A dissipative force between colliding viscoelastic bodies: Rigorous approach

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Abstract –A collision of viscoelastic bodies is analysed within a mathematically rigorous approach. We develop a perturbation scheme to solve continuum mechanics equation, which deals simultaneously with strain and strain rate in the bulk of the bodies' material. We derive dissipative force that acts between particles and express it in terms of particles' deformation, deformation rate and material parameters. It differs noticeably from the currently used dissipative force, found within the quasi-static approximation and does not suffer from inconsistencies of this approximation. The proposed approach may be used for other continuum mechanics problems where the bulk dissipation is addressed.

Introduction. – Granular materials are abundant in nature and play an important role in industry. Properties of these systems are very unusual and depend on the applied load: for a small load a granular medium behaves as a solid, for a larger load it flows like a liquid, while at still larger excitations, a gas-like behavior may be observed. Such rich behavior is a consequence of the dissipative nature of the interaction forces between particles comprising a granular system. Therefore for an adequate description of granular media it is crucial to develop a quantitative model for the dissipative forces at particles' contacts.

While the elastic component of the inter-particle force is known for more than a century from the famous work of Hetrz [1], where a mathematically rigorous theory has been developed, a rigorous derivation for the dissipative component is still lacking. The existing phenomenological expressions for the dissipative force used either linear, e.g. [2, 3] or quadratic [4] dependence on the deformation rate; these however do not agree with the experimental data, e.g. [2, 5]. An attempt to obtain a dissipative force from the basic principles, has been undertaken in [6]; only a limited class of deformations has been addressed there.

A first complete derivation of the dissipative force between viscoelastic bodies from the continuum mechanics equations has been done only recently [7]. In this work a so called *quasi-static* approximation has been introduced. The functional dependence of the dissipative force on the deformation and deformation rate, found in Ref. [7], has been already proposed (without any mathematical derivation) in the earlier work of Kuwabara and Kono [8]. In later studies [9, 10] a flaw in the derivation of the dissipative force in Ref. [7] was corrected; still the restrictive assumption of the quasi-static approximation was used [9, 10].

In the quasi-static approximation it is assumed that the displacement field in the deformed material completely coincides with that for the static case. That is, an immediate response of the particles' material to the external load is supposed. More precisely, the quasi-static approximation implies that: (i) the characteristic deformation rate is much smaller than the speed of sound in the system and (ii) the microscopic relaxation time of the particle's material is negligibly small as compared to the duration of the impact. The precise definition of the former quantity will be given below, physically, however it characterizes the response of the material to the applied load. In the present study we develop a *mathematically rigorous* perturbative approach, which allows to go beyond the quasi-static approximation; we demonstrate that this approximation, although being physically plausible, is not mathematically complete. This happens because the deviations from the static deformations, neglected in the quasi-static approximation, ultimately yield a contribution to the dissipative force, comparable to the force itself in this approximation.

The proposed approach may be also used to analyze other time-dependent impact problems.

Perturbation scheme for the continuum mechanics equation. – To find a force acting between the bodies in a contact with a given deformation at their surfaces, one needs to solve continuum mechanics equation for the stress tensor. Integration of the obtained stress over the contact area yields the inter-particle force. The contact mechanics equation, that is, the equation of motion for a body material, generally reads, e.g. [11],

$$\rho\ddot{\mathbf{u}} = \nabla\cdot\hat{\sigma} = \nabla\cdot\left(\hat{\sigma}^{el}+\hat{\sigma}^{v}\right)\,, \tag{1}$$

where ρ is the material density, $\mathbf{u} = \mathbf{u}(\mathbf{r})$ is the displacement field in a point $\mathbf{r}$ and $\hat{\sigma}$ is the stress tensor, comprised of the elastic $\hat{\sigma}^{el}$ and viscous $\hat{\sigma}^{v}$ parts. The elastic stress linearly depends on the strain tensor $u_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right)$ [11],

$$\sigma^{el}_{ij}(\mathbf{u}) = 2E_1\left(u_{ij} - \frac{1}{3}\delta_{ij}u_{ll}\right) + E_2\delta_{ij}u_{ll}\,; \tag{2}$$

correspondingly, the viscous stress depends on the strain *rate* tensor [11]:

$$\sigma^{v}_{ij}(\dot{\mathbf{u}}) = 2\eta_1\left(\dot{u}_{ij} - \frac{1}{3}\delta_{ij}\dot{u}_{ll}\right) + \eta_2\delta_{ij}\dot{u}_{ll}\,. \tag{3}$$

Here $E_1 = \frac{Y}{2(1+\nu)}$, $E_2 = \frac{Y}{3(1-2\nu)}$, with Y and ν being respectively the Young modulus and Poisson ratio, and η_1 and η_2 are respectively shear and bulk viscosities of the bodies' material; i, j, l denote Cartesian coordinates and the Einstein's summation rule is applied.

Let us estimate the magnitude of the different terms in Eq.(1). This may be easily done using the dimensionless units. For the length scale we take R, which corresponds to the characteristic size of colliding bodies, while for the time scale we use the collision duration τ_c. Then $v_0 = R/\tau_c$ is the characteristic velocity at the impact. Taking into account that differentiation with respect to a coordinate yields for dimensionless quantities the factor $1/R$ and with respect to time the factor $1/\tau_c$, we obtain

$$\nabla\sigma^{v} \sim \lambda_1\,\nabla\sigma^{el} \qquad \lambda_1 = \tau_{rel}/\tau_c \tag{4}$$

$$\rho\ddot{u} \sim \lambda_2\,\nabla\sigma^{el} \qquad \lambda_2 = v_0^2/c^2\,. \tag{5}$$

Here $c^2 = Y/\rho$ and $\tau_{rel} = \eta/Y$ characterize respectively the speed of sound and the microscopic relaxation time in the material and $\eta \sim \eta_1 \sim \eta_2$ [7]. Hence, the term associated with the viscous stress is smaller by factor λ_1 than the one corresponding to the elastic stress, while the term associated with the inertial effects is smaller by the factor λ_2.

Neglecting terms, of the order of λ_1 and λ_2, that is, the terms $\nabla\sigma^{v}$ and $\rho\ddot{u}$, Eq. (1) simplifies to

$$\nabla\cdot\hat{\sigma}^{el}(\mathbf{u}) = 0, \tag{6}$$

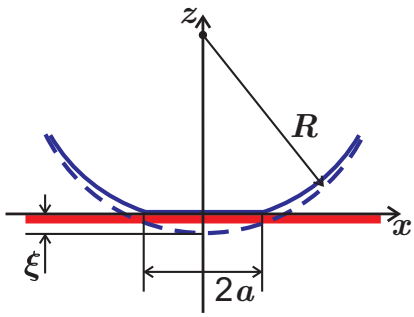

Fig. 1: Collision of visco-elastic sphere of radius R with undeformable plane. Here a is the radius of the contact zone and ξ is the deformation. The contact plane is located at $z = 0$.

which yields the static displacement field $\mathbf{u} = \mathbf{u}(\mathbf{r})$. This approximation corresponds to the quasi-static approximation, used in the literature [7, 9, 10, 12, 13]. Neglecting terms of the order λ_2 but keeping terms of the order of λ_1 yields:

$$\nabla\cdot\hat{\sigma} = \nabla\cdot\left(\hat{\sigma}^{el}(\mathbf{u}) + \hat{\sigma}^{v}(\dot{\mathbf{u}})\right) = 0. \tag{7}$$

Physically, the above equation describes the over-damped motion of a material when the inertial effects, proportional to λ_2, are negligible; the excitation of elastic waves in this case may be ignored. Such conditions are important for many applications, especially for slow collisions.

To go beyond the quasi-static approximation one has to solve Eq. (7) which contains both the displacement field $\mathbf{u}$ as well as its time derivative, $\dot{\mathbf{u}}$. Eq. (7) needs to be supplemented by the boundary conditions. These correspond to vanishing stress on the free surface of the bodies and given displacement $\mathbf{u}$ at the contact area. For simplicity we consider here a collision of a sphere of radius R with a hard undeformable plane located at $z = 0$, Fig. 1. The generalization for a contact of two arbitrary convex bodies of different materials is straightforward, but leads to cumbersome notations; it will be addressed elsewhere [14]. Let $\xi = R - z_O$ be the deformation, where z_O is z-coordinate of the center of mass of the sphere, then z-component of the displacement on the contact plane reads for small deformations [11]:

$$u_z(x, y) = \xi - \frac{1}{2R}(x^2 + y^2)\,. \tag{8}$$

In a vast majority of applications $\lambda_1 = \tau_{rel}/\tau_c \ll 1$, which implies that the viscous stress is small as compared to the elastic stress. This allows to solve Eq. (7) perturbatively, as a series in a small parameter $\lambda_1 \propto \eta$. Here we follow the standard perturbation scheme, e.g. [15]: To notify the order of different terms, we introduce a "technical" small parameter λ, which at the end of computations is to be taken as unity. Hence one can write,

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}^{(0)}(\mathbf{r}) + \lambda\mathbf{u}^{(1)}(\mathbf{r}) + \lambda^2\mathbf{u}^{(2)}(\mathbf{r}) + \ldots \tag{9}$$

and

$$\hat{\sigma} = \hat{\sigma}^{el} + \lambda\hat{\sigma}^{v} \tag{10}$$

Substituting the Eqs. (9) and (10) into Eq. (7) and collecting terms of the same order in λ yields a hierarchic set of equations. The zero-order equation reads,

$$\nabla \cdot \hat{\sigma}^{el}\left(\mathbf{u}^{(0)}\right) = 0 \tag{11}$$
$$u_z^{(0)}\Big|_{z=0} = \xi - \frac{1}{2R}(x^2+y^2)\,.$$

The first-order (that is, proportional to λ) equation is

$$\nabla \cdot \left(\hat{\sigma}^{el}(\mathbf{u}^{(1)}) + \hat{\sigma}^{v}(\dot{\mathbf{u}}^{(0)})\right) = 0 \tag{12}$$
$$u_z^{(1)}\Big|_{z=0} = 0\,,$$

and so on, where the expressions for $\hat{\sigma}^{el}$ and $\hat{\sigma}^{v}$ are given by Eqs. (2) and (3). In all these equations the stress tensor vanishes on the free surface. Note that in the proposed perturbation scheme, only zero-order equation (11) has non-zero boundary conditions, corresponding to the boundary conditions (8) of the initial problem; all other, high-order perturbation equations, have homogeneous boundary conditions. Such partition of the boundary conditions is justified due to linearity of the problem.

The zero-order solution $\mathbf{u}^{(0)}$ of Eq. (11) of the above perturbative approach is to be substituted into Eq. (12) to find the first-order solution $\mathbf{u}^{(1)}$, which may be further used to obtain $\mathbf{u}^{(2)}$ from the second-order equation, etc. Hence the series

$$\hat{\sigma} = \hat{\sigma}^{(0)} + \lambda\hat{\sigma}^{(1)} + \lambda^2\hat{\sigma}^{(2)} + \dots \tag{13}$$

is generated, where $\hat{\sigma}^{(0)} = \hat{\sigma}^{el\,(0)} = \hat{\sigma}^{el}\left(\mathbf{u}^{(0)}\right)$ is the zero-order term, $\hat{\sigma}^{el\,(1)} = \hat{\sigma}^{el}\left(\mathbf{u}^{(1)}\right)$ and $\hat{\sigma}^{v\,(1)} = \hat{\sigma}^{v}\left(\dot{\mathbf{u}}^{(0)}\right)$ are the first-order terms, $\hat{\sigma}^{el\,(2)} = \hat{\sigma}^{el}\left(\mathbf{u}^{(2)}\right)$ is the second-order term with respect to $\lambda_1 \propto \eta$, etc.

Zero-order solution. Hertz theory. – To illustrate the approach we start with the zero-order Eq. (11). It corresponds to the quasi-static approximation (6), which solution is known. In the above notations Eq. (11) reeds

$$\nabla_j \sigma_{ij}^{el\,(0)} = E_1 \Delta u_i^{(0)} + \left(E_2 + \frac{1}{3}E_1\right)\nabla_i \nabla_j u_j^{(0)} = 0\,. \tag{14}$$

To solve Eq. (14) we use the approach of Ref. [11] and write the solution as

$$\mathbf{u}^{(0)} = f^{(0)}\mathbf{e}_z + \nabla\varphi^{(0)}\,, \tag{15}$$

where $\varphi^{(0)} = K^{(0)} z f^{(0)} + \psi^{(0)}$, $K^{(0)}$ is some constant to be found and $f^{(0)}$ and $\psi^{(0)}$ are unknown harmonic functions ($\Delta f^{(0)} = 0$ and $\Delta\psi^{(0)} = 0$). We assume the lack of tangential stress at the interface, which is e.g. fulfilled when the bodies at a contact are of the same material. Physically, the substitute (15) is dictated by the symmetry of the problem: The main displacement of the material occurs along z-axes. Taking into account that

$$\Delta\mathbf{u}^{(0)} = \Delta\nabla\varphi^{(0)} = 2K^{(0)}\nabla\frac{\partial f^{(0)}}{\partial z} \tag{16}$$

and

$$\nabla\cdot\mathbf{u}^{(0)} = (1+2K^{(0)})\frac{\partial f^{(0)}}{\partial z}, \tag{17}$$

as it follows from Eq. (15), we recast Eq. (14) into the form:

$$\begin{aligned}\nabla_j \sigma_{ij}^{el(0)} &= \Big[2E_1K^{(0)} + \\ &+ (1+2K^{(0)})\left(E_2 + \frac{E_1}{3}\right)\Big]\nabla_i\frac{\partial f^{(0)}}{\partial z} = 0\,,\end{aligned} \tag{18}$$

which implies (for non-zero $f^{(0)}$) that

$$K^{(0)} = -\frac{1}{2}\frac{3E_2+E_1}{3E_2+4E_1}\,. \tag{19}$$

Consider now the boundary condition for the stress tensor. Obviously, on the free boundary all components of the stress vanish. In the contact region, located at the surface, $z=0$, the tangential components of the stress tensor σ_{zx} and σ_{zy} vanish as well, while the normal component of the stress tensor equals (up to the sign) to the normal component of the external pressure $P_z^{(0)}$, e.g. [11]:

$$\sigma_{zx}^{el(0)}\Big|_{z=0}=0;\quad \sigma_{zy}^{el(0)}\Big|_{z=0}=0;\quad \sigma_{zz}^{el(0)}\Big|_{z=0}=-P_z^{(0)}. \tag{20}$$

Using the Eq. (2) for the elastic part of the stress tensor, together with the displacement vector (15) we recast the boundary conditions (20) into the form:

$$\frac{\partial}{\partial x}\left(\frac{3E_1}{4E_1+3E_2}f^{(0)} + 2\frac{\partial\psi}{\partial z}\right)\Big|_{z=0} = 0 \tag{21}$$
$$\frac{\partial}{\partial y}\left(\frac{3E_1}{4E_1+3E_2}f^{(0)} + 2\frac{\partial\psi}{\partial z}\right)\Big|_{z=0} = 0 \tag{22}$$
$$\frac{\partial}{\partial z}\left(\frac{3E_1}{4E_1+3E_2}f^{(0)} + 2\frac{\partial\psi}{\partial z}\right)\Big|_{z=0} = -\frac{P_z^{(0)}}{E_1}. \tag{23}$$

From Eqs. (21) and (22) follows the relation between $f^{(0)}$ and $\frac{\partial\psi}{\partial z}$ at $z=0$:

$$\left(\frac{\partial\psi}{\partial z} + \frac{3}{2}\frac{E_1}{4E_1+3E_2}f^{(0)}\right)\Big|_{z=0} = \text{const} = 0\,. \tag{24}$$

The constant in the above relation equals to zero, since it holds true independently on the coordinate that is, also at the infinity; at the infinity, however, the deformation and thus the above functions vanish. Since $f^{(0)}$, ψ and $\partial\psi/\partial z$ are harmonic functions, the condition that their linear combination vanishes on the boundary, Eq. (24), implies that this combination is zero in the total domain, that is,

$$\frac{\partial\psi}{\partial z} = -\frac{3}{2}\frac{E_1}{4E_1+3E_2}f^{(0)}\,. \tag{25}$$

Substituting the last relation into (23) yields

$$\frac{\partial f^{(0)}}{\partial z}\Big|_{z=0} = -\frac{4E_1+3E_2}{E_1(E_1+3E_2)}P_z^{(0)}. \tag{26}$$

Since $f^{(0)}$ is a harmonic function, one can use the relation between the normal derivative of a harmonic function on a surface and its value in the bulk, as it follows from the theory of harmonic functions (see e.g. [11,16]), hence we find:

$$f^{(0)}(\mathbf{r}) = \frac{4E_1 + 3E_2}{2\pi E_1(E_1 + 3E_2)} \iint_S \frac{P_z^{(0)}(x', y')\,\mathrm{d}x'\mathrm{d}y'}{|\mathbf{r} - \mathbf{r}'|}, \quad (27)$$

where S is the contact area.

Using Eq. (15) we can write z-component of the zero-order displacement at $z = 0$ as

$$u_z^{(0)}\Big|_{z=0} = (1 + K^{(0)})\, f^{(0)}\Big|_{z=0} + \frac{\partial \psi}{\partial z}\Big|_{z=0},$$

which together with (25) and definition of $K^{(0)}$ (Eq. (19)) yields,

$$u_z^{(0)}\Big|_{z=0} = \frac{1}{2}\, f^{(0)}\Big|_{z=0}. \quad (28)$$

If we now express E_1 and E_2 in terms of ν and Y, we obtain from Eqs. (28), (27) and (20):

$$u_z^{(0)}\Big|_{z=0} = -\frac{(1-\nu^2)}{\pi Y} \iint_S \frac{\sigma_{zz}^{el(0)}(x', y', z = 0)\,\mathrm{d}x'\mathrm{d}y'}{|\mathbf{r} - \mathbf{r}'|}. \quad (29)$$

Eq. (29) is a standard relation of the static continuum theory, e.g. [11]. Physically it relates the distribution of the normal displacement and normal stress at the contact zone. The distribution of the normal pressure there follows from the Hertz theory (see e.g. [11]):

$$-\sigma_{zz}^{el(0)}\Big|_{z=0} = P_z^{(0)} = \frac{2Y}{\pi R(1-\nu^2)}\sqrt{a^2 - (x^2 + y^2)}, \quad (30)$$

where a is the radius of the contact circle. Substituting Eq. (30) into (29) and performing integration over the contact zone we obtain, as expected, the displacement (8). Moreover, since $\xi = u_z^{(0)}(x = 0, y = 0)\Big|_{z=0}$, we find the relation between deformation and the radius of the contact circle, $\xi = a^2/R$. Integrating the stress (30) over the contact we obtain the elastic Hertzian force, e.g. [11]:

$$F_H = F_z^{el(0)} = B\xi^{3/2}, \qquad B = \frac{4Y\sqrt{R}}{3(1-\nu^2)}. \quad (31)$$

Obviously, the zero-order terms refer to the static case and do not describe dissipation. As it follows from the above discussion [see Eq. (12)] there are two first-order terms, $\hat{\sigma}^v\left(\dot{\mathbf{u}}^{(0)}\right) = \hat{\sigma}^{v\,(1)}$ and $\hat{\sigma}^{el}\left(\mathbf{u}^{(1)}\right) = \hat{\sigma}^{el\,(1)}$. The former one depends on the *known* zero-order solutions $\mathbf{u}^{(0)}(\mathbf{r})$ and hence may be related to the zero-order stress $\hat{\sigma}^{el(0)} = \hat{\sigma}^{el}(\mathbf{u}^{(0)})$ as

$$\sigma_{ij}^{v(1)} = \frac{\eta_1}{E_1}\dot{\sigma}_{ij}^{el(0)} + \left(\eta_2 - \eta_1\frac{E_2}{E_1}\right)(1 + 2K^{(0)})\frac{\partial \dot{f}^{(0)}}{\partial z}\delta_{ij}, \quad (32)$$

where we use Eqs. (2), (3) and (17). If we now apply Eq. (26) for $\partial f^{(0)}/\partial z$ and Eq. (19) for the constant $K^{(0)}$, we find the zz-component of this tensor at the contact plane, $z = 0$:

$$\begin{aligned} \sigma_{zz}^{v(1)}(x, y, 0) &= \alpha_0 \dot{\sigma}_{zz}^{el(0)}(x, y, 0) \quad (33) \\ \alpha_0 &= \frac{3\eta_2 + \eta_1}{3E_2 + E_1} = \frac{(2+2\nu)(1-2\nu)(3\eta_2 + \eta_1)}{3Y} \end{aligned}$$

where the definitions of E_1 and E_2 have been used.

The other first-order term, $\hat{\sigma}^{el}\left(\mathbf{u}^{(1)}\right) = \hat{\sigma}^{el\,(1)}$ depends on the first-order displacement $\mathbf{u}^{(1)}(\mathbf{r})$ which is still to be found. Neglecting this term and keeping only one first-order term (33) corresponds to the quasi-static approximation for the dissipative force [7] discussed above. The expression for α_0 coincides with the result of [9,10], where the necessary corrections have been implemented.

First-order solution. Beyond the quasi-static approximation. – Turn now to the first-order equation (12), which is actually an equation for the function $\mathbf{u}^{(1)}$ that describes deviations of the displacement from the static case. We write this equation as

$$\nabla_j \sigma_{ij}^{el(1)} = -\nabla_j \sigma_{ij}^{v(1)}, \quad (34)$$

where the left-hand side contains the unknown function $\mathbf{u}^{(1)}$, while the right-hand side depends on $\mathbf{u}^{(0)}$ and is therefore known. Using Eqs. (16), (17) and Eq. (19) for $K^{(0)}$ we obtain for the r.h.s. of Eq. (34):

$$\begin{aligned} \nabla_j \sigma_{ij}^{v(1)} &= \left[2\eta_1 K^{(0)} + (1 + 2K^{(0)})\left(\eta_2 + \frac{\eta_1}{3}\right)\right]\nabla_i \frac{\partial \dot{f}^{(0)}}{\partial z} \\ &= \frac{3(E_1\eta_2 - E_2\eta_1)}{(4E_1 + 3E_2)}\nabla_i \frac{\partial \dot{f}^{(0)}}{\partial z}. \quad (35) \end{aligned}$$

To proceed with the solution of Eq. (34) for $\mathbf{u}^{(1)}$ we reduce it to the solution of two simpler equations. Namely, due to linearity of the problem, one can represent the first-order displacement field as a sum of two parts, $\mathbf{u}^{(1)} = \bar{\mathbf{u}}^{(1)} + \tilde{\mathbf{u}}^{(1)}$, which correspond to the two parts of the elastic stress tensor, $\sigma_{ij}^{el(1)} = \tilde{\sigma}_{ij}^{el(1)} + \bar{\sigma}_{ij}^{el(1)}$. Here the first part of $\sigma_{ij}^{el(1)}$ is the solution of the *inhomogeneous* equation with homogeneous boundary conditions:

$$\nabla_j \tilde{\sigma}_{ij}^{el(1)} = -\nabla_j \sigma_{ij}^{v(1)} \quad (36)$$

$$\tilde{\sigma}_{xz}^{el(1)}\Big|_{z=0} = \tilde{\sigma}_{yz}^{el(1)}\Big|_{z=0} = \tilde{\sigma}_{zz}^{el(1)}\Big|_{z=0} = 0, \quad (37)$$

while the second part is the solution of the *homogeneous* equation with the given boundary conditions for the displacement $\bar{u}_z^{(1)}$ at the contact plane:

$$\begin{aligned} \nabla_j \bar{\sigma}_{ij}^{el(1)} &= 0 \quad (38) \\ \bar{u}_z^{(1)} &= u_z^{(1)} - \tilde{u}_z^{(1)} = -\tilde{u}_z^{(1)}. \end{aligned}$$

Here we use the boundary conditions (12), that is, $u_z^{(1)}\Big|_{z=0} = 0$. The boundary problem (38) is exactly the

same as the above problem (11) for the zero-order functions. Hence the same relation (29) holds true for the first-order functions, that is,

$$\bar{u}_z^{(1)}\Big|_{z=0} = -\frac{(1-\nu^2)}{\pi Y}\iint_S \frac{\bar{\sigma}_{zz}^{el(1)}(x',y',z=0)\,\mathrm{d}x'\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{39}$$

To solve Eq. (36) we write the displacement field $\tilde{\mathbf{u}}^{(1)}$ in a form, similar to this for the zero-order solution (15):

$$\tilde{\mathbf{u}}^{(1)} = f^{(1)}\mathbf{e}_z + \nabla\varphi^{(1)}\,, \tag{40}$$

where $\varphi^{(1)} = K^{(1)}zf^{(1)} + \psi^{(1)}$, $K^{(1)}$ is some constant and $f^{(1)}$ and $\psi^{(1)}$ are harmonic functions. Then we can write the first-order elastic stress tensor $\tilde{\sigma}_{ij}^{el(1)}$ as

$$\begin{aligned}\tilde{\sigma}_{ij}^{el(1)} &= (1+2K^{(1)})\Big[E_1(\delta_{jz}\nabla_i f^{(1)} + \delta_{iz}\nabla_j f^{(1)}) \\ &+ \left(E_2 - \frac{2}{3}E_1\right)\frac{\partial f^{(1)}}{\partial z}\delta_{ij}\Big] + 2E_1K^{(1)}z\nabla_i\nabla_j f^{(1)} \\ &+ 2E_1\nabla_i\nabla_j\psi^{(1)}. \end{aligned}\tag{41}$$

Choosing $K^{(1)} = -\frac{1}{2}$ the above stress tensor simplifies to

$$\tilde{\sigma}_{ij}^{el(1)} = -zE_1\nabla_i\nabla_j f^{(1)} + 2E_1\nabla_i\nabla_j\psi^{(1)} \tag{42}$$

and the boundary conditions (37) read:

$$\tilde{\sigma}_{xz}^{el(1)}\Big|_{z=0} = \frac{\partial}{\partial x}\left(\frac{\partial\psi^{(1)}}{\partial z}\right)\Big|_{z=0} = 0 \tag{43}$$

$$\tilde{\sigma}_{yz}^{e(1)}\Big|_{z=0} = \frac{\partial}{\partial y}\left(\frac{\partial\psi^{(1)}}{\partial z}\right)\Big|_{z=0} = 0\,. \tag{44}$$

Therefore we conclude,

$$\frac{\partial\psi^{(1)}}{\partial z}\Big|_{z=0} = \mathrm{const} = 0\,, \tag{45}$$

where the last relation follows from the condition that $\psi^{(1)}$ vanishes at the infinity, x, $y \to \infty$, where the deformation is zero. Since $\psi^{(1)}$ is a harmonic function, we conclude that the vanishing normal derivative on the boundary, Eq. (45), implies that this function vanishes everywhere, that is, $\psi^{(1)}(x,y,z) \equiv 0$ (see e.g. [16]). Hence

$$\tilde{\sigma}_{ij}^{el(1)} = -E_1 z\nabla_i\nabla_j f^{(1)} \tag{46}$$

and the third boundary condition in Eq. (37), $\tilde{\sigma}_{zz}^{el(1)} = 0$ at $z = 0$, is automatically fulfilled. Taking into account that the function $f^{(1)}$ is harmonic, we obtain,

$$\nabla_j\tilde{\sigma}_{ij}^{el(1)} = -E_1\nabla_i\frac{\partial f^{(1)}}{\partial z}\,. \tag{47}$$

Substituting the above relation for $\nabla_j\tilde{\sigma}_{ij}^{el(1)}$ and Eq. (35) for $\nabla_j\sigma_{ij}^{v(1)}$ into Eq. (36), we recast this equation into the form,

$$E_1\nabla_i\frac{\partial f^{(1)}}{\partial z} = -\frac{3(E_2\eta_1 - E_1\eta_2)}{(4E_1+3E_2)}\nabla_i\frac{\partial\dot{f}^{(0)}}{\partial z}$$

which implies the relation between functions $f^{(1)}$ and $\dot{f}^{(0)}$:

$$\begin{aligned} f^{(1)} &= -\alpha_1\dot{f}^{(0)} \\ \alpha_1 &= \frac{3(E_2\eta_1 - E_1\eta_2)}{E_1(3E_2+4E_1)} = \frac{(1+\nu)(1-2\nu)}{(1-\nu)Y}\left[\frac{2+2\nu}{3-6\nu}\eta_1 - \eta_2\right]. \end{aligned}\tag{48}$$

The function $f^{(1)}$ may be now exploited to express the displacement $\tilde{u}_z^{(1)}$ on the contact plane. Using Eq. (40) with $K^{(1)} = -\frac{1}{2}$ we write for $\tilde{u}_z^{(1)}$:

$$\tilde{u}_z^{(1)} = \frac{1}{2}f^{(1)} - \frac{z}{2}\frac{\partial f^{(1)}}{\partial z}\,; \tag{49}$$

substituting there $f^{(1)}$ from Eq. (48) we arrive at

$$\tilde{u}_z^{(1)} = -\frac{1}{2}\alpha_1\left(\dot{f}^{(0)} - z\frac{\partial\dot{f}^{(0)}}{\partial z}\right)\,, \tag{50}$$

where $f^{(0)}$ is given by Eq. (27). Thus, the above relation presents the solution for the displacement $\tilde{u}_z^{(1)}$. For the contact plane $z = 0$ it yields the boundary condition for Eq. (38):

$$\bar{u}_z^{(1)}\Big|_{z=0} = -\,\tilde{u}_z^{(1)}\Big|_{z=0} = \frac{1}{2}\alpha_1\dot{f}^{(0)}\Big|_{z=0}\,. \tag{51}$$

Taking into account that $\frac{1}{2}\dot{f}^{(0)}\Big|_{z=0} = \dot{u}_z^{(0)}\Big|_{z=0}$, according to Eq. (28), we obtain, expressing $\dot{u}_z^{(0)}$ in terms of $\dot{\sigma}_{zz}^{el(0)}$, as it follows from Eq. (29):

$$\bar{u}_z^{(1)}\Big|_{z=0} = -\frac{(1-\nu^2)}{\pi Y}\iint_S \frac{\alpha_1\dot{\sigma}_{zz}^{el(0)}(x',y',z=0)\,\mathrm{d}x'\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{52}$$

Comparing then Eqs. (39) and (52) we conclude that the first-order stress tensor $\bar{\sigma}_{zz}^{el(1)}$ at the contact plane reads,

$$\bar{\sigma}_{zz}^{el(1)}\Big|_{z=0} = \alpha_1\dot{\sigma}_{zz}^{el(0)}\Big|_{z=0}\,. \tag{53}$$

Finally we obtain for the total first-order stress tensor $\sigma_{zz}^{(1)}$:

$$\begin{aligned}\sigma_{zz}^{(1)}\Big|_{z=0} &= \left(\bar{\sigma}_{zz}^{el(1)} + \tilde{\sigma}_{zz}^{el(1)} + \sigma_{zz}^{v(1)}\right)\Big|_{z=0} \\ &= (\alpha_0+\alpha_1)\dot{\sigma}_{zz}^{el(0)}\Big|_{z=0}\,, \end{aligned}\tag{54}$$

where we use Eqs. (33) and (53) and take into account that $\tilde{\sigma}_{zz}^{el(1)} = 0$ on the contact plane (see Eq. (37)).

The dissipative force. – The elastic inter-particles force refers to the zero-order term in the perturbation expansion (13), while the remaining terms quantify dissipation. Hence, in the linear with respect to the dissipative constants approximation, the total dissipative force reads

$$F_z^{v(1)} = \iint_S \sigma_{zz}^{(1)}(x,y)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y\,,$$

so that Eq. (54) yields,

$$F_z^{v(1)} = A\frac{\partial}{\partial t}\iint_S \sigma_{zz}^{el(0)}(x,y)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y = A\dot{F}_z^{el(0)}$$
$$A = \alpha_0 + \alpha_1, \tag{55}$$

where $F_z^{el(0)}$ is the normal force corresponding to the elastic reaction of the medium. It is equal to the Hertzian force, Eq. (31). Using the expressions (33) and (48) for α_0 and α_1 and Eq. (31) for the Hertzian force, we arrive at the final result for the dissipative force:

$$F_z^{v(1)} = \frac{3}{2}AB\sqrt{\xi}\dot{\xi} \tag{56}$$
$$A = \frac{1}{Y}\frac{1+\nu}{1-\nu}\left[\frac{4}{3}\eta_1(1-\nu+\nu^2)+\eta_2(1-2\nu)^2\right].$$

Here the constant B depends on the geometry of the colliding bodies and their material properties; for the simple case of a collision of a sphere with a hard plane, it is given by Eq. (31). For a collision of two spheres of radii R_1 and R_2 of the same material it reads [7, 11],

$$B = \frac{2Y}{3(1-\nu^2)}\sqrt{R_{\text{eff}}} \qquad R_{\text{eff}} = \frac{R_1R_2}{R_1+R_2}.$$

Generally, B depends on the local curvatures of the bodies at the contact, e.g. [7,11,14]. Although the derivation has been illustrated for the simple case, it remains valid for the bodies of any convex shapes and different materials [14].

Note that the new result (56) for the dissipative force has been obtained by a rigorous perturbation approach. It contains all first-order terms with respect to the small parameter λ_1, proportional to the material viscosities $\eta_{1/2}$ which guarantees the physical consistency of the theory. On the contrary, the previous result, based on the quasi-static approximation suffers from the incomplete account of the first-order stress terms. Indeed, this approximation takes into account $\hat{\sigma}^{v(1)}$ but ignores $\hat{\sigma}^{el(1)}$. Physically, $\hat{\sigma}^{v(1)}$ is the component of the stress associated with the strain rate (i.e. with the relative motion of different parts of the material) and thus has a "purely dissipative" nature. This stress causes an additional strain in the bulk, and the respective displacement field $\mathbf{u}^{(1)}(\mathbf{r})$ which gives rise to the excess elastic stress $\hat{\sigma}^{el(1)}$; both first-order stress terms are of the same order of magnitude as it follows from Eqs. (33), (53) and (48). Hence the quasi-static approximation is not generally valid. It manifests its inconsistency for the case of $\nu = 1/2$, which corresponds to materials with very small elastic shear module (like rubber). Although this approximation predicts vanishing dissipation in such materials, there are no physical mechanisms that could assure the energy conservation. At the same time, our new theory is free from such inconsistencies.

Conclusion. – We develop a mathematically rigorous method to describe the dissipative force acting between viscoelastic bodies during a collision. It is based on a perturbation scheme, applied to the over-damped continuum mechanics equation, with the inertial effects neglected. We use the small parameter, which is the ratio of microscopic relaxation time and the characteristic time of a collision and is proportional to the dissipative constants of the material. Applying the perturbation approach we obtain the dissipative force, linear with respect to this small parameter. The presented method is rather general and may be further developed to take into account the inertial effects as well as the high-order corrections with respect to the small parameter. The obtained dissipation force is expressed in terms of the time derivative of the elastic force, as it follows from the Hertz theory, and elastic and viscous material constants. It noticeably differs from the one obtained previously within the quasi-static approximation and demonstrates physically correct behavior for the whole range of material parameters. Finally, we wish to stress that the proposed approach may be also applied for similar continuum mechanics problems, where dissipation in a bulk due to the strain rate is addressed.

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