# A NOTE ON (CO)HOMOLOGIES OF ALGEBRAS FROM UNPUNCTURED SURFACES 

YADIRA VALDIVIESO-DÍAZ


#### Abstract

In a previous paper, the author compute the dimension of Hochschild cohomology groups of Jacobian algebras from (unpunctured) triangulated surfaces, and gave a geometric interpretation of those numbers in terms of the number of internal triangles, the number of vertices and the existence of certain kind of boundaries. The aim of this note is computing the cyclic (co)homology and the Hochschild homology of the same family of algebras and giving an interpretation of those dimensions through elements of the triangulated surface.


## 1. Introduction

A surface with marked points, or simply a surface, is a pair $(S, M)$, where $S$ is a compact connected Riemann surface with (possibly empty) boundary, and $M$ is a non-empty finite subset of $S$ containing at least one point from each connected component of the boundary of $S$. We said that $(S, M)$ is an unpunctured surface if $M$ is contained in the boundary of $S$. We define a triangulation as a maximal collection of non-crossing arcs with endpoints in $M$.

Given an (tagged) triangulation $\mathbb{T}$, it is possible to construct a finite dimensional algebra $A_{\mathbb{T}}$, which turns out to be gentle if ( $S, M$ ) is an unpunctured surface (see [7, 1, 8, 12]), and in particular quadratic monomial. In this note, we compute three different (co)homologies of those gentle algebras coming from unpunctured surfaces, namely: Hochschild homology and cyclic homology and cohomology, and we show that there is a combinatorial interpretation of those (co)homologies through the elements of the surface and the triangulation.

Given an associative algebra $A$ over a field $\mathbf{k}$ and $M$ an $A$-bimodule, we define the Hochschild cohomology of $A$, with coefficients in $M$, as the graded vector space $\mathrm{HH}^{*}(A, M)=$ $\operatorname{Ext}_{A \otimes_{\mathbf{k}} A^{\text {op }}}^{*}(A, M)$, where $A^{\mathrm{op}}$ is the algebra $A$ with the opposite multiplication. The original definition was introduced by Hochschild in [6] using a resolution of $A$ as bimodule. Later, Cartan and Eilenberg in [2, Chapter 9] extended it to algebras over more general rings, and also dualized it, giving the definition of Hochschild homology.

The cyclic cohomology can be defined in several ways, but the original definition was given by Connes in [3], as a variation of the de Rham homology in spaces with bad behaviour. He used a sub-complex of the Hochschild complex when $M=\operatorname{Hom}_{\mathbf{k}}(A, \mathbf{k})$, called cyclic complex. In this note, we use the definition of cyclic (co)homology given in [9, Theorem 4.1.13], for algebras over rings of characteristic zero. As in the Hochschild cohomology case, the dual definition of cyclic cohomology was given later by several authors: Loday, Kassel, Quillen and Tsygan.

To compute those (co)homologies, we use the computation of the Hochschild homology of quadratic monomial algebras given by Skölderberg in [10], and two results of Loday

[^0][9. Theorem 4.1.13, Section 2.4.8], which relate the Hochschild homology and the cyclic homology by one hand, and the last one with the cyclic cohomology by other hand.

The main result of this note is the following:
Theorem. Let $(S, M, \mathbb{T})$ be a triangulated surface and $A_{\mathbb{T}}$ be the algebra associated to ( $S, M, \mathbb{T}$ ). Then,

$$
\begin{aligned}
& \text { i) } \operatorname{HH}_{n}\left(A_{\mathbb{T}}\right) \simeq \begin{cases}\mathbf{k}\left(Q_{\mathbb{T}}\right)_{0} & \text { if } n=0 \\
\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n} & \text { if } n \equiv 3(\bmod 6) \\
\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n+1} & \text { if } n \equiv 2(\bmod 6) \\
0 & \text { otherwise }\end{cases} \\
& \text { ii) } \operatorname{HC}_{n}\left(A_{\mathbb{T}}\right) \simeq \begin{cases}\mathbf{k}\left(Q_{\mathbb{T}}\right)_{0} & \text { if } n=0 \\
\mathbf{k}\left(Q_{\mathbb{T}}\right)_{0} \oplus\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n+1} & \text { if } n \equiv 2(\bmod 6) \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

Moreover, the dimension of the quotients $\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{k}$ appearing in $\mathrm{HH}_{n}\left(A_{\mathbb{T}}\right)$ and $\mathrm{HC}_{n}\left(A_{\mathbb{T}}\right)$, are equal to the number of internal triangles of $\mathbb{T}$.

As consequence of the main Theorem, it follows that the dimension of the cyclic cohomology is computed as follows.
Corollary. Let $A_{\mathbb{T}}$ be the algebra associated to the triangulated surface $(S, M, \mathbb{T})$ and denote by $\operatorname{int}(\mathbb{T})$ the set of internal triangles of $\mathbb{T}$. Then

$$
\operatorname{dim}_{\mathbf{k}}\left(\operatorname{HC}^{n}\left(A_{\mathbb{T}}\right)\right)=\left\{\begin{array}{ll}
\left|\left(Q_{\mathbb{T}}\right)_{0}\right| & \text { if } n=0 \\
\left|\left(Q_{\mathbb{T}}\right)_{0}\right|+|\operatorname{int}(\mathbb{T})| & \text { if } n \equiv 2 \\
0 & \text { otherwise }
\end{array}(\bmod 6)\right.
$$

According to the previous results, the dimensions of the cyclic (co)homology and Hochschild homology of the algebra $A_{\mathbb{T}}$ depend on $|\operatorname{int}(\mathbb{T})|$ and the number of vertices of the quiver $Q_{\mathbb{T}}$, then it is easy to observe those (co)homologies are not invariant under flips of arcs, and therefore they are not invariant under mutation of quivers with potentials. See [5] for definition of flips of arcs in triangulations, 4] for definitions of mutations of quivers with potentials and [7] for details about the relation between algebras from surfaces and quivers with potentials and its mutations.

## 2. Results

In the first part of this section, we recall some definitions and notations of path algebras and surfaces with marked points and we include the computation of the Hochschild homology given by Sköldberg in [10] for completeness.

Let $Q=\left(Q_{0}, Q_{1}\right)$ be a finite quiver with a set of vertices $Q_{0}$ and a set of arrows $Q_{1}$. We denote the source and the target of an arrow $a \in Q_{1}$ by $s(a)$ and $t(a)$, respectively. A path $w$ of length $l$ is a sequence of $l$ arrows $a_{1} \cdots a_{l}$, such that $t\left(a_{k}\right)=s\left(a_{k+1}\right)$ for every $k=1, \ldots, l-1$, we say that its source $s(w)$ is $s\left(\alpha_{1}\right)$ and its target $t(w)$ is $t\left(\alpha_{l}\right)$. We denote by $|w|$ the length of the path $w$. We write $\left[e_{i}\left|a_{1} \cdots a_{l}\right| e_{j}\right]$ instead of $a_{1} \cdots a_{l}$ to emphasize that the source of the path $a_{1} \cdots a_{l}$ is $e_{i}$ and its target is $e_{j}$.

Let $\mathbf{k}$ be an algebraically closed field. The path algebra $k Q$ is the $\mathbf{k}$-vector space with basis the set of paths in $Q$ and the product of the basis elements is given by the concatenations of
the sequences of arrows of the paths $w$ and $w^{\prime}$ if they form a path and zero otherwise. Let $F$ be the two-sided ideal of $k Q$ generated by the arrows of $Q$. A two-sided ideal $I$ is said to be admissible if there exists an integer $m_{0} \geq 2$ such that $F^{m_{0}} \subseteq I \subseteq F^{2}$ and its elements are called relations. The pair $(Q, I)$ is called a bounded quiver.

The quotient algebra $k Q / I$ is said a quadratic monomial algebra if the admissible ideal $I$ is generated by paths of length 2 . We associate its Koszul dual $A^{!}$which is the quotient algebra $A^{!}=\mathbf{k} Q / J$ where $J$ is generated by all paths $w$ of length 2 such that $w \notin I$.

In the next few paragraphs, we give a construction of a quadratic monomial algebra from an unpunctured surface.

Let $(S, M)$ be an unpunctured surface. An $\operatorname{arc} \tau$ in $(S, M)$ is a not self-crossing curve in $S$ with endpoints in $M$ and not isotopic to a point or to a boundary segment.

For any two arcs $\tau$ and $\tau^{\prime}$ in $S$, let $e\left(\tau, \tau^{\prime}\right)$ be the minimal number of crossings of $\tau$ and $\tau^{\prime}$, that is, $e\left(\tau, \tau^{\prime}\right)$ is the minimum of numbers of crossings of curves $\sigma$ and $\sigma^{\prime}$, where $\sigma$ is isotopic to $\tau$ and $\sigma$ is isotopic to $\tau^{\prime}$. Two arcs $\tau$ and $\tau^{\prime}$ are called non-crossing if $e\left(\tau, \tau^{\prime}\right)=0$. A triangulation $\mathbb{T}$ is a maximal collection of non-crossing arcs. The arcs of a triangulation $\mathbb{T}$ cut the surface into triangles. A triangle $\triangle$ in $\mathbb{T}$ is called an internal triangle if none of its sides is a boundary segment. We refer to the triple $(S, M, \mathbb{T})$ as a triangulated surface.

If $\mathbb{T}=\left\{\tau_{1}, \cdots \tau_{m}\right\}$ is a triangulation of an unpunctured surface ( $S, M$ ), we define a quiver $Q_{\mathbb{T}}$ as follows: $Q_{\mathbb{T}}$ has $m$ vertices, one for each arc in $\mathbb{T}$. We will denote the vertex corresponding to $\tau_{i}$ by $e_{i}$ (or $i$ if there is no ambiguity). The number of arrows from $i$ to $j$ is the number of triangles $\triangle$ in $\mathbb{T}$ such that the arcs $\tau_{i}, \tau_{j}$ form two sides of $\triangle$, with $\tau_{j}$ following $\tau_{i}$ when going around the triangle $\triangle$ in the counter-clockwise orientation. Note that the interior triangles in $\mathbb{T}$ correspond to oriented 3-cycles in $Q_{\mathbb{T}}$.

Following [1, 7, in the unpunctured case, the algebra $A_{\mathbb{T}}$ is the quotient of the path algebra of the quiver $Q_{\mathbb{T}}$ by the two-sided ideal generated by the subpaths of length two of each oriented 3 -cycle in $Q_{\mathbb{T}}$, then $A_{\mathbb{T}}$ is a quadratic monomial algebra. It is easy to see that $A_{\mathbb{T}}$ is also a gentle algebra.

Since $A_{\mathbb{T}}$ is a quadratic monomial algebra, for any triangulated surface $(S, M, \mathbb{T})$, we use the computations of Sköldberg in [10, Corollary 1] to compute the Hochschild homology of $A_{\mathbb{T}}$. Then, as we mention before, we use [9, Theorem 4.1.13 and Section 2.4.8] to compute the dimension of the cyclic homology and cohomology. Before give the result of Sköldberg we need to introduce some definitions and notations.

Let $A$ be a quadratic monomial algebra, observe that the algebra $A=\amalg_{n \in \mathbb{N}} A_{n}\left(A^{!}=\right.$ $\left.\amalg_{n \in \mathbb{N}} A_{n}^{!}\right)$is $\mathbb{N}$-graded, where $A_{n}\left(A_{n}^{!}\right)$is the $\mathbf{k}$ vector space generated by all paths of length $n$ of $A$ (and $A$ ! respectively). This $\mathbb{N}$-graded is called the internal grading. For a $\mathbb{N}$-graded vector space $V$, we define $V_{\geq i}$ by $\amalg_{i \geq n} V_{i}$ for a natural $n$.

We give both $A$ and its dual $A^{!}$an other $\mathbb{N}$-graded, called homological grading, by assigning to a basis element $a_{1} \cdots a_{l}$ of $A$ homological degree 0 , and to a basis element $b_{1} \cdots b_{n}$ of $A^{!}$ homological degree $n$.

For two elements $x, y$ homogeneous with respect to the homological grading, in a bigraded algebra, we define their graded commutator $[x, y]=x y-(-1)^{\operatorname{homdeg}(x) \operatorname{homdeg}(y)} y x$, where $\operatorname{homdeg}(y)$ is the homological degree, and we extend this definition bi-linearly to any pair of elements. If $A$ is a bi-graded algebra we define the vector space of commutators $[A, A]$ as

$$
[A, A]=\operatorname{span}_{\mathbf{k}}\{[x, y] \mid x, y \in A\}
$$

Remark 1. Denote by $\mathcal{C}_{m}$ the set of cycles $Q$ of length $m$. Observe that the cyclic group $C_{n}=\langle g\rangle$ acts on $\mathcal{C}_{n}$ by the action $g a_{1} \cdots a_{n}=a_{n} a_{1} \cdots a_{n-1}$. We say that two cycles $w$ and $w^{\prime}$ are cyclically equivalent if $w$ and $w^{\prime}$ are in the same orbit. Moreover, we say that any two cycles $w, w^{\prime}$ are $[A, A]$-equivalent if $w$ and $w^{\prime}$ are not elements of $[A, A]$ and they are cyclically equivalent, we denote this situation in the follow way $w \equiv w^{\prime}(\bmod [A, A])$.

According to Sköldberg [10, Corollary 1] the Hochschild homology of a quadratic monomial algebra is computed as follows.

Theorem 2. The Hochschild homology of any quadratic monomial algebra $A=\mathbf{k} Q / I$ is given by

$$
H H_{n}(A) \simeq \begin{cases}\mathbf{k} Q_{0} \oplus(A /[A, A])_{\geq 1} & \text { if } n=0 \\ \left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{2} \oplus(A /[A, A])_{\geq 1} & \text { if } n=1 \\ \left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{n} \oplus\left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{n+1} & \text { if } n \geq 2\end{cases}
$$

The computation of the cyclic homology of a quadratic monomial algebra over a field, not necessarily of characteristic zero, was given by Sköldberg in [11]. However, as consequence of Theorem 2 and the short exact sequence

$$
0 \longrightarrow \widetilde{\mathrm{HC}}_{n-1} \longrightarrow \widetilde{\mathrm{HH}}_{n} \longrightarrow \widetilde{\mathrm{HC}}_{n} \longrightarrow 0,
$$

where $\widetilde{\mathrm{HH}}_{n}(A)$ is the quotient $\mathrm{HH}_{n}(A) / \mathrm{HH}_{n}\left(A_{0}\right)$ and $\widetilde{\mathrm{HC}}_{n}(A)$ is the quotient $\mathrm{HC}_{n}(A) / \mathrm{HC}_{n}\left(A_{0}\right)$, see [9, Theorem 4.1.13], it is possible to compute the cyclic homology of a quadratic monomial algebra using an inductive argument, as follows.

Theorem 3. [11, Theorem 4] Let $A$ be a quadratic monomial algebra, where $\mathbf{k}$ is a field of characteristic 0 , the cyclic homology of $A$ is given by

$$
H C_{n}(A) \simeq\left\{\begin{array}{ll}
\mathbf{k} Q_{0} \oplus(A /[A, A])_{\geq 1} & \text { if } n=0 \\
\mathbf{k} Q_{0} \oplus\left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{n+1} & \text { if } n \equiv 0 \\
\left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{n+1} & \text { otherwise }
\end{array}(\bmod 2) \text { and } n \neq 0\right.
$$

As shown in Theorem 2 and Theorem 3, to compute the cyclic and Hochschild homology of any algebra $A$, we need to compute the groups $A /[A, A]$ and $A^{!} /\left[A^{!}, A^{!}\right]$. In the following lemmas we show that those groups, for algebras coming from triangulated surfaces, are related to the internal triangles of the triangulation. In order to do that, we need to introduce some notation. Denote by $\operatorname{Int}(\mathbb{T})=\left\{\Delta_{1}, \cdots, \Delta_{t}\right\}$ the set of internal triangles of the triangulated surface $(S, M, \mathbb{T})$ and by $Q\left(\Delta_{i}\right)$ the subquiver of $Q_{\mathbb{T}}$ associated to the internal triangle $\Delta_{i}$ for each $i=1, \ldots, t$. By construction of $Q_{\mathbb{T}}$, the quiver $Q\left(\Delta_{i}\right)$ is a 3-cycle.

Remark 4. By definition $[A, A]$ is generated by the elements $[x, y]$ such that $x, y \in A$. In particular, if $w=\left[e_{i}\left|a_{1} a_{2} \ldots a_{l}\right| e_{j}\right]$ is a path such that $e_{i} \neq e_{j}$, we have that $w=\left[w, e_{j}\right]$. Then any path which is not a cycle is an element of $[A, A]$.

Lemma 5. Let $(S, M, \mathbb{T})$ be a triangulated surface and $A_{\mathbb{T}}$ be the algebra associated to $(S, M, \mathbb{T})$. Then the quotient $\left(A_{\mathbb{T}} /\left[A_{\mathbb{T}}, A_{\mathbb{T}}\right]\right)_{\geq 1}$ in trivial.

Proof. Since any path which is not a cycle is an element of $\left[A_{\mathbb{T}}, A_{\mathbb{T}}\right]$, it is enough to show that any cycle, non-zero in $A_{\mathbb{T}}$ of positive length, is an element of the commutator $\left[A_{\mathbb{T}}, A_{\mathbb{T}}\right]$.

Suppose $w=\left[e_{i}\left|a_{1} a_{2} \ldots a_{l}\right| e_{i}\right]$ is a cycle. Since $Q_{\mathbb{T}}$ has no loops, we have that $|w| \geq 2$. Moreover, the algebra $A_{\mathbb{T}}$ is a finite dimensional quadratic monomial algebra, then $a_{l} a_{i}$ is an element of the ideal $I_{\mathbb{T}}$, therefore $w=\left[a_{1} \cdots a_{l-1}, a_{l}\right]$, as we claim.
Lemma 6. Let $(S, M, \mathbb{T})$ be a triangulated surface and $A_{\mathbb{T}}$ be the algebra associated to $(S, M, \mathbb{T})$ and $n \neq 3(2 t+1)$ for any $t \in \mathbb{Z}^{\geq 0}$. Then the quotient $\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n}$ is trivial. Moreover, if $n=3(2 t+1)$ for some $t \in \mathbb{Z}^{\geq 0}$, then $\operatorname{dim}_{\mathbf{k}}\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n}=|\operatorname{int}(\mathbb{T})|$
Proof. By definition $A^{!}$is the path algebra $k Q_{\mathbb{T}} / J$, where $J$ is the ideal generated by the paths $m$ of length 2 such that $m \notin I$. Before give a basis for $A^{!} /\left[A^{!}, A^{!}\right]$, we first give the generators of $J$ and the non-zero paths in $A^{!}$.

Recall any path $w=\left[e_{i}\left|a_{1} a_{2} \ldots a_{l}\right| e_{j}\right]$, non-zero in $A$, is coming from arcs attached to a marked point $x$ as in Figure 1, and each arrow is opposite to $x$. Denote by $w_{x}$ the maximal non-zero path coming from the arcs attached to the marked point $x$. Then $J$ is generated by the subpaths of length 2 of each maximals non-zero paths, therefore element of the basis of $A^{!}$is a sequence of consecutive arrows of a 3 -cycle $Q(\Delta)$ associated to an internal triangle $\Delta$ of $\mathbb{T}$ or an arrow in $Q_{\mathbb{T}}$.


## Figure 1. Non-zero path

Since the length of any cycle in $A^{!}$is always a multiple of 3 and any path which is not a cycle is element of $\left[A^{!}, A^{!}\right]$, it is clear that $\left[A^{!}, A^{!}\right] \cap A_{n}^{!}=A_{n}^{!}$for any $n$ which is not multiple of 3 .

Now, suppose $n$ is multiple of 3 and even. We claim that $\left[A^{!}, A^{!}\right] \cap A_{n}^{!}=A_{n}^{!}$. Let $w$ be path of length $n$. If $w$ is not a cycle, then $w$ is an element of $\left[A^{!}, A^{!}\right] \cap A_{n}$, see Remark 4. Suppose $w$ is a cycle, then $w=\left(a_{1} a_{2} a_{3}\right)^{t}$ for some $t \in \mathbb{N}$ such that $n=3 t$. Therefore $\left[a_{1} a_{2} a_{3},\left(a_{1} a_{2} a_{3}\right)^{t-1}\right]=\left(a_{1} a_{2} a_{3}\right)^{t}-(-1)^{(3)(3 t-3)}\left(a_{1} a_{2} a_{3}\right)^{t}$, observe that $(-1)^{(3)(3 t-3)}=-1$, then $\left[a_{1} a_{2} a_{3},\left(a_{1} a_{2} a_{3}\right)^{t-1}\right]=2\left(a_{1} a_{2} a_{3}\right)^{t}$, hence $\left[A^{!}, A^{!}\right] \cap A_{n}^{!}=A_{n}^{!}$.

Then, in both cases, when $n$ is not multiple of 3 or $n$ is multiple of 3 and even, we have that $\left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{n}=0$.

Finally, suppose $n$ is multiple of 3 and odd. By definition

$$
\left[w_{1}, w_{2}\right]=w_{1} w_{2}-(-1)^{\left|w_{1}\right|\left|w_{2}\right|} w_{2} w_{1}
$$

is a generator of $\left[A^{!}, A^{!}\right] \cap A_{n}^{!}$, if $w_{1}$ and $w_{2}$ are paths such that the length $\left|w_{1} w_{2}\right|=n$ or $\left|w_{2} w_{1}\right|=n$. Moreover, since $n$ is odd and $n=\left|w_{1}\right|+\left|w_{2}\right|$, we have that $(-1)^{\left|w_{1}\right|\left|w_{2}\right|}=1$, then

$$
\left[w_{1}, w_{2}\right]=w_{1} w_{2}-w_{2} w_{1} .
$$

Let $g$ be a generator of the cyclic group $C_{n}$. We claim that any generator of $\left[A^{!}, A^{!}\right] \cap A_{n}^{!}$ is an element of the form:
(i) $\left(c_{1} c_{2} c-3\right)^{t}-g^{r}\left(c_{1} c_{2} c_{3}\right)^{t}$ where $c_{1} c_{2} c_{3}$ is a 3 -cycle coming from an internal triangle $\Delta_{k}$ of $\mathbb{T}$ and $r \in\{1,2, \ldots, n-1\}$ or
(ii) a path $w^{\prime}$ which is not a cycle, of length $n$.

Let $w_{1}=a_{1} a_{2} \ldots a_{l_{1}}$ and $w_{2}=b_{1} b_{2} \ldots b_{l_{2}}$ paths such that the length $\left|w_{1} w_{2}\right|=n$ or $\left|w_{2} w_{1}\right|=n$. Suppose $\left[w_{1}, w_{2}\right]$ is not a path, then $w_{1} w_{2}$ and $w_{2} w_{1}$ are non-zero cycles in $A^{!}$, and therefore both of them are sequences of consecutive arrows of the same 3 -cycle $Q(\Delta)$ associated to an internal triangle $\Delta$ of $\mathbb{T}$, hence $w_{2} w_{1}=g^{r} w_{1} w_{2}$ for some $r \in\{1,2 \ldots, n-1\}$ and $w_{1} w_{2}=\left(c_{1} c_{2} c_{3}\right)^{t}$, for some $c_{1} c_{2} c_{3}$ is a 3 -cycle coming from an internal triangle $\Delta_{k}$ of $\mathbb{T}$, as we claim.

And, as consequence, $\operatorname{dim}_{\mathbf{k}}\left(\left(A^{!} /\left[A^{!}, A^{!}\right]\right)_{n}\right)=|\operatorname{int}(\mathbb{T})|$.

Proof of Main Theorem. Let $A_{\mathbb{T}}$ be the algebra associated to the triangulated surface ( $S, M, \mathbb{T}$ ).
We first compute the cyclic homology groups $\mathrm{HC}_{n}\left(A_{\mathbb{T}}\right)$. Since $\left(A_{\mathbb{T}} /\left[A_{\mathbb{T}}, A_{\mathbb{T}}\right]\right) \geq 1$ is trivial by Lemma 5, we have that $\operatorname{HC}_{0}\left(A_{\mathbb{T}}\right)=\mathbf{k} Q_{0}$. Let $n \geq 1$, by Lemma 6 we have that the quotient $\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n+1}$ is also trivial for any $n+1 \neq 3(2 k+1)$, then by Theorem 3 we have that the $n$-cyclic homology group of $A_{\mathbb{T}}$ is given by

$$
\operatorname{HC}_{n}\left(A_{\mathbb{T}}\right) \simeq\left\{\begin{array}{ll}
\mathbf{k} Q_{0} & \text { if } n=0 \\
\mathbf{k} Q_{0} \oplus\left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n+1} & \text { if } n \equiv 2 \\
0 & \text { otherwise }
\end{array}(\bmod 6)\right.
$$

Similarly by Theorem 2 and also Lemmas 5and 6, we have that the Hochschild homology groups of $A_{\mathbb{T}}$ is given by

$$
\operatorname{HH}_{n}\left(A_{\mathbb{T}}\right) \simeq \begin{cases}\mathbf{k}\left(Q_{\mathbb{T}}\right)_{0} & \text { if } n=0 \\ \left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n} & \text { if } n \equiv 3 \quad(\bmod 6) \\ \left(A_{\mathbb{T}}^{!} /\left[A_{\mathbb{T}}^{!}, A_{\mathbb{T}}^{!}\right]\right)_{n+1} & \text { if } n \equiv 2 \quad(\bmod 6) \\ 0 & \text { otherwise }\end{cases}
$$

It follows from Loday [9, Section 2.4.8], that for any finite dimensional algebra $A$ with unit the $n$-cyclic homology group $\mathrm{HC}^{n}(A)$ of $A$ and the $n$-cyclic cohomology group $\operatorname{Hom}_{k}\left(\mathrm{HC}_{n}(A), k\right)$ of $A$ are isomorphic. Since $A_{\mathbb{T}}$ is a finite dimensional algebra for any triangulated surface $(S, M, \mathbb{T})$, the Corollary follows from the main Theorem.

To conclude this note, we compute the dimension of the cyclic (co)homologies groups and dimension of the Hochschild homology groups of two algebras from surfaces, which are closed related.

Example 7. Consider the triangulated surface $(S, M, \mathbb{T})$ of the Figure 2, where $S$ is a surface of genus zero with 3 boundaries components and four marked points.

Observe that the quiver $Q_{\mathbb{T}}$ has 7 vertices and there are 3 internal triangles: $\triangle_{1}\left(\tau_{1}, \tau_{5}, \tau_{7}\right)$, $\triangle_{2}\left(\tau_{1}, \tau_{6}, \tau_{2}\right)$ and $\triangle_{3}\left(\tau_{6}, \tau_{5}, \tau_{4}\right)$. Then, according to our main Result, the dimension of the Hochschild homology groups $\mathrm{HH}_{n}\left(A_{\mathbb{T}}\right)$ are computed as follows:


Figure 2. The triangulated surface $(S, M, \mathbb{T})$

$$
\operatorname{dim}_{\mathbf{k}} \operatorname{HH}_{n}\left(A_{\mathbb{T}}\right)=\left\{\begin{array}{ll}
7 & \text { if } n=0 \\
3 & \text { if } n \equiv 3 \\
0 & \text { otherwise }
\end{array}(\bmod 6) \text { or } n \equiv 2 \quad(\bmod 6)\right.
$$

Since $\operatorname{dim}_{\mathbf{k}}\left(\mathrm{HC}^{n}\left(A_{\mathbb{T}}\right)\right)=\operatorname{dim}_{\mathbf{k}}\left(\mathrm{HC}_{n}\left(A_{\mathbb{T}}\right)\right)$ by Corollary for any $n \in \mathbb{Z}^{\geq 0}$, the dimension of the cyclic (co)homology groups $\mathrm{HC}_{n}\left(A_{\mathbb{T}}\right)$ and $\mathrm{HC}^{n}\left(A_{\mathbb{T}}\right)$ are computed as follows:

$$
\operatorname{dim}_{\mathbf{k}}\left(\operatorname{HC}^{n}\left(A_{\mathbb{T}}\right)\right)=\operatorname{dim}_{\mathbf{k}}\left(\operatorname{HC}_{n}\left(A_{\mathbb{T}}\right)\right)=\left\{\begin{array}{ll}
7 & \text { if } n=0 \\
10 & \text { if } n \equiv 2 \\
0 & \text { otherwise }
\end{array}(\bmod 6)\right.
$$

Finally, consider the triangulated surface $\left(S, M, \mathbb{T}^{\prime}\right)$ of Figure 3, which is obtained by removing the arc $\tau_{7}$ of the triangulation $\mathbb{T}$ and replacing it by $\tau_{7}^{\prime}$, that is, a flip of arc $\tau_{7}$.


Figure 3. The triangulated surface $\left(S, M, \mathbb{T}^{\prime}\right)$

In this case, the quiver $Q_{\mathbb{T}}^{\prime}$ has also 7 vertices, which is actually an invariant of $(S, M)$, but there are only 2 internal triangles: $\Delta_{2}\left(\tau_{1}, \tau_{6}, \tau_{2}\right)$ and $\Delta_{3}\left(\tau_{6}, \tau_{5}, \tau_{4}\right)$, then:

$$
\operatorname{dim}_{\mathbf{k}} \operatorname{HH}_{n}\left(A_{\mathbb{T}}^{\prime}\right)= \begin{cases}2 & \text { if } n \equiv 3(\bmod 6) \text { or } n \equiv 2(\bmod 6) \\ \operatorname{dim}_{\mathbf{k}} \operatorname{HH}_{n}\left(A_{\mathbb{T}}\right) & \text { otherwise }\end{cases}
$$

and

$$
\operatorname{dim}_{\mathbf{k}}\left(\operatorname{HC}^{n}\left(A_{\mathbb{T}}^{\prime}\right)\right)=\operatorname{dim}_{\mathbf{k}}\left(\operatorname{HC}_{n}\left(A_{\mathbb{T}}^{\prime}\right)\right)= \begin{cases}9 & \text { if } n \equiv 2(\bmod 6) \\ \operatorname{dim}_{\mathbf{k}}\left(\operatorname{HC}^{n}\left(A_{\mathbb{T}}\right)\right) & \text { otherwise }\end{cases}
$$

Therefore, the (co)homologies computed in this note are not invariant under flips and mutations of quivers with potentials.

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E-mail address: yadira@matem.unam.mx
Instituto de Matemáticas, UNAM Área de la Investigación Científica, Circuito exterior, Ciudad Universitaria, CDMX, 04510, México.


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