

# Cointegrated VARIMA models: specification and simulation

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## Abstract

In this note we show how specify cointegrated vector autoregressive-moving average models and how they can be used to generate cointegrated time series.

## 1 Introduction

Available tests for cointegration in pure VAR or mixed VARMA models (Johansen , 1988, 1991; Yap and Reinsel , 1995; Lüktephohl and Claessen , 1997) are built on the assumption that the data generating process is homogeneously non-stationary under the null hypothesis. Data simulation for Monte Carlo experiments is performed by assuming that all the presample values are given. In this approach, the presence of cointegration imposes a set of restrictions on the AR matrix polynomial, but not affect to the MA matrix polynomial whose only role is to provide parsimony.

In this note, we show that there is an alternative specification for cointegrated VARIMA models based on a data generating process that is stationary under the null hypothesis. Data

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simulation proceeds in two stages: (1) stationary realizations are generated without needing to fix presample values and (2) integrated realizations are then obtained by cumulating the stationary ones with given presample. The presence of cointegration in our representation imposes a similar set of restrictions on the MA matrix polynomial, but does not affect to the AR matrix polynomial, which also plays a role in the definition of the cointegration matrix.

It can be illustrative to relate both approaches to the two strategies commonly followed to identify the degree of differencing in the applied analysis of univariate time series, namely, the underdifferencing and overdifferencing strategies underlying the type Dickey and Fuller (1979) and Nyblom and Mäkeläinen (1983) tests, respectively. However, we emphasize that our cointegrated VARIMA model is an homogeneously non-invertible but non-overdifferenced representation that can be estimated using standard methods based on the exact maximum likelihood function.

The main contribution of the note is to specify a cointegrated VARIMA model compatible with univariate representations widely used to forecast. Such representations include IMA(1,1) structures that provide adaptive terms in the forecast function, being reasonable its consideration when modeling multiple time series. The algorithm here proposed to generate cointegrated time series can be used to evaluate the performance of different tests for multivariate cointegration, specially those that claim to be independent of the data generating process (e.g. Stock and Watson , 1988; Nyblom and Harvey , 2000). Another useful experiment is to compare the performance of these multivariate tests with their univariate analogous versions.

The paper is structured as follows. Section 2 describes the data generating process and how the cointegrating relations can be recovered from the moving average in a VARIMA model. Section 3 illustrates the simulation procedure in a given example and section 4 concludes.

## 2 Cointegration in Vector ARIMA models

The Engle and Granger (1987) representation theorem is based on the Wold representation of a first order integrated  $m$ -dimensional vector stochastic process

$$\nabla \mathbf{z}_t = \Psi(B) \mathbf{a}_t, \quad (1)$$

where  $B$  and  $\nabla = 1 - B$  are the backshift and first difference operators respectively,  $\Psi(B) = \mathbf{I}_m + \Psi_1 B + \Psi_2 B^2 + \dots$  is a matrix polynomial in  $B$  of infinite order,  $\Psi(1) = \mathbf{I}_m + \Psi_1 + \Psi_2 + \dots$  is the sometimes called gain of the linear filter  $\Psi(B)$  whose determinantal equation has all its zeros on or outside the unit circle ( $|\Psi(B)| = 0$  for  $|B| \leq 1$ ), and  $\mathbf{a}_t$  is a multivariate white noise process with zero mean vector and covariance matrix  $\Omega_a$ . Under these assumptions, they shown that an  $r \times m$  matrix of coefficients  $\mathbf{C}$  will define a set of  $r < m$  cointegration relations when  $\mathbf{C}\Psi(1) = 0$ , that is, when  $\Psi(1)$  has rank  $m - r$ . The proof follows easily from the well-known multivariate Beveridge-Nelson decomposition of  $\Psi(B) = \Psi(1) + \Psi^*(B)\nabla$  that reveals the stationarity of  $\mathbf{C}\mathbf{z}_t = \mathbf{C}\Psi^*(B)\mathbf{a}_t$ , where  $\Psi^*(B) = \Psi_0^* + \Psi_1^* B + \Psi_2^* B^2 + \dots$  and  $\Psi_j^* = \Psi_{j+1} + \Psi_{j+2} + \dots$  so that  $\Psi^*(B)$  is invertible. The choice of  $\mathbf{C}$  is obtained from the reformulation of (1) into an error-correction model.

We follow now a similar reasoning to determine the cointegrating restrictions in the vector ARIMA(p,1,q) process as the parsimonious representation of (1),

$$\Phi(B)\nabla \mathbf{z}_t = \Theta(B)\mathbf{a}_t, \quad (2)$$

where  $\Phi(B) = \mathbf{I}_m - \Phi_1 B - \dots - \Phi_p B^p$  and  $\Theta(B) = \mathbf{I}_m - \Theta_1 B - \dots - \Theta_q B^q$  are the finite order AR and MA matrix polynomials, respectively, such that  $\Phi(B)\Psi(B) = \Theta(B)$ ,  $|\Phi(B)| = 0$  for  $|B| < 1$ , and  $|\Theta(B)| = 0$  for  $|B| \leq 1$ . By replacing  $\Phi(B)$  and  $\Theta(B)$  by their respective

Beveridge-Nelson decompositions, (2) can be written as

$$\Phi(1)\nabla\mathbf{z}_t = \Phi^*(B)\nabla^2\mathbf{z}_t + \Theta(1)\mathbf{a}_t + \Theta^*(B)\nabla\mathbf{a}_t, \quad (3)$$

where  $\Phi(B)^* = \Phi_0^* + \Phi_1^*B + \dots + \Phi_{p-1}^*B^{p-1}$  and  $\Theta^*(B) = \Theta_0^* + \Theta_1^*B + \dots + \Theta_{q-1}^*B^{q-1}$ . It is now clear from (3) that  $\mathbf{z}_t$  will exhibit cointegration if there exists an  $r \times m$  matrix  $\mathbf{C}$  that annihilates  $\Theta(1)$ ,  $\mathbf{C}\Theta(1) = 0$ . In such a situation, the cancelation of the common  $\nabla$  operator on both sides of the equation implies that the linear transformation  $\mathbf{D}\mathbf{z}_t$ , with  $\mathbf{D} = \mathbf{C}\Phi(1)$ , is the sum of two stationary processes and therefore, the rows of  $\mathbf{D}$  can be interpreted as the  $r$  cointegrating relations of the system. Hence, we can deduce the role that each operator of the vector ARMA model plays in the cointegration analysis. While the gain of the MA matrix polynomial determines the existence or non-existence of cointegration, that of the AR matrix polynomials appears only as a weighing factor in the definition of the cointegrating vectors.

From the discussion above, a necessary condition for the existence of cointegration is that the gain matrix of the moving average filter has rank  $(m - r) < m$ . Therefore, matrix  $\mathbf{C}$ , if it exists, can be found from the spectral decomposition of  $\Theta(1) = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$  which, would reduce to

$$\Theta(1) = \mathbf{P}_2\mathbf{\Lambda}_2\mathbf{Q}_2',$$

where the diagonal submatrix  $\mathbf{\Lambda}_2$  contains the  $m - r$  non-null eigenvalues, all of which are inside the unit circle, the  $m \times (m - r)$  submatrix  $\mathbf{P}_2$  contains the corresponding eigenvectors and the  $m \times (m - r)$  submatrix  $\mathbf{Q}_2$  contains the rows in the conformable partition of  $\mathbf{P}^{-1}$ . Hence, if  $\mathbf{P}_1$  is the  $m \times r$  submatrix of eigenvectors associated to the null eigenvalue with multiplicity  $r$  and  $\mathbf{Q}_1$  is the  $m \times r$  submatrix with the corresponding rows of  $\mathbf{P}^{-1}$ , then it is clear that  $\mathbf{C} = \mathbf{Q}_2'$ .

As an example consider the simple vector IMA(1,1) model

$$\nabla\mathbf{z}_t = (\mathbf{I}_m - \Theta B)\mathbf{a}_t,$$

which will exhibit cointegration when  $\mathbf{I} - \Theta$  is rank deficient or, equivalently, if  $\Theta$  has some unit eigenvalues. Thus, a testing strategy for the cointegrating rank of the system can be based on the transformed model

$$\nabla \mathbf{P}^{-1} \mathbf{z}_t = (\mathbf{I}_m - \Lambda B) \mathbf{b}_t,$$

where  $\mathbf{b}_t = \mathbf{P}^{-1} \mathbf{a}_t$  and  $\Lambda = \{\lambda_1, \dots, \lambda_m\}$  is the diagonal matrix containing the eigenvalues of the moving average matrix. If the system is reordered such that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ , the null hypothesis of multiple cointegration implies testing  $H_0 : \lambda_1 = \dots = \lambda_r = 0$  against the alternative hypothesis  $H_1 : \lambda_1 = \dots = \lambda_{r-1} = 0$  by extending the tests for noninvertibility in univariate models proposed by Tanaka (1990) and Saikkonen and Luukkonen (1993) in a similar way to the extension of the Nyblom and Mäkeläinen (1983) and Kwiatkowski et al. (1992) test given by Nyblom and Harvey (2000).

### 3 Simulating cointegrated time series

To specify a cointegrated vector ARIMA model that can be used as a data generating process we must worry mainly of the choice of the MA coefficient matrices. To this end, we create directly the gain matrix  $\Theta(1)$  from its spectral decomposition  $\mathbf{P}\Lambda\mathbf{P}^{-1}$ , remembering that  $r$  eigenvalues are null and that the cointegrating matrix  $\mathbf{C}$  fixes  $r$  rows of the  $\mathbf{P}^{-1}$ . The  $m - r$  remaining eigenvalues (real and/or complex conjugates) are chosen arbitrarily inside unit circle and determine if the corresponding rows of  $\mathbf{P}^{-1}$  are either real or complex vectors, which can also be chosen arbitrarily. Once  $\Lambda$  and  $\mathbf{P}^{-1}$  have been created, we calculate  $\mathbf{P}$  and  $\Theta(1)$  and impose the cointegration restriction by fixing  $\Theta_1 = \mathbf{I}_m - \Theta_2 - \dots - \Theta_q - \Theta(1)$ , where the MA coefficient matrices  $\Theta_2, \dots, \Theta_q$  can be chosen arbitrarily as long as the resulting MA(q) polynomial has all its zeros on or outside unit circle. If the model has autoregressive part, any stationary polynomial  $\Phi(B)$  is suitable. Once chosen the parameters of the VARMA model, the stationary series  $\mathbf{w}_t$  is simulated using exact initial conditions as described by Hillmer and Tiao

(1979). Then, the I(1) series is obtained by aggregation as  $\mathbf{y}_t = \mathbf{w}_t + \mathbf{y}_{t-1}$  where  $\mathbf{y}_0$  is fixed.

The following example illustrates the procedure of simulation of a cointegrated bivariate VIMA(1,2) model with cointegrating vector  $\mathbf{Q}'_1 = (1, -1.5)$ . Assuming  $\lambda_2 = 0.3$  and  $\mathbf{Q}'_2 = (0.7635, 1.4069)$ , so that  $\mathbf{P}^{-1} = [\mathbf{Q}_1 \ \mathbf{Q}_2]'$ , the gain matrix can be recovered as

$$\Theta(1) = \begin{pmatrix} 0.1346 & 0.2481 \\ 0.0897 & 0.1654 \end{pmatrix}, \quad \text{with} \quad \Omega_a = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix},$$

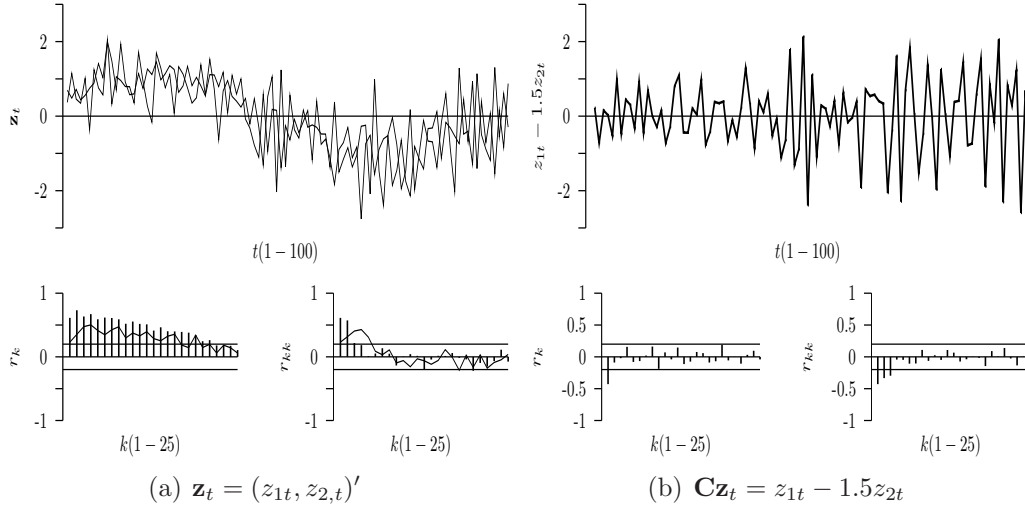
Next, we set an arbitrary  $\Theta_2$  matrix and recover the following MA(2) model

$$\begin{pmatrix} \nabla z_{1t} \\ \nabla z_{2t} \end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.1368 & -0.6129 \\ -0.2882 & 1.5009 \end{pmatrix} B - \begin{pmatrix} -0.2714 & 0.3648 \\ 0.1985 & -0.6663 \end{pmatrix} B^2 \right] \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix},$$

which has one unit root, a real root equal to 4.9976 and a pair of complex conjugates,  $1.3263 \pm 0.2932i$ , with moduli equal to 1.3584. The eigenvalues of the gain matrix  $\Theta(1) = \mathbf{I}_2 - \Theta_1 - \Theta_2$  are 0.3 and 0, and the cointegrating vector, which arises from the row of the inverse matrix of eigenvectors corresponding to the zero eigenvalue is, conveniently normalized, equal to its assumed value of (1, -1.5).

The left panel of figure 1 shows a realization of this cointegrated bivariate VIMA(1,2) model along with the simple and partial autocorrelation functions shown as bars for  $z_1$  and a straight line for  $z_2$ . The slow decay of the simple autocorrelation function show that both series are I(1). As assumed, imposing the cointegrating relation this decay disappears, as shown in the right panel of figure 1, and the correlation structure of the cointegrating relation resembles that of the the remaining MA(1) structure, as implied by equation (3).

Figure 1: Simulated bivariate system



## 4 Conclusions

In this paper we have shown how to embed the cointegration analysis into the Tiao-Box modeling approach by defining the restrictions that these type of relationships between non-stationary time series imposes on the MA parameters. Both the order and the cointegration matrix can be obtained from the conventional estimation of a VARIMA model. Notwithstanding, the development of a strategy for testing multivariate cointegration requires extending the univariate tests for noninvertibility proposed by Tanaka (1990) and Saikkonen and Luukkonen (1993), which is also part of our research agenda. We also give a general procedure to simulate a cointegrated VARIMA model with known cointegrating vectors. Further extensions of the work will show a full testing procedure for detecting the presence of cointegration in a model and the number of cointegrating relations. Our approach can be easily extended to handle with seasonal cointegration related to the seasonal operator or some of its simplifying factors.

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