_i the arrow in the path ρ_{i-1} such that
- $t(\beta_i) = s(\beta_{i+1})$ if i is even or;
- $s(\beta_i) = t(\beta_{i+1})$ if i is odd.

and by ρ'_i the shortest path anti-parallel to α_i in $\alpha_{i-1}\rho_{i-1}$. If there exists a shortest path ρ_i anti-parallel to α_i different to ρ'_i, then the element $i+1$ of the sequence is $(\alpha_i, \rho_i, \rho'_i)$. Otherwise, $(\alpha_n, \rho_n, \rho_n) = (\alpha_i, 0, \rho'_i)$ for each $n \geq i$.

The sequence $\{(\alpha_n, \rho_n, \rho'_n)\}_{n \in \mathbb{N} \cup \{0\}}$ is called *cyclic sequence* of c. We say that the cyclic sequence $\{(\alpha_n, \rho_n, \rho'_n)\}_{n \in \mathbb{N} \cup \{0\}}$ is *finite* if there exists $m \in \mathbb{N}$ such that $(\alpha_n, \rho_n, \rho_n) = (\alpha_m, 0, \rho'_m)$ for every $n \geq m$.

Remark 6 Consider the quiver Q in Figure 3, which is associated to a triangulation of a sphere with 4 punctures. Observe that the cyclic sequence of any oriented chordless cycle in Q is infinite, because there are exactly two shortest paths anti-parallel to each arrow of Q.

In the following theorem, we give combinatorial conditions on a quiver Q that guarantee the finite-dimensionality of the Jacobian algebra that results from taking a primitive potential on Q.

Theorem 2 *Let Q be a cyclically oriented quiver such that:*

i) for any arrow α there are at most 2 shortest path anti-parallel to α;
ii) the cyclic sequence of any oriented chordless cyclic c is finite.

If W is a primitive potential of Q, then the Jacobian algebra $\mathcal{P}(Q,W)$ is a finite dimensional algebra.

Proof (Proof of Theorem 2) Let

$$W = \sum_{\substack{c \\ \text{min. cycle}}} x_c c$$

be a primite potential. It is enough to show that any non-intersecting oriented cycle c is zero in $\mathcal{P}(Q,W)$. To fix notation denote by

$$c = (x_1 \stackrel{\beta_0}{\to} x_2 \stackrel{\beta_1}{\to} \dots x_{L-1} \stackrel{\beta_L}{\to} x_1).$$

Suppose c is chordless, then by hypothesis the cyclic sequence $\{(\alpha_n, \rho_n, \rho'_n)\}_{n\in\mathbb{N}\cup\{0\}}$ of c is finite. Let $m \in \mathbb{N}$ be the minimal number such that $(\alpha_n, \rho_n, \rho'_n) = (\alpha_m, 0, \rho'_m)$ for every $n \geq m$. Then we have

$$\partial_{\alpha_n}(W) = x_{c_n}\rho_n + x_{c_{n-1}}\rho'_n$$

for every $n = 1, \dots, m-1$ and

$$\partial_{\alpha_m}(W) = x_{c_{m-1}}\rho'_m$$

because there is only one shortest path anti-parallel to α_m.

By construction of the cyclic sequence of c we have that $\rho'_0 = \beta_1 \dots \beta_L$ and ρ_0 are shortest paths anti-parallel to α_0, then $\beta_1 \dots \beta_L = -\frac{x_{c_0}}{x_c}\rho_0$ in $\mathcal{P}(Q,W)$, therefore,

$$\begin{aligned} c &= \alpha_0\beta_1 \dots \beta_L \\ &= -\frac{x_{c_0}}{x_c}\alpha_0\rho_0 \end{aligned} \tag{6}$$

Recall that ρ'_1 is the shortest path anti-parallel of α_1 in the cycle $\alpha_0\rho_0$, then

$$c = -\frac{x_{c_0}}{x_c}\rho'_1\alpha_1.$$

Repeating this process for each triple of sequence we have

$$\begin{aligned} c &= \left(-\frac{x_{c_0}}{x_c}\right)\left(-\frac{x_{c_1}}{x_{c_0}}\right)\rho_1\alpha_1 = \frac{x_{c_1}}{x_c}\rho_1\alpha_1 \\ &= \frac{x_{c_1}}{x_c}\alpha_2\rho'_2 \\ &= \left(\frac{x_{c_1}}{x_c}\right)\left(-\frac{x_{c_2}}{x_{c_1}}\right)\alpha_2\rho_2 = -\frac{x_{c_2}}{x_c}\alpha_2\rho_2 \\ &= -\frac{x_{c_2}}{x_c}\rho'_3\alpha_3 \\ &\vdots \\ &= \begin{cases} \frac{x_{c_{m-1}}}{x_c}\rho_{m-1}\alpha_{m-1} & \text{if } m-1 \text{ is odd} \\ -\frac{x_{c_{m-1}}}{x_c}\alpha_{m-1}\rho_{m-1} & \text{if } m-1 \text{ is even} \end{cases} \end{aligned} \tag{7}$$

Since ρ'_m is the shortest path anti-parallel to α_m in the cycle $\rho_{m-1}\alpha_{m-1}$ or $\alpha_{m-1}\rho_{m-1}$, the expression 7 can be rewritten in the following way:

$$c = \begin{cases} \frac{x_{c_{m-1}}}{x_c}\alpha_m\rho'_m & \text{if } m-1 \text{ is odd} \\ -\frac{x_{c_{m-1}}}{x_c}\rho'_m\alpha_m & \text{if } m-1 \text{ is even} \end{cases} \tag{8}$$

Then $c = 0$ in $\mathcal{P}(Q, W)$ because ρ'_l is in the Jacobian ideal $J(W)$. Therefore any oriented chordless cycle is zero in $\mathcal{P}(Q, W)$.

Suppose c is non-chordless, then there exists a chord $\beta_1 : x_i \to x_j$ with vertex in c such that $c_1 = \gamma_1\beta_{j+1}\ldots\beta_i$ is a oriented chordless cycle. Consider j the minimal number of the subset $\{0, 1, 2, \ldots, L-1\}$ such that c_1 is a oriented chordless cycle and there is not a chord in the path $\beta_1\ldots\beta_j$. Denote by $i_0 = j$, $i_1 = i$ and by $\tilde{c_1}$ the walk which is obtained by replacing the path $\beta_{i_0+1}\ldots\beta_{i_1}$ by the arrow γ_1 in the cycle c, namely,

$$\tilde{c_1} = (x_0 \xrightarrow{\beta_1} x_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{i_0}} x_{i_0} \xleftarrow{\gamma_1} x_{i_1} \xrightarrow{\beta_{i_1+1}} \ldots x_{n-1} \xrightarrow{\beta_n} x_0).$$

Since $\tilde{c_1}$ is a non-oriented cycle, then $\tilde{c_1}$ is non-chordless, because Q is a cyclically oriented quiver, then there exists a chord $\gamma_2 : x_{i_3} \to x_{i_2}$ such that $c_2 = \gamma_2\beta_{i_2+1}\ldots\beta_{i_3}$ is oriented chordless cycle and there is not a chord in the path $\beta_{i_1+1}\ldots\beta_{i_2}$.

Let $\tilde{c_2}$ be the walk which is obtained by replacing the path $\beta_{i_2+1}\ldots\beta_{i_3}$ by the arrow γ_2 in the walk $\tilde{c_1}$, namely,

$$\tilde{c_2} = (x_0 \xrightarrow{\beta_1} x_2 \cdots \xrightarrow{\beta_{i_0}} x_{i_0} \xleftarrow{\gamma_1} x_{i_1} \xrightarrow{\beta_{i_1+1}} \cdots \xrightarrow{\beta_{i_2}} x_{i_2} \xleftarrow{\gamma_2} x_{i_3} \xrightarrow{\beta_{i_3+1}} \ldots x_{n-1} \xrightarrow{\beta_n} x_1)$$

which is again not oriented and therefore not chordless, then exists a chord $\gamma_3 : x_{i_5} \to x_{i_4}$, with the same properties of the arrows γ_1 and γ_2, and a oriented chordless cycle $c_3 = \gamma_3\beta_{i_3+1}\ldots\beta_{i_4}$ and a walk $\tilde{c_3}$.

Observe that the vertex of the arrows γ_i are elements of an increasingly smaller subset of $\{0, 1, \ldots, L-1\}$, then there is a natural number r such that $\tilde{c_r}$ is oriented chordless cycle and in particular $s(\gamma_i) = t(\gamma_{i+1})$ for every $i = 1, \ldots, r$.

Then

$$\partial_{\gamma_2}(W) = x_{c_1}\beta_{i_2+1}\ldots\beta_{i_3} + x_{c_2}\gamma_1\gamma_k\ldots\gamma_3$$

where $x_c, x'_c \in k$, therefore c can be rewritten as the following

$$-\frac{x_{c_2}}{x_{c_1}}\beta_1\ldots\beta_{i_0}\gamma_1\gamma_k\ldots\gamma_3\beta_{i_3+1}\ldots\beta_{i_4}\ldots\beta_n = -\frac{x_{c_2}}{x_{c_1}}\beta_1\ldots\beta_{i_0}\gamma_1\gamma_k\ldots c_3\ldots\beta_n \tag{9}$$

Then c is a zero path, because c_3 is oriented chordless cycle. □

Remark 7 Given a 2-acyclic quiver Q satisfying the combinatorial conditions stated in Theorem 2 and given a primitive potential W on Q, the authors do not known if (Q, W) is non-degenerate.

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