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Applications of Quaternionic Holomorphic Geometry to minimal surfaces

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Abstract: In this paper we give a survey of methods of Quaternionic Holomorphic Geometry and of applications of the theory to minimal surfaces. We discuss recent developments in minimal surface theory using integrable systems. In particular, we give the Lopez–Ros deformation and the simple factor dressing in terms of the Gauss map and the Hopf differential of the minimal surface. We illustrate the results for well–known examples of minimal surfaces, namely the Riemann minimal surfaces and the Costa surface.

1 Introduction

The study of immersions $f : M \to \mathbb{R}^3$ from a 2-dimensional Riemannian manifold into 3-space is a classical topic. Local results were obtained around the end of the 19th century by constructing local solutions of the Gauss–Codazzi equations for special surface classes, e.g. [1, 5, 6, 15], usually given by conditions on the curvature or by variational properties arising from physical properties. For example, minimal surfaces are the critical points of the area functional with fixed boundary, and thus appear physically as soap films. Minimal surfaces also occur in other settings in nature, for example in string theory where the apparent horizon is a minimal hypersurface (thus linking the theory of black holes to minimal surfaces and the Plateau problem [17]), or in biological systems where triply periodic minimal surfaces give the structure of photonic crystals and their optical properties, [30]. Other examples of classically studied surfaces are isothermic surfaces, that is, surfaces which allow a conformal curvature line parametrisation: given a thermally isolated surface of constant heat conduction, the constant coordinate lines are isotherms if and only if the coordinates are isothermal. Examples of isothermic surfaces include surfaces of revolution. Finally, the so–called Willmore surfaces give another important surface class relevant for this paper: these surfaces are given as the critical points of the bending energy.

One classical method of constructing new examples of surfaces of a given surface class is to use a suitable (geometric) transformation which simplifies the underlying PDEs and allows to construct new surfaces from a given simple one. A classical example is the Darboux transformation [15] for isothermic surfaces: if two conformal immersions are enveloped by a conformal Ribaucour sphere congruence then both surfaces are isothermic and are called Darboux transforms of each other. In particular, solutions of a Riccati equation allow to construct a Darboux transform of a given isothermic surface.

Research in surface theory gathered new impetus in the second half of 20th century: tools from geometric analysis and integrable systems allow to study global properties. In this paper, we are interested in the methods from integrable systems: Starting with the work of Uhlenbeck [39] integrable systems methods have been highly successful in the geometric study of harmonic maps from Riemann surfaces into suitable spaces, e.g., [21], [40], [3], [9], [16], [38]. In particular, the theory can be used to describe the moduli spaces of surface

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classes which are given in terms of a harmonicity condition, such as CMC surfaces and Willmore surfaces, e.g., [4, 8, 12, 20, 33, 36].

On the other hand, when comparing to the theory of algebraic curves, surface theory is still lacking in terms of tools, examples and classification results: the study of holomorphic maps $f : M \to \mathbb{C}^n$ where M is a Riemann surface is a far richer and more developed theory. The underlying Cauchy–Riemann equations can be solved globally, and algebraic curves can be studied via their holomorphic line bundles. Powerful tools include the classical Kodaira embedding, the Riemann–Roch Theorem and the Plücker theorem.

The aim of Quaternionic Holomorphic Geometry is to combine these two theories by replacing complex numbers with quaternions. The main ingredient is the observation that conformal maps $f : M \to S^4$ from a Riemann surface into the 4-sphere can serve as an analogue of a complex holomorphic map: the 4-sphere can be identified with the quaternionic projective space $\mathbb{HP}^1$ so that conformal maps $f : M \to \mathbb{HP}^1$ can be studied in a similar way to complex holomorphic maps $f : M \to \mathbb{CP}^1$. In this paper we survey some of the results in this area [2, 19], e.g. the quaternionic versions of the Kodaira–, Plücker– and Riemann–Roch theorem. We also will discuss the link to integrable systems: the spectral curve of a conformal torus [7] is given in terms of a generalisation of the classical Darboux transform.

To link these results to minimal surfaces in 3-space, we first give a short but comprehensive description of geometric data of a conformal immersion into 3-space (including the mean curvature, the Gaussian curvature and the Hopf differential) in the language of Quaternionic Holomorphic Geometry and explore the consequences for minimal surfaces. In particular, we show that the conformal Gauss map of a minimal surface can be identified with the Gauss map and the support function of the minimal surface.

Since minimal surfaces in Euclidean 3-space give rise to various harmonic maps, that is, the immersion itself, its Gauss map and its conformal Gauss map, the study of minimal surfaces fits well within the theory of integrable systems — even though this link has not been studied in detail yet. However, in the recent study of properly embedded minimal planar domains [27, 28] algebro-geometric properties of the hierarchy of the Korteweg–de Vries equation have been used in an essential way whereas the same Lamé potentials appear in the study of the spectral curve of an Euclidean minimal torus with two planar ends and translational periods [11]. A further recent link is the observation that the well-known Lopez–Ros deformation of minimal surfaces is indeed a special case of the simple factor dressing of harmonic maps, [24].

In this paper, we give the Lopez–Ros deformation and the simple factor dressing in terms of the Gauss map and the Hopf differential of the minimal surface. We illustrate the results for the Riemann minimal surfaces and the Costa surface.

2 Quaternionic Holomorphic Geometry

In this section we will give a short introduction of Quaternionic Holomorphic Geometry, for details and more results see [2, 19, 32].

An immersion $f : M \to \mathbb{R}^3$ of a Riemann surface M into 3-space is called conformal if $|df(X)| = |df(JX)|$ and $< df(X), df(JX) >= 0$ for all $X \in TM$ where J is the complex structure on the Riemann surface M. If N is the Gauss map of f then the conformality condition can be expressed equivalently as

$$df(JX) = N \times df(X)\,, \qquad X \in TM\,.$$

Denoting by $*$ the negative Hodge star operator, that is, $*\omega(X) = \omega(JX)$ for $X \in TM$, this can be written as

$$*df = N \times df\,. \tag{2.1}$$

We now want to use the algebra of quaternions for computations. Recall that $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$ where $i^2 = j^2 = k^2 = -1$ and $ij = k$. The imaginary quaternions are given by $\mathrm{Im}\,\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{i, j, k\}$ and can be identified with $\mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$. Similarly $\mathrm{Re}\,\mathbb{H} = \mathbb{R}$ so that $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$. With this in mind, we can write the quaternionic multiplication for $a, b \in \mathrm{Im}\,\mathbb{H}$ as

$$ab = -< a, b > + a \times b \tag{2.2}$$

where <, > is the standard inner product on $\mathbb{R}^3$. In particular, the 2-sphere in $\mathbb{R}^3$ is given by

$$S^2 = \{n \in \operatorname{Im}\mathbb{H} : n^2 = -1\}.$$

Therefore, the Gauss map $N : M \to S^2$ of an immersion f is a complex structure on the trivial line bundle $\underline{\mathbb{H}} = M \times \mathbb{H}$ over M.

Moreover, (2.1) and (2.2) show that an immersion $f : M \to \mathbb{R}^3$ is conformal if and only if

$$*df = Ndf.$$

Recall that a map $f : M \to \mathbb{C}$ is holomorphic if and only if df is complex linear, that is, if $*df = idf$. Thus, a conformal immersion $f : M \to \mathbb{R}^3$ is an analogue of a complex holomorphic map. In particular, if the Gauss map of a conformal immersion f is constant, say $N(p) = i$ for all $p \in M$, then f takes values in the jk-plane. We can identify the jk-plane with the complex bundle $\mathbb{C}k$ where the complex structure is given by left multiplication by the quaternion i and $\mathbb{C} = \operatorname{span}_{\mathbb{R}}\{1, i\}$. Thus, f is complex holomorphic since $*df = Ndf = idf$.

More generally, we will consider conformal immersions $f : M \to \mathbb{R}^4$ into 4-space, that is, immersions with

$$*df = Ndf = -dfR,$$

where the maps $N, R : M \to S^2$ give the Gauss map of the immersion f after identifying the Grassmannian of 2-planes in $\mathbb{R}^4$ with $\operatorname{Gr}_2(\mathbb{R}^4) = S^2 \times S^2$. The tangent space and normal space of f are uniquely determined by N and R via, [2],

$$\begin{aligned} df_p(T_pM) &= \{v \in \mathbb{H} \mid Nv + vR = 0\} \\ \perp_f &= \{v \in \mathbb{H} \mid Nv - vR = 0\}. \end{aligned} \tag{2.3}$$

Note that in general the *left* and *right normals* N and R are not perpendicular to the tangent space of f, and that $N = R$ if and only if $f : M \to \operatorname{Im}\mathbb{H}$ (up to translation in $\mathbb{H}$). To construct a conformal theory, one compactifies $S^4 = \mathbb{R}^4 \cup \{\infty\} = \mathbb{H} \cup \{\infty\} = \mathbb{HP}^1$. In this case, since a point in $\mathbb{HP}^1$ is a line in $\mathbb{H}^2$, an immersion $f : M \to \mathbb{HP}^2$ can be identified with a quaternionic line bundle L of the trivial $\mathbb{H}^2$ bundle $\underline{\mathbb{H}}^2 = M \times \mathbb{H}^2$ over M via

$$L_p = f(p), \quad p \in M.$$

(We will choose all quaternionic bundles to have quaternions act from the right).

The derivative of f is given by

$$\delta = \pi d|_L : L \to \underline{\mathbb{H}}^2/L.$$

where $\pi : \underline{\mathbb{H}}^2 \to \underline{\mathbb{H}}^2/L$ is the canonical projection. The immersion f is conformal if there exists a complex structure J on $\underline{\mathbb{H}}^2/L$, that is, a quaternionic endomorphism $J \in \operatorname{End}(\underline{\mathbb{H}}^2/L)$ with $J^2 = -1$, such that

$$*\delta = J\delta.$$

To simplify notations complex structures will be denoted by the same symbol J, e.g. on TM and $\underline{\mathbb{H}}^2/L$, unless it is not clear from the context where the complex structure is defined. If $f : M \to \mathbb{H}$ is an immersion then we consider f as map into $\mathbb{HP}^1$ via its line bundle

$$L = \psi\mathbb{H}, \qquad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

In this case, the point at infinity is given by

$$\infty = e\mathbb{H}, \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so that $\underline{\mathbb{H}}^2 = L \oplus \infty$. Note that $\pi_L : e\mathbb{H} \to \underline{\mathbb{H}}^2/L$ is an isomorphism and in the following, we will identify in this case $\underline{\mathbb{H}}^2/L = e\mathbb{H} = \infty$. In particular, $J\pi e = Ne$ defines a complex structure on $\underline{\mathbb{H}}^2/L$ and

$$\delta\psi = \pi d\psi = \pi \begin{pmatrix} df \\ 0 \end{pmatrix} = edf$$

shows that f is indeed conformal in $\mathbb{HP}^1$ exactly when $*df = Ndf$.

To motivate further our idea that conformal immersions play the role of quaternionic holomorphic functions, we introduce a holomorphic structure on quaternionic bundles V over M. Given a complex structure J on V we can decompose every 1-form $\omega = \omega' + \omega'' \in \Omega^1(V)$ into K and $\bar{K}$-parts by

$$\omega' = \frac{1}{2}(\omega - J * \omega) \in \Gamma(KV), \quad \text{and} \quad \omega'' = \frac{1}{2}(\omega - J * \omega) \in \Gamma(\bar{K}V),$$

where

$$\Gamma(KV) = \{\omega \in \Omega^1(V) : *\omega = J\omega\}, \quad \Gamma(\bar{K}V) = \{\omega \in \Omega^1(V) : *\omega = -J\omega\}.$$

A *(quaternionic) holomorphic structure* on a complex quaternionic bundle (V, J) is a first order quaternionic linear operator $D : \Gamma(V) \to \Gamma(\bar{K}V)$ satisfying

$$D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)''$$

where $\psi \in \Gamma(V)$ and $\lambda : M \to \mathbb{H}$.

If we denote by E the $+i$ eigenspace of J then the quaternionic bundle $V = E \oplus E$ is a double of the complex bundle E. The holomorphic structure D in general does not commute with J: decomposing $D = \bar{\partial} + Q$ into J–commuting and anti-commuting parts where

$$\bar{\partial} = \frac{1}{2}(D - JDJ), \qquad Q = \frac{1}{2}(D + JDJ)$$

the J–commuting part $\bar{\partial}$ induces a complex holomorphic structure on E. In fact, $\bar{\partial}$ is the double of a complex $\bar{\partial}$-operator on E. The *Willmore energy* of the quaternionic holomorphic structure D is the L^2 norm of the *Hopf field* Q,

$$W(D) = 2\int < Q \wedge *Q >,$$

where $< Q >:=$ Re tr(Q). The Willmore energy measures thus the difference from our theory to the complex theory.

A section $\psi \in \Gamma(V)$ is called *(quaternionic) holomorphic* if $D\psi = 0$. The set of holomorphic sections will be denoted by $H^0(V, D)$ or $H^0(V)$ for short if the holomorphic structure is known in the setting. For a 2-dimensional linear subspace $H \subset H^0(L, D)$ of holomorphic sections of a line bundle L with complex structure J and holomorphic structure D, any choice of basis ψ, φ of H will give a quotient $f : M \to \mathbb{H}$, $\psi = \varphi f$, provided that φ never vanishes. Defining $N : M \to S^2$ by $J\varphi = \varphi N$, we see by the Leibniz rule for D that

$$*df = Ndf.$$

A different choice of basis results in a Moebius transformation of f. Moreover, the Willmore energy of the quaternionic holomorphic bundle coincides with the Willmore energy of the (possibly branched) immersion f.

This correspondence between linear subspaces of holomorphic sections and (branched) conformal immersions is a special case of the *Kodaira embedding* for quaternionic holomorphic bundles [19].

If (V, J, D) is a complex quaternionic holomorphic bundle, then there is [32] a unique quaternionic holomorphic structure on KV^{-1} (where V^{-1} is the dual bundle of V) such that

$$d < \omega, \psi > = < D\omega, \psi > - < \omega \wedge D\psi >$$

for $\omega \in \Gamma(KV^{-1})$ and $\psi \in \Gamma(V)$ where $<,>$ denotes the standard pairing $KV^{-1} \times V \to T^*M \otimes \mathbb{H}$. The degree of a complex quaternionic bundle V is the degree of the underlying complex bundle. With this at hand, the Riemann–Roch theorem of the complex holomorphic setting translates verbatim to the quaternionic setting by using the invariance of the index of an elliptic operator under continuous deformations:

Theorem 2.1 (Riemann–Roch Theorem, [19]). *Let V be a quaternionic holomorphic vector bundle with holomorphic structure D over a compact Riemann surface M of genus g. Then*

$$\dim H^0(V) - \dim H^0(KV^{-1}) = \deg V - (g-1)\,\mathrm{rankV}\,.$$

This theorem gives a first estimate on the number of holomorphic sections, and thus the number of conformal immersions with a given left normal, on a compact surface.

Complex holomorphic bundles of negative degree do not admit complex holomorphic section whereas quaternionic holomorphic bundles generally do (in which case, the Hopf field has to be non-trivial). This suggests that the quaternionic version of the Plücker formula will have to involve the Willmore energy; indeed, the Plücker estimate shows that the Willmore energy of a quaternionic holomorphic line bundle has to grow quadratically in the number of holomorphic section if the line bundle has more holomorphic sections than allowed in the complex holomorphic theory:

Theorem 2.2 (Plücker formula, [19]). *Let $H \subset H^0(V)$ be an $n+1$-dimensional linear subspace of holomorphic sections of a quaternionic holomorphic line bundle (L, D) of degree d. Then the Willmore energy satisfies*

$$\frac{1}{4\pi}(W - W^\star) = (n+1)(n(1-g) - d) + \operatorname{ord} H$$

where W and $W^\star$ denote the Willmore energies of L and its dual curve $L^\star$ respectively.

Here, roughly speaking, $\operatorname{ord} H$ counts the singularities of all the higher osculating curves of the Kodaira embedding of V. The dual curve is the highest osculating curve; in case when L is a line subbundle of $\underline{\mathbb{H}}^2$, the dual curve is given by $L^\star = L^\perp = \{\alpha \in \mathbb{H}^\star : \alpha|_L = 0\}$.

The Plücker formula has ample applications, for example, it can be used to give a quantitive lower bound for the eigenvalues of the Dirac operator on surfaces or an area estimate for constant mean curvature tori $f : T^2 \to \mathbb{R}^2$ in terms of the genus of its spectral curve, [19].

To discuss the spectral curve of a conformal torus, we will consider the complex quaternionic bundle $V = \underline{\mathbb{H}}^2/L$ given by a conformal immersion $f : M \to \mathbb{HP}^1$ with complex structure J. We can define a holomorphic structure on V by setting

$$D\varphi = (\pi d\hat{\varphi})'' \tag{2.4}$$

where $\hat{\varphi} \in \Gamma(\underline{\mathbb{H}}^2)$ is an arbitrary lift of $\varphi \in \Gamma(V)$, i.e., $\pi\hat{\varphi} = \varphi$. Any other lift $\hat{\phi} = \hat{\varphi} + \Psi$, $\Psi \in \Gamma(L)$ satisfies since $\star\delta = J\delta$

$$(\pi d\hat{\phi})'' = (\pi d\hat{\varphi})'' + (\pi d\Psi)'' = D\hat{\varphi} + (\delta\Psi)'' = D\hat{\varphi}$$

which shows that D is well-defined. It is easy to verify that D is a holomorphic structure on $\underline{\mathbb{H}}^2/L$. If $f : M \to \mathbb{H}$ then $ef \in H^0(V)$ when identifying $e\mathbb{H} = V$; indeed, f is the quotient of the two holomorphic sections e, ef, that is, we see an instance of the Kodaira theorem we discussed before.

If W is a bundle over M we denote by $\tilde{W}$ the pullback of W to the universal cover $\tilde{M}$ of M. The computation (2) then can be used to show that every holomorphic section $\varphi \in H^0(\tilde{V})$ has a unique lift $\hat{\varphi} \in \Gamma(\tilde{M} \times \mathbb{H}^2)$, the *prolongation of* φ, such that

$$\pi d\hat{\varphi} = 0\,.$$

Then $\hat{\varphi}\mathbb{H} : \tilde{M} \to S^4$ defines a branched conformal immersion on $\tilde{M}$, a so-called *Darboux transform* of f. This generalises [7] the classical definition of a Darboux transform, which only can be defined for isothermic surfaces, to general conformal surfaces. *Isothermic surfaces* are surfaces for which the principal curvature lines of the surface are conformal coordinates. Note that if there exists a Darboux pair such that $f, \hat{f} : M \to \mathbb{R}^3$ are in 3-space, then both definitions of Darboux transform coincide, and both $f, \hat{f}$ are isothermic. This explains why one has to study surfaces in S^4 even if one is mainly concerned about surfaces in 3-space. The Darboux transformations is invariant under Moebius transformations and satisfies a Bianchi permutability condition.

If $\varphi \in H^0(\tilde{V})$ is a holomorphic section with multiplier, that is, if

$$D\varphi = 0 \quad \text{and} \quad \gamma^\star\varphi = \varphi h_\gamma$$

for some representation $h : \pi_1(M) \to \mathbb{H}_\star$ of the fundamental group $\pi_1(M)$ of M, then the Darboux transform $\hat{f} : M \to S^4$ is defined on M (rather than $\tilde{M}$). Intuitively, the space of Darboux transforms $\hat{f} : T^2 \to S^4$ of a conformal torus $f : T^2 \to S^4$ (with zero normal bundle degree) can be now viewed as the spectral curve of f.

More precisely, in the case of a torus, the fundamental group gives a lattice $\Gamma \subset \mathbb{C}$ and every representation $h \in \mathrm{Hom}(\Gamma, \mathbb{H}_\star)$ can be conjugated to a complex representation. The subspace of possible complex multipliers of holomorphic sections is a 1-dimensional analytic variety, and its normalisation to a Riemann surface is the spectral curve Σ of the conformal torus. In particular, there is a smooth map

$$F : T^2 \times \Sigma \to S^4$$

such that $F(\cdot, \sigma) : T^2 \to S^4$ is a Darboux transform of f for $\sigma \in \Sigma$ and $F(p, \cdot) : \Sigma \to S^4$ lifts, via the twistor projection $\mathbb{CP}^3 \to \mathbb{HP}^1$, to a complex holomorphic curve in $\mathbb{CP}^3$. This way, the spectral curve parametrises generic Darboux transforms of f and the conformal immersion f can be reconstructed, [7], from meromorphic functions on the Riemann surface Σ.

For a complex structure $S \in \Gamma(\mathrm{End}(\underline{\mathbb{H}}^2))$, $S^2 = -1$, we decompose the trivial connection $d = d' + d''$ on $\underline{\mathbb{H}}^2$ into type where

$$d'' = \frac{1}{2}(d + S \star d) \quad \text{and} \quad d' = \frac{1}{2}(d - S \star d)\,. \tag{2.5}$$

Then d'' is a quaternionic holomorphic structure on $\underline{\mathbb{H}}^2$. Decomposing d', d'' further into

$$d'' = \bar{\partial} + Q \quad \text{and} \quad d' = \partial + A$$

where the S-commuting parts of d'' and d' are given by

$$\bar{\partial}\phi = \frac{1}{2}(d''\phi - Sd''(S\phi)) \quad \text{and} \quad \partial\phi = \frac{1}{2}(d'\phi - Sd'(S\phi))$$

respectively. The S anti-commuting parts are the *Hopf fields* of S:

$$Q\phi = \frac{1}{2}(d''\phi + Sd''(S\phi)) = \frac{1}{4}(SdS - {\star}dS)\phi \quad \text{and} \quad A\phi = \frac{1}{2}(d'\phi + Sd'(S\phi)) = \frac{1}{4}(SdS + {\star}dS)\phi\,. \tag{2.6}$$

Put differently, the Hopf fields give a decomposition

$$dS = 2({\star}Q - {\star}A) \tag{2.7}$$

of the differential of the complex structure S into type, so that $(dS)' = -2 \star A$ and $(dS)'' = 2 \star Q$.

In particular, if $L \subset \underline{\mathbb{H}}^2$ is a line subbundle and $SL = L$ then S induces a complex structure J on $\underline{\mathbb{H}}^2/L$ via $J\pi = \pi S$ and the corresponding holomorphic structure (2.4) on $\underline{\mathbb{H}}^2/L$ is given by

$$D\varphi = \frac{1}{2}(\pi d\hat{\varphi} + J\pi \star d\hat{\varphi}) = \pi(d''\hat{\varphi}) \tag{2.8}$$

where $\pi\hat{\varphi} = \varphi$.

A special choice of complex structure is the *conformal Gauss map* of a conformal immersion $f : M \to S^4$: it is the unique complex structure $S \in \Gamma(\mathrm{End}(\underline{\mathbb{H}}^2))$, $S^2 = -1$, such that $SL = L$, $dS(L) \subset \Omega^1(L)$, ${\star}\delta = S\delta = \delta S$ and $Q|_L = 0$ where L is the line bundle of f. The first three conditions show that $S : M \to S^4$ defines a sphere congruence so that $S(p)$ goes through $f(p)$ and whose tangent space at p coincides with the tangent space of f at p for all $p \in M$. The condition on the compatibility of S and δ implies that the orientations of the tangent spaces coincide. The final condition on the Hopf field shows that the mean curvature vectors of f and $S(p)$ coincide at p, [2]. Put differently, the conformal Gauss map can be seen geometrically as the mean curvature sphere congruence of f.

3 Conformal immersions in $\mathbb{R}^3$

Since we are interested in conformal surfaces in 3-space we will derive some basic geometric quantities in terms of the quaternionic model we use. See [2] for the corresponding data for conformal immersion in 4-space.

If $f : M \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ is conformal, and $z = x + iy$, $*dz = idz$, a conformal coordinate on M, then the *conformal factor* e^u of f is given by

$$e^u(dx^2 + dy^2) = |df|^2 \,.$$

We will identify 2-forms ω with quadratic forms via $\omega(X) = \omega(X, JX)$. In particular, if $\omega = \eta \wedge \theta$ then

$$\omega(X) = \eta \wedge \theta(X, JX) = \eta(x)\theta(JX) - \eta(JX)\theta(X)$$

and the quadratic form of the 2-form ω is $\eta * \theta - *\eta\theta$. Since $*df = Ndf = -dfN$, $*(*df) = -df$, and $\overline{df} = -df$ we thus see that

$$df \wedge *df = -df^2 - *df^2 = -2df^2 = 2|df|^2 \,,$$

so that the conformal factor is in the quaternionic language given by

$$2e^u(dx^2 + dy^2) = 2e^u dx \wedge dy = df \wedge *df \,.$$

The second fundamental form $II(X, Y) = (X \cdot df(Y))^\perp$ computes with (2.3) to

$$II(X, Y) = \frac{1}{2}(X \cdot df(Y) - N(X \cdot df(Y))N) \,,$$

where $N : M \to S^2$ is the Gauss map of f with $*df = Ndf = -dfN$. Differentiating $df = NdfN$ gives

$$X \cdot df(Y) = -dN(X) * df(Y) + N(X \cdot df(Y))N + *df(Y)dN(X)$$

so that

$$II(X, Y) = -\frac{1}{2}(dN(X) * df(Y) - *df(Y)dN(X)) \,. \tag{3.1}$$

The *mean curvature vector* is $\mathcal{H} = \frac{1}{2} \operatorname{tr} II = HN$ where $H : M \to \mathbb{R}$ denotes the mean curvature. Note that this choice of sign for H differs from the one in [2]. For a conformal immersion the mean curvature satisfies

$$\Delta f = 2e^u HN \,. \tag{3.2}$$

Since z is a conformal coordinate we have $*dz = idz$ and thus $d * df = d(-f_x dy + f_y dx) = -\Delta f dx \wedge dy$. Therefore,

$$dN \wedge df = d * df = -\Delta f dx \wedge dy = -2e^u HNdx \wedge dy = -HNdf \wedge *df = -Hdf \wedge df$$

and

$$(dN + Hdf) \wedge df = 0 \,.$$

Since f is an immersion and

$$\begin{aligned} (dN + Hdf) \wedge df &= dN * df + Hdf * df - *dNdf - H * dfdf \\ &= dN * df + Hdf * df + *dNN * df + HdfNdf \\ &= (dN - N * dN + 2Hdf) * df \end{aligned}$$

we see that (3.2) is equivalent to

$$-Hdf = \frac{1}{2}(dN - N * dN) = (dN)' \,. \tag{3.3}$$

The *Gaussian curvature* is given by

$$K = \det II = \frac{1}{|df(X)|^4}\left(< II(X, X), II(JX, JX) > - < II(JX, X), II(X, JX) >\right)$$

A straightforward computation [2] using (3.1), the conformality $*df = Ndf = -dfN$ of f, $< Na, b >= - < a, Nb >$ and $< adf, bdf >=< dfa, dfb >=< a, b > |df|^2$, $a, b \in \mathbb{H}$, shows that

$$K|df|^2 = < *dN, NdN > \,. \tag{3.4}$$

Using the complexification $\mathfrak{C}^3 = \mathbb{R}^3 \oplus \mathbf{i}\,\mathbb{R}^3$, where we denote the complex structure by $\mathbf{i}$ to differentiate it from the quaternion i, we consider the complex derivatives

$$g_z = \frac{1}{2}(g_x - \mathbf{i}\,g_y)\,, \quad g_{\bar z} = \frac{1}{2}(g_x + \mathbf{i}\,g_y)$$

for smooth functions g on M. With this notation, the conformal factor of f is given by

$$< f_z, f_{\bar z} >= \frac{1}{2}e^u$$

and the mean curvature, since $\Delta f = 4f_{z\bar z}$, by

$$\frac{1}{2}He^u =< f_{z\bar z}, N > \ .$$

Here we denote by $<,>$ the standard symmetric bilinear form on $\mathfrak{C}^3$ that is $< a, b >= \sum_{i=1}^3 a_i b_i$ where $a_i, b_i \in \mathfrak{C}$ denote the coordinates of $a, b \in \mathfrak{C}^3$ respectively.

The *Hopf differential* of f is the (complex) quadratic form:

$$\mathcal{Q} = -2 < f_{zz}, N > dz^2\,.$$

Since $< f_{zz}, N >= - < N_z, f_z >= \operatorname{Re}(N_z f_z) = \frac{1}{2}(N_z f_z + f_z N_z)$ we see that $\mathcal{Q}$ is twice the real part (with respect to the quaternions) of $-dN^{(1,0)}df^{(1,0)}$, that is,

$$\mathcal{Q} = -(dN^{(1,0)}df^{(1,0)} + df^{(1,0)}dN^{(1,0)})\,,$$

where $\omega^{(1,0)} = \frac{1}{2}(\omega - \mathbf{i} * \omega)$ is the $(1,0)$-part of a 1-form ω with respect to the complex structure $\mathbf{i}$. In particular, the Hopf differential is independent of the choice of conformal coordinate.

The Willmore energy of f is given by

$$W(f) = \int (H^2 - K)|df|^2$$

and the integrand relates to the Hopf differential by

$$(H^2 - K)|df|^4 = |\mathcal{Q}|^2\,.$$

Consider the complex structure

$$S = G\begin{pmatrix} N & 0 \\ H & -N \end{pmatrix} G^{-1}\,, \qquad G = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\,. \tag{3.5}$$

We will now show that S is the conformal Gauss map of f. Using (3.3),

$$dS = G\begin{pmatrix} (dN)'' & 0 \\ dH & -(dN)'' \end{pmatrix} G^{-1}$$

and the Hopf fields (2.6) compute to

$$4 * Q = S * dS + dS = G\begin{pmatrix} 2(dN)'' & 0 \\ 2dH + \omega & 0 \end{pmatrix} G^{-1} \tag{3.6}$$

and

$$4 * A = S * dS - dS = G\begin{pmatrix} 0 & 0 \\ \omega & 2(dN)'' \end{pmatrix} G^{-1} \tag{3.7}$$

where $\omega = -dH - N * dH + H * (dN)''$. In particular,

$$SL = S\begin{pmatrix} f \\ 1 \end{pmatrix}\mathbb{H} = G\begin{pmatrix} N & 0 \\ H & -N \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{H} = L\,,$$

and similarly, $dSL \subset \Omega^1(L)$. For $\psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(L)$ we have, when identifying $e\mathbb{H} = \underline{\mathbb{H}}^2/L$, $\delta\psi = \begin{pmatrix} df \\ 0 \end{pmatrix} = edf$ so that $*\delta\psi = eNdf = -edfN$. Since

$$S\delta\psi = S\pi edf = \pi Sedf = \pi(eN + H\psi)df = eNdf$$

and

$$\delta S\psi = \pi d(-\psi N) = -\pi(d\psi)N = -edfN$$

we see that

$$*\delta = S\delta = \delta S\,.$$

Finally, $Q\psi = 0$ so that $Q|_L = 0$, and S is the conformal Gauss map of f.

Using (3.3) and (3.4) we have

$$(H^2 - K)|df|^2 = |dN'|^2 - < *dN, NdN >= |dN'|^2 + < N * dN, dN >= |dN''|^2\,.$$

On the other hand,

$$\text{Re tr}\, A \wedge *A = -\frac{1}{4}\,\text{Re} * dN'' \wedge dN'' = \frac{1}{2}|dN|''$$

In particular, the Willmore energy of f is gien by

$$W(f) = \int (H^2 - K)|df|^2 = 2\int < A \wedge *A >$$

where as before $< B >= \text{Re tr}\, B$. A surface $f : M \to \mathbb{R}^3$ is a *Willmore surface*, that is, a critical point of the Willmore energy under compactly supported variations (without fixing the conformal structure on M), if and only if the conformal Gauss map is harmonic [10, 18, 35]. The conformal Gauss map S is harmonic [2] if the Hopf fields are co-closed, that is, if

$$d * A = 0$$

(which by (2.7) is equivalent to $d * Q = 0$).

4 Minimal surfaces

In this section we will collect the main properties of minimal surfaces, that is, surfaces with vanishing mean curvature, needed for this paper. To apply tools from Quaternionic Holomorphic Geometry, we will consider conformal immersions $f : M \to \mathbb{R}^3$.

Recall (3.2) that a conformal immersion $f : M \to \mathbb{R}^3$ satisfies

$$\Delta f = 2e^u HN$$

where H is the mean curvature, e^u is the conformal factor and N is the Gauss map of f. In particular, if f is conformal then f is a *minimal surface* if and only if f is harmonic. Put differently, a conformal immersion is minimal if there is a holomorphic function $\Phi : \tilde{M} \to \mathfrak{C}^3$ from the universal cover $\tilde{M}$ to complex 3-space $\mathfrak{C}^3$ such that $\text{Re}\,\Phi = f$. Note that this implies that there are no minimal surfaces on a compact Riemann surface M. Since f is conformal, Φ is a null curve that is

$$< \Phi', \Phi' >= 0$$

where $<, >$ denotes the standard symmetric bilinear form on $\mathfrak{C}^3$. Conversely, a holomorphic null curve $\Phi : \tilde{M} \to \mathfrak{C}^3$ gives rise to a conformal minimal immersion $f = \text{Re}(\Phi) : \tilde{M} \to \mathbb{R}^3$ provided $|\Phi_1'|^2 + |\Phi_2'|^2 + |\Phi_3'|^2 \neq 0$ where Φ_i are the coordinate functions of Φ. In particular, if $f = \text{Re}(\Phi) : M \to \mathbb{R}^3$ is a minimal surface with holomorphic null curve $\Phi : \tilde{M} \to \mathfrak{C}^3$ then the *associated family*

$$f_\theta = \text{Re}(e^{-\mathbf{i}\theta}\Phi) : \tilde{M} \to \mathbb{R}^3$$

is a family of minimal surfaces. The surface $f^* = f_{\frac{\pi}{2}}$ is called the *conjugate surface* of f and the associated family is given in terms of f and its conjugate f^* as

$$f_\theta = \cos\theta f + \sin\theta f^* .$$

The *Enneper-Weierstrass representation* allows to construct holomorphic null curves, and thus minimal surfaces, from the Weierstrass data (g, ω) where g is meromorphic function and ω a holomorphic 1-form by

$$\Phi = \int (\frac{1}{2}(1-g^2), \frac{\mathbf{i}}{2}(1+g^2), g)\omega .$$

Here a pole p of g of order m has to be a zero of order $2m$ of ω. The holomorphic function g is the stereographic projection of the Gauss map N of f:

$$N = \frac{1}{1+|g|^2}\begin{pmatrix} 2\,\mathrm{Re}\,(g) \\ 2\,\mathrm{Im}\,(g) \\ |g|^2 - 1 \end{pmatrix} .$$

The Weierstrass data of a minimal surface can be recovered from the holomorphic null curve

$$\omega = d\Phi_1 - \mathbf{i}\, d\Phi_2 , \qquad g = \frac{d\Phi_3}{d\Phi_1 - \mathbf{i}\, d\Phi_2} \tag{4.1}$$

where Φ_i are the coordinate functions of Φ.

Since Φ is holomorphic we have $f_x = f^*_y$ and $f_y = -f^*_x$ since $f = \mathrm{Re}\,(\Phi)$ and $f^* = \mathrm{Im}\,(\Phi)$. Put differently, $\Phi' = 2f_z$. In particular, the conformal factor $e^u = 2 < f_z, f_{\bar z} >$ is given in terms of the Weierstrass data (g, ω) as

$$e^u = \frac{1}{4}(1+|g|^2)^2|\omega|^2$$

and the Hopf differential $\mathcal{Q} = -2 < f_{zz}, N > dz^2$ by

$$\mathcal{Q} = dg\omega .$$

The Gaussian curvature $K|df|^4 = -|\mathcal{Q}|^2$ is given by

$$K = -\left(\frac{2}{1+|g|^2}\right)^4 \left|\frac{dg}{\omega}\right|^2 .$$

Since the holomorphic null curve $\Phi_\theta = e^{-\mathbf{i}\theta}\Phi$ of the associated family f_θ is given by the holomorphic null curve Φ of f, the Weierstrass data (4.1) of f_θ compute to

$$g_\theta = g \quad \text{and} \quad \omega_\theta = e^{-\mathbf{i}\theta}\omega .$$

In particular, the Gauss map $N_\theta = N$ is preserved for the associated family f_θ whereas the Hopf differential is changed by

$$\mathcal{Q}_\theta = e^{-\mathbf{i}\theta}\mathcal{Q} .$$

Note that the Gauss map and the Hopf differential determine a minimal surface uniquely (up to translation).

From (3.3) we see that the Gauss map of a minimal surface is conformal with

$$\star dN = -NdN = dNN .$$

In particular, N is harmonic since the Laplacian of N is normal:

$$d \star dN = -dN \wedge dN = dN \wedge \star dNN = 2|dN|^2 N .$$

Theorem 4.1. *Let $f : M \to \mathbb{R}^3$ be a minimal surface. The conformal Gauss map S of f is given by*

$$S = \begin{pmatrix} N & 2u \\ 0 & -N \end{pmatrix}$$

where u is the support function of f.

Proof. Since $H = 0$ by (3.5) we have $2u = -Nf - fN = 2 < N, f >$ since $f, N : M \to \operatorname{Im}\mathbb{H}$. Thus, u is the support function [29] of f. □

The support function $u =< N, f >$ of a minimal surface f is a Jacobi function, see [23] for a discussion of Jacobi functions of a minimal surface in the quaternionic context. The Hopf field (3.7) of S computes

$$2 * A = G \begin{pmatrix} 0 & 0 \\ 0 & dN \end{pmatrix} G^{-1} = \begin{pmatrix} 0 & f \\ 0 & 1 \end{pmatrix} dN\,.$$

Since $df \wedge dN = df * dN - *dfdN = 0$ we see that

$$2d * A = \begin{pmatrix} 0 & df \wedge dN \\ 0 & 0 \end{pmatrix} = 0$$

and the conformal Gauss map is harmonic. Therefore, every minimal surface $f : M \to \mathbb{R}^3$ is a Willmore surface, and its Willmore energy

$$W(f) = -\int K|df|^2$$

is the negative of the total curvature of the minimal surface f.

By a result by Osserman, [31], a complete minimal surface $f : M \to \mathbb{R}^3$ has finite total curvature if M is conformally equivalent to a compact Riemann surface $\bar{M}$, punctured at finitely many points $\{p_1, \ldots, p_r\}$, such that $d\Phi$ extends meromorphically into the punctures p_i.

If z is a conformal coordinate centered at p and f is complete near p, then p is an embedded finite total curvature end if and only $d\Phi$ has a pole of order 2 at $z = 0$ and the residue of $d\Phi$ at $z = 0$ is real, [24]. If the residue is zero, the end is called *planar*, otherwise, it is called *catenoidal*.

5 Minimal surfaces and integrable systems

As we have seen, minimal surfaces are given by complex holomorphic functions, and one of the reasons for the comparable success of the theory of minimal surfaces is due to the wealth of methods from Complex Analysis that can be used to study minimal surfaces. On the other hand, in the recent study of properly embedded minimal planar domains [27, 28] algebro-geometric properties of the hierarchy of the Korteweg-de Vries equation have been used in an essential way.

A further reason that we are interested in studying minimal surfaces with the tools of integrable systems is the important open question whether there exists a complete, embedded minimal surface of genus $g \geq 1$ of finite total curvature with more than $g+2$ ends. It is conjectured, this is part of the Finite Topology Conjecture by Hoffman and Meeks [22], that there are no such minimal surfaces. In the case of genus $g = 0$ the Finite Topology Conjecture is true: using a deformation on the Weierstrass data, López and Ros [25] showed that the only complete, embedded, finite total curvature minimal surfaces of genus $g = 0$ are the catenoid and the plane, thus showing that in this case the number of ends is less or equal to $g+2$. In a recent paper the López–Ros deformation [24] has been identified as a special case of a well-known operation in integrable systems, the simple factor dressing, applied to the conformal Gauss map of the minimal surface. We will summarise the key results in [24] and give a new view of the simple factor dressing in terms of the Gauss map and the Hopf differential.

Given a harmonic map from a Riemann surface to an appropriate target space, there is an associated family of flat connections, given by the spectral parameters $\mu \in \mathbb{C}_*$. Conversely, under reality and holomorphy assumptions flat connections of the appropriate form give rise to harmonic maps. To obtain new harmonic maps from a given one, one can thus investigate which gauge matrices r_λ give new suitable families of flat connections. The resulting new harmonic maps are called *dressings*. A particular simple choice is for r_λ to have a simple pole away from $0, \infty$: in this case, every parallel bundle of the original family of flat connections gives a dressing matrix r_λ, and one can explicitly compute the new harmonic map. In case when $N : M \to S^2$ is the Gauss map of a minimal surface, the associated family of flat connections on $\underline{\mathbb{H}}$ is given by

$$d_\mu\beta = d\beta + \frac{1}{2}dN(-N\beta(a-1) + \beta b)\,, \quad \beta \in \Gamma(\underline{\mathbb{H}})\,,$$

with $a = \frac{\mu+\mu^{-1}}{2}$, $b = i\frac{\mu^{-1}-\mu}{2}$. Note that d_μ is a complex connection with respect to the complex structure given by right multiplication by i, and d_μ is quaternionic if and only if $\mu \in S^1$. A section $\beta \in \Gamma(\underline{\mathbb{H}})$ is d_μ–parallel if and only if $\beta = Nm + m\frac{i(1+\mu)}{1-\mu}$, $m \in \mathbb{H}_\star$. The simple factor dressing of N is then [24]

$$\hat{N} = (N+\rho)N(N+\rho)^{-1}\,, \quad \rho = m\frac{i(1+\mu)}{1-\mu}m^{-1}\,,$$

where $\mu \in \mathbb{C}_\star$ gives the pole of the dressing matrix and $m \in \mathbb{H}_\star$ the parallel complex line bundle $\beta\mathbb{C}$ of the simple factor dressing.

Note however, that the harmonic Gauss map does not uniquely determine the minimal surface: the catenoid and the Enneper surface have Weierstrass data $(g(z) = z, \omega = \frac{dz}{z^2})$ and $(g(z) = z, \omega = dz)$ respectively so that the Gauss maps coincide.

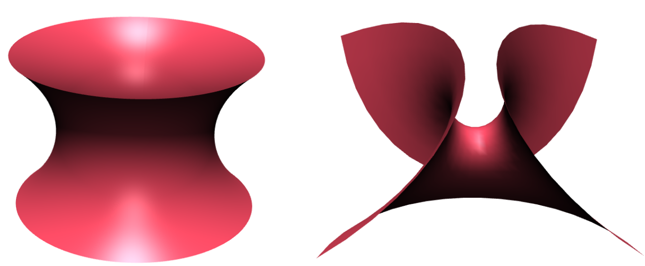

Figure 1: Catenoid and Enneper surface

In [24] we thus investigated the simple factor dressing of a different harmonic map: since a minimal surface is Willmore, its conformal Gauss map S is harmonic. Again the parallel sections $\varphi \in \Gamma(\underline{\mathbb{H}}^2)$ of a flat connection d_μ^S, $\mu \in \mathbb{C}_\star$, of the associated family of S can be computed explicitly to be

$$\varphi = en \quad \text{or} \quad \varphi = e\alpha + \psi\beta$$

where $n \in \mathbb{H}_\star$ and $\alpha = -f^\star m - fm\frac{i(1+\mu)}{1-\mu}$, $\beta = Nm + m\frac{i(1+\mu)}{1-\mu}$.

In fact, the $(0,1)$-part $(d_\mu^S)''$ is the holomorphic structure d'' given in (2.5) and (2.8) shows that $e\alpha \in H^0(\underline{\mathbb{H}}^2/L)$. Moreover, the section $e\alpha + \psi\beta$ is the prolongation of $e\alpha$ and therefore, $\varphi\mathbb{H}$ is a Darboux transform of f. This way, the simple factor dressing relates to the spectral data we discussed before.

When restricting to the case when the dressed surface takes values in 3-space the *simple factor dressing* with *parameter* (μ, m) is given by [24]

$$\hat{f} = -f\frac{m(a-1)m^{-1}}{2} + f^\star\frac{mbm^{-1}}{2} - m\frac{b}{a-1}m^{-1}\left(f\frac{mbm^{-1}}{2} + f^\star\frac{m(a-1)m^{-1}}{2}\right)$$

where $a = \frac{\mu+\mu^{-1}}{2}$, $b = i\frac{\mu^{-1}-\mu}{2}$ and $m \in \mathbb{H}_\star$ gives again the complex parallel bundle $\text{span}_\mathbb{C}\{em, e\alpha+\psi\beta\}$ whereas $\mu \in \mathbb{C}_\star$ gives the pole of the simple factor dressing. The simple factor dressing $\hat{f} : \tilde{M} \to \mathbb{R}^3$ with parameter (μ, m) is a minimal surface in 3-space which is complete if f is complete.

The Gauss map of a simple factor dressing of f is the simple factor dressing of the Gauss map N of f (with the same parameters). The case when $m = 1$ is called a simple factor dressing with parameter μ and is denoted by f^μ. It is given by

$$f^\mu = \begin{pmatrix} f_1 \\ f_2\cosh s - f_3^\star \sinh s \\ f_3 \cosh s + f_2^\star \sinh s \end{pmatrix}$$

where $s = -\ln|\mu|$ and $f_i, f_i^\star$ are the coordinate functions of f and a conjugate $f^\star$ of f.

Note that a simple factor dressing $\hat{f}$ with parameter (μ, m) is obtained from the simpler case $m = 1$ by rotations

$$\hat{f} = \mathcal{R}_m((\mathcal{R}_m^{-1}(f))^\mu)$$

where $\mathcal{R}_m = mfm^{-1}$ and $(\mathcal{R}_m^{-1}(f))^\mu$ is the simple factor dressing with parameter μ of $\mathcal{R}_m^{-1}(f) = \mathcal{R}_{m^{-1}}(f) = m^{-1}fm$.

This allows to easily investigate the periods of the simple factor dressing: a simple factor dressing is closed if all flux, that is, the periods of the conjugate surface, can be rotated simultaneously into the line $i\mathbb{R}$. Moreover, the end behaviour can be controlled:

Theorem 5.1 ([24]). *The simple factor dressing is a complete surface if f is complete. If f has a planar end at p then $\hat{f}$ is single-valued on M and p is a planar end of $\hat{f}$. If f has a catenoidal end at p, and $\hat{f}$ is single-valued then $\hat{f}$ has a catenoidal end at p.*

If there exists $m \in \mathbb{H}_\star$ such that $m\tau_\gamma^\star m^{-1} \in i\mathbb{R}$ for all $\gamma \in \pi_1(M)$ where $\tau_\gamma^\star$ are the periods of the conjugate surface, i.e., $\gamma^* f^\star = f^\star + \tau_\gamma^\star$, then $\hat{f}$ has the same periods as f.

The *López–Ros deformation* f^r with parameter r is given by the Weierstrass data $(rg, \frac{\omega}{r})$, $r > 0$, where (g, ω) is the Weierstrass data of f, [25]. The Lopez–Ros deformation is, [24], a special case of the simple factor dressing:

$$f^r = \mathcal{R}_n((\mathcal{R}_{n-1}f)^\mu) = \begin{pmatrix} f_1 \cosh s - f_2^\star \sinh s \\ f_1 \cosh s + f_2^\star \sinh s \\ f_3 \end{pmatrix}$$

where $n = 1 - i - j - k$ and $\mu = -\frac{1}{r}$ and $r = e^s$. Conversely, all simple factor dressings are obtained, after rotation, by applying a Lopez–Ros deformation with complex parameter σ to a rotation of f.

Since $\mathcal{Q} = dg\omega$ the Hopf differential is preserved under the Lopez–Ros deformation, and so is the Hopf differential of a simple factor dressing: if $\tilde{f} = \mathcal{R}_m f$ then $\tilde{N} = \mathcal{R}_m N$ and the Hopf differential is given by

$$\tilde{\mathcal{Q}} = -2 < \mathcal{R}_m f_{zz}, \mathcal{R}_m N >= \mathcal{Q}\,.$$

Put differently, the simple factor dressing of a minmal surface is given by performing a simple factor dressing of the Gauss map while fixing the Hopf differential (whereas the assocciated family fixes the Gauss map and rotates the Hopf differential).

We summarise:

Theorem 5.2. *Let $f : M \to \mathbb{R}^3$ be a minimal surface with Gauss map N and Hopf differential $\mathcal{Q}$. The simple factor dressing with parameters (μ, m) is the unique minimal surface $\hat{f} : \tilde{M} \to \mathbb{R}^3$ in 3-space with Gauss map*

$$\hat{N} = (N + \rho)N(N + \rho)^{-1}\,, \quad \rho = m\frac{i(1+\mu)}{1-\mu}m^{-1}\,,$$

and Hopf differential $\hat{\mathcal{Q}} = \mathcal{Q}$.

6 The simple factor dressing: examples

We conclude this paper with a discussion of the simple factor dressing of some well-known minimal surfaces, the Riemann minimal surfaces and the Costa surface. We discuss whether it is possible to close periods of the simple factor dressing in the examples.

6.1 The Riemann minimal surfaces

Riemann described in [34] a 1-parameter family of complete, embedded, singly periodic minimal surfaces which are foliated by circles in parallel planes. Recently, it has been shown [27, 28] that every properly embedded, minimal planar domain $M \subset \mathbb{R}^3$ with infinite topology is a Riemann minimal example.

The *Shiffman function* of a minimal surface measures the curvature variation of the parallel sections of the surface. Thus, the Shiffman function vanishes identically if and only if the minimal surface is foliated by circles and straight lines in parallel planes, [37]. The Shiffman function is a Jacobi function, and thus gives rise to a holomorphic section of $\underline{\mathbb{H}}^2/L$, see [23]. In particular, the integrable hierarchy given by the Shiffman function for a properly embedded minimal planar domain in [28] links to the spectral data of a minimal surface, that is, the possible monodromy of holomorphic sections $\varphi \in H^0(\underline{\mathbb{H}}^2/L)$.

The holomorphic null curve of the Riemann minimal surface with parameter σ is given by (see [26] for the parametrisation and details on the following implementation)

$$\tilde{\Phi} = \begin{pmatrix} \frac{1}{\sqrt{\sigma}}\left(\sqrt{\frac{z-1}{z}}\sqrt{z+\sigma} + \frac{2}{\sqrt{1+\sigma}}\left(-(1+\sigma)E(\arcsin\sqrt{\frac{\sigma+z}{\sigma}}, \frac{\sigma}{1+\sigma}) + F(\arcsin\sqrt{\frac{\sigma+z}{\sigma}}, \frac{\sigma}{1+\sigma})\right)\right) \\ -\sqrt{\frac{z-1}{-\sigma z}}\sqrt{z+\sigma} \\ -2\frac{F(\arcsin\sqrt{\frac{\sigma}{\sigma+z}}, \frac{1+\sigma}{\sigma})}{\sqrt{\sigma}} \end{pmatrix}$$

on the fundamental domain $\Sigma_\sigma = \{z \in \mathbb{C} \mid |z - \frac{1-\sigma}{2}| \le \frac{1+\sigma}{2}, \operatorname{Im} z \ge 0\} \setminus \{0\}$. Here $F(\phi, m)$ and $E(\phi, m)$ denote the incomplete elliptic integral of the first and second kind respectively with Jacobi amplitude ϕ and elliptic modulus m.

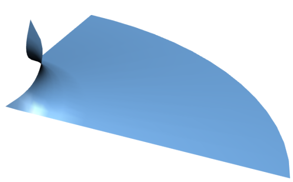

Figure 2: Fundamental piece of the Riemann minimal surface with $\sigma = 2$.

To simplify the computations, we will consider the translated holomorphic null curve $\Phi = \tilde{\Phi} - \operatorname{Re}(\tilde{\Phi}(1)) - \operatorname{Im}(\tilde{\Phi}(-\sigma))$, and use the symmetries of the surface to construct the complete surface from the fundamental domain, for details, see [26]:

In particular, the translational period τ^* of the conjugate surface is

$$\tau^* = \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \operatorname{Im}(\Phi(-\sigma) - \Phi(i\sqrt{\sigma})) + \operatorname{Im}\Phi(i\sqrt{\sigma})$$

so that the rotation $v \mapsto mvm^{-1}$ with

$$m = \frac{1}{2\sqrt{a^2+c^2}}(-ci + j(a - \sqrt{a^2+c^2}))$$

rotates τ^* via $m\tau^* m^{-1} = i\sqrt{a^2+c^2}$ into a multiple of i. Then the simple factor dressing with parameters (μ, m) has the same translational period as the Riemann minimal surface by Theorem 5.1. Since the Riemann minimal surface has planar ends, so does the simple factor dressing. However, the simple factor dressing, as the Lopez-Ros deformation, in general does not preserve embeddedness:

Note that the Lopez-Ros deformation in this case will not close the periods along the catenoidal necks.

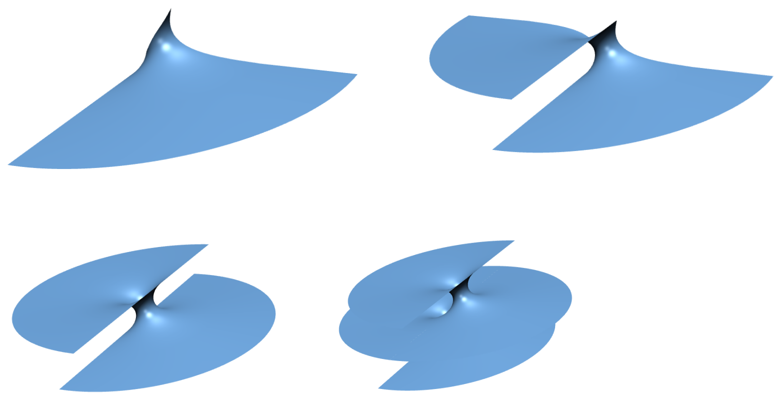

Figure 3: Construction of the Riemann minimal surface with $\sigma = 2$ by its symmetries.

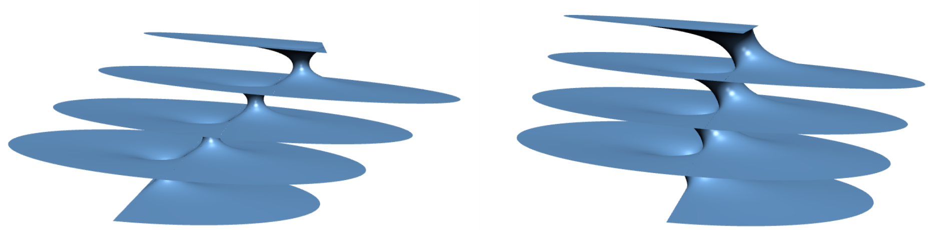

Figure 4: Riemann minimal surface with $\sigma = 8$ and $\sigma = 0.4$.

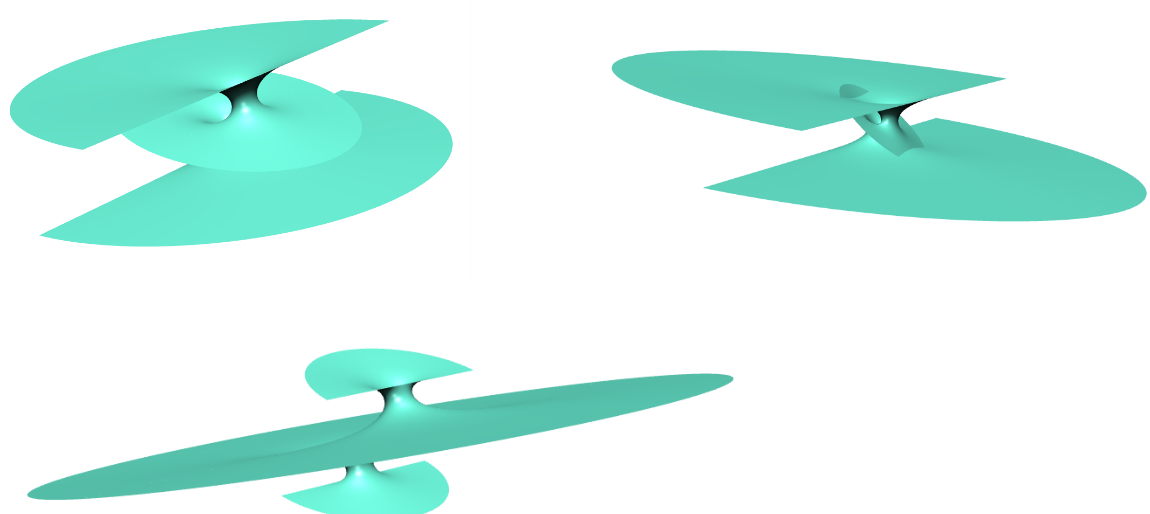

Figure 5: Simple factor dressings with parameters (μ, m) of the Riemann minimal surface with $\sigma = 2$ whee $m = \frac{1}{2\sqrt{a^2+c^2}}(-ci + j(a - \sqrt{a^2+c^2}))$ as above and $\mu = \frac{1}{2}(1+i)$, $\mu = 0.4$ and $\mu = 2i$ respectively.

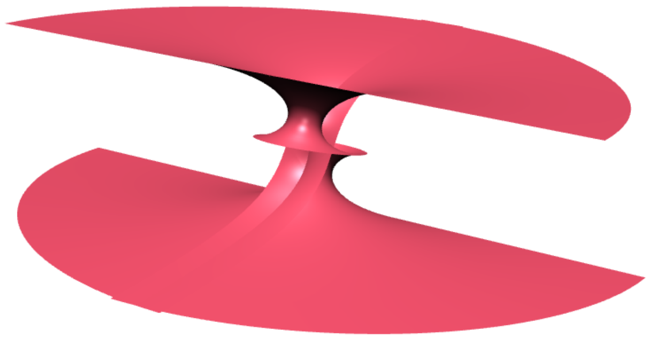

Figure 6: Lopez-Ros deformation with parameter $r = \frac{3}{2}$ of the Riemann minimal surface with $\sigma = 2$.

6.2 The Costa surface

This surface was discovered by Costa [14] and was the first minimal embedded punctured torus with finite total curvature. The Costa surface played an important role in introducing computer based experiments to geometry: J. Hoffmann, see [13], developed computer programs to visualise Costa's surface based on theoretical work of Hoffman and Meeks. This allowed performing mathematical experiments to obtain insights on properties of the Costa surface. These observations could later [22] be used to prove results rigidly, e.g., the embeddedness of the Costa surface and its symmetries. In view of the Finite Topology Conjecture, the Costa surface exhibits the conjectured maximal numbers of ends for a complete, embedded, finite total curvature minimal torus.

We use the parametrisation in [41] to implement the Costa surface which is defined on the square torus $w^2 = z(z^2 - 1)$. The Weierstrass data of the Costa surface is given by

$$g(z) = \frac{\rho}{\sqrt{z(z^2-1)}}, \qquad \omega = \frac{\sqrt{z}}{\rho\sqrt{z^2-1}}dz\,.$$

where the Lopez-Ros parameter $\rho = \frac{\Gamma(\frac{3}{4})}{\sqrt{2}\Gamma(\frac{5}{4})}$ is chosen so that the periods on the torus close. Here Γ is the Euler Gamma function. Note that the residues of $d\Phi$ are real so that the ends are indeed embedded finite total curvature ends. Then the holomorphic null curve of the Costa surface can be integrated explicitly in terms of the hypergeometric function ${}_2F_1(a, b, c, z)$ as

$$\Phi = \frac{1}{2}\begin{pmatrix} \phi_2(z) - \phi_1(z) \\ \mathbf{i}\,(\phi_1(z) + \phi_2(z)) \\ \log\frac{z-1}{z+1} \end{pmatrix}$$

where ϕ_1, ϕ_2 are, see [41],

$$\phi_1(z) = (2\rho\mathbf{i}\sqrt{z})\,{}_2F_1(\frac{1}{4}, \frac{3}{2}, \frac{5}{4}, z^2)$$

and

$$\phi_2(z) = -\frac{2\mathbf{i}\,z^{\frac{3}{2}}}{3\rho}\,{}_2F_1(\frac{1}{2}, \frac{3}{4}\frac{7}{4}, z^2)\,,$$

(and z is in the first quadrant).

By using the symmetries we obtain the full Costa surface.

The catenoidal ends are at $z = \pm 1$ and the planar end is at $z = \infty$. Since the periods of the conjugate surface around the ends $z = \pm 1$ is given by $\tau^*_{\pm 1} = 2\pi k$, we see that the Lopez-Ros deformation closes the periods around the catenoidal ends ± 1, and the Lopez-Ros deformation is defined on $\mathbb{C} \setminus \{\pm 1\}$ with two catenoidal ends and one planar end. However, the Lopez-Ros deformation is not defined on the torus.

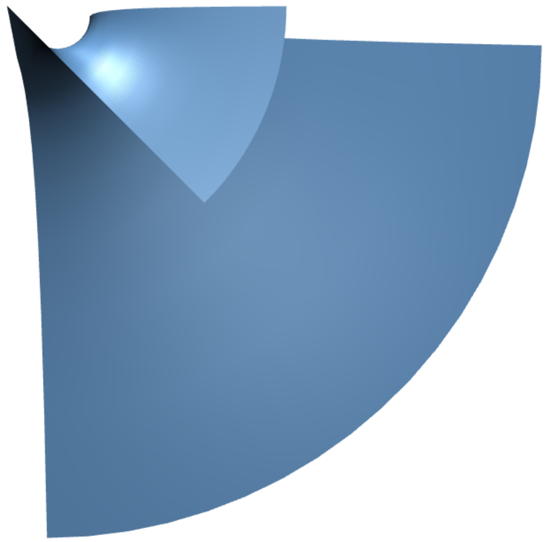

Figure 7: Fundamental piece of the Costa surface.

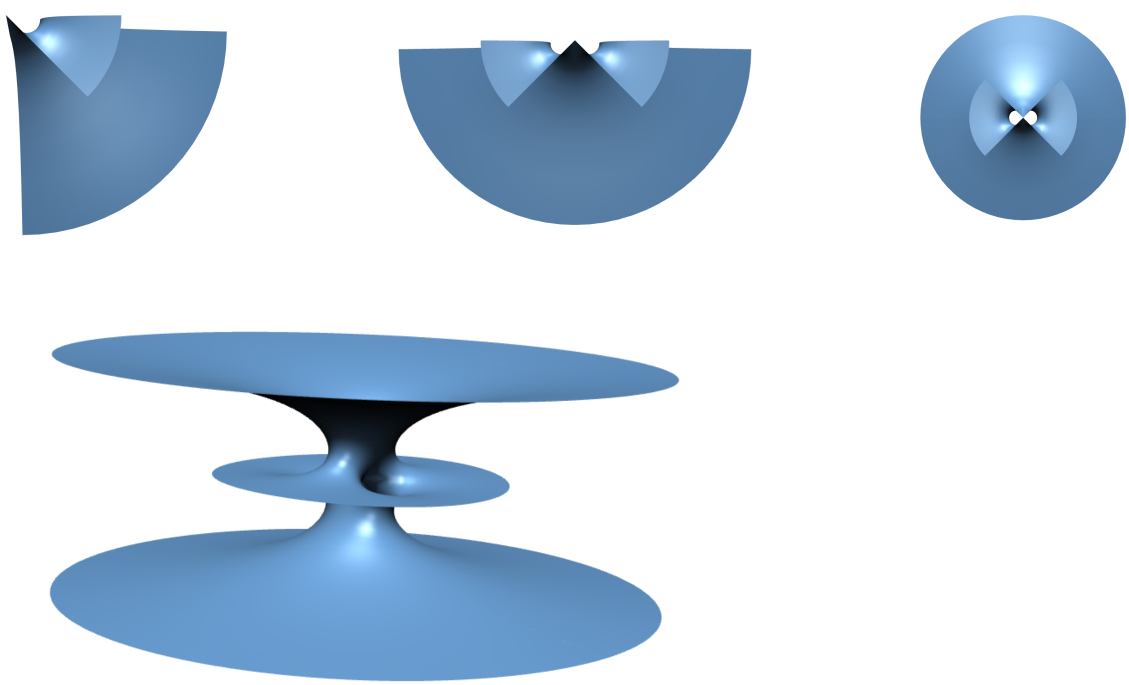

Figure 8: Construction of the Costa surface by its symmetries.

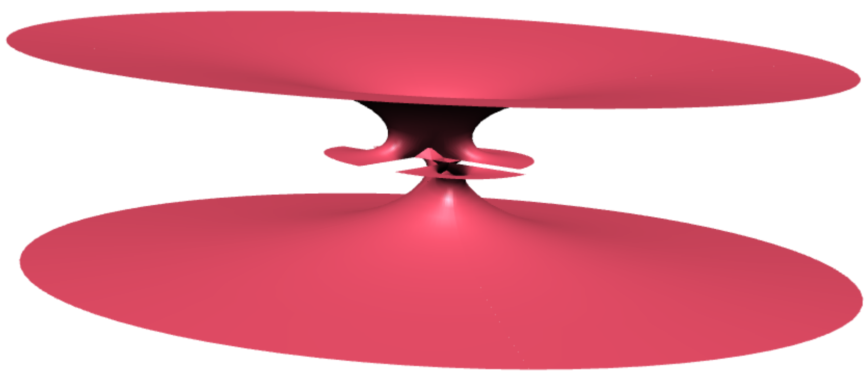

Figure 9: Lopez-Ros deformation with parameter $r = \frac{3}{2}$ of the Costa surface.

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