The equally spaced energy levels of the quantum harmonic oscillator revisited: A back-to-front reconstruction of an n-body Hamiltonian

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Abstract

The "back-to-front" derivation of the properties of the quantum harmonic oscillator, starting with its equally spaced energy levels, is re-examined. A new derivation that exploits the natural rotational symmetry of the quantum harmonic oscillator is proposed. The new approach allows the "back-to-front" idea to be extended further by showing that it is possible to derive the Hamiltonian of a system of particles from the starting point that the population is represented by a natural number. This involves the symmetry properties of phasors and Schwinger's theory of angular momentum. The analysis is also extended to multi-mode bosonic systems and fermionic systems. It is suggested that these results offer an alternative way to formulate physics, based on discreteness.

1 Introduction

The harmonic oscillator is ubiquitous in nature and of universal importance in physics. Well known examples of physical systems of this type are, simple pendula, atomic lattice vibrations, electromagnetic waves and so on. The harmonic oscillator is also of profound importance in quantum theory. The

conventional analysis of the characteristics of the quantum harmonic oscillator (QHO) is a common example of the use of the Schrödinger wave equation and is widely studied in this form in undergraduate courses and it can be found in most introductory textbooks (e.g., [1]). As is well known, the Schrödinger wave equation employs a Hamiltonian operator in which the momentum operators are differentials with respect to co-ordinates and the potential operator is just a scalar function of co-ordinates. In the case of the one dimensional QHO, the potential is quadratic in a Cartesian co-ordinate. This results in the simple linear relationship between energy, E_n, and the quantum number, n, of the form

$$E_n = \hbar\omega_0(n + \frac{1}{2}),\qquad(1)$$

where $\hbar$ is the reduced Planck constant, ω_0 is the fundamental frequency of the oscillator and n is a natural number, $0, 1, 2, 3, \ldots$, etc. The appearance of the half in the bracket in Eq.(1) indicates that the lowest energy state of the QHO is $\hbar\omega_0/2$, but the energy levels above this are equally spaced. This conventional approach to the QHO has also been elaborated in a variety of ways, for example, by factorizing the Hamiltonian into raising and lowering operators [2], using Fourier integrals [3] and Feynman path integrals [4]. However, it is the linear relationship between the energy and n that is arguably its most striking feature. Not only does this take us back to the origins of quantum mechanics with Planck's theory of blackbody radiation, but it also allows the reinterpretation of single particle energy levels as excitation energies of an oscillating field and n then is reinterpreted as the number of particles excited. This interpretation is the basis of modern quantum optics and quantum field theory of fundamental particles. It is arguable that it is in this form that the QHO finds its most important applications.

In an interesting and thought provoking paper, Andrews and Romero [5] have suggested that it is possible to derive the properties of the QHO by starting with the equal energy spacing, in what they call a "back-to-front" derivation (BTFD). They point to some major advantages to such an approach, particularly from a pedagogical point of view. For example, the BTFD means that one can derive the quadratic form of the Hamiltonian and, assuming the n in the energy represents the number of photons in an electromagnetic field, that the energy density must be quadratic in the electric and magnetic fields. One of the aims of the present paper is to propose a new BTFD that takes a different approach to that of Andrews and Romero [5]. The new method arguably relies on fewer assumptions than the previous method, but more importantly can be extended to a BTFD of an n-body Hamiltonian, among other new results.

In the new BTFD, presented below, the condition that the natural numbers are non-negative, that was not considered by Andrews and Romero is utilized. The new method also involves a phase or cyclic symmetry [6] that shows the strong connection between the harmonic oscillator and motion in a circle. This symmetry is possessed by both the classical and quantum harmonic oscillators, as was recognised from the early days of quantum mechanics [7]. Indeed, Royer [8] has shown that it is precisely this symmetry that is ultimately responsible for the equal energy spacing. It should also be pointed out that there is another system, besides the QHO, with equal energy spacing and that is one in which a charged particle is placed in a magnetic field. The resulting energy levels are referred to as Landau levels [9]. However, unlike the QHO case, Landau levels are associated with shifts, due to the presence of a magnetic field, that are proportional to integers, both positive and negative ones, and not just the natural numbers. This is not surprising, since the Landau energy shifts are proportional to the eigenvalues of angular momentum for the particle, about the magnetic field direction, the so-called magnetic quantum numbers, that take on integer values. However, as Schwinger [10] showed, angular momentum can be interpreted in terms of a pair of QHOs. This result has an important bearing on the second aim of this paper.

A second aim of the present paper is to show that we can go beyond the BTFD of the properties of the QHO. The phase (cyclic) symmetry that is the basis of both the QHO and Landau levels allows the BTFD analysis to be extended to second quantization. It will be shown below that, with a starting assumption that there exists a system of n particles, then one can derive the basic dynamical properties of the system, including the properties of the creation and annihilation operators, and the Hamiltonian of the system, without any further assumptions, including any preconceived notion of energy. The dynamical properties are introduced through a shift operator, which is an algebraic form of differentiation. As is well known, the time shift operator is identified with the Hamiltonian. The QHO is unique in that shifts in time and spatial co-ordinates correspond to phase shifts, so that its dynamic properties can all be recovered from a phase shift operator. As we will argue, the importance of this result means that it is possible to use discreteness as a starting point for deriving the dynamical equations of physical systems. The usual starting point for physical theory is classical physics [11]. Once the scalar dynamical variables of classical physics are established, they can be turned into operator form to get quantum mechanics. Finally, the wave functions of quantum mechanics are turned into operators to get quantum field theory [12], a process commonly referred to as *second quantization*. The results presented below, in the spirit of Andrews' and Romero's

BTFD, suggest that we could start with the idea that the universe is made up of countable items. We just treat the number of items as the eigenvalue spectrum of an operator and derive the equations of motion from that. The point here is that it is a simple matter to see the relevance of operators in this context, because it is straightforward to find operators with discrete eigenvalues that can be interpreted as a number of particles. This is tantamount to an atomistic viewpoint. In contrast, if the number of particles is taken as a scalar variable, there is no way of constraining its value to whole numbers when it is incorporated into the differential equations that constitute a dynamical theory. This constraint is obvious for the number of particles, but not so obvious for an amount of energy. In many ways, the discreteness of matter is a more natural starting point than the discreteness of energy.

Another major advantage of the BTFD is that the fundamental quantization rule that involves the commutation relationship between a co-ordinate and its corresponding momentum component emerges quite naturally, which is unlike what is to be found in most quantum mechanics textbooks where the quantization rule is merely assumed to be the case. This means that we can, in principle, derive the equations of motion of systems with potentials other than the quadratic form of the QHO without relying on classical mechanical principles. This is a powerful result that allows us to develop physical theory quite generally from an atomistic starting point.

The paper is set out as follows. In section 2, a valid BTFD of QHO physics is presented. In section 3, we outline the relationship between phase shifts, integers and natural numbers. A BTFD for second quantization is proposed in section 4. The analysis is extended to multi-mode systems in section 5 and to fermionic systems in section 6. The implications for the development of physical theory from an atomistic starting point are discussed in section 7.

2 Back-to-front derivation of QHO physics from equally spaced energy levels

Our goal in this section is essentially the same as that of Andrews and Romero [5], i.e., to show that a system with a set of energy eigenvalues that are proportional to the set, $\mathbb{N}$, of natural numbers, represents quantum harmonic oscillator physics. In operator terms the set $\mathbb{N}$ can be represented as the eigenvalue spectrum of a number operator, $\hat{n}$, such that

$$\hat{n}|n\rangle = n|n\rangle, \tag{2}$$

where n is a natural number and $|n\rangle$ represents a state with an eigenvalue of n. These states are assumed to form a basis space of orthogonal unit vectors (indexed by a single label), generating the space, $\mathcal{N}$. This orthonormality condition means that the scalar product

$$\langle n|m\rangle = \delta_{nm}, \tag{3}$$

where both n and m are natural numbers. Now at this stage, we have no way of representing $\hat{n}$, in fact we are looking for such a representation. We do know that one of the properties of natural numbers is that they are all real and non-negative. This suggests that we try to find an operator, $\hat{a}$ that is defined on $\mathcal{N}$, such that

$$\hat{n} = \hat{a}^\dagger \hat{a}, \tag{4}$$

where $\hat{a}^\dagger$ is the adjoint[1]of $\hat{a}$, that satisfies Eq.(4). The reason for this is that we know that $\hat{a}^\dagger \hat{a}$ is Hermitian, since $(\hat{a}^\dagger \hat{a})^\dagger = \hat{a}^\dagger \hat{a}$, and consequently has real eigenvalues, and also that $\langle n|\hat{n}|n\rangle = n = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = ||\hat{a}|n\rangle||^2 \geq 0$, implying that the eigenvalues of $\hat{n}$ are non-negative. However, we should note that Bender et al.[13] have pointed out that, under certain special symmetry conditions, it is possible to find operators with real eigenvalues that are not Hermitian and also ones with real, non-negative eigenvalues that are not Hermitian. So care must be taken to check that no inconsistencies arise due to the form of Eq.(4). However, to be clear, these considerations in no way prevent us from seeking a pair of operators, $\hat{a}$ and $\hat{a}^\dagger$, that satisfy Eq.(4), which we may treat as a definition of $\hat{a}$ and $\hat{a}^\dagger$. Our task now is to determine their properties. At this stage we know nothing about $\hat{a}$ apart from Eq.(4) and we do not yet have an explicit representation for $\hat{n}$. Now, using Eq.(2) we can write the Hamiltonian, $\hat{H}$, that operates on $\mathcal{N}$ to yield the eigenvalues in Eq.(1), in the form

$$\hat{H}|n\rangle = \hbar\omega_0(\hat{n} + \frac{1}{2})|n\rangle = \hbar\omega_0(\hat{a}^\dagger \hat{a} + \frac{1}{2})|n\rangle = \hbar\omega_0(n + \frac{1}{2})|n\rangle. \tag{5}$$

To make progress we notice an important symmetry that applies to Eq.(4), i.e., $\hat{n}$ is invariant to a phase shift in $\hat{a}$, i.e. $\hat{a} \to \hat{a}\exp(-i\theta)$ and $\hat{a}^\dagger \to \hat{a}^\dagger \exp(i\theta)$, where θ is a scalar phase angle [6]. Associating a phase rotation with $\hat{a}$ means that it becomes a phasor and immediately shows a connection to oscillator-like characteristics and simple harmonic motion. To proceed we need to be able to handle operator valued functions of phase. There is a well known procedure for constructing operator valued functions in quantum mechanics. It involves the construction of a unitary transformation operator $\hat{U}$

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[1]The adjoint, if it exists, is defined on $\mathcal{N}$ by $\langle m|\hat{a}^\dagger|n\rangle = (\langle n|\hat{a}|m\rangle)^*$. Also, $\langle m|n\rangle = (\langle n|m\rangle)^*$, where $(\langle n|m\rangle)^*$ is the complex conjugate of $\langle n|m\rangle$.

that generates shifts in an appropriate system variable. Students of quantum mechanics are familiar with this type of transformation where Hamiltonians generate time shifts and the momentum operator generates linear co-ordinate shifts [1]. One can follow a very similar procedure to construct operator valued functions of phase (see section 3 and ref. [6]). However, for the moment we will adopt a simpler and quicker path and utilize the familiar Heisenberg form of time dependent operators. We can do this by simply parameterizing the phase by assuming that the phase shift, $\theta(t) = \omega_a t$, where ω_a is an arbitrary, but fixed, scalar valued angular frequency that we take as positive and t is time. When we say that ω_a is fixed we mean that it is not dependent on which eigenstate the system is in and it should not be confused with an eigenfrequency at this stage. Then, as t increases from some initial value the phase does too. Thus changes in phase are mapped unambiguously into changes in time. It remains to find the properties of the operators $\hat{n}$ and $\hat{a}$ in Eqs. (2) and (4).

A time dependent operator $\hat{g}(t)$ for a system with a Hamiltonian, $\hat{H}$, has a Heisenberg form[1]

$$\hat{g}(t) = \exp(it\hat{\Omega})\hat{g}(0)\exp(-it\hat{\Omega}), \tag{6}$$

where $\hat{g}(0)$ is $\hat{g}(t)$ evaluated at $t = 0$, $\exp(-it\hat{\Omega}) = \hat{U}(t)$ is unitary and $\hat{\Omega} = \hat{H}/\hbar$. This leads directly to the Heisenberg equation

$$i\frac{\mathrm{d}\hat{g}}{\mathrm{d}t} = [\hat{g}, \hat{\Omega}], \tag{7}$$

where the RHS of Eq.(7) is a conventional commutation bracket. We may assume that $\hat{a}(t) = \hat{a}(0)\exp(-i\omega_a t)$ satisfies the Heisenberg equation with the Hamiltonian in Eq.(5). Then, with $\hat{\Omega} = \omega_0 \hat{n}$

$$\omega_a \hat{a} = \omega_0 [\hat{a}, \hat{n}]. \tag{8}$$

Notice, we have dropped the term containing $\frac{1}{2}$ in the Hamiltonian from the definition of $\hat{\Omega}$, because, being a scalar, it has no effect on the Heisenberg equation. Rearranging Eq.(8) and operating on $\mathcal{N}$ yields

$$\hat{n}(\hat{a}|n\rangle) = (n - \lambda_0)\hat{a}|n\rangle, \tag{9}$$

where $\lambda_0 = \omega_a/\omega_0$. From the definitions of ω_a and ω_0, we know that λ_0 is a positive scalar constant. Now Eq.(9) implies that $\hat{a}|n\rangle$ is also an eigenstate of $\hat{n}$, assuming that it is in $\mathcal{N}$, with an eigenvalue of $n - \lambda_0$. Thus we can assume that

$$\hat{a}|n\rangle = \alpha_n|n - \lambda_0\rangle, \tag{10}$$

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where α_n is a scalar that could depend on n. Leaving aside for the moment the implications for the value of λ_0 that the state being in $\mathcal{N}$ brings, we note that Eq.(10) implies that $\|\hat{a}|n\rangle\|^2 = \alpha_n^2$. However we know that $\|\hat{a}|n\rangle\|^2 = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = n$, so $\alpha_n = \sqrt{n}$. Let us now look at the value of λ_0. If $|n - \lambda_0\rangle$ is a state in $\mathcal{N}$, then λ_0 must certainly be a natural number less than or equal to n, so that $n - \lambda_0$ is a natural number. In fact we must have $\lambda_0 = 1$. To see this we note that $\hat{a}^p|n\rangle = \sqrt{n(n - \lambda_0)(n - 2\lambda_0)\ldots(n - (p-1)\lambda_0)}|n - p\lambda_0\rangle$, where p is a natural number. The only way we can be sure that, whatever the value of n, continued operation with $\hat{a}$ will not yield a state that in not in $\mathcal{N}$ is if $\lambda_0 = 1$. This is because, with $\lambda_0 = 1$, the value of the square root will be exactly zero for $p \geq n$, because in this case it will contain the factor, $n - n = 0$ when the value of p reaches n. Thus, there is a state $|0\rangle$ such that $\hat{a}|0\rangle = 0$, beyond which the system cannot be taken by further applications of $\hat{a}$. This result is important as it ensures that no inconsistencies arise concerning the Hermiticity and non-negative eigenvalues of $\hat{n}$. Also, the state $|0\rangle$ is then the ground state of our system. Note that the above result also implies that we must have $\omega_a = \omega_0$. So we can conclude that

$$\hat{a}|n\rangle = \sqrt{n}|n - 1\rangle. \tag{11}$$

Then, with the aid of Eq.(4), it is easy to show that we must also have

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n + 1\rangle \tag{12}$$

and hence that

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{13}$$

Thus we have arrived at the properties of the QHO in a BTFD fashion, as required. Notice also that, unlike in the approach of Andrews and Romero[5], it was not necessary to *assume* initially that $\hat{a}|n\rangle = \alpha_n|n - 1\rangle$ [2].

The results above depend crucially on the phase symmetry of the number operator. In the next section we explore the relationship between phase symmetry, integers and natural numbers in a little more detail.

3   Integers, natural numbers and phase

Below we will outline the operator mathematics for generating the set of integers, $\mathbb{Z}$, and the set of natural numbers, $\mathbb{N}$, using phase symmetry. To do

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[2] Andrews and Romero[5] begin by assuming the form $\hat{a}|n\rangle = \alpha_n|n - 1\rangle$ and use time reversal arguments to get $\hat{a}^\dagger|n\rangle = \beta_n|n + 1\rangle$. From these results they deduce that $(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a})|n\rangle = (|\alpha_{n+1}|^2 - |\alpha_n|^2)|n\rangle$. They then take $|\alpha_{n+1}|^2 - |\alpha_n|^2$ as a constant independent of n. This cannot be justified on the basis of their previous assumptions and must be taken entirely as supposition, thus weakening their case somewhat.

this we require a phase shift operator. We first look for the eigenfunctions of the operator $\hat{Z}$ that is defined by

$$i\frac{\mathrm{d}}{\mathrm{d}\theta}|\Phi\rangle = \hat{Z}|\Phi\rangle. \tag{14}$$

where $|\Phi\rangle$ is a state in a Hilbert space, $\mathcal{H}_z$ on which $\hat{Z}$ operates. The parameter θ is the phase, i.e. a rotation about the origin in the complex plane. The states in $\mathcal{H}_z$ are defined in terms of the eigenfunctions that satify Eq.(14). So, we need to solve

$$i\frac{\mathrm{d}}{\mathrm{d}\theta}|\Phi_\lambda\rangle = \lambda|\Phi_\lambda\rangle, \tag{15}$$

where λ is an eigenvalue. The solutions are $|\Phi_\lambda(\theta)\rangle = \exp(-i\lambda\theta)|\Phi_\lambda(0)\rangle$. However, $\exp(-i\lambda\theta)$ is only single valued in the complex plane if $\lambda = z$, where z is any member of $\mathbb{Z}$. We let $\Phi_z(0) = (2\pi)^{-1/2}$, and define the scalar product, $\langle\Phi_p|\Phi_q\rangle$, by

$$\langle\Phi_p|\Phi_q\rangle = \int\limits_0^{2\pi} \Phi_p(\theta)^*\Phi_q(\theta)\mathrm{d}\theta = \delta_{pq}, \tag{16}$$

where $\langle\Phi_p| = (|\Phi_p\rangle)^*$ and p and q are integers. This shows that the $|\Phi_p\rangle$ constitute a set of orthonormal vectors of the Hilbert space, $\mathcal{H}_z$. Also, the fact that $\hat{Z}$ has real eigenvalues means that it is reasonable to take it as an Hermitian operator. It is actually essential to treat $\hat{Z}$ as such since we require it for the definition of a unitary operator (see below). Again, given the previously mentioned work of Bender et al. [13], we should make sure that no inconsistencies arise due to this, as we did previously with the definition involving $\hat{n}$ in Eq.(4). A superposition state, $|\Phi\rangle$, may be defined on $\mathcal{H}_z$ by $|\Phi\rangle = \sum_n c_n|\Phi_n\rangle$, where the c_n are a set of complex scalar valued coefficients. Then, $|\Phi\rangle$ is said to be normalized if $\langle\Phi|\Phi\rangle = 1$, which is the case as long as $\sum_n\|c_n\|^2 = 1$.

Eq.(14) may be formally integrated to yield,

$$|\Phi(\theta)\rangle = \exp(-i\theta\hat{Z})|\Phi(0)\rangle, \tag{17}$$

where $|\Phi(0)\rangle$ is $|\Phi(\theta)\rangle$ at $\theta = 0$. Here we can see that the unitary operator, $\hat{U}_\theta = \exp(-i\theta\hat{Z})$ acts as a phase shift operator, given that we have taken $\hat{Z}$ as Hermitian. It can be used to define operator functions of θ in the following way. Suppose an operator $\hat{Q}$ is not an explicit function of θ, so we write it as $\hat{Q}(0)$. Then its expectation value with respect to the normalised Hilbert space vector, $|\Phi(\theta)\rangle$, is

$$\langle\Phi(\theta)|\hat{Q}(0)|\Phi(\theta)\rangle = \langle\Phi(0)|\exp(i\theta\hat{Z})\hat{Q}(0)\exp(-i\theta\hat{Z})|\Phi(0)\rangle, \tag{18}$$

where, $\langle\Phi(0)|\exp(i\theta\hat{Z}) = (\exp(-i\theta\hat{Z})|\Phi(0)\rangle)^*$, where we have used $\hat{Z}^\dagger = \hat{Z}$, since we have taken $\hat{Z}$ as Hermitian. We can now define $\hat{Q}(\theta)$ by

$$\hat{Q}(\theta) = \exp(i\theta\hat{Z})\hat{Q}(0)\exp(-i\theta\hat{Z}), \tag{19}$$

which maintains the meaning of the expectation value since

$$\langle\Phi(0)|\hat{Q}(\theta)|\Phi(0)\rangle = \langle\Phi(\theta)|\hat{Q}(0)|\Phi(\theta)\rangle. \tag{20}$$

The operator in Eq.(19) can be formally differentiated with respect to θ to give

$$i\frac{\mathrm{d}\hat{Q}(\theta)}{\mathrm{d}\theta} = \hat{Q}(\theta)\hat{Z} - \hat{Z}\hat{Q}(\theta) = [\hat{Q}(\theta),\hat{Z}], \tag{21}$$

where $[\hat{Q}(\theta),\hat{Z}]$ is the commutation bracket for the two operators involved.

We can demonstrate the close relationship between the phase shift operator, and integers by applying Eq.(21) to a phasor operator defined by $\hat{A}(\theta) = \hat{A}(0)\exp(-i\theta)$. Then, with $i\frac{\mathrm{d}\hat{A}}{\mathrm{d}\theta} = \hat{A}$ substituted into Eq.(21) for $\hat{A}$, we get

$$\hat{A} = \hat{A}\hat{Z} - \hat{Z}\hat{A}. \tag{22}$$

Operating on the eigenstates of $\mathcal{H}_z$ with Eq.(22) and rearranging we get

$$\hat{Z}\hat{A}|z\rangle = (z-1)\hat{A}|z\rangle, \tag{23}$$

where we have written $|z\rangle$ for $|\Psi_z\rangle$ and used $\hat{Z}|z\rangle = z|z\rangle$. Eq.(23) implies that $\hat{A}|z\rangle$ is also an eigenstate of $\hat{Z}$, with an eigenvalue of $z-1$ and hence proportional to a state $|z-1\rangle$. It is easy to see that operating with $\hat{A}$ on $|z-1\rangle$ will yield $|z-2\rangle$ and so on. Applying a similar procedure to the adjoint $\hat{A}^\dagger$ [1] yields

$$\hat{Z}\hat{A}^\dagger|z\rangle = (z+1)\hat{A}^\dagger|z\rangle, \tag{24}$$

implying that $\hat{A}^\dagger|z\rangle$ is also an eigenstate of $\hat{Z}$, with an eigenvalue of $z+1$ and hence proportional to a state $|z+1\rangle$. From these results it is clear that repeated application of $\hat{A}$ and $\hat{A}^\dagger$ generate a set of eigenstates of $\hat{Z}$ whose eigenvalues are the set of integers, $\mathbb{Z}$, to within an arbitrary constant, which can be eliminated without loss of generality, in agreement with the eigenvalues found from Eq.(15). This cyclic symmetry that the phasor possesses is also present in the quantum theory of angular momentum, although there, there is a crucial difference in the interpretation of θ. In the case of angular momentum, θ is an angle in a plane in configuration space, rather than, as here, in the complex plane.

The Hilbert space, $\mathcal{H}_z$, of integer states can be mapped to a Fock-Darwin space [10, 9], by replacing $|z\rangle$ by $|n,m\rangle$, where n and m are natural numbers.

The Fock-Darwin space is a tensor product of two Fock spaces[3] $\mathcal{F}_n \otimes \mathcal{F}_m$, where the $|n\rangle$ represent vectors in $\mathcal{F}_n$ and $|m\rangle$ represent vectors in $\mathcal{F}_m$. Then

$$\hat{Z} = \hat{n} - \hat{m}, \tag{25}$$

where $\hat{n}$ and $\hat{m}$ are number operators. Now, invoking the previous argument about the non-negativity of natural numbers, we let $\hat{a}^\dagger \hat{a} = \hat{n}$ and $\hat{b}^\dagger \hat{b} = \hat{m}$ with the proviso that $\hat{n}$ and $\hat{a}$ only operate on $\mathcal{F}_n$ and that $\hat{m}$ and $\hat{b}$ only operate on $\mathcal{F}_m$. In other words $\hat{a}$ and $\hat{b}$ are treated as being independent of one another. This implies $[\hat{a}, \hat{b}] = [\hat{a}, \hat{m}] = 0$ etc. Our aim is to deduce the properties of $\hat{a}$ and $\hat{a}^\dagger$. Operating on the Fock-Darwin space with $\hat{Z}$ then yields

$$\hat{Z}|n, m\rangle = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})|n, m\rangle = (n - m)|n, m\rangle. \tag{26}$$

This clearly shows that the spectrum of $\hat{Z}$ is the set $\mathbb{Z}$. Notice that the results in Eqs.(23), (24), and (26) are consistent with the Hermitian nature of $\hat{Z}$ that was dealt with ealier.

We now make use of the fact that $\hat{n}$ is invariant to a phase shift in $\hat{a}$ and let $\hat{a}(\theta) = \hat{a}(0) \exp(-i\theta)$. Substituting this into Eq.(21) and at the same time using Eq.(26) yields

$$\hat{a} = [\hat{a}, \hat{a}^\dagger \hat{a}] = [\hat{a}, \hat{n}], \tag{27}$$

since $[\hat{a}, \hat{b}^\dagger \hat{b}] = 0$. Eq.(27) contains no $\hat{b}$ terms so we can just operate on $\mathcal{F}_n$ with it. After rearranging the result we get

$$\hat{n}\hat{a}|n\rangle = (n - 1)\hat{a}|n\rangle. \tag{28}$$

Now we can see that $\hat{a}|n\rangle$ is an eigenstate of $\hat{n}$ with eigenvalue $n - 1$, so we can again assume that $\hat{a}|n\rangle = \alpha_n|n - 1\rangle$, where α_n is a scalar which may depend on n. We also know $\|\hat{a}|n\rangle\|^2 = n$, and so, $\alpha_n = \sqrt{n}$. So we can conclude that

$$\hat{a}|n\rangle = \sqrt{n}|n - 1\rangle. \tag{29}$$

Then from $\hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$ it is then easy to show that we must also have

$$\hat{a}^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \tag{30}$$

and that

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{31}$$

Thus we have obtained the properties of of $\hat{a}$ and $\hat{a}^\dagger$, purely from phase symmetry.

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[3]A Fock space is the name given to the space we previously referred to as $\mathcal{N}$, when it is applied to particle numbers.

The above results may be generalized by noting that the sum of a set of integers is an integer, so the sum of a number of integer operators is just an integer operator. So, let

$$\hat{Z} = \sum_{i=1}^{S} \hat{Z}_i, \tag{32}$$

where $\hat{Z}_i$, can be represented in terms of a phase differential, like Eq.(14), as

$$i\frac{\mathrm{d}}{\mathrm{d}\theta_i}|\Psi_i\rangle = \hat{Z}_i|\Psi_i\rangle, \tag{33}$$

where θ_i is a phase angle in the complex plane. We assume that there are a finite number, S, of sets, subscript, i. Integrating Eq.(33) yields

$$|\Psi_i(\theta_i)\rangle = \exp(-i\theta_i\hat{Z}_i)|\Psi_i(0)\rangle. \tag{34}$$

We can solve the eigenvalue equation for each $\hat{Z}_i$ as we did with Eq.(15) and obtain $\Psi_{z_i}(\theta_i) = \Psi_{z_i}(0)\exp(-iz_i\theta_i)$, where z_i is an integer and θ_i is a phase. The $\Psi_{z_i}(\theta_i)$ can be individually normalized as before and then will constitute a Hilbert space, $\mathcal{H}_{z_i}$. Again this can be mapped onto a Fock-Darwin space with orthonormal state vectors, $|n_i, m_i\rangle$, and again $\hat{Z}_i$ can be represented by the difference of two number operators such that

$$\hat{Z}_i = \hat{a}_i^\dagger\hat{a}_i - \hat{b}_i^\dagger\hat{b}_i, \tag{35}$$

where $\hat{a}_i^\dagger\hat{a}_i = \hat{n}_i$ and $\hat{b}_i^\dagger\hat{b}_i = \hat{m}_i$. Noting that the only non-commutation is between $\hat{a}_i$ and $\hat{a}_i^\dagger$, with the same subscript i, we end up with the results

$$\hat{a}_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle,$$

$$\hat{a}_i^\dagger|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle,$$

$$\hat{a}_i^\dagger\hat{a}_i|n_i\rangle = \hat{n}_i|n_i\rangle = n_i|n_i\rangle,$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{36}$$

The last relation in Eqs.(36) simply follows from $\hat{a}_i^\dagger\hat{a}_j|\ldots, n_i, \ldots, n_j, \ldots\rangle = \hat{a}_j\hat{a}_i^\dagger|\ldots, n_i, \ldots, n_j, \ldots\rangle = \sqrt{(n_i + 1)n_j}|\ldots, n_i + 1, \ldots, n_j - 1, \ldots\rangle$, for $j \neq i$.

Finally, from Eqs.(32) and (35), we can also define a pair of global integers, $\hat{n}$ and $\hat{m}$ by

$$\hat{n} = \sum_{i=1}^{S} \hat{n}_i = \sum_{i=1}^{S} \hat{a}_i^\dagger\hat{a}_i, \tag{37}$$

with a corresponding result for $\hat{m}$.

4 Back-to-front derivation of the Hamiltonian for an n-body system

One obvious question arises when considering the BTFD of QHO physics that we considered in section 2, and that is, what would motivate the equal energy spacing starting point, without prior knowledge that such a system exists? Unless we had already solved the Schrödinger equation for a quadratic potential, how would we know? Actually, there is a system where equal spacing is obvious without any prior knowledge of the Hamiltonian and that is a system of n particles. We know unequivocally that we can have 0 or 1 or 2 etc, particles. In other words, the number of particles is just a counting number, including zero, that again is a member of the set, $\mathbb{N}$. But what has this to do with the BTFD? The equal spacing we are talking about here is not (yet) referring to energy steps. The steps are simply the consequence of the discrete nature of the population number, which we can take for granted. If anything it reflects a primitive notion of *atomism* in its most basic form. The question is then, is knowing that a primitive universe consists of n countable discrete items (*atoms*), sufficient information to derive its Hamiltonian, without any further assumptions? This is not an unrealistic question to ask, since we know there is a strong connection between the properties of the QHO and second quantization, which is the basis of quantum field theory in which particles are seen as excitations of a QHO field, where the quantum number of the energy of the QHO is re-interpreted as the number of particles excited in the field. Here there is clear a priori knowledge of the meaning of n, unlike in the QHO case dealt with in the previous section. Notice that we used the term *atom* above, not in its modern sense of a collection of protons, electrons and neutrons, but rather as an item that is a primitive constituent member of a population whose only property is the population number.

Now the answer to the question might seem pretty obvious. If the energy of one particle is ϵ, then the energy of n particles, each with the same energy, should be $n\epsilon$. This just takes us back to Eq.(1) does it not? So where is the problem? This actually presupposes that we already know what energy is and somehow all the particles have the same energy. We want to go a little deeper and make no assumptions about energy. Indeed, in section 3 the properties of the operators $\hat{a}$ and $\hat{a}^{\dagger}$ have been established without any reference to energy or assumptions about a system Hamiltonian. We showed in section 3 that the results in Eqs.(2), (4), (11), (12) and (13) can be derived without any prior knowledge of the system Hamiltonian. Below we show how to deduce the properties of the Hamiltonian for a system knowing only that it has a quantum number, n that we can interpret a a number of particles (or

items or atoms). We define the Hamiltonian in terms of a time shift operator, $\hat{\Omega}$, such that any operator of the system obeys Eqs.(6) and (7). Then we ask the question, what must $\hat{\Omega}$ be if it is to account for the kinematics of our primitive universe of n particles? The state vectors are still of the form $|n\rangle$, but now the n refers to particle number and the particle number space is referred to as a Fock space, $\mathcal{F}$.

We again observe that $\hat{n}$ is invariant to a phase shift in $\hat{a}$, so we again assume $\hat{a}(t) = \hat{a}(0) \exp(-i\omega_0 t)$ and substitute this in Eq.(7) to yield

$$i\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = \omega_0 \hat{a} = [\hat{a}, \hat{\Omega}]. \tag{38}$$

Our task is to deduce $\hat{\Omega}$. Now from Eq.(27) we can write

$$\omega_0 \hat{a} = [\hat{a}, \omega_0 \hat{n}]. \tag{39}$$

Subtracting Eq.(39) from Eq.(38) yields

$$[\hat{a}, \hat{\Omega} - \omega_0 \hat{n}] = 0. \tag{40}$$

Eq.(40) implies that $\hat{\Omega} - \omega_0 \hat{n} = F(\hat{a})$, where $F(\hat{a})$ is an arbitrary function of $\hat{a}$. By applying a similar procedure to $\hat{a}^\dagger(t) = \hat{a}^\dagger(0) \exp(i\omega_0 t)$, we get

$$[\hat{a}^\dagger, \hat{\Omega} - \omega_0 \hat{n}] = 0, \tag{41}$$

and so $\hat{\Omega} - \omega_0 \hat{n} = G(\hat{a}^\dagger)$, where $G(\hat{a}^\dagger)$ is an arbitrary function of $\hat{a}^\dagger$. The only way we can reconcile Eqs.(40) and (41) is if $F(\hat{a}) = G(\hat{a}^\dagger) = \gamma$, where γ is an arbitrary scalar that is independent of $\hat{a}$ and $\hat{a}^\dagger$. So we may conclude that

$$\hat{\Omega} = \omega_0 \hat{n} + \gamma. \tag{42}$$

We can also see how to identify the time shift operator, $\hat{\Omega}$, with energy, by letting $\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P})$, where $\hat{Q}$ and $\hat{P}$ are Hermitian operators. Substituting into Eq.(31) then gives

$$[\hat{P}, \hat{Q}] = -i, \tag{43}$$

which shows that $\hat{Q}$ and $\hat{P}$ can be interpreted as a pair of canonical variables. Note that Eq.(21) is not an assumption nor is it necessary to postulate it. This quantization condition comes naturally from the foregoing analysis. Writing $\hat{\Omega}$ in terms of $\hat{Q}$ and $\hat{P}$ and taking $\gamma = \frac{\omega_0}{2}$ gives

$$\hat{\Omega} = \frac{\omega_0}{2}(\hat{P}^2 + \hat{Q}^2), \tag{44}$$

which exhibits the quadratic form of the QHO, which also may be associated with the energy density of a field, such as the electromagnetic field. Thus we have arrived at second quantization by BTFD.

It is also revealing to replace $\hat{Q}^2$ in Eq.(44) by a more general function of $\hat{Q}$. So if $\hat{\Omega} = \frac{\omega_0}{2}\hat{P}^2 + V(\hat{Q})$, then

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}t} = -i[\hat{Q},\hat{\Omega}] = \omega_0\hat{P}, \tag{45}$$

and

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}t} = -i[\hat{P},\hat{\Omega}] = -\frac{\partial V(\hat{Q})}{\partial\hat{Q}} \tag{46}$$

where we have used Eq.(43) to give $[\hat{P},V(\hat{Q})] = -i\frac{\partial V(\hat{Q})}{\partial\hat{Q}}$. We can recognize Eq.(45) and (46) as a scaled form of Ehrenfest's theorem [1] which represents Newton's laws in operator form. However, when $V(\hat{Q}) \neq \hat{Q}^2$, then $\hat{n}$ no longer commutes with $\hat{\Omega}$ and so is not a constant in time. There are two distinct interpretations of this situation. The first is to abandon the simple n-particle picture and adopt a representation in which $\hat{Q} \to q$ is a system coordinate and $\hat{P} \to -i\frac{\mathrm{d}}{\mathrm{d}q}$, is a momentum operator in differential form. This leads to the Schrödinger picture. The second interpretation is to extend the picture to a multi-mode case where the potential causes interaction between modes and leads to time dependent population numbers. We examine this second view next.

5 Generalization to multi-mode systems

It is clear from the results of the previous section that we are dealing with a collection of bosons, since the population number, n, is not limited. It is however confined to a single mode with a fundamental frequency, ω_0. We note that within a single mode there is nothing to distinguish two bosons of the same state, i.e., possessing the same fundamental frequency. Now we want to consider a multi-moded case, that is one in which we have more than one set of bosons and the sets are distinguishable.

Let us assume that $\hat{n}$ represents a global population that is divided into subsets of like particles given by Eq.(37). We will assume that the number of sets, S, is finite, so that the sum in Eq.(37) has a finite number of terms. Clearly each $\hat{n}_i$ is a member of $\mathbb{N}$, so we can look for an operator $\hat{a}_i$ such that $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$. These operators will operate on a state that is represented in so-called occupation number notation by $|n_1, n_2, \ldots, n_S\rangle$. We can assume that

each $\hat{n}_i$ is subject individually to a phase symmetry of the kind introduced above, i.e., $\hat{a}_i(\theta_i) = \hat{a}_i(0)\exp(-i\theta_i)$.

In order to distinguish the different set of particles and at the same time provide a universal system variable that will allow a single Heisenberg equation for the whole system, we let $\theta_i = \omega_i t$. If $\hat{\Omega}$ is again the Hamiltonian for the system. Then

$$i\frac{\mathrm{d}\hat{a}_i}{\mathrm{d}t} = \omega_i\hat{a}_i = [\hat{a}_i, \hat{\Omega}]. \tag{47}$$

However,

$$\omega_i\hat{a}_i = [\hat{a}_i, \omega_i\hat{n}_i], \tag{48}$$

So, subtracting Eq.(48) from Eq.(47), we get $[\hat{a}_i, \hat{\Omega} - \omega_i\hat{n}_i] = 0$ and hence, using Eqs.(36), we can conclude that $\hat{\Omega} - \omega_i\hat{n}_i = K(\hat{a}_i, \hat{a}_j, \hat{a}_j^\dagger)$ with $j \neq i$. Similarly from the time derivative of $\hat{a}_i^\dagger(t) = \hat{a}_i(0)\exp(-i\omega_i t)$ we find $\hat{\Omega} - \omega_i\hat{n}_i = L(\hat{a}_i^\dagger, \hat{a}_j, \hat{a}_j^\dagger)$ with $i \neq j$. The only way these relations can be satisfied for all i is if

$$\hat{\Omega} = \sum_{i=1}^{S} \omega_i\hat{n}_i + \Gamma = \sum_{i=1}^{S} \omega_i\hat{a}_i^\dagger\hat{a}_i + \Gamma, \tag{49}$$

where Γ is an arbitrary scalar constant. From the above relations, one can show that $\frac{\mathrm{d}\hat{n}_i}{\mathrm{d}t} = 0$. This clearly implies that for the system as a whole, $\frac{\mathrm{d}\hat{n}}{\mathrm{d}t} = 0$. Clearly we recover the single mode case when $S = 1$.

The multimode case above can be explored further in a telling way. We can apply a linear transformation to the set of annihilation operators $\hat{a}_i$, by treating them as a column vector. Then we let $\hat{a}_i = \sum_j \gamma_{ij}\hat{c}_j$, where the γ_{ij} are the constant scalar elements of a $S \times S$ matrix. Substituting for $\hat{a}_i$ into the expression for $\hat{\Omega}$ yields (letting Γ be zero)

$$\hat{\Omega} = \sum_i u_i\hat{c}_i^\dagger\hat{c}_i + \sum_{jk} v_{jk}\hat{c}_j^\dagger\hat{c}_k, \tag{50}$$

where the sums are over the number of categories and u_i and v_{jk} are constant coefficients that depend on the γ_{ij}. As long as the matrix with elements, γ_{jk}, is chosen to be unitary, then one finds

$$\hat{n} = \sum_{i=1}^{S} \hat{r}_i = \sum_{i=1}^{S} \hat{c}_i^\dagger\hat{c}_i, \tag{51}$$

where $\hat{r}_i = \hat{c}_i^\dagger\hat{c}_i$ and

$$\hat{c}_i|r_i\rangle = \sqrt{r_i}|r_i - 1\rangle,$$
$$\hat{c}_i^\dagger|r_i\rangle = \sqrt{r_i + 1}|r_i + 1\rangle,$$

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$$\hat{c}_i^\dagger \hat{c}_i |r_i\rangle = \hat{r}_i |r_i\rangle = r_i |r_i\rangle,$$
$$[\hat{c}_i, \hat{c}_i^\dagger] = 1. \tag{52}$$

In Eqs.(52), the $|r_i\rangle$ are vectors in a new Fock space that is made up of the eigenstates of the new number operator, $\hat{r}_i = \hat{c}_i^\dagger \hat{c}_i$. The eigenvalues of $\hat{r}_i$ are the set of natural numbers once more, but now, $\hat{r}_i$ no longer commutes with $\hat{\Omega}$ in Eq.(50), so $\frac{d\hat{r}_i}{dt} \neq 0$, however it is straightforward to show that we still have $\frac{d\hat{n}}{dt} = 0$ for the system as a whole. So we can still interpret Eqs.(52) as a system of n particles divided into S modes, but the sum over jk in Eq.(50) indicates that a particle in mode-k is scattered into mode-j. This is indicative of the effect of a potential that changes the states of the particles. There is a close relation between this result and the generalization of $V(\hat{Q})$ in the previous section and it can be shown that $v_{jk} = \langle r_j | V | r_k \rangle$ [14]. This allows a powerful interpretation of the nature of potentials in physics and the result can be extended to particle-particle interactions via particle exchange. It is also worth pointing out that the steps from Eq.(50) to Eq.(52) constitute a BTFD of particle interactions, since one usually starts with the Hamiltonian in the form in Eq.(50) (see ref. [14]). Reversing the transform by using the inverse of the matrix, γ_{ij}, then yields the Hamiltonian in the form of Eq.(49), from which the energy eigenvalues of the system can be determined. This reverse transform is actually the usual Bogoliubov transformation that is well known in many-body quantum physics [14].

6   Fermions

It is also possible to deduce alternatives to the bosonic commutation relations Eqs.(52) for the creation and annihilation operators, that are still consistent with $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$, the time shift operator, Eq.(49) and the Heisenberg equation for $\hat{a}_i(t) = \hat{a}_i(0)\exp(-i\omega_i t)$. We can treat $[\hat{a}_i, \hat{a}_j] = 0$ as an ordering rule that implies $\hat{a}_i \hat{a}_j = \hat{a}_j \hat{a}_i$, and then generalize this to $\hat{a}_i \hat{a}_j = \mu \hat{a}_j \hat{a}_i$, where μ is an as yet unknown constant scalar factor. Reversing the order once more gives $\hat{a}_i \hat{a}_j = \mu^2 \hat{a}_i \hat{a}_j$, so $\mu = \pm 1$. Similarly, if we assume that $\hat{a}_i \hat{a}_j^\dagger = \rho \hat{a}_j^\dagger \hat{a}_i + \sigma$, where ρ and σ are scalar constants, then we get

$$\omega_i \hat{a}_i = [\hat{a}_i, \sum_{j=1}^{S} \omega_j \hat{a}_j^\dagger \hat{a}_j] = \sum_{j=1}^{S} \omega_j (\hat{a}_j^\dagger \hat{a}_j \hat{a}_i (\mu\rho - 1) + \sigma \hat{a}_j). \tag{53}$$

Thus, since this expression has to be true for all i and j, we must have, $\mu = \rho = \pm 1$ and $\sigma = \delta_{ij}$. When $\mu = \rho = 1$ we recover our previous (bosonic) results, but when $\mu = \rho = -1$ we get something new, i.e.

$$\{\hat{a}_i, \hat{a}_j\} = 0, \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0, \{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}, \tag{54}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. It is then a straightforward matter to show that $\hat{a}_i^2 = \hat{a}_i^{\dagger 2} = 0$ and $\hat{n}_i^2 = \hat{n}_i$, so $n_i = 0$ or 1. The first and third rules in Eqs.(36) still apply, but the second needs a slight modification to $\hat{a}_i^{\dagger}|n_i\rangle = \sqrt{1-n_i}|n_i+1\rangle$. This new set of relations describes the properties of fermions.

7 Comments and conclusions

This paper has two closely related aims. The first is to propose a new "back-to-front" derivation of the properties of the QHO, starting from the knowledge that its energy levels are equally spaced above a ground-state level. This has been achieved by noting that the Hamiltonian of such a system can be written in terms of an operator, $\hat{n}$, with a spectrum of eigenvalues that are natural numbers. The reality and non-negativity of such eigenvalues means that the operator can be equated to the product of an operator multiplied by its adjoint, since such a combination has eigenvalues that must be real and non-negative. We showed, via a phase symmetry argument [6], that the adjoint pair, $\hat{a}$ and $\hat{a}^{\dagger}$, must have the properties of the creation and annihilation operators, that define the dynamical properties of the QHO. The method succeeds because the relationship between $\hat{a}$ and $\hat{a}^{\dagger}$ and $\hat{n}$ shows that $\hat{a}$ must naturally be a phasor, which intrinsically incorporates QHO behaviour.

The second aim was to go a step further and to show that a similar "back-to-front" argument could be used to obtain the properties of $\hat{a}$ and $\hat{a}^{\dagger}$ without first knowing the Hamiltonian of an n-particle system, from which the Hamiltonian itself could then be inferred. The motivation here was to see if the dynamics of a system of particles could be deduced without any preconceived notions of energy that come from classical physics. To do this we defined the Hamiltonian simply in terms of a time shift operator. Again, phase symmetry plays a key role. One can then show that an arbitrary phasor has a Hamiltonian with integer eigenvalues. By invoking Schwinger's idea [10] that an integer is just the difference between two natural numbers, one can then show that the Hamiltonian that applies to the special phasor, $\hat{a}$ in $\hat{n} = \hat{a}^{\dagger}\hat{a}$, must be proportional to $\hat{n}$. This approach has major advantages over conventional ones which involve quantizing a classical theory, as advocated by Dirac [11] among others. First, we have shown that quantum theory can be developed on quantum concepts, in this case on the idea that we start with n particles and does not need analogies with classical physics. This is already a proto-quantum idea in the sense that the number of particles is discrete. Discreteness can readily be associated with the eigenvalues of an operator. Where discreteness is obvious with particle numbers, it is much less so when it comes to energy. In the author's opinion, the discreteness argument is the

simplest justification for the role of operators at all, in physical theory. Second, the commutation relations that are fundamental to quantum behaviour and for the identification of canonical variables emerge quite naturally from the approach adopted here. This allows physical concepts such as energy and momentum also to emerge naturally from the operator formalism.

Generalizing the system to a multi-mode case is also straightforward. Importantly, this approach shows that on can derive the idea that the quantity we identify with energy is a linear superposition of individual energies. It is also then a straightforward matter to introduce the idea of inter-modal or inter-particle interactions which leads naturally to the notion of potential scattering of particles between different states. Although the initial approach to n-body systems naturally leads to bosonic behaviour, we have shown how to extend the approach to yield fermionic rules too.

In conclusion, the results above show that the development of physics can be based on the idea of the discreteness of nature i.e. *atomism* in its original sense. It is worth noting here that concept of atomism predates that of quantum physics by several centuries. It was Leucippus and his pupil Democritus [15], in the fourth century BC, who first realized that it was possible to understand the nature of the universe by breaking it into bits. The approach above uses operator mathematics to explore the consequences of this fundamental idea. Clearly, it puts number at the centre of physics. This may be contrasted with much of modern physical thinking which has geometry at its heart. The advantage of number as a fundamental conceptual basis is that it links physics to information and, in particular, to the increasingly important field of quantum information [16, 17]. The approach adopted here can provide insight into the physical consequences of the idea of a universe of bits and qubits. All that is required to understand this approach is the method of generating natural numbers as a spectrum of eigenvalues, together with a knowledge of operator valued functions that can be found in standard undergraduate level textbooks on quantum mechanics.

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