ADAPTIVE DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC INTERFACE PROBLEMS

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ABSTRACT. An interior-penalty discontinuous Galerkin (dG) method for an elliptic interface problem involving, possibly, curved interfaces, with flux-balancing interface conditions, e.g., modelling mass transfer of solutes through semi-permeable membranes, is considered. The method allows for *extremely general curved* element shapes employed to resolve the interface geometry exactly. A residual-type a posteriori error estimator for this dG method is proposed and upper and lower bounds of the error in the respective dG-energy norm are proven. The a posteriori error bounds are subsequently used to prove a basic a priori convergence result. The theory presented is complemented by a series of numerical experiments. The presented approach applies immediately to the case of curved domains with non-essential boundary conditions, too.

1. INTRODUCTION

Interface conditions are used in the modelling of various engineering applications and physical, chemical, biological phenomena, in particular, ones involving multiple distinct materials with different diffusion, density, permeability or conductivity properties. Such interface conditions are typically used to close systems of partial differential equations (PDEs) posed on multi-compartment distinct material regions. As a result, these interface problems often admit solutions having jump discontinuities of the state variable and/or of some of its derivatives across the interface. In other words, their solutions may have higher regularity in individual material regions than in the entire physical domain. The analytical regularity theory for interface problems is far less advanced than for respective standard (one-compartment) initial/boundary-value problems. Therefore, the reliable and efficient numerical approximation of such problems is desirable. Furthermore, such a development has the potential to be used to inform on the underlying local analytical regularity properties, too. However, in many applications interfaces arise in the form of general, curved, manifolds of co-dimension one, thus making their numerical treatement challenging.

A class of interface problems, which is still relatively unexplored, are problems with flux-balancing interface conditions, resulting in discontinuities of the state variable itself. This class of interface conditions model, among other things, the mass transfer of solutes through semi-permeable membranes in a number of engineering applications and biological processes such as in filtering, electrophysiology, and cell biology; see, e.g., [15, 8, 19, 17] for more details on the modelling. The design of practical high-order numerical methods for this class of problems poses a number of challenges, most important being the discontinuity of the solution across the interface, and the geometric approximation of the, possibly curved, interface itself.

In the context of finite element methods (FEMs), when the interface is a general manifold of co-dimension one, the geometry cannot be described exactly by the mesh, as even isoparametric elements can only exactly resolve interfaces with polynomial level-sets. A number of methods to address this shortcoming have been

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proposed over the years, such as the unfitted FEM [4, 28, 32, 5], mortar elements [23], immersed interface methods [31, 33, 36], fictitious domain methods [6, 10, 11], composite FEM [37], cut-cell techniques [34, 38, 9, 27], etc.

Although many of the aforementioned works also provide a priori error analysis of the proposed methods and/or goal-oriented error estimation techniques, the lack of availability of rigorous a posteriori bounds may appear somewhat surprising at first sight. This is, nonetheless, an important open question for interface problems, as their solutions often admit rich, a priori unknown, structures in the vicinity of the interface and/or in the intersection of interfaces and physical boundaries. Observing, however, that, upon interface approximation, the exact solution is defined on a different domain to its finite element approximation, the standard approach of proving a posteriori bounds, i.e., using PDE stability results linking the error with the residual, becomes cumbersome. Few a posteriori bounds for curved domains exist, focusing on the related (but simpler) problem of proving a posteriori error bounds for elliptic problems posed on one-compartment curved domains [21, 2]; see also [20].

To address the challenge of general curved interface geometry, in this work we present a *fitted* interior-penalty discontinuous Galerkin (dG) method for an elliptic interface problem involving elements with extremely general curved faces. The elliptic interface problem considered here is posed on a multi-compartment domain and the specific flux-balancing interface conditions have been proposed in the modelling of mass transfer through semi-permeable membranes [15, 19, 17]. Such interface problems, yielding discontinuous solutions across the interface, can be easily implemented within an existing dG code simply by modifying the interior penalty dG numerical fluxes accordingly [26, 17, 18]. Moreover, the extremely general element shapes allowed in the proposed method are able to resolve very general interface geometries exactly, up to quadrature errors. The optimal approximation of the finite element spaces and good conditioning of the respective stiffness matrices are ensured by the use of *physical coordinate* basis functions, as opposed to standard mapped ones from a reference element; this idea was utilised in [5, 16], where efficient techniques for the assembly step are presented. Furthermore, an alternative construction using *parametric maps* of reference elements with *extremely general reference element shapes* is also proposed. The latter may prove to be useful in the context of high-order finite element spaces.

We prove residual-type a posteriori bounds for the proposed dG method whose fitted nature crucially avoids some of the aforementioned theoretical challenges. At the same time, however, generalisations of standard approximation, inverse and conforming-nonconforming recovery estimates (in the spirit of [29]) to elements with curved faces required for the proof are derived in detail. The latter may be of independent interest. For a posteriori error bounds for conforming FEM for elliptic interface problems we refer to [13, 14], for dG methods to [12], and to [35] for a finite volume scheme. Using the derived a posteriori error bounds, we also give a basic a priori convergence result for the proposed method using the efficiency of the a posteriori estimator, under minimal regularity assumptions, in the spirit of the seminal work [25]. We prefer to do so, since the regularity theory for elliptic interface problems is far from being well developed.

We stress that the developments presented below also apply naturally to the case of elliptic problems with non-essential (Robin or Neumann-type) boundary conditions over a *single* curved domain. Since elliptic problems over domains with piecewise curved boundaries are relevant in applications, we believe that the results presented below may be of wider interest.
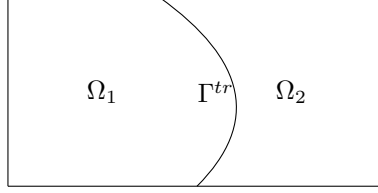
FIGURE 1. Ω subdivided into two sub-domains Ω_1 and Ω_2 by the interface Γ^{tr}.

The fitted approach proposed below may appear cumbersome at first sight, if employed as spatial discretisation in the context of evolutionary PDEs involving moving interfaces. This is, in fact, not necessarily the case. The theoretical developments presented below appear to be generalisable, at least in principle, to a cut-cell-type setting, whereby a mesh is not subordinate to the interface location a priori. In this context, some implementation issues have already been tackled in [5]. This is not done here, however, in the interest of simplicity of the presentation of the key ideas, and will be considered in detail elsewhere. An interesting attribute of the fitted approach presented below is that, using curved elements, it is possible to represent the geometry accurately without necessarily resorting to the standard practice of mesh-refinement in the vicinity of the interface, cf., for instance [28].

The remainder of this work is organised as follows. In Section 2, the model problem is introduced. The discontinuous Galerkin method, along with the admissible curved element shapes are discussed in Section 3. Some necessary approximation, trace, and inverse estimates for general curved elements are presented in Section 4. An extension of the conforming-nonconforming recovery operator from [29] to curvilinear elements is proven in Section 5. Upper and lower a posteriori error bounds for the proposed dG method are shown in Section 7. A basic convergence result of the dG method under minimal regularity is presented in Section 8. In Section 9, we comment on the use of general curved elements in conjunction with parametric finite element mappings. In Section 10, a series of numerical experiments investigating the performance of the a posteriori error bounds, implemented using the `deal.II` finite element library are presented. Finally, we draw some conclusions and discuss a number of further directions of research in Section 11.

2. Model problem

Let Ω be a bounded open polygonal/polyhedral domain with Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^d$, $d = 2, 3$. Ω is split into two sub-domains Ω_1 and Ω_2, such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^{tr}$, with $\Gamma^{tr} := (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega$ being also Lipschitz continuous with bounded curvature; see Figure 1 for an illustration.

We consider the model problem:

$$
\begin{aligned}
-\Delta u &= f, & &\text{in } \Omega_1 \cup \Omega_2, \\
u &= 0, & &\text{on } \partial\Omega, \\
\mathbf{n}^1 \cdot \nabla u_1 &= C_{tr}(u_2 - u_1)|_{\Omega_1}, & &\text{on } \bar{\Omega}_1 \cap \Gamma^{tr}, \\
\mathbf{n}^2 \cdot \nabla u_2 &= C_{tr}(u_1 - u_2)|_{\Omega_2}, & &\text{on } \bar{\Omega}_2 \cap \Gamma^{tr},
\end{aligned}
\tag{2.1}
$$

with $f : \Omega_1 \cup \Omega_2 \to \mathbb{R}$ known function, $u_i = u|_{\bar{\Omega}_i}$, $i = 1, 2$, $C_{tr} > 0$ a given interface transmission (e.g., *permeability*) constant and $\mathbf{n}^i$, $i = 1, 2$ denoting the respective outward unit normal vectors. This is a simplified model for mass transfer of a solute through a semi-permeable membrane through, e.g., osmosis, but it is rich enough in

highlighting the aforementioned challenges posed for the numerical analysis of this class of problems. We note that, setting $\Omega_2 = \emptyset$, we trivially recover the classical elliptic problem for u_1 with non-essential boundary conditions: setting $C_{tr} = 0$, we retrieve the homogeneous Neumann problem, while for $C_{tr} \neq 0$, we recover the homogeneous Robin problem.

Let $L_p(\omega)$, $1 \leq p \leq \infty$ and $H^r(\omega)$, $r \in \mathbb{R}$, denote the standard Lebesgue and Hilbertian Sobolev spaces on a domain $\omega \subset \Omega$. The norm of $L_2(\omega) \equiv H^0(\omega)$, $\omega \subset \Omega$, will be denoted by $\| \cdot \|_\omega$, and is induced by the standard $L_2(\omega)$-inner product, denoted by $\langle \cdot, \cdot \rangle_\omega$; when $\omega = \Omega$, we shall use the abbreviations $\| \cdot \| \equiv \| \cdot \|_\Omega$ and $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_\Omega$. Also, we set $\mathcal{H}^1 := H^1(\Omega_1 \cup \Omega_2)$, and

$$\mathcal{H}_0^1 := \{v \in \mathcal{H}^1 : v = 0 \text{ on } \partial\Omega\}.$$

Upon integrating by parts on each sub-domain and applying the interface condition, we arrive to (2.1) in weak form, which reads: find $u \in \mathcal{H}_0^1$ such that

$$(2.2) \qquad D(u, v) := \int_\Omega \nabla u \cdot \nabla v dx + \int_{\Gamma^{tr}} C_{tr} [\![u]\!] \cdot [\![v]\!] ds = \int_\Omega fv dx,$$

for all $v \in \mathcal{H}_0^1$, where $[\![u]\!] := v_1|_K \mathbf{n}^1 + v_2|_K \mathbf{n}^2$ is the jump across the interface and $f \in L_2(\Omega)$. Problem (2.2) is well posed in $\mathcal{H}_0^1$ by the Lax-Milgram Lemma.

3. Discontinuous Galerkin method

We introduce an interior penalty discontinuous Galerkin (dG) finite element method for the discretization of the elliptic interface problem (2.2). The dG method employs elements with, possibly, curved faces, able to resolve the interface geometry exactly. The method is closely related to the spatial discretization for parabolic interface problems introduced in [17], with the latter assuming exact interface resolution using standard (non-curved) simplicial or box-type elements only.

A key attribute of the proposed method is the use of physical frame basis functions, i.e., the elemental bases consist of polynomials on the elements themselves, rather than mapped polynomials through a mapping from a reference element. Crucially, the lack of conformity of the dG method allows for such physical frame polynomial basis functions to be used on very general element shapes. The implementation challenges arising from this non-standard choice will be discussed below.

3.1. **The mesh.** Let $\mathcal{T} = \{K\}$ be a locally quasi-uniform subdivision of Ω, possibly containing regular hanging nodes, with K a generic, possibly curved, open simplicial, box-type, or prismatic element of diameter h_K. More specifically, we shall assume that the mesh consists of triangular or quadrilateral elements when $d = 2$, and of tetrahedral or prismatic elements with triangular bases when $d = 3$. We stress that the prismatic elements considered here are *not* assumed to have parallel bases, in general. The mesh skeleton $\Gamma := \cup_{K \in \mathcal{T}} \partial K$ is subdivided into three disjoint subsets $\Gamma = \partial\Omega \cup \Gamma^{int} \cup \Gamma^{tr}$, where $\Gamma^{int} := \Gamma \backslash (\partial\Omega \cup \Gamma^{tr})$.

We shall assume that elements with curved faces will be employed *only* to resolve the interface geometry, i.e., only elements $K \in \mathcal{T}$ such that $\partial K \cap \Gamma^{tr} \neq \emptyset$ are curved, see Figure 2 for an illustration. This is also realistic from a practical perspective, as the global use of curved elements is more computationally demanding (with no immediate advantage) during assembly. We stress that, with minor modifications only, the theory presented below extends to the case were curved elements are used to resolve curved boundaries with non-essential boundary conditions in addition to interfaces or, indeed, in the context of single element-long domains separated by multiple interfaces of the form (2.1); in both cases elements with more than one curved faces are present in the mesh (see Remark 7.6 below for a more detailed

discussion). We refer to Section 11 for a discussion on the use of curved elements for different settings and how the present developments can be used.
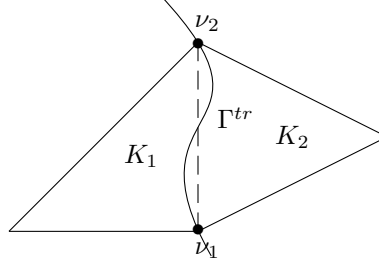

FIGURE 2. Curved elements K_1 and K_2 (solid lines/curves) from either side of the interface Γ^{tr}, resolving the geometry of Γ^{tr}.

We make some further assumptions on the admissible meshes near the (curved) interface. We assume that every element $K \in \mathcal{T}$ is contained in either Ω_1 or Ω_2 so that the set $\{K \in \mathcal{T} : K \cap \Omega_i \neq \emptyset\}$, $i = 1, 2$, forms a subdivision of Ω_i. Moreover, for simplicity (and with no essential loss of generality,) we assume that the set $\partial K \cap \Gamma^{tr} \neq \emptyset$ is one whole face of K, or one vertex of K only. Hence, when $d = 3$, we shall only consider (possibly curved) tetrahedral or prismatic elements with triangular bases $K \in \mathcal{T}$ such that $\partial K \cap \Gamma^{tr} \neq \emptyset$, so that a unique cut plane passes through the 3 vertices of K lying on Γ^{tr}. Elsewhere in the mesh, box-type elements when $d = 3$ are also allowed. Moreover, we assume that the mesh is constructed in such a way that each element K is a Lipschitz domain.

Assumption 3.1. For all elements $K \in \mathcal{T}^{tr}$, we assume that:

a) *(star-shapedness)* each element $K \in \mathcal{T}^{tr}$, having the face $E \subset \Gamma^{tr}$, is star-shaped with respect to all vertices opposite this face E; note that we have one such vertex when K is simplicial, or more than one such vertices when K is box-type or prismatic. Furthermore, we assume that each element $K \in \mathcal{T}^{tr}$ is also star-shaped with respect to all the midpoints of the edges sharing a common vertex with the face $E \subset \Gamma^{tr}$ and are not (edges of) $E \subset \Gamma^{tr}$ itself; we refer to Figure 3 for an illustration for $d = 2$.

b) *(shape-regularity)* we have $\mathbf{m}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \geq c|\mathbf{m}(\mathbf{x})|$ uniformly across the mesh, for every vector $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x_0}$, with $\mathbf{x} \in E$ and $\mathbf{x_0}$ any vertex opposite $E \in \Gamma$, and $\mathbf{n}(\mathbf{x})$ the respective unit outward normal vector to E at $\mathbf{x}$. Moreover, we assume that $|\mathbf{m}(\mathbf{x})| \sim h_K$ uniformly.

Note that Assumption 3.1 b) is trivially satisfied by shape-regular elements K with straight faces. It is a natural condition in view of proving trace estimates, cf. Lemma 4.1 below (see also [1, Theorem 3.10] and [22, Section 3] for illuminating expositions). Assumption 3.1 a) can always be fulfilled on sufficiently fine meshes, given that the curvature of Γ^{tr} is bounded.

We denote the set of, possibly curved, *interface elements* by

$$\mathcal{T}^{tr} := \{K \in \mathcal{T} : \text{meas}_{d-1}(\partial K \cap \Gamma^{tr}) > 0\};$$

with $\text{meas}_r(\omega)$ denoting the r-dimensional Hausdorff measure of a set $\omega \subset \mathbb{R}^d$; see Figure 2 for an illustration of such elements. Note that elements having just one vertex on Γ^{tr} do *not* belong to $\mathcal{T}^{tr}$.

Definition 3.2. For each $K \in \mathcal{T}^{tr}$, we define the simplicial or box-type *related element* $\tilde{K}$ to be the element with straight/planar faces having the same vertices as K. Let also $\underline{K} \subset K$ be the largest sub-element with straight/planar faces and all faces parallel to the faces of the related element $\tilde{K}$.
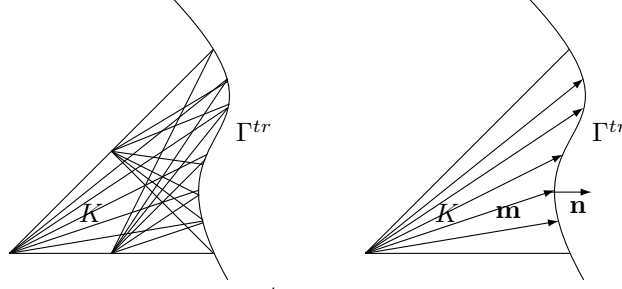
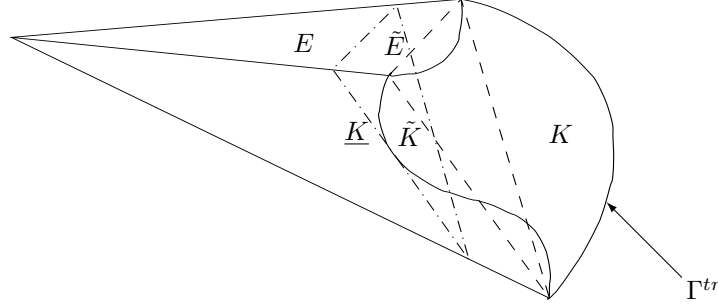FIGURE 3. Elements $K \in \mathcal{T}^{tr}$ are assumed to satisfy Assumption 3.1 a) (left) and b) (right)

FIGURE 4. A three-dimentional curved element $K \in \mathcal{T}^{tr}$ (enclosed by the solid lines and curve), its related elements $\tilde{K}$ having the same vertices as K and straight faces (two same faces and the third depicted by a dashed line,) and $\underline{K}$ (having two same faces and the third depicted by a dashed-dotted line.) Although it does not belong to Γ^{tr}, the face E (enclosed by the solid lines and curve,) has a curved edge while the related face $\tilde{E}$ (two same faces and the third depicted by a dashed line,) is a straight triangle.

For two adjacent elements $K, K' \in \mathcal{T}^{tr}$ sharing a common face $E \in \Gamma^{int} \cup \Gamma^{tr}$, we shall denote by $\tilde{E} := \partial \tilde{K} \cap \partial \tilde{K}'$ the related common face of the two (also adjacent) related simplicial or prismatic elements $\tilde{K}, \tilde{K}'$.

Notice that in general, $K \neq \tilde{K}$ when $\partial K \cap \Gamma^{tr}$ is curved; see Figure 4 for an illustration.

Next, we define
$$\Gamma^{int}_{tr} := \{E \in \Gamma^{int} : E \neq \tilde{E}\},$$
i.e., the subset of Γ^{int} containing all the faces $E \in \Gamma^{int}$ with different related faces $\tilde{E}$; see again Figure 4 for an illustration. Notice that $E \neq \tilde{E}$ is possible only when $d = 3$.

The above star-shapedness assumptions effectively imply that the angles between the faces $E \subset \Gamma^{tr}$ and those faces in Γ^{int}_{tr} cannot be arbitrarily small and that the Jacobian of the function parametrising $E \subset \Gamma^{tr}$ on a local coordinate system, as defined above, cannot be very large. Satisfying these assumptions may require a small number of refinements of the elements $K \in \mathcal{T}^{tr}$ of a given initial mesh.

3.2. **Finite element space and the dG method.** We define the discontinuous finite element space S^p_h, subortinate to the mesh $\mathcal{T} = \{K\}$, by

(3.1) $$S^p_h = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_p(K)\},$$

where $\mathcal{P}_p(K)$ denotes the space of polynomials of *total* degree p on an element K.

For each element face $E \subset \Gamma^{int} \cup \Gamma^{tr}$, there are two elements K_1 and K_2 such that $E \subset \partial K_1 \cap \partial K_2$. The outward unit normal vectors on E of ∂K_1 and ∂K_2 are denoted by $\mathbf{n}_{K_1}$ and $\mathbf{n}_{K_2}$, respectively. For a function $v : \Omega \to \mathbb{R}$ that may be discontinuous across Γ, we set $v_i = v|_{K_i}$, and we define the jump $[\![v]\!]$ and the average $\{v\}$ of v across E by

$$(3.2) \qquad [\![v]\!] = v|_{K_1}\mathbf{n}_{K_1} + v|_{K_2}\mathbf{n}_{K_2}, \quad \{v\} = \frac{1}{2}\left(v|_{K_1} + v|_{K_2}\right).$$

Similarly, for a vector valued function $\mathbf{w}$, piecewise smooth on $\mathcal{T}$ with $\mathbf{w}_i = \mathbf{w}|_{K_i}$, we define

$$[\![\mathbf{w}]\!] = \mathbf{w}|_{K_1} \cdot \mathbf{n}_{K_1} + \mathbf{w}|_{K_2} \cdot \mathbf{n}_{K_2}, \quad \{\mathbf{w}\} = \frac{1}{2}\left(\mathbf{w}|_{K_1} + \mathbf{w}|_{K_2}\right).$$

When $E \subset \partial\Omega$, we set $\{v\} = v$, $[\![v]\!] = v\mathbf{n}$ and $[\![\mathbf{w}]\!] = \mathbf{w}\mathbf{n}$ with $\mathbf{n}$ denoting the outward unit normal to the boundary $\partial\Omega$.

We introduce the meshsize function $\mathbf{h} : \Omega \to \mathbb{R}$, where $\mathbf{h}|_K = h_K$, $K \in \mathcal{T}$ and $\mathbf{h} = \{\mathbf{h}\}$ on each $(d-1)$-dimensional open face $E \subset \Gamma$. We also define $h_{\max} := \max_{x\in\Omega} \mathbf{h}$ and $h_{\min} := \min_{x\in\Omega} \mathbf{h}$. Without loss of generality, we shall assume that $h_{\max}$ remains uniformly bounded throughout this work, thus, avoiding having estimation constants dependent on $\max\{1, h_{\max}\}$.

The interior penalty discontinuous Galerkin method for (2.2) is defined as: find $u_h \in S_h^p$ such that

$$(3.3) \qquad D_h(u_h, v_h) = \langle f, v_h \rangle,$$

for all $v_h \in S_h^p$, where

$$(3.4) \quad \begin{aligned} D_h(u_h, v_h) = &\sum_{K\in\mathcal{T}} \int_K \nabla u_h \cdot \nabla v_h dx - \int_{\Gamma\setminus\Gamma^{tr}} (\{\nabla u_h\} \cdot [\![v_h]\!] + \{\nabla v_h\} \cdot [\![u_h]\!])ds \\ &+ \int_{\Gamma\setminus\Gamma^{tr}} \frac{\gamma_0}{\mathbf{h}}[\![u_h]\!] \cdot [\![v_h]\!]ds + \int_{\Gamma^{tr}} C_{tr}[\![u_h]\!] \cdot [\![v_h]\!]ds; \end{aligned}$$

here $\gamma_0 > 0$ is the discontinuity-penalization parameter (to be defined precicely below,) and $C_{tr} > 0$ is the permeability coefficient. We note carefully that there is *no* discontinuity penalization on the interface. As we shall see below, the penalty parameter has to be chosen large enough in order to ensure the stability of the discontinuous Galerkin discretization.

4. Approximation, trace, and inverse estimates

For the proof of upper and lower a posteriori error bounds, we shall require approximation, trace, and inverse estimates for the elements with curved boundaries $K \in \mathcal{T}^{tr}$, with uniform constants, i.e., constants that are independent of the particular shape of K. We begin by extending the standard trace estimate to elements with curved faces.

Lemma 4.1. *Let $v \in \mathcal{H}_0^1$ and $K \in \mathcal{T}^{tr}$. Then, under the above assumptions on the mesh, we have*

$$(4.1) \qquad \|v\|_{\partial K \cap \Gamma^{tr}}^2 \leq C\left(h_K^{-1}\|v\|_K^2 + h_K\|\nabla v\|_K^2\right),$$

with $C > 0$, independent of the shape and size of K and of v.

Proof. Since $K \in \mathcal{T}^{tr}$ is star-shaped with respect to any given vertex $\nu_{E_{tr}}$ opposite the face $E_{tr} = \partial K \cap \Gamma^{tr}$, let $\mathbf{m}(s)$ be the vector pointing from the vertex $\nu_{E_{tr}}$ to all points $s \in K$, thereby defining $\mathbf{m} : K \to \mathbb{R}^d$, cf. Figure 3 (right). Without loss of generality, we assume that $K \in \mathcal{T}^{tr}$ is simplicial. Indeed, if $K \in \mathcal{T}^{tr}$ is prismatic, let $K_0 \subset K$ to be the (curved) simplex defined by $\nu_{E_{tr}}$ and E_{tr} and follow the argument presented below for K_0 instead.

Defining the vector field $F = \mathbf{m}v^2$, the divergence theorem implies

$$\int_{E_{tr}} (\mathbf{m} \cdot \mathbf{n}) v^2 \, ds = \int_{\partial K} F \cdot \mathbf{n} \, ds = \int_K \nabla \cdot F \, dx = \int_K (\nabla \cdot \mathbf{m}) v^2 \, dx + 2 \int_K v \nabla v \cdot \mathbf{m} \, dx,$$

noting that $\mathbf{m}(s) \cdot \mathbf{n}(s) = 0$ for all $s \in \partial K \backslash E_{tr}$, which, in turn, yields

$$(4.2) \qquad \min_{E_{tr}} |\mathbf{m} \cdot \mathbf{n}| \|v\|_{E_{tr}}^2 \leq d\|v\|_K^2 + h_K \|v\|_K \|\nabla v\|_K,$$

noting that $\|\nabla \cdot \mathbf{m}\|_{L_\infty(K)} = d$ and $\|\,|\mathbf{m}|\,\|_{L_\infty(K)} \leq h_K$. The result already follows by Assumption 3.1 b). $\qquad \square$

Next, let $\Pi_0 : L^2(\Omega) \to S_h^0$ denote the orthogonal L^2-projection operator onto the element-wise constant functions, given by

$$\Pi_0 v|_K := |K|^{-1} \int_K v \, dx, \qquad \text{for } K \in \mathcal{T},$$

with $|\cdot| \equiv \text{meas}_d(\cdot)$ denoting the volume. We have the following approximation result.

Lemma 4.2. *Given the assumptions on the mesh, for each $v \in H^1(K)$, $K \in \mathcal{T}$, we have the bounds*

$$(4.3) \qquad \|v - \Pi_0 v\|_K \leq C h_K \|\nabla v\|_K,$$

and

$$(4.4) \qquad \|v - \Pi_0 v\|_{\partial K} \leq C \sqrt{h_K} \|\nabla v\|_K,$$

with $C > 0$ constant independent of the shape of $K \in \mathcal{T}$, on v and on h_K.

Proof. Due to the general, possibly curved, shape of the elements $K \in \mathcal{T}^{tr}$, a simple application of a standard Bramble-Hilbert type result (cf., e.g., [7]) and scaling is *not* sufficient to provide uniform constant C with respect to the shape of K. Instead, we work as follows. For $K \in \mathcal{T} \backslash \mathcal{T}^{tr}$, the results are well-known. For $K \in \mathcal{T}^{tr}$, the Friedrichs-type inequality proven in [40, Theorem 3.2], with explicit constant with respect to the domain, along with shape-regularity, yields (4.3). The bound (4.4) follows by combining (4.1) with (4.3). $\qquad \square$

For each $K \in \mathcal{T}^{tr}$, we shall require special sub-simplices contained in K, with certain properties, having, in particular, straight/planar faces.

Lemma 4.3. *Let $K \in \mathcal{T}^{tr}$. For each $v \in \mathcal{P}_p(K)$, there exists a simplex $K_\flat(v) \subset K$ with straight/planar faces such that*

$$|K| \leq C_\flat |K_\flat(v)| \qquad \text{with} \qquad \|v\|_{L^\infty(K)} = \|v\|_{L^\infty(K_\flat(v))},$$

where the positive constant $C_\flat$ is independent of v, h_K, and p, but depends, however, on the shape-regularity constant of K.

Proof. Let $K \in \mathcal{T}^{tr}$ and fix $v \in \mathcal{P}(K)$. Define $\mathbf{x}_K \in K$ to be a point where the maximum of v in K is attained, viz.,

$$\|v\|_{L^\infty(K)} = |v(\mathbf{x}_K)|.$$

To prove the result, it is sufficient to show that there exists a simplex $K_\flat(v) \subset K$ with straight/planar faces containing $\mathbf{x}_K \in K$ such that $|K| \leq C_\flat |K_\flat(v)|$. Recalling Definition 3.2, we observe that, for $\underline{K}$, we have $|\underline{K}| \sim |\tilde{K}|$ from shape-regularity. If $\mathbf{x}_K \in \underline{K}$, then we can take $K_\flat(v) := \underline{K}$. If $\mathbf{x}_K \in K \backslash \underline{K}$, the star-shapedness of K with respect to the midpoints of the faces (when $d = 2$) or the edges (when $d = 3$) allows for the construction of a simplex $K_\flat(v)$ with faces (when $d = 2$) or edges (when $d = 3$) defined by the line-segments connecting $\mathbf{x}_K$ with these midpoints. Given that the distance between $\mathbf{x}_K$ and these midpoints is equivalent to $h_{\underline{K}}/2$, the
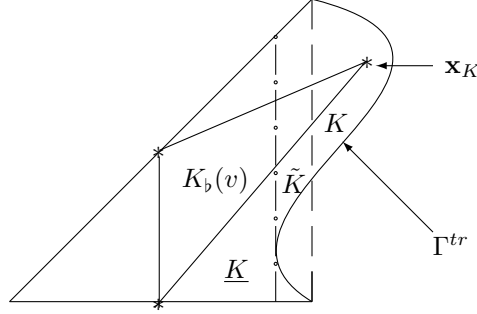
FIGURE 5. A curved element $K \in \mathcal{T}^{tr}$ (enclosed by the solid lines and curve,) its related element $\tilde{K}$ having the same vertices as K and straight faces (two same faces and the third depicted by a dashed line,) $\underline{K}$ (having two same faces and the third depicted by a dashed-dottedline,) and $K_\flat(v)$ for some $v \in \mathcal{P}_p(K)$ (enclosed by the solid lines with endpoints denoted by $*$.) Here, $\mathbf{x}_K$ is the point where the maximum of v in K is attained.

result follows. Since we have established that exists at least one $K_\flat(v)$ per element, we may define $K_\flat(v)$ as the one with the maximum area, to minimize the constant $C_\flat$. Notice that $C_\flat$ can be taken independent of the polynomial v, as the area of $K_\flat(v)$ is always bounded from below by a multiple of h_K^2 and K is compact. $\qquad\square$

We refer to Figure 5 for an illustration of the elements $K, \tilde{K}, \underline{K}$, and $K_\flat(v)$, for some $v \in \mathcal{P}_p(K)$. Notice that we have $K = \tilde{K} = \underline{K} = K_\flat(v)$, for all $v \in \mathcal{P}_p(K)$, when the face $E \subset \Gamma^{tr}$ of a $K \in \mathcal{T}^{tr}$ is *not* curved.

The above result is required to show the following crucial inverse-type estimates between L^2-norms of polynomials on curved elements $K \in \mathcal{T}^{tr}$ and their related elements $\tilde{K}$.

Lemma 4.4. *Let $K \in \mathcal{T}^{tr}$ and assume that the related element $\tilde{K}$ is such that*

$$(4.5) \qquad c_{inv} C_\flat p^{2d} |K \backslash \tilde{K}| < |K|,$$

with $c_{inv} > 0$ the constant of the inverse estimate $\|v\|_{L_\infty(\tilde{K})}^2 \le c_{inv} p^{2d} |\tilde{K}|^{-1} \|v\|_{\tilde{K}}^2$, for all $v \in \mathcal{P}_p(K)$. Then, the following estimate holds

$$\|v\|_K^2 \le \theta_{inv}(K) \|v\|_{K \cap \tilde{K}}^2,$$

where $\theta_{inv}(K) := |K| / \left(|K| - c_{inv} C_\flat p^{2d} |K \backslash \tilde{K}| \right)$.

Proof. Let $v \in \mathcal{P}_p(K)$. We have, respectively,

$$\begin{aligned}
\|v\|_K^2 &= \|v\|_{K \cap \tilde{K}}^2 + \|v\|_{K \backslash \tilde{K}}^2 \\
&\le \|v\|_{K \cap \tilde{K}}^2 + |K \backslash \tilde{K}| \|v\|_{L_\infty(K \backslash \tilde{K})}^2 \\
&\le \|v\|_{K \cap \tilde{K}}^2 + |K \backslash \tilde{K}| \|v\|_{L_\infty(K_\flat(v))}^2 \\
&\le \|v\|_{K \cap \tilde{K}}^2 + c_{inv} p^{2d} |K_\flat(v)|^{-1} |K \backslash \tilde{K}| \|v\|_{K_\flat(v)}^2 \\
&\le \|v\|_{K \cap \tilde{K}}^2 + c_{inv} C_\flat p^{2d} |K|^{-1} |K \backslash \tilde{K}| \|v\|_K^2,
\end{aligned}$$

as $K_\flat(v) \subset K$, using Lemma 4.3; the result is already implied. $\qquad\square$

Lemma 4.5. *Let $K \in \mathcal{T}^{tr}$ and let $\underline{K} \subset K$ and $\tilde{K}$ as in Definition 3.2 be such that*

$$(4.6) \qquad c_{inv} p^{2d} |\tilde{K} \backslash \underline{K}| < |\tilde{K}|,$$

for $c_{inv} > 0$ as in Lemma 4.4. Then, for each $v \in \mathcal{P}_p(K)$, the following estimate holds

$$\|v\|_{\tilde{K}}^2 \leq \eta_{inv}(K)\|v\|_{\underline{K}}^2,$$

where $\eta_{inv}(K) := |\tilde{K}|/\big(|\tilde{K}| - c_{inv}p^{2d}|\tilde{K}\backslash\underline{K}|\big)$.

Proof. Let $v \in \mathcal{P}_p(K)$. We have, respectively

$$\|v\|_{\tilde{K}}^2 = \|v\|_{\underline{K}}^2 + \|v\|_{\tilde{K}\backslash\underline{K}}^2 \leq \|v\|_{\underline{K}}^2 + |\tilde{K}\backslash\underline{K}|\|v\|_{L_\infty(\tilde{K}\backslash\underline{K})}^2$$

$$\leq \|v\|_{\underline{K}}^2 + |\tilde{K}\backslash\underline{K}|\|v\|_{L_\infty(\tilde{K})}^2 \leq \|v\|_{\underline{K}}^2 + c_{inv}p^{2d}|\tilde{K}|^{-1}|\tilde{K}\backslash\underline{K}|\|v\|_{\tilde{K}}^2,$$

which already implies the result. $\square$

Notice that, when $K \in \mathcal{T}^{tr}$ is convex, we have $\tilde{K} = \underline{K}$ and, thus, $\eta_{inv}(K) = 1$. Also, when $K \in \mathcal{T}^{tr}$ is not curved, we have $K = \tilde{K} = \underline{K}$ and, therefore, $\theta_{inv}(K) = 1 = \eta_{inv}(K)$.

Remark 4.6. It is possible to extend the applicability of the above estimates by replacing p^{2d} by p^2 in (4.6) at the expense of a, more involved to estimate, constant c_{inv}. We refer to [24, Lemma 3.7] for a similar construction. This remark also applies to (4.5) for the case where $K_\flat(v)$ and K have parallel faces.

Remark 4.7. A close inspection of the proof of Lemma 4.3 reveals that the shape-regularity assumption of K can be relaxed to requiring that there exists a, uniform across the mesh, constant $c_{alt} > 0$ such that $|\tilde{K}| \leq c_{alt}|\underline{K}|$. The constant $C_\flat$ will then depend on c_{alt} instead of the shape-regularity constant of K as stated in Lemma 4.3. This, in turn, implies the validity of the inverse estimates in Lemmata 4.4 and 4.5 in this setting also.

For the remaining of this work, we shall require the above inverse-type estimates, hence we make the following saturation assumption which can always be satisfied after a finite number of refinements of an original coarse mesh.

Assumption 4.8. We assume that the conditions (4.5) and (4.6) are satisfied for all elements $K \in \mathcal{T}^{tr}$.

We continue with a generalization of the standard inverse-type estimate from a face of an element to the element itself; here the face is allowed to be curved.

Lemma 4.9. *Let $K \in \mathcal{T}^{tr}$ such that a whole face of K, say E_{tr}, is contained in Γ^{tr}, and is, in general, curved. Then, for each $v \in \mathcal{P}_p(K)$, the inverse estimate*

$$\|v\|_{E_{tr}}^2 \leq C_{inv}\frac{p^2}{h_K}\|v\|_K^2,$$

with $C_{inv} > 0$ constant, independent of v, p, h_K and K, but dependent on the shape-regularity constant of K.

Proof. We partition E_{tr} into m $(d-1)$-dimensional pieces of equal measure, denoted by E_j, $j = 1,\ldots,m$. Further, we construct a partition of K into (curved) sub-elements K_j, by considering the simplices with one face E_j and the remaining vertex being the vertex of K opposite E_{tr}, when K is simplicial or by considering the prismatic elements obtained by extrusion of E_j orthogonally to the face of K opposite E_{tr}, when K is prismatic. We refer to Figure 6 for an illustration when $d = 2$.

Denoting by $\tilde{E}_j$ the straight/planar face of the related element $\tilde{K}_j$ approximating E_j, we have

$$(4.7) \qquad \|v\|_{\tilde{E}_j}^2 \leq Cp^2\frac{|\tilde{E}_j|}{|\tilde{K}_j|}\|v\|_{\tilde{K}_j}^2 \leq C\frac{p^2}{h_K}\|v\|_{\tilde{K}_j}^2 \leq C\eta_{inv}(K_j)\frac{p^2}{h_K}\|v\|_{K_j}^2,$$
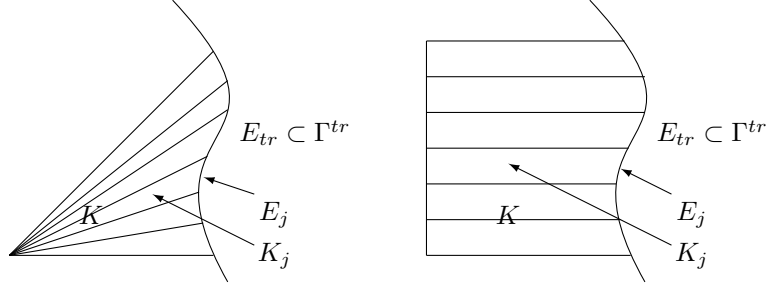
FIGURE 6. Curved elements $K \in \mathcal{T}^{tr}$ with their partitions.

as, in view of Remark 4.7, it is possible to apply Lemma 4.5 on each K_j (for sufficiently large m.)

As $m \to \infty$, K_j becomes infinitesimal in $(d-1)$-dimensions, and so, approximating K_j by $\tilde{K}_j$ produces arbitrarily small error in the geometry, resulting to $\eta_{inv}(K_j) \to 1$. Moreover, since E_{tr} admits a differentiable parametrization, we have $\lim_{m \to \infty} \sum_{j=1}^{m} \|v\|_{\tilde{E}_j}^2 = \|v\|_{E_{tr}}^2$. Therefore, summing (4.7) over j and taking $m \to \infty$, we arrive at the required result. $\qquad\square$

Lemma 4.10. *Let $K \in \mathcal{T}^{tr}$ and let E a face of K, such that $E \subset \partial K \backslash \Gamma^{tr}$. Then, for each $v \in \mathcal{P}_p(K)$, the inverse estimate*

$$\|v\|_E^2 \le C_{inv} \frac{p^2}{h_K} \|v\|_K^2,$$

with $C_{inv} > 0$ constant, independent of v, p, h_K and K, but dependent on the shape-regularity constant of K.

Proof. Fix $K \in \mathcal{T}^{tr}$ and a face $E_* \subset \partial K \backslash \Gamma^{tr}$. For $d = 2$, the star-shapedness with respect to the midpoints of the faces $E \subset \partial K \backslash \Gamma^{tr}$, allows for the existence of a straight-edged triangle $K_* \subset K$ having E_* as one face and as remaining vertex the midpoint of the other face $E \subset \partial K \backslash \Gamma^{tr}$ opposite to E_*. From the shape-regularity of K, we infer that $|K| \sim |K_*|$. On this triangle K_*, we can apply the standard inverse inequality to deduce

$$\|v\|_{E_*}^2 \le C \frac{p^2}{h_{K_*}} \|v\|_{K_*}^2 \le C_{inv} \frac{p^2}{h_K} \|v\|_K^2,$$

as required.

For $d = 3$ and K with $\text{meas}_2(\partial K \cap \Gamma^{tr}) > 0$, we approximate $E_* \subset \partial K \backslash \Gamma^{tr}$ by a quasiuniform triangulation consisting of m triangles, denoted by $\tilde{E}_j$, $j = 1, \dots, m$. Let x_{E_*} be the midpoint of an edge of K which is not an edge of E_* and consider the straight-faced pyramids $\tilde{K}_j$, $j = 1, \dots, m$, having $\tilde{E}_j$ as one base and x_{E_*} as remaining vertex. On each $\tilde{K}_j$, we can apply the standard inverse estimate

$$\sum_{j=1}^{m} \|v\|_{\tilde{E}_j}^2 \le C p^2 \sum_{j=1}^{m} \frac{|\tilde{E}_j|}{|\tilde{K}_j|} \|v\|_{\tilde{K}_j}^2 \le C \frac{p^2}{h_K} \sum_{j=1}^{m} \|v\|_{\tilde{K}_j}^2 \le C \eta_{inv}(K_j) \frac{p^2}{h_K} \sum_{j=1}^{m} \|v\|_{K_j}^2,$$

working as before, with K_j the pyramid with (curved) base on E_* and vertex x_{E_*} corresponding to the (straight) pyramid $\tilde{K}_j$. Taking $m \to \infty$ gives the result. $\qquad\square$

5. RECOVERY OPERATOR

An important tool for the a posteriori analysis will be a conforming recovery operator in the spirit of the original construction by Karakashian and Pascal [29]. In particular, we shall modify the construction from [29] to allow for discontinuous

functions across Γ^{tr} and for curved elemental faces and edges on Γ^{tr}, under the following assumption.

Assumption 5.1. We define the positive function $\theta : L_2(\Omega_1 \cup \Omega_2) \to \mathbb{R}$ with $\theta|_K := \theta_{inv}(K)$, for $K \in \mathcal{T}^{tr}$, $\theta|_K := 1$, for $K \in \mathcal{T} \backslash \mathcal{T}^{tr}$, and $\theta := \{\theta\}$ on $\Gamma \backslash \Gamma^{tr}$. We also define a function $\eta : L_2(\Omega_1 \cup \Omega_2) \to \mathbb{R}$ analogously starting from $\eta_{inv}(K)$, for $K \in \mathcal{T}^{tr}$. For the remaining of this work, we shall assume that θ and η are locally quasi-uniform.

Lemma 5.2. *Given the above mesh assumptions, there exists a recovery operator* $\mathcal{E} : S_h^p \to \mathcal{H}_0^1$, *such that*

$$(5.1) \qquad \sum_{K \in \mathcal{T}} \|\nabla^\alpha (v_h - \mathcal{E}(v_h))\|_K^2 \leq C_\alpha \sum_{E \subset \Gamma \backslash \Gamma^{tr}} \|\sqrt{\theta \eta} \mathbf{h}^{1/2-\alpha} [\![v_h]\!]\|_E^2,$$

for all $v_h \in S_h^p$, $C_\alpha > 0$, $\alpha = 0, 1$, *independent of* v_h, θ *and* $\mathbf{h}$.

Proof. The proof is based on the one of [29, Theorem 2.2]; particular care is given in dealing with the additional challenges posed by the, possibly curved, interface elements. Without loss of generality, we assume that the mesh is conforming on each Ω_i, $i = 1, 2$, i.e., no hanging nodes are present; for, otherwise, we perform a finite number of 'green' refinements to remove the hanging nodes, and we consider the new refined mesh in the place of the original $\mathcal{T}$ in what follows.

We begin by choosing a (Lagrange) basis for S_h^p. For each $K \in \mathcal{T} \backslash \mathcal{T}^{tr}$, we consider the standard Lagrange degrees of freedom. For each $K \in \mathcal{T}^{tr}$, we choose respective the Lagrange basis of $\tilde{K}$. Let $\mathcal{N}$ denote the set of all Lagrange nodes of S_h^p, and we define five of its subsets:

- $\mathcal{N}_0$ the set of all internal elemental nodes;
- $\mathcal{N}_{int}$ the set of all nodes situated on Γ^{int};
- $\mathcal{N}_{\partial\Omega}$ the set of all nodes situated on $\partial\Omega$;
- $\mathcal{N}_{tr}$ the set of all nodes situated on Γ^{tr};
- $\mathcal{N}_{out}$ the set of nodes belonging to each element $K \in \mathcal{T}^{tr}$, situated *outside* K. (E.g., the node of K_1 situated at the midpoint of the linear segment $\nu_1 \nu_2$ in Figure 7.)
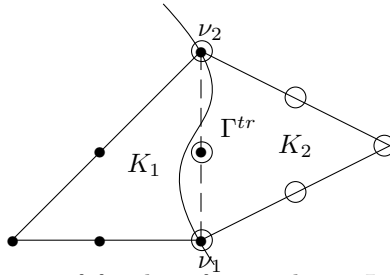

FIGURE 7. Degrees of freedom for quadratic Lagrange interface elements $K_1, K_2 \in \mathcal{T}^{tr}$, denoted by $\bullet$ and $\circ$, respectively. Note that one degree of freedom of K_1 is situated *outside* K_1 (i.e., in $\bar{\tilde{K}}_1 \backslash \bar{K}_1$,) at the midpoint of the linear segment $\nu_1 \nu_2$.

Evidently, we have $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_{int} \cup \mathcal{N}_{\partial\Omega} \cup \mathcal{N}_{tr} \cup \mathcal{N}_{out}$. Note, however, that, $\mathcal{N}_{out} \cap \mathcal{N}_0 \neq \emptyset$, in general; for an illustration consider the node ν situated at the midpoint of the linear segment $\nu_1 \nu_2$ in Figure 7: ν viewed as a node for K_2 implies $\nu \in \mathcal{N}_0$ and viewed as a node for K_1 implies $\nu \in \mathcal{N}_{out}$.

Further, let $\mathcal{N}_{tr}^i$ and $\mathcal{N}_{out}^i$ denote the two subsets of the interface nodes $\mathcal{N}_{tr}$, and the 'outer' nodes $\mathcal{N}_{out}$ associated with the Lagrange basis functions from the

respective elements belonging to Ω_1 and Ω_2 only, respectively. Note that if non-matching grids are used across the interface Γ^{tr}, $\mathcal{N}_{tr}^1$ and $\mathcal{N}_{tr}^2$ are, in general different and strict subsets of $\mathcal{N}_{tr}$; if, on the other hand, there are no hanging nodes on the interface, the $\mathcal{N}_{tr}^i$, $i = 1, 2$, are each a copy of $\mathcal{N}_{tr}$. Completely analogous properties characterise $\mathcal{N}_{out}^i$, $i = 1, 2$ also.

For each node $\nu \in \mathcal{N} \backslash (\mathcal{N}_{tr} \cup \mathcal{N}_{out})$, we define its element-neighbourhood

$$\omega_\nu := \{K \in \mathcal{T} : \nu \in \bar{K}\},$$

along with its cardinality $|\omega_\nu|$. Note that, when $\nu \in \mathcal{N}_0$, we have $|\omega_\nu| = 1$. Also, for each node $\nu \in \mathcal{N}_{tr}^i$, $i = 1, 2$, we define its 'one-sided' element neighbourhood

$$\omega_\nu^i := \{K \in \mathcal{T} : K \subset \Omega_i, \nu \in \bar{K}\}, \qquad i = 1, 2,$$

along with its cardinality $|\omega_\nu^i|$, $i = 1, 2$, while for each node $\nu \in \mathcal{N}_{out}$, we define its 'one-sided' element neighbourhood

$$\omega_\nu^i := \{K \in \mathcal{T}^{tr} : K \subset \Omega_i, \nu \in \bar{\bar{K}}\}, \qquad i = 1, 2,$$

along with its cardinality $|\omega_\nu^i|$. Finally, for each $K \in \mathcal{T}$, let $\mathcal{N}_K := \{\nu : \nu \in \bar{\bar{K}}\}$, the set of Lagrange nodes of K.

The recovery operator $\mathcal{E} : S_h^p \to \mathcal{H}_0^1$ is defined by determining its nodal values N_ν at each of the Lagrange nodes $\nu \in \mathcal{N}$:

$$(5.2) \qquad N_\nu(\mathcal{E}(v_h)) := \begin{cases} 0, & \text{if } \nu \in \mathcal{N}_{\partial\Omega}; \\ \dfrac{1}{|\omega_\nu|} \displaystyle\sum_{K \in \omega_\nu} N_\nu(v_h|_K), & \text{if } \nu \in \mathcal{N}_{int} \cup \mathcal{N}_0; \\ \dfrac{1}{|\omega_\nu^i|} \displaystyle\sum_{K \in \omega_\nu^i} N_\nu(v_h|_K), & \text{if } \nu \in \mathcal{N}_{tr}^i \cup \mathcal{N}_{out}^i, \ i = 1, 2. \end{cases}$$

Note that $\mathcal{E}(v_h)$ will be, in general, discontinuous across Γ^{tr}. Therefore, denoting by ϕ_ν the conforming Lagrange basis function at the node ν, (which may, nonetheless, be discontinuous across Γ^{tr},) we have

$$(5.3) \qquad \mathcal{E}(v_h) = \sum_{\nu \in \mathcal{N}} N_\nu(\mathcal{E}(v_h))\phi_\nu,$$

allowing for the regular nodes on $\nu \in \mathcal{N}_{tr}$ to be counted twice in the summation, i.e., once for each $i = 1, 2$; here we have used the convention that ϕ_ν signifies its *restriction* onto the respective element K for all nodes $\nu \in \mathcal{N}_{out}$. Hence, we have $\mathcal{E}(v_h) \in \mathcal{H}_0^1$.

From this, we deduce, respectively,

$$\sum_{K \in \mathcal{T}} \|\nabla(v_h - \mathcal{E}(v_h))\|_K^2 \leq \sum_{K \in \mathcal{T}} \theta_{inv}(K) \|\nabla(v_h - \mathcal{E}(v_h))\|_{K \cap \tilde{K}}^2$$

$$\leq \sum_{K \in \mathcal{T}} \sum_{\nu \in \mathcal{N}_K} \theta_{inv}(K) |N_\nu(v_h|_K) - N_\nu(\mathcal{E}(v_h))|^2 \|\nabla\phi_\nu\|_{K \cap \tilde{K}}^2$$

$$\leq C \sum_{\nu \in \mathcal{N} \backslash \mathcal{N}_0} \theta(\nu) \mathbf{h}^{d-2}(\nu) \sum_{K : \nu \in \mathcal{N}_K} |N_\nu(v_h|_K) - N_\nu(\mathcal{E}(v_h))|^2$$

$$=: C \sum_{\nu \in \mathcal{N} \backslash \mathcal{N}_0} I_\nu,$$

using the standard bound $\|\nabla\phi_\nu\|_{K \cap \tilde{K}}^2 \leq \|\nabla\phi_\nu\|_{\tilde{K}}^2 \leq Ch_{\tilde{K}}^{d-2}$, where $\mathbf{h}(\nu)$ and $\theta(\nu)$ are given by extending the definitions of the functions $\mathbf{h}$ and θ (to include the mesh

nodes also)

$$
\mathbf{h}(\nu) := \begin{cases} \dfrac{1}{|\omega_\nu|} \displaystyle\sum_{K \in \omega_\nu} h_K, & \text{if } \nu \in \mathcal{N}_{int} \cup \mathcal{N}_{\partial\Omega}; \\[2ex] \dfrac{1}{|\omega_\nu^i|} \displaystyle\sum_{K \in \omega_\nu^i} h_{\tilde{K}}, & \text{if } \nu \in \mathcal{N}_{tr}^i \cup \mathcal{N}_{out}^i, \ i = 1, 2, \end{cases}
$$

and $\theta(\nu) := 1$, if $\nu \in \mathcal{N}_{int} \cup \mathcal{N}_{\partial\Omega}$, while $\theta(\nu) := \frac{1}{|\omega_\nu^i|} \sum_{K \in \omega_\nu^i} \theta_{inv}(K)$, if $\nu \in \mathcal{N}_{tr}^i \cup \mathcal{N}_{out}^i$, $i = 1, 2$. We have

$$
\sum_{\nu \in \mathcal{N}_{\partial\Omega}} I_\nu = \sum_{\nu \in \mathcal{N}_{\partial\Omega}} \theta(\nu) \mathbf{h}^{d-2}(\nu) \sum_{K : \nu \in \mathcal{N}_K} \left| N_\nu(v_h|_K) \right|^2 \leq C \sum_{E \subset \partial\Omega} \|\sqrt{\theta} \mathbf{h}^{\frac{d-2}{2}} v_h\|_{L_\infty(E)}^2.
$$

Also,

$$
\sum_{\nu \in \mathcal{N}_{int} \cup \mathcal{N}_{tr} \cup \mathcal{N}_{out}} I_\nu \leq C \sum_{E \subset \Gamma^{int}} \sum_{\nu \in \tilde{E}} \theta(\nu) \mathbf{h}^{d-2}(\nu) \left| N_\nu(v_h|_{K_1}) - N_\nu(v_h|_{K_2}) \right|^2
$$
$$
\leq C \sum_{E \subset \Gamma^{int}} \|\sqrt{\theta} \mathbf{h}^{\frac{d-2}{2}} [\![v_h]\!]\|_{L_\infty(\tilde{E})}^2,
$$

with $\tilde{E} := E$ when $E \notin \Gamma_{tr}^{int}$. Note that $\tilde{E} \neq E$ is possible only for $d = 3$. The first inequality follows from applying the crucial [29, Lemma 2.2] and working as in the proof of [29, Theorem 2.2], while the last inequality follows from the shape-regularity property. We remark that, for $d = 2$, we have $|N_\nu(v_h|_K) - N_\nu(\mathcal{E}(v_h))| = 0$ as $|\omega_\nu^i| = 1$ when $\nu \in \mathcal{N}_{out}^i$, $i = 1, 2$, while for $d = 3$ and for $p = 1$, we have $\mathcal{N}_{out} = \emptyset$. For $d = 3$, and $p \geq 2$, we may have $|\omega_\nu^i| > 1$ and, thus, the above calculation is non-trivial for $\nu \in \mathcal{N}_{out}$.

Combining the above bounds, we arrive at

$$
\sum_{K \in \mathcal{T}} \|\nabla(v_h - \mathcal{E}(v_h))\|_K^2 \leq C \sum_{E \subset \Gamma \backslash \Gamma^{tr}} \|\sqrt{\theta} \mathbf{h}^{\frac{d-2}{2}} [\![v_h]\!]\|_{L_\infty(\tilde{E})}^2.
$$

Finally, applying the standard inverse estimate

$$
\|v\|_{L_\infty(\tilde{E})}^2 \leq C h_K^{-d+1} \|v\|_{\tilde{E}}^2,
$$

for all polynomials $v \in \mathcal{P}_p(\tilde{E})$, and using Lemma 4.5 with $K = E$, we deduce

$$
\|v\|_{\tilde{E}}^2 \leq \eta_{inv}(E) \|v\|_E^2.
$$

Hence, we obtain the required bound for $\alpha = 1$. The proof for $\alpha = 0$ is completely analogous. $\qquad\square$

Remark 5.3. When Γ^{tr} is not curved, i.e., when the mesh $\mathcal{T}$ does *not* contain any elements with curved faces, we have $\theta = 1 = \eta$ on $\Gamma \backslash \Gamma^{tr}$ in (5.1), thereby retrieving the known bound of Karakashian and Pascal [29, Theorem 2.2].

6. Stability of the method

We begin by assessing the stability of the discontinuous Galerkin method (3.3). To this end, on S_h^p, we introduce the dG-energy norm

$$
(6.1) \qquad \|\!|v_h|\!\| := \left(\sum_{K \in \mathcal{T}} \|\nabla v_h\|_K^2 + \|\sqrt{\gamma_0/\mathbf{h}} [\![v_h]\!]\|_{\Gamma \backslash \Gamma^{tr}}^2 + C_{tr} \|[\![v_h]\!]\|_{\Gamma^{tr}}^2 \right)^{\frac{1}{2}}.
$$

Theorem 6.1. *The discrete problem (3.3) admits a unique solution* $u_h \in S_h^p$ *for* $\gamma_0 := \gamma\theta\eta \geq \gamma$ *with* $\gamma > 0$ *a sufficiently large constant depending on* p*. Moreover, we have the stability bound*

$$
(6.2) \qquad\qquad\qquad \|\!|u_h|\!\| \leq C \|f\|,
$$

with $C = C(p, C_{inv}, C_0, C_1) > 0$ constant, with C_{inv} as in Lemma 4.10 and C_0, C_1 as in (5.1), but independent of the curvature of Γ^{tr} and of $\mathbf{h}$.

Proof. We begin by assessing the coercivity of the bilinear form (3.4). From the definition, we have

$$D_h(v_h, v_h) = \sum_{K \in \mathcal{T}} \|\nabla v_h\|_K^2 - 2 \int_{\Gamma \setminus \Gamma^{tr}} \{\nabla v_h\} \cdot [\![v_h]\!] ds + \|\sqrt{\gamma_0/\mathbf{h}}[\![v_h]\!]\|_{\Gamma \setminus \Gamma^{tr}}^2 + C_{tr} \|[\![v_h]\!]\|_{\Gamma^{tr}}^2,$$

for all $v_h \in S_h^p$. To bound the second term in the right-hand side of the above equation we use the inverse estimate of Lemma 4.10 to deduce, in a standard fashion,

$$(6.3) \quad \int_{\Gamma \setminus \Gamma^{tr}} \{\nabla u_h\} \cdot [\![u_h]\!] \, ds \leq C \sum_{K \in \mathcal{T}} C_{inv} p^2 \|\sqrt{1/\gamma_0}\, \nabla u_h\|_K^2 + \frac{1}{4} \|\sqrt{\gamma_0/\mathbf{h}}[\![v_h]\!]\|_{\Gamma \setminus \Gamma^{tr}}^2.$$

Hence,

$$D_h(v_h, v_h) \geq (1 - \frac{2CC_{inv}p^2}{\gamma}) \sum_{K \in \mathcal{T}} \|\nabla v_h\|^2 + \frac{1}{2} \|\sqrt{\gamma_0/\mathbf{h}}[\![v_h]\!]\|_{\Gamma \setminus \Gamma^{tr}}^2 + C_{tr} \|[\![v_h]\!]\|_{\Gamma^{tr}}^2,$$

and we conclude that D_h is coercive if $\gamma > 2CC_{inv}p^2$. The proof of the continuity of D_h is standard and is omitted for brevity, and the existence of a unique u_h solving (3.3) now follows by the Lax-Milgram Lemma. To prove (6.2), from (3.3) and (6.3), we have

$$\|\|u_h\|\|^2 = 2 \int_{\Gamma \setminus \Gamma^{tr}} \{\nabla u_h\} \cdot [\![u_h]\!] \, ds + \langle f, u_h \rangle$$

$$(6.4) \qquad \leq \frac{2CC_{inv}p^2}{\gamma} \sum_{K \in \mathcal{T}} \|\nabla u_h\|_K^2 + \frac{1}{2} \|\sqrt{\gamma_0/\mathbf{h}}[\![v_h]\!]\|_{\Gamma \setminus \Gamma^{tr}}^2 + \frac{1}{\epsilon} \|f\|^2 + \frac{\epsilon}{4} \|u_h\|^2,$$

for any $\epsilon > 0$. To bound the last term on the right-hand side of (6.4), we use the Poincaré inequality and Lemma 5.2, respectively:

$$\|u_h\|^2 \leq \|u_h - \mathcal{E}(u_h)\|^2 + C \sum_{i=1,2} \|\nabla \mathcal{E}(u_h)\|_{\Omega_i}^2$$

$$\leq C \Big(\sum_{\alpha=0,1} \sum_{K \in \mathcal{T}} \|\nabla^\alpha (u_h - \mathcal{E}(u_h))\|_K^2 + \sum_{K \in \mathcal{T}} \|\nabla u_h\|_K^2 \Big)$$

$$\leq C \Big(\frac{1}{\gamma} \sum_{E \subset \Gamma \setminus \Gamma^{tr}} \|\sqrt{\gamma_0/\mathbf{h}}[\![v_h]\!]\|_E^2 + \sum_{K \in \mathcal{T}} \|\nabla u_h\|_K^2 \Big),$$

having used the assumption that $\mathbf{h} < 1$ and $\gamma_0 = \gamma \theta \eta$. The required bound now follows by fixing $\epsilon < \frac{4}{C} \max\{\frac{\gamma}{2}, 1 - \frac{2CC_{inv}p^2}{\gamma}\}$. $\qquad \square$

7. A POSTERIORI ERROR BOUND

Letting $\Pi : L_2(\Omega) \to S_h^p$ denote the orthogonal L_2-projection operator onto the discontinuous finite element space, we begin by defining the a posteriori error indicator

$$(7.1) \quad \Upsilon := \Big(\sum_{K \in \mathcal{T}} \Upsilon_K^2 \Big)^{1/2}, \quad \text{with} \quad \Upsilon_K := \Big(\Upsilon_{R_K}^2 + \Upsilon_{E_K}^2 + \Upsilon_{J_K}^2 + \Upsilon_{Tr_K}^2 \Big)^{1/2},$$

comprising of the interior, normal flux, jump and interface residuals

$$\Upsilon_{R_K} := \|\mathbf{h}(\Pi f + \Delta u_h)\|_K, \quad \Upsilon_{E_K} := \|\sqrt{\mathbf{h}}[\![\nabla u_h]\!]\|_{\partial K \cap \Gamma^{int}},$$

$$\Upsilon_{J_K} := \sqrt{\gamma} \|\sqrt{\gamma_0/\mathbf{h}}[\![u_h]\!]\|_{\partial K \cap \Gamma \setminus \Gamma^{tr}}, \quad \Upsilon_{Tr_K} := \sum_{i=1}^2 \|\sqrt{\mathbf{h}}(C_{tr}[\![u_h]\!] + \nabla u_h) \cdot \mathbf{n}^i\|_{\partial K \cap \Gamma^{tr}}.$$

We also define the data oscillation term

$$\Theta_1 := \|\mathbf{h}(f - \Pi f)\|,$$

along its restriction on each K, $\Theta_{1,K} := \|\mathbf{h}(f - \Pi f)\|_K$.

7.1. Upper bound.

For the proof of an a posteriori bound, we use the conforming recovery operator. To this end, we decompose the error into conforming and non-conforming parts $u - u_h = e^c + u_h^d$, with $e^c := u - \mathcal{E}(u_h)$ and $u_h^d := \mathcal{E}(u_h) - u_h$, noting that $e^c \in \mathcal{H}_0^1$.

We consider an (inconsistent) extension $\hat{D}_h : (\mathcal{H}_0^1 + S_h^p) \times (\mathcal{H}_0^1 + S_h^p) \to \mathbb{R}$ of the bilinear form D_h, given by

$$\hat{D}_h(w, v) = \sum_{K \in \mathcal{T}} \int_K \nabla w \cdot \nabla v \, dx - \int_{\Gamma \setminus \Gamma^{tr}} (\{\nabla \Pi w\} \cdot [\![v]\!] + \{\nabla \Pi v\} \cdot [\![w]\!]) ds$$
$$+ \int_{\Gamma \setminus \Gamma^{tr}} \frac{\gamma_0}{\mathbf{h}} [\![w]\!] \cdot [\![v]\!] ds + \int_{\Gamma^{tr}} C_{tr} [\![w]\!] \cdot [\![v]\!] ds.$$

Then, we have immediately that $\hat{D}_h(w_h, v_h) = D_h(w_h, v_h)$ for all $w_h, v_h \in S_h^p$, and also $\hat{D}_h(w, v) = D(w, v)$ for all $w, v \in \mathcal{H}_0^1$. We also note the continuity property

$$(7.2) \qquad \hat{D}_h(w, v) \leq C \||w\|\| \||v\|\|, \qquad \text{for all } w, v \in \mathcal{H}_0^1 + S_h^p,$$

which can be shown in a standard fashion once equipped with the inverse estimate from Lemma 4.10.

The error equation can be derived as follows

$$\||e^c\|\|^2 = D(e^c, e^c) = D(u, e^c) - \hat{D}_h(u_h, e^c) - \hat{D}_h(u_h^d, e^c)$$
$$= \int_\Omega f e^c \, dx - \hat{D}_h(u_h, e^c) - \hat{D}_h(u_h^d, e^c) + D_h(u_h, \Pi_0 e^c) - \int_\Omega f \Pi_0 e^c \, dx$$
$$= \left(\int_\Omega f(e^c - \Pi_0 e^c) \, dx - \hat{D}_h(u_h, e^c - \Pi_0 e^c) \right) - \hat{D}_h(u_h^d, e^c) =: I - II.$$

We estimate the terms I and II above in the following lemmata.

Lemma 7.1. *We have*

$$I \leq C (\Upsilon^2 + \Theta_1^2)^{1/2} \|\nabla e^c\|,$$

for $\gamma_0 = \gamma \theta \eta$ for some $\gamma > 1$ as in the proof of Theorem 6.1.

Proof. Integration by parts yields

$$I = \sum_{K \in \mathcal{T}} \int_K (f + \Delta u_h)(e^c - \Pi_0 e^c) \, dx - \int_{\Gamma^{int}} [\![\nabla u_h]\!] \cdot \{e^c - \Pi_0 e^c\} ds$$
$$(7.3) \qquad + \int_{\Gamma \setminus \Gamma^{tr}} [\![u_h]\!] \cdot \{\nabla \Pi(e^c - \Pi_0 e^c)\} ds - \int_{\Gamma \setminus \Gamma^{tr}} \frac{\gamma_0}{\mathbf{h}} [\![u_h]\!] \cdot [\![e^c - \Pi_0 e^c]\!] ds$$
$$- \int_{\Gamma^{tr}} \left(C_{tr} [\![u_h]\!] \cdot [\![e^c - \Pi_0 e^c]\!] + [\![\nabla u_h (e^c - \Pi_0 e^c)]\!] \right) ds.$$

We focus on estimating the third term on the right-hand side of (7.3), which can be bounded as

$$(7.4) \qquad \|\sqrt{\mathbf{h}/(\gamma \gamma_0)} \{\nabla \Pi(e^c - \Pi_0 e^c)\}\|_{\Gamma \setminus \Gamma^{tr}} \|\sqrt{\gamma \gamma_0/\mathbf{h}} [\![u_h]\!]\|_{\Gamma \setminus \Gamma^{tr}}.$$

The term involving e^c can be further bounded by

$$\sum_{E\in\Gamma\backslash\Gamma^{tr}}\|\sqrt{\mathbf{h}/(\gamma\gamma_0)}\{\nabla\Pi(e^c-\Pi_0 e^c)\}\|_E^2 \le C\frac{C_{inv}p^2}{\gamma}\sum_{K\in\mathcal{T}}\|\sqrt{1/\gamma_0}\nabla\Pi(e^c-\Pi_0 e^c)\|_K^2$$

$$\le C\sum_{K\in\mathcal{T}}\|\sqrt{\theta/\gamma_0}\nabla\Pi(e^c-\Pi_0 e^c)\|_{\tilde{K}}^2 \le C\sum_{K\in\mathcal{T}}\|\sqrt{\theta/\gamma_0}\mathbf{h}^{-1}\Pi(e^c-\Pi_0 e^c)\|_{\tilde{K}}^2$$

$$\le C\sum_{K\in\mathcal{T}}\|\sqrt{\theta\eta/\gamma_0}\mathbf{h}^{-1}\Pi(e^c-\Pi_0 e^c)\|_K^2 \le C\sum_{K\in\mathcal{T}}\|\nabla e^c\|_K^2,$$

using Lemma 4.10, Lemma 4.4, a standard inverse estimate on $\tilde{K}$, and Lemma 4.5, respectively and observing that $\theta\eta/\gamma_0 = 1/\gamma < 1$; hence, we get

$$\int_{\Gamma\backslash\Gamma^{tr}}[\![u_h]\!]\cdot\{\nabla\Pi(e^c-\Pi_0 e^c)\}ds \le C\gamma^{1/2}\|\sqrt{\gamma_0/\mathbf{h}}[\![u_h]\!]\|_{\Gamma\backslash\Gamma^{tr}}\|\nabla e^c\|.$$

Working in a standard fashion for the remaining terms, we deduce

$$I \le C\Big(\sum_{K\in\mathcal{T}}h_K^2\|f+\Delta u_h\|_K^2 + \|\sqrt{\mathbf{h}}[\![\nabla u_h]\!]\|_{\Gamma^{int}}^2 + \gamma\|\sqrt{\gamma_0/\mathbf{h}}[\![u_h]\!]\|_{\Gamma\backslash\Gamma^{tr}}^2\Big)^{\frac{1}{2}}\|\nabla e^c\|$$

$$+ C\sum_{i=1}^{2}\sum_{K\in\mathcal{T}^{tr}}\|\sqrt{\mathbf{h}}(C_{tr}[\![u_h]\!]+\nabla u_h)\cdot\mathbf{n}^i\|_{\partial K\cap\Gamma^{tr}}\|\nabla e^c\|_K,$$

using the approximation bounds given in Lemma 4.2. The result already follows. □

To estimate II, we use (7.2) for $\hat{D}_h$, along with the following bound.

Lemma 7.2. *With the above mesh assumptions and choice of γ_0, we have*

$$(7.5) \qquad \||u_h^d\||^2 \le C\sum_{K\in\mathcal{T}}\big(1+\gamma^{-1}(1+h_K C_{tr})\big)\Upsilon_{J_K}^2,$$

for $C > 0$ generic constant, independent of $\mathbf{h}$ and of u_h.

Proof. Using Lemma 5.2, we have

$$\sum_{K\in\mathcal{T}}\|\nabla u_h^d\|_K^2 + \|\sqrt{\gamma_0/\mathbf{h}}[\![u_h^d]\!]\|_{\Gamma\backslash\Gamma^{tr}}^2 \le C\sum_{K\in\mathcal{T}}(C_1\gamma^{-1}+1)\Upsilon_{J_K}^2.$$

For the third term on the right hand side of (6.1), we use (4.1) and Lemma 5.2 once more to deduce

$$C_{tr}\|[\![u_h^d]\!]\|_{\Gamma^{tr}}^2 \le C_{tr}\sum_{j=1}^{2}\|u_h^d|_{\Omega_j}\|_{\Gamma^{tr}}^2 \le CC_{tr}\sum_{K\in\mathcal{T}^{tr}}\big(h_K\|\nabla u_h^d\|_K^2 + h_K^{-1}\|u_h^d\|_K^2\big)$$

$$\le CC_{tr}\Big((C_1+C_0)\sum_{E\subset\Gamma^{int}}\|\sqrt{\theta\eta}[\![u_h]\!]\|_E^2\Big)$$

$$\le CC_{tr}\Big(\gamma^{-1}\sum_{K\in\mathcal{T}}h_K\Upsilon_{J_K}^2\Big).$$

Combining the last two bounds already yields the result. □

Theorem 7.3 (Upper bound). *Let u be the solution of (2.1) and let $u_h \in S_h^p$ be its dG approximation with γ_0 as in the statement of Lemma 7.1. Then, we have the following a posteriori error bound*

$$(7.6) \qquad \||u-u_h\||^2 \le C\left(\Upsilon^2+\Theta_1^2\right) + C\sum_{K\in\mathcal{T}}\big(1+\gamma^{-1}(1+h_K C_{tr})\big)\Upsilon_{J_K}^2.$$

Proof. The proof follows immediately from the error equation, the bounds on I and on II, along with the triangle inequality $\||u-u_h\|| \le \||e^c\|| + \||u_h^d\||$. □

7.2. **Lower bound.** We employ standard bubble functions, [39], to show lower bounds for the above a posteriori estimator. A key challenge to overcome is the shape of interface elements $K \in \mathcal{T}^{tr}$, for which we do not posses any stability properties of respective elemental or face bubble functions. We shall overcome this by employing bubble functions on $\underline{K}$ and on $\underline{E}$ instead.

To this end, we denote by ω_E the union of the elements sharing an interior face $E \in \Gamma^{int} \backslash \Gamma_{tr}^{int}$, and by ψ_K and ψ_E the (standard) element and face bubble functions [39]. The functions $\psi_K \in H_0^1(K)$ and $\psi_E \in H_0^1(\omega_E)$ are such that $\|\psi_K\|_{L_\infty(K)} = 1$, and $\|\psi_E\|_{L_\infty(E)} = 1$. Moreover, for each $v \in S_h^p$, there exist positive constants c_1, c_2, independent of $\mathbf{h}$ and of v, such that

$$(7.7) \qquad \|v\|_K^2 \leq c_1 \|\sqrt{\psi_K} v\|_K^2, \quad \|v\|_E^2 \leq c_2 \|\sqrt{\psi_E} v\|_E^2,$$

for all $K \in \mathcal{T} \backslash \mathcal{T}^{tr}$ and $E \subset \Gamma^{int} \backslash \Gamma_{tr}^{int}$. When $K \in \mathcal{T}^{tr}$ or when $E \in \Gamma_{tr}^{int}$, (7.7) holds for $\underline{K}$ or $\underline{E}$ instead, respectively, so that, by Lemma 4.4 and Lemma 4.5,

$$(7.8) \qquad \|v\|_K^2 \leq \theta\eta(K) \|v\|_{\underline{K}}^2 \leq c_1 \theta\eta(K) \|\sqrt{\psi_{\underline{K}}} v\|_{\underline{K}}^2,$$

where $\theta\eta(K) := \theta_{inv}(K)\eta_{inv}(K)$ and

$$(7.9) \qquad \|v\|_E^2 \leq \theta\eta(E) \|v\|_{\underline{E}}^2 \leq c_2 \theta\eta(E) \|\sqrt{\psi_{\underline{E}}} v\|_{\underline{E}}^2,$$

with $\theta\eta(E) := \theta_{inv}(E)\eta_{inv}(E)$. To treat both the cases $K \in \mathcal{T} \backslash \mathcal{T}^{tr}$ and $K \in \mathcal{T}^{tr}$ simultaneously, we shall carry over the θ and η terms (along with $\underline{K}$ and $\underline{E}$) for all $K \in \mathcal{T}$, recalling that $\theta_{inv}(K) = 1 = \eta_{inv}(K)$ (and $\underline{K} = K$, $\underline{E} = E$) when $K \in \mathcal{T} \backslash \mathcal{T}^{tr}$.

We can now show a lower bound for the a posteriori error estimator.

Theorem 7.4 (lower bound). *Let u be the solution of (2.1) and let $u_h \in S_h^p$ the dG solution given by (3.3). Then, for all $K \in \mathcal{T}$, we have the following bound*

$$(7.10) \qquad \Upsilon_{R_K}^2 + \Upsilon_{E_K}^2 \leq C \sum_{K' \in \omega_K} (\theta\eta(K'))^2 \big(\|\nabla(u - u_h)\|_{K'}^2 + \Theta_{1,K'}^2 \big),$$

where $\omega_K := \{K' \in \mathcal{T} : meas_{d-1}((\partial K \cap \partial K') \backslash \Gamma^{tr}) \neq 0\}$. Further, for two elements $K_i \in \mathcal{T}^{tr}$ sharing a face $E \subset \Gamma^{tr}$, we have the bound
$$(7.11)$$
$$\sum_{i=1}^2 \|\sqrt{\mathbf{h}}(C_{tr}[\![u_h]\!] + \nabla u_h) \cdot \tilde{\mathbf{n}}^i\|_{\tilde{E}_i}^2 \leq C \sum_{i=1}^2 \Big((\theta\eta(K_i))^2 \big(\|\nabla(u - u_h)\|_{K_i}^2 + \Theta_{1,K_i}^2 \big) + \Theta_{2,K_i}^2 \Big),$$

where $\tilde{E}_i := \tilde{E} \cap \partial \tilde{K}_i$, $i = 1, 2$, represent the related faces $\tilde{E}$, signifying that the values of a function on $\tilde{E}_i$ are taken from within $\tilde{K}_i$. Also, $\tilde{\mathbf{n}}^i$ denote the respective outward normal to $\tilde{E}_i$. Finally, $\Theta_{2,K_i} := |\tilde{K}_i \triangle K_i| h_{K_i}^{-d} \|C_{tr}[\![u_h]\!] + \nabla u_h\|_{\tilde{E}_i}$ is the interface oscillation term, with $P \triangle Q := (P \backslash Q) \cup (Q \backslash P)$ denoting the symmetric difference between two sets P and Q.

Proof. We first prove (7.10). For the interior residual, for $K \in \mathcal{T}$, we set $R|_K := (\Pi f + \Delta u_h)|_K$, and $M|_K := h_K^2 R \psi_{\underline{K}}$. Then, using (7.8), we have

$$(7.12) \quad \Upsilon_{R_K}^2 = h_K^2 \|R\|_K^2 \leq c_1 \theta\eta(K) h_K^2 \|\sqrt{\psi_{\underline{K}}} R\|_{\underline{K}}^2 = c_1 \theta\eta(K) \langle \Pi f + \Delta u_h, M \rangle_{\underline{K}}.$$

Using integration by parts along with (2.2) yields

$$(7.13) \qquad \begin{aligned} \langle \Pi f + \Delta u_h, M \rangle_{\underline{K}} &= \langle \nabla(u - u_h), \nabla M \rangle_{\underline{K}} + \langle \Pi f - f, M \rangle_{\underline{K}} \\ &\leq \|\nabla(u - u_h)\|_{\underline{K}} \|\nabla M\|_{\underline{K}} + h_K \|\Pi f - f\|_K h_K^{-1} \|M\|_K. \end{aligned}$$

Further, as $h_K \sim h_{\underline{K}}$, we have $\|\nabla M\|_{\underline{K}}^2 \leq C h_K^{-2} \|M\|_{\underline{K}}^2 \leq C h_K^2 \|R\|_K^2$, which, used on (7.13) and in view of (7.12), implies

$$(7.14) \qquad \Upsilon_{R_K}^2 \leq C\theta\eta(K) \big(\|\nabla(u - u_h)\|_K + \|\mathbf{h}(\Pi f - f)\|_K \big) \Upsilon_{R_K},$$

which already gives the required bound.

For the normal flux residual, for $E \subset \Gamma^{int}$, we set $\omega_E := K_1 \cup K_2 \cup E$ with $K_1, K_2 \in \mathcal{T}$ such that $E = \partial K_1 \cap \partial K_2$ and, on ω_E, we define the function $\tau_E := h(E)[\![\nabla u_h]\!]\psi_{\underline{E} \cap \partial \underline{K}_1 \cap \partial \underline{K}_2}$. Here, $[\![\nabla u_h]\!]$ in ω_E is understood as its constant extension in the normal direction to E. (Notice that $E \in \Gamma \backslash \Gamma^{tr}$ is *not* curved, so there is a unique normal direction to E.) Since $[\![\nabla u]\!] = 0$ on Γ^{int}, and $\tau_E|_{\partial \omega_E} = 0$, with $\omega_E = K_1 \cup K_2 \cup E$ we have, for $K \in \mathcal{T}$,

(7.15)
$$\Upsilon^2_{E_K} \leq c_2 \sum_{E \subset \partial K \cap \Gamma^{int}} \theta\eta(E)\langle [\![\nabla u_h]\!], \tau_E\rangle_E = c_2 \sum_{E \subset \partial K \cap \Gamma^{int}} \theta\eta(E)\langle [\![\nabla(u_h - u)]\!], \tau_E\rangle_E.$$

Integration by parts and (2.2) imply

(7.16)
$$\begin{aligned} \langle [\![\nabla(u_h - u)]\!], \tau_E\rangle_E &= \langle \Delta u_h + f, \tau_E\rangle_{\omega_E} + \langle \nabla(u_h - u), \nabla\tau_E\rangle_{\omega_E} \\ &= \langle \Pi f + \Delta u_h, \tau_E\rangle_{\omega_E} + \langle f - \Pi f, \tau_E\rangle_{\omega_E} + \langle \nabla(u_h - u), \nabla\tau_E\rangle_{\omega_E}. \end{aligned}$$

Observing that $\tau_E = 0$ on $(K_1 \cup K_2 \cup E)\backslash(\underline{K}_1 \cup \underline{K}_2 \cup (\underline{E} \cap \partial \underline{K}_1 \cap \partial \underline{K}_2))$, a standard inverse estimate implies $\|\nabla\tau_E\|^2_{\underline{K}_i} \leq Ch_{K_i}^{-2}\|\tau_E\|^2_{\underline{K}_i} \leq Ch_{K_i}\|[\![\nabla u_h]\!]\|^2_{\underline{E} \cap \partial \underline{K}_i}$, which, used on (7.16) and in view of (7.15), implies

(7.17)
$$\Upsilon^2_{E_K} \leq Cc_2\bigg(\sum_{K' \in \omega_K} \big(\Upsilon^2_{R_{K'}} + \Theta^2_{K'}\big)^{\frac{1}{2}} \bigg)\Upsilon_{E_K}.$$

We now prove the bound on the interface residual (7.11). For $E \subset \partial K_1 \cap \partial K_2 \cap \Gamma^{tr}$, we consider the related face $\tilde{E}$ of $\tilde{K}_i$, $i = 1, 2$, and let also $\tilde{\mathbf{n}}$ signify the normal vector to $\tilde{E}$. We consider the face bubble $\psi^i_{\tilde{E}}$ supported in $\tilde{K}_i$, $i = 1, 2$, respectively. We shall also make use of the extension and/or restriction of $\psi^i_{\tilde{E}}$ onto K_i, $i = 1, 2$, denoted for simplicity also by $\psi^i_{\tilde{E}}$. Therefore, $\psi^i_{\tilde{E}} = 0$ on $\partial K_i \backslash E$ also, since $\psi^i_{\tilde{E}}$ is constructed to vanish on the $(d-1)$-dimensional hyperplanes containing the (straight) faces of K_i not belonging to the interface. We define

$$r^i_E := h(E)\big((C_{tr}[\![u_h]\!] + \nabla u_h) \cdot \tilde{\mathbf{n}}\,\psi^i_{\tilde{E}}\big)|_{K_i},$$

for $i = 1, 2$, where $(C_{tr}[\![u_h]\!] + \nabla u_h) \cdot \tilde{\mathbf{n}}$ in K_i is understood as its constant extension in the $\tilde{\mathbf{n}}$-direction. Setting $r_E|_{K_i} := r^i_E$, $i = 1, 2$, we have $r_E \in \mathcal{H}^1_0$ by construction. Using the interface conditions in (2.1) along with (2.2), we deduce

(7.18)
$$\begin{aligned} &\sum_{i=1}^2 \langle (C_{tr}[\![u_h]\!] + \nabla u_h) \cdot \mathbf{n}^i, r_E\rangle_{E_i} \\ &= \langle C_{tr}[\![u_h - u]\!], [\![r_E]\!]\rangle_E + \langle \Delta u_h + f, r_E\rangle_{K_1 \cup K_2} - \langle \nabla(u_h - u), \nabla r_E\rangle_{K_1 \cup K_2}, \end{aligned}$$

with $E_i := E \cap \partial K_i$, and $\mathbf{n}^i := \mathbf{n}|_{\Omega_i}$. Setting $N := C_{tr}[\![u_h]\!] + \nabla u_h$ for brevity, and using (7.18), we get

(7.19)
$$\begin{aligned} c_2^{-1} \sum_{i=1}^2 \|\sqrt{\mathbf{h}}N \cdot \tilde{\mathbf{n}}^i\|^2_{\tilde{E}_i} &\leq \sum_{i=1}^2 \langle N \cdot \tilde{\mathbf{n}}^i, r_E\rangle_{\tilde{E}_i} \\ &= \langle C_{tr}[\![u_h - u]\!], [\![r_E]\!]\rangle_E + \langle \Delta u_h + f, r_E\rangle_{K_1 \cup K_2} - \langle \nabla(u_h - u), \nabla r_E\rangle_{K_1 \cup K_2} \\ &\quad + \sum_{i=1}^2 \Big(\langle N \cdot \tilde{\mathbf{n}}^i, r_E\rangle_{\tilde{E}_i} - \langle N \cdot \mathbf{n}^i, r_E\rangle_{E_i}\Big). \end{aligned}$$

Note that $\sqrt{\mathbf{h}}N \cdot \tilde{\mathbf{n}}^i$ is a polynomial and, therefore, the constant c_2 in the first inequality above is *independent* of E. Now, recalling that E and $\tilde{E}$ have the same

endpoints, for the last two terms on the right-hand side of (7.19), we have

$$(7.20) \quad \langle N \cdot \tilde{\mathbf{n}}^i, r_E \rangle_{\tilde{E}_i} - \langle N \cdot \mathbf{n}^i, r_E \rangle_{E_i} = \oint_{\tilde{E}_i \cup E_i} (N r_E) \cdot \mathbf{n} \, ds = \int_{\tilde{K}_i \triangle K_i} \nabla \cdot (N r_E) \, dx,$$

from the divergence theorem.

Combining (7.19) and (7.20), along with the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
(7.21) \qquad & c_2^{-1} \sum_{i=1}^{2} \| \sqrt{\mathbf{h}} N \cdot \tilde{\mathbf{n}}^i \|_{\tilde{E}_i}^2 \\
& \leq \| C_{tr} [\![u_h - u]\!] \|_E \| [\![r_E]\!] \|_E + \sum_{i=1}^{2} \Big(\| \nabla(u_h - u) \|_{K_i} \| \nabla r_E^i \|_{K_i} \\
& \quad + \| \mathbf{h}(\Delta u_h + f) \|_{K_i} \| \mathbf{h}^{-1} r_E^i \|_{K_i} + \| \nabla \cdot (N r_E) \|_{L^1(\tilde{K}_i \triangle K_i)} \Big).
\end{aligned}
$$

Now, Lemma 4.3, and a standard inverse estimate give, respectively,

$$
\begin{aligned}
\| \nabla r_E^i \|_{K_i}^2 & \leq C h_{K_i}^d \| \nabla r_E^i \|_{L^\infty(K_i)}^2 = C h_{K_i}^d \| \nabla r_E^i \|_{L^\infty((K_i)_\flat(\nabla r_E^i))}^2 \\
& \leq C h_{K_i}^{d-2} \| r_E^i \|_{L^\infty((K_i)_\flat(\nabla r_E^i))}^2 \leq C h_{K_i}^{d-2} \| r_E^i \|_{L^\infty(K_i)}^2 \\
& = C h_{K_i}^{d-2} \| r_E^i \|_{L^\infty(\tilde{E}_i)}^2 \leq C h_{K_i}^{-1} \| r_E^i \|_{\tilde{E}_i}^2 \leq C \| \sqrt{\mathbf{h}} N \cdot \tilde{\mathbf{n}}^i \|_{\tilde{E}_i}^2,
\end{aligned}
$$

since r_E^i is constant in the direction of $\tilde{\mathbf{n}}^i$. Also, from Lemma 4.9, we have

$$\| r_E^i \|_{E_i}^2 \leq C h_{K_i}^{-1} \| r_E^i \|_{K_i}^2 \leq C h_{K_i}^{d-1} \| r_E^i \|_{L^\infty(\tilde{E}_i)}^2 \leq C h_{K_i} \| \sqrt{\mathbf{h}} N \cdot \tilde{\mathbf{n}}^i \|_{\tilde{E}_i}^2.$$

Finally, using Lemma 4.3, along with standard inverse estimates, we have

$$
\begin{aligned}
\| \nabla \cdot (N r_E) \|_{L^1(\tilde{K}_i \triangle K_i)} & \leq | \tilde{K}_i \triangle K_i | \| \nabla \cdot (N r_E) \|_{L^\infty(\tilde{K}_i \triangle K_i)} \\
& \leq | \tilde{K}_i \triangle K_i | \| \nabla \cdot (N r_E) \|_{L^\infty(\tilde{K}_i \cup K_i)} \\
& = | \tilde{K}_i \triangle K_i | \| \nabla \cdot (N r_E) \|_{L^\infty((\tilde{K}_i \cup K_i)_\flat(N r_E))} \\
& \leq C | \tilde{K}_i \triangle K_i | h_{K_i}^{-1} \| N r_E \|_{L^\infty((\tilde{K}_i \cup K_i)_\flat(N r_E))} \\
& \leq C | \tilde{K}_i \triangle K_i | h_{K_i}^{-1} \| N r_E \|_{L^\infty(\tilde{E}_i)} \\
& \leq C | \tilde{K}_i \triangle K_i | h_{K_i}^{-d} \| N r_E \|_{L^1(\tilde{E}_i)} \\
& \leq C | \tilde{K}_i \triangle K_i | h_{K_i}^{-d} \| N(\sqrt{\mathbf{h}} N \cdot \tilde{\mathbf{n}}^i) \|_{L^1(\tilde{E}_i)} \\
& \leq C | \tilde{K}_i \triangle K_i | h_{K_i}^{-d} \| N \|_{\tilde{E}_i} \| (\sqrt{\mathbf{h}} N \cdot \tilde{\mathbf{n}}^i) \|_{\tilde{E}_i}.
\end{aligned}
$$

Combining the above bounds and using (7.14), we deduce from (7.21):

$$
\begin{aligned}
\sum_{i=1}^{2} \| \sqrt{\mathbf{h}} N \cdot \tilde{\mathbf{n}}^i \|_{\tilde{E}_i}^2 & \leq C \Big(\sum_{i=1}^{2} \| \sqrt{\mathbf{h}} C_{tr} [\![u_h - u]\!] \|_{E_i}^2 + | \tilde{K}_i \triangle K_i |^2 h_{K_i}^{-2d} \| N \|_{\tilde{E}_i}^2 \\
& \quad + (\theta \eta(K_i))^2 \big(\| \nabla(u - u_h) \|_{K_i}^2 + \| \mathbf{h}(\Pi f - f) \|_{K_i}^2 \big) \Big),
\end{aligned}
$$

which implies the result.

$\square$

Remark 7.5. For the jump residual, we trivially have

$$(7.22) \qquad \Upsilon_{J_K}^2 = \sqrt{\gamma} \| \sqrt{\gamma_0 / \mathbf{h}} [\![u - u_h]\!] \|_{\partial K \cap \Gamma \setminus \Gamma^{tr}}^2,$$

so it is omitted in the lower bound.

We observe that the interface oscillation term $\Theta_{2,K}$ is equal to zero when $K = \tilde{K}$, i.e., on non-curved elements. When $K \neq \tilde{K}$, the size of $\Theta_{2,K}$ depends on the ratio

between the d-dimensional volume of the symmetric difference between K and $\tilde{K}$, divided by $h_K^d \sim |K|$.

Remark 7.6. The dG method (3.3) and the a posteriori bounds presented above remain valid when quadrilateral elements with two curved faces with no common vertex/edge are present in the mesh. Indeed, observing that the recovery $\mathcal{E}$ may be defined on a refinement of the original mesh, we can split such elements with more than one curved faces into sub-elements with one curved face only; the remaining analysis remains essentially intact. We stress that the actual computation would take place on the original mesh which may involve elements with more than one curved faces. Elements with more than one curved faces may arise when cascades of interfaces separated by very thin, one-element-wide, compartments, or in the presence of both curved interfaces and non-essential boundaries.

Remark 7.7. An interesting further development would be the proof of L_2-norm and/or localized a posteriori error bounds. Such a result typically requires a duality argument and the availability of H^2-stability a priori PDE estimates. The increased rate of convergence is then a result of estimates of the form $\|v - v_h\|_K \leq Ch^2 |v|_{H^2(K)}$, where v_h has to be at least an element-wise linear approximation of $v \in H^2(K)$. Such approximation results are not available on curved domains K, to the best of our knowledge, with explicit dependence on the element shape. Of course, local bounds based on a dual weighted residual approach are always possible at the expense of calculating explicitly a (finer) solution of the dual problem.

8. A priori error bound

Since no a priori error bound is available for the (fitted) discontinuous Galerkin method proposed above for the elliptic interface problem, we use the above a posteriori bounds to show a basic a priori convergence result in the spirit of the celebrated work of Gudi [25]. Here, however, we need to account also for the oscillation term arising from the treatment of the interface.

Theorem 8.1. *Let $u \in \mathcal{H}_0^1$ and $u_h \in S_h^p$ be the solutions of (2.2) and (3.3) respectively. Then, the error bound*

$$(8.1) \qquad \|u - u_h\| \leq C \inf_{v_h \in S_h^p} \left(\|u - v_h\| + \Theta_1 + \Theta_2 \right),$$

holds with $\Theta_2|_K := \Theta_{2,K}$, $K \in \mathcal{T}$.

Proof. Let $v_h \in S_h^p$ with $v_h \neq u_h$ and set $\psi := u_h - v_h \in S_h^p$. Coercivity, (2.2), (3.3), and continuity imply

$$\frac{1}{2} \|u_h - v_h\|^2 \leq \hat{D}_h(u_h - v_h, \psi) = \langle f, \psi \rangle - \hat{D}_h(v_h, \psi)$$
$$= \hat{D}_h(u - v_h, \mathcal{E}(\psi)) + \langle f, \psi - \mathcal{E}(\psi) \rangle - \hat{D}_h(v_h, \psi - \mathcal{E}(\psi))$$
$$\leq C \|u - v_h\| \|\mathcal{E}(\psi)\| + \langle f, \psi - \mathcal{E}(\psi) \rangle - \hat{D}_h(v_h, \psi - \mathcal{E}(\psi)).$$

Noting that Lemma 5.2 implies $\|\mathcal{E}(\psi)\| \leq C \|\psi\|$, for some constant $C > 0$ depending on $\theta\eta$, after division by by $\|\psi\|$, we arrive at

$$(8.2) \qquad \|u_h - v_h\| \leq C \|u - v_h\| + 2 \frac{\langle f, \psi - \mathcal{E}(\psi) \rangle - \hat{D}_h(v_h, \psi - \mathcal{E}(\psi))}{\|\psi\|}.$$

Now, to estimate the second term on the right-hand side of (8.2), integration by parts gives

$$
\begin{aligned}
\mathcal{R} &:= \int_{\Omega} f(\psi - \mathcal{E}(\psi))\, dx - \hat{D}_h(v_h, \psi - \mathcal{E}(\psi)) \\
&= \sum_{K \in \mathcal{T}} \int_K (f + \Delta v_h)(\psi - \mathcal{E}(\psi))\, dx - \int_{\Gamma^{int}} [\![\nabla v_h]\!] \cdot \{\psi - \mathcal{E}(\psi)\}\, ds \\
&\quad + \int_{\Gamma \backslash \Gamma^{tr}} [\![v_h]\!] \cdot \{\nabla \Pi(\psi - \mathcal{E}(\psi))\}\, ds - \int_{\Gamma \backslash \Gamma^{tr}} \frac{\gamma_0}{\mathbf{h}} [\![v_h]\!] \cdot [\![\psi]\!]\, ds \\
&\quad - \int_{\Gamma^{tr}} \Big(C_{tr} [\![v_h]\!] \cdot [\![\psi - \mathcal{E}(\psi)]\!] + [\![\nabla v_h(\psi - \mathcal{E}(\psi)]\!] \Big)\, ds.
\end{aligned}
\tag{8.3}
$$

Working in a standard fashion to estimate the terms on the right-hand side of (8.3), we have

$$
\begin{aligned}
\mathcal{R} &\le C \Big(\sum_{K \in \mathcal{T}} h_K^2 \|f + \Delta v_h\|_K^2 + h_K \|[\![\nabla v_h]\!]\|_{\partial K \cap \Gamma^{int}}^2 + h_K^{-1} \|[\![v_h]\!]\|_{\partial K \cap \Gamma \backslash \Gamma^{tr}}^2 \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\sum_{K \in \mathcal{T}} h_K^{-2} \|\psi - \mathcal{E}(\psi))\|_K^2 \Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_{K \in \mathcal{T}^{tr}} \sum_{i=1}^2 h_K \|(C_{tr}[\![v_h]\!] + \nabla v_h) \cdot \mathbf{n}^i\|_{\partial K \cap \Gamma^{tr}}^2 \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\sum_{j=1}^2 \|\mathbf{h}^{-1/2}(\psi - \mathcal{E}(\psi))|_{\Omega_j}\|_{\Gamma_{tr}}^2 \Big)^{\frac{1}{2}}.
\end{aligned}
\tag{8.4}
$$

For the last term on the right-hand side of (8.4), we use Lemmata 4.9 and 5.2 to deduce

$$
\sum_{j=1}^2 \|\mathbf{h}^{-1/2}(\psi - \mathcal{E}(\psi))|_{\Omega_j}\|_{\Gamma_{tr}}^2 \le C \sum_{K \in \mathcal{T}^{tr}} \|\mathbf{h}^{-1}(\psi - \mathcal{E}(\psi))\|_K^2 \le C \|\sqrt{\theta \eta} \mathbf{h}^{-1/2} [\![\psi]\!]\|_{\Gamma \backslash \Gamma^{tr}}^2.
$$

Noting that the fact that u_h is the dG solution was not used in the proof of the lower bound (Theorem 7.4 above), it can be replaced by any $v_h \in S_h^p$. Therefore, Theorem 7.4 (with v_h replacing u_h) and the triangle inequality yield the result. □

The above result offers a basic convergence proof for the proposed (fitted) discontinuous Galerkin method for interface problems. Note that the regularity of solutions to such interface problems, which may involve *piecewise smooth* interface manifolds, is not well understood in the literature. Therefore, such basic convergence results, not requiring any regularity of the underlying solution, are desirable.

9. HIGHER ORDER INTERFACE APPROXIMATION

The saturation of the approximation of the geometry by the mesh, (4.5) and (4.6), is required to be satisfied for the above a posteriori error bounds to hold. One way of achieving this in practice is an initial refinement step in the vicinity of $\mathcal{T}^{tr}$. This approach is expected to deliver optimal convergence rates for the respective adaptive algorithm when $p = 1, 2$. Indeed, from the a priori error analysis of finite element methods with local basis of degree p and with boundary and/or interface approximation, we can expect optimal convergence rates when the curved boundaries/interfaces are approximated locally by interpolants of degree $p - 1$.

To ensure that the above interface approximation requirements (4.5), (4.6) do not result to potentially excessive and unnecessary refinement in the vicinity of

Γ^{tr}, when higher order elements are used, we can employ a (non-standard) *fitted* approach based on parametric elemental mappings, which we shall now describe.

Each element $K \in \mathcal{T}^{tr}$ is assumed to be constructed via a parametric elemental mapping $F_K : \hat{K} \to K$ of polynomial degree $q \in \mathbb{N}$, from a *curved* reference element $\hat{K}$ with $|\hat{K}| \sim O(1)$.
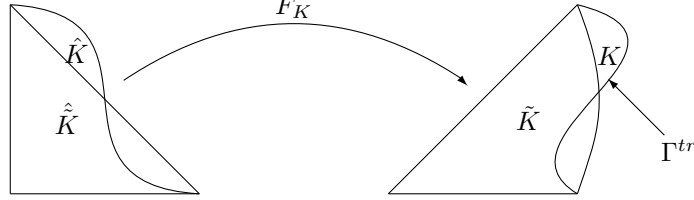

FIGURE 8. A curved element $K \in \mathcal{T}^{tr}$ and the related q-degree parametric element $\tilde{K}$ as the mapping of the respective reference elements $\hat{K}$ and $\hat{\tilde{K}}$.

More precisely, we begin by considering a parametric mesh of degree $q \in \mathbb{N}$, whose skeleton approximates the interface Γ^{tr} with a piecewise interpolant of degree q. Setting $\tilde{K}$ to be one such (unfitted) parametric element with non-trivial intersection with Γ^{tr}, we consider the q-degree parametric mapping $F_{\tilde{K}} : \hat{\tilde{K}} \to \tilde{K}$ with $\hat{\tilde{K}}$ being the (classical) reference element, i.e., it may be the d-simplex, the reference d-hypercube or the reference d-prism, the latter constructed as tensor-product of the reference $(d-1)$-simplex and the interval $[0,1]$. By considering the extension of $F_{\tilde{K}}$ on a larger domain $\hat{Y} \supset \hat{\tilde{K}}$ with the same (polynomial) formula, we can precisely define $\hat{Y}$ as the $F_{\tilde{K}}$-pre-image of $K \cup \tilde{K}$ where $K \in \mathcal{T}^{tr}$ is the *fitted* element related to $\tilde{K}$; we denote the extension of $F_{\tilde{K}}$ to $\hat{Y}$ by F_K. We refer to Figure 8 for an illustration. Hence, the reference element $\hat{K} := F_K^{-1}(K)$ will, in general, be curved.

We, now, define the *mapped* discontinuous finite element space $S_h^{p,q}$, subortinate to the mesh $\mathcal{T} = \{K\}$, by

(9.1) $$S_h^{p,q} = \{v \in L^2(\Omega) : v \circ F_K^{-1}|_K \in \mathcal{R}_p(\hat{K})\},$$

where $\mathcal{R}_p(\hat{K}) \in \{\mathcal{P}_p(\hat{K}), \mathcal{Q}_p(\hat{K})\}$, and $\mathcal{Q}_p(\hat{K})$ denotes space of tensor-product polynomials of degree p in each variable; when $\hat{\tilde{K}}$ is a d-simplex, we may select $\mathcal{R}_p(\hat{K}) = \mathcal{P}_p(\hat{K})$, while we select $\mathcal{R}_p(\hat{K}) = \mathcal{Q}_p(\hat{K})$, in general, otherwise to ensure optimal approximation rates.

The key motivation for the above construction is that we can re-use the above developments in this fitted mapped setting also, by first applying the elemental mappings F_K and then use the results from Section 4 on the curved reference element $\hat{K}$ instead. An inspection of the proofs from Section 4 shows that *all* results hold true in this setting also, with the constants in the estimates now depending also on the nature of the mapping F_K, as is standard in parametric finite elements.

10. NUMERICAL EXPERIMENTS

We shall now illustrate the performance of the a posteriori error estimator within a standard adaptive algorithm, through an implementation based on the `deal.II` finite element library [3]. This allows for the use of curved elements via high-order parametric mappings of tensor-product reference elements. We take advantage of this capability and approximate curved interfaces via tensor-product elements defined through parametric mappings of degree *higher* than linear, as described in Section 9.

DoFs	estimator	rate	$\|\|\|.\|\|\|$-error	rate	L^2-error	H^1-error
192	1.3381e+01	—	1.7550e+00	—	6.2516e-02	1.3886e+00
768	7.6174e+00	0.81	8.4388e-01	1.06	2.0201e-02	7.2406e-01
3072	3.9838e+00	0.94	3.9950e-01	1.08	5.7567e-03	3.6434e-01
12288	2.0259e+00	0.95	1.9147e-01	1.06	1.5192e-03	1.8187e-01
49152	1.0202e+00	0.99	9.3267e-02	1.04	3.8765e-04	9.0744e-02
196608	5.1826e-01	0.98	4.6010e-02	1.02	9.7624e-05	4.5323e-02

TABLE 1. Example 1. Convergence of the estimator and errors; quadratic mapping with $p = 1$.

DoFs	estimator	rate	$\|\|\|.\|\|\|$-error	rate	L^2-error	H^1-error
432	1.8278e+00	—	4.5728e-01	—	7.6800e-03	1.9028e-01
1728	3.8913e-01	2.2	8.4015e-02	2.44	6.2879e-04	3.3034e-02
6912	9.2506e-02	2.07	1.8457e-02	2.19	5.9084e-05	6.8618e-03
27648	2.2127e-02	2.06	4.2318e-03	2.12	5.7411e-06	1.5135e-03
110592	5.5103e-03	2.00	1.0017e-03	2.08	5.7413e-07	3.4841e-04

TABLE 2. Example 1. Convergence of estimator and errors; quadratic mapping with $p = 2$.

DoFs	estimator	rate	$\|\|\|.\|\|\|$-error	rate	L^2-error	H^1-error
768	6.5932e-02	–	4.3641e-03	—	8.5525e-05	3.0614e-03
3072	8.1666e-03	3.01	4.6166e-04	3.24	5.7611e-06	3.7515e-04
12288	1.0233e-03	2.99	5.2890e-05	3.12	3.8178e-07	4.6925e-05
49152	1.2937e-04	2.98	6.3781e-06	3.05	2.4791e-08	5.9393e-06
196608	1.7042e-05	2.92	8.0091e-07	2.99	1.5886e-09	7.5685e-07

TABLE 3. Example 1. Convergence of estimator and errors; quadratic mapping with $p = 3$.

Although not discussed above merely for simplicity of the presentation, the extension of the proposed dG method to problems with non-homogeneous Dirichlet boundary conditions is straightforward; the a posteriori bound is then modified accordingly [30]. In all cases considered below the interface residual term $\Theta_{2,K}$ was omitted due to its insignificant magnitude (Example 1 below) or simply because it is equal to zero (Example 2 below). We set $\gamma = 10$ throughout.

10.1. **Example 1.** We consider the problem (2.2) with $\Omega = (-1, 1)^2$ and the interface Γ^{tr} being a circle of radius $r = 0.5$, centred at the origin. The Dirichlet boundary conditions and the source term f are determined by the exact solution

$$u = \begin{cases} r^3, & \text{in } \Omega_1; \\ r^3 + 1, & \text{in } \Omega_2, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ and $C_{tr} = 0.75$.

Upon satisfactory approximation of the interface geometry, the above problem does not admit singular behaviour and we expect to observe optimal convergence rates. To simulate the fitted approach we use parametric maps of degree higher than linear for the interface elements.

We begin by assessing the decay of the estimators under uniform refinement, using quadratic parametric mappings ($q = 2$) for the elements on Γ^{tr}: in Tables 1, 2, and 3, the convergence of the a posteriori estimator, of the energy norm, of

the H^1-seminorm and of the L^2-norm of the error are reported, along with the respective convergence rates for the estimator and for the energy error, for $p = 1, 2$, and 3, respectively. The estimator and the dG-norm of the actual error are plotted in Fig. 10(a), for both adaptive and uniform refinement; for the adaptive refinement a standard bulk criterion is used. Optimal convergence rates are observed in all cases.
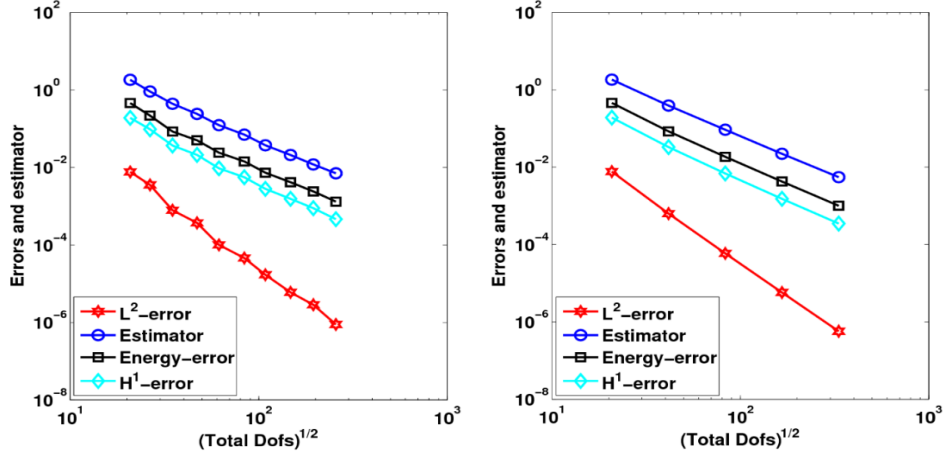

FIGURE 9. Example 1. Convergence of the estimator and errors for quadratic mapping and $p = 2$: adaptive (left) and uniform (right) refinements.

10.2. **Example 2.** Let, now, $\Omega = (-1, 1) \times (0, 1)$, subdivided into $\Omega_1 = (-1, 0) \times (0, 1)$, $\Omega_2 = (0, 1)^2$, i.e., interfacing at $x = 0$. The Dirichlet boundary conditions and the source term f are determined by the exact solution

$$
u = \begin{cases} (x^2 + y^2)^{3/4} + x, & \text{in} \quad \Omega_1; \\ 1 + y^{3/2} + x, & \text{in} \quad \Omega_2, \end{cases}
$$

which has a point singularity at $(0, 0)$. This example studies the interaction of the interface discontinuity, with the point singularity at one interface endpoint. The convergence under uniform as well as adaptive refinement is given in Fig. 11 for $p = 1, 2$, while the respective effectivity indices (i.e., the ratio between estimator and exact solution) and adapted meshes are given in Fig. 12.

11. CONCLUSIONS

A fitted interior penalty discontinuous Galerkin finite element method for the solution of flux-balancing elliptic interface problems with general interface geometry has been presented and its stability has been proven. A posteriori bounds for the energy norm and a basic convergence result under minimal solution regularity assumptions have been shown. The proofs require a number of new results on direct and inverse approximation for curved elements, which may be of independent interest. Numerical experiments showed the good performance of the a posteriori bound in practice.

(a) Refinement 3

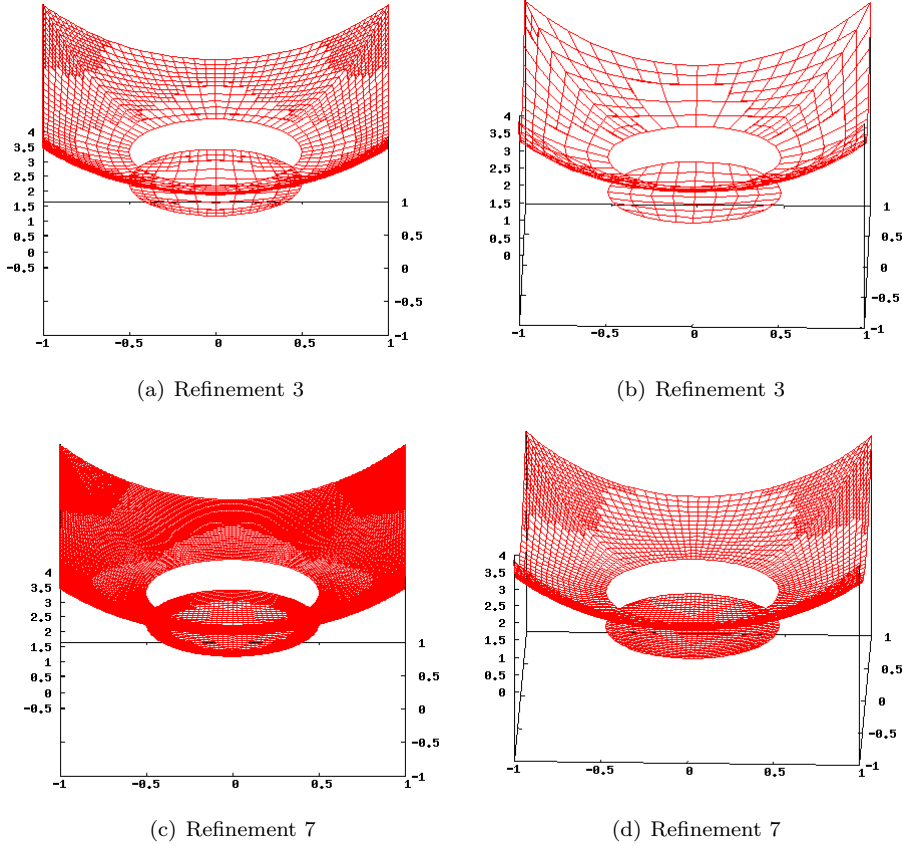(b) Refinement 3

(c) Refinement 7

(d) Refinement 7

FIGURE 10. Example 1. Meshes produced by the adaptive algorithm with quadratic mapping, for $p = 1$ (left) and $p = 2$ (right)

A current, challenging, yet very relevant direction of research is the extension of the proposed "fitted" dG methods in the context of the classical transmission/interface problems of the form

$$
\begin{aligned}
-\Delta u &= f, && \text{in } \Omega_1 \cup \Omega_2, \\
u &= 0, && \text{on } \partial\Omega, \\
[\![u]\!] &= 0, && \text{on } \Gamma^{tr}, \\
[\![\nabla u]\!] &= 0, && \text{on } \Gamma^{tr}.
\end{aligned}
$$

(11.1)

There is no difficulty in defining a dG scheme for this problem involving curved elements in the spirit of the seminal work [28]. However, the derivation of a posteriori bounds and respective adaptive algorithms for this class of problems remains out of reach at the moment. This is due to the following key theoretical challenge. Since for (11.1), we have $u \in H_0^1(\Omega)$ (under the domain assumptions of this work), to prove a posteriori error estimates we would need to construct a respective recovery operator as done in Section 5. Such a construction remains a challenge for it is not clear how to construct a continuous finite element function across a general curved face. This problem will be discussed elsewhere. Nonetheless, many of the theoretical tools proved in this work (such as approximation and inverse estimates), are expected to be useful in the derivation of a posteriori bounds for these classical interface problems.
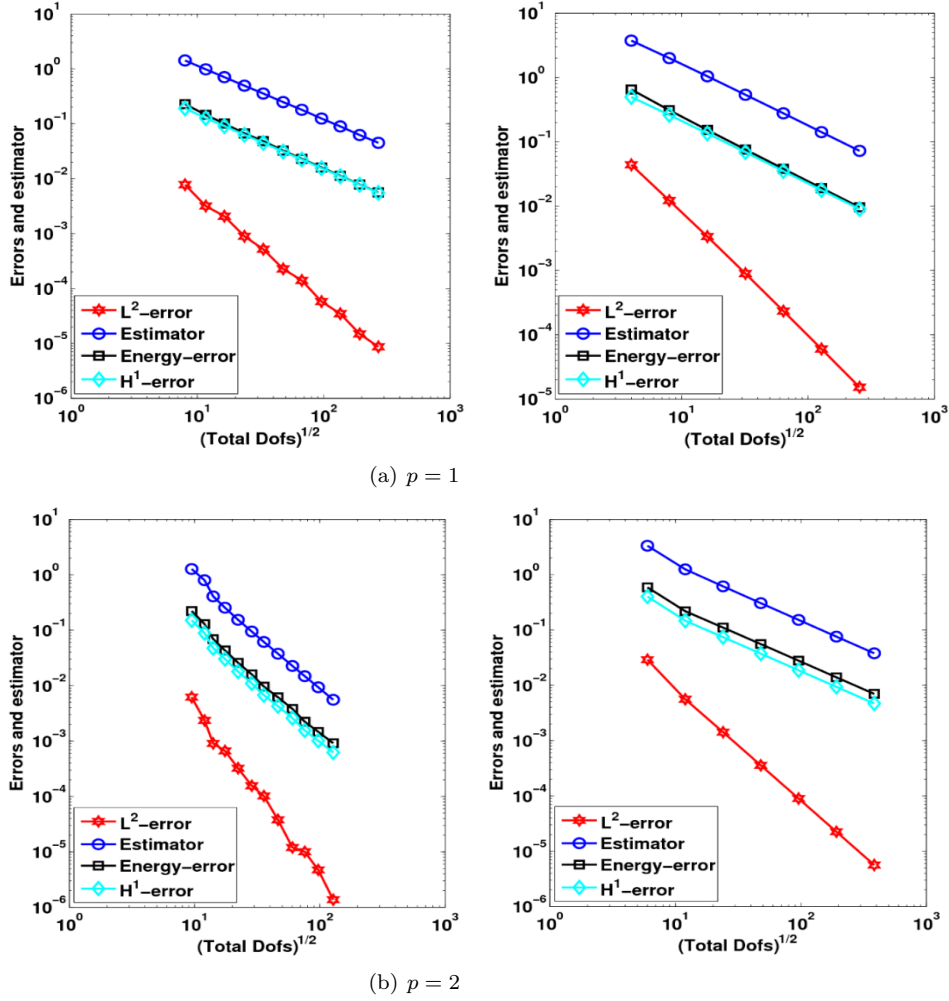
(a) $p = 1$

(b) $p = 2$

FIGURE 11. Example 2. Convergence of the estimator and of the actual dG-norm errors for adaptive (left) and uniform (right) refinement for $p = 1, 2$.

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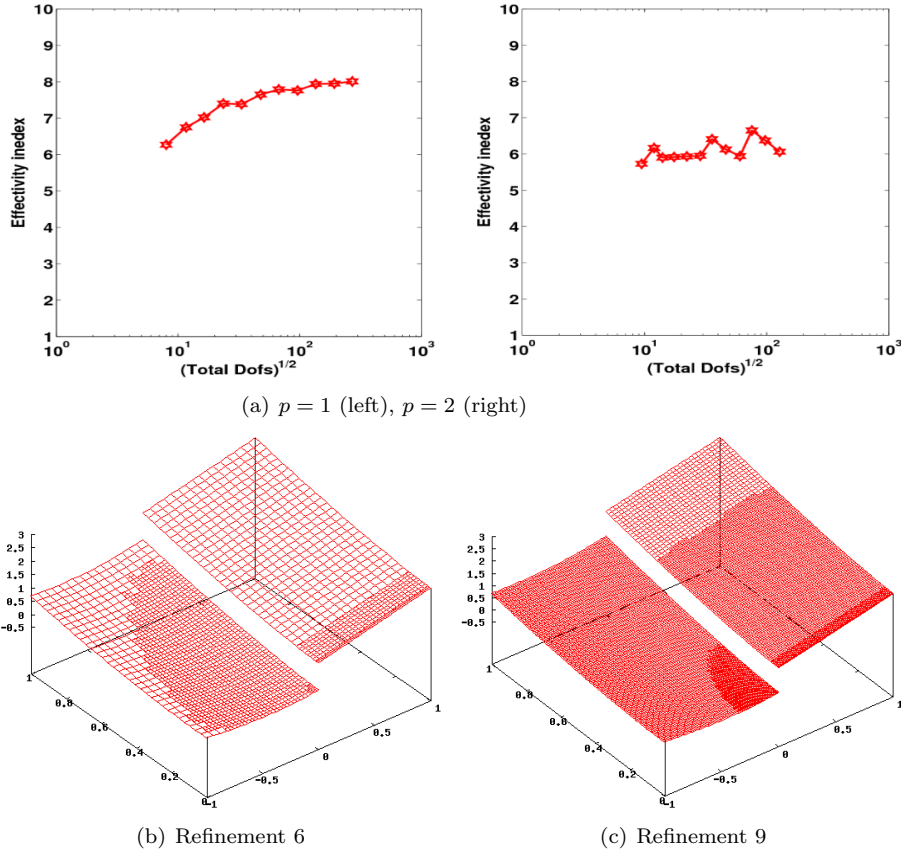
(a) $p = 1$ (left), $p = 2$ (right)

(b) Refinement 6   (c) Refinement 9

FIGURE 12. Example 2. Effectivity indices for $p = 1, 2$ and meshes produced ny the adaptive algorithm for $p = 1$.

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