May 2019
SAGE

Switched $\mathcal{H}_2-$state-feedback control with application to a fighter aircraft

Emre Kemer[1], Hasan Başak[2] and Emmanuel Prempain[3]

Abstract

This paper proposes two different $\mathcal{H}_2-$state-feedback controller synthesis algorithms for uncertain linear, time-varying, switched systems. The synthesis algorithms are based on a dwell-time approach which makes use of time-varying parameter-dependent Lyapunov functions. The control laws consist of state-feedback controllers that are switched according to external signals. The proposed synthesis algorithms are then employed to design switched $\mathcal{H}_2-$state-feedback control laws for the longitudinal dynamics of the ADMIRE fighter benchmark model.The results obtained in simulation show the merits of the proposed approach.

Keywords

Dwell time; $\mathcal{H}_2-$switched control; Switched Lyapunov functions; Longitudinal aircraft dynamics; Load factor control.

[1] Corresponding author: Dept. of Electrical and Electronics Engineering, Faculty of Eng., Uşak University, Uşak, TURKEY, Email: emre.kemer@usak.edu.tr

[2] Dept. of Electrical and Electronics Engineering, Faculty of Eng., Artvin Çoruh University, Artvin, TURKEY,

[3] Control Systems Research Group, Dept. of Engineering, University of Leicester, Leicester LE1 7RH, UK

Introduction

Switching in control has many applications in communication networks, space applications, automotive engine control, robotics, power electronics and so on see e.g. Yuan et al. (2017). Various stability analysis tools for switched systems are given in the literature as reported in the survey paper of Lin and Antsaklis (2009). Common and piecewise quadratic Lyapunov functions are used in Johansson and Rantzer Johansson and Rantzer (1998); Rantzer and Johansson (2000) to guarantee the stability of linear switched systems that are switched according to the system states. Briat (2017) uses the concepts of constant, minimum, maximum and range dwell-times and linear co-positive Lyapunov functions to analyse the stability and the stabilization of linear positive impulsive switched systems. Geromel and Colaneri (2006); Allerhand and Shaked (2010, 2013-1) use the so-called minimum dwell-time concept to prove stability of linear switched systems under slow switching. From performance and stability analysis, it is often possible to derive state-feedback controller synthesis conditions. Allerhand and Shaked (2011),Allerhand and Shaked (2013-2) obtained state-feedback synthesis conditions for uncertain switched linear systems subject to dwell-time constraints that permit the design of a switched state-feedback control law that meets a closed-loop $\mathcal{L}_2$-performance objective. Similarly, piecewise linear quadratic Lyapunov functions are used in Allerhand and Shaked (2013-3) for the design of robust state estimation filters.

This paper tackles the problem of $\mathcal{H}_2$ state-feedback controller synthesis for uncertain switched linear systems subject to a dwell-time constraint. The paper follows the line of research of Allerhand and Shaked (2013-3) and is closely related to the work published in Kemer and Prempain (2018) by Kemer and Prempain. Here, the aim is to reduce the effects of a zero mean white noise disturbance input on the output of the switched system. To this end, state-feedback synthesis algorithms, based on the minimum dwell-time approach of Geromel and Colaneri (2006); Allerhand and Shaked (2010, 2013-1) and on the time-varying parameter-dependent Lyapunov functions, are proposed. Two switched $\mathcal{H}_2-$state-feedback controller synthesis algorithms (namely the conditions of Theorems 1 and 2) are given. In both cases, the state-space data of each sub-system are assumed to belong to a polytope, that is, the state-space matrices of each sub-subsystem are written as a convex combinations of the polytope vertices. In the first algorithm, the state-feedback controller are designed with vertex-independent Lyapunov functions, while, in the second theorem, these are assumed to be vertex-dependent for reduced conservatism. In both cases, the control law consists of linear time-varying (possibly time-invariant) state-feedback controllers that are switched according to external signals and for which switching between state-feedback controllers is subject to a dwell-time constraint. The conditions of the Theorems are given in terms of linear matrix inequalities (LMIs).

The proposed switched $\mathcal{H}_2-$state-feedback synthesis algorithms are applied to the design a longitudinal switched state-feedback laws for the ADMIRE fighter benchmark model Forssell and Nilsson (2005). There is an extensive literature on feedback controller designs for aircraft systems. To cite a few: Allerhand and Shaked (2013-2) applied a switched control technique to control the short-period mode of a F4E fighter aircraft. Cheng et al. (2018) proposes an asynchronous switching technique that enables switching with lag. Finite-time sliding mode and super-twisting control for fighter aircraft control are used in Raj et al. (2018). Sidoryuk et al. (2007); Ameho and Prempain (2011) present linear parameter-varying controllers for the ADMIRE benchmark model.

In this paper, switched $\mathcal{H}_2-$ control laws are designed to track a load factor command for the ADMIRE aircraft model. These are based on state-feedback controllers which are switched according to Mach number and altitude so that the overall switched control laws cover a large portion of the flight envelope.

The paper is organized as follows. The minimum dwell-time stability principles which will be used in the sequel are presented in the next section. New switched $\mathcal{H}_2$ state-feedback controller synthesis conditions are given in the third section. The fourth section introduces the aircraft fighter longitudinal control problem and describes the design of a switched integral state-feedback control law. Simulation results are given and discussed in the fifth section. Concluding remarks are given at the end of the paper.

Notation: $Tr(.)$ denotes matrix trace. The hermitian operator $He\{.\}$ is defined as $He\{A\} = A + A'$. For symmetric matrices, $P > 0 (\geq 0)$ indicates that P is positive definite (semi-definite). A symmetric matrix $\begin{bmatrix} P & Q \\ Q' & P \end{bmatrix}$ is denoted by $\begin{bmatrix} P & Q \\ * & P \end{bmatrix}$. $\mathbb{R}^n$ stands for the n-dimensional Euclidean space. $\mathbb{R}^{p \times q}$ is the set of $p \times q$ real matrices.

Preliminaries

Throughout we consider the switched, time-varying, system defined as:

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}(t)x(t) + B_{u,\sigma(t)}(t)u(t) + B_{w,\sigma(t)}(t)w(t), \quad x(0)=0,\\ z(t) &= C_{\sigma(t)}(t)x(t) + D_{u,\sigma(t)}(t)u(t) + D_{w,\sigma(t)}(t)w(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^l$ is an exogenous disturbance input and $z(t) \in \mathbb{R}^p$ is the output error. $\sigma(t)$ is the switching rule defined for $t \geq 0$. There are M time-varying subsystems and so the time-dependent switching $\sigma(t)$ takes values in $\{1, \ldots, M\}$. The system matrices of the switched system are assumed to reside in the union of the subsystems' polytopes $\Omega_i(t)$:

$$\Omega(t) = \bigcup_{i=1}^{M} \Omega_i(t) \tag{2}$$

where the $\Omega_i(t)$ are defined as:

$$\Omega_i(t) = \left[\begin{array}{c|cc} A_i(t) & B_{u,i}(t) & B_{w,i}(t) \\ \hline C_i(t) & D_{u,i}(t) & D_{w,i}(t) \end{array}\right] = \sum_{j=1}^{N} \eta_j(t)\Omega_i^{(j)}, \tag{3}$$

where

$$\sum_{j=1}^{N} \eta_j(t) = 1,\ \eta_j(t) \geq 0,\ t \geq 0 \tag{4}$$

and

$$\Omega_i^{(j)} = \left[\begin{array}{c|cc} A_i^{(j)} & B_{u,i}^{(j)} & B_{w,i}^{(j)} \\ \hline C_i^{(j)} & D_{u,i}^{(j)} & D_{w,i}^{(j)} \end{array}\right] \tag{5}$$

In the definition above, i indexes the sub-polytopes and ranges from 1 to M. Each sub-polytope is assumed to have N vertices indexed by integer j. The sub-system matrices are expressed as convex combinations of the sub-polytope vertices. We will assume that the sub-polytopes overlap and that the polytopic coordinates $\eta_j(t) > 0$, $j = 1, \ldots, N$, time rates are bounded.

Definition 1. *If the system's switching instants* $(t_1, t_2, \dots)$ *satisfy*

$$t_{k+1} - t_k \geq T_d,\ k = 0, 1, \dots \tag{6}$$

for some $T_d > 0$ *then* T_d *is called a dwell-time for the system. The smallest* T_d *for which the global stability of the switched can be guaranteed, for any possible switching sequence, is called the minimum dwell-time and is denoted* T*.*

Definition 2. *The time interval between the switching instant* t_k *and* $t_k + T$ *is divided into* $H + 1$ *equally spaced instants, denoted* $t_{k,h}$*, as shown in Figure 1.*

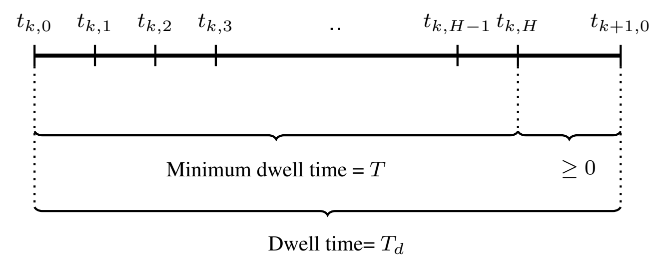

Figure 1. Definition of the time instants $t_{k,h} := t_k + hT/H$, $h = 0, \dots, H$, $t_{k,0} := t_k$ for all k

Stability Analysis

Throughout, we will use Lyapunov functions of the form:

$$V(t, x) = x'(t) P_i(t) x(t), \qquad i = 1, \dots, M \tag{7}$$

where the matrices $P_i(t)$ are assumed to vary linearly in the intervals $t_{k,h} \leq t \leq t_{k,h+1}$, for all $h = 0, \dots, H-1$ (Figure 1). The Lyapunov function $V(t, x)$ must satisfy the following two conditions:

Condition 1. Positive-Definiteness. *If the matrices* $P_i(t)$*,* $i = 1, \dots, M$*, are symmetric positive-definite then the positive-definiteness of* $V(t, x)$ *is automatically guaranteed.*

Condition 2. Time decreasing. *Over the time interval* $t_k \leq t < t_{k+1}$*, for which the* i*-th system is active, the condition* $\dot{V}(t, x) < 0$ *for all* $x \in \mathbb{R}^n$*,* $x \neq 0$*, reduces to the system of matrix* *inequalities:*

$$\dot{P}_i(t) + He\{P_i(t)A_i^{(j)}\} < 0,\, j = 1, \ldots, N \tag{8}$$

The matrix $P_i(t)$, for $t_{k,h} \le t < t_{k,h+1}$, can be written in terms of a convex sum of two constant matrices $P_{i,h}$ and $P_{i,h+1}$ as follows:

$$\begin{aligned} &P_i(t) = r_1(t)P_{i,h} + r_2(t)P_{i,h+1}, \quad 0 \le r_1(t) \le 1, \\ &0 \le r_2(t) \le 1, \quad r_1(t) + r_2(t) = 1, \quad r_2(t) \triangleq t - t_{k,h}/(t_{k,h+1} - t_{k,h}) \end{aligned} \tag{9}$$

From (9) one can see that $P_i(t)$ is positive-definite if both $P_{i,h}$ and $P_{i,h+1}$ are positive-definite matrices. The time derivative of $P_i(t)$ is the constant matrix:

$$\dot{P}_i(t) = \frac{P_{i,h+1} - P_{i,h}}{t_{k,h+1} - t_{k,h}} = \frac{P_{i,h+1} - P_{i,h}}{\frac{T}{H}}. \tag{10}$$

Thus, for each $i \in \{1, \ldots, N\}$, satisfaction of (8) reduces to

$$\frac{P_{i,h+1} - P_{i,h}}{\frac{T}{H}} + He\{P_{i,h}A_i^{(j)}\} < 0, \tag{11}$$

$$\frac{P_{i,h+1} - P_{i,h}}{\frac{T}{H}} + He\{P_{i,h+1}A_i^{(j)}\} < 0, \tag{12}$$

$$j = 1, \ldots, M \tag{13}$$

$$P_{i,h+1} > 0,\, P_{i,h} > 0,\, h = 0, \ldots, H-1 \tag{14}$$

Between two consecutive switching instants t_k and t_{k+1}, in which the ith sub-system is active, the Lyapunov function (7) can be written as:

$$V(t,x) = x'(t)P_i(t)x(t) \tag{15}$$

with

$$P_i(t) = \begin{cases} P_{i,h} + (P_{i,h+1} - P_{i,h})\frac{(t - t_{k,h})}{T/H} & t \in [t_{k,h}, t_{k,h+1}), \\ P_{i,H} & t \in [t_{k,H}, t_{k+1}). \end{cases} \tag{16}$$

for $h = 0, \ldots, H-1$. The matrix $P_i(t)$ changes linearly from $P_{i,h}$ to $P_{i,h+1}$ for $t \in [t_{k,h}, t_{k,h+1})$. After $t = t_{k,H}$ and before the next switching instant, the matrix $P_i(t)$ is held constant and equal to $P_{i,H}$. Note that a large H provides less conservative conditions but at the expense of an increased computational complexity Allerhand and Shaked (2011). Based on the time-varying Lyapunov function (15), the following lemma provides the basic stability analysis tool for system (1)-(5).

Lemma 1. Allerhand and Shaked (2011). *For a given $T > 0$, if there exist positive definite matrices $P_{i,h}$, $i = 1, \dots, M$, that satisfy the following system of LMIs:*

$$\begin{aligned} &\tfrac{P_{i,h+1}-P_{i,h}}{T/H} + He\{P_{i,h}A_i^{(j)}\} < 0, \\ &\tfrac{P_{i,h+1}-P_{i,h}}{T/H} + He\{P_{i,h+1}A_i^{(j)}\} < 0, \end{aligned} \tag{17a}$$

where $h = 0, \dots, H-1,$

$$He\{P_{i,H}A_i^{(j)}\} < 0, \tag{17b}$$

$$P_{i,H} - P_{s,0} \geq 0,\ s \in \{1, \dots, M\},\ s \neq i, \tag{17c}$$

for all $j = 1, \dots, N$, then the uncertain system (1) - (5) *is globally asymptotically stable for any time-dependent switching law with dwell-time larger than or equal to T.*

Note that, before the first switching instant, the Lyapunov function decreases thanks to conditions (17b). During the time interval $t_k \leq t \leq t_k + T$, the conditions (17a) which come from (11), ensure that the Lyapunov function decreases monotonically. Then, the conditions (17b) guarantee that $V(t, x)$ decreases after $t_k + T$ and before the next switching instant. The Lyapunov function is ensured to be non-increasing between any arbitrary switching instants by virtue of the conditions (17c).

In Lemma 1, the same Lyapunov matrices (16) are used for each polytopic subsystem. A less conservative approach consists of using *parameter or vertex-dependent Lyapunov matrices* which depend both on time and on the subsystems vertices. To this end, we consider the following time-varying and parameter dependent Lyapunov function, $V(t, x)$ defined as:Allerhand and Shaked (2011)

$$\begin{aligned} &V(t, x) = x'(t)P_{\sigma(t)}x(t), \\ &P_{\sigma(t)} = \textstyle\sum_{j=1}^{N} \eta_j P_{\sigma(t)}^{(j)}, \quad \sum_{j=1}^{N} \eta_j = 1, \quad \eta_j \geq 0, \end{aligned} \tag{18}$$

where $P_i^{(j)}(t)$ is defined as:

$$P_i^{(j)}(t) = \begin{cases} P_{i,h}^{(j)} + (P_{i,h+1}^{(j)} - P_{i,h}^{(j)})\frac{(t-t_{k,h})}{T/H} & t \in [t_{k,h}, t_{k,h+1}), \\ P_{i,H}^{(j)} & t \in [t_{k,H}, t_{k+1}). \end{cases} \quad (19)$$

The stability of the uncertain switched system (1)-(5) can be tested with the following lemma:

Lemma 2. Allerhand and Shaked (2011). *For a given $T > 0$, if there exist matrices $S_{i,h}$, $G_{i,h}$ of compatible dimensions and positive-definite matrices $P_{i,h}^{(j)}$, $i = 1, \ldots, M$, $h = 0, \ldots, H$, $j = 1, \ldots, N$ such that, for all $i = 1, \ldots, M$, $j = 1, \ldots, N$ and $h = 0, \ldots, H-1$:*

$$\begin{bmatrix} \frac{P_{i,h+1}^{(j)} - P_{i,h}^{(j)}}{T/H} + He\{S_{i,h}A_i^{(j)}\} & P_{i,h}^{(j)} - S_{i,h} + A_i^{(j)\prime} G_{i,h}' \\ * & -G_{i,h}' - G_{i,h} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \frac{P_{i,h+1}^{(j)} - P_{i,h}^{(j)}}{T/H} + He\{S_{i,h+1}A_i^{(j)}\} & P_{i,h+1}^{(j)} - S_{i,h+1} + A_i^{(j)\prime} G_{i,h+1}' \\ * & -G_{i,h+1}' - G_{i,h+1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} He\{S_{i,H}A_i^{(j)}\} & P_{i,H}^{(j)} - S_{i,H} + A_i^{(j)\prime} G_{i,H}' \\ * & -G_{i,H}' - G_{i,H} \end{bmatrix} < 0,$$

$$P_{i,H}^{(j)} - P_{s,0}^{(j)} \geq 0, \qquad \forall s \in \{1, \ldots, M\} \text{ and } s \neq i,$$

then the uncertain system (1)-(5) is globally asymptotically stable for any time-dependent switching rule with dwell-time greater than or equal to T.

The proof is straightforward and based on the application of the Finsler Lemma with the parameter-dependent Lyapunov function (18). The next section uses the preceding lemmas, namely Lemmas 1 and 2, to construct switched sub-optimal $\mathcal{H}_2$ state-feedback control laws for the uncertain switched system (1)-(5).

Robust Controller Design

We introduce the following well-known $\mathcal{H}_2-$ result (see e.g. Scherer, Weiland (2000)).

Lemma 3. $\mathcal{H}_2-$optimal state-feedback control. *Consider the linear time-invariant closed-loop system:*

$$\begin{aligned} \dot{x} &= (A + B_u K)x + B_w w, \\ z &= (C + D_u K)x, \end{aligned} \quad (21)$$

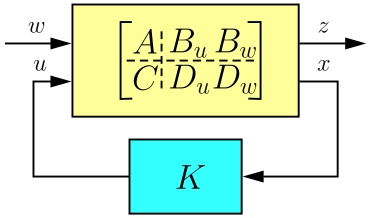

Figure 2. State-feedback control configuration.

where w is the exogenous input, z is the system output error and K is a constant state-feedback gain. If there exist symmetric positive-definite matrices $Q > 0$, $Z > 0$, matrix Y of compatible dimensions and a scalar υ such that the following inequalities hold:

$$
\begin{gathered}
Tr(Z) < \upsilon^2, \begin{bmatrix} Z & B'_w \\ * & Q \end{bmatrix} > 0, \\
\begin{bmatrix} He\{AQ + B_u Y\} & QC' + Y'D'_u \\ * & -I \end{bmatrix} < 0
\end{gathered}
\tag{22}
$$

then the state-feedback control law $u = Kx$ with $K = YQ^{-1}$ is stabilizing and guarantees that the closed-loop $\mathcal{H}_2$-gain is less than or equal to $\sqrt{Tr(Z)}$.

Remark: $Tr(Z)$ is an upper bound for the closed-loop system's $\mathcal{H}_2-$ gain.

The next result generalises Lemma 3 to the switched system (1)-(5).

Theorem 1. *For a given $T > 0$, if there exist matrices $Y_{i,h}$ of compatible dimensions, symmetric positive-definite matrices $Z > 0$ and $Q_{i,h} > 0$ that solve for all $i = 1, \ldots, M$ and*

$j = 1, \ldots, N$, the following LMIs minimisation program:

$$
\begin{aligned}
& minimise\, Tr(Z) \\
& \qquad subject\ to \\
& \qquad\qquad \begin{bmatrix} Z & B_{w,i}^{(j)\prime} \\ * & Q_{i,h} \end{bmatrix} > 0, \\
& \qquad\qquad \begin{bmatrix} \frac{Q_{i,h}-Q_{i,h+1}}{T/H} + He\{A_i^{(j)} Q_{i,h} + B_{u,i}^{(j)} Y_{i,h}\} & Q_{i,h} C_i^{(j)\prime} + Y_{i,h}^{\prime} D_{u,i}^{(j)\prime} \\ * & -I \end{bmatrix} < 0, \\
& \qquad\qquad \begin{bmatrix} \frac{Q_{i,h}-Q_{i,h+1}}{T/H} + He\{A_i^{(j)} Q_{i,h+1} + B_{u,i}^{(j)} Y_{i,h+1}\} & Q_{i,h+1} C_i^{(j)\prime} + Y_{i,h+1}^{\prime} D_{u,i}^{(j)\prime} \\ * & -I \end{bmatrix} < 0, \\
& \qquad\qquad \text{where } h = 0, \ldots, H-1, \\
& \qquad\qquad \begin{bmatrix} He\{A_i^{(j)} Q_{i,H} + B_{u,i}^{(j)} Y_{i,H}\} & Q_{i,H} C_i^{(j)\prime} + Y_{i,H}^{\prime} D_{u,i}^{(j)\prime} \\ * & -I \end{bmatrix} < 0, \\
& \qquad\qquad Q_{i,H} - Q_{s,0} \leq 0,\ s = 1, \ldots, M,\ s \neq i
\end{aligned}
$$

then the switched state-feedback control law, $u = K_{\sigma(t)} x$, with $K_i(t) \in \{K_1(t), \ldots, K_M(t)\}$ defined as:

$$
K_i(t) = \begin{cases} \bar{Y}_{i,h}\, \bar{Q}_{i,h}^{-1} & t \in [t_{k,h}, t_{k,h+1}) \\ Y_{i,H}\, Q_{i,H}^{-1} & t \in [t_{k,H}, t_{k+1,0}) \end{cases} \tag{24}
$$

where $\bar{Y}_{i,h} = Y_{i,h} + (Y_{i,h+1} - Y_{i,h})\frac{(t-t_{k,h})}{T/H}$ and $\bar{Q}_{i,h} = Q_{i,h} + (Q_{i,h+1} - Q_{i,h})\frac{(t-t_{k,h})}{T/H}$, is globally stabilizing and guarantees that the closed-loop $\mathcal{H}_2$-gain (i.e. the feedback association of the uncertain switched system (1)-(5) with the feedback gains given in (24)) is less than or equal to $\sqrt{Tr(Z)}$ for any time-dependent switching rule with dwell-time greater than T.

Proof. The proof is based on the Lyapunov function given in (7). Closed-loop stability implies that:

$$
\dot{P}_i(t) + He\{P_i(t) A_{cl,i}^{(j)}\} < 0
$$

where $A_{cl,i}^{(j)}$ denotes the vertices of the evolution matrix of the closed-loop system obtained from the feedback interconnection of sub-system i and the gain $K_i(t)$ defined above. To avoid couplings between the Lyapunov and the controller matrix variables, the above inequalities are multiplied by $P_i^{-1}(t) := Q_i(t)$ from the left- and right-hand sides and become

$$- \dot{Q}_i(t) + He\{A_i^{(j)} Q_i(t) + B_{u,i}^{(j)} Y_i(t)\} < 0 \tag{25}$$

where $Y_i(t) = K_i(t)Q_i(t)$ and where the time-varying, positive-definite matrices, $Q_i(t) \in \{Q_1(t), \ldots, Q_M(t)\}$, are defined as:

$$Q_i(t) = \begin{cases} Q_{i,h} + (Q_{i,h+1} - Q_{i,h})\frac{(t-t_{k,h})}{T/H} & t \in [t_{k,h}, t_{k,h+1}), \\ Q_{i,H} & t \in [t_{k,H}, t_{k+1}) \end{cases} \tag{26}$$

where $h = 0, 1, \ldots, H-1$. Substituting these into Lemma 1 provides conditions that guarantee closed-loop stability and these conditions used in conjunction with Lemma 3 conditions' structure provide the conditions of the theorem. This completes the proof.

Remark: Theorem 1 provides time-varying state-feedback gains $K_i(t)$ which in most cases might not be desirable practically. Time-invariant state-feedback gains are often preferred. These simpler time-invariant state-feedback gains can constructed if $Y_{i,h}$ and $Q_{i,h}$ are selected independent of h.

The next result is similar but uses vertex-dependent Lyapunov matrices for reduced conservatism.

Theorem 2. *For given scalars $\beta > 0$ and $T > 0$, if there exist matrices $S_{i,h}$, $Y_{i,h}$ and symmetric positive-definite matrices Z and $Q_{i,h}^{(j)}$ that solve for all $i = 1, \ldots, M$ and $j = 1, \ldots, N$ the* *following LMIs minimisation program:*

$$
\begin{aligned}
& minimise\, Tr(Z) \\
& \quad subject\ to \\
& \begin{bmatrix} Z & B_{w,i}^{(j)\,\prime} \\ * & Q_{i,h}^{(j)} \end{bmatrix} > 0, \qquad \forall h \in \{0,\dots,H\}, \\
& \left[\begin{array}{ccc} \frac{Q_{i,h}^{(j)}-Q_{i,h+1}^{(j)}}{T/H} + He\{A_i^{(j)}S'_{i,h} + B_{u,i}^{(j)}Y_{i,h}\} & S'_{i,h}C_i^{(j)\,\prime} + Y'_{i,h}D_{u,i}^{(j)\,\prime} & \dots \\ * & -I & \\ * & * & \end{array} \right. \\
& \qquad \left. \begin{array}{c} Q_{i,h}^{(j)} - S'_{i,h} + \beta A_i^{(j)}S_{i,h} + \beta B_{u,i}^{(j)}Y_{i,h} \\ \beta C_i^{(j)}S_{i,h} + \beta D_{u,i}^{(j)}Y_{i,h} \\ -\beta S'_{i,h} - \beta S_{i,h} \end{array} \right] < 0, \\
& \left[\begin{array}{ccc} \frac{Q_{i,h}^{(j)}-Q_{i,h+1}^{(j)}}{T/H} + He\{A_i^{(j)}S_{i,h+1} + B_{u,i}^{(j)}Y_{i,h+1}\} & S'_{i,h+1}C_i^{(j)\,\prime} + Y'_{i,h+1}D_{u,i}^{(j)\,\prime} & \dots \\ * & -I & \\ * & * & \end{array} \right. \\
& \qquad \left. \begin{array}{c} Q_{i,h+1}^{(j)} - S'_{i,h+1} + \beta A_i^{(j)}S_{i,h+1} + \beta B_{u,i}^{(j)}Y_{i,h+1} \\ \beta C_i^{(j)}S_{i,h+1} + \beta D_{u,i}^{(j)}Y_{i,h+1} \\ -\beta S'_{i,h+1} - \beta S_{i,h+1} \end{array} \right] < 0,
\end{aligned}
$$

where $h = 0,\dots,H-1$

$$
\begin{aligned}
& \left[\begin{array}{ccc} He\{A_i^{(j)}S_{i,H} + B_{u,i}^{(j)}Y_{i,H}\} & S'_{i,H}C_i^{(j)\,\prime} + Y'_{i,H}D_{u,i}^{(j)\,\prime} & \dots \\ * & -I & \\ * & * & \end{array} \right. \\
& \qquad \left. \begin{array}{c} Q_{i,H}^{(j)} - S'_{i,H} + \beta A_i^{(j)}S_{i,H} + \beta B_{u,i}^{(j)}Y_{i,H} \\ \beta C_i^{(j)}S_{i,H} + \beta D_{u,i}^{(j)}Y_{i,H} \\ -\beta S'_{i,H} - \beta S_{i,H} \end{array} \right] < 0, \\
& Q_{i,H}^{(j)} - Q_{s,0}^{(j)} \le 0. \qquad \forall s \in \{1,\dots,M\} \text{ and } s \neq i
\end{aligned}
$$

then the switched state-feedback controller $K_{\sigma(t)}(t)$ *with* $K_i \in \{K_1(t),\dots,K_M(t)\}$ *defined as:*

$$
K_i(t) = \begin{cases} \bar{Y}_{i,h}\,\bar{S}_{i,h}^{-1} & t \in [t_{k,h}, t_{k,h+1}) \\ Y_{i,H}\,S_{i,H}^{-1} & t \in [t_{k,H}, t_{k+1}) \end{cases} \tag{28}
$$

where $\bar{Y}_{i,h} = Y_{i,h} + (Y_{i,h+1} - Y_{i,h})\frac{(t-t_{k,h})}{T/H}$ *and* $\bar{S}_{i,h} = S_{i,h} + (S_{i,h+1} - S_{i,h})\frac{(t-t_{k,h})}{T/H}$, *guarantees that the closed-loop system - i.e. the interconnection between system (1)-(5) with the controller (28)- is globally stable and such that the closed-loop* $\mathcal{H}_2$*-gain from* w *to* z *is less than or equal to* $\sqrt{Tr(Z)}$ *for any switching rule with dwell-time greater than or equal to* T.

Remark: The minimisation of $Tr(Z)$ provides an upper bound on the system H_2-norm. Again the theorem provides time-varying state-feedback gains. If $Y_{i,h}$ and $S_{i,h}$ are selected independent of h then the gains become constant and are given by $K_i = Y_i S_i^{-1}$.

Proof. The proof is similar to that of Theorem 1 where the time-varying and parameter-dependent positive-definite Lyapunov matrices $Q^{(j)}_{\sigma(t)}(t) \in \{Q^{(j)}_1(t), \ldots, Q^{(j)}_M(t)\}$ defined as:

$$Q_i^{(j)}(t) = \begin{cases} Q_{i,h}^{(j)} + (Q_{i,h+1}^{(j)} - Q_{i,h}^{(j)})\frac{(t-t_{i,h})}{T/H} & t \in [t_{k,h}, t_{k,h+1}), \\ Q_{i,H}^{(j)} & t \in [t_{k,H}, t_{k+1}) \end{cases} \tag{29}$$

are used instead.

The robust switched state-feedback synthesis procedures are summarised below:

1. A solution of the LMIs of Theorem 1 provides the matrices $Q_{i,h}$ and $Y_{i,h}$; substituting these matrices into (24) gives the time-varying state-feedback controller gains $K_i(t)$.
2. A solution of the LMIs of Theorem 2 provides the matrices $S_{i,h}$ and $Y_{i,h}$ which substituted into (28) give the time-varying state-feedback gains $K_i(t)$.
3. The matrices $S_{i,h}$ and $Y_{i,h}$ of Theorem 2 can be replaced by S_i and Y_i to obtain constant state-feedback controller gains; then solving the modified LMIs of Theorem 2 provides the matrices S_i and Y_i from which the constant state-feedback gains $K_i = Y_i S_i^{-1}$ are obtained.

Application to a Fighter Aircraft

Longitudinal Dynamics of the ADMIRE Model

ADMIRE is a freely and publicly available advanced simulation model of a generic fighter aircraft Forssell and Nilsson (2005). ADMIRE was developed and maintained by the Swedish Defense Research Agency. The aircraft features delta wing, actuated canard configuration, inboard and outboard elevons and thrust vectoring. The model incorporates actuator and sensor models and the simulation package includes trim and linearisation routines. ADMIRE has been used to demonstrate various control techniques such as LPV and switched control in Sidoryuk et al. (2007); Ameho and Prempain (2011); Kemer and Prempain (2014).

We focus attention on the control of the load factor which involves the short-period mode states, namely, the angle of attack, α, and pitch rate q (Table 1). At a given equilibrium flight condition, the linearised short-period model state-space equations are:

$$\begin{aligned} \dot{x}_p &= A_{sp}x_p + B_{sp,u}u + B_{sp,w}w, \\ y &= C_{sp}x_p + D_{sp,u}u + D_{sp,w}w, \end{aligned} \tag{30}$$

where $x_p = [\alpha,\ q]'$ is the model state vector, $u = [\delta_e,\ t_{ss}]'$ is the control input, $y := n_z$ is the load factor output to be controlled and $w = [u_{dist},\ w_{dist}]'$ are the wind disturbance components along the x and z aircraft body axes. The subscript sp in the above stands for short-period. The linear models (30) are obtained with the trim and linearisation tools of the benchmark and are reduced according to the model reduction technique proposed by Queinnec et al. (2002).

Table 1. The longitudinal dynamic parameters.

Symbol	Definition	Unit
α	angle of attack	(deg)
q	pitch rate	(deg/s)
n_z	load factor	(g)
δ_e	elevon deflection	(deg)
t_{ss}	throttle stick setting	(-)

Integral Controller

In order to track the load factor with zero steady-state error to step demands, an integrator is introduced, as shown in Figure 3, so that the tracking error, defined as $e = \int (r - n_z)dt$, satisfies

$$\dot{e} \; := \; r - n_z \; = \; r - C_{sp}x_p - D_{sp,u}u - D_{sp,w}w \tag{31}$$

where r denotes the load factor command. Combining (30) with (31) gives the augmented plant

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A_{sp} & 0 \\ -C_{sp} & 0 \end{bmatrix} x + \begin{bmatrix} B_{sp,u} \\ -D_{sp,u} \end{bmatrix} u + \begin{bmatrix} B_{sp,w} \\ -D_{sp,w} \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ e &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{aligned} \tag{32}$$

with state-vector $x = [\,x_p',\ e\,]'$.

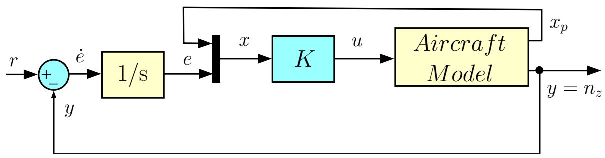

Figure 3. Generic state-feedback with integrator control structure

To construct a switched, polytopic state-space model that covers the flight envelope, the flight envelope is divided into 8 overlapping cells as shown in Figure 4. The number of polytopic sub-systems is thus $M = 8$. Each cell has 4 vertices in which each vertex corresponds to a Mach number and altitude pair, about which an equilibrium state and a linearised aircraft model are computed. Thus, the sub-systems' vertices of the augmented model are:

$$\Omega_i^{(j)} = \left[\begin{array}{c|cc} A_i^{(j)} & B_{u,i}^{(j)} & B_{w,i}^{(j)} \\ \hline C_i^{(j)} & D_{u,i}^{(j)} & D_{w,i}^{(j)} \end{array}\right] \tag{33}$$

$$= \left[\begin{array}{c|cc} \begin{bmatrix} A_{sp,i}^{(j)} & 0 \\ -C_{sp,i}^{(j)} & 0 \end{bmatrix} & \begin{bmatrix} B_{sp,u,i}^{(j)} \\ -D_{sp,u,i}^{(j)} \end{bmatrix} & \begin{bmatrix} B_{sp,w,i}^{(j)} \\ -D_{sp,w,i}^{(j)} \end{bmatrix} \\ \hline C_i^{(j)} & D_{u,i}^{(j)} & 0 \end{array}\right] \tag{34}$$

where, for instance, $A_{sp,i}^{(j)}$ is the evolution matrix of the linearised short-period aircraft model obtained at the altitude and airspeed vertex j of cell i (Figure 4). $C_i^{(j)}$ and $D_{u,i}^{(j)}$ are the performance index weighting matrices selected by the designer and which can be chosen to be the same over a cell.

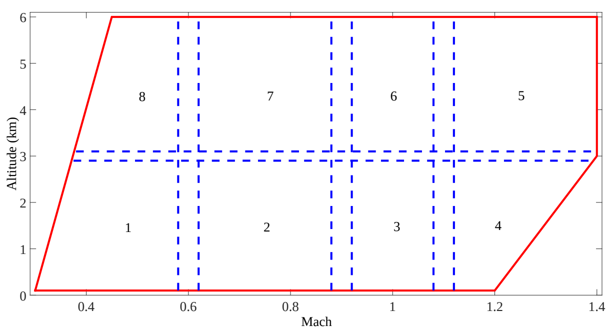

Figure 4. The flight envelope (solid red lines) with overlapping cells (dotted blue lines)

Based on the 32, 3-state vertex models given above (34), the switched $\mathcal{H}_2-$state feedback control laws can now be designed with the synthesis conditions of Theorems 1 and 2. The theorems conditions are programmed and solved using the Matlab YALMIP toolbox Lofberg (2004).

Numerical results

The state-feedback gains of the switched control law are computed by solving the minimisation LMI programs of Theorems 1 and 2. In both cases, the dwell-time was chosen as $T = 0.4\,s$ with $H = 30$ and the line search parameter $\beta = 0.04$ (Theorem 2 only). For these values, the optimisation programs are feasible and give the $\mathcal{H}_2$ performance given in Table 2. Both theorems produce control laws of almost similar performance and, surprisingly, the vertex-dependent approach with constant switched state-feedback gains produces the best performance. Logically, the best performance should have been obtained with the time-varying state-feedback controllers. This unexpected result is probably due numerical errors when solving the LMIs which involve a large number of decision variables and constraints.

Table 2. $\mathcal{H}_2$-gain and Signal to Noise Ratio (SNR).

	$\mathcal{H}_2$-gain	SNR (dB)
Theorem 1	0.7076	9.5515
Theorem 2 time-varying gains	0.7081	9.5899
Theorem 2 constant gains	0.671	9.5563

Non-linear simulation results

The switched state-feedback integral controllers are implemented according to the diagram of Figure 5. Mach number and altitude, which rule the switching, are assumed to be measurable. The non-linear longitudinal model includes actuators dynamics and measurement noise. Here, the sensors measurements are corrupted with white noises of power spectral densities $P_\alpha = 1 \times 10^{-4}$ deg^2/Hz and $P_q = 1 \times 10^{-4}$ deg^2/s.

From an implementation standpoint, when the aircraft altitude and Mach number pair crosses the boundary between adjacent cells and provided that the dwell-time switching constraint is satisfied, then a will switch occur. Such a switch often introduces a discontinuity in the control signal. Thus, to ensure the continuity of the control signal at a switching instant, a constant, v, is added to control signal:

$$u = K(t)x + v, \quad v = u^{+} - u^{-}.$$

where u^{+}, (respectively u^{-}), is the control signal after, (respectively before), the switching instant.

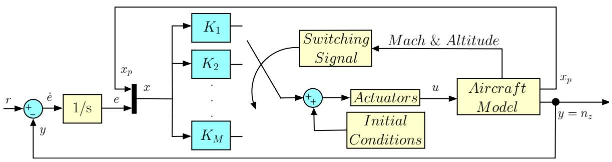

Figure 5. Switched state-feedback control law

The non-linear aircraft model is initialised level-flight at an altitude of 2900m and a Mach number of 0.7 (cells 2, 7 in Figure 4) with the controllers corresponding to cell 2. The output, states and control responses obtained with the switched control laws computed with Theorems 1 and 2 are given in Figures 6, 7 and 8, respectively.

Figures 6 and 7 show very similar closed-loop responses for the three switched control laws. A controllers switch, corresponding to the transition from cell 2 to cell 7, occurs at $t = 4.3$s

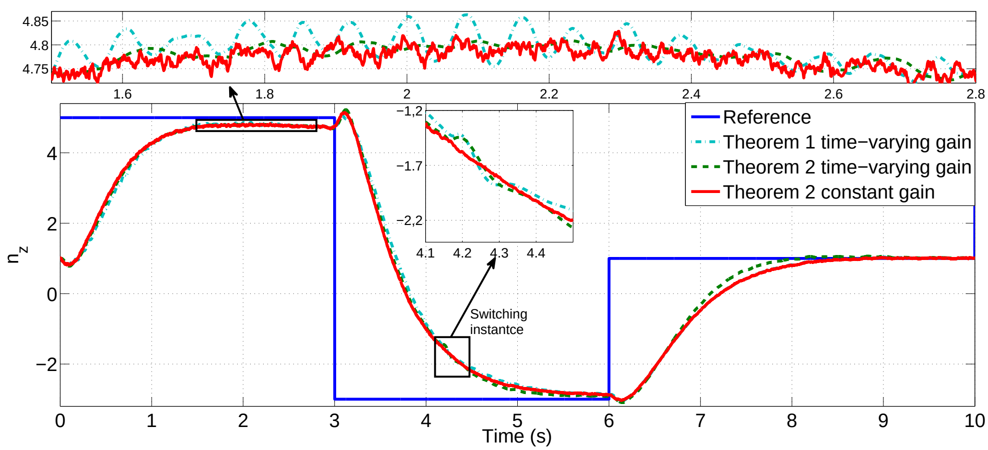

Figure 6. Closed-loop load factor responses with the switched state-feedback control laws

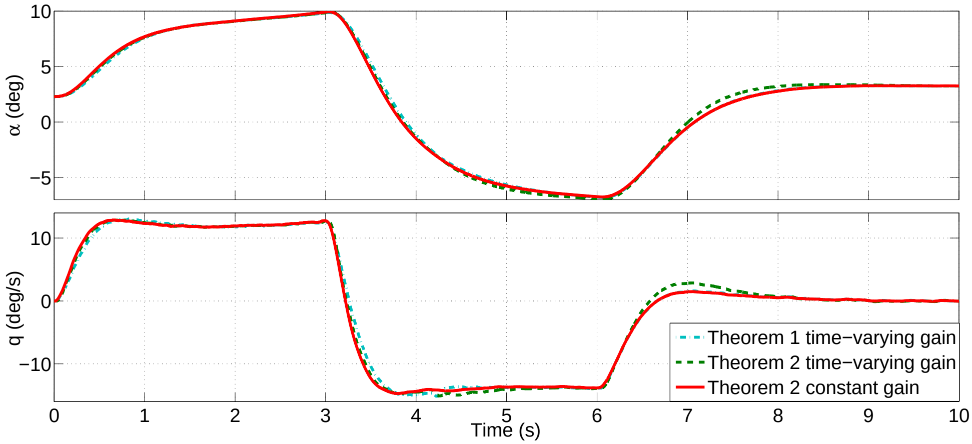

Figure 7. Short-period state responses with the switched state-feedback control laws

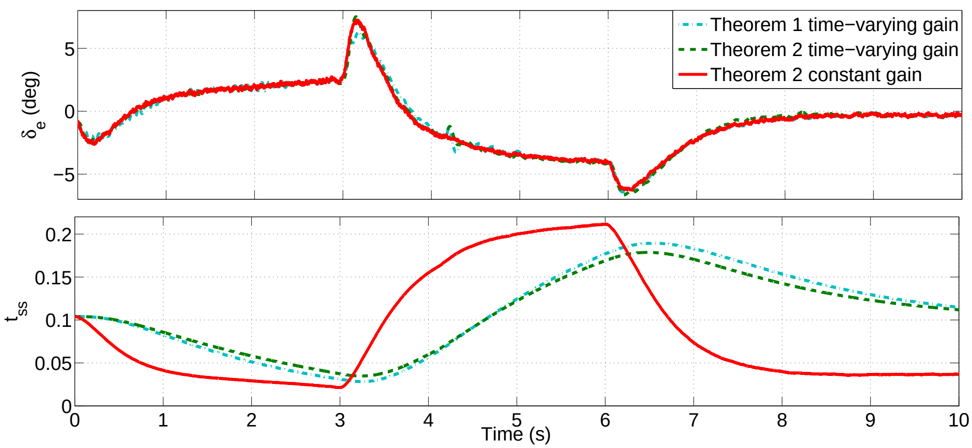

Figure 8. Elevator and throttle responses corresponding to the load factor command of Figure 6.

as indicated in Figure 6. Pitch rate and elevator deflection are slightly affected by the switch as observed in Figures 7 and 8. The law designed with Theorem 1 presents a slightly higher sensitivity to noise than the others as observed in the magnified responses plot at the top of Figure 6. This result is also consistent with the figures of Table 2.

The laws using the time-varying state-feedback controllers exhibit some oscillatory tendencies just after the switching instant, $t = 4.3$s. This issue may be problematic for higher disturbances and/or noise levels. In this case, the time-varying state-feedback gains do not provide any advantage over their time-invariant counterparts. The best switched law, from an implementation viewpoint and also from a computational design complexity, is the one obtained with Theorem 2 which uses constant state-feedback controller gains.

Conclusion

$\mathcal{H}_2$ state-feedback controller synthesis algorithms for uncertain switched linear systems subject to a dwell-time constraint are proposed following the approach of Allerhand and Shaked (2013-3). The novel switched $\mathcal{H}_2-$state-feedback synthesis algorithms are applied to the design a longitudinal switched state-feedback control laws for the ADMIRE fighter benchmark model so that to cover a large portion of the flight envelope. For the ADMIRE benchmark problem, the use of time-varying state-feedback controllers did not bring better performance over the law using constant state-feedback gains. Time domain simulations show the potential of the proposed switched control laws as these laws enable the aircraft to track a load factor command accurately while guaranteeing attenuation against disturbances modelled as white noise signals, over a large flight envelop.

In the future, we will consider delays in the switching signal as these can affect the control performance significantly.

References

Chengzhi Y and Fen W. Almost output regulation of switched linear dynamics with switched exosignals. *International Journal of Robust and Nonlinear Control, Wiley Online Library* 2017; 27(16): 3197-3217.

Lin H and Antsaklis P. Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results. *Automatic control, IEEE Transactions on* 2009;54(2): 308-322.

Johansson M and Rantzer A. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control* 1998; 43(4): 555-559.

Rantzer A and Johansson M. Piecewise linear quadratic optimal control. *IEEE Transactions on Automatic Control* 2000; 45(4): 629-637.

Briat C. Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems. *Nonlinear Analysis: Hybrid Systems,Elsevier* 2017; 24: 198-226.

Geromel JC and Colaneri P. Stability and Stabilization of Continuous-time Switched Linear Systems. *SIAM Journal on Control and Optimization* 2006; 45(5): 1915-1930.

Allerhand LI and Shaked U. Robust State-Dependent Switching of Linear Systems With Dwell Time. *IEEE Transactions on Automatic Control* April 2013; 58(4): 994-1001.

Allerhand LI and Shaked U. Robust Control of Switched Linear Systems with Dwell Time. In: *Electrical and Electronics Engineers in Israel (IEEEI), 26th Convention of IEEE*, 2010, pp.198-201.

Boyarski S and Shaked U. Time-convexity and time-gain-scheduling in finite-horizon robust H_∞-control. In: *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009, Proceedings of the 48th IEEE Conference on. IEEE*, pp. 2765-2770.

Allerhand LI and Shaked U. Robust stability and stabilization of linear switched systems with dwell time. *IEEE Transactions on Automatic Control* 2011; 56(2): 381-386.

Allerhand LI and Shaked U. Robust Control of Linear Systems via Switching. *IEEE Transactions on Automatic Control* 2013; 58(2): 506-512.

Allerhand LI and Shaked U. Robust estimation of linear switched systems with dwell time. *International Journal of Control Taylor & Francis* 2013; 86(11): 2067-2074.

Kemer E and Prempain E. Switched H_2 State-feedback Control for Polytopic Systems Based on Dwell Time. *Journal of Selcuk-Technic* 2018; Special Issue: 117-132

Ameho Y and Prempain E. Linear Parameter Varying Controllers for the ADMIRE Aircraft Longitudinal Dynamics. In: *American Control Conference (ACC)*, 2011, pp. 1315-1320.

Sidoryuk M, Goman M, Kendrick S, et al. An LPV Control Law Design and Evaluation for the ADMIRE Model. In: *Nonlinear Analysis and Synthesis Techniques for Aircraft Control*: Springer-Verlag Berlin Heidelberg, 2007, pp. 197-229.

Kemer E and Prempain E. Switched Control for a Fighter Aircraft. In: *Control (CONTROL), 2014 UKACC International Conference on, IEEE*, pp. 110-114.

Zhou C, Zhu J, Lei H, et al. Robust constraint backstepping control for high-performance aircraft with account of unsteady aerodynamic effects. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering* 2016; 230(13): 2484-2503.

Oktay T and Sultan C. Comfortable Helicopter Flight Via Passive/Active Morphing. *IEEE Transactions On Aerospace and Electronic Systems* 2015; 51: 2876-2886.

Kodhanda A, Kolhe JP, Zeru T, et al. Robust aircraft control based on UDE theory. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering* 2017; 231(4):728-742.

Lee J, Kim S, Jung S, at al. Linear Parameter-Varying Control of Variable Span-Sweep Morphing Aircraft *AIAA Scitech 2019 Forum* 2019; doi: 10.2514/6.2019-0106.

Wen N, Liu Z and Zhu L. Linear-parameter-varying-based adaptive sliding mode control with bounded L_2 gain performance for a morphing aircraft. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering* 2018; doi: 10.1177/0954410018764472.

Oktay T, Çelik H and Türkmen I. Maximizing Autonomous Performance of Fixed-Wing Unmanned Aerial Vehicle to Reduce Motion Blur in Taken Images. *Proceedings Of The Institution Of Mechanical Engineers Part I-Journal Of Systems and Control Engineering* 2018; 232:857-868.

Forssell L and Nilsson U. *ADMIRE the aero-data model in a research environment version 4.0, model description*: FOI-Totalförsvarets forskningsinstitut, 2005.

Queinnec I, Tarbouriech S and Garcia G. *A state-space approach for stability analysis of a combat aircraft in presence of rate limits involving pilot-induced-oscillation*. Technical report, 2002.

Raj K, Muthukumar V, Singh SN, at al. Finite-time sliding mode and super-twisting control of fighter aircraft. *Aerospace Science and Technology* 2018; 82: 487-498.

Cheng H, Fu W, Dong C, at al. Asynchronously finite-time H_∞ control for morphing aircraft. *Transactions of the Institute of Measurement and Control* 2018; 40(16): 4330–4344. doi:10.1177/0142331217746737

Pogromsky AY, Jirstrand M and Spangeus P. On stability and passivity of a class of hybrid systems. In: *IEEE Conference on Decision and Control, IEEE*, 1998, vol. 4, pp. 3705-3710.

Finsler P. Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formem. *Commentarii Mathematici Helvetici* 1937; 9.

Scherer C and Weiland S. *Linear matrix inequalities in control*. Lecture Notes in Dutch Institute for Systems and Control, Delft, The Netherlands, vol. 3. 2000.

Lofberg J. YALMIP: A toolbox for modeling and optimization in MATLAB. In: *Computer Aided Control Systems Design, 2004 IEEE International Symposium on, IEEE*, pp. 284-289.