MAXIMAL ZERO PRODUCT SUBRINGS AND INNER IDEALS OF SIMPLE RINGS

ALEXANDER BARANOV AND ANTONIO FERNÁNDEZ LÓPEZ

ABSTRACT. Let Q be a (non-unital) simple ring. A nonempty subset S of Q is said to have *zero product* if $S^2 = 0$. We classify all maximal zero product subsets of Q by proving that the map $\mathcal{R} \mapsto \mathcal{R} \cap \mathrm{LeftAnn}(\mathcal{R})$ is a bijection from the set of all proper nonzero annihilator right ideals of Q onto the set of all maximal zero product subsets of Q. We also describe the relationship between the maximal zero product subsets of Q and the maximal inner ideals of its associated Lie algebra.

1. Introduction

Let Q be a (not necessarily unital) associative ring. A nonempty subset S of Q is said to have *zero product* if $S^2 = 0$. By Zorn's Lemma, any zero product subset is contained in a maximal one, which is obviously a zero-product subring. Note also that 0 is the unique maximal zero product subset of a ring Q if and only if Q has no nonzero nilpotent elements.

In this paper we describe the maximal zero product subsets of a prime ring Q with nonzero core, in particular, of a simple ring, by proving that the map $\mathcal{R} \mapsto \mathcal{R} \cap \mathrm{LeftAnn}(\mathcal{R})$ is a bijection from the set of all proper nonzero annihilator right ideals of Q onto the set of all maximal zero product subsets of Q. In particular, if Q is a simple unital Baer ring (e.g. a simple Artinian ring), all maximal zero product subsets of Q are of the form $eQ(1-e)$, where e is a nontrivial idempotent of Q. Moreover, $e_1Q(1-e_1) = e_2Q(1-e_2)$, for e_1, e_2 idempotents of Q, if and only if $e_1Q = e_2Q$.

In the case when Q is a simple ring coinciding with its socle, the maximal zero product subsets are classified in terms of the associated geometry.

Finally, we describe the relationship between the maximal zero product subsets of a simple ring and the inner ideal structure of its associated Lie algebra.

2010 *Mathematics Subject Classification.* 16D30, 17B60.

Supported by the Spanish MEC and Fondos FEDER, MTM2010-19482.

2. Preliminaries and notation

Throughout this section Q is a (not necessarily unital) associative ring; $\mathcal{L}$ denotes a left and $\mathcal{R}$ a right ideal of Q; $\mathcal{I}_r(Q)$ and $\mathcal{I}_l(Q)$ are the lattices of all right and left ideals of Q, respectively. By an ideal we mean a two-sided ideal.

2.1. For a subset S of Q we denote by

$$\operatorname{lann}(S) = \operatorname{LeftAnn}(S) := \{a \in Q : aS = 0\}$$

the left annihilator of S. Note that $\operatorname{lann}(S)$ is a left ideal of Q (an ideal if S is a left ideal). A left ideal $\mathcal{L}$ is said to be an *annihilator left ideal* if $\mathcal{L} = \operatorname{lann}(S)$ for some subset S of Q. Similarly, one defines the right annihilator $\operatorname{rann}(S) := \{a \in Q : Sa = 0\}$, which is an *annihilator right ideal*.

2.2. A ring Q is said to be *semiprime* if $I^2 = 0$ implies $I = 0$ for any ideal I of Q; equivalently, $aQa = 0$ implies $a = 0$ for every $a \in Q$. If Q is semiprime, then $\operatorname{lann}(I) = \operatorname{rann}(I)$ and $I \cap \operatorname{rann}(I) = 0$ for any ideal I of Q.

2.3. A ring Q is said to be *prime* if $IJ = 0$ implies $I = 0$ or $J = 0$ for I, J ideals of Q. For a ring Q the following conditions are equivalent:

(i) Q is prime.
(ii) $\operatorname{lann}(I) = 0$ for any nonzero ideal I of Q.
(iii) $aIb = 0$ implies $a = 0$ or $b = 0$, for any nonzero ideal I of Q and any a, b in Q.

2.4. Let Q be a ring. The *core* of Q, denoted by $\operatorname{core}(Q)$, is defined as the intersection of all nonzero ideals of Q. If Q has nonzero core, then $\operatorname{core}(Q)$ is a minimal ideal. Moreover, a prime ring has nonzero core if and only if it contains a minimal ideal. It is also clear that any simple ring is prime and equal to its core, and that if Q is prime with nonzero socle, then $\operatorname{core}(Q) = \operatorname{soc}(Q)$.

3. Orthogonal pairs of one-sided ideals

Throughout this section Q will denote an arbitrary associative ring.

3.1. We have a Galois connection between the lattice $\mathcal{I}_r(Q)$ of all right ideals of Q and the lattice $\mathcal{I}_l(Q)$ of all left ideals of Q given by $\mathcal{R} \mapsto \operatorname{lann}(\mathcal{R})$ and $\mathcal{L} \mapsto \operatorname{rann}(\mathcal{L})$, that is,

(i) $\mathcal{L}_1 \subseteq \mathcal{L}_2 \Rightarrow \operatorname{rann}(\mathcal{L}_2) \subseteq \operatorname{rann}(\mathcal{L}_1)$ and $\mathcal{R}_1 \subseteq \mathcal{R}_2 \Rightarrow \operatorname{lann}(\mathcal{R}_2) \subseteq \operatorname{lann}(\mathcal{R}_1)$,
(ii) $\mathcal{L} \subseteq \operatorname{lann}(\operatorname{rann}(\mathcal{L}))$ and $\mathcal{R} \subseteq \operatorname{rann}(\operatorname{lann}(\mathcal{R}))$,

for all $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{I}_l(Q)$ and $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2 \in \mathcal{I}_r(Q)$.

Denote by $\overline{\mathcal{L}} := \operatorname{lann}(\operatorname{rann}(\mathcal{L}))$ and $\overline{\mathcal{R}} := \operatorname{rann}(\operatorname{lann}(\mathcal{R}))$ the corresponding closures of $\mathcal{L}$ and $\mathcal{R}$. It follows from (i) and (ii) that

$$\operatorname{rann}(\mathcal{L}) \subseteq \overline{\operatorname{rann}(\mathcal{L})} = \operatorname{rann}(\operatorname{lann}(\operatorname{rann}(\mathcal{L}))) = \operatorname{rann}(\overline{\mathcal{L}}) \subseteq \operatorname{rann}(\mathcal{L})$$

and similarly for $\operatorname{lann}(\mathcal{R})$. Therefore we have

(iii) $\operatorname{rann}(\mathcal{L}) = \overline{\operatorname{rann}(\mathcal{L})} = \operatorname{rann}(\overline{\mathcal{L}})$,
(iv) $\operatorname{lann}(\mathcal{R}) = \overline{\operatorname{lann}(\mathcal{R})} = \operatorname{lann}(\overline{\mathcal{R}})$.

In particular, a right (respectively, left) ideal of Q is closed if and only if it is an annihilator right (respectively, left) ideal.

3.2. By an *orthogonal pair* of Q we mean a pair $(\mathcal{R},\ \mathcal{L})$, where $\mathcal{R}$ is a nonzero right and $\mathcal{L}$ is a nonzero left ideals of Q such that $\mathcal{L}\mathcal{R} = 0$.

Lemma 3.3. *For an orthogonal pair* $(\mathcal{R},\ \mathcal{L})$ *the following conditions are equivalent:*

(i) $\mathcal{R} = \overline{\mathcal{R}}$ *and* $\mathcal{L} = \operatorname{lann}(\mathcal{R})$,
(ii) $\mathcal{R} = \operatorname{rann}(\mathcal{L})$ *and* $\mathcal{L} = \operatorname{lann}(\mathcal{R})$,
(iii) $\mathcal{L} = \overline{\mathcal{L}}$ *and* $\mathcal{R} = \operatorname{rann}(\mathcal{L})$.

Proof. It suffices to prove that (i)$\Leftrightarrow$(ii). The proof of (ii)$\Leftrightarrow$(iii) is similar. Suppose that $\mathcal{L} = \operatorname{lann}(\mathcal{R})$. Then $\operatorname{rann}(\mathcal{L}) = \operatorname{rann}(\operatorname{lann}(\mathcal{R}) = \overline{\mathcal{R}}$, as required. $\square$

We say that $(\mathcal{R}_1,\ \mathcal{L}_1) \subseteq (\mathcal{R}_2,\ \mathcal{L}_2)$ if and only if $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\mathcal{L}_1 \subseteq \mathcal{L}_2$. This gives a partial order on the set of orthogonal pairs.

Proposition 3.4. *Let* $(\mathcal{R},\ \mathcal{L})$ *be an orthogonal pair of* Q*. Then:*

(i) $(\overline{\mathcal{R}},\ \operatorname{lann}(\mathcal{R}))$ *and* $(\operatorname{rann}(\mathcal{L}),\ \overline{\mathcal{L}})$ *are maximal orthogonal pairs.*
(ii) *Any orthogonal pair is contained in a maximal one.*
(iii) $(\mathcal{R},\ \mathcal{L})$ *is maximal if and only if it satisfies the equivalent conditions of Lemma 3.3.*

Proof. (i) Since $\mathcal{R} \subseteq \overline{\mathcal{R}}$ and $\mathcal{L} \subseteq \operatorname{lann}(\mathcal{R})$, both $\overline{\mathcal{R}}$ and $\operatorname{lann}(\mathcal{R})$ are nonzero; and since $\operatorname{lann}(\mathcal{R}) = \operatorname{lann}(\overline{\mathcal{R}})$, we have that $(\overline{\mathcal{R}},\ \operatorname{lann}(\mathcal{R}))$ is an orthogonal pair. Suppose now that $(\overline{\mathcal{R}},\ \operatorname{lann}(\mathcal{R}))$ is contained in an orthogonal pair $(\mathcal{R}',\ \mathcal{L}')$. Then $\overline{\mathcal{R}} \subseteq \mathcal{R}'$ implies $\operatorname{lann}(\mathcal{R}') \subseteq \operatorname{lann}(\overline{\mathcal{R}}) = \operatorname{lann}(\mathcal{R})$, so

$$\mathcal{L}' \subseteq \operatorname{lann}(\mathcal{R}') \subseteq \operatorname{lann}(\mathcal{R}) \subseteq \mathcal{L}',$$

which proves that $\mathrm{lann}(\mathcal{R}) = \mathcal{L}'$. Hence

$$\mathcal{R}' \subseteq \mathrm{rann}(\mathcal{L}') = \overline{\mathcal{R}} \subseteq \mathcal{R}',$$

which proves that $\overline{\mathcal{R}} = \mathcal{R}'$. Therefore the orthogonal pair $(\overline{\mathcal{R}},\ \mathrm{lann}(\mathcal{R}))$ is maximal. Similarly, one can prove that $(\mathrm{rann}(\mathcal{L}),\ \overline{\mathcal{L}})$ is a maximal orthogonal pair.

(ii) Let $(\mathcal{R},\ \mathcal{L})$ be an orthogonal pair. As noted in the proof of (i), $(\mathcal{R},\ \mathcal{L})$ is contained in the maximal orthogonal pair $(\overline{\mathcal{R}},\ \mathrm{lann}(\mathcal{R}))$. Similarly, $(\mathcal{R},\ \mathcal{L})$ is also contained in the maximal orthogonal pair $(\mathrm{rann}(\mathcal{L}),\ \overline{\mathcal{L}})$.

(iii) Suppose that $(\mathcal{R},\ \mathcal{L})$ is maximal. Then $(\mathcal{R},\ \mathcal{L}) \subseteq (\overline{\mathcal{R}},\ \mathrm{lann}(\mathcal{R}))$ implies $\mathcal{R} = \overline{\mathcal{R}}$ and $\mathcal{L} = \mathrm{lann}(\mathcal{R})$. □

Proposition 3.5. *Let B be an additive subgroup of Q. Then the following conditions are equivalent:*

(i) $BQB \subseteq B$ *and* $B^2 = 0$.

(ii) *There exist* $\mathcal{L} \in \mathcal{I}_l(Q)$ *and* $\mathcal{R} \in \mathcal{I}_r(Q)$ *such that* $\mathcal{R}\mathcal{L} \subseteq B \subseteq \mathcal{R} \cap \mathcal{L}$ *and* $\mathcal{L}\mathcal{R} = 0$.

Proof. (i) $\Rightarrow$ (ii): Taking $\mathcal{L} = B + QB$ and $\mathcal{R} = B + BQ$, it is easily seen that (ii) holds.

(ii) $\Rightarrow$ (i): Clearly, $B^2 \subseteq \mathcal{L}\mathcal{R} = 0$ and $BQB \subseteq \mathcal{R}Q\mathcal{L} \subseteq \mathcal{R}\mathcal{L} \subseteq B$. □

3.6. Following [1], we say that an additive subgroup B of Q is a *regular inner ideal* of Q if it satisfies the equivalent conditions of the above proposition. We note the following properties of regular inner ideals.

(i) If B is nonzero in Proposition 3.5, then both $\mathcal{R}$ and $\mathcal{L}$ are nonzero and therefore $(\mathcal{R}, \mathcal{L})$ is an orthogonal pair.

(ii) If Q is a prime ring and $(\mathcal{R}, \mathcal{L})$ is an orthogonal pair, then any additive subgroup B of Q with $\mathcal{R}\mathcal{L} \subseteq B \subseteq \mathcal{R} \cap \mathcal{L}$ is a *nonzero* regular inner ideal, since $B = 0$ would imply $\mathcal{R}Q\mathcal{L} \subseteq \mathcal{R}\mathcal{L} \subseteq B = 0$, which is a contradiction by 2.3(iii).

(iii) If Q is a von Neumann regular ring, then any orthogonal pair $(\mathcal{R}, \mathcal{L})$ gives rise to a *unique* regular inner ideal $B = \mathcal{R}\mathcal{L} = \mathcal{R} \cap \mathcal{L}$, since

$$\mathcal{R} \cap \mathcal{L} = (\mathcal{R} \cap \mathcal{L})Q(\mathcal{R} \cap \mathcal{L}) \subseteq \mathcal{R}\mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}.$$

4. Regular inner ideals in prime rings with nonzero core

Throughout this section Q will be a prime ring with nonzero core containing nonzero nilpotent elements. We set $H := \mathrm{core}(Q)$. Recall that any simple ring is prime and equals to its core.

The following lemma shows that the orthogonal pair $(\mathcal{R},\ \mathcal{L})$ associated to a nonzero regular inner ideal B of Q is defined by B almost uniquely.

Lemma 4.1. *Let B be a nonzero regular inner ideal of Q with associated orthogonal pair $(\mathcal{R},\ \mathcal{L})$. Then $BH = \mathcal{R}H$ and $HB = H\mathcal{L}$. In particular, if Q is simple and unital, then $BQ = \mathcal{R}$ and $QB = \mathcal{L}$.*

Proof. By 2.3(iii), the ideal $\mathcal{L}H\mathcal{R}$ is nonzero. Since $\mathcal{L}H\mathcal{R} \subseteq H$ and H is minimal, we have that $\mathcal{L}H\mathcal{R} = H$. Hence

$$\mathcal{R}H = \mathcal{R}\mathcal{L}H\mathcal{R} \subseteq BH\mathcal{R} = BH \subseteq \mathcal{R}H,$$

which proves that $BH = \mathcal{R}H$. Similarly, one proves that $HB = H\mathcal{L}$. □

Lemma 4.2. *Let $(\mathcal{R}_1,\ \mathcal{L}_1)$ and $(\mathcal{R}_2,\ \mathcal{L}_2)$ be maximal orthogonal pairs. Then $\mathcal{R}_1 \cap \mathcal{L}_1 \subseteq \mathcal{R}_2 \cap \mathcal{L}_2$ implies $(\mathcal{R}_1,\ \mathcal{L}_1) = (\mathcal{R}_2,\ \mathcal{L}_2)$.*

Proof. Set $\mathcal{L} := \mathcal{L}_1 + \mathcal{L}_2$ and $B_j = \mathcal{R}_j \cap \mathcal{L}_j$, $j = 1, 2$. By Lemma 4.1,

$$H\mathcal{L} = H\mathcal{L}_1 + H\mathcal{L}_2 = HB_1 + HB_2 \subseteq HB_2 = H\mathcal{L}_2 \subseteq \mathcal{L}_2.$$

We claim that $\mathcal{L}\mathcal{R}_2 = 0$. Otherwise, $H = H\mathcal{L}\mathcal{R}_2$ (as $\mathcal{L}\mathcal{R}_2$ is a two-sided ideal and H is minimal) and hence, by the formula displayed above,

$$H = H\mathcal{L}\mathcal{R}_2 \subseteq \mathcal{L}_2\mathcal{R}_2 = 0,$$

which is a contradiction. Thus $\mathcal{L}\mathcal{R}_2 = 0$ and hence

$$\mathcal{L}_1 \subseteq \mathcal{L} \subseteq \operatorname{lann}(\mathcal{R}_2) = \mathcal{L}_2.$$

Similarly, $\mathcal{R}_1 \subseteq \mathcal{R}_2$. But then $(\mathcal{R}_1,\ \mathcal{L}_1) = (\mathcal{R}_2,\ \mathcal{L}_2)$ by maximality of $(\mathcal{R}_1,\ \mathcal{L}_1)$. □

Theorem 4.3. *Let Q be a prime ring with nonzero core containing nonzero nilpotent elements and let S be a subset of Q. Then the following are equivalent.*

(i) *S is a maximal zero product subset of Q.*
(ii) *S is a maximal regular inner ideal of Q.*
(iii) *$S = \mathcal{R} \cap \mathcal{L}$, where $(\mathcal{R},\ \mathcal{L})$ is a maximal orthogonal pair, i.e. $\mathcal{R} = \operatorname{rann}(\mathcal{L})$ and $\mathcal{L} = \operatorname{lann}(\mathcal{R})$.*

Proof. (i)⇒(ii): Suppose that S is a maximal zero product subset. Since its span is a zero product subset, S is actually an additive subgroup of Q. Put $B = SQS + S$. Then $S \subseteq B$ and $B^2 = 0$. Since S is maximal, one has $S = B$. Therefore

$$BQB = SQS \subseteq B,$$

so $B = S$ is a regular inner ideal of Q. Clearly, B is maximal as S is maximal.

(ii)⇒(iii): By Proposition 3.5, there is an orthogonal pair $(\mathcal{R},\ \mathcal{L})$ such that $S \subseteq \mathcal{R}\cap\mathcal{L}$. By taking closures if necessary, one can assume that the pair $(\mathcal{R},\ \mathcal{L})$ is maximal. By Proposition 3.5, $\mathcal{R}\cap\mathcal{L}$ is a regular inner ideal of Q, so $S = \mathcal{R}\cap\mathcal{L}$ as S is maximal.

(iii)⇒(i): Let $S = \mathcal{R}\cap\mathcal{L}$ where $(\mathcal{R},\ \mathcal{L})$ is a maximal orthogonal pair. Then $S^2 \subseteq \mathcal{L}\mathcal{R} = 0$, so S is a zero product subset. Let S' be a maximal zero product subset of Q containing S. Then by the arguments above, $S' = \mathcal{R}'\cap\mathcal{L}'$ where $(\mathcal{R}',\ \mathcal{L}')$ is a maximal orthogonal pair. By Lemma 4.2, $(\mathcal{R},\ \mathcal{L}) = (\mathcal{R}',\ \mathcal{L}')$, so $S = S'$, as required. □

Corollary 4.4. *The map $\mathcal{R} \mapsto \mathcal{R}\cap \operatorname{lann}(\mathcal{R})$ is a bijection from the set of all proper nonzero annihilator right ideals of Q onto the set of all maximal zero product subsets of Q.*

Proof. This follows from Theorem 4.3 and Lemma 4.2. □

4.5. Recall that Q is a *Baer ring* if every left annihilator of any subset of Q is generated (as a left ideal) by an idempotent element. If Q is unital then it is known that this definition is left-right symmetric. Note that every simple Artinian ring is a unital Baer ring.

Corollary 4.6. *Let Q be a simple unital Baer ring with nonzero nilpotent elements. Then $S \subset Q$ is a maximal zero product subset if and only if $S = eQ(1-e)$ where $e \neq 0, 1$ is a nontrivial idempotent of Q. Moreover, $e_1Q(1-e_1) = e_2Q(1-e_2)$, for e_1, e_2 idempotents of Q, if and only if $e_1Q = e_2Q$.*

Proof. By Theorem 4.3, the maximal zero product subsets of Q are the intersections $\mathcal{R}\cap\operatorname{lann}(\mathcal{R})$ where $\mathcal{R}$ runs over all proper nonzero annihilator right ideals of Q. Since Q is Baer, $\mathcal{R} = eQ$ for some idempotent e. Then $\operatorname{lann}(\mathcal{R}) = \operatorname{lann}(eQ) = Q(1-e)$. Indeed, one has $a \in \operatorname{lann}(eQ)$ if and only if $ae = 0$, or equivalently, $a = a(1-e) \in Q(1-e)$. It is also clear that

$$\mathcal{R}\cap\operatorname{lann}(\mathcal{R}) = eQ \cap Q(1-e) = eQ(1-e).$$

Finally, by Corollary 4.6, $e_1Q(1-e_1) = e_2Q(1-e_2)$ if and only if $e_1Q = e_2Q$. □

4.7. By [6, IV.8], a ring Q is simple with minimal one-sided ideals if and only $Q \cong Y \otimes_\Delta X$, where $(X, Y, \langle\cdot,\cdot\rangle)$ is a pair of dual vectors spaces over a division ring Δ, and where the product is given by

$$(y_1 \otimes x_1)(y_2 \otimes x_2) = y_1 \otimes \langle x_1, y_2\rangle x_2$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$. According to this representation of Q, we have (see [6, IV.16.Theorem 1]:

(i) The map $W \mapsto W \otimes X$ is a lattice isomorphism of the lattice $\mathcal{S}(Y)$ of all subspaces of Y onto the lattice $\mathcal{I}_r(Q)$ of all right ideals of Q.
(ii) The map $V \mapsto Y \otimes V$ is a lattice isomorphism of the lattice $\mathcal{S}(X)$ of all subspaces of X onto the lattice $\mathcal{I}_l(Q)$ of all left ideals of Q.

4.8. It is easy to check that if $\mathcal{R} = W \otimes X$ is a right ideal of Q, then $\mathrm{lann}(\mathcal{R}) = Y \otimes W^{\perp}$, where $W^{\perp} = \{x \in X : \langle x, W\rangle = 0\}$. Similarly, for any left ideal $\mathcal{L} = Y \otimes V$ of Q, $\mathrm{rann}(Y \otimes V) = V^{\perp} \otimes X$. Thus $\mathcal{R}$ is an annihilator right ideal if and only $\mathcal{R} = W \otimes Y$, where W is a *closed* subspace of Y, i.e., $W^{\perp\perp} = W$. (We have a Galois connection between the lattice $\mathcal{S}(X)$ of all subspaces of X and the lattice $\mathcal{S}(Y)$ of all subspaces of Y given by $V \to V^{\perp}$ and $W \to W^{\perp}$. Note also that finite dimensional subspaces are closed.)

Corollary 4.9. *Let $Q = Y \otimes_{\Delta} X$ be a simple ring with minimal one-sided ideals. Then the map $W \mapsto W \otimes W^{\perp}$ is a bijection from the set of nonzero proper closed subspaces of Y onto the set of maximal zero product subsets of Q.*

Proof. By Corollary 4.6, any maximal zero product subset S of Q is of the form $S = \mathcal{R} \cap \mathrm{lann}(\mathcal{R})$ for a unique proper nonzero annihilator right ideal $\mathcal{R}$ of Q. Now it follows from (4.7) and (4.8) that $\mathcal{R} = W \otimes X$ for a unique nonzero proper closed subspace W of Y and $\mathrm{lann}(W \otimes X) = Y \otimes W^{\perp}$. Hence

$$S = \mathcal{R} \cap \mathrm{lann}(\mathcal{R}) = (W \otimes X) \cap (Y \otimes W^{\perp}) = W \otimes W^{\perp}$$

as required. □

We finish with an application to the Lie inner ideal structure of simple rings.

4.10. Recall that every associative ring Q becomes a Lie ring $Q^{(-)}$ under the product $[x, y] = xy - yx$. An additive subgroup B of a Lie ring L is called an *inner ideal* if $[[B, L], B] \subseteq B$. An inner ideal B is said to be *abelian* if $[B, B] = 0$. Inner ideals were first systematically studied by Benkart [2], see also [3, 4] for some recent development.

Corollary 4.11. *Let Q be a non-unital simple associative ring of characteristic not* 2 *or* 3*. For an additive subgroup B of Q the following conditions are equivalent.*

(i) *B is a maximal zero product subset of Q.*
(ii) *B is a maximal regular inner ideal of Q.*

(iii) *B is a maximal abelian inner ideal of $Q^{(-)}$.*

Proof. (i) $\Rightarrow$ (ii). This is proved in Theorem 4.3.

(ii) $\Rightarrow$ (iii). If B is a maximal regular inner ideal of Q, then $[B, B] \subseteq B^2 = 0$ and $[[B, Q], B] \subseteq BQB \subseteq B$, so B is an abelian inner ideal of the Lie algebra $Q^{(-)}$.

(iii) $\Rightarrow$ (i). By [5, Theorem 5.4] (applied to the case of a non-unital simple ring), every maximal abelian inner ideal of $Q^{(-)}$ is a zero product subset of Q, and hence a maximal zero product subset by (i) $\Rightarrow$ (iii). □

References

[1] A.A. Baranov, J. Rowley, Inner ideals of simple locally finite Lie algebras, *J. Algebra* **379** (2013), 11–30.

[2] G. Benkart, The Lie inner structure of associative rings, *J. Algebra* **43** (1976) 561–584.

[3] G. Benkart, A. Fernández López, The Lie inner ideal structure of associative rings revisited, *Communications in Algebra* **37** (2009), 3833–3850.

[4] J. Brox, A. Fernández López, M. Gómez Lozano, Inner ideals of Lie algebras of skew elements of prime rings with involution, *Proc. AMS* **144** (2016), 2741-2751.

[5] A. Fernández López, Lie inner ideals are nearly Jordan inner ideals, *Proc. AMS* **142** (2014), 795–804.

[6] N. Jacobson, *Structure of Rings*. Amer. Math. Soc. Colloquium Publ., Providence, 1964.

Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK
E-mail address: ab155@le.ac.uk

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071, Málaga, Spain
E-mail address: emalfer@uma.es