Stochastic Calculations with Applications to Finance

Kuo Wang

Department of Mathematics

University of Leicester

This thesis is submitted for the degree of

Doctor of Philosophy

August 2019

Acknowledgements

I would like to give my greatest thank to my supervisor, Professor Sergey Utev, who had been a supervisor of mine since my postgraduate studies at University of Leicester and guided me to the world of stochastic calculus. With his incredible breadth and depth of knowledge and intuition, he helped me with understanding knowledge, structuring my thought and suggesting valuable insightful comments. This thesis would not have been possible to write without his help and support.

Meanwhile, I would like to thank numerous friends for their help and support whilst writing this thesis, as well as everyone in the Department of Mathematics, and the College of Science and Engineering for providing me with such a wonderful five years at University of Leicester.

Finally, I would like to thank my parents for always supporting me in every aspect and believing in me. I would not be where I am today without their love and support.

Abstract

This thesis presents a variety of probabilistic and stochastic calculations related to the Ornstein-Uhlenbeck process, the weighted self-normalized sum of exchangeable variables, various operators defined on the Wiener space and Greeks in mathematical finance.

First, we discuss some properties of the weighted self-normalized sum of exchangeable variables. Then we show two methods to compute the different order moments of the Brownian motion via the definition of expactation and the so-called Malliavin calculus, repectively. We also show how to compute the different order moments of the Ornstein-Uhlenbeck process by using Itô calculus and generlize it to the Itô processes of the Ornstein-Uhlenbeck type.

Finally we show how to apply the Malliavin calculus to compute different operators defined on the Wiener space such as the derivative opertor, the divergence opertor, the infinitesimal generator of the Ornstein-Uhlenbeck semigroup and the associated characteristics. We also apply Malliavin calculus to compute Greeks for European options as well as exotic options, where the integration by parts formula provides a powerful tool. In addition, we demonstrate the computation of Greeks for the models where we treat share price Itô martingale models such as W_t and $W_t^2 - t$.

Contents

Chapter 1

Introduction

The purposes of this dissertation are to state a variety of probability calculations and stochastic calculus related to the Ornstein-Uhlenbeck process, the weighted self-normalized sum of exchangeable variables, various operators defined on the Wiener space and Greeks in finance.

The Ornstein-Uhlenbeck process (OUP) has been used in numerous fields, including biology [3, 13], finance [2], and energy market [5], since it was firstly introduced by Uhlenbeck, G. and Ornstein, L. in 1930 [40]. See [14, 9, 39] for a good account on the stochastic calculus for the Ornstein-Uhlenbeck process.

The reason why the weighted self-normalized sum of exchangeable variables is discussed in this thesis, is that first it is motivated by the paper of S. Y. Novak and the supervisor S. Utev [28]. And the second reason is that the ratio of sums of random variables is natural in Greeks, which implies that the techniques used to compute the ratio will be useful. See [7, 12] for more details about the ratio of sums of random variables and weighted sums of random variables.

The Malliavin calculus is an infinite-dimensional differential calculus on the Wiener space, also known as the stochastic calculus of variations. This theory was initiated by Malliavin [25] in 1976, and further developed by Stroock [21–23], Bismut [24], Watanabe [41], Bells [4] and others. The original motivation is based on how to give a probabilistic proof of Hormander's 'sum of squares' theorem. The proof of Hormander's theorem is considered as the most important application of Malliavin calculus.

The discussion about the Malliavin calculus in this thesis can be divided into two parts. The first part is the theory of various differential operators defined on the Wiener space. The second part is some applications of the Malliavin calculus in mathematical finance, that is the computation of Greeks via the Integration by parts formula.

1.1 Structure of the Thesis

The first Chapter of this thesis reviews some basic concepts and preliminary knowledge of probability theory, mainly based on [6, 35, 29, 42, 17].

By presenting several definitions, examples, and selected proofs in Chapter 2, it covers the probability spaces, the random variables and its the distribution function, the expectation of a random variable in terms of the integration and the convergence for a sequence of random variables in Section 2.1. It covers the stochastic processes and the filtrations, the Brownian motion also known as Wiener process, the stochastic integral and its properties, including mean-zero property, isometry and linearity, Itô process, Itô formula and Itô isometry in Section 2.2.

In Chapter 2, we also review the analysis on the Wiener space, it covers isonormal Gaussian process and its properties, the Hermite polynomial and Wiener chaos in Section 2.3.1, the iterated Itô integrals in Section 2.3.2, the derivative operator D, its associated characteristics and the operator D^h in Section 2.3.3. In addition, an integration-by-parts formula which plays a fundamental role throughtout whole computations in the thesis is presented in Section 2.3.3. In Section 2.3.4, we review the divergence operator δ, which is the adjoint of the derivative operator D and several relative lemmas and propositions. It covers constucting the Ornstein-Uhlenbeck semigroup and Mehler's formula in Section 2.3.5, the generator of the Ornstein-Uhlenbeck semigroup, operator L and the associated characteristics in Section 2.3.6.

Finally, in Chapter 2, we review the financial modelling, especially the Black-Scholes model in Section 2.4.1 and 2.4.2. The Integration by parts formula with its application in computation of price sensitivities (Greeks) is presented in Section 2.4.3. Generally, computation of Greeks is considered as an important application of Malliavin calculus in mathematical finance.

The Chapter 3 discuss the expectation of a randomly weighted self-normalized sum, the value of up to forth moment of an exchangeable random variable and the proof of the convergence in distribution of the the weighted self-normalized sum of exchangeable variables.

In Chapter 4, we present the computations of moments via the definition of expectation in Section 4.1, via the properties of divergence operator and Skorohod integral in Section 4.2, and via the Itô formula in Section 4.3.1. An open question about computing the moments of an Ornstein-Uhlenbeck type process is demonstrated in Section 4.3.2.

In Chapter 5, we present the computations of operator D and D^h in Section 5.1, operator L in Section 5.2, and the norms $\|\cdot\|_L$ as well as $\|\cdot\|_{2,2}$ in Section 5.3. These operators will play essential roles in computing the Greeks in Chapter 6.

In Chapter 6, we present the computations of Greeks for European options in Section 6.1, Greeks for exotic options in Section 6.2, and Greeks for the models where we treat share price Itô martingale models such as W_t and $W_t^2 - t$ in Section 6.3.

1.2 Results Stated in the Thesis

Some calculations and proofs of the convergence in distribution ralated to the weighted self-normalized sum of exchangeable variables are stated in Chapter 3.

Several examples of calculations of various moments of the Brownian motion are stated in Section 4.1 and 4.2.

The properties of the Ornstein-Uhlenbeck process are stated in Section 4.3.1, including its variance, covariance, and the values of up to the forth moment.

Some additional calculations for the Ornstein-Uhlenbeck type process are stated in Section 4.3.2.

Several examples of calculations of the derivative operator D and the operator D^h are stated in Section 5.1.

Several examples of calculations of the operator L, which coincides with the infinitesimal generator of the Ornstein-Uhlenbeck semigroupthe, are stated in Section 5.2.

Several examples of calculations of the the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ are stated in Section 5.3.

Several examples of calculations of Greeks for European options and exotic options (i.e. Delta, Gamma and Vega) are stated in Section 6.1 and 6.2.

Some additional calculations of Greeks for the models where we treat share price Itô martingale models such as W_t and $W_t^2 - t$ are stated in Section 6.3.

Chapter 2

Background and Terminology

2.1 Basic concepts of probability theory

In this section we will recall some basic concepts of the probability theory, based on [6, 35, 29, 42, 17].

Definition 2.1.1. (Probability space) *A probability space associated with a random experiment is a triple* $(\Omega, \mathscr{F}, P)$ *which satisfies:*

1. *The sample space* Ω *is the set of all possible outcomes of the random experiment.*
2. *The* $\sigma-$*algebra* $\mathscr{F}$ *is a set of subsets of* Ω *which satisfies:*
   (a) $\emptyset \in \mathscr{F}$, $\Omega \in \mathscr{F}$;
   (b) *If* $A \in \mathscr{F}$*, then its complement* A^c *belongs to* $\mathscr{F}$*;*
   (c) $A_1, A_2, \ldots \in \mathscr{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathscr{F}$.
3. *The probability measure* P *on the space* $(\Omega, \mathscr{F})$ *is a function which associates a number* $P(A)$ *to each set* $A \in \mathscr{F}$ *with the following properties:*
   (a) $0 \leq P(A) \leq 1$;
   (b) $P(\Omega) = 1$;

(c) *For any sequence $A_1, A_2, \ldots$ of disjoints sets in $\mathscr{F}$ (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$),*

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

Definition 2.1.2. *Each $A \in \mathscr{F}$ will be called an event and $P(A)$ implies the probability that the even A occurs. The set $\emptyset$ is called the empty event with a probability of zero. The set Ω is also called the certain set and its probability is 1.*

Definition 2.1.3. *The probability space $(\Omega, \mathscr{F}, P)$ is called a complete probability space if for each set $A \in \mathscr{F}$ with zero probability $P(A) = 0$, any subset of A is in $\mathscr{F}$.*

Example 2.1.4. (Incomplete space) Consider a sample space $\Omega = \{1,2,3,4,5\}$, and the $\sigma-$algebra $\mathscr{F}$ is generated as follows:

Consider three events $B_i, i = 1,2,3$, where $B_1 = \{1,2,3\}$, $B_2 = \{4\}$ and $B_3 = \{5\}$. And the corresponding probability of each event $B_i, i = 1,2,3$ is given as $P(B_1) = 0$, $P(B_2) = \frac{1}{2}$ and $P(B_3) = \frac{1}{2}$.

And therefore, the $\sigma-$algebra generated by $B_i, i = 1,2,3$ is

$$\mathscr{F} = \{\emptyset, \Omega, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, \{4\}, \{5\}, \{4,5\}\}.$$

As any subsets of $B_1 \notin \mathscr{F}$, then by the definition of complete probability space, we can obtain that the probability space $(\Omega, \mathscr{F}, P)$ is not complete.

Definition 2.1.5. (Random variable) *Consider a function X from the space Ω to the real line $\mathbb{R}$*

$$\Omega \xrightarrow{X} \mathbb{R},$$

where the point ω is mapped to $X(\omega)$

$$\omega \longrightarrow X(\omega).$$

If the function X is $\mathscr{F}$-measurable from $(\Omega, \mathscr{F})$ into $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, where $\mathscr{B}_{\mathbb{R}}$ is the Borel $\sigma-$field, then X is called a random variable on $(\Omega, \mathscr{F})$ with values in $\mathbb{R}$, that is, $X^{-1}(B) \in \mathscr{F}$, for any Borel set B in $\mathbb{R}$.

Note that each value $X(\omega)$ is assigned to the outcome ω in Ω by the random variable X . For any set $A \in \mathscr{B}_{\mathbb{R}}$, $(X \in A)$ or $\{X \in A\}$ will denote the event $X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\}$.

Definition 2.1.6. *Under the measurability condition of the random variables, given two real numbers $a \leq b$, the set of all outcomes ω for which $a \leq X(\omega) \leq b$ is an event. Similarly, the event $\{\omega \in \Omega \mid a \leq X(\omega) \leq b\}$ will be denoted by $(a \leq X \leq b)$ or $\{a \leq X \leq b\}$.*

Definition 2.1.7. *A random variable X defines a $\sigma-$field $\{X^{-1}(B), B \in \mathscr{B}_{\mathbb{R}}\} \subset \mathscr{F}$ called the σ-field generated by X, denoted by $\sigma(X)$, which is the smallest $\sigma-$field which makes X measurable. Moreover the assertion that X is a random variable is equivalent to saying that $\sigma(X) \subset \mathscr{F}$.*

Definition 2.1.8. (Distribution function) *A random variable X with values in $\mathbb{R}$ defines a probability measure on the Borel $\sigma-$field $\mathscr{B}_{\mathbb{R}}$ by $P_X = P \circ X^{-1}$, that is*

$$P_X(B) = P(X^{-1}(B)) = P(\{\omega \ : \ X(\omega) \in B\}).$$

And the probability measure P_X is called the law or the distribution of X. The function $F \ : \ \mathbb{R} \longrightarrow [0,1]$ defined by

$$F(x) = P(X \leq x) = P_X((-\infty, x)),$$

is called the distribution function of the random variable X.

Definition 2.1.9. (Density function) *We will say that a random variable X has a probability density function $f \ : \ \mathbb{R} \longrightarrow \mathbb{R}$, if the function $f(x)$ is non-negative on $\mathbb{R}$, measurable with respect to the Borel $\sigma-$field $\mathscr{B}_{\mathbb{R}}$ and with the property*

$$\int_{\mathbb{R}} f(x) dx = 1,$$

and

$$P(a < X < b) = \int_a^b f(x)dx.$$

The distribution function F is non-decreasing, right continuous and with

$$\lim_{x \longrightarrow -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \longrightarrow +\infty} F(x) = 1.$$

If the random variable X is absolutely continuous with density f, the distribution function F has the property that

$$F(x) = \int_{-\infty}^{x} f(s)ds,$$

and $F'(x) = f(x)$ if the density is continuous.

Definition 2.1.10. (Expactation) *The expected value or mean of a random variable X on $(\Omega, \mathscr{F}, P)$ is defined as the Lebesgue integral of X with respect to the probability measure P:*

$$E(X) = \int_{\Omega} X(\omega)P(d\omega) = \int_{\Omega} X(\omega)dP(\omega),$$

or simply

$$E(X) = \int_{\Omega} XdP.$$

Example 2.1.11. (Indicator function) Suppose that A is an event in a probability space Ω, the random variable

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \text{if} \quad \omega \in A, \\ 0, & \text{if} \quad \omega \notin A. \end{cases}$$

is called the indicator function of A. The probability law of this indicator function is called the Bernoulli distribution with parameter $p = P(A)$. And the expected value of this random variable is $E(\mathbf{1}_A(\omega)) = P(A)$.

Theorem 2.1.12. (Fubini's Theorem) *Suppose that $(X, \mathscr{F}_1, \mu)$ and $(Y, \mathscr{F}_2, \nu)$ are σ–finite measure spaces, $\pi = \mu \otimes \nu$ is the product measure on $\mathscr{F}_1 \otimes \mathscr{F}_2$ and $f : X \times Y \to \mathbb{R}$ is a $\mathscr{F}_1 \otimes \mathscr{F}_2$–measurable function. Then the following three conditions are equivalent:*

$$\int_{X \times Y} |f| d\pi < \infty, \quad i.e. \quad f \in L^1(\pi),$$

$$\int_X \Big(\int_Y |f(x,y)| d\nu(y) \Big) d\mu(x) < \infty \quad and$$
$$\int_Y \Big(\int_X |f(x,y)| d\mu(x) \Big) d\nu(y) < \infty,$$

where $L^1(\pi)$ denotes the Lebesgue space [1] of functions for which the absolute value is Lebesgue integrable with respect to mearure π.

See in [1].

Example 2.1.13. Consider a random variable $X \geq 0$, by definition we know that

$$X = \int_0^{+\infty} \mathbf{1}_{\{X>t\}} dt.$$

The indicator function $\mathbf{1}_{\{X>t\}}$ is in L^1, then by applying Fubini's theorem, we have

$$E(X) = \int_\Omega X dP = \int_\Omega \Big(\int_0^{+\infty} \mathbf{1}_{\{X>t\}} dt \Big) dP = \int_0^{+\infty} \Big(\int_\Omega \mathbf{1}_{\{X>t\}} dP \Big) dt.$$

By using the fact

$$P(X > t) = \int_\Omega \mathbf{1}_{\{X>t\}} dP,$$

we obtain

$$E(X) = \int_0^{+\infty} P(X > t) dt.$$

Lemma 2.1.14. *Given random variable $X : \Omega \longrightarrow \mathbb{R}$ with law P , let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel measurable function and $E(|g(X)|) < \infty$, then it holds*

$$E(g(X)) = \int_\Omega g(X(\omega)) dP(\omega) = \int_{-\infty}^{\infty} g(x) dP_X(x).$$

Moreover if X has the probability density function f, it holds

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

Definition 2.1.15. *A random variable X is said to have a finite moment of order $p \geq 1$, if $E(|X|^p) < \infty$. The pth moment of X is defined by*

$$E(X^p) = \int_{-\infty}^{\infty} x^p dP_X(x), \quad p \in \mathbb{N}.$$

The set of random variables with finite pth moment is denoted by $L^p(\Omega, \mathscr{F}, P)$.

Definition 2.1.16. *The variance of a random variable X is defined by*

$$\text{var}(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2,$$

if the first two moments of X exist and are finite.

The variance of X measures the deviation of X from its expected value.

Definition 2.1.17. *If X and Y are two random variables, the covariance of X and Y is defined by*

$$\begin{aligned} \text{cov}(X,Y) &= E[(X - E(X))(Y - E(Y))] \\ &= E(XY) - E(X)E(Y), \end{aligned}$$

provided $E(|X|^2) < \infty$ and $E(|Y|^2) < \infty$.

By the linearity of the expactation, we obtain

$$\begin{aligned} \text{var}(X+Y) &= E[(X+Y)^2] - (E(X+Y))^2 \\ &= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 + 2E(XY) - 2E(X)E(Y)) \\ &= \text{var}(X) + \text{var}(Y) + 2\text{cov}(X,Y). \end{aligned}$$

Example 2.1.18. Suppose that X is a real valued random variable with $X \geq 0$ almost surely and $p \in \mathbb{N}$, assume that X has the probability density function $f \in L^1$, then by using the definition of expactation and applying the simple fact

$$x^p = \int_0^x p t^{p-1} dt,$$

we derive

$$\begin{aligned}
E(X^p) &= \int_0^\infty x^p f(x)dx \\
&= \int_0^\infty \Big(\int_0^x p t^{p-1} dt \Big) f(x)dx \\
&= \int_0^\infty \Big(\int_0^\infty p t^{p-1} \mathbf{1}_{\{t \le x\}} dt \Big) f(x)dx.
\end{aligned}$$

Then, as the function $f \in L^1$, applying Fubini's theorem, we have

$$\begin{aligned}
E(X^p) &= \int_0^\infty p t^{p-1} \Big(\int_0^\infty \mathbf{1}_{\{t \le x\}} f(x)dx \Big) dt \\
&= \int_0^\infty p t^{p-1} \Big(\int_t^\infty f(x)dx \Big) dt \\
&= \int_0^\infty p t^{p-1} P(X \ge t)dt.
\end{aligned}$$

Definition 2.1.19. *The vector $X = \{X_1, \cdots, X_n\}$ is an $n-$dimensional random vector if its components $X_1, \cdots, X_n$ are random variables. That is, X is a random variable with values in $\mathbb{R}^n$. Then the mathematical expectation of an $n-$dimensional random vector X is the vector*

$$E(X) = (E(X_1), \cdots, E(X_n)).$$

And the covariance matrix of an $n-$dimensional random vector X is the matrix

$$\Gamma_X = (\mathrm{cov}(X_i, X_j))_{1 \le i,j \le n}.$$

Note that the matrix Γ_X is symmetric. Moreover, we know that by the definition of variance and the linearity of expactations

$$\begin{aligned}
\mathrm{var}\Big(\sum_{i=1}^n a_i X_i\Big) &= E\Big(\Big(\sum_{i=1}^n a_i X_i\Big)^2\Big) - \Big(E\Big(\sum_{i=1}^n a_i X_i\Big)\Big)^2 \\
&= E\Big(\sum_{i,j=1}^n a_i a_j X_i X_j\Big) - \sum_{i,j=1}^n E(a_i X_i) E(a_j X_j).
\end{aligned}$$

Then by separating cases $i = j$ and $i \neq j$, we have

$$\begin{aligned}\operatorname{var}\Big(\sum_{i=1}^{n} a_i X_i\Big) &= \sum_{i=1}^{n} (E(a_i^2 X_i^2) - (E(a_i X_i))^2) + \sum_{i \neq j} (E(a_i a_j X_i X_j) - E(a_i X_i) E(a_j X_j)) \\ &= \sum_{i=1}^{n} \operatorname{var}(a_i X_i) + \sum_{i \neq j} a_i a_j \operatorname{cov}(X_i, X_j).\end{aligned}$$

Finally, by rearranging

$$\begin{aligned}\operatorname{var}\Big(\sum_{i=1}^{n} a_i X_i\Big) &= \sum_{i,j=1}^{n} a_i a_j \operatorname{cov}(X_i, X_j) \\ &= \sum_{i,j=1}^{n} \Gamma_X(i,j) a_i a_j \geq 0,\end{aligned}$$

for all real numbers $a_i, \cdots, a_n$. That is, the matrix Γ_X is non-negative definite.

The following definitions are some different types of convergence for a sequence of random variables $X_n, n = 1, 2, 3, \cdots$.

Definition 2.1.20. (Almost sure convergence) $X_n \xrightarrow{a.s.} X$, *if*

$$\lim_{n\to\infty} X_n(\omega) = X(\omega),$$

for all $\omega \notin N$, *where* $P(N) = 0$.

Definition 2.1.21. (Convergence in probability) $X_n \xrightarrow{P} X$, *if*

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0,$$

for all $\varepsilon > 0$.

Definition 2.1.22. (Convergence in mean of order $p \geq 1$**)** $X_n \xrightarrow{L^p} X$, *if*

$$\lim_{n\to\infty} E(|X_n - X|^p) = 0.$$

Definition 2.1.23. (Convergence in law) $X_n \xrightarrow{\mathscr{L}} X$, *if*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for any point x where the distribution function F_X is continuous.

2.2 Stochastic Processes, Brownian Motion, Stochastic integral and Itô calculus

This section will continue with the preliminary knowledge on stochastic processes, based on [6, 35, 29, 42, 17].

Definition 2.2.1. (Stochastic process) *Let $(\Omega, \mathscr{F}, P)$ be the reference probability space, a stochastic process $X = \{X_t, t \in T\}$ with values in the space $E = \mathbb{R}$ is a collection of random variables $X_t : \Omega \longrightarrow E$, $t \in T$ on the same probability space $(\Omega, \mathscr{F}, P)$. The parameter set T is a subset of the real line, and the index $t \in T$ is meant to represent time. The space E is called the state space.*

The stochastic process can also be considered as a measurable mapping:

$$X = X(t, \omega) : T \times \Omega \longrightarrow \mathbb{R}.$$

Definition 2.2.2. *Let $\{X_t, t \in T\}$ be a real-valued stochastic process and $\{t_1, t_2, \cdots, t_n\} \subset T$ satisfying $t_1 < t_2 < \cdots < t_n$, then the probability distribution $P_{t_1,\cdots,t_n} = P \circ (X_{t_1}, \cdots, X_{t_n})^{-1}$ of the random vector*

$$(X_{t_1}, \cdots, X_{t_n}) : \Omega \longrightarrow \mathbb{R}^n$$

is called a finite-dimensional marginal distribution of the process $\{X_t, t \in T\}$.

Definition 2.2.3. *A real-valued process $\{X_t, t \in T\}$ is called a second order process if $E(X_t^2) < \infty$ for all $t \in T$.*

Definition 2.2.4. *Let $\{\mathscr{F}_t, t \in T\}$ be a filtration, i.e., a family of sub-σ-fields of $\mathscr{F}$ increasing in time. That is, if $s < t$ then $\mathscr{F}_s \subset \mathscr{F}_t$.*

In this way, the following interpretation holds true:

"$\mathscr{F}_t$ contains all the information which are available up to time t",

i.e., all the events whose occurrence can be established up to time t.

Definition 2.2.5. *Given a filtration $\{\mathscr{F}_t, t \in T\}$, the σ-algebra*

$$\mathscr{F}_\infty = \bigvee_{t \in T} \mathscr{F}_t = \sigma\Big(\bigcup_{t \in T} \mathscr{F}_t\Big)$$

stands for the limit at infinity, which is determined by the minimal σ-field which contains all $\mathscr{F}_t$. Here, $\sigma(\cdot)$ denotes the σ-algebra generated by a collection of sets.

Definition 2.2.6. (Adapted process) *Given a stochastic process $X = X_t, t \in T$ and a filtration $\{\mathscr{F}_t, t \in T\}$ on $(\Omega, \mathscr{F}, P)$, the process X is said to be adapted to the filtration $\{\mathscr{F}_t, t \in T\}$ if, for any $t \in T$ fixed, the random variable X_t is $\mathscr{F}_t$-measurable on Ω, that is, $X^{-1}(B) \in \mathscr{F}_t$, for any Borel set B in $\mathbb{R}$. Equivalently, we say that X is adapted to $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$.*

Notice that it is always possible to construct a filtration with respect to which the process is adapted, by setting $\mathscr{F}_t^X = \sigma(\mathscr{F}_s, s \leq t)$. $\mathscr{F}_t^X$ is called the natural filtration of X.

Example 2.2.7. (Unadapted case) Consider a sample space $\Omega = \{1,2,3,4,5\}$, and two $\sigma-$algebras $\mathscr{F}_1$ and $\mathscr{F}_2$ on Ω, such that

$$\mathscr{F}_1 = \{\emptyset, \Omega, \{1,2,3\}, \{4,5\}\}$$

and

$$\mathscr{F}_2 = \{\emptyset, \Omega, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, \{4\}, \{5\}, \{4,5\}\}.$$

Then we have $\mathscr{F}_1 \subset \mathscr{F}_2$.

Consider two random variables X and Z, provided that $X(4) = 11$, $X(5) = 22$ and $Z(\{4,5\}) = 33$.

Consider a stochastic process $\{Y_i, i = 1,2\}$ which satisfies $Y_1 = X$ and $Y_2 = Z$, then we can see that $Y^{-1}(11) = \{4\} \not\subset \mathscr{F}_1$ and $Y^{-1}(22) = \{5\} \not\subset \mathscr{F}_1$. This implies that the stochastic process $\{Y_i, i = 1,2\}$ is not adapted to the filtration $\{\mathscr{F}_i, i = 1,2\}$.

Definition 2.2.8. (Brownian motion or Wiener process) *An adapted stochastic process $W = \{W(t), t \geq 0\}$ on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t, t \geq 0\}, P)$ is called a (standard) Brownian motion or a Wiener process if it satisfies:*

1. $W(0) = 0$;
2. *For every $0 \leq s \leq t$ the random variable $W(t) - W(s)$ is independent of $\mathscr{F}_s$;*
3. *For every $0 \leq s \leq t$ the random variable $W(t) - W(s)$ has Gaussian distribution $N(0, t-s)$ with mean zero and variance $t - s$.*

Notice that the time dependence is usually denoted as a subscript, so that $W_t \equiv W(t)$.

Remark 2.2.9.

1. By property 2 we can say that for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ the increments $W_{t_n} - W_{t_{n-1}}, \cdots, W_{t_2} - W_{t_1}$ are independent random variables.
2. The Brownian motion or Wiener process $\{W_t, t \geq 0\}$ is a Gaussian process. Its mean is $E(W_t) = 0$ and the variance is

$$\mathrm{var}(W_t) = E(W_t^2) - (E(W_t))^2 = E(W_t^2) = t.$$

Using the split $W_t = W_t - W_s + W_s$ in to the sum of two independent variables, with the linearity of the expactation, we have the autocovariance functions of the Brownian motion:

$$\begin{aligned} E(W_t W_s) &= E[(W_t - W_s + W_s)W_s] \\ &= E[(W_t - W_s)W_s] + E(W_s^2). \end{aligned}$$

Using the independence and properties of Brownian motion, we obtain

$$E(W_tW_s) \quad = \quad E(W_t - W_s)E(W_s) + E(W_s^2) = s,$$

if $s \leq t$. That is, $E(W_tW_s) = \min(s,t)$.

Definition 2.2.10. **($L^2_{a,T}$ space)** *Suppose that the process $u = u_t, t \in [0,T]$ is a stochastic process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t, t \geq 0\}, P)$. Denote by $L^2_{a,T}$ the space of stochastic processes $u = u_t, t \in [0,T]$, such that:*

1. *u is adapted to $\mathscr{F}_t$ and the mapping $(s, \omega) \longrightarrow u_s(\omega)$ is measurable on the product space $[0,T] \times \Omega$ with respect to the product σ -field $\mathscr{B}_{[0,T]} \times \mathscr{F}$;*
2. $E\left(\int_0^T u_t^2 dt\right) < \infty$.

Note that, the condition 1 requires that random variables of the form $\int_0^t u_s ds$ are $\mathscr{F}_t$-measurable. The condition 2 means that the moment of second order of the process is integrable on the time interval $[0,T]$. In fact, by Fubini's theorem we have

$$E\Big(\int_0^T u_t^2 dt\Big) = \int_0^T E(u_t^2) dt.$$

Also, the condition 2 means that the stochastic process u is a function of two variables (t, ω), which belongs to the Hilbert space $L^2([0,T] \times \Omega)$.

Definition 2.2.11. **(Stochastic integral)** *A simple process u in $L^2_{a,T}$ is a stochastic process of the form:*

$$u_t = \sum_{k=1}^{n} \phi_k \mathbf{1}_{(t_{k-1},t_k]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0,T]$ and ϕ_k are square integrable $\mathscr{F}_{t_{k-1}}$-measurable random variables. The stochastic integral of u with respect to the Brownian motion W_t is defined as

$$\int_0^T u_t dW_t = \sum_{j=1}^{n} \phi_j (W_{t_j} - W_{t_{j-1}}).$$

Note that by Fubini's theorem, for a stochastic process $u = u_t, t \in [0,T]$, if any one of $E\Big(\int_0^T u_t dt\Big)$ and $\int_0^T E(u_t)dt$ is finite then

$$E\Big(\int_0^T u_t dt\Big) = \int_0^T E(u_t)dt.$$

Lemma 2.2.12. *The stochastic integral $\int_0^T u_t dW_t$ defined on the space $L^2_{a,T}$ of simple processes u has the following properties:*

1. *Mean-zero:*
$$E\Big(\int_0^T u_t dW_t\Big) = 0;$$
2. *Isometry:*
$$E\Big[\Big(\int_0^T u_t dW_t\Big)^2\Big] = E\Big(\int_0^T u_t^2 dt\Big);$$
3. *Linearity:*
$$\int_0^T (au_t + v_t)dW_t = a\int_0^T u_t dW_t + b\int_0^T v_t dW_t,$$

   if a, b are constant.

See the lemma in [29] p97 and the proof is modified.

Proof. By the definition, the stochastic integral of u with respect to the Brownian motion W_t is
$$\int_0^T u_t dW_t = \sum_{j=1}^n \phi_j (W_{t_j} - W_{t_{j-1}}).$$

Set random variables $\Delta W_j = W_{t_j} - W_{t_{j-1}}$, which has the Gaussian distribution $N(0, t_j - t_{j-1})$. From the properties of Brownian motion, we know that the random variables $\phi_i\phi_j$, ΔW_i and ΔW_j are independent if $i \neq j$, and the random variable ϕ_i and ΔW_i, ϕ_i^2 and $(\Delta W_i)^2$ are independent if $i = j$, and $E((\Delta W_i)^2) = \text{var}(\Delta W_i) = t_j - t_{j-1}$. That is,

$$E(\phi_i\phi_j\Delta W_i\Delta W_j) = \begin{cases} 0, & i \neq j, \\ E(\phi_j^2(t_j - t_{j-1})), & i = j. \end{cases}$$

Then, we obtain

$$
\begin{aligned}
E\Big(\int_0^T u_t dW_t\Big) &= E\Big(\sum_{j=1}^n \phi_j \Delta W_j\Big) \\
&= \sum_{j=1}^n E(\phi_j)\cdot E(\Delta W_j) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
E\Big[\Big(\int_0^T u_t dW_t\Big)^2\Big] &= E\Big(\sum_{j=1}^n \phi_j \Delta W_j\Big)^2 \\
&= \sum_{i,j=1}^n E(\phi_i\phi_j \Delta W_i \Delta W_j).
\end{aligned}
$$

By independence and rearranging

$$
\begin{aligned}
E\Big[\Big(\int_0^T u_t dW_t\Big)^2\Big] &= \sum_{i,j=1}^n E(\phi_i\phi_j)\cdot E(\Delta W_i \Delta W_j) \\
&= E[\sum_{j=1}^n \phi_j^2 (t_j - t_{j-1})] \\
&= E\Big(\int_0^T u_t^2 dt\Big).
\end{aligned}
$$

Therefore, the mean-zero and isometry properties for simple processes are proved. The linearity can be obtained by simple algebra.

□

Moreover, by applying similar technique for simple processes

$$
u_t = \sum_{k=1}^n \phi_k \mathbf{1}_{(t_{k-1},t_k]}(t)
$$

and

$$
v_t = \sum_{k=1}^n \psi_k \mathbf{1}_{(t_{k-1},t_k]}(t),
$$

we have:

$$E\Big(\int_0^T u_t dW_t \int_0^T v_t dW_t\Big) = \sum_{i,j=1}^{n} E(\phi_i \psi_j \Delta W_i \Delta W_j)$$
$$= E\Big[\sum_{j=1}^{n} \phi_j \psi_j (t_j - t_{j-1})\Big].$$

Finally, by definition we deduce

$$E\Big(\int_0^T u_t dW_t \int_0^T v_t dW_t\Big) = E\Big(\int_0^T u_t v_t dt\Big).$$

Lemma 2.2.13. *If u is a process in the space $L^2_{a,T}$, then there exists a sequence of simple processes $u^{(n)}$ such that*

$$\lim_{n\to\infty} E(\int_0^T |u_t - u_t^{(n)}|^2 dt) = 0.$$

See the lemma in [29] p95 - 96.

Definition 2.2.14. **($L_{a,T}$ space)** *Suppose that the process $u = u_t, t \in [0,T]$ is a stochastic process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t, t \geq 0\}, P)$. Denote by $L_{a,T}$ the space of stochastic processes $u = u_t, t \in [0,T]$, such that:*

1. *u is adapted to $\mathscr{F}_t$ and the mapping $(s,\omega) \longrightarrow u_s(\omega)$ is measurable on the product space $[0,T] \times \Omega$ with respect to the product σ -field $\mathscr{B}_{[0,T]} \times \mathscr{F}$;*

2. *$P(\int_0^T u_t^2 dt < \infty) = 1$.*

Definition 2.2.15. **($L^1_{a,T}$ space)** *Suppose that the process $u = u_t, t \in [0,T]$ is a stochastic process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t, t \geq 0\}, P)$. Denote by $L^1_{a,T}$ the space of stochastic processes $u = u_t, t \in [0,T]$, such that:*

1. *u is adapted to $\mathscr{F}_t$ and the mapping $(s,\omega) \longrightarrow u_s(\omega)$ is measurable on the product space $[0,T] \times \Omega$ with respect to the product σ -field $\mathscr{B}_{[0,T]} \times \mathscr{F}$;*

2. *$P(\int_0^T |u_t| dt < \infty) = 1$.*

Definition 2.2.16. (Itô process) *A continuous and adapted stochastic process $\{X_t, 0 \leq t \leq T\}$ is called an Itô process if it can be represented in the form*

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds,$$

where u belongs to the space $L_{a,T}$ and v belongs to the space $L^1_{a,T}$. The differential form can be written as

$$dX_t = u_t dW_t + v_t dt.$$

See in [29].

Note that the term $\int_0^t u_s dW_s$ is called *Itô integral*, and it satisfies

$$E\Big(\int_0^t u_s dW_s\Big) = 0.$$

Lemma 2.2.17. (Itô formula) *Suppose that X is an Itô process. Let $f(t,x)$ be a function twice defferentiable with respect to the variable x and once differentiable with respect to t, with continuous partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$ (we say that f is of class $C^{1,2}$). Then, the process $Y_t = f(t, X_t)$ is again an Itô process with the representation*

$$\begin{aligned} Y_t = f(0, X_0) &+ \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) u_s dW_s \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, X_s) v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) u_s^2 ds. \end{aligned}$$

And in differential notation, the process Y_t can be written as

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2$ can be computed by using $(dt)^2 = 0$, $dt dW_t = dW_t dt = 0$ and $(dW_t)^2 = dt$.

See the lemma in [29] p106.

Notice that if the process X_t is the Brownian motion W_t, the Itô formula can be represented in the following simple version

$$f(t,W_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s,W_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,W_s)ds.$$

And the corresponding differential version is

$$df(t,W_t) = \frac{\partial f}{\partial t}(t,W_t)dt + \frac{\partial f}{\partial x}(t,W_t)dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,W_t)dt.$$

Lemma 2.2.18 (Itô isometry). *The Itô integral satisfies*

$$E\Big[\Big(\int_0^t u_s dW_s\Big)^2\Big] = E\Big(\int_0^t u_s^2 ds\Big),$$

where u belongs to the space $L_{a,T}$.

See in [29].

Lemma 2.2.19 (Itô formula for two variables[20]**).** *The Itô formula for functions in two variables applied in the computation is the following:*

Suppose that the stochastic process $\{S_t, 0 \le t \le T\}$ is an Itô process of the form

$$S_t = S_0 + \int_0^t u_s dW_s + \int_0^t v_s ds,$$

where u belongs to the space $L_{a,T}$ and v belongs to the space $L^1_{a,T}$.

Let $f(t,x)$ be a function twice differentiable with respect to the variable x and once differentiable with respect to t, with continuous partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$.

Then, the process $Y_t = f(t,S_t)$ is again an Itô process with the differential representation

$$df(t,S_t) = \frac{\partial f}{\partial t}(t,S_t)dt + \frac{\partial f}{\partial S}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(t,S_t)(dS_t)^2.$$

2.3 Analysis on the Wiener Space

This section describes the basic framework for the Malliavin Calculus on the Wiener space. The general context consists of a complete probability space $(\Omega, \mathscr{F}, P)$ and a Gaussian subspace $\mathscr{H}_1$ of $L^2(\Omega, \mathscr{F}, P)$. That is,$\mathscr{H}_1$ is a closed subspace whose elements are zero-mean Gaussian random variables. Often it will be convenient to assume that $\mathscr{H}_1$ is isometric to an L^2 space of the form $L^2(T, \mathscr{B}, \mu)$,where $\mathscr{B}$ is the σ-algebra and μ is a σ-finite measure without atoms. In this way the elements of $\mathscr{H}_1$ can be interpreted as stochastic integrals of functions in $L^2(T, \mathscr{B}, \mu)$ with respect to a random Gaussian measure on the parametor space T (Gaussian white noise).

The section presents several definitions, propositions, lemmas and detailed proofs motivated by Nualart [30, 31], Giulia Di Nunno, Bernt Oksendal, Frank Proske [8] and others [33, 32, 26, 11, 18, 19], and furture explanations and numerous examples are demonstrated in chapter 5. Moreover, they will be illustrated by numerous calculations of Greeks in financial applications in chapter 6.

2.3.1 Wiener Chaos

Suppose that H is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_H$. The norm of an element $h \in H$ will be denoted by $\|h\|_H$. The Hilbert space H which is associated to the Gaussian process W is a general Hilbert space.

Definition 2.3.1. *We say that a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathscr{F}, P)$ is isonormal Gaussian process (or a Gaussian process on H) if W is a centered Gaussian family of random variables such that*

$$E[W(h)W(g)] = \langle h, g \rangle_H$$

for all $h, g \in H$.

From the definition, we have the following properties, see in [30] p4, the proof of these properties is modified.

1. $W(h)$ can be written as the form of $W(h) = \int_0^T h(t)dW_t$, where dW_t is Wiener process. And therefore by Itô isometry we have:

$$\begin{aligned} E[W(h)W(g)] &= E[\int_0^T h(t)dW_t \int_0^T g(t)dW_t] \\ &= E[\int_0^T h(t)g(t)dt] \\ &= \langle h, g\rangle_H. \end{aligned}$$

2. The mapping $h \longrightarrow W(h)$ is linear. In fact, for any $\lambda, \mu \in \mathbb{R}$ and $h, g \in H$, we can obtain that

$$\begin{aligned} &E[(W(\lambda h + \mu g) - \lambda W(h) - \mu W(g))^2] \\ = &\|\lambda h + \mu g\|_H^2 + \lambda^2 \|h\|_H^2 + \mu^2 \|g\|_H^2 \\ &- 2\lambda\langle \lambda h + \mu g, h\rangle_H - 2\mu\langle \lambda h + \mu g, g\rangle_H + 2\mu\lambda\langle h, g\rangle_H \\ = &0. \end{aligned}$$

   The mapping $h \longrightarrow W(h)$ provides a linear isometry of H onto a closed subspace of $L^2(\Omega, \mathscr{F}, P)$ that we will denote by $\mathscr{H}_1$. The elements of $\mathscr{H}_1$ are zero-mean Gaussian random variables.

3. From definition, we can say that each random variables $W(h)$ is Gaussian and centered.

4. By Kolmogorov's theorem [34], given the Hilbert space H we can always construct a probability space and a Gaussian process $\{W(h)\}$ verifying the above conditions.

Given a function $F(x,t) = \exp\left(tx - \frac{t^2}{2}\right)$, by Taylor's theorem, the expansion in powers of t at $t = 0$ is:

$$\begin{aligned} F(x,t) &= \exp\left(\frac{x^2}{2} - \frac{1}{2}(x-t)^2\right) \\ &= 1 + e^{\frac{x^2}{2}} \sum_{n=1}^{\infty} \frac{t^n}{n!}\left(\frac{d^n}{dt^n} e^{-\frac{(x-t)^2}{2}}\right)\Big|_{t=0}. \end{aligned}$$

And therefore we have the definition of the Hermite polynomial.

Definition 2.3.2. (Hermite polynomial) *The nth Hermite polynomial, denoted by $H_n(x)$, is defined in the following way:*

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, n \geq 1,$$

and $H_0(x) = 1$.

See in [30] p4.

Proposition 2.3.3. *For $n \geq 1$, the following properties hold:*

1. $H_n'(x) = H_{n-1}(x)$;
2. $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$;
3. $H_n(-x) = (-1)^n H_n(x)$.

The proof is modified from [30] p5.

Proof. In fact, for the function

$$\begin{aligned} F(x,t) &= \exp\Big(tx - \frac{t^2}{2}\Big) \\ &= 1 + e^{\frac{x^2}{2}} \sum_{n=1}^{\infty} \frac{t^n}{n!} \Big(\frac{d^n}{dt^n} e^{-\frac{(x-t)^2}{2}}\Big)\Big|_{t=0}, \end{aligned}$$

by the definition of the Hermite polynomial, we have

$$F(x,t) = \sum_{n=0}^{\infty} t^n H_n(x).$$

Also we have $H_1(x) = x$ and $H_2(x) = \frac{1}{2}(x^2 - 1)$.

For $n \geq 1$, from the equation of partial derivative

$$\frac{\partial F}{\partial x} = t \cdot \exp\Big(tx - \frac{t^2}{2}\Big) = tF,$$

by plugging in $F(x,t) = \sum_{n=0}^{\infty} t^n H_n(x)$ and rearranging the summation, we can obtain

$$\begin{aligned}\Big(\sum_{n=0}^{\infty} t^n H_n(x)\Big)'_x &= \sum_{n=1}^{\infty} t^n H'_n(x) \\ &= \sum_{n=0}^{\infty} t^{n+1} H_n(x) \\ &= \sum_{n=1}^{\infty} t^n H_{n-1}(x),\end{aligned}$$

which yields

$$H'_n(x) = H_{n-1}(x).$$

Also, we can obtain the equation of partial derivative

$$\frac{\partial F}{\partial t} = (x-t)\cdot \exp\Big(tx - \frac{t^2}{2}\Big) = (x-t)F,$$

that is, by plugging in $F(x,t) = \sum_{n=0}^{\infty} t^n H_n(x)$ and rearranging the summation

$$\begin{aligned}\Big(\sum_{n=0}^{\infty} t^n H_n(x)\Big)'_t &= (n+1)\sum_{n=1}^{\infty} t^n H_{n+1}(x) \\ &= x\sum_{n=1}^{\infty} t^n H_n(x) - \sum_{n=1}^{\infty} t^n H_{n-1}(x),\end{aligned}$$

which yields

$$(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x).$$

Finally, we can obtain the equation

$$F(-x,t) = F(x,-t),$$

that is

$$\begin{aligned}\sum_{n=0}^{\infty} t^n H_n(-x) &= \sum_{n=0}^{\infty} (-t)^n H_n(x) \\ &= \sum_{n=0}^{\infty} (-1)^n t^n H_n(x),\end{aligned}$$

which yields

$$H_n(-x) = (-1)^n H_n(x) \quad .$$

□

Moreover, if n is odd we have $H_n(0) = 0$ and $H_{2k}(0) = \dfrac{(-1)^k}{2^k k!}$ for all $k \geq 1$.

Lemma 2.3.4. *Let X, Y be two random variables with joint Gaussian distribution such that $E(X) = E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$. Then for all $n, m \geq 0$, we have*

$$E(H_n(X)H_m(Y)) = \begin{cases} 0, & n \neq m, \\ \dfrac{1}{n!}(E(XY))^n, & n = m. \end{cases}$$

The proof is modified from [30] p5.

Proof. For fixed $s, t \in \mathbb{R}$, let Z denote the random variable defined by

$$Z = sX - \frac{s^2}{2} + tY - \frac{t^2}{2}$$

By using $E(X) = E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$, we have

$$\begin{aligned} E(Z) &= E\Big(sX - \frac{s^2}{2} + tY - \frac{t^2}{2}\Big) \\ &= -\frac{s^2}{2} - \frac{t^2}{2}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{var}(Z) &= \mathrm{var}(sX + tY) \\ &= s^2 \mathrm{var}(X) + t^2 \mathrm{var}(Y) + 2\mathrm{cov}(sX, tY) \\ &= s^2 + t^2 + 2stE[(X - E(X))(Y - E(Y))] \\ &= s^2 + t^2 + 2stE(XY). \end{aligned}$$

Then we know that $Z \sim N\Big(-\frac{s^2}{2} - \frac{t^2}{2}, s^2 + t^2 + 2stE(XY)\Big)$.

Also by the definition of expectation and the property of density function we know that

$$\begin{aligned}
E(e^R) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^r \cdot e^{-\dfrac{(r-\mu)^2}{2\sigma^2}} dr \\
&= e^{\mu+\frac{\sigma^2}{2}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{(r-(\mu+\sigma^2))^2}{2\sigma^2}} dr \\
&= e^{\mu+\frac{\sigma^2}{2}},
\end{aligned}$$

if the random variable $R \sim N(\mu, \sigma^2)$.

Then we have

$$\begin{aligned}
E\Big[\exp\Big(sX - \frac{s^2}{2}\Big)\exp\Big(tY - \frac{t^2}{2}\Big)\Big] &= E(e^Z) \\
&= \exp\Big(-\frac{s^2}{2} - \frac{t^2}{2} + \frac{1}{2}(s^2 + t^2 + 2stE(XY))\Big) \\
&= \exp(stE(XY)).
\end{aligned}$$

Taking the $(n+m)$th partial derivative $\dfrac{\partial^{n+m}}{\partial s^n \partial t^m}$ at $s = t = 0$ on both sides of the above equation, we can obtain

$$E(n!H_n(X)m!H_m(Y)) = \frac{\partial^m}{\partial t^m}\Big(t^n(E(XY))^n \exp(stE(XY)\Big).$$

That is

$$E(n!m!H_n(X)H_m(Y)) = \begin{cases} 0, & n \neq m, \\ n!\,(E(XY))^n, & n = m. \end{cases}$$

$\square$

From the orthogonality of Hermite polynomial $H_n(x)$, for each $n \geq 1$, the random variables $\{H_n(W(h)), h \in H, \|h\|_H = 1\}$ can generate a closed linear subspaces of $L^2(\Omega, \mathscr{F}, P)$, denote by $\mathscr{H}_n$. The space $\mathscr{H}_0$ will be the set of constants. For $n = 1$, the space $\mathscr{H}_1$ coincides with the set of random variables $\{W(h), h \in H\}$. Again from Lemma 2.3.4 we know that the subspaces $\mathscr{H}_n$ and $\mathscr{H}_m$ are orthogonal if $n \neq m$. The space $\mathscr{H}_n$ is called the Wiener chaos of order n.

2.3.2 Iterated Itô Integrals

Definition 2.3.5. *A real function $g : [0,T]^n \to \mathbb{R}$ is called symmetric if*

$$g(t_{\sigma_1},...,t_{\sigma_n}) = g(t_1,...,t_n)$$

for all permutations $\sigma = (\sigma_1,...,\sigma_n)$ of $(1,2,...,n)$.

See in [8] p8.

Let $L^2([0,T]^n)$ be the standard space of square integrable Borel real functions on $[0,T]^n$ such that

$$\|g\|^2_{L^2([0,T]^n)} \quad := \quad \int_{[0,T]^n} g^2(t_1,...,t_n)dt_1,...,dt_n < \infty.$$

Let $\tilde{L}^2([0,T]^n) \subset L^2([0,T]^n)$ be the space of symmetric square integrable Borel real functions on $[0,T]^n$. Consider the set

$$S_n = \{(t_1,...,t_n) \in [0,T]^n \quad : \quad 0 \leq t_1 \leq ... \leq t_n \leq T\}.$$

Notice that this set S_n occupies the fraction $\dfrac{1}{n!}$ of the whole $n-$dimensional box $[0,T]^n$. Therefore, if $g \in \tilde{L}^2([0,T]^n)$ then $g|S_n \in \tilde{L}^2(S_n)$ and

$$\begin{aligned}\|g\|^2_{L^2([0,T]^n)} &= n! \int_{S_n} g^2(t_1,...,t_n)dt_1,...,dt_n \\ &= n! \, \|g\|^2_{L^2(S_n)}\end{aligned}$$

where $\|\cdot\|_{L^2(S_n)}$ denotes the norm induced by $L^2([0,T]^n)$ on $L^2(S_n)$, the space of the square integrable functions on S_n.

Definition 2.3.6. *If f is a real function on $[0,T]^n$, then its symmetrization $\tilde{f}$ is defined by*

$$\tilde{f}(t_1,...,t_n) = \frac{1}{n!}\sum_{\sigma} f(t_{\sigma_1},...,t_{\sigma_n})$$

where the sum is taken over all permutations $\sigma = (\sigma_1, ..., \sigma_n)$ *of* $(1, 2, ..., n)$.

Then we know $\tilde{f} = f$ if and only if f is symmetric.

Example 2.3.7. The symmetrization of the function

$$f(t_1, t_2) = t_1^3 + t_1 t_2, \quad (t_1, t_2) \in [0, T]^2$$

is

$$\begin{aligned} \tilde{f}(t_1, t_2) &= \frac{1}{2}(f(t_1, t_2) + f(t_2, t_1)) \\ &= \frac{1}{2}(t_1^3 + t_2^3 + 2t_1 t_2), \quad (t_1, t_2) \in [0, T]^2. \end{aligned}$$

The symmetrization of the funtion

$$f(t_1, t_2, t_3) = t_1^2 + t_2 \sin t_3, \quad (t_1, t_2, t_3) \in [0, T]^3$$

is

$$\tilde{f}(t_1, t_2) = \frac{1}{6}(t_1^2 + t_2^2 + t_3^2 + t_1 \sin t_2 + t_2 \sin t_1 + t_1 \sin t_3 + t_3 \sin t_1 + t_2 \sin t_3 + t_3 \sin t_2)$$

where $(t_1, t_2, t_3) \in [0, T]^3$.

Definition 2.3.8. (Iterated Itô integral) *Let* f *be a deterministic function defined on* S_n $(n \geq 1)$ *such that*

$$\|f\|^2_{L^2(S_n)} = \int_{S_n} f^2(t_1, ..., t_n) dt_1, ..., dt_n < \infty.$$

Then we can define the n-fold iterated Itô integral as

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, ..., t_n) dW(t_1) dW(t_2) \cdots dW(t_{n-1}) dW(t_n).$$

See in [8] p8.

Notice that at each iteration $i = 1, 2, ..., n$, the integrand

$$\int_0^{t_i} \cdots \int_0^{t_2} f(t_1, ..., t_n) dW(t_1) \cdots dW(t_{i-1}), \quad t_i \in [0, t_{i+1}]$$

is a stochastic process which is $\mathscr{F}-$adapted ($\mathscr{F}$ is the σ-algebra) and square integrable with respect to $dP \times dt_i$, then the corresponding Itô integral with respect to $dW(t_i)$ is well-defined. And therefore $J_n(f)$ is well-defined.

By the construction of the Itô integral we know that $J_n(f)$ belong to $L^2(P)$, which is the space of square integrable random variables. The norm of $X \in L^2(P)$ is denoted by

$$\|X\|_{L^2(P)} := [E(X^2)]^{\frac{1}{2}} = (\int_\Omega X^2(\omega) P(d\omega))^{\frac{1}{2}}.$$

By appying the Itô isometry iteratively, we can obtain the following result.

Lemma 2.3.9. *The following relations hold true:*

$$E(J_n(g) J_m(f)) = \begin{cases} 0, & n \neq m, \\ (g, f)_{L^2(S_n)}, & n = m. \end{cases}$$

where

$$(g, f)_{L^2(S_n)} = \int_{S_n} g(t_1, ..., t_n) f(t_1, ..., t_n) dt_1, ..., dt_n$$

is the inner product of $L^2(S_n)$. In particular, we have

$$\|J_n(f)\|_{L^2(P)} = \|f\|_{L^2(S_n)}.$$

See proof in [8] p9.

Note that it is straightforward to see that the n-fold iterated Itô integral is a linear operator. That is,

$$J_n(af + bg) = aJ_n(f) + bJ_n(g)$$

for $f, g \in L^2(S_n)$ and $a, b \in \mathbb{R}$.

Definition 2.3.10. *If $f \in \tilde{L}^2([0,T]^n)$, we define*

$$I_n(f) := \int_{[0,T]^n} f(t_1,...,t_n)dW(t_1)\cdots dW(t_n) = n!J_n(f).$$

The $I_n(f)$ is also called the n-fold iterated Itô integral.

See in [8] p10.

Then by definition, we have

$$\begin{aligned}
\|J_n(f)\|_{L^2(P)} &= E(I_n^2(f)) \\
&= E((n!)^2 J_n^2(f)) \\
&= (n!)^2 \|f\|^2_{L^2(S_n)} \\
&= n!\,\|f\|^2_{L^2([0,T]^n)}.
\end{aligned}$$

By the relationship between $J_n(f)$ and $I_n(f)$, from Lemma 2.3.9 , we have the following result.

Lemma 2.3.11. *If $g \in \tilde{L}^2([0,T]^n)$ and $f \in \tilde{L}^2([0,T]^m)$ the following relations hold true:*

$$E(I_n(g)I_m(f)) = \begin{cases} 0, & n \neq m, \\ n!\,(g,f)_{L^2([0,T]^n)}, & n = m. \end{cases}$$

where

$$(g,f)_{L^2([0,T]^n} = \int_{[0,T]^n} g(t_1,...,t_n)f(t_1,...,t_n)dt_1,...,dt_n$$

2.3.3 The Derivative Operator

This section will review the definition and several properties of the derivative operator, based on [30].

Denote $W = \{W(h), h \in H\}$ as an isonormal Gaussian process associated with the Hilbert space H, which defined in a complete probability space $(\Omega, \mathscr{F}, P)$, and $\mathscr{F}$ is generated by W.

Denote $C_p^\infty(\mathbb{R}^n)$ as the set of all infinitely continuously differentiable function $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth.

Definition 2.3.12. (Derivative Operator) *Let $\mathscr{S}$ denote the class of smooth random variables such that a random variable $F \in \mathscr{S}$ has the form*

$$\begin{aligned} F &= f(W(h_1),\cdots,W(h_n)) \\ &= f\Big(\int_0^T h_1(t)dW_t,...,\int_0^T h_n(t)dW_t\Big), \end{aligned}$$

where $W(h_1)=\int_0^T h_1(t)dW_t$ and f belongs to $C_p^\infty(\mathbb{R}^n)$, $h_1,\cdots,h_n$ are in H, and $n \geq 1$.

Then, the derivative of a smooth random variable F is the H-valued random variables given by

$$DF = \sum_{i=1}^n \partial_i f(W(h_1),...,W(h_n))h_i,$$

or

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1(t)),...,W(h_n(t)))h_i(t),$$

where the notation represents $\partial_i f = \dfrac{\partial f}{\partial x_i}$, whenever $f \in C^1(\mathbb{R}^n)$.

Notice that The derivative operator DF can be considered as the derivative of a square integrable random variable $F:\Omega \longrightarrow \mathbb{R}$ with respect to the chance parameter $\omega \in \Omega$. And the derivative DF is defined as the process $\{D_tF, t \geq 0\}$.

The product rule for the derivative operator D is given as follows.

Lemma 2.3.13. *If F,G are smooth random variables, then we can obtain the derivative operator of the product FG*

$$D(FG) = FDG + GDF.$$

By using the integration by parts technique, we can obtain the following result.

Lemma 2.3.14. *Suppose that F is a smooth random variable and $h \in H$, then*

$$E(\langle DF,h\rangle_H) = E(FW(h)).$$

The proof is modified from [30] p26.

Proof. First we can normalize the equation and assume that there exist orthonomal elements of H, $e_1, \cdots, e_n$ such that $h = e_1$ and F is a smooth random variable of the form

$$F = f(W(e_1), \cdots, W(e_n)),$$

where f belongs to $C_p^\infty(\mathbb{R}^n)$. Set $\mathbf{W} = (W(e_1), ..., W(e_n))$, and by definition we have

$$\begin{aligned}\langle W(e_i), W(e_j)\rangle_H &= E[W(e_i)W(e_j)] \\ &= \langle e_i, e_j\rangle_H \\ &= \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}\end{aligned}$$

And

$$E(W(e_i)) = 0, \qquad \text{var}(W(e_i)) = 1, \qquad \text{cov}(W(e_i), W(e_j)) = 0, \quad i \neq j.$$

Therefore, $\mathbf{W} = (W(e_1), ..., W(e_n))$ are independent and identically distributed standard Gaussian random variables.

Set $\mathbf{W} = \mathbf{x} = (x_1, ..., x_n)$ and let $\phi(\mathbf{x})$ denote the density of the standard Gaussian distribution on $\mathbb{R}^n$, that is

$$\phi(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2}\sum_{i=1}^n x_i^2\Big).$$

By definition

$$\begin{aligned}\langle DF, e_1\rangle_H &= \Big\langle \sum_{i=1}^n \partial_i f(W(e_1), ..., W(e_n)) e_i, e_1 \Big\rangle_H \\ &= \sum_{i=1}^n \partial_i f(W(e_1), ..., W(e_n)) \langle e_i, e_1\rangle_H \\ &= \partial_1 f(W(e_1), ..., W(e_n)).\end{aligned}$$

On the other hand, notice that

$$
\begin{aligned}
E[f'(x_1)] &= \int_{-\infty}^{+\infty} f'(x_1)\phi(x_1)dx_1 \\
&= \int_{-\infty}^{+\infty} \phi(x_1)df(x_1) \\
&= 0 - \int_{-\infty}^{+\infty} f(x_1)\phi'(x_1)dx_1 \\
&= \int_{-\infty}^{+\infty} f(x_1)x_1\phi(x_1)dx_1 \\
&= E[f(x_1)x_1].
\end{aligned}
$$

So, we have first by using the definition and $h = e_1$, then by the integration by parts technique,

$$
\begin{aligned}
E(\langle DF, h\rangle_H) &= E(\langle DF, e_1\rangle_H) \\
&= E(\partial_1 f(W(e_1), ..., W(e_n))) \\
&= E(\partial_1 f(\mathbf{x})) \\
&= \int_{\mathbb{R}^n} \partial_1 f(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \\
&= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} \partial_1 f(\mathbf{x})\phi(x_1)dx_1 \Big) \phi(x_2, .., x_n)dx_2...dx_n.
\end{aligned}
$$

By rearranging

$$
\begin{aligned}
E(\langle DF, h\rangle_H) &= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} f(\mathbf{x})x_1\phi(x_1)dx_1 \Big) \phi(x_2, .., x_n)dx_2...dx_n \\
&= \int_{\mathbb{R}^n} f(\mathbf{x})x_1\phi(\mathbf{x})d\mathbf{x} \\
&= E(FW(e_1)) = E(FW(h)).
\end{aligned}
$$

The proof of the lemma is complete. □

By combining the product rule with Lemma 2.3.14, we have the following result.

Lemma 2.3.15. *Suppose that F and G are smooth random variables, and let $h \in H$. Then we have*

$$E(G\langle DF, h\rangle_H) = E(-F\langle DG, h\rangle_H + FGW(h)).$$

The proof is modified from [30] p26.

Proof. By using Lemma 2.3.14, we have

$$\begin{aligned}
E(FGW(h)) &= E(\langle D(FG), h\rangle_H) \\
&= E(\langle FDG + GDF, h\rangle_H) \\
&= E(\langle GDF, h\rangle_H) + E(\langle FDG, h\rangle_H),
\end{aligned}$$

which implies

$$E(G\langle DF, h\rangle_H) = E(-F\langle DG, h\rangle_H + FGW(h)).$$

The proof is complete . □

The following part is the constructure of norm space. See more in [30] p27.

Definition 2.3.16 (Norm space). *For any $p \geq 1$ we will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables* **S** *with respect to the norm*

$$\|F\|_{1,p} = [E(|F|^p) + E(\|DF\|_H^p)]^{\frac{1}{p}}.$$

For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with the scalar product

$$\langle F, G\rangle_{1,2} = E(FG) + E(\langle DF, DG\rangle_H).$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the iterated derivative $D^k F$ is a random variable with values in $H^{\otimes k}$.

Definition 2.3.17. (**Seminorm**) *For every $p \geq 1$ and any natural number $k \geq 1$, the seminorm on* **S** *is defined by*

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{j=1}^{k} E(\|D^j F\|_{H^{\otimes j}}^p)\right]^{\frac{1}{p}}.$$

Lemma 2.3.18. *The family of seminorms verifies the followng properties:*

1. *Monotonicity:* $\|F\|_{k,p} \leq \|F\|_{j,q}$ *for any* $F \in \mathbf{S}$*, if* $p \leq q$ *and* $k \leq j$*.*

2. *Closability: The operator* D^k *is closable from* $\mathbf{S}$ *into* $L^p(\Omega; H^{\otimes k})$ *for all* $p \geq 1$*.*

3. *Compatibility: Let* $p, q \geq 1$ *be real numbers and* k, j *be natural numbers. Suppose that* F_n *is a sequence of smooth random variables such that* $\|F_n j\|_{k,p}$ *converges to zero as* n *tends to infinity, and* $\|F_n - F_m\|_{j,q}$ *converges to zero as* n, m *tend to infinity. Then* $\|F_n\|_{j,q}$ *tends to zero as* n *tends to infinity.*

See the lemma and proof in [30] p27.

We will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathbf{S}$ with respect to the norm $\|\cdot\|_{k,p}$. From the property 1' it follows that $\mathbb{D}^{k+1,p} \subset \mathbb{D}^{k,q}$ if $k \geq 0$ and $p > q$. For $k = 0$ we put $\|\cdot\|_{0,p} = \|\cdot\|_p$ and $\mathbb{D}^{0,p} = L^p(\Omega)$.

Definition 2.3.19. **(D^h operator)** *For a fixed element* h *in the Hilbert space* H*, the operator* D^h *on the set* $\mathbf{S}$ *of smooth random variables is defined as*

$$D^h F = \langle DF, h \rangle_H.$$

This operator is closable from $L^p(\Omega)$ into $L^p(\Omega)$, for any $p \geq 1$, and it has a domain that we will denote by $\mathbb{D}^{h,p}$.

The chain rule for the derivative operator [30] p28 is presented below.

Proposition 2.3.20. *Let* $\phi : \mathbb{R}^m \to \mathbb{R}$ *be a continuously differentiable function with bounded derivatives, and fixed* $p \geq 1$*. Suppose that* $F = (F^1, ..., F^m)$ *is a random vector whose components belong to the space* $\mathbb{D}^{1,p}$*. Then* $\phi(F) \in \mathbb{D}^{1,p}$*, and*

$$D(\phi(F)) = \sum_{i=1}^{m} \partial_i \phi(F) DF^i.$$

See proof in [30] p28.

2.3.4 The Divergence Operator

We will first review the divergence operator in the framework of a Gaussian isonormal process $W = W(h), h \in H$ associated with the Hilbert space H. We assume that W is defined on a complete probability space $(\Omega, \mathscr{F}, P)$, and that $\mathscr{F}$ is generated by W.

Definition 2.3.21. (Divergence Operator) *We denote by δ the adjoint of the operator D.That is, δ is an unbounded operator on $L^2(\Omega;H)$ with values in $L^2(\Omega)$ such that*:

1. *The domain of δ, denoted by* Domδ*, is the set of $H-$valued square integrable random variables $u \in L^2(\Omega;H)$ such that*

$$|E(\langle DF, u\rangle_H)| \leq c\|F\|_2,$$

   for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

2. *If u belongs to* Dom$delta$*, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by*

$$E(F\delta(u)) = E(\langle DF, u\rangle_H)$$

   for any $F \in \mathbb{D}^{1,2}$.

   The operator δ is called the divergence operator and is closed as the adjoint of an unbounded and densely defined operator.

See in [30] p36 - 37.

Notice that δ is a linear operator, that is

$$\delta(au + bv) = a\delta(u) + b\delta(v),$$

if $a, b \in \mathbb{R}$ and $u, v \in \text{Dom}\delta$.

For the equation $E(F\delta(u)) = E(\langle DF, u\rangle_H)$, the following property holds true

$$E(\delta(u)) = E(\langle 0, u\rangle_H) = 0,$$

if $u \in \mathrm{Dom}\delta$, by taking $F = 1$.

Denote by $\mathbf{S}_H$ the class of smooth elementary elements of the form

$$u = \sum_{j=1}^{n} F_j h_j,$$

where the F_j are smooth random variables, and the h_j are elements of H.

Lemma 2.3.22. *Let F_j be smooth random variables, h_j be elements of H. Then the element $u = \sum_{j=1}^{n} F_j h_j$ belongs to $\mathrm{Dom}\delta$. Moreover,*

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H.$$

See in [30] p37 - 38, and the proof is modified.

Proof. We will apply the method of integration-by-parts stated in Lemma 2.3.15. Given any smooth random variables $G \in \mathbf{S}_0$. First by linearity of $\langle \cdot, \cdot \rangle_H$, we have

$$\begin{aligned} E(\langle DG, u \rangle_H) &= E\Big(\sum_{j=1}^{n} \langle DG, F_j h_j \rangle_H \Big). \\ &= E\Big(\sum_{j=1}^{n} F_j \langle DG, h_j \rangle_H \Big). \end{aligned}$$

By using the Lemma 2.3.15,

$$E(K \langle DF, h \rangle_H) = E(FKW(h)) - E(F \langle DK, h \rangle_H).$$

we obtain

$$E(\langle DG, u \rangle_H) = E\Big(\sum_{j=1}^{n} GF_j W(h_j) \Big) - E\Big(\sum_{j=1}^{n} G \langle DF_j, h_j \rangle_H) \Big),$$

Overall, we have

$$E(\langle DG, u \rangle_H) = E\Big[G \Big(\sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H \Big) \Big],$$

for any smooth random variables $G \in \mathbf{S}_0$, which implies that $u = \sum_{j=1}^{n} F_j h_j$ belong to Domδ.

Then by definition of the adjoint operator δ, we have

$$E(G\delta(u)) = E(\langle DG, u\rangle_H)$$

That is

$$E(G\delta(u)) = E\Big[G\Big(\sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j\rangle_H\Big)\Big],$$

for any smooth random variables $G \in \mathbf{S}_0$.

The proof is complete. □

By Lemma 2.3.22, we can represent the δ-operator in another possible way, which will be applied in the following proofs. Note that if $u \in \mathbb{D}^{1,2}(H)$ then the derivative Du is a square integrable random variable with values in the Hilbert space $H \otimes H$, which can be identified with the space of Hilbert-Schmidt operators from H to H.

Lemma 2.3.23. *Let $u \in \mathbf{S}_H$, $F \in \mathbf{S}$ and $h \in H$, then*

$$\delta(D^h u) = \sum_{j=1}^{n} D^h F_j W(h_j) - \sum_{j=1}^{n} \langle D(D^h F_j), h_j\rangle_H.$$

See in [30] p38, and the proof is modified.

Proof. Let $u \in \mathbf{S}_H$, $F \in \mathbf{S}$ and $h \in H$, suppose $u = \sum_{j=1}^{n} F_j h_j$, then the derivative operator is given as

$$Du = \sum_{j=1}^{n} DF_j h_j,$$

and then by definition of D^h operator and linearity of $\langle \cdot, \cdot \rangle_H$, we have

$$\begin{aligned} D^h u &= \Big\langle \sum_{j=1}^{n} DF_j h_j, h \Big\rangle_H \\ &= \sum_{j=1}^{n} h_j \langle DF_j, h\rangle_H \\ &= \sum_{j=1}^{n} (D^h F_j) h_j. \end{aligned}$$

Hence, by straightforward application of Lemma 2.3.22, the target is achieved. □

Lemma 2.3.24. *Let $u \in \mathbf{S}_H$, $F \in \mathbf{S}$ and $h \in H$, then*

$$D^h(\delta(u)) = \langle u,h\rangle_H + \delta(D^h u).$$

See in [30] p38 and the proof is modified from [30] p37 - 38.

Proof. By using

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j\rangle_H,$$

we obtain

$$\begin{aligned} D^h(\delta(u)) &= \langle D(\delta(u)), h\rangle_H \\ &= \Big\langle D\big(\sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j\rangle_H\big), h\Big\rangle_H. \end{aligned}$$

By the linearity of $\langle\cdot,\cdot\rangle_H$ and rearranging, we have

$$\begin{aligned} D^h(\delta(u)) &= \sum_{j=1}^{n} \langle F_j h_j + DF_j W(h_j) - D(\langle DF_j, h_j\rangle_H), h\rangle_H \\ &= \sum_{j=1}^{n} (F_j \langle h_j, h\rangle_H + \langle DF_j, h\rangle_H W(h_j) - \langle D(D^h F_j), h_j\rangle_H) \\ &= \Big\langle \sum_{j=1}^{n} F_j h_j, h\Big\rangle_H + \Big(\sum_{j=1}^{n} D^h F_j W(h_j) - \sum_{j=1}^{n} \langle D(D^h F_j), h_j\rangle_H\Big). \end{aligned}$$

Apply Lemma 2.3.23, we obtain

$$D^h(\delta(u)) = \langle u,h\rangle_H + \delta(D^h u).$$

The proof is complete.

□

Proposition 2.3.25. *The space $\mathbb{D}^{1,2}(H)$ is included in the domain of δ. If $u,v \in \mathbb{D}^{1,2}(H)$. then*

$$E(\delta(u)\delta(v)) = E(\langle u,v\rangle_H) + E(\mathrm{Tr}(Du \circ Dv)).$$

The proof is modified from [30] p38.

Proof. First suppose that $u,v \in \mathbf{S}_H$. Let $e_i, i \geq 1$ be a complete orthonormal systerm on H. We have

$$\langle e_i, e_i\rangle_H = 1,$$

and therefore

$$\begin{aligned} D(\delta(u)) &= D(\delta(u))\langle e_i, e_i\rangle_H \\ &= e_i\langle D(\delta(u)), e_i\rangle_H \\ &= e_i D^{e_i}(\delta(u)). \end{aligned}$$

Then, by definition of the divergence operator, we have

$$\begin{aligned} E(\delta(u)\delta(v)) &= E(\langle v, D(\delta(u))\rangle_H) \\ &= E\Big[\sum_{i=1}^{\infty}\langle v, e_i\rangle_H D^{e_i}(\delta(u))\Big]. \end{aligned}$$

By using Lemma 2.3.24, we deduce

$$\begin{aligned} &= E\Big[\sum_{i=1}^{\infty}\langle v, e_i\rangle_H(\langle u, e_i\rangle_H + \delta(D^{e_i}u))\Big] \\ &= E(\langle u,v\rangle_H) + E\Big(\sum_{i,j=1}^{\infty} D^{e_i}\langle u, e_j\rangle_H D^{e_j}\langle u, e_i\rangle_H\Big) \\ &= E(\langle u,v\rangle_H) + E(\mathrm{Tr}(Du \circ Dv)). \end{aligned}$$

Then, we obtain

$$E(\delta(u)^2) \leq E(\|u\|_h^2) + E(\|Du\|_{H\otimes H}^2).$$

As the definition of seminorm on $\mathbf{S}_V$ [30] p31 is given as

$$\|F\|_{k,p,V}= \left[E(\|F\|_V^p)+\sum_{j=1}^{k}E(\|D^jF\|_{H^{\otimes j}\otimes V}^p)\right]^{\frac{1}{p}}.$$

Overall we have

$$E(\delta(u)^2)\leq \|u\|_{1,2,H}^2.$$

The space $\mathbb{D}^{1,2}(H)$ is defined as the completion of $\mathbf{S}_V$ with repect to the norm $\|\cdot\|_{k,p,V}$. The above condition implies that the space $\mathbb{D}^{1,2}(H)$ is included in the domain of δ. In fact, if $u\in\mathbb{D}^{1,2}(H)$, there exists a sequence $u^n\in\mathbf{S}_H$ such that u^n converges to u in $L^2(\Omega)$ and Du^n converges to Du in $L^2(\Omega;H\otimes H)$. Therefore, $\delta(u^n)$ converges in $L^2(\Omega)$ and its limit is $\delta(u)$.

Moreover,

$$E(\delta(u)\delta(v))=E(\langle u,v\rangle_H)+E(\mathrm{Tr}(Du\circ Dv))$$

holds for any $u,v\in\mathbb{D}^{1,2}(H)$. □

Lemma 2.3.26. *Let G be a square integrable random variable. Suppose there exists $Y\in L^2(\Omega)$ such that*

$$E(G\delta(hF))=E(YF),$$

for all $F\in\mathbb{D}^{1,2}$. Then $G\in\mathbb{D}^{h,2}$ and $D^hG=Y$.

The proof is modified from [30] p39.

Proof. Recall the definition of Wiener chaos $\mathscr{H}_n$: for each $n\geq 1$, $\mathscr{H}_n$ is the closed linear subspaces of $L^2(\Omega,\mathscr{F},P)$ generated by the random variables $\{H_n(W(h)),h\in H,\|h\|_H=1\}$. We denote by J_n the projection on the nth Wiener chaos $\mathscr{H}_n$. Then the random variable $F\in\mathbb{D}^{1,2}$ has the Wiener chaos expansion

$$F=\sum_{n=0}^{\infty}J_nF.$$

We have

$$\begin{aligned} E(YF) &= E(G\delta(hF)) \\ &= \sum_{n=1}^{\infty} E((J_nG)\delta(hF). \end{aligned}$$

By the definition of the divergence operator and rearranging, we obtain

$$\begin{aligned} E(YF) &= \sum_{n=1}^{\infty} E(\langle DJ_nG, Fh\rangle_H) \\ &= \sum_{n=1}^{\infty} E(F\langle DJ_nG, h\rangle_H) \\ &= \sum_{n=1}^{\infty} E(FD^h(J_nG)). \end{aligned}$$

Hence, $J_{n-1}Y = D^h(J_nG)$ for each $n \geq 1$.

And this implies that $G \in \mathbb{D}^{h,2}(H)$ and $D^hG = Y$. □

Proposition 2.3.27. *Suppose that $u \in \mathbb{D}^{h,2}(H)$, and D^hu belongs to the domain of the divergence. Then $\delta(u) \in \mathbb{D}^{h,2}(H)$ and the commutation relation holds as*

$$D^h(\delta(u)) = \langle u, h\rangle_H + \delta(D^hu).$$

See the proposition in [30] p38 and the proof is modified.

Proof. For all $F \in \mathbb{D}^{1,2}$, by using the definition of the divergence operator and Proposition 2.3.25, we have

$$\begin{aligned} E(\delta(u)\delta(hF)) &= E(\langle u, hF))\rangle_H + \langle Tr(Du \circ hDF)\rangle_H) \\ &= E(F\langle u, h\rangle_H + \langle D^hu, DF\rangle_H) \\ &= E[F(\langle u, h\rangle_H + \delta(D^hu)]. \end{aligned}$$

Set $G = \delta(u)$ and $Y = \langle u,h\rangle_H + \delta(D^h u)$, that is $E(G\delta(hF)) = E(YF)$. Then from Lemma 2.3.22, we can deduce that $\delta(u) \in \mathbb{D}^{h,2}(H)$ and

$$D^h(\delta(u)) = D^h G = Y = \langle u,h\rangle_H + \delta(D^h u).$$

□

Proposition 2.3.28. *Let $F \in \mathbb{D}^{1,2}$ and u be in the domain of δ such that $Fu \in L^2(\Omega;H)$. Then Fu belongs to the domain of δ and the following equation holds*

$$\delta(Fu) = F\delta(u) - \langle DF,u\rangle_H,$$

provided that $F\delta(u) - \langle DF,u\rangle_H$ is square integrable.

The proof is modified from [30] p39.

Proof. For any smooth random variable $G \in \mathbf{S}_0$, by using

$$D(FG) = FDG + GDF,$$

we have

$$\begin{aligned} E(\langle DG,Fu\rangle_H) &= E(\langle FDG,u\rangle_H) \\ &= E(\langle D(FG) - GDF,u\rangle_H). \end{aligned}$$

By using the definition of the divergence operator and rearranging, we have

$$\begin{aligned} E(\langle DG,Fu\rangle_H) &= E(FG\delta(u) - G\langle DF,u\rangle_H) \\ &= E[G(F\delta(u) - \langle DF,u\rangle_H)], \end{aligned}$$

which implies that $Fu \in \mathrm{Dom}\delta$. Hence for any smooth random variable $G \in \mathbf{S}_0$ we have

$$\begin{aligned} E(G\delta(Fu)) &= E(\langle DG,Fu\rangle_H) \\ &= E[G(F\delta(u) - \langle DF,u\rangle_H)]. \end{aligned}$$

That is

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H.$$

□

If we replace u by a deterministic element $h \in H$, then the Proposition 2.3.28 can be represented by the following version.

Proposition 2.3.29. *Let $h \in H$ and $F \in \mathbb{D}^{h,2}$. Then Fh belongs to the domain of δ and the following equation holds*

$$\delta(Fh) = FW(h) - D^h F.$$

See proof in [30] p39.

The following is the definition of the Skorohod stochastic integral of the process u, see in [30] p40.

Definition 2.3.30. (Skorohod integral) *Suppose that the separable Hilbert space H is an L^2 space of the form $H = L^2(T, \mathscr{B}, \mu)$,where T is the parameter space, $\mathscr{B}$ is the σ-algebra and μ is a σ-finite measure without atoms. Then the elements of $Dom\delta \subset L^2(T \times \Omega)$ are square integrable processes, and we will call the divergence $\delta(u)$ as the Skorohod stochastic integral of the process u. And the notation is as follows:*

$$\delta(u) = \int_0^T u_t dW_t.$$

The Skorohod stochastic integral will play an important role in the computation of operators in Chapter 5 and 6.

2.3.5 The Semigroup of Ornstein-Uhlenbeck

In this section, we will review the main property of the Ornstein-Uhlenbeck semigroup, based on [30].

We assume that $W = \{W(h), h \in H\}$ is an isonormal Gaussian process associated to the Hilbert space H defined in a complete probability space $(\Omega, \mathscr{F}, P)$, and $\mathscr{F}$ is generated by W. We recall that J_n denotes the orthogonal projection on the nth Wiener chaos.

Definition 2.3.31. (Ornstein-Uhlenbeck semigroup) *The Ornstein-Uhlenbeck semigroup is the one-parameter semigroup $T_t, t \geq 0$ of contraction operators on $L^2(\Omega)$ defined by*

$$T_t(F) = \sum_{n=0}^{\infty} e^{-nt} J_n F,$$

for any $F \in L^2(\Omega)$.

See in [30] p54.

Suppose that the process $W' = \{W'(h), h \in H\}$ is an independent copy of W. We will assume that W and W' are defined on the product probability space $(\Omega \times \Omega', \mathscr{F} \otimes \mathscr{F}', P \times P')$. For any $t > 0$ we consider the process $Z = \{Z(h), h \in H\}$ defined by

$$Z(h) = e^{-t} W(h) + \sqrt{1 - e^{-2t}} W'(h), \quad h \in H.$$

From the definition we have that

$$E(Z(h)) = E(e^{-t} W(h) + \sqrt{1 - e^{-2t}} W'(h)) = 0,$$

and

$$\begin{aligned} \mathrm{cov}(Z(h_1), Z(h_2)) &= E(Z(h_1) Z(h_2)) \\ &= E[(e^{-t} W(h_1) + \sqrt{1 - e^{-2t}} W'(h_1))(e^{-t} W(h_2) + \sqrt{1 - e^{-2t}} W'(h_2))]. \end{aligned}$$

By rearranging, that is

$$\begin{aligned} \mathrm{cov}(Z(h_1), Z(h_2)) &= e^{-2t} \langle h_1, h_2 \rangle_H + (1 - e^{-2t}) \langle h_1, h_2 \rangle_H \\ &= \langle h_1, h_2 \rangle_H \\ &= E(W(h_1) W(h_2)), \end{aligned}$$

which implies that $Z = \{Z(h), h \in H\}$ is Gaussian process, and it has the same covariance function as W.

Definition 2.3.32. (Mehler's formula) *Let $W : \Omega \to \mathbb{R}^H$ and $W' : \Omega' \to \mathbb{R}^H$ be the canonical mappings associated with the processes $\{W(h), h \in H\}$ and $\{W'(h), h \in H\}$, respectively. Given a random variables $F \in L^2(\Omega)$, we can write $F = \psi_F \circ W$, where ψ_F is a measurable mapping from $\mathbb{R}^H$ to $\mathbb{R}$, determined $P \circ W^{-1}$ a.s. Hence, the random variable $\psi_F(Z(\omega, \omega')) = \psi_F(e^{-t}W(\omega) + \sqrt{1-e^{-2t}}W'(\omega')$ is well defined $P \times P'$ a.s. Then, for any $t \geq 0$, we have the equation called Mehler's formula in the form of*

$$T_t(F) = E'(\psi_F(e^{-t}W + \sqrt{1-e^{-2t}}W)),$$

where E' denotes mathematical expected value with respect to the probability P'.

See in [30] p54-55.

In following part, the equivalence between the Definition 2.3.31 and Mehler's formula is illustrated, which is modified from [30] p55.

First of all, we know that both definitions give rise to a linear contraction operator on $L^2(\Omega)$.

This is clear in the Definition 2.3.31, and on the other hand, in Mehler's formula it defines a linear contraction operator on $L^2(\Omega)$ for and $p \geq 1$ because the following inequation holds:

$$\begin{aligned} E(|T_t(F)|^p) &= E(|E'(\psi_F(e^{-t}W + \sqrt{1-e^{-2t}}W))|^p) \\ &\leq E(E'(|\psi_F(e^{-t}W + \sqrt{1-e^{-2t}}W)|^p)) \\ &= E(|F|^p). \end{aligned}$$

The second step suffices that to check that both the Definition 2.3.31 and Mehler's formula coincide when $F = \exp\Big(W(h) - \frac{1}{2}\|h\|_H^2\Big)$, $h \in H$.

By the definition of E', we have

$$\begin{aligned} &E'\Big(\exp\Big(e^{-t}W(h) + \sqrt{1-e^{-2t}}W'(h) - \frac{1}{2}\|h\|_H^2\Big)\Big) \\ &= \exp\Big(e^{-t}W(h) - \frac{1}{2}\|h\|_H^2\Big)E'(\exp(\sqrt{1-e^{-2t}}W'(h))). \end{aligned}$$

Notice that for a random variable x with a standard Gaussian distribution $N(0,1)$, if C is a constant and the second derivative of function f is continuous, the following equation can be proved:

$$E(\exp(Cx)) = \exp(C^2/2).$$

By definition we have

$$\begin{aligned}
E(\exp(Cx)) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{Cx} e^{-x^2/2} dx \\
&= e^{C^2/2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-C)^2/2} dx \\
&= e^{C^2/2}.
\end{aligned}$$

And then by using $W'(h) \sim \|h\| N(0,1)$ and rearranging, we obtain

$$\begin{aligned}
&E'\Big(\exp\Big(e^{-t}W(h) + \sqrt{1-e^{-2t}}W(h) - \frac{1}{2}\|h\|_H^2\Big)\Big) \\
&= \exp\Big(e^{-t}W(h) - \frac{1}{2}\|h\|_H^2\Big)\exp\Big(\frac{1-e^{-2t}}{2}\|h\|_H^2\Big) \\
&= \exp\Big(e^{-t}W(h) - \frac{e^{-2t}}{2}\|h\|_H^2\Big) \\
&= \exp\Big((e^{-t}\|h\|_H)\frac{W(h)}{\|h\|_H} - \frac{(e^{-t}\|h\|_H)^2}{2}\Big).
\end{aligned}$$

Recall the Hermite polynomial

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, n \geq 1,$$

if the function is $F(x,s) = \exp(sx - \frac{s^2}{2})$, we have

$$F(x,s) = \sum_{n=0}^{\infty} s^n H_n(x).$$

Let $x = \dfrac{W(h)}{\|h\|_H}$ and $s = e^{-t}\|h\|_H$, we deduce

$$\begin{aligned}
&E'\Big(\exp\Big(e^{-t}W(h) + \sqrt{1-e^{-2t}}W(h) - \frac{1}{2}\|h\|_H^2\Big)\Big) \\
&= \sum_{n=0}^{\infty} e^{-nt}\|h\|_H^n H_n\Big(\frac{W(h)}{\|h\|_H}\Big).
\end{aligned}$$

By the definition of Hermite polynomial, we have

$$\|h\|_H^n H_n\Big(\frac{W(h)}{\|h\|_H}\Big) = \frac{1}{n!} I_n(h^{\otimes n}).$$

And therefore, we deduce

$$E'\Big(\exp\Big(e^{-t}W(h) + \sqrt{1-e^{-2t}}W(h) - \frac{1}{2}\|h\|_H^2\Big)\Big) = \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} I_n(h^{\otimes n}).$$

On the other hand, by using the fact [30] p28

$$I_n(h^{\otimes n}) = n! J_n(h^{\otimes n}),$$

we have

$$\begin{aligned}
T_t(F) &= \sum_{n=0}^{\infty} e^{-nt} J_n(h^{\otimes n}) \\
&= \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} I_n(h^{\otimes n}),
\end{aligned}$$

which yields the desired equality.

Proposition 2.3.33. *The operators T_tF have the following porperties:*

1. *T_tF is non-negative (i.e. $F \geq 0$ implies $T_tF \geq 0$).*
2. *T_tF is symmetric:*

$$E(GT_tF) = E(FT_tG) = \sum_{n=0}^{\infty} e^{-nt} E(J_n(F)J_n(G)).$$

See in [30] p55 and the proof is modified.

Proof. By using the fact $G = \sum_{n=0}^{\infty} J_n(G)$ and the orthogonality of J_n, we deduce

$$\begin{aligned} E(GT_tF) &= E\Big[\sum_{n=0}^{\infty} J_n(G) \sum_{m=0}^{\infty} e^{-mt} J_m(F)\Big] \\ &= \sum_{n=0}^{\infty} e^{-nt} E(E(J_n(F)J_n(G)), \end{aligned}$$

which yields the symmetry of the operators T_tF. □

2.3.6 The Generator of the Ornstein-Uhlenbeck Semigroup

In this section, we will review the properties of the infinitesimal generator of the Ornstein-Uhlenbeck semigroup, based on [30].

Definition 2.3.34. **(L operator)** *Let $F \in L^2(\Omega)$ be a square integrable random variable. The operator L is defined in the following way:*

$$LF = \sum_{n=0}^{\infty} -nJ_nF,$$

provided this series converges in $L^2(\Omega)$. J_n denotes the orthogonal projection on the nth Wiener chaos.

Definition 2.3.35. *The domain of L operator is the set*

$$\mathrm{Dom}L = \Big\{F \in L^2(\Omega), F = \sum_{n=0}^{\infty} I_n(f_n) : \sum_{n=0}^{\infty} n^2 \|J_nF\|_2^2 < \infty\Big\}.$$

In particular, $\mathrm{Dom}L \subset \mathbb{R}^{1,2}$.

Proposition 2.3.36. *For all $F, G \in \mathrm{Dom}L$, we have*

$$E(FLG) = E(GLF),$$

which implies that L is an unbounded symmectric operator on $L^2(\Omega)$.

Proposition 2.3.37. *The operator L coincides with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup* $\{T_t, t \geq 0\}$.

See proof in [30].

The following proposition will be usefull, as it gives a explanation of the relationship between the operator D, δ and L, and provides a possible way for computing the operator L by computing $\delta(DF)$.

Proposition 2.3.38. *The equation* $\delta(DF) = -LF$ *holds true, that is, for* $F \in L^2(\Omega)$ *the statement* $F \in \mathrm{Dom}L$ *is equivalent to* $F \in \mathrm{Dom}\delta L$ *(i.e.,* $F \in \mathbb{D}^{1,2}$ *and* $DF \in \mathrm{Dom}\delta$*), and in this case* $\delta(DF) = -LF$.

The proof is modified from [30] p59.

Proof. First suppose that $F \in \mathbb{D}^{1,2}$ and that $DF \in \mathrm{Dom}\delta$. Let G be a random variable in the nth Wiener chaos $\mathscr{H}_n$, by using the definitions of the derivative operator and the divergence operator, we have

$$\begin{aligned} E(G\delta(DF)) &= E(\langle DG, DF\rangle_H) \\ &= n^2(n-1)!\,\langle g, f_n\rangle_{H^{\otimes n}} \\ &= nE(GJ_nF). \end{aligned}$$

By using the fact $F = \sum_{n=0}^{\infty} J_nF$ and the orthogonality of the Wiener chaos, we have

$$\begin{aligned} E(G\delta(DF)) &= E\Big(G\sum_{n=1}^{\infty} J_n\delta(DF)\Big) \\ &= E(GJ_n\delta(DF)). \end{aligned}$$

That is,

$$J_n\delta(DF) = nJ_nF.$$

which implies $F \in \mathrm{Dom}L$ and by summing up from $n = 0$ to $n = \infty$ and the definition of the operator L, we have

$$LF = -\delta(DF).$$

Conversely, if $F \in \mathrm{Dom}L$, then$F \in \mathbb{D}^{1,2}$ and for any $G \in \mathbb{D}^{1,2}$, $G = \sum_{n=0}^{\infty} I_n(g_n)$, we have

$$\begin{aligned} E(G\delta(DF)) &= \sum_{n=0}^{\infty} nE(J_nGJ_nF) \\ &= -E(GLF). \end{aligned}$$

Therefore, $DF \in \mathrm{Dom}\delta$, and $LF = -\delta(DF)$.

The proof is complete. □

Proposition 2.3.39. *It holds that* $\mathbf{S} \subset \mathrm{Dom}L$*, and for any* $F \in \mathbf{S}$ *of the form* $F = f(W(h_1),...,W(h_n))$*,* $f \in C_p^{\infty}(\mathbb{R}^n)$*, we have*

$$\begin{aligned} LF = &\sum_{i,j=1}^{n} \partial_i\partial_j f(W(h_1),...,W(h_n))\langle h_i, h_j\rangle_H \\ &- \sum_{i=1}^{n} \partial_i f(W(h_1),...,W(h_n))W(h_i). \end{aligned}$$

The proof is modified from [30].

Proof. As $F \in \mathbf{S}$, that is $F \in \mathbb{D}^{1,2}$. And by the definition

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1),...,W(h_n))h_i,$$

we know that $DF \in \mathbf{S}_H \subset \mathrm{Dom}\delta$.

Set $F_i = \partial_i f(W(h_1),...,W(h_n))$ and by

$$\delta(u) = \sum_{i=1}^{n} F_iW(h_i) - \sum_{i=1}^{n} \langle DF_i, h_i\rangle_H,$$

we have

$$\begin{aligned}\delta(DF) &= \sum_{i=1}^{n} \partial_i f(W(h_1),...,W(h_n))W(h_i) - \sum_{i=1}^{n} \langle D(\partial_i f(W(h_1),...,W(h_n))), h_i\rangle_H \\ &= \sum_{i=1}^{n} \partial_i f(W(h_1),...,W(h_n))W(h_i) - \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} \partial_j \partial_i f(W(h_1),...,W(h_n))h_j, h_i \right\rangle_H \\ &= \sum_{i=1}^{n} \partial_i f(W(h_1),...,W(h_n))W(h_i) - \sum_{i,j=1}^{n} \partial_j \partial_i f(W(h_1),...,W(h_n))\langle h_j, h_i\rangle_H.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}LF &= -\delta(DF) \\ &= \sum_{i,j=1}^{n} \partial_i \partial_j f(W(h_1),...,W(h_n))\langle h_i, h_j\rangle_H \\ &\quad - \sum_{i=1}^{n} \partial_i f(W(h_1),...,W(h_n))W(h_i).\end{aligned}$$

This proof is complete.

□

Proposition 2.3.40. *Suppose that $F = (F^1,...,F^m)$ is a random vector whose components belong to $\mathbb{D}^{2,4}$. Let φ be a function in $C^2(\mathbb{R}^m)$ with bounded first and second partial derivatives. Then $\varphi \in \mathrm{Dom}L$, and*

$$L(\varphi(F)) = \sum_{i,j=1}^{m} \partial_i \partial_j \varphi(F)\langle DF^i, DF^j\rangle_H + \sum_{i=1}^{m} \partial_i \varphi(F) LF^i.$$

See proof in [30] p60.

The following is the definition of the norm $\|\cdot\|_L$, see in [30] p60.

Definition 2.3.41. **($\|\cdot\|_L$ norm)** *The norms $\|\cdot\|_L$ on* **S** *is defined as*

$$\|F\|_L = [E(F^2) + E(|LF|^2)]^{\frac{1}{2}}.$$

From Definition 2.3.41, we know that $\mathrm{Dom}L = \mathbb{D}^{2,2}$.

By the definition of operator $LF = \sum_{n-0}^{\infty} -nJ_nF$, and the property of J_nF

$$E(J_nF \cdot J_mF) = 0, \quad \text{if} \quad n \neq m,$$

we deduce

$$E(|LF|^2) = \sum_{n-0}^{\infty} n^2 \|J_nF\|^2.$$

By using the fact

$$E\Big(\|DF\|_H^2\Big) = \sum_{n=1}^{\infty} n\|J_nF\|_2^2,$$

and

$$E\Big(\|D^2F\|_{H\otimes H}^2\Big) = \sum_{n=1}^{\infty} n(n-1)\|J_nF\|_2^2,$$

we have

$$\begin{aligned} E(F^2)+E(|LF|^2) &= E(F^2)+\sum_{n=1}^{\infty} n^2\|J_nF\|_2^2 \\ &= E(F^2)+\sum_{n=1}^{\infty} n\|J_nF\|_2^2+\sum_{n=1}^{\infty}(n^2-n)\|J_nF\|_2^2 \\ &= E(F^2)+E\Big(\|DF\|_H^2\Big)+E\Big(\|D^2F\|_{H\otimes H}^2\Big). \end{aligned}$$

Recall the definition of the seminorms on **S**:

$$\|F\|_{k,p}= \Big[E(|F|^p)+\sum_{j=1}^{k} E\Big(\|D^jF\|_{H^{\otimes j}}^p\Big)\Big]^{\frac{1}{p}}.$$

We can obtain that the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ coincide. Furture explanations and detailed examples will be demonstrated in Section 5.3.

2.4 Malliavin Calculus in finance

In this section, in order to demonstrate the applications of Malliavin Calculus to mathematical finance in Chapter 6, first we review the financial modelling especially the Black-Scholes model. Then, we review a probability method for numerical calculations of price sensitivities (Greeks) by using the Integration by parts formula.

2.4.1 Financial Modelling

Continuous Time Markets There are many types of financial modelling [37, 10, 38, 8]. Here we work with continuous time martingale type construction.

We use the following two constructions :

1' Black-Scholes model.

$$S_t = S_0 e^{H_t}, t \in [0,T]$$

where

$$H_t = \int_0^t (\mu_s - \frac{\sigma_s^2}{2}) ds + \int_0^t \sigma_s dW_s,$$

and W_t is a Wiener process.

In calculation of Greeks, we work with a classical Black-Scholes model, where

$$S_t = S_0 \exp\Big[\Big(a - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big], \quad t \in [0,T]$$

2' Itô martingale modelling.

$$M_t = \int_0^t Y_s dW_s$$

where Y_s is a adapted process.

Two particular examples we will treat in calculation of Greeks:

$$M_t = Y_0 + \sigma W_t \qquad \text{(Bachelier model(1900))}$$

and

$$M_t = Y_0 + \sigma(W_t^2 - t)$$

Discrete time markets Discrete time martingale type construction is based on a random walk.

$$M_n = \sum_{j=1}^{n} Y_j$$

The classical Binomial model is constructed as a Geometric random walk.

$$S_n = S_0 e^{M_n}$$

When Y's has two values, the market is complete in general, which implies that it can be hedged.

$$\begin{array}{lll} & & S_1 = S_0 u \;; \quad \text{share is going up} \\ & \nearrow & \\ S_0 & & \\ & \searrow & \\ & & S_1 = S_0 d \;; \quad \text{share is going down} \end{array}$$

When Y's has three or more values, the market is incomplete in general, which implies that it can not be hedged.

$$\begin{array}{lll} & & S_1 = S_0 u \;; \quad \text{share is going up} \\ & \nearrow & \\ S_0 & \rightarrow & \qquad S_1 = S_0 c \\ & \searrow & \\ & & S_1 = S_0 d \;; \quad \text{share is going down} \end{array}$$

Overall, the issue of completeness and incompleteness comes in for the valuation of payoffs.

In the case of complete markets, there exists a uniquely defined risk-neutral probability measure, also known as a martingale measure, so the value of future payoffs are simply discounted mathematical expectations with respect to this unique martingale measure.

In the case of incomplete markets, the corresponding unique martingale measure is undefined, so the risk-neutral pricing is no longer appropriate.

See [16] for more details about complete and incomplete market cases.

2.4.2 Black-Scholes Model

This section gives a brief review of the Black-Scholes model, based on [30].

Definition 2.4.1. (Black-Scholes model) *Consider a market consisting of one risky asset (stock) and one risk-free asset (bond):*

The price process of the risky asset is assumed to be a geometric Brownian motion (GBM), which has the form $S_t = S_0 e^{H_t}, t \in [0,T]$, *with*

$$H_t = \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s,$$

where $W = \{W_t, t \in [0,T]\}$ *is a Brownian motion defined in a complete probability space* $(\Omega, \mathscr{F}, P)$. *The filtration generated by the Brownian motion and completed by the* $P-null$ *sets is denoted as* $\{\mathscr{F}_t, t \in [0,T]\}$. S_0 *is the initial stock price,* μ_t *is the rate of growth of the price* ($E(S_t) = S_0 e^{\mu t}$)*, and* σ_t *is called the volatility process. The mean rate of return* μ_t *and the volatility process* σ_t *are supposed to be measurable and adapted processes satisfying the following integrability conditions*

$$\int_0^T |\mu_t| dt < \infty \qquad \text{and} \qquad \int_0^T \sigma_t^2 dt < \infty,$$

almost surely.

The price of the bond is denote by $G_t \in [0,T]$ *, and the following differential equation holds*

$$dG_t = r_t G_t dt, \quad G_0 = 1,$$

where the interest rate process is a nonnegative measurable and adapted process satisfying the integrability condition $\int_0^T r_t dt < \infty$, almost surely. That is,

$$G_t = \exp\Big(\int_0^t r_s ds\Big).$$

Fix a time interval $[0,T]$, imagine an investor invests in the assets by owning a certain amount of non-risky assets and stocks respectively. Let α_t be the number of non-risky assets and β_t be the number of stocks at time t.

Definition 2.4.2. (Portfolio) *A portfolio or trading strategy is a couple*

$$\phi_t = \{(\alpha_t, \beta_t), t \in [0,T]\}$$

such that the components α_t and β_t are measurable and adapted process such that

$$\int_0^T |\alpha_t| r_t dt < \infty, \quad \int_0^T |\beta_t \mu_t| dt < \infty, \quad \int_0^T \beta_t^2 \sigma_t^2 dt < \infty,$$

almost surely.

See in [30] p322.

The portfolio ϕ is said to be self-financing if there is no fresh investment and there is no consumption. And all the following portfolios are considered to be self-financing from now on.

Definition 2.4.3. (Value of portfolio) *The investor's initial wealth is given as*

$$x = \alpha_0 + \beta_0 S_0.$$

And investor's wealth at time t, which is also considered as the value of the portfolio, is

$$V_t(\phi) = \alpha_t G_t + \beta_t S_t,$$

or

$$V_t(\phi) = x + \int_0^t \alpha_s dG_s + \int_0^t \beta_s dS_s.$$

Moreover, the value process $V_t(\phi)$ of any self-financing portfolio can be proved as a local martingale. See proof in [30] p323.

Definition 2.4.4. (Derivative) *A derivative is a contract on the risky asset that produces a payoff H at maturity time T. Generally, the payoff H is an $\mathscr{F}_T$-measurable non-negative random variable.A non-negative $\mathscr{F}_T$-measurable payoff H is said to be replicated if there exists a self-financing portfolio ϕ such that $V_T(\phi) = H$.*

The following proposition shows that any derivative $E(W_T^{-2} Z_T^2 H^2) < \infty$ is replicable, where the process Z_t is defined by

$$Z_t = \exp\Big(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds \Big).$$

Proposition 2.4.5. *Let H be a non-negative $\mathscr{F}_T$-measurable random variable such that $E(W_T^{-2} Z_T^2 H^2) < \infty$. Then, there exists a self-financing portfolio ϕ such that $V_T(\phi) = H$.*

See the proposition and proof in [30] p326.

Under the assumptions of Proposition 2.4.5, the price of a derivative can be obtained by the following proposition.

Proposition 2.4.6. *The price of a derivative with payoff H at time $t < T$ is given by the value at time t of a portfolio wich replicates H. And the value of a portfolio at time t is given as*

$$V_t(\phi) = Z_t^{-1} E\Big(Z_T e^{-\int_t^T r_s ds} H | \mathscr{F}_t \Big).$$

Moreover, the value of a portfolio can be obtained by the following proposition.

Proposition 2.4.7. (Value of a portfolio) *The value of the arbitrage free portfolio at time t is given as*

$$V_t(\phi) = E_Q\Big(e^{-\int_t^T r_s ds} H | \mathscr{F}_t \Big),$$

where the measure Q is given by $\frac{dQ}{dP} = Z_T$.In particular, the value of the portfolio at time $t = 0$ is given as

$$V_0(\phi) = E_Q\Big(e^{-\int_0^T r_s ds} H\Big).$$

See the proposition and proof in [30] p327.

And this equation will play a very important role in the computation of Greeks in Chapter 6.

2.4.3 Integration by Parts Formula and Computation of Greeks

Recall that $W = \{W(h), h \in H\}$ denotes an isonormal Gaussian process associated with the Hilbert space H. We assume that W is defined on a complete probability space $(\Omega, \mathscr{F}, P)$, and that $\mathscr{F}$ is generated by W.

In the following part, we will review a general integration by parts formula, which plays a fundamental role in the computaion of Greeks. See in [30] p330.

Proposition 2.4.8. (Integraltion by parts formula) *Let F, G be two random variables such that $F \in \mathbb{D}^{1,2}$. Consider an H-valued random variable u such that $D^u F = \langle DF, u\rangle_H \neq 0$ a.s. and $Gu(D^u F)^{-1} \in \mathrm{Dom}\delta$. Then, for any continuously differentiable function f with bounded derivative we have*

$$E(f'(F)G) = E(f(F)H(F,G)),$$

where $H(F,G) = \delta(Gu(D^u F)^{-1})$.

The proof is modified from [30] p331.

Proof. Recall the fact that

$$\begin{aligned} D^u(f(F)) &= \langle Df(F), u\rangle_H \\ &= f'(F)\langle DF, u\rangle_H \\ &= f'(F) D^u F. \end{aligned}$$

By $D^u F \neq 0$ a.s. we have

$$\begin{aligned} f'(F) &= D^u(f(F))(D^u F)^{-1}. \\ &= \langle Df(F), u\rangle_H (D^u F)^{-1}. \end{aligned}$$

Hence by rearranging

$$\begin{aligned} E(f'(F)G) &= E(\langle Df(F), u\rangle_H (D^u F)^{-1} G) \\ &= E(\langle Df(F), Gu(D^u F)^{-1}\rangle_H). \end{aligned}$$

Recall the duality relationship for any $F \in \mathbb{D}^{1,2}$

$$E(F\delta(u)) = E(\langle DF, u\rangle_H),$$

if $u \in \mathrm{Dom}\delta$.

Finally, we can deduce that

$$E(f'(F)G) = E(f(F)\delta(Gu(D^u F)^{-1})).$$

$\square$

Suppose that the parameters appeared in the Black-Scholes model from section 2.4.2 are constants, that is $\sigma_t = \sigma$, $\mu_t = \mu$ and $r_t = r$. Then the stock price can be denoted by

$$S_t = S_0 \exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big),$$

which is a geometric Brownian motion (GBM).

Consider an option with payoff H such that $E_Q(H^2) < \infty$. Recall its price at time $t = 0$ is determined by

$$V_0 = E_Q(e^{-rT} H).$$

Suppose that we can write the payoff function as $H = f(F_\alpha)$, where α is one of the parameters of the model, that is, S_0, σ or r. Then, computing the derivative of the expected value $E_Q(e^{-rT}H)$ with respect to the parameter α, we have

$$\frac{\partial V_0}{\partial \alpha} = e^{-rT} E_Q\left(f'(F_\alpha)\frac{dF_\alpha}{d\alpha}\right).$$

Using Proposition 2.4.8 we can deduce

$$\frac{\partial V_0}{\partial \alpha} = e^{-rT} E_Q\Big(f(F_\alpha) H\Big(F_\alpha, \frac{dF_\alpha}{d\alpha}\Big)\Big).$$

A Greek is a derivative of a financial quantity, usually an option price, with respect to any of the parameters of the model. The derivative of the option price at time $t = 0$ with respect to the initial price of the stock S_0 is called Delta, which is considered as the most important Greek. Denote Delte by Δ. The Gamma, denoted by Γ, is the second derivative of the option price V_0 with respect to the initial stock price S_0. That is, $\Gamma = \dfrac{\partial^2 V_0}{\partial S_0^2}$. The derivative of V_0 with respect to the volatility σ is called Vega, denoted by ϑ. That is, $\vartheta = \dfrac{\partial V_0}{\partial \sigma}$.

These Greeks are useful to measure the stability of this quantity under variations of the parameter.

Chapter 3

Weighted Self-normalized Sum of Exchangeable Variables

Assume that $Y = \{Y_i, i \geq 1\}$ is a sequence of independent, identically distributed random variables, where Y is non-negative, and let $X = \{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables independent of $Y = \{Y_i, i \geq 1\}$, where X satisfies

$$E(X^2) < \infty \quad \text{and} \quad E(X) = 0.$$

Let R_n denote the randomly weighted self-normalized sum

$$R_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}.$$

and S_n denote $\sum_{i=1}^n Y_i$.

This chapter is motivated by the following theorem [27].

Theorem 3.0.1. *The ratio R_n converges in distribution to a non-degenerate variable if and only if Y belongs to a domain of attraction of the positive stable law with characteristics $0 \leq \alpha < 1$.*

The result has been proved in the mason and Zinn paper [27] under the condition

$$E(|X|^p) < \infty, \quad \text{for } p > 2.$$

We only have a discussion on the truncation argument.

If we do a truncation for X_i such that

$$X_i^{(m)} = \begin{cases} X_i, & |X_i| \leq m \\ 0, & \text{else} \end{cases} \quad \text{for } 1 \leq i \leq n$$

where m is a constant, $m > 0$.

Then we have the truncated weighted self-normalized sum

$$R_n^{(m)} = \sum_{i=1}^{n} \frac{X_i^{(m)} Y_i}{S_n},$$

and the weighted self-normalized sum R_n is

$$\begin{aligned} R_n &= R_n^{(m)} + (R_n - R_n^{(m)}) \\ &= \sum_{i=1}^{n} \frac{X_i^{(m)} Y_i}{S_n} + \sum_{i=1}^{n} \frac{(X_i - X_i^{(m)}) Y_i}{S_n}, \end{aligned}$$

where S_n denotes $\sum_{i=1}^{n} Y_i$.

Thus, by triangular inequality and since $|X_i^{(m)}| \leq m$

$$\begin{aligned} |R_n^{(m)}| &\leq \sum_{i=1}^{n} \frac{|X_i^{(m)}| Y_i}{S_n} \\ &\leq m \cdot \sum_{i=1}^{n} \frac{Y_i}{S_n} \\ &= m, \end{aligned}$$

which implies that $R_n^{(m)}$ is bounded by m.

Denote that $\Delta_n^{(m)} = R_n - R_n^{(m)}$. Observe that

$$(X_i - X_i^{(m)}) = X_i \mathbf{1}(|X_i| > m).$$

Then and by independence

$$\begin{aligned} E\Big(\Delta_n^{(m)})^2 &= E(\sum_{i=1}^{n} \frac{(X_i - X_i^{(m)})Y_i}{S_n}\Big)^2 \\ &= E\Big(\sum_{i=1}^{n} \frac{X_i \mathbf{1}(|X_i| > m)Y_i}{S_n}\Big)^2. \end{aligned}$$

Recall the following fact. See in Appendix for the proof.
For an exchangeable variable $\delta_i = \dfrac{X_i Y_i}{\sum_{i=1}^{n} Y_i}$, where X_i and Y_i are i.i.d., we have

$$\begin{aligned} E\Big(\sum_{i=1}^{n} \delta_i\Big)^2 &= E\Big(\sum_{1 \le i,j \le n} \delta_i \delta_j\Big) \\ &= \sum_{i=1}^{n} E(\delta_i^2) + \sum_{i \ne j}^{n} E(\delta_i \delta_j), \end{aligned}$$

moreover we have

$$E\Big(\sum_{i=1}^{n} \delta_i\Big)^2 = nE(\delta^2) + n(n-1)E(\delta_1 \delta_2).$$

Therefore, we obtain

$$E(\Delta_n^{(m)})^2 = nE(X^2 \mathbf{1}(|X| > m)) \cdot E\frac{Y^2}{S_n^2} + n(n-1)[E(X\mathbf{1}(|X| > m))]^2 \cdot E\frac{Y_1 Y_2}{S_n^2}.$$

Notice that since Y, Y_i are iid, we have

$$\begin{aligned} nE\Big(\frac{Y^2}{S_n^2}\Big) &= \sum_{i=1}^{n} E\Big(\frac{Y_i^2}{S_n^2}\Big) \\ &= E\Big(\sum_{i=1}^{n} \frac{Y_i^2}{S_n^2}\Big) \\ &\le 1. \end{aligned}$$

The last line follows because Y_i are non-negative.

Similarly, we have

$$n(n-1)E\left(\frac{Y_1Y_2}{S_n^2}\right) = \sum_{i\neq j}^{n} E\left(\frac{Y_iY_j}{S_n^2}\right).$$

Then by non-negativity of Y_i

$$\begin{aligned}\sum_{i\neq j}^{n} E\left(\frac{Y_iY_j}{S_n^2}\right) &\leq \sum_{i,j=1} E\left(\frac{Y_iY_j}{S_n^2}\right)\\ &= E\left(\frac{S_n^2}{S_n^2}\right)\\ &= 1.\end{aligned}$$

Finally, we deduce that

$$E(\Delta_n^{(m)})^2 \leq E(X^2\mathbf{1}(|X|> m)) + [E(X\mathbf{1}(|X|> m))]^2.$$

From the definition of variance of X I(|X|>m), we have

$$\mathrm{var}(X\mathbf{1}(|X|> m)) = E(X^2\mathbf{1}(|X|> m)) - [E(X\mathbf{1}(|X|> m))]^2 \geq 0,$$

and therefore

$$E(\Delta_n^{(m)})^2 \leq 2E(X^2\mathbf{1}(|X|> m)),$$

which implies that $E(\Delta_n^{(m)})^2$ tends to 0, as m goes to infinity, uniformly in n.

For fixed m, as $R_n^{(m)}$ is bounded by m, denote the sequence $R_n^{(m)}$ by W_m, then by Lemma 2.2.13 there exists $W_{m'}$ such that $\|W_{m'} - W\| \longrightarrow 0$ in L_2, as $m' \longrightarrow \infty$.

Thus by triangular inequality and

$$R_n - W = R_n - R_n^{(m')} + R_n^{(m')} - W_{m'} + W_{m'} - W,$$

we have

$$\|R_n - W\| \leq \|R_n - R_n^{(m')}\| + \|R_n^{(m')} - W_{m'}\| + \|W_{m'} - W\|.$$

As the L_2 norm $\|R_n - R_n^{(m')}\| \longrightarrow 0$ uniformly in n, as $m' \longrightarrow \infty$, by taking a limsup as $n \longrightarrow \infty$ first, then as $m' \longrightarrow \infty$, we have

$$\limsup_{m' \to \infty} \limsup_{n \to \infty} \|R_n - W\| \leq \limsup_{m' \to \infty} \left[\limsup_{n \to \infty} \|R_n - R_n^{(m')}\| + \|W_{m'} - W\| \right].$$

Hence we have

$$\|R_n - W\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

which implies $ER_N^2 \leq K < \infty$, where K is constant, $K > 0$.

The proof of the truncation case is complete.

Chapter 4

Computing Moments of Stochastic Processes

In this chapter, we will illustrate some calculations of moments of stochastic processes via different methods.

4.1 By Using the Definition of Expectation

First of all, we will apply the traditional way to compute the expectations, by using the definition of the expectation.

Recall that if a random variable X has the probability density function f, then the corresponding expectation of $g(X)$ is given as

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

In the following example, we will apply this method to compute the even moments of the Brownian motion.

Example 4.1.1. For fixed time t , since the Brownian motion W_t has a Gaussian distribution $N(0,t)$, for any natural number $n \in \mathbb{N}$ we have

$$\begin{aligned}
E(W_t^{2n}) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} x^{2n} e^{-x^2/2t} dx \\
&= t^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^{2n} e^{-y^2/2} dy \quad (\text{using} \quad x = \sqrt{t} y).
\end{aligned}$$

By using integration by parts, we have

$$\begin{aligned}
E(W_t^{2n}) &= t^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-1) y^{2n-1} d(e^{-y^2/2}) \\
&= t^n \frac{1}{\sqrt{2\pi}} \Big(\int_{-\infty}^{+\infty} (2n-1) y^{2n-2} e^{-y^2/2} dy - y^{2n-1} e^{-y^2/2} \Big|_{y=-\infty}^{y=+\infty} \Big).
\end{aligned}$$

As the fact that

$$y^{2n-1} e^{-y^2/2} \Big|_{y=-\infty}^{y=+\infty} = 0,$$

by straightforward algebra, we obtain

$$\begin{aligned}
E(W_t^{2n}) &= t^n (2n-1) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^{2n-2} e^{-y^2/2} dy \\
&\cdots \\
&= t^n (2n-1)(2n-3) \cdots 1 \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.
\end{aligned}$$

By plugging in

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 1,$$

and

$$\begin{aligned}
(2n-1)(2n-3) \cdots 1 &= \frac{2n(2n-1)(2n-2) \cdots 1}{2n(2n-2)(2n-4) \cdots 2} \\
&= \frac{(2n)!}{2^n n!},
\end{aligned}$$

we have

$$E(W_t^{2n}) = \frac{(2n)!}{2^n n!} t^n.$$

Similarly, the increment $W_t - W_s$ has the Gaussian distribution $N(0, t-s)$, for any natural number $n \in \mathbb{N}$ we have

$$E[(W_t - W_s)^{2n}] = \frac{(2n)!}{2^n n!}(t-s)^n.$$

In particular, for $n = 1$ and $n = 2$, we have

$$E(W_t^2) = t, \quad E(W_t^4) = 3t^2$$

and

$$E[(W_t - W_s)^2] = t - s, \quad E[(W_t - W_s)^4] = 3(t-s)^2.$$

Similarly, we can compute the odd moments of the Brownian motion.

Example 4.1.2. We will show that $E(W_t^{2n+1}) = 0$, for fixed t and $n \in \mathbb{N}$.

Let Z be a random variable with a Gaussian distribution $N(0,t)$ and function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel measurable function and $E(|f(X)|) < \infty$. We can prove that $E(f'(Z)) = \frac{1}{t}E(Zf(Z))$:

By using integration by parts, we have

$$\begin{aligned} E(f'(Z)) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f'(x) e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-x^2/2t} d(f(x)) \\ &= \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} f(x) \Big|_{x=-\infty}^{x=+\infty} + \frac{1}{t} \int_{-\infty}^{+\infty} x f(x) \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx. \end{aligned}$$

As

$$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} f(x) \Big|_{x=-\infty}^{x=+\infty} = 0,$$

we obtain

$$E(f'(Z)) = \frac{1}{t} E(Zf(Z)).$$

Let $f(Z) = Z^{2n}$, by using the previous equation, we have

$$2nE(Z^{2n-1}) = \frac{1}{t} E(Z^{2n+1}).$$

That is,

$$
\begin{aligned}
E(Z^{2n+1}) &= 2ntE(Z^{2n-1}) \\
&= 2^2n(n-1)t^2E(Z^{2n-3}) \\
&\vdots \\
&= 2^n n! t^n E(W_t).
\end{aligned}
$$

And therefore, by the fact $E(W_t) = 0$, we have

$$E(W_t^{2n+1}) = 0,$$

for fixed t and $n \in \mathbb{N}$.

4.2 By Using Properties of Divergence Operator and Skorohod Integral

In this section, we will still try to compute the moments of stochastic processes, while we will apply some properties in Malliavin calculus.

Recall that $W = \{W(h), h \in H\}$ is an isonormal Gaussian process associated with the hilbert space H, which defined in a complete probability space $(\Omega, \mathscr{F}, P)$, and $\mathscr{F}$ is generated by W.

From Definition 2.3.21, we know that the domain of the divergence operator δ, denoted by Domδ, is the set of $H-$valued square integrable random variables $u \in L^2(\Omega; H)$ such that

$$|E(\langle DF, u\rangle_H)| \leq c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

And the divergence operator δ has the following properties:

1. If u belongs to Domδ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

$$E(F\delta(u)) = E(\langle DF, u\rangle_H)$$

for any $F \in \mathbb{D}^{1,2}$.

2. In the case that the elements of Dom$\delta \in L^2(T \times \Omega)$ are square integrable processes, the divergence operator $\delta(u)$ is named as the Skorohod stochastic integral of the process u, and it hols:

$$\delta(u) = \int_T u_t dW_t.$$

In the following example, we will apply the above properties to compute $E(W_T^3)$.

Example 4.2.1. By the definition of the skorohod integral

$$\delta(u) = \int_T u_t dW_t,$$

we can obtain that $\delta(1) = W_T$.

Then we can write $E(W_T^3)$ as $E(W_T^2\delta(1))$.

By using the properties of the divergence operator δ:

$$E(F\delta(u)) = E(\langle DF, u\rangle_H),$$

and

$$\langle h_t, g_t\rangle_H = \int_T h_t g_t dt,$$

we have

$$\begin{aligned} E(W_T^3) &= E(W_T^2\delta(1)) \\ &= E(\langle D_t W_T^2, 1\rangle_H) \\ &= E\Big(\int_0^T D_t W_T^2 dt\Big). \end{aligned}$$

Recall the definition of the derivative operator: suppose that a smooth random variables F has the form

$$\begin{aligned} F &= f(W(h_1),\cdots,W(h_n)) \\ &= f\Big(\int_0^T h_1(t)dW_t,...,\int_0^T h_n(t)dW_t\Big), \end{aligned}$$

where $W(h_1)=\int_0^T h_1(t)dW_t$ is an isonormal Gaussian process associated with the Hilbert space H and f belongs to $C_p^\infty(\mathbb{R}^n)$, $h_1,\cdots,h_n$ are in the Hilbert space H, and $n\geq 1$.

Then, the derivative of a smooth random variable F is the H-valued random variables given by

$$DF=\sum_{i=1}^n \frac{\partial_i f}{\partial W(h_i)}(W(h_1),...,W(h_n))h_i$$

or

$$D_tF=\sum_{i=1}^n \frac{\partial_i f}{\partial W(h_i(t))}(W(h_1(t)),...,W(h_n(t)))h_i(t).$$

Then by definition and the fact $W_T=\int_0^T h(t)dW_t$,where $h(t)=1,t\leq T$, we have

$$\begin{aligned} D_tW_T^2 &= \frac{\partial W_T^2}{\partial W_T}\cdot 1 \\ &= 2W_T. \end{aligned}$$

Plugging in $D_tW_T^2=2W_T$ and $E(W_T)=0$, we have

$$\begin{aligned} E(W_T^3) &= E\Big(\int_0^T 2W_T dt\Big). \\ &= E(2TW_T) \\ &= 0. \end{aligned}$$

Example 4.2.2. By using similar algebra, we obtain

$$\begin{aligned}
E(W_T^4) &= E(W_T^3\delta(1)) \\
&= E(\langle D_t W_T^3, 1\rangle_H) \\
&= E(\int_0^T D_t W_T^3 dt).
\end{aligned}$$

By plugging in $D_t W_T^3 = 3W_T^2$, we have

$$\begin{aligned}
E(W_T^4) &= E\Big(\int_0^T 3W_T^2 dt\Big) \\
&= 3TE(W_T^2).
\end{aligned}$$

By using the fact $E(W_T^2) = T$, we have

$$E(W_T^4) = 3T^2.$$

Example 4.2.3. In the case of $E(W_T^k)$, $k = 2, 3, \cdots$, we have

$$\begin{aligned}
E(W_T^k) &= E(W_T^{k-1}\delta(1)) \\
&= E\Big(\int_0^T D_t W_T^{k-1} dt\Big) \\
&= (k-1)TE(W_T^{k-2}).
\end{aligned}$$

By using the fact $E(W_T) = 0$ and $E(W_T^2) = T$, for $n \in \mathbb{N}$, we obtain

$$E(W_T^{2n+1}) = 2^n n! T^n E(W_T) = 0,$$

and

$$
\begin{aligned}
E(W_T^{2n}) &= (2n-1)TE(W_T^{2n-2}) \\
&\vdots \\
&= \frac{(2n)!}{2^n n!}T^{n-1}E(W_T^2) \\
&= \frac{(2n)!}{2^n n!}T^n.
\end{aligned}
$$

which implies the same result we deduced in section 4.1.

4.3 By Using Itô Formula

In this section, we will show how to compute the expectations by applying Itô formula. Espectially, we will calculate up to 4th moment of the Ornstein-Uhlenbeck process as well as the expectation of exponential Ornstein-Uhlenbeck process.

Assume the stochastic process x_t can be modelled as an Ornstein-Uhlenbeck process, then it follows :

$$dx_t = \theta(m - x_t)dt + \sigma dW_t,$$

with $t \geq 0$ and the initial value x being denoted by x_0. The parameter θ is the rate of this reversion, the parameter m is the average level, also known as the long-run average value and the parameter σ, ($\sigma \geq 0$), represents the volatility. dW_t is the increment of a standard Brownian Motion, $W_t \sim N(0,t)$.

Thus, this one-factor Ornstein-Uhlenbeck process is formed as a drift term $\theta(m - x_t)dt$, plus a stochastic term σdW_t . When the current value of x_t is greater than the average level m, the drift term is negative, leading to the result of pulling the value down towards the average level , also known as its place of equilibrium; otherwise, the drift is positive, pulling the value up towards its average level.

4.3.1 Properties of Ornstein-Uhlenbeck Process

By applying Itô calculus, the following properties of Ornstein-Uhlenbeck process can be obtained.

Lemma 4.3.1. *An Ornstein-Uhlenbeck process x_t of the form*

$$dx_t = \theta(m - x_t)dt + \sigma dW_t,$$

is a Gasussian process, and the following properties hold true:

1. $E(x_t) = m + (x_0 - m)e^{-\theta t}$;
2. $E(x_t^2) = [m + (x_0 - m)e^{-\theta t}]^2 + \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$;
3. $\mathrm{var}(x_t) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$ *and* $\mathrm{cov}(x_t, x_s) = \frac{\sigma^2}{2\theta}(e^{-\theta|t-s|} - e^{-\theta(t+s)})$;
4. $E(x_t^3) = [m + (x_0 - m)e^{-\theta t}]^3 + \frac{3\sigma^2}{2\theta}(1 - e^{-2\theta t})[m + (x_0 - m)e^{-\theta t}]$;
5. $E(x_t^4) = [m + (x_0 - m)e^{-\theta t}]^4 + \frac{3\sigma^2}{\theta}(1 - e^{-2\theta t})[m + (x_0 - m)e^{-\theta t}]^2 + \frac{3\sigma^4}{4\theta^2}(1 - e^{-2\theta t})^2$;
6. $E[\exp(Zx_t)] = \exp\Big[Zm + Z(x_0 - m)e^{-\theta t} + \frac{\sigma^2 Z^2}{4\theta}(1 - e^{-2\theta t})\Big]$ *for a constant* $Z > 0$.

See in [36] p84-85 for the Gaussianity and the moments of Ornstein-Uhlenbeck process, the following proof is modified.

Proof. 1.

Define a new process $f(t, x_t) = x_t e^{\theta t}$. As $f_t^{'} = \theta x_t e^{\theta t}$, $f_x^{'} = e^{\theta t}$, $f_{xx}^{''} = 0$ and $(dx_t)^2 = \sigma^2 dt$, by applying Itô formula, we have the following differential representation

$$\begin{aligned}
d(x_t e^{\theta t}) &= f_t^{'} dt + f_x^{'} dx + \frac{1}{2} f_{xx}^{''}(dx_t)^2 \\
&= \theta x_t e^{\theta t} dt + e^{\theta t}(\theta(m - x_t)dt + \sigma x_t dW_t) \\
&= \theta m e^{\theta t} dt + \sigma e^{\theta t} dW_t.
\end{aligned}$$

Thus, the process $f(t,x_t) = x_t e^{\theta t}$ in integral notation is

$$\begin{aligned} x_t e^{\theta t} &= x_0 + \int_0^t \theta m e^{\theta s} ds + \int_0^t \sigma e^{\theta s} dW_s \\ &= x_0 + m(e^{\theta t} - 1) + \int_0^t \sigma e^{\theta s} dW_s, \end{aligned}$$

which implies

$$x_t = m + (x_0 - m)e^{-\theta t} + \int_0^t \sigma e^{\theta(s-t)} dW_s.$$

As the integral $\int_0^t \sigma e^{\theta(s-t)} dW_s$ is an Itô integral, we know that

$$E\Big(\int_0^t \sigma e^{\theta(s-t)} dW_s\Big) = 0,$$

and therefore by taking expactations on both sides, we obtain

$$E(x_t) = m + (x_0 - m)e^{-\theta t}.$$

2. Similarly, define the process $g(t,x_t) = x_t^2 e^{2\theta t}$, and therefore

$$g_t^{'} = 2\theta x_t^2 e^{2\theta t} \ , \ g_x^{'} = 2x_t e^{2\theta t} \ , \ g_{xx}^{''} = 2e^{2\theta t}.$$

Hence, apply the Itô formula we obtain

$$\begin{aligned} d(x_t^2 e^{2\theta t}) &= g_t^{'} dt + g_x^{'} dx_t + \frac{1}{2} g_{xx}^{''} (dx_t)^2 \\ &= e^{2\theta t}(2\theta m x_t + \sigma^2) dt + 2\sigma e^{2\theta t} x_t dW_t. \end{aligned}$$

Thus, by integrate both sides from 0 to t, we have

$$x_t^2 e^{2\theta t} = x_0^2 + \int_0^t e^{2\theta s}(2\theta m x_s + \sigma^2) ds + 2\sigma \int_0^t e^{2\theta s} x_s dW_s,$$

and times the exponential $e^{-2\theta t}$ yields

$$x_t^2 = x_0^2 e^{-2\theta t} + e^{-2\theta t} \int_0^t e^{2\theta s}(2\theta m x_s + \sigma^2) ds + 2\sigma e^{-2\theta t} \int_0^t e^{2\theta s} x_s dW_s.$$

Note that the last term $2\sigma e^{-2\theta t}\int_0^t e^{2\theta s}x_s dW_s$ is an Itô integral, we know its expactation is zero.

Then by taking expactations on both sides and Fubini's theorem, we have

$$E(x_t^2) = x_0^2 e^{-2\theta t} + e^{-2\theta t}\int_0^t e^{2\theta s}(2\theta m E(x_s) + \sigma^2)ds.$$

To compute the integral, plug in $E(x_s) = m + (x_0 - m)e^{-\theta s}$ and rearrange, we obtain:

$$\begin{aligned}
E(x_t^2) &= x_0^2 e^{-2\theta t} + e^{-2\theta t}\int_0^t e^{2\theta s}\left\{2\theta m[m + (x_0 - m)e^{-\theta s}] + \sigma^2\right\}ds \\
&= e^{-2\theta t}\left[x_0^2 + m^2(e^{2\theta t} - 1) + 2m(x_0 - m)(e^{\theta t} - 1) + \frac{\sigma^2}{2\theta}(e^{2\theta t} - 1)\right] \\
&= (m + (x_0 - m)e^{-\theta t})^2 + \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}).
\end{aligned}$$

3. From $\mathrm{var}(x_t) = E(x_t^2) - (E(x_t))^2$ and the results of 1 and 2, we have

$$\mathrm{var}(x_t) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}).$$

This together with the form of x_t

$$x_t = m + (x_0 - m)e^{-\theta t} + \int_0^t \sigma e^{\theta(s-t)}dW_s$$

and

$$E(x_t) = m + (x_0 - m)e^{-\theta t},$$

means that the law of the Gaussian stochastic process x_t is the normal distribution

$$N\Big(m + (x_0 - m)e^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\Big),$$

Moreover, we can compute the covariances of the process x_t in the following way

$$\begin{aligned}\mathrm{cov}(x_t,x_s) &= E[(x_t-E(x_t))(x_s-E(x_s))]\\ &= E\Big[\sigma^2 e^{-\theta(t+s)}\Big(\int_0^t e^{\theta h}dW_h\Big)\Big(\int_0^s e^{\theta h}dW_h\Big)\Big].\end{aligned}$$

By applying the Itô isometry

$$\begin{aligned}\mathrm{cov}(x_t,x_s) &= \sigma^2 e^{-\theta(t+s)}E\Big[\Big(\int_0^t e^{\theta h}dW_h\Big)\Big(\int_0^s e^{\theta h}dW_h\Big)\Big]\\ &= \sigma^2 e^{-\theta(t+s)}\int_0^{t\wedge s} e^{2\theta h}dh\\ &= \frac{\sigma^2}{2\theta}(e^{-\theta|t-s|}-e^{-\theta(t+s)}).\end{aligned}$$

4. Let $\varphi(t,x_t)=x_t^3e^{3\theta t}$, then $\varphi_t^{'}=3\theta x_t^3 e^{3\theta t}$, $\varphi_x^{'}=3x_t^2e^{3\theta t}$ and $\varphi_{xx}^{''}=6x_te^{3\theta t}$. Thus by using Itô formula, we have

$$\begin{aligned}d(x_t^3e^{3\theta t}) &= \varphi_t^{'}dt+\varphi_x^{'}dx_t+\frac{1}{2}\varphi_{xx}^{''}(dx_t)^2\\ &= e^{3\theta t}(3\theta m x_t^2+3\sigma^2x_t)dt+3\sigma x_t^2e^{3\theta t}dW_t.\end{aligned}$$

Integrate on both sides we obtain

$$x_t^3e^{3\theta t}=x_0^3+\int_0^t e^{3\theta s}(3\theta m x_s^2+3\sigma^2x_s)ds+\int_0^t 3\sigma x_s^2e^{3\theta s}dW_s,$$

again times the exponential $e^{-3\theta t}$ yields

$$x_t^3=x_0^3e^{-3\theta t}+e^{-3\theta t}\int_0^t e^{3\theta s}(3\theta m x_s^2+3\sigma^2x_s)ds+e^{-3\theta t}\int_0^t 3\sigma x_s^2e^{3\theta s}dW_s.$$

By taking the expectation and Fubini's theorem, then by plugging the values of the second moment, we have

$$\begin{aligned}E(x_t^3) &= x_0^3 e^{-3\theta t} + e^{-3\theta t}\int_0^t e^{3\theta s}(3\theta m E(x_s^2) + 3\sigma^2 E(x_s))ds \\ &= x_0^3 e^{-3\theta t} + e^{-3\theta t}\int_0^t e^{3\theta s} 3\theta m\{[m+(x_0-m)e^{-\theta s}]^2 \\ &\quad + \frac{\sigma^2}{2\theta}(1-e^{-2\theta s})) + 3\sigma^2[m+(x_0-m)e^{-\theta s}]\}ds,\end{aligned}$$

and therefore by rearranging

$$E(x_t^3) = [m+(x_0-m)e^{-\theta t}]^3 + \frac{3\sigma^2}{2\theta}(1-e^{-2\theta t})[m+(x_0-m)e^{-\theta t}].$$

5. Define a new process $\psi(t,x_t) = x_t^4 e^{4\theta t}$, as $\psi_t' = 4\theta x_t^4 e^{4\theta t}$, $\psi_x' = 4x_t^3 e^{4\theta t}$ and $\psi_{xx}'' = 12x_t^2 e^{4\theta t}$, by Itô formula we have

$$\begin{aligned}d(x_t^4 e^{4\theta t}) &= \psi_t' dt + \psi_x' dx_t + \frac{1}{2}\psi_{xx}''(dx_t)^2 \\ &= e^{4\theta t}(4\theta m x_t^3 + 6\sigma^2 x_t^2)dt + 4\sigma x_t^3 e^{4\theta t} dW_t.\end{aligned}$$

Thus, Integrate on both sides and again times the exponential $e^{-4\theta t}$ yields

$$x_t^4 e^{4\theta t} = x_0^4 + \int_0^t e^{4\theta s}(4\theta m x_s^3 + 6\sigma^2 x_s^2)ds + \int_0^t 4\sigma x_s^3 e^{4\theta s} dW_s,$$

and then

$$x_t^4 = x_0^4 e^{-4\theta t} + e^{-4\theta t}\int_0^t e^{4\theta s}(4\theta m x_s^3 + 6\sigma^2 x_s^2)ds + \int_0^t 4\sigma x_s^3 e^{4\theta s} dW_s.$$

By taking expectation on both sides and Fubini's theorem, then by plugging the values of the third moment, we have

$$
\begin{aligned}
E(x_t^4) &= x_0^4 e^{-4\theta t} + e^{-4\theta t}\int_0^t e^{4\theta s}[4\theta m E(x_s^3) + 6\sigma^2 E(x_s^2)]ds \\
&= x_0^4 e^{-4\theta t} + \int_0^t 4\theta m e^{4\theta(s-t)}\left\{[m+(x_0-m)e^{-\theta t}]^3 + \frac{3\sigma^2}{2\theta}(1-e^{-2\theta t})[m+(x_0-m)e^{-\theta t}]\right\}ds \\
&\quad + \int_0^t 6\sigma^2 e^{4\theta(s-t)}\left\{[m+(x_0-m)e^{-\theta t}]^2 + \frac{\sigma^2}{2\theta}(1-e^{-2\theta t})\right\}ds.
\end{aligned}
$$

Finally, by computing integrals and rearranging

$$
E(x_t^4) = [m+(x_0-m)e^{-\theta t}]^4 + \frac{3\sigma^2}{\theta}(1-e^{-2\theta t})[m+(x_0-m)e^{-\theta t}]^2 + \frac{3\sigma^4}{4\theta^2}(1-e^{-2\theta t})^2.
$$

6. Set $\omega(t,x_t) = \exp(Zx_t)$ and therefore we have

$$
\omega_t' = 0,\ \omega_x' = Z\exp(Zx_t).\ \omega_{xx}'' = Z^2\exp(Zx_t),
$$

Then by Itô formula we have

$$
\begin{aligned}
d\omega(t,x_t) &= \omega_t' dt + \omega_x' dx_t + \frac{1}{2}\omega_{xx}''(dx_t)^2 \\
&= \left[\theta Z(m-x_t)\exp(Zx_t) + \frac{1}{2}\sigma^2 Z^2\exp(Zx_t)\right]dt + \sigma Z\exp(Zx_t)dW_t,
\end{aligned}
$$

which implies

$$
\exp(Zx_t) = \exp(Zx_0) + \int_0^t \left[\theta Z(m-x_s)\exp(Zx_s) + \frac{1}{2}\sigma^2 Z^2\exp(Zx_s)\right]ds + \int_o^t \sigma Z\exp(Zx_s)dW_s.
$$

By taking expectations on both sides and Fubini's theorem, we have

$$
E[\exp(Zx_t)] = \exp(Zx_0) + \int_0^t \left\{Z\left(\theta m + \frac{1}{2}\sigma^2 Z\right)E[\exp(Zx_s)] - \theta Z E[x_s\exp(Zx_s)]\right\}ds.
$$

If we set $u(t,Z) = E[\exp(Zx_t)]$, the partial derivative of u with respect to Z is given as

$$u'_Z = E[x_t \exp(Zx_t)].$$

Rewriting the equation on $E[\exp(Zx_t)]$ we have

$$u(t,Z) = \exp(Zx_0) + \int_0^t \Big[Z\Big(\theta m + \frac{1}{2}\sigma^2 Z\Big)u(s,Z) - \theta Z u'_Z(s,Z)\Big]ds$$

By using Leibniz's integral rule [15] for differentiating an integral

$$\Big(\int_{a(t)}^{b(t)} k(t,x)dx\Big)'_t = \int_{a(t)}^{b(t)} k'_t(t,x)dx + b'(t)k(t,b(t)) - a'(t)k(t,a(t)),$$

where $k(t,x)$ is a function such that both $k(t,x)$ and its partial derivative $k'_t(t,x)$ are continuous in x and t in some region of the (t,x)-plane, the function $a(t)$ and $b(t)$ are both continuous and both have continuous derivatives for $t \in [0,T]$, and $a(t) \leq x \leq b(t)$, we can partial derivative $u(t,Z)$ with respect to t, and obtain a PDE

$$u'_t + \theta Z u'_Z - Z\Big(\theta m + \frac{1}{2}\sigma^2 Z\Big)u = 0.$$

To find the characteristics curves of this PDE, we solve the ordinary equation $\frac{dZ}{dt} = \theta Z$ and get

$$\ln Z - \theta t = \text{const}, \quad \text{for} \quad Z \neq 0.$$

Using separation of variables, we set

$$\xi(t,Z) = t \quad , \quad \eta(t,Z) = \ln Z - \theta t,$$

and the corresponding Jacobian is found by straightforward calculations

$$\begin{aligned}\frac{\partial(\xi,\eta)}{\partial(t,Z)} &= \det\begin{pmatrix}\xi_t & \xi_Z \\ \eta_t & \eta_Z\end{pmatrix} \\ &= \xi_t\eta_Z - \xi_Z\eta_t \\ &= \frac{1}{Z} \neq 0.\end{aligned}$$

Hence, the transformation of the coordinates $(t,Z) \leftrightarrow (\xi,\eta)$ is both non-singular and smooth. The inverse transformation is given by

$$t(\xi,\eta) = \xi \quad \text{and} \quad Z(\xi,\eta) = e^{(\theta\xi+\eta)}.$$

And therefore, the transformed PDE is

$$v_\xi - Z\Big(\theta m + \frac{1}{2}\sigma^2 Z\Big)v = 0.$$

where $u(t,Z) = v(\xi,\eta)$.

By rearranging and plugging in $Z(\xi,\eta) = e^{(\theta\xi+\eta)}$, we have

$$(\ln v)_\xi = \theta m e^{(\theta\xi+\eta)} + \frac{1}{2}\sigma^2 e^{2(\theta\xi+\eta)}.$$

Integrating both sides with respect to ξ, we have

$$\ln v = m e^{(\theta\xi+\eta)} + \frac{1}{4\theta}\sigma^2 e^{2(\theta\xi+\eta)} + K_1(\eta),$$

where the function $K_1(\eta)$ is a function that only depends on variable η.

That is,

$$v(\xi,\eta) = \exp\Big[m e^{(\theta\xi+\eta)} + \frac{1}{4\theta}\sigma^2 e^{2(\theta\xi+\eta)}\Big] \cdot K_2(\eta),$$

where the function $K_2(\eta)$ is a function that only depends on variable η.

By using the condition $v(0,\eta) = u(0,Z) = \exp(e^{\eta} x_0)$, we have

$$\exp(e^{\eta} x_0) = \exp\Big(me^{\eta} + \frac{1}{4\theta}\sigma^2 e^{2\eta}\Big) \cdot K_2(\eta).$$

By rearranging, we obtain

$$K_2(\eta) = \exp\Big[(x_0 - m)e^{\eta} - \frac{1}{4\theta}\sigma^2 e^{2\eta}\Big].$$

And therefore, the function $v(\xi,\eta)$ can be represented as follows

$$v(\xi,\eta) = \exp\Big[(x_0 - m + me^{\theta\xi})e^{\eta} + \frac{1}{4\theta}\sigma^2 e^{2\eta}(e^{2\theta\xi} - 1)\Big].$$

Hence, by the relation between functions $u(t,Z)$ and $v(\xi,\eta)$, and plugging in $\xi = t$ and $\eta = \ln Z - \theta t$, we deduce

$$\begin{aligned} u(t,Z) &= v(\xi,\eta) \\ &= \exp\Big[(x_0 - m + me^{\theta\xi})e^{\eta} + \frac{1}{4\theta}\sigma^2 e^{2\eta}(e^{2\theta\xi} - 1)\Big] \\ &= \exp\Big[(x_0 - m + me^{\theta t})Ze^{-\theta t} + \frac{1}{4\theta}\sigma^2 Z^2 e^{-2\theta t}(e^{2\theta t} - 1)\Big]. \end{aligned}$$

By rearranging, we have

$$\begin{aligned} E[\exp(Zx_t)] &= u(t,Z) \\ &= \exp\Big[Zm + Z(x_0 - m)e^{-\theta t} + \frac{\sigma^2 Z^2}{4\theta}(1 - e^{-2\theta t})\Big]. \end{aligned}$$

□

Observing from property 1 and 3, the Ornstein-Uhlenbeck process x_t has the Gaussian distribution of $N\Big(m + (x_0 - m)e^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\Big)$. Then as time t increases, the variance increases. When the time t goes to infinity, the expectation and the variance of x_t tend to converge to m and $\dfrac{\sigma^2}{2\theta}$, respectively, which implies that when t tends to infinity, the law of x_t converges to the normal law $N\Big(m, \dfrac{\sigma^2}{2\theta}\Big)$.

As for each $t > 0$ the law of x_t will always be $N\left(m, \dfrac{\sigma^2}{2\theta}\right)$, if the initial condition x_0 has distribution $N\left(m, \dfrac{\sigma^2}{2\theta}\right)$, this distribution is called stationary.

Also, as θ tends to zero, the process becomes $dx_t = \sigma dW_t$, which is a Brownian motion with the standard deviation of $\sigma\sqrt{t}$.

If θ goes to infinity, then the variance tends to be zero, showing that the process x_t cannot escape from its place of equilibrium, which is the average level m, even for a moment.

4.3.2 Additional Calculations for Ornstein-Ulenbeck Type Process

This part is motivated by the open question to calculate the expectation of $E(x_t^k)$ of an Itô process which has a similar representation as an Ornstein-Ulenbeck process.

More exactly, we assume the stochastic processx_t follows the Ornstein-Uhlenbeck type stochastic differential equation as below:

$$dx_t = \theta(m - x_t^2)dt + \sigma dW_t,$$

with $x_t = 0$ and W_t is a Brownian motion.

In order to calculate the expectation of x_t, we define another process $f(t, x_t) = x_t^k$, and let function $g(t, k)$ denote $E(x_t^k)$.

When $k = 0$, $f(t, x_t) = x_t^0 = 1$, that is $g(t, 0) = 1$.

When $k = 1$, by integrating on both sides, we have

$$x_t = \int_0^t \theta(m - x_s^2)ds + \int_0^t \sigma dW_s,$$

Taking expectations on both sides, we have

$$E(x_t) = \int_0^t \theta(m - E(x_s^2))ds,$$

which is

$$g(t, 1) = \int_0^t \theta(m - g(s, 2))ds.$$

When $k \geq 2$, we get that $f_t^{'} = 0$, $f_x^{'} = kx_t^{k-1}$, $f_{xx}^{''} = k(k-1)x_t^{k-2}$ and $(dx_t)^2 = \sigma^2 dt$. Then, by Itô lemma

$$dx_t^k = kx_t^{k-1}\theta(m - x_t^2)dt + kx_t^{k-1}\sigma dW_t + \frac{1}{2}k(k-1)x_t^{k-2}\sigma^2 dt.$$

Integrating on both sided, we deduce

$$x_t^k = \int_0^t \Big[\theta mkx_s^{k-1} - \theta kx_s^{k+1} + \frac{1}{2}\sigma^2 k(k-1)x_s^{k-2}\Big]ds + \int_0^t \sigma kx_s^{k-1}dW_s,$$

Then by using

$$E\Big(\int_0^t \sigma kx_s^{k-1}dW_s\Big) = 0,$$

and by taking expectations and Fubini's theorem, we have

$$E(x_t^k) = \int_0^t \Big[\theta mkE(x_s^{k-1}) - \theta kE(x_s^{k+1}) + \frac{1}{2}\sigma^2 k(k-1)E(x_s^{k-2})\Big]ds.$$

That is,

$$g(t,k) = \int_0^t \Big[\theta mkg(s,k-1) - \theta kg(s,k+1) + \frac{1}{2}\sigma^2 k(k-1)g(s,k-2)\Big]ds.$$

Partial differentiating both sides with respect to t, we get that

$$g_t'(t,k) = \theta mkg(t,k-1) - \theta kg(t,k+1) + \frac{1}{2}\sigma^2 k(k-1)g(t,k-2).$$

Let constants C_1, C_2 and C_3 represent θm, $-\theta$ and $\frac{1}{2}\sigma^2$, respectively, then the equation becomes

$$g_t'(t,k) = C_1 kg(t,k-1) + C_2 kg(t,k+1) + C_3 k(k-1)g(t,k-2).$$

Suppose that z is a complex number frequency parameter $z = a + ib$, a and b are real numbers, by taking Laplace transform on both sides, we get that

$$\begin{aligned}\int_0^{+\infty} g_t'(t,k)e^{-zt}dt = C_1 k \int_0^{+\infty} g(t,k-1)e^{-zt}dt \\ + C_2 k \int_0^{+\infty} g(t,k+1)e^{-zt}dt \\ + C_3 k(k-1) \int_0^{+\infty} g(t,k-2)e^{-zt}dt.\end{aligned}$$

As the fact

$$\begin{aligned}\int_0^{+\infty} g_t'(t,k)e^{-zt}dt &= \int_0^{+\infty} e^{-zt}dg(t,k) \\ &= e^{-zt}g(t,k)\Big|_{t=0}^{t=+\infty} + z\int_0^{+\infty} g(t,k)e^{-zt}dt \\ &= z\int_0^{+\infty} g(t,k)e^{-zt}dt,\end{aligned}$$

($g(t,0) = 1$, $g(0,k) = 0$, $g(0,0) = 0$ as we define $0/0 = 0$.) we have that

$$\begin{aligned}z\int_0^{+\infty} g(t,k)e^{-zt}dt = C_1 k \int_0^{+\infty} g(t,k-1)e^{-zt}dt \\ + C_2 k \int_0^{+\infty} g(t,k+1)e^{-zt}dt \\ + C_3 k(k-1) \int_0^{+\infty} g(t,k-2)e^{-zt}dt,\end{aligned}$$

Set $\mathscr{L}_g(k) = \int_0^{+\infty} g(t,k)e^{-zt}dt$, we have

$$z\mathscr{L}_g(k) = C_1 k\mathscr{L}_g(k-1) + C_2 k\mathscr{L}_g(k+1) + C_3 k(k-1)\mathscr{L}_g(k-2),$$

with

$$\begin{aligned}\mathscr{L}_g(0) &= \int_0^{+\infty} e^{-zt}dt \\ &= -\frac{1}{z}e^{-zt}\Big|_{t=0}^{t=+\infty} \\ &= \frac{1}{z}.\end{aligned}$$

Let sequences a_k denote $\mathscr{L}_g(k)$, and set $R_1 = \frac{C_1}{z}$, $R_2 = \frac{C_2}{z}$ and $R_3 = \frac{C_3}{z}$, we have

$$a_k = R_1 k a_{k-1} + R_2 k a_{k+1} + R_3 k(k-1) a_{k-2}, \quad a_0 = \frac{1}{z}.$$

Let u be a variable, and by multiplying u^k and taking the sums of both sides, we get

$$\sum_{k=2}^{\infty} a_k u^k = R_1 \sum_{k=2}^{\infty} k a_{k-1} u^k + R_2 \sum_{k=2}^{\infty} k a_{k+1} u^k + R_3 \sum_{k=2}^{\infty} k(k-1) a_{k-2} u^k.$$

Now, we apply several simple properties of the power series.

Given $0 < p < 1$, the following properties can be obtained:

1. $\sum_{n=0}^{\infty} p^n = \lim_{n \longrightarrow \infty} \frac{1-p^n}{1-p} = \frac{1}{1-p} = C$, where C is constant.
2. $\sum_{n=N}^{\infty} p^n = p^N \sum_{n=0}^{\infty} p^n = \frac{p^N}{1-p} = p^N \cdot C$.
3. $\sum_{n=0}^{\infty} n p^n = p \sum_{n=0}^{\infty} n p^{n-1} = p\left(\sum_{n=0}^{\infty} p^n\right)' = p\left(\frac{1}{1-p}\right)' = \frac{p}{(1-p)^2}$.
4. $\sum_{n=N}^{\infty} n p^n = p^N \sum_{n=0}^{\infty} (n+N) p^n = p^N \left(\sum_{n=0}^{\infty} n p^n + N \sum_{n=0}^{\infty} p^n\right)$
$= p^N \left[\frac{p}{(1-p)^2} + N \cdot \frac{1}{1-p}\right] = N \cdot \frac{p^N}{1-p} + \frac{p^{N+1}}{(1-p)^2}$.

The techniques used here will be applied in the following part.

Let $\mathscr{A}(u)$, which is a function of u, denote $\sum_{k=0}^{\infty} a_k u^k$.

Then we will rewrite the equation

$$\sum_{k=2}^{\infty} a_k u^k = R_1 \sum_{k=2}^{\infty} k a_{k-1} u^k + R_2 \sum_{k=2}^{\infty} k a_{k+1} u^k + R_3 \sum_{k=2}^{\infty} k(k-1) a_{k-2} u^k.$$

as an equation with respect to $\mathscr{A}(u)$.

For the first term on the right side, by algebra we have

$$\begin{aligned}\sum_{k=2}^{\infty} ka_{k-1}u^k &= \sum_{i=1}^{\infty}(i+1)a_i u^{i+1}\\ &= u\sum_{i=1}^{\infty}(i+1)a_i u^i\\ &= u\Big(\sum_{i=1}^{\infty} a_i u^{i+1}\Big)'_u.\end{aligned}$$

By plugging in $\sum_{k=1}^{\infty} a_k u^k = \mathscr{A}(u) - a_0$, we have

$$\sum_{k=2}^{\infty} ka_{k-1}u^k = u[u(\mathscr{A}(u) - a_0)]'_u.$$

For the second term on the right side, by algebra we have

$$\begin{aligned}\sum_{k=2}^{\infty} ka_{k+1}u^k &= \sum_{j=3}^{\infty}(j-1)a_j u^{j-1}\\ &= u\sum_{j=3}^{\infty}(j-1)a_j u^{j-2}\\ &= u\Big(\sum_{j=3}^{\infty} a_j u^{j-1}\Big)'_u\\ &= u\Big(\frac{1}{u}\sum_{j=3}^{\infty} a_j u^j\Big)'_u\end{aligned}$$

By plugging in $\sum_{k=3}^{\infty} a_k u^k = \mathscr{A}(u) - a_0 - a_1 u - a_2 u^2$, we have

$$\sum_{k=2}^{\infty} ka_{k+1}u^k = u\Big[\frac{1}{u}(\mathscr{A}(u) - a_0 - a_1 u - a_2 u^2)\Big]'_u.$$

For the last term on the right side, by algebra we have

$$\begin{aligned}
\sum_{k=2}^{\infty} k(k-1)a_{k-2}u^k &= \sum_{l=0}^{\infty}(l+1)(l+2)a_l u^{l+2} \\
&= u^2\sum_{l=0}^{\infty}(l+1)(l+2)a_l u^l \\
&= u^2\Big(\sum_{l=0}^{\infty} a_l u^{l+2}\Big)''_{uu} \\
&= u^2\Big(u^2\sum_{l=0}^{\infty} a_l u^l\Big)''_{uu}.
\end{aligned}$$

That is

$$\sum_{k=2}^{\infty} k(k-1)a_{k-2}u^k = u^2[u^2\mathscr{A}(u)]''_{uu}.$$

Hence the equation

$$\sum_{k=2}^{\infty} a_k u^k = R_1\sum_{k=2}^{\infty} k a_{k-1}u^k + R_2\sum_{k=2}^{\infty} k a_{k+1}u^k + R_3\sum_{k=2}^{\infty} k(k-1)a_{k-2}u^k$$

becomes a second order ODE

$$\begin{aligned}
\mathscr{A}(u) - a_0 - a_1 u = R_1 u[u(\mathscr{A}(u)-a_0)]'_u& \\
+ R_2 u\Big[\frac{1}{u}(\mathscr{A}(u) - a_0 - a_1 u - a_2 u^2)\Big]'_u &+ R_3 u^2[u^2\mathscr{A}(u)]''_{uu},
\end{aligned}$$

with initial condition $\mathscr{A}(0) = 0$.

The future work is to solve the second order ODE to obtain Ex_t^k.

Chapter 5

Computing operators D, δ, L and associated characteristics

In this chapter, we will illustrate the computation of the operators D, δ, L and associated characteristics in Malliavin calculus. Not only it will help us to gain the understanding about the analysis on Wiener space, which can be considered as the fundamental part of Malliavin calculus, but also will make a contribution in the computation of Greeks in Chapter 6. As these operators will play essential roles in computing the Greeks.

5.1 The Derivative Operator D

First of all, we start by computing the derivative operator D, which is defined as follows:

Recall that $W = \{W(h), h \in H\}$ is an isonormal Gaussian process associated with the Hilbert space H, which defined in a complete probability space $(\Omega, \mathscr{F}, P)$, and $\mathscr{F}$ is generated by W.

If a smooth random variable F has the form

$$\begin{aligned} F &= f(W(h_1), \cdots, W(h_n)) \\ &= f\Big(\int_0^T h_1(t)dW_t, ..., \int_0^T h_n(t)dW_t\Big), \end{aligned}$$

where $W(h_1) = \int_0^T h_1(t)dW_t$ and f belongs to $C_p^\infty(\mathbb{R}^n)$, $h_1, \cdots, h_n$ are in the Hilbert space H, and $n \geq 1$.

Then, the derivative of a smooth random variable F is the H-valued random variables given by

$$DF = \sum_{i=1}^{n} \frac{\partial_i f}{\partial W(h_i)}(W(h_1), ..., W(h_n))h_i$$

or

$$D_t F = \sum_{i=1}^{n} \frac{\partial_i f}{\partial W(h_i(t))}(W(h_1(t)), ..., W(h_n(t)))h_i(t).$$

Recall that the notations are equivalent

$$W_t = W_t = B(t).$$

Example 5.1.1. In order to compute the derivative of the isonormal Gaussian process $W(h)$, where h belongs to the associated Hilbert space H.

First we know that for the smooth random variable $W(h) = f(W(h))$, the function f is given as $f(x) = x$.

Then by using $f' = 1$ and the Definition 2.3.12, we have

$$D(W(h)) = f' \cdot h = h.$$

As for the smooth random variable $W^2(h) = f(W(h))$, the function f is given as $f(x) = x^2$ and $f' = 2x$. By applying the same definition, we have

$$D(W^2(h)) = f' \cdot h = 2W(h)h.$$

Example 5.1.2. Consider the Brownian motion W_T, as the fact $W_T = \int_0^T h(t)dW_t$, where $h(t) = 1, t \leq T$, the stochastic process W_T^2 can be considered as

$$W_T^2 = f\Big(\int_0^T h(t)dW_t\Big),$$

where $f(x) = x^2$.

Then we can obtain the derivative is given as

$$D_t(W_T^2) = f' \cdot h(t) = 2W_T.$$

Similarly, we can obtain $D_t(W_T^k) = kW_T^{k-1}$.

Example 5.1.3. (Related to stock price). Given $S_T = S_0 \exp\Big(\int_0^T \Big(r - \frac{1}{2}\sigma\Big) dt + \int_0^T \sigma dW_t\Big)$,we have

$$\begin{aligned} S_T &= S_0 \exp\Big(\int_0^T (r + \frac{1}{2}\sigma) dt + \int_0^T \sigma dW_t\Big) \\ &= f\Big(\int_0^T h(t) dW_t\Big), \end{aligned}$$

where $h(t) = \sigma, t \leq T$.

Then the derivative is

$$\begin{aligned} D_t S_T &= \frac{\partial f}{\partial x}\Big(\int_0^T \sigma dW_t\Big) h(t) \\ &= S_0 \exp(\int_0^T \Big(r + \frac{1}{2}\sigma) dt + \int_0^T \sigma dW_t\Big) \cdot \sigma \\ &= \sigma S_T. \end{aligned}$$

Moreover, we have compute the D^h operator for S_T by using the definition $D^h F = \langle DF, h \rangle_H$, that is

$$\begin{aligned} D^h S_T &= \langle D_t S_T, h \rangle_H \\ &= \int_0^T D_t S_T dt. \end{aligned}$$

By plugging in the previous result $D_t S_T = \sigma S_T$, we obtain

$$\begin{aligned} D^h S_T &= \int_0^T \sigma S_T dt \\ &= \sigma T S_T. \end{aligned}$$

Example 5.1.4. Given $F = f(W(h_1), W(h_2) = W(h_1) \times W(h_2)$, then

$$\begin{aligned} D_t F &= \frac{\partial f}{\partial x_1} h_1(t) + \frac{\partial f}{\partial x_2} h_2(t) \\ &= W(h_2(t))h_1(t) + W(h_1(t))h_2(t). \end{aligned}$$

The following part is motivated by calculations about the derivative operator of a stochastic integral $\int_0^T u_s dW_s$.

Recall the method we use in Section 2.3.3 to define a stochastic process:

If a stochastic process u in $L^2_{a,T}$ is a simple process, we will define it as the form of:

$$u_s = \sum_{k=1}^{n} \phi_k \mathbf{1}_{(s_{k-1}, s_k]}(s),$$

where $0 = s_0 < s_1 < \cdots < s_n = T$ is a sequence of partitions of an time interval $[0, T]$, ϕ_k are square integrable $\mathscr{F}_{s-1}$- measurable random variables and $\mathbf{1}_{(s_{k-1}, s_k]}(s)$ is the indicator function.

Then the stochastic integral of u with respect to the Brownian motion B is defined by

$$\int_0^T u_s dW_s = \sum_{j=1}^{n} \phi_j (W_{s_j} - W_{s_{j-1}}).$$

Then for $t \leq s \leq T$, by the definition of the derivative operator and differential by parts method, we obtain

$$\begin{aligned} D_t\Big(\int_0^T u_s dW_s\Big) &= D_t\Big(\sum_{j=1}^{n} \phi_j (W_{s_j} - W_{s_{j-1}})\Big) \\ &= \sum_{j=1}^{n} D_t \phi_j (W_{s_j} - W_{s_{j-1}}) + \sum_{j=1}^{n} \phi_j D_t (W_{s_j} - W_{s_{j-1}}). \end{aligned}$$

By using the fact

$$\begin{aligned} D_t(W_{s_j} - W_{s_{j-1}}) &= D_t\Big(\int_0^T \mathbf{1}_{(s_{j-1}, s_j]}(t) dW_t\Big) \\ &= \mathbf{1}_{(s_{j-1}, s_j]}(t), \end{aligned}$$

and therefore

$$\begin{aligned}
&\sum_{j=1}^{n} D_t\phi_j(W_{s_j}-W_{s_{j-1}})+\sum_{j=1}^{n}\phi_j D_t(W_{s_j}-W_{s_{j-1}})\\
&=\sum_{j=1}^{n} D_t\phi_j(W_{s_j}-W_{s_{j-1}})+\sum_{j=1}^{n}\phi_j\mathbf{1}_{(s_{j-1},s_j]}(t)\\
&=\int_0^T D_t u_s dW_s+u_t.
\end{aligned}$$

This implies the following lemma.

Lemma 5.1.5. *If u is a stochastic process in $L^2_{a,T}$, for $t\le s\le T$ the derivative of the stochastic integral of u satisfies:*

$$D_t\Big(\int_0^T u_s dW_s\Big)=u_t+\int_0^T D_t u_s dW_s.$$

Example 5.1.6. Consider the stochastic integral $\int_0^T W_s dW_s$, we have

$$\begin{aligned}
D_t\Big(\int_0^T W_s dW_s\Big)&=W_t+\int_0^T D_t W_s dW_s\\
&=W_t+\int_t^T dW_s\\
&=W_t+W_T-W_t\\
&=W_T.
\end{aligned}$$

By using $W_s dW_s=d\Big(\frac{W_s^2-s}{2}\Big)$, we know

$$\int_0^T W_s dW_s=\frac{W_T^2-T}{2}.$$

Then by definition of the derivative operator, we have

$$\begin{aligned}
D_t\Big(\int_0^T W_s dW_s\Big)&=D_t\Big(\frac{W_T^2-T}{2}\Big)\\
&=W_T,
\end{aligned}$$

which yields the same result.

More generally, if f is a continuously differentiable function, we have

$$\begin{aligned} D_t(\int_0^T f(W_s)dW_s) &= f(W_t) + \int_0^T D_t f(W_s)dW_s \\ &= f(W_t) + \int_0^T f'(W_s)D_tW_sdW_s \\ &= f(W_t) + \int_t^T f'(W_s)dW_s. \end{aligned}$$

Moreover, for a stochastic process Y_T defined by

$$Y_T = \int_{s=0}^{s=T}\int_{u=0}^{u=s} \sigma_u dW_u dW_s,$$

the derivative of Y_T is given as

$$\begin{aligned} D_tY_T &= \int_0^t \sigma_u dW_u + \int_{s=0}^{s=T} D_t\Big(\int_{u=0}^{u=s}\sigma_u dW_u\Big)dW_s \\ &= \int_0^t \sigma_u dW_u + \int_{s=0}^{s=T}\Big(\sigma_t + \int_{u=0}^{u=s} D_t\sigma_u dW_u\Big)dW_s \\ &= \int_0^t \sigma_u dW_u + \int_0^T \sigma_t dW_s + \int_{s=0}^{s=T}\int_{u=0}^{u=s} D_t\sigma_u dW_u dW_s. \end{aligned}$$

5.2 The Operator L

In this section, furture explainations and several examples about the infinitesimal generator of the Ornstein-Uhlenbeck semigroup are demonstrated.

By Proposition 2.3.37, we know that the operator L coincided with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T_t, t \geq 0\}$.

Let $F \in L^2(\Omega)$ be a square integrable random variable. Recall that the operator L is defined as:

$$LF = \sum_{n=0}^{\infty} -nJ_nF,$$

provided this series converges in $L^2(\Omega)$. J_n denotes the orthogonal projection on the nth Wiener chaos.

And the domain of this operator will be the set

$$\mathrm{Dom}L = \left\{ F \in L^2(\Omega), F = \sum_{n=0}^{\infty} I_n(f_n) : \sum_{n=0}^{\infty} n^2 \|J_n F\|_2^2 < \infty \right\}$$

Recall the Proposition 2.3.38, which explains the relationship between the operator D, δ and L:

For $F \in L^2(\Omega)$ the statement $F \in \mathrm{Dom}L$ is equivalent to $F \in \mathrm{Dom}\delta L$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$), and in this case $\delta(DF) = -LF$.

Combining with the Proposition 2.3.28:

Let $F \in \mathbb{D}^{1,2}$ and u be in the domain of δ such that $Fu \in L^2(\Omega; H)$. Then Fu belongs to the domain of δ and the following equation holds

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H,$$

provided that $F\delta(u) - \langle DF, u \rangle_H$ is square integrable.

We will have an achievable way for computing the operator L by computing $\delta(DF)$, which will be demonstrated in these following examples.

Example 5.2.1. Consider the case that $F = W(h)$, $h \in H$. We can compute the derivative operator

$$DF = h,$$

and by using the Skorohod integral we have

$$\delta(DF) = W(h).$$

Then by using the Proposition 2.3.38, the operator L is given as

$$LF = -\delta(DF) = W(h).$$

Example 5.2.2. Consider the case that $F = W(h_1)W(h_2)$, $h_1, h_2 \in H$. We can deduce

$$DF = W(h_1)h_2 + W(h_2)h_1$$

and by using Proposition 2.3.28 we have

$$\begin{aligned}\delta(DF) &= W(h_1)\int_T h_2 dW - \langle DW(h_1), h_2\rangle_H + W(h_2)\int_T h_1 dW - \langle DW(h_2), h_1\rangle_H \\ &= 2W(h_1)W(h_2) - 2\langle h_1, h_2\rangle_H.\end{aligned}$$

Then by using the Proposition 2.3.38, the operator L is given as

$$\begin{aligned}LF &= -\delta(DF) \\ &= 2\langle h_1, h_2\rangle_H - 2W(h_1)W(h_2).\end{aligned}$$

Example 5.2.3. Consider the case that $F = W(h_1)W(h_2)W(h_3)$, $h_1, h_2, h_3 \in H$. We can deduce

$$DF = W(h_2)W(h_3)h_1 + W(h_1)W(h_3)h_2 + W(h_1)W(h_2)h_3.$$

By applying Proposition 2.3.38 and the linearity of the divergence operator, we have

$$\begin{aligned}LF &= -\delta(DF) \\ &= -[\delta(W(h_2)W(h_3)h_1) + \delta(W(h_1)W(h_3)h_2) + \delta(W(h_1)W(h_2)h_3)].\end{aligned}$$

By using Proposition 2.3.28 we have

$$\begin{aligned}\delta(W(h_2)W(h_3)h_1) &= W(h_1)W(h_2)W(h_3) - \langle D(W(h_2)W(h_3)), h_1\rangle_H \\ &= W(h_1)W(h_2)W(h_3) - \langle W(h_2)h_3 + W(h_3)h_2, h_1\rangle_H \\ &= W(h_1)W(h_2)W(h_3) - W(h_3)\langle h_1, h_2\rangle_H - W(h_2)\langle h_1, h_3\rangle_H.\end{aligned}$$

By applying similar technique, we deduce

$$\delta(W(h_1)W(h_3)h_2) = W(h_1)W(h_2)W(h_3) - W(h_3)\langle h_1, h_2\rangle_H - W(h_1)\langle h_2, h_3\rangle_H,$$

and

$$\delta(W(h_1)W(h_2)h_3) = W(h_1)W(h_2)W(h_3) - W(h_2)\langle h_1,h_3\rangle_H - W(h_1)\langle h_2,h_3\rangle_H.$$

Then the operator L is given as

$$\begin{aligned} LF &= 2W(h_1)\langle h_2,h_3\rangle_H + 2W(h_2)\langle h_1,h_3\rangle_H + 2W(h_3)\langle h_1,h_2\rangle_H \\ &\quad - 3W(h_1)W(h_2)W(h_3). \end{aligned}$$

5.3 Characteristics of Operator L

This section gives a discussion about the associated characteristics of operator L. In particular, we illustrate the fact that the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ coincide by computing $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ respectively.

Recall that the norms $\|\cdot\|_L$ on $\mathbf{S}$ is defined as

$$\|F\|_L = [E(F^2) + E(|LF|^2)]^{\frac{1}{2}},$$

and Dom$L = \mathbb{D}^{2,2}$.

The technique to compute the operator L is same as that in section 5.2.

We alse recall the definition of the seminorms on $\mathbf{S}$:

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{j=1}^{k} E(\|D^jF\|^p_{H^{\otimes j}})\right]^{\frac{1}{p}}.$$

5.3.1 Example: Case: $F = (W(h))^2$

Consider the case $F = (W(h))^2$, $h \in H$ by the definition of derivative operator, we have

$$DF = 2W(h)h,$$

and by using Proposition 2.3.38 we have

$$\begin{aligned} LF &= -\delta(DF) \\ &= -2\delta(W(h)h). \end{aligned}$$

By using the Proposition 2.3.29

$$\delta(Fh) = FW(h) - D^h F,$$

we obtain

$$\begin{aligned} LF &= -2[(W(h))^2 - \langle DW(h), h\rangle_H] \\ &= -2[(W(h))^2 - \|h\|_H^2] \\ &= 2[\|h\|_H^2 - (W(h))^2]. \end{aligned}$$

Then the $\|\cdot\|_L$ norm is given as

$$\begin{aligned} \|F\|_L &= [E(F^2) + E(|LF|^2)]^{\frac{1}{2}} \\ &= [E((W(h))^4) + 4E((\|h\|_H^2 - (W(h))^2)^2)]^{\frac{1}{2}}. \end{aligned}$$

By the definition of Wiener space, we have $E((W(h))^2) = \|h\|_H^2$, and therefore by rearranging we have

$$\begin{aligned} \|F\|_L &= [4\|h\|_H^4 - 8\|h\|_H^2 E((W(h))^2) + 5E((W(h))^4)]^{\frac{1}{2}} \\ &= [-4\|h\|_H^4 + 5E((W(h))^4)]^{\frac{1}{2}}. \end{aligned}$$

Finally, by using the properties of SBM $E(W_t^4) = 3t^2$, that is $E((W(h))^4) = 3\|h\|_H^4$ and by plugging in we have

$$\|F\|_L = \sqrt{11}\|h\|_H^2.$$

In order to compute the norm $\|\cdot\|_{2,2}$, we first compute the second order derivative. By definition, we have

$$D^2F = 2h(u)h(t).$$

Then the $\|\cdot\|_{2,2}$ norm can be computed as

$$\begin{aligned}
\|F\|_{2,2} &= [E(F^2) + E(\|DF\|_H^2) + E(\|D^2F\|_{H\otimes H}^2)]^{\frac{1}{2}} \\
&= [E((W(h))^4) + 4E(\|h\|_H^2(W(h))^2) + 4E(\|h(u)h(t)\|_{H\otimes H}^2)]^{\frac{1}{2}} \\
&= [7\|h\|_H^4 + 4E(\|h(u)h(t)\|_{H\otimes H}^2)]^{\frac{1}{2}}.
\end{aligned}$$

By Fubini we have

$$\begin{aligned}
\|h(u)h(t)\|_{H\otimes H}^2 &= \int_T \int_T (h(u)h(t))^2 dt du \\
&= \int_T (h(t))^2 dt \int_T (h(u))^2 du \\
&= \|h\|_H^4.
\end{aligned}$$

Then by plugging in, we obtain

$$\|F\|_{2,2} = \sqrt{11}\|h\|_H^2,$$

which implies that the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ coincide.

5.3.2 Example: Case: $F = S_0 \exp(m + \sigma W(h))$

Consider the case $F = S_0 \exp(m + \sigma W(h))$, $h \in H$, by the definition of derivative operator, we have

$$DF = \sigma F h,$$

and by using proposition 2.3.38 we have

$$\begin{aligned} LF &= -\delta(DF) \\ &= -\sigma\delta(Fh). \end{aligned}$$

By using the Proposition

$$\delta(Fh) = FW(h) - D^h F,$$

we obtain

$$\begin{aligned} LF &= -\sigma(FW(h) - \langle DF, h\rangle_H) \\ &= -\sigma(FW(h) - \sigma F\|h\|_H^2) \\ &= \sigma F(\sigma\|h\|_H^2 - W(h)). \end{aligned}$$

Then the $\|\cdot\|_L$ norm is given as

$$\begin{aligned} \|F\|_L &= [E(F^2) + E(|LF|^2)]^{\frac{1}{2}} \\ &= [E(F^2) + \sigma^2 E(F^2(\sigma\|h\|_H^2 - W(h))^2)]^{\frac{1}{2}} \\ &= [(1+\sigma^4\|h\|_H^4)E(F^2) - 2\sigma^3\|h\|_H^2 E(F^2 W(h)) + \sigma^2 E(F^2(W(h))^2)]^{\frac{1}{2}}. \end{aligned}$$

Notice that for a random variable x with a standard Gaussian distribution $N(0,1)$, if C is a constant we have

$$E(\exp(Cx)) = \exp(C^2/2), \qquad \text{eq(1)}$$

and if the second derivative of function f is continuous, the following property can be proved:

$$E(xf(x)) = E(f'(x)). \qquad \text{eq(2)}$$

By using integration by parts, we have

$$\begin{aligned}
E(f'(x)) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} f'(x) e^{-x^2/2} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} df(x) \\
&= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} f(x)|_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x f(x) dx.
\end{aligned}$$

By using the fact

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} f(x)|_{x=-\infty}^{x=+\infty} = 0,$$

we can deduce eq(2). By applying eq(2) twice, we obtain

$$E(x^2 f(x)) = E(f(x)) + E(f''(x)) \qquad \text{eq(3)}$$

Notice that $W(h) \sim \|h\|_H N(0,1)$, that is $W(h) = \|h\|_H x$.

By applying eq(1), we have

$$\begin{aligned}
E(F^2) &= S_0^2 e^{2m} E[\exp(2\sigma \|h\|_H x)] \\
&= S_0^2 e^{2m} \exp(2\sigma^2 \|h\|_H^2) \\
&= S_0^2 \exp(2m + 2\sigma^2 \|h\|_H^2).
\end{aligned}$$

Set $A = S_0^2 \exp(2m + 2\sigma^2 \|h\|_H^2)$.

By applying eq(2) and eq(1), we have

$$\begin{aligned}
E(F^2 W(h)) &= S_0^2 e^{2m} \|h\|_H E[\exp(2\sigma \|h\|_H x) x] \\
&= 2\sigma S_0^2 e^{2m} \|h\|_H^2 E[\exp(2\sigma \|h\|_H x)] \\
&= 2\sigma S_0^2 e^{2m} \|h\|_H^2 \exp(2\sigma^2 \|h\|_H^2) \\
&= 2\sigma \|h\|_H^2 A.
\end{aligned}$$

By applying eq(3), eq(2) and eq(1), we have

$$\begin{aligned}
E(F^2(W(h))^2) &= S_0^2 e^{2m}\|h\|_H^2 E[\exp(2\sigma\|h\|_H x)x^2] \\
&= S_0^2 e^{2m}\|h\|_H^2 [E(\exp(2\sigma\|h\|_H x)) + 4\sigma^2\|h\|_H^2 E(\exp(2\sigma\|h\|_H x))] \\
&= S_0^2 e^{2m}\|h\|_H^2 (1+4\sigma^2\|h\|_H^2)\exp(2\sigma^2\|h\|_H^2) \\
&= \|h\|_H^2(1+4\sigma^2\|h\|_H^2)A.
\end{aligned}$$

Then the $\|\cdot\|_L$ norm can be computed as

$$\begin{aligned}
\|F\|_L &= [(1+\sigma^4\|h\|_H^4)A - 4\sigma^4\|h\|_H^4 A + \sigma^2\|h\|_H^2(1+4\sigma^2\|h\|_H^2)A]^{\frac{1}{2}} \\
&= [1+\sigma^2\|h\|_H^2 + \sigma^4\|h\|_H^4]^{\frac{1}{2}} A^{\frac{1}{2}} \\
&= [1+\sigma^2\|h\|_H^2 + \sigma^4\|h\|_H^4]^{\frac{1}{2}} S_0 \exp(m+\sigma^2\|h\|_H^2).
\end{aligned}$$

In order to compute the norm$\|F\|_{2,2}$, we will first compute D^2F by using the first derivative $DF = \sigma F h$:

$$D^2F = \sigma^2 F h(s)h(u),$$

then we have

$$\begin{aligned}
\|F\|_{2,2} &= [E(F^2) + E(\|DF\|_H^2) + E(\|D^2F\|_{H\otimes H}^2)]^{\frac{1}{2}} \\
&= [E(F^2) + \sigma^2\|h\|_H^2 E(F^2) + \sigma^4\|h\|_H^4 E(F^2)]^{\frac{1}{2}}.
\end{aligned}$$

Finally, by plugging in $E(F^2) = A$ and rearranging, we obtain the $\|\cdot\|_{2,2}$ norm

$$\begin{aligned}
\|F\|_{2,2} &= [1+\sigma^2\|h\|_H^2 + \sigma^4\|h\|_H^4]^{\frac{1}{2}} A^{\frac{1}{2}} \\
&= [1+\sigma^2\|h\|_H^2 + \sigma^4\|h\|_H^4]^{\frac{1}{2}} S_0 \exp(m+\sigma^2\|h\|_H^2),
\end{aligned}$$

which implies that the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ coincide.

5.3.3 Example: Case: $F = W(h_1)W(h_2)$

Consider the case that $F = W(h_1)W(h_2)$, $h_1, h_2 \in H$. We have

$$DF = W(h_1)h_2 + W(h_2)h_1,$$

and

$$LF = 2\langle h_1, h_2\rangle_H - 2W(h_1)W(h_2).$$

Then the $\|\cdot\|_L$ norm is given as

$$\begin{aligned}\|F\|_L &= [E(F^2) + E(|LF|^2)]^{\frac{1}{2}} \\ &= [E((W(h_1)W(h_2))^2) + 4(\langle h_1, h_2\rangle_H^2 - 2\langle h_1, h_2\rangle_H E(W(h_1)W(h_2)) \\ &+ E((W(h_1)W(h_2))^2))]^{\frac{1}{2}}.\end{aligned}$$

By plugging in $E(W(h_1)W(h_2)) = \langle h_1, h_2\rangle_H$ and rearranging we obtain

$$\|F\|_L = [5E((W(h_1)W(h_2))^2) - 4\langle h_1, h_2\rangle_H^2]^{\frac{1}{2}}.$$

In order to compute $E((W(h_1)W(h_2))^2)$, we recall the Itô formula for functions in two variables [20]:

Let $f(x,y)$ be a function with continuous partial derivatives up to order two. Let X and Y be Itô process with

$$dX_t = \mu_t^X dt + \sigma_t^X dW_t,$$

$$dY_t = \mu_t^Y dt + \sigma_t^Y dW_t.$$

Then

$$\begin{aligned} df(X_t,Y_t) &= \frac{\partial f}{\partial x}(X_t,Y_t)dX_t + \frac{\partial f}{\partial y}(X_t,Y_t)dY_t \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t,Y_t)(\sigma_t^X)^2 dt + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(X_t,Y_t)(\sigma_t^Y)^2 dt \\ &+ \frac{\partial^2 f}{\partial x \partial y}(X_t,Y_t)\sigma_t^X \sigma_t^Y dt. \end{aligned}$$

By using $W(h_1) = \int_0^T h_1(t)dW_t$ and $W(h_2) = \int_0^T h_2(t)dW_t$, we have

$$dW(h_1) = h_1(t)dW_t,$$

and

$$dW(h_2) = h_2(t)dW_t.$$

By applying the Itô formula for functions in two variables, we obtain

$$\begin{aligned} d(W(h_1)W(h_2))^2 &= 2W(h_1)(W(h_2))^2 dW(h_1) + 2W(h_2)(W(h_1))^2 dW(h_2) \\ &+ (W(h_2))^2(h_1(t))^2 dt + (W(h_1))^2(h_2(t))^2 dt \\ &+ 4W(h_1)W(h_2)h_1(t)h_2(t)dt. \end{aligned}$$

Then by integrating we have

$$\begin{aligned} (W(h_1)W(h_2))^2 &= \int_0^T 2W(h_1)(W(h_2))^2 dW(h_1) + \int_0^T 2W(h_2)(W(h_1))^2 dW(h_2) \\ &+ \int_0^T (W(h_2))^2(h_1(t))^2 dt + \int_0^T (W(h_1))^2(h_2(t))^2 dt \\ &+ \int_0^T 4W(h_1)W(h_2)h_1(t)h_2(t)dt. \end{aligned}$$

Taking the expectations, we have

$$\begin{aligned} E[(W(h_1)W(h_2))^2] &= \int_0^T E[(W(h_2))^2](h_1(t))^2 dt + \int_0^T E[(W(h_1))^2](h_2(t))^2 dt \\ &+ \int_0^T 4E[W(h_1)W(h_2)]h_1(t)h_2(t)dt. \end{aligned}$$

By plugging in $E[(W(h_1))^2] = \int_0^t (h_1(s))^2 ds$, $E[(W(h_2))^2] = \int_0^t (h_2(s))^2 ds$ and $E[W(h_1)W(h_2)] = \int_0^t h_1(s)h_2(s)ds$, we have

$$\begin{aligned} E[(W(h_1)W(h_2))^2] = &\int_0^T \int_0^t (h_2(s))^2 ds (h_1(t))^2 dt + \int_0^T \int_0^t (h_1(s))^2 ds (h_2(t))^2 dt \\ &+ 4\int_0^T \Big(\int_0^t h_1(s)h_2(s)ds \Big) h_1(t)h_2(t)dt. \end{aligned}$$

By using the fact

$$\int_0^T \Big(\int_0^t H(s)ds \Big) H(t)dt = \frac{1}{2}\Big(\int_0^T H(t)dt \Big)^2,$$

we deduce

$$\begin{aligned} E[(W(h_1)W(h_2))^2] &= \int_0^T (h_1(t))^2 dt \int_0^T (h_2(s))^2 ds + 2\Big(\int_0^T h_1(s)h_2(s)ds \Big)^2 \\ &= \|h_1\|_H^2 \|h_2\|_H^2 + 2\langle h_1, h_2 \rangle_H^2. \end{aligned}$$

And therefore the $\|\cdot\|_L$ norm is given as

$$\begin{aligned} \|F\|_L &= [5E((W(h_1)W(h_2))^2) - 4\langle h_1, h_2 \rangle_H^2]^{\frac{1}{2}} \\ &= [5\|h_1\|_H^2 \|h_2\|_H^2 + 6\langle h_1, h_2 \rangle_H^2]^{\frac{1}{2}}. \end{aligned}$$

In order to compute the norm$\|F\|_{2,2}$, we will first compute D^2F by using $DF = W(h_1)h_2 + W(h_2)h_1$:

$$D^2F = h_1(t)h_2(u) + h_1(u)h_2(t),$$

then we have

$$\begin{aligned} E(\|DF\|_H^2) &= \|h_2\|_H^2 E[(W(h_1))^2] + \|h_1\|_H^2 E[(W(h_2))^2] + 2\langle h_1, h_2 \rangle_H E[W(h_1)W(h_2)] \\ &= 2\|h_1\|_H^2 \|h_2\|_H^2 + 2\langle h_1, h_2 \rangle_H^2, \end{aligned}$$

and

$$\begin{aligned} E(\|D^2F\|^2_{H\otimes H}) &= E(\|h_1(t)h_2(u)+h_1(u)h_2(t)\|^2_{H\otimes H}) \\ &= 2\|h_1\|^2_H\|h_2\|^2_H+2\langle h_1,h_2\rangle^2_H. \end{aligned}$$

By plugging in and rearranging, we know that the norm$\|F\|_{2,2}$ is

$$\begin{aligned} \|F\|_{2,2} &= [E(F^2)+E(\|DF\|^2_H)+E(\|D^2F\|^2_{H\otimes H})]^{\frac{1}{2}} \\ &= [5\|h_1\|^2_H\|h_2\|^2_H+6\langle h_1,h_2\rangle^2_H]^{\frac{1}{2}}, \end{aligned}$$

which implies that the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$ coincide.

Chapter 6

Computing Greeks

In this chapter, we apply Malliavin calculus to compute Greeks for variety financial models. In most cases, the Integration by parts formula will play a very important role.

Recall that the Integration by parts formula (Proposition 2.4.8):

Let F, G be two random variables such that $F \in \mathbb{D}^{1,2}$. Consider an H-valued random variable u such that $D^u F = \langle DF, u \rangle_H \neq 0$ a.s. and $Gu(D^u F)^{-1} \in \mathrm{Dom}\delta$. Then, for any continuously differentiable function f with bounded derivative we have

$$E(f'(F)G) = E(f(F)H(F,G)),$$

where $H(F,G) = \delta(Gu(D^u F)^{-1})$.

Then for an option with payoff H such that $E_Q(H^2) < \infty$. Recall its price at time $t = 0$ is determined by

$$V_0 = E_Q(e^{-rT} H).$$

Greeks is defined as the derivative of the expected value $E_Q(e^{-rT} H)$ with respect to one of the parameters of the model, such as S_0, σ or r.

Moreover if we can write the payoff function as $H = f(F_\alpha)$, where α is one of S_0, σ or r. By applying Proposition 2.4.8, Greeks can be computed as

$$\begin{aligned}\frac{\partial V_0}{\partial \alpha} &= e^{-rT} E_Q(f'(F_\alpha)\frac{dF_\alpha}{d\alpha}) \\ &= e^{-rT} E_Q(f(F_\alpha) H(F_\alpha, \frac{dF_\alpha}{d\alpha})).\end{aligned}$$

6.1 Computation of Greeks for European Options

Consider the price process of the stock is a GBM $S_t = S_0 e^{H_t}, t \in [0,T]$, with

$$H_t = \int_0^t (r - \frac{\sigma^2}{2}) ds + \int_0^t \sigma dW_s,$$

where $W = \{W_t, t \in [0,T]\}$ is a Brownian motion defined in a complete probability space $(\Omega, \mathscr{F}, P)$.

Suppose the payoff H only depends on the price of the stock at the maturity time T. That is, $H = \Phi(S_T)$. We call these financial derivative products satisfying $H = \Phi(S_T)$ European options.

Recall the the option price at time $t = 0$ is $V_0 = E_Q(e^{-rT}\Phi(S_T))$. And the Greeks can be computed as follows.

Lemma 6.1.1. *Suppose that Φ is a Lipschitz function and the stock price is $S_T = S_0 \exp\Big(\Big(r - \frac{1}{2}\sigma^2\Big)T + \sigma W_T\Big)$. Given $V_0 = E_Q(e^{-rT}\Phi(S_T))$, then the first derivative of V_0 with respect to S_0 is*

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} E_Q(\Phi(S_T) W_T).$$

This Lemma was stated in [30] p332 and the following proof is modified from [30] p332.

Proof. By Φ is a Lipschitz function we can derive

$$\begin{aligned}\Delta &= \frac{\partial V_0}{\partial S_0} \\ &= E_Q\Big(e^{-rT}\Phi'(S_T)\frac{\partial S_T}{\partial S_0}\Big).\end{aligned}$$

By using the fact $\dfrac{\partial S_T}{\partial S_0} = \dfrac{S_T}{S_0}$ and rearranging we have

$$\Delta = \frac{e^{-rT}}{S_0} E_Q(\Phi'(S_T)S_T).$$

Consider the case that $u = 1$, $F = S_T$, and $G = S_T$. Remind $D_t S_T = \sigma S_T$. We have

$$D^u S_T = \int_0^T D_t S_T dt = \sigma T S_T.$$

Hence, all the conditions appearing in Integration by parts formula are satisfied in this case, and therefore applying Proposition 2.4.8 we can obtain

$$\Delta = \frac{e^{-rT}}{S_0} E_Q(\Phi(F)H(F,G)),$$

and

$$\begin{aligned} H(F,G) &= \delta(Gu(D^u F)^{-1}) \\ &= \frac{\delta(1)}{\sigma T}. \end{aligned}$$

By using

$$\delta(1) = \int_0^T dW_t = W_T,$$

we obtain

$$H(F,G) = \frac{W_T}{\sigma T}.$$

Finally, we have

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} E_Q(\Phi(S_T)W_T).$$

The proof is complete. □

Lemma 6.1.2. *Suppose that Φ is a Lipschitz function and the stock price is $S_T = S_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma W_T)$. Given $V_0 = E_Q(e^{-rT}\Phi(S_T))$, then the second derivative of V_0 with respect* *to S_0 is*

$$\Gamma = \frac{e^{-rT}}{S_0^2 \sigma T} E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\Big)\Big).$$

The lemma was stated in [30] p333, while the following proof is different from [30] p333.

Proof. By definition we have

$$\begin{aligned}\Gamma &= \frac{\partial^2 V_0}{\partial S_0^2} \\ &= \frac{\partial \Delta}{\partial S_0} \\ &= -\frac{e^{-rT}}{S_0^2 \sigma T} E_Q(\Phi(S_T) W_T) + \frac{e^{-rT}}{S_0^2 \sigma T} E_Q(\Phi'(S_T) S_T W_T).\end{aligned}$$

Applying Proposition 2.4.8 with $u = 1$, $F = S_T$ and $G = S_T W_T$, we have

$$\begin{aligned}E_Q(\Phi'(S_T) S_T W_T) &= E_Q\Big(\Phi(S_T)\delta\Big(\frac{W_T}{\sigma T}\Big)\Big) \\ &= E_Q\Big(\Phi(S_T)\frac{1}{\sigma T}\delta(W_T)\Big).\end{aligned}$$

Recall the proposition

$$\delta(Lu) = L\delta(u) - \langle DL, u\rangle_H,$$

if $L \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta$.

That is

$$\delta(Lu) = L\int_0^T u(t)dW_t - \int_0^T D_t L u(t)dt,$$

by using the fact

$$\delta(u) = \int_0^T u(t)dW_t.$$

Therefore, we have

$$\begin{aligned}\delta(W_T) &= W_T\int_0^T dW_t - \int_0^T dt \\ &= W_T^2 - T.\end{aligned}$$

That is

$$E_Q(\Phi'(S_T)S_TW_T) = E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma}\Big)\Big).$$

Finally by rearranging we obtain

$$\begin{aligned}\Gamma &= -\frac{e^{-rT}}{S_0^2\sigma T}E_Q(\Phi(S_T)W_T) + \frac{e^{-rT}}{S_0^2\sigma T}E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma}\Big)\Big)\\ &= \frac{e^{-rT}}{S_0^2\sigma T}E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\Big)\Big).\end{aligned}$$

The proof is complete. □

Lemma 6.1.3. *Suppose that Φ is a Lipschitz function and the stock price is $S_T = S_0\exp((r - \frac{1}{2}\sigma^2)T + \sigma W_T)$. Given $V_0 = E_Q(e^{-rT}\Phi(S_T))$, then the derivative of V_0 with respect to σ is*

$$\vartheta = e^{-rT}E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\Big)\Big).$$

This Lemma was stated in [30] p333 and the following proof is modified from [30] p333.

Proof. By definition we know that

$$\begin{aligned}\vartheta &= \frac{\partial V_0}{\partial \sigma}\\ &= E_Q\Big(e^{-rT}\Phi'(S_T)\frac{\partial S_T}{\partial \sigma}\Big).\end{aligned}$$

By using

$$\frac{\partial S_T}{\partial \sigma} = S_T(W_T - \sigma T),$$

we have

$$\vartheta = e^{-rT}E_Q(\Phi'(S_T)S_T(W_T - \sigma T)).$$

Applying Proposition 2.4.8 with $u = 1$, $F = S_T$ and $G = S_T(W_T - \sigma T)$ we have

$$\vartheta = e^{-rT}E_Q(\Phi(F)H(F,G)),$$

and

$$\begin{aligned} H(F,G) &= \delta(S_T(W_T - \sigma T)(\sigma T S_T)^{-1}) \\ &= \delta\Big(\frac{W_T}{\sigma T} - 1\Big). \end{aligned}$$

By using

$$\begin{aligned} \delta\Big(\frac{W_T}{\sigma T} - 1\Big) &= \Big(\frac{W_T}{\sigma T} - 1\Big)W_T - \int_0^T \frac{1}{\sigma T} dt \\ &= \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}, \end{aligned}$$

we have

$$H(F,G) = \Big(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\Big).$$

And therefore

$$E_Q(\Phi'(S_T)S_T(W_T - \sigma T)) = E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\Big)\Big).$$

Finally we obtain

$$\vartheta = e^{-rT} E_Q\Big(\Phi(S_T)\Big(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\Big)\Big).$$

The proof is complete. $\square$

6.2 Computation of Greeks for Exotic Options

Consider the price process of the stock is $S_t = S_0 e^{H_t}, t \in [0,T]$, with

$$H_t = \int_0^t (r - \frac{\sigma^2}{2}) ds + \int_0^t \sigma dW_s,$$

where $W = \{W_t, t \in [0,T]\}$ is a Brownian motion defined in a complete probability space $(\Omega, \mathscr{F}, P)$.

Suppose a option whose payoff function is a function of the average of the stock price $\frac{1}{T}\int_0^T S_t dt$, that is

$$H = \Phi\Big(\frac{1}{T}\int_0^T S_t dt\Big).$$

For instance, the payoff function of an Asiatic call-option with exercise price K is

$$H = \Big(\frac{1}{T}\int_0^T S_t dt - K\Big)^+,$$

which implies that it is a derivative of this type. In this case, there is no closed formula for the density of the random variable $\frac{1}{T}\int_0^T S_t dt$.

Recall the price of this option at time $t = 0$ is given by

$$V_0 = e^{-rT} E_Q\Big(\Phi\Big(\frac{1}{T}\int_0^T S_t dt\Big)\Big).$$

In order to compute the Delta Δ for this type of options, we set $\bar{S}_T = \frac{1}{T}\int_0^T S_t dt$.

Lemma 6.2.1. *Suppose that Φ is a Lipschitz function, the stock price is $S_T = S_0 \exp\Big(\Big(r - \frac{1}{2}\sigma^2\Big)T + \sigma W_T\Big)$ and $\bar{S}_T = \frac{1}{T}\int_0^T S_t dt$. Given $V_0 = e^{-rT} E_Q\Big(\Phi\Big(\frac{1}{T}\int_0^T S_t dt\Big)\Big)$, then the first derivative of V_0 with respect to S_0 is*

$$\Delta = \frac{2e^{-rT}}{S_0\sigma^2} E_Q\Big(\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big),$$

where $m = r - \frac{\sigma^2}{2}$.

This Lemma was stated in [30] p334 and the following proof is modified from [30] p334.

Proof. By definition we have

$$\begin{aligned}\Delta &= \frac{\partial V_0}{\partial S_0} \\ &= E_Q\Big(e^{-rT}\Phi'(\bar{S}_T)\frac{\partial \bar{S}_T}{\partial S_0}\Big).\end{aligned}$$

By using the fact

$$\frac{\partial \bar{S}_T}{\partial S_0} = \frac{\bar{S}_T}{S_0},$$

we have

$$\Delta = \frac{e^{-rT}}{S_0} E_Q(\Phi'(\bar{S}_T)\bar{S}_T).$$

Set $u = S_t$, $F = \bar{S}_T$, $G = \bar{S}_T$, we have

$$\begin{aligned} D_t F &= D_t \frac{1}{T} \int_0^T S_r dr \\ &= \frac{1}{T} \int_0^T D_t S_r dr. \end{aligned}$$

By using the fact

$$D_t S_r = \sigma S_r \mathbf{1}_{\{r \geq t\}},$$

we have

$$\begin{aligned} D_t F &= \frac{1}{T} \int_0^T \sigma S_r \mathbf{1}_{\{r \geq t\}} dr \\ &= \frac{\sigma}{T} \int_t^T S_r dr. \end{aligned}$$

And therefore

$$\begin{aligned} D^u F &= \int_0^T S_t D_t F dt \\ &= \int_0^T S_t \Big(\frac{\sigma}{T} \int_t^T S_r dr \Big) dt \\ &= \frac{\sigma}{T} \int_0^T S_t \Big(\int_t^T S_r dr \Big) dt. \end{aligned}$$

Set $v(t) = \int_t^T S_r dr = \int_0^T S_r dr - \int_0^t S_r dr$, differential v with respect to t, we have

$$v_t' = S_t,$$

that is,

$$dv(t) = S_t dt.$$

Hence, the integral $\int_0^T S_t \Big(\int_t^T S_r dr \Big) dt$ can be written as

$$\begin{aligned}
\int_0^T S_t \Big(\int_t^T S_r dr \Big) dt &= \int_0^T v(t) dv(t) \\
&= \frac{1}{2} (v(t))^2 |_{t=0}^{t=T} \\
&= \frac{1}{2} ((v(T))^2 - (v(0))^2).
\end{aligned}$$

As $v(T) = 0$ and $v(0) = \int_0^T S_r dr$, we obtain

$$\int_0^T S_t \Big(\int_t^T S_r dr \Big) dt = \frac{1}{2} \Big(\int_0^T S_r dr \Big)^2.$$

That is

$$D^u F = \frac{\sigma}{2T} \Big(\int_0^T S_r dr \Big)^2.$$

Applying Proposition 2.4.8 we have

$$\Delta = \frac{e^{-rT}}{S_0} E_Q(\Phi(F) H(F,G)),$$

and

$$\begin{aligned}
H(F,G) &= \delta(Gu(D^u F)^{-1}) \\
&= \delta\Bigg(\frac{S_t \frac{1}{T} \int_0^T S_r dr}{\frac{\sigma}{2T} \Big(\int_0^T S_r dr \Big)^2} \Bigg) \\
&= \frac{2}{\sigma} \delta \Big(\frac{S_t}{\int_0^T S_t dt} \Big).
\end{aligned}$$

Recall the proposition

$$\delta(Lu) = L \int_0^T u(t) dW_t - \int_0^T D_t L u(t) dt,$$

if $L \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta$.

We have

$$\delta\Big(\frac{S_t}{\int_0^T S_t dt}\Big) = \frac{1}{\int_0^T S_t dt}\int_0^T S_t dW_t - \int_0^T D_t\Big(\frac{1}{\int_0^T S_r dr}\Big)S_t dt$$
$$= \frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + \int_0^T \frac{D_t \int_0^T S_r dr}{\Big(\int_0^T S_r dr\Big)^2} S_t dt.$$

By using

$$D_t \int_0^T S_r dr = \int_t^T \sigma S_r dr,$$

we deduce

$$\delta\Big(\frac{S_t}{\int_0^T S_t dt}\Big) = \frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + \frac{\int_0^T \Big(\int_t^T \sigma S_r dr\Big) S_t dt}{\Big(\int_0^T S_r dr\Big)^2}.$$

Plugging in

$$\int_0^T \Big(\int_t^T S_r dr\Big) S_t dt = \frac{1}{2}\Big(\int_0^T S_r dr\Big)^2,$$

we deduce

$$\delta\Big(\frac{S_t}{\int_0^T S_t dt}\Big) = \frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + \frac{\sigma}{2}.$$

And therefore

$$H(F,G) = \frac{2}{\sigma}\delta\Big(\frac{S_t}{\int_0^T S_t dt}\Big)$$
$$= \frac{2}{\sigma}\frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + 1.$$

By definition we have

$$S_T = S_0 + r\int_0^T S_t dt + \sigma \int_0^T S_t dW_t,$$

that is,

$$\int_0^T S_t dW_t = \frac{1}{\sigma}\Big(S_T - S_0 - r\int_0^T S_t dt\Big).$$

Hence

$$H(F,G) = \frac{2}{\sigma^2}\frac{S_T - S_0}{\int_0^T S_t dt} + 1 - \frac{2r}{\sigma^2}$$
$$= \frac{2}{\sigma^2}\Big(\frac{S_T - S_0}{\int_0^T S_t dt} - \Big(r - \frac{\sigma^2}{2}\Big)\Big)$$
$$= \frac{2}{\sigma^2}\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big),$$

where $m = r - \frac{\sigma^2}{2}$.

Finally, we obtain the following expression for the Delta

$$\Delta = \frac{e^{-rT}}{S_0}E_Q(\Phi(F)H(F,G))$$
$$= \frac{2e^{-rT}}{S_0\sigma^2}E_Q\Big(\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big),$$

where $m = r - \frac{\sigma^2}{2}$.

The proof is complete. □

Lemma 6.2.2. *Suppose that Φ is a Lipschitz function, the stock price is $S_T = S_0\exp\Big(\Big(r - \frac{1}{2}\sigma^2\Big)T + \sigma W_T\Big)$ and $\bar{S}_T = \frac{1}{T}\int_0^T S_t dt$. Given $V_0 = e^{-rT}E_Q\Big(\Phi\Big(\frac{1}{T}\int_0^T S_t dt\Big)\Big)$, then the second derivative of V_0 with respect to S_0 is*

$$\Gamma = \frac{4e^{-rT}}{S_0\sigma^4}E_Q\Big[\Phi(\bar{S}_T)\Big(\Big(\frac{S_T - S_0}{T\bar{S}_T} - r\Big)^2 - \frac{\sigma^2 S_0}{T\bar{S}_T} - \frac{r\sigma^2}{2}\Big)\Big].$$

Proof. By definition we know

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2}$$
$$= \frac{\partial \Delta}{\partial S_0}.$$

Pluggung in

$$\Delta = \frac{2e^{-rT}}{S_0\sigma^2}E_Q\Big(\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big),$$

where $m = r - \frac{\sigma^2}{2}$, we deduce

$$\begin{aligned}
\Gamma &= -\frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big] + \frac{2e^{-rT}}{S_0\sigma^2}\frac{\partial E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big]}{\partial S_0} \\
&= -\frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big] \\
&\quad + \frac{2e^{-rT}}{S_0\sigma^2}E_Q\Big[\Phi'(\bar{S}_T)\frac{\partial \bar{S}_T}{\partial S_0}\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big) + \Phi(\bar{S}_T)\frac{\partial\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)}{\partial S_0}\Big].
\end{aligned}$$

As we know that

$$\begin{aligned}
\frac{\partial\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)}{\partial S_0} &= \frac{\Big(\frac{\partial S_T}{\partial S_0} - 1\Big)T\bar{S}_T - (S_T - S_0)T\frac{\partial \bar{S}_T}{\partial S_0}}{T^2\bar{S}_T^2}. \\
&= 0
\end{aligned}$$

By using $\frac{\partial S_T}{\partial S_0} = \frac{S_T}{S_0}$ and $\frac{\partial \bar{S}_T}{\partial S_0} = \frac{\bar{S}_T}{S_0}$, we can obtain

$$\begin{aligned}
\Gamma &= -\frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big] + \frac{2e^{-rT}}{S_0\sigma^2}E_Q\Big[\Phi'(\bar{S}_T)\frac{\bar{S}_T}{S_0}\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big] \\
&= -\frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big)\Big] + \frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big[\Phi'(\bar{S}_T)\Big(\frac{S_T - S_0}{T} - m\bar{S}_T\Big)\Big].
\end{aligned}$$

Set $F = \bar{S}_T$, $G = \dfrac{S_T - S_0}{T} - m\bar{S}_T$ and $u_t = S_t$, we have

$$D^u F = \frac{\sigma}{2T}\Big(\int_0^T S_r dr\Big)^2.$$

Applying Proposition 2.4.8 we have

$$E_Q\Big[\Phi'(\bar{S}_T)\Big(\frac{S_T - S_0}{T} - m\bar{S}_T\Big)\Big] = E_Q(\Phi(F)H(F,G)),$$

and

$$\begin{aligned}
H(F,G) &= \delta(Gu(D^u F)^{-1}) \\
&= \delta\Big(\frac{S_t\Big(\frac{S_T - S_0}{T} - m\bar{S}_T\Big)}{\frac{\sigma}{2T}\Big(\int_0^T S_r dr\Big)^2}\Big) \\
&= \frac{2}{\sigma}\delta\Big(\Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{m}{\int_0^T S_t dt}\Big)S_t\Big).
\end{aligned}$$

Recall the proposition

$$\delta(Lu) = L\int_0^T u(t)dW_t - \int_0^T D_t L u(t)dt,$$

if $L \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta$.

Set $L = \dfrac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \dfrac{m}{\int_0^T S_t dt}$.

By using $D_t S_T = \sigma S_T$, $D_t(\int_0^T S_r dr) = \sigma\int_t^T S_r dr$ and rearranging we have

$$\begin{aligned}
D_t L &= \frac{\sigma S_T\Big(\int_0^T S_t dt\Big)^2 - (S_T - S_0)2\int_0^T S_t dt\,\sigma\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^4} + \frac{m\sigma\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^2} \\
&= \sigma\Big(\frac{S_T}{\Big(\int_0^T S_t dt\Big)^2} - 2(S_T - S_0)\frac{\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^3} + m\frac{\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^2}\Big).
\end{aligned}$$

And therefore

$$\begin{aligned}
\int_0^T D_t L u(t)dt &= \sigma\int_0^T\Big(\frac{S_T}{\Big(\int_0^T S_t dt\Big)^2} - 2(S_T - S_0)\frac{\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^3} + m\frac{\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^2}\Big)S_t dt \\
&= \sigma\Big(\frac{S_T}{\int_0^T S_t dt} - 2(S_T - S_0)\frac{\int_0^T S_t\Big(\int_t^T S_r dr\Big)dt}{\Big(\int_0^T S_t dt\Big)^3} + m\frac{\int_0^T S_t\Big(\int_t^T S_r dr\Big)dt}{\Big(\int_0^T S_t dt\Big)^2}\Big).
\end{aligned}$$

By using the fact

$$\int_0^T S_t\Big(\int_t^T S_r dr\Big)dt = \frac{1}{2}\Big(\int_0^T S_r dr\Big)^2,$$

and rearranging we can obtain

$$\int_0^T D_t Lu(t)dt = \sigma\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{m}{2}\Big).$$

That is,

$$\delta(Lu) = \Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{m}{\int_0^T S_t dt}\Big)\int_0^T S_t dW_t - \sigma\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{m}{2}\Big).$$

By plugging in

$$\int_0^T S_t dW_t = \frac{1}{\sigma}\Big(S_T - S_0 - r\int_0^T S_t dt\Big),$$

and rearranging we deduce

$$\begin{aligned}\delta(Lu) &= \Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{m}{\int_0^T S_t dt}\Big)\frac{1}{\sigma}\Big(S_T - S_0 - r\int_0^T S_t dt\Big) - \sigma\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{m}{2}\Big)\\ &= \frac{1}{\sigma}\Big[\frac{(S_T - S_0)^2}{\Big(\int_0^T S_t dt\Big)^2} - (m+r)\frac{(S_T - S_0)}{\int_0^T S_t dt} - \frac{\sigma^2 S_0}{\int_0^T S_t dt} + m^2\Big].\end{aligned}$$

Then, we have

$$H(F,G) = \frac{2}{\sigma^2}\Big[\frac{(S_T - S_0)^2}{\Big(\int_0^T S_t dt\Big)^2} - (m+r)\frac{(S_T - S_0)}{\int_0^T S_t dt} - \frac{\sigma^2 S_0}{\int_0^T S_t dt} + m^2\Big].$$

And therefore

$$\begin{aligned}E_Q[\Phi'(\bar{S}_T)(\frac{S_T - S_0}{T} - m\bar{S}_T)] &= E_Q(\Phi(F)H(F,G))\\ &= \frac{2}{\sigma^2}E_Q[\Phi(\bar{S}_T)(\frac{(S_T - S_0)^2}{(\int_0^T S_t dt)^2} - (m+r)\frac{(S_T - S_0)}{\int_0^T S_t dt}\\ &\quad - \frac{\sigma^2 S_0}{\int_0^T S_t dt} + m^2)].\end{aligned}$$

Finally, by plugging in we deduce

$$\begin{aligned}
\Gamma &= -\frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{S_T-S_0}{T\bar{S}_T}-m\Big)\Big] \\
&\quad+\frac{4e^{-rT}}{S_0^2\sigma^4}E_Q\Big[\Phi(\bar{S}_T)\Big(\frac{(S_T-S_0)^2}{\Big(\int_0^T S_t dt\Big)^2}-(m+r)\frac{(S_T-S_0)}{\int_0^T S_t dt}-\frac{\sigma^2 S_0}{\int_0^T S_t dt}+m^2\Big)\Big] \\
&= \frac{4e^{-rT}}{S_0^2\sigma^4}E_Q\Big[\Phi(\bar{S}_T)\Big(-\frac{\sigma^2}{2}\frac{S_T-S_0}{\Big(\int_0^T S_t dt\Big)^2}+\frac{\sigma^2 m}{2}+\frac{(S_T-S_0)^2}{\Big(\int_0^T S_t dt\Big)^2}-(m+r)\frac{(S_T-S_0)}{\int_0^T S_t dt} \\
&\quad-\frac{\sigma^2 S_0}{\int_0^T S_t dt}+m^2\Big)\Big].
\end{aligned}$$

By rearranging, we obtain

$$\Gamma = \frac{4e^{-rT}}{S_0^2\sigma^4}E_Q\Big[\Phi(\bar{S}_T)\Big(\Big(\frac{S_T-S_0}{T\bar{S}_T}-r\Big)^2-\frac{\sigma^2 S_0}{T\bar{S}_T}-\frac{r\sigma^2}{2}\Big)\Big].$$

Another way to compute the Gamma is

$$\begin{aligned}
\Gamma &= \frac{\partial^2 V_0}{\partial S_0^2} \\
&= e^{-rT}E_Q\Big(\Phi''(\bar{S}_T)\Big(\frac{\partial \bar{S}_T}{\partial S_0}\Big)^2\Big).
\end{aligned}$$

By using $\dfrac{\partial \bar{S}_T}{\partial S_0}=\dfrac{\bar{S}_T}{S_0}$, we have

$$\Gamma = \frac{e^{-rT}}{S_0^2}E_Q(\Phi''(\bar{S}_T)\bar{S}_T^2).$$

Assuming that Φ' is Lipschitz, set $u=S_t$, $F=\bar{S}_T$ and $G=\bar{S}_T^2$, we know that

$$D^u F = \frac{\sigma}{2T}\Big(\int_0^T S_r dr\Big)^2.$$

Then, by applying Proposition 2.4.8 we have

$$\Gamma = \frac{e^{-rT}}{S_0}E_Q(\Phi'(F)H(F,G)),$$

and

$$\begin{aligned} H(F,G) &= \delta(Gu(D^uF)^{-1}) \\ &= \frac{2}{\sigma}\delta\Big(\frac{S_t}{T}\Big). \end{aligned}$$

By using

$$\begin{aligned} \delta\Big(\frac{S_t}{T}\Big) &= \frac{1}{T}\int_0^T S_t dW_t \\ &= \frac{1}{\sigma T}\Big(S_T - S_0 - r\int_0^T S_t dt\Big) \\ &= \frac{1}{\sigma}\Big(\frac{S_T - S_0}{T} - r\bar{S}_T\Big), \end{aligned}$$

we obtain

$$H(F,G) = \frac{2}{\sigma^2}\Big(\frac{S_T - S_0}{T} - r\bar{S}_T\Big).$$

Therefore we have

$$\Gamma = \frac{2e^{-rT}}{S_0^2\sigma^2}E_Q\Big(\Phi'(\bar{S}_T)\Big(\frac{S_T - S_0}{T} - r\bar{S}_T\Big)\Big).$$

Again applying Proposition 2.4.8 with $u = S_t$, $F = \bar{S}_T$ and $G = \dfrac{S_T - S_0}{T} - r\bar{S}_T$ we have

$$\Gamma = \frac{2e^{-rT}}{S_0^2\sigma^2}E_Q(\Phi(F)H(F,G)),$$

and

$$\begin{aligned} H(F,G) &= \delta(Gu(D^uF)^{-1}) \\ &= \delta\Big(\frac{2}{\sigma}S_t\Big(\frac{S_T - S_0}{T^2\bar{S}_T^2} - \frac{r}{T\bar{S}_T}\Big)\Big) \\ &= \frac{2}{\sigma}\delta\Big(S_t\Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{r}{\int_0^T S_t dt}\Big)\Big). \end{aligned}$$

Then,

$$\begin{aligned}
\Gamma &= \frac{2e^{-rT}}{S_0^2\sigma^2} E_Q(\Phi(\bar{S}_T)H(F,G)) \\
&= \frac{4e^{-rT}}{S_0^2\sigma^3} E_Q\Big[\Phi(\bar{S}_T)\delta\Big(S_t\Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{r}{\int_0^T S_t dt}\Big)\Big)\Big].
\end{aligned}$$

Set $L = \frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{r}{\int_0^T S_t dt}$.Then Γ can be written as

$$\Gamma = \frac{4e^{-rT}}{S_0^2\sigma^3} E_Q[\Phi(\bar{S}_T)\delta(LS_t)].$$

Recall the proposition

$$\delta(Lu) = L\int_0^T u(t)dW_t - \int_0^T D_t Lu(t)dt,$$

if $L \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta$.

Remind the fact

$$\int_0^T D_t\Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{m}{\int_0^T S_t dt}\Big)S_t dt = \sigma\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{m}{2}\Big),$$

we can obtain

$$\begin{aligned}
\int_0^T D_t L S_t dt &= \int_0^T D_t\Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{r}{\int_0^T S_t dt}\Big)S_t dt \\
&= \sigma\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{r}{2}\Big).
\end{aligned}$$

That is,

$$\delta(LS_t) = \Big(\frac{S_T - S_0}{\Big(\int_0^T S_t dt\Big)^2} - \frac{r}{\int_0^T S_t dt}\Big)\int_0^T S_t dW_t - \sigma\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{r}{2}\Big).$$

By plugging in

$$\int_0^T S_t dW_t = \frac{1}{\sigma}\Big(S_T - S_0 - r\int_0^T S_t dt\Big),$$

and rearranging we deduce

$$\delta(LS_t) = \frac{1}{\sigma}\Big[\Big(\frac{S_T - S_0 - r\int_0^T S_t dt}{\int_0^T S_t dt}\Big)^2 - \sigma^2\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{r}{2}\Big)\Big].$$

Finally we can deduce

$$\begin{aligned}
\Gamma &= \frac{4e^{-rT}}{S_0^2\sigma^3} E_Q[\Phi(\bar{S}_T)\delta(LS_t)] \\
&= \frac{4e^{-rT}}{S_0^2\sigma^4} E_Q\Big[\Phi(\bar{S}_T)\Big(\Big(\frac{S_T - S_0 - r\int_0^T S_t dt}{\int_0^T S_t dt}\Big)^2 - \sigma^2\Big(\frac{S_0}{\int_0^T S_t dt} + \frac{r}{2}\Big)\Big] \\
&= \frac{4e^{-rT}}{S_0^2\sigma^4} E_Q\Big[\Phi(\bar{S}_T)\Big(\Big(\frac{S_T - S_0}{T\bar{S}_T} - r\Big)^2 - \frac{S_0\sigma^2}{T\bar{S}_T} - \frac{r\sigma^2}{2}\Big)\Big].
\end{aligned}$$

which implies the same result.

The proof is complete. □

Lemma 6.2.3. *Suppose that Φ is a Lipschitz function, the stock price is $S_T = S_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma W_T)$ and $\bar{S}_T = \frac{1}{T}\int_0^T S_t dt$. Given $V_0 = e^{-rT}E_Q(\Phi(\frac{1}{T}\int_0^T S_t dt))$, then the derivative of V_0 with respect to σ is*

$$\vartheta = \frac{2e^{-rT}}{\sigma^2} E_Q\Big[\Phi(\bar{S}_T)\Big((W_T - \sigma T)\Big(\frac{S_T - S_0}{T\bar{S}_T} - m\Big) - \sigma\Big)\Big],$$

where $m = r - \dfrac{\sigma^2}{2}$.

Proof. By definition we know that

$$\begin{aligned}
\vartheta &= \frac{\partial V_0}{\partial \sigma} \\
&= E_Q\Big(e^{-rT}\Phi'(\bar{S}_T)\frac{\partial \bar{S}_T}{\partial \sigma}\Big).
\end{aligned}$$

By using

$$\frac{\partial S_T}{\partial \sigma} = (W_T - \sigma T)\bar{S}_T,$$

we have

$$\vartheta = e^{-rT} E_Q(\Phi'(\bar{S}_T)(W_T - \sigma T)\bar{S}_T).$$

Applying Proposition 2.4.8 with $u = S_t$, $F = \bar{S}_T$ and $G = (W_T - \sigma T)\bar{S}_T$ we have

$$\vartheta = e^{-rT} E_Q(\Phi(F)H(F,G)),$$

and

$$H(F,G) = \delta(Gu(D^u F)^{-1}).$$

Remind that $D^u F = \frac{\sigma}{2T}\left(\int_0^T S_r dr\right)^2$, we deduce

$$\begin{aligned} H(F,G) &= \delta(Gu(D^u F)^{-1}) \\ &= \delta\Big(\frac{2T(W_T - \sigma T)\bar{S}_T S_t}{\sigma\left(\int_0^T S_r dr\right)^2}\Big) \\ &= \frac{2}{\sigma}\delta\Big(\frac{(W_T - \sigma T)S_t}{\int_0^T S_r dr}\Big). \end{aligned}$$

Then,

$$\begin{aligned} \vartheta &= e^{-rT} E_Q(\Phi(\bar{S}_T)H(F,G)) \\ &= \frac{2e^{-rT}}{\sigma} E_Q\Big(\Phi(\bar{S}_T)\delta\Big(\frac{(W_T - \sigma T)S_t}{\int_0^T S_r dr}\Big)\Big). \end{aligned}$$

Recall the proposition

$$\delta(Lu) = L\int_0^T u(t)dW_t - \int_0^T D_t L u(t)dt,$$

if $L \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta$.

Set $L = \dfrac{W_T - \sigma T}{\int_0^T S_r dr}$.

By using $D_t W_T = 1$, $D_t\Big(\int_0^T S_r dr\Big) = \sigma\int_t^T S_r dr$ and rearranging we have

$$\begin{aligned}
D_t L &= \frac{\int_0^T S_t dt - (W_T - \sigma T)\sigma\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^2} \\
&= \frac{1}{\int_0^T S_t dt} - \sigma(W_T - \sigma T)\frac{\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^2}.
\end{aligned}$$

And therefore

$$\begin{aligned}
\int_0^T D_t L S_t dt &= \int_0^T \Big(\frac{1}{\int_0^T S_t dt} - \sigma(W_T - \sigma T)\frac{\int_t^T S_r dr}{\Big(\int_0^T S_t dt\Big)^2}\Big) S_t dt \\
&= 1 - \sigma(W_T - \sigma T)\frac{\int_0^T S_t\Big(\int_t^T S_r dr\Big) dt}{\Big(\int_0^T S_t dt\Big)^2}.
\end{aligned}$$

By using the fact

$$\int_0^T S_t\Big(\int_t^T S_r dr\Big) dt = \frac{1}{2}\Big(\int_0^T S_r dr\Big)^2,$$

and rearranging we can obtain

$$\int_0^T D_t L S_t dt = 1 - \frac{\sigma(W_T - \sigma T)}{2}.$$

That implies

$$\delta(LS_t) = \Big(\frac{W_T - \sigma T}{\int_0^T S_r dr}\Big)\int_0^T S_t dW_t - \Big(1 - \frac{\sigma(W_T - \sigma T)}{2}\Big).$$

By plugging in

$$\int_0^T S_t dW_t = \frac{1}{\sigma}\Big(S_T - S_0 - r\int_0^T S_t dt\Big),$$

and rearranging we deduce

$$\delta\Big(\Big(\frac{W_T-\sigma T}{\int_0^T S_r dr}\Big)S_t\Big) = \Big(\frac{W_T-\sigma T}{\int_0^T S_r dr}\Big)\frac{1}{\sigma}\Big(S_T - S_0 - r\int_0^T S_t dt\Big) - \Big(1 - \frac{\sigma(W_T-\sigma T)}{2}\Big)$$
$$= \frac{1}{\sigma}\Big[(W_T-\sigma T)\Big(\frac{S_T-S_0}{T\bar{S}_T} - \Big(r - \frac{\sigma^2}{2}\Big)\Big) - \sigma\Big].$$

Finally we obtain

$$\vartheta = \frac{2e^{-rT}}{\sigma^2}E_Q\Big[\Phi(\bar{S}_T)\Big((W_T-\sigma T)\Big(\frac{S_T-S_0}{T\bar{S}_T} - m\Big) - \sigma\Big]$$

where $m = r - \frac{\sigma^2}{2}$.

The proof is complete. □

6.3 Greeks for Other Itô Martingales Modelling

In this part, we will discuss some calculations of Greeks for these models formed of Itô martingale. Especially, in some cases the Integration by parts formula can not be applied because of some 'bad points' (i.e. W_0).

6.3.1 Example on Brownian Motion Market

Let the stock price at maturity time T be

$$S_T = S_0 + \sigma W_T,$$

and the payoff H only depends on S_T. That is, $H = \Phi(S_T)$.

Suppose that the option price at time $t = 0$ is

$$V_0 = e^{-rT}E_Q(\phi(S_T)).$$

The derivative of V_0 with respect to the parameter σ can be computed as follows:

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= e^{-rT}E_Q\Big(\phi'(S_T)\frac{\partial S_T}{\partial \sigma}\Big)\\ &= e^{-rT}E_Q(\phi'(S_T)W_T)\end{aligned}$$

by using $\dfrac{\partial S_T}{\partial \sigma} = W_T$.

Set $F = S_T$, $G = W_T$ and $u = 1$, we deduce

$$D_tF = \sigma,$$

and therefore

$$\begin{aligned}D^uF &= \int_0^T \sigma dt\\ &= \sigma T.\end{aligned}$$

By applying Proposition 2.4.8 we can obtain

$$\frac{\partial V_0}{\partial \sigma} = e^{-rT}E_Q(\phi(S_T)H(F,G)),$$

and

$$\begin{aligned}H(F,G) &= \delta(Gu(D^uF)^{-1})\\ &= \frac{\delta(W_T)}{\sigma T}.\end{aligned}$$

By plugging in

$$\delta(W_T) = W_T^2 - T,$$

we obtain

$$H(F,G) = \frac{W_T^2 - T}{\sigma T}.$$

Finally, we have

$$\frac{\partial V_0}{\partial \sigma} = e^{-rT}E_Q\Big(\phi(S_T)\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma}\Big)\Big).$$

6.3.2 Example on $W_t^2 - t$ Martingale Market

Let the stock price at maturity time T be

$$S_T = S_0 + \sigma(W_T^2 - T),$$

and the paroff H only depends on S_T. That is, $H = \Phi(S_T)$.

Suppose that the option price at time $t = 0$ is

$$V_0 = e^{-rT} E_Q(\phi(S_T)).$$

The derivative of V_0 with respect to the parameter σ can be computed as follows.

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= e^{-rT} E_Q\Big(\phi'(S_T)\frac{\partial S_T}{\partial \sigma}\Big)\\ &= e^{-rT} E_Q(\phi'(S_T)(W_T^2 - T)),\end{aligned}$$

by using $\dfrac{\partial S_T}{\partial \sigma} = W_T^2 - T$.

Set $F = S_T$, $G = W_T^2 - T$ and $u = 1$, we deduce

$$D_t F = 2\sigma W_T,$$

and therefore

$$\begin{aligned}D^u F &= \int_0^T 2\sigma W_T dt\\ &= 2\sigma T W_T.\end{aligned}$$

Then we have

$$Gu(D^u F)^{-1} = \frac{W_T}{2\sigma T} - \frac{1}{2\sigma W_T}.$$

Recall the proof of Proposition 2.4.8 we know that the equation

$$E(\langle Df(F), Gu(D^u F)^{-1}\rangle_H) = E(f(F)\delta(Gu(D^u F)^{-1}))$$

holds if $Gu(D^uF)^{-1}$ belongs to Domδ.

As the expectation $E\left(\frac{1}{W_T}\right)$ is not defined and $\frac{W_T}{2\sigma T} - \frac{1}{2\sigma W_T}$ does not belong to the domain of δ, we can not apply Proposition 2.4.8 to compute this Greek.

Set $H(x) = \phi(S_0 + \sigma(x - T))$, we have

$$H'(x) = \sigma\phi'(S_0 + \sigma(x - T)).$$

Define function $Q(x)$ by

$$Q(x) = H(x) - H(0) - H'(0)x,$$

we deduce

$$\begin{aligned} Q'(x) &= H'(x) - H'(0) \\ &= \sigma\phi'(S_0 + \sigma(x - T)) - H'(0). \end{aligned}$$

Then the derivative of V_0 with respect to the parameter σ is

$$\begin{aligned} \frac{\partial V_0}{\partial \sigma} &= e^{-rT} E_Q(\phi'(S_T)(W_T^2 - T)) \\ &= \frac{e^{-rT}}{\sigma} [E_Q(Q'(W_T^2)(W_T^2 - T)) + H'(0)E(W_T^2 - T). \end{aligned}$$

By using

$$E(W_T^2 - T) = 0$$

we obtain

$$\begin{aligned} \frac{\partial V_0}{\partial \sigma} &= \frac{e^{-rT}}{\sigma} E_Q(Q'(W_T^2)(W_T^2 - T)) \\ &= \frac{e^{-rT}}{\sigma} \int_{-\infty}^{+\infty} Q'(x^2)(x^2 - T) \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx. \end{aligned}$$

Set $x^2 = y$, we have $dx = \dfrac{dy}{2\sqrt{y}}$, then

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_0^{+\infty} Q'(y)(y-T)e^{-\frac{y}{2T}} \frac{dy}{2\sqrt{y}} \\ &= \frac{e^{-rT}}{2\sigma\sqrt{2\pi T}} \int_0^{+\infty} \frac{(y-T)}{\sqrt{y}} e^{-\frac{y}{2T}} dQ(y).\end{aligned}$$

By using integration by parts, we obtain

$$\frac{\partial V_0}{\partial \sigma} = \frac{e^{-rT}}{2\sigma\sqrt{2\pi T}} \Big[\frac{(y-T)}{\sqrt{y}} e^{-\frac{y}{2T}} Q(y) \Big|_{y=0}^{y=+\infty} - \int_0^{+\infty} \Big(\frac{(y-T)}{\sqrt{y}} e^{-\frac{y}{2T}} \Big)' Q(y) dy \Big].$$

By using $Q(0) = 0$, we have $\dfrac{(y-T)}{\sqrt{y}} e^{-\frac{y}{2T}} Q(y) \Big|_{y=0}^{y=+\infty} = 0$, then

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= \frac{e^{-rT}}{2\sigma\sqrt{2\pi T}} \int_0^{+\infty} \Big(\frac{\sqrt{y}}{2T} - \frac{1}{\sqrt{y}} - \frac{T}{2y^{3/2}} \Big) Q(y) e^{-\frac{y}{2T}} dy \\ &= \frac{e^{-rT}}{2\sigma\sqrt{2\pi T}} \int_{-\infty}^{+\infty} \Big(\frac{x^2}{T} - 2 - \frac{T}{x^2} \Big) Q(x^2) e^{-\frac{x^2}{2T}} dx \\ &= \frac{e^{-rT}}{2\sigma} E_Q \Big(Q(W_T^2) \Big(\frac{W_T^2}{T} - 2 - \frac{T}{W_T^2} \Big) \Big).\end{aligned}$$

As we know that $\left| \dfrac{Q(W_T^2)}{W_T^2} \right| <$ constant, the expectation above is well defined.

Another way to compute this Greek is stated as follows.

First by applying the perturbation

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= e^{-rT} E_Q \Big(\phi'(S_T) \frac{\partial S_T}{\partial \sigma} \Big) \\ &= \lim_{\varepsilon \to 0} e^{-rT} E_Q(\phi'(S_T)(W_T^2 - T)\mathbf{1}(|W_T| > \varepsilon)).\end{aligned}$$

Then, by applying the D^u operator

$$\frac{\partial V_0}{\partial \sigma} = \lim_{\varepsilon \to 0} e^{-rT} E_Q \Big(\frac{D^u \phi(S_T)}{D^u S_T} (W_T^2 - T)\mathbf{1}(|W_T| > \varepsilon) \Big).$$

Then, by integration by parts formula

$$\frac{\partial V_0}{\partial \sigma} = \lim_{\varepsilon \to 0} \frac{e^{-rT}}{2\sigma T} E_Q\Big[\phi(S_T)\delta\Big(\Big(W_T - \frac{T}{W_T}\Big)\mathbf{1}(|W_T|> \varepsilon)\Big)\Big].$$

Then, by Proposition 2.3.28 and similar technique

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= \lim_{\varepsilon \to 0} \frac{e^{-rT}}{2\sigma T} E_Q\Big[\phi(S_T)((W_T^2 - T)\mathbf{1}(|W_T|> \varepsilon) - \int_0^T D_t\Big(\Big(W_T - \frac{T}{W_T}\Big)\mathbf{1}(|W_T|> \varepsilon)dt\Big)\Big] \\ &= \lim_{\varepsilon \to 0} \frac{e^{-rT}}{2\sigma T} E_Q\Big[\phi(S_T)\Big((W_T^2 - T)\mathbf{1}(|W_T|> \varepsilon) - T\Big(1 + \frac{T}{W_T^2}\Big)\mathbf{1}(|W_T|> \varepsilon)dt\Big)\Big].\end{aligned}$$

Finally, by straightforward arguments

$$\begin{aligned}\frac{\partial V_0}{\partial \sigma} &= \lim_{\varepsilon \to 0} \frac{e^{-rT}}{2\sigma T} E_Q\Big[\phi(S_T)\Big(W_T^2 - 2T - \frac{T^2}{W_T^2}\Big)\mathbf{1}(|W_T|> \varepsilon)\Big] \\ &= \frac{e^{-rT}}{2\sigma T} E_Q\Big[(\phi(S_T) - \phi(S_0 - \sigma T))\Big(W_T^2 - 2T - \frac{T^2}{W_T^2}\Big)\Big].\end{aligned}$$

Chapter 7

Conclusion

Brief summary of the thesis

In this dissertation we presented variety of probabilistic and stochastic calculations related to the weighted self-normalized sum of exchangeable variables, the Ornstein-Uhlenbeck process, various operators defined on the Wiener space and Greeks in mathematical finance.

In particular, several properties of the weighted self-normalized sum of exchangeable variables are discussed.

Different order moments of the Ornstein-Uhlenbeck process are computed by using Itô calculus.

Various operators defined on the Wiener space, such as the derivative operator, the divergence operator, the infinitesimal generator of the Ornstein-Uhlenbeck semigroup and its characteristics are computed via the Malliavin calculus.

We also apply Malliavin calculus to compute Greeks where in addition to the classical Black-Scholes model we also treat share price Itô martingale models such as B_t and $B_t^2 - t$.

Main results

We generalize examples of calculations of various moments of the Brownian motion and Ornstein-Uhlenbeck process to the Itô processes of the Ornstein-Uhlenbeck type.

We presented the variety of examples

(i) on calculations of the derivative operator D and the operator D^h;

(ii) on calculations of the operator L ;

(iii) on calculations of the the norms $\|\cdot\|_L$ and $\|\cdot\|_{2,2}$

In addition, the integration by parts formula is modified to calculate Greeks for the Itô martingale stock markets.

Future works

It would be interesting to solve the second order ODE stated in section 4.3.2 to obtain Ex_t^k for Ornstein-Uhlenbeck type process.

It would be interesting to compute higher order derivatives of option prices for Black-Scholes model.

It would be interesting to consider the relative financial markets and compute Greeks for the cases:

(i) Itô martingales with Y_t being an Orntein-Uhlenbeck process.

(ii) Discrete time martingales with Y_i being exchangeable.

Bibliography

[1] Apostol, T. M. (1974). *Mathematical Analysis, Second Edition*. Pearson.

[2] Barndorff-Nielsen, O. E. and Shephard, N. (2001). Non-gaussian ornstein–uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(2):167–241.

[3] Bartoszek, K., Glémin, S., Kaj, I., and Lascoux, M. (2017). Using the ornstein–uhlenbeck process to model the evolution of interacting populations. *Journal of theoretical biology*, 429:35–45.

[4] Bell, D. R. (1987). *The Malliavin calculus*, volume 73. Springer Berlin et al.

[5] Benth, F. E., Kallsen, J., and Meyer-Brandis, T. (2007). A non-gaussian ornstein–uhlenbeck process for electricity spot price modeling and derivatives pricing. *Applied Mathematical Finance*, 14(2):153–169.

[6] Bonaccorsi, S. and Priola, E. (2006). From brownian motion to stochastic differential equations. https://www.math.tecnico.ulisboa.pt/~czaja/ISEM/10internetseminar200607.pdf.

[7] Cohn, H. and Hall, P. (1982). On the limit behaviour of weighted sums of random variables. *Probability Theory and Related Fields*, 59(3):319–331.

[8] Di Nunno, G., Øksendal, B. K., and Proske, F. (2009). *Malliavin calculus for Lévy processes with applications to finance*, volume 2. Springer.

[9] Elworthy, K. D., Li, X.-M., and Yor, M. (1999). The importance of strictly local martingales; applications to radial ornstein–uhlenbeck processes. *Probability theory and related fields*, 115(3):325–355.

[10] Fournié, E., Lasry, J.-M., Lebuchoux, J., Lions, P.-L., and Touzi, N. (1999). Applications of malliavin calculus to monte carlo methods in finance. *Finance and Stochastics*, 3(4):391–412.

[11] Friz, P. K. (2002). An introduction to malliavin calculus. http://dl.icdst.org/pdfs/files/ae7e5cfe7c59bd5c21e98ceb6a8275af.pdf.

[12] Fuchs, A., Joffe, A., and Teugels, J. (2002). Expectation of the ratio of a sum of squares to the square of the sum: Exact and asymptotic results. *Theory of Probability & Its Applications*, 46(2):243–255.

[13] Garamszegi, L. Z. (2014). *Modern Phylogenetic Comparative Methods and Their Application in Evolutionary Biology*. Springer.

[14] Graczyk, P. and Jakubowski, T. (2012). Analysis of ornstein–uhlenbeck and laguerre stochastic processes. In *Séminaires & Congrès*, volume 25, pages 195–249.

[15] Haile, P. (2018). Differentiating an integral. http://www.econ.yale.edu/~pah29/409web/leibniz.pdf.

[16] Hak-man, T. A., David, N., and Gang, Y. G. (2011). *Stochastic analysis, stochastic systems, and applications to finance*. World Scientific.

[17] Karatzas, I. and Shreve, S. E. (1998). Brownian motion. In *Brownian Motion and Stochastic Calculus*, pages 47–127. Springer.

[18] Karlin, S. (2014). *A first course in stochastic processes*. Academic press.

[19] Karlin, S. and Taylor, H. E. (1981). *A second course in stochastic processes*. Elsevier.

[20] Klein, I. (2000s). Ito processes and stochastic differentials. https://homepage.univie.ac.at/irene.klein/fvm_2_ito_calc_G.pdf.

[21] Kusuoka, S. and Stroock, D. (1984). Applications of the malliavin calculus, part i. In *North-Holland Mathematical Library*, volume 32, pages 271–306. Elsevier.

[22] Kusuoka, S. and Stroock, D. (1985). Applications of the malliavin calculus, part ii. *J. Fac. Sci. Univ. Tokyo*, 32(1-76).

[23] Kusuoka, S. and Stroock, D. (1987). Applications of the malliavin calculus, part iii. *J. Fac. Sci. Univ. Tokyo*, 34(391-442).

[24] M. Bismut, J. (1984). Large deviations and the malliavin calculus. *Birkhauser Prog. Math.*

[25] Malliavin, P. (1978). Stochastic calculus of variations and hypoelliptic operators. In *Proc. Internat. Symposium on Stochastic Differential Equations, Kyoto Univ., Kyoto, 1976*. Wiley.

[26] Malliavin, P. and Thalmaier, A. (2006). *Stochastic calculus of variations in mathematical finance*. Springer Science & Business Media.

[27] Mason, D., Zinn, J., et al. (2005). When does a randomly weighted self-normalized sum converge in distribution? *Electronic Communications in Probability*, 10:70–81.

[28] Novak, S. Y. and Utev, S. (1990). Asymptotics of the distribution of the ratio of sums of random variables. *Siberian Mathematical Journal*, 31(5):781–788.

[29] Nualart, D. (2000s). Stochastic Processes. http://nualart.faculty.ku.edu/StochasticCalculus.pdf.

[30] Nualart, D. (2006). *The Malliavin calculus and related topics*, volume 1995. Springer.

[31] Nualart, D. and Saussereau, B. (2009). Malliavin calculus for stochastic differential equations driven by a fractional brownian motion. *Stochastic Processes and their Applications*, 119(2):391–409.

[32] Nualart, E. (2009). Lectures on malliavin calculus and its applications to finance. *Lecture notes*.

[33] Øksendal, B. (2010). An introduction to malliavin calculus with applications to economics,(1997). *Lecture Notes, Norwegian School of Economics and Business Administration*.

[34] Oksendal, B. (2013). *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media.

[35] Papoulis, A. and Pillai, S. U. (2002). *Probability, random variables, and stochastic processes*. Tata McGraw-Hill Education.

[36] Pavliotis, G. A. (2015). Stochastic process and applications. http://wwwf.imperial.ac.uk/~pavl/PavliotisBook.pdf.

[37] Petrou, E. et al. (2008). Malliavin calculus in lévy spaces and applications to finance. *Electronic Journal of Probability*, 13:852–879.

[38] Pinsky, M. and Karlin, S. (2010). *An introduction to stochastic modeling*. Academic press.

[39] Thierfelder, C. (2015). The trending ornstein-uhlenbeck process and its applications in mathematical finance. *Mathematical Finance*.

[40] Uhlenbeck, G. E. and Ornstein, L. S. (1930). On the theory of the brownian motion. *Physical review*, 36(5):823.

[41] Watanabe, S., Nair, M. G., and Rajeev, B. (1984). *Lectures on stochastic differential equations and Malliavin calculus*, volume 73. Springer Berlin et al.

[42] Zitkovic, G. (2010). Introduction to Stochastic Processes - Lecture Notes. https://web.ma.utexas.edu/users/gordanz/notes/introduction_to_stochastic_processes.pdf.

Appendix A

Exchangeable variable

Assume $\{Y_i\}_{i\geq 1}$ is a sequence of independent, identically distributed random variables, where Y is non-negative, and let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. X random variables independent of $\{Y_i\}_{i\geq 1}$, where X satisfies

$$E(X^2) < \infty \quad \text{and} \quad E(X) = 0.$$

Let R_n denote the randomly weighted self-normalized sum

$$R_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}.$$

By definition and swapping expectation and summation

$$\begin{aligned} E(R_n) &= E\Big(\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}\Big) \\ &= \sum_{i=1}^n E\Big(\frac{X_i Y_i}{\sum_{i=1}^n Y_i}\Big). \end{aligned}$$

Then, by independence

$$E(R_n) = \sum_{i=1}^n E\Big(X_i \cdot E\frac{Y_i}{\sum_{i=1}^n Y_i}\Big).$$

As Y_i are i.i.d., we deduce

$$\begin{aligned} 1 &= E\Big(\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n Y_i}\Big) \\ &= \sum_{i=1}^n E\Big(\frac{Y_i}{\sum_{i=1}^n Y_i}\Big) \\ &= n \cdot E\Big(\frac{Y_i}{\sum_{i=1}^n Y_i}\Big), \end{aligned}$$

which is equal to

$$E\Big(\frac{Y_i}{\sum_{i=1}^n Y_i}\Big) = \frac{1}{n}.$$

And therefore we have

$$E(R_n) = \sum_{i=1}^n E(X_i) \cdot \frac{1}{n} = E(X) = 0.$$

Set $\delta_i = \dfrac{X_i Y_i}{\sum_{i=1}^n Y_i}$, notice that δ is an exchangeable variable.

Then we have

$$\begin{aligned} E\Big(\sum_{i=1}^n \delta_i\Big)^2 &= E\Big(\sum_{1\le i,j\le n} \delta_i \delta_j\Big) \\ &= \sum_{i=1}^n E(\delta_i^2) + \sum_{i\neq j}^n E(\delta_i \delta_j), \end{aligned}$$

again as X_i and Y_i are i.i.d., we get

$$E\Big(\sum_{i=1}^n \delta_i\Big)^2 = nE(\delta^2) + n(n-1)E(\delta_1 \delta_2).$$

Furthermore, we can use this method similarly on the situation of $E\Big(\sum_{i=1}^n \delta_i\Big)^3$ and $E\Big(\sum_{i=1}^n \delta_i\Big)^4$:

Case $n = 3$ (third moment). By a straightforward arguments and then splitting the sum into equal and not equal indices we derive

$$E\Big(\sum_{i=1}^{n}\delta_i\Big)^3 = E\Big(\sum_{1\le i,j,k\le n}\delta_i\delta_j\delta_k\Big)$$
$$= \sum_{i=1}^{n}E\Big(\delta_i^3) + \sum_{\substack{i=j\neq k;\\ i=k\neq j;\\ j=k\neq i}}E(\delta_i\delta_j\delta_k) + \sum_{i\neq j,j\neq k,i\neq k}E(\delta_i\delta_j\delta_k\Big).$$

Then, by independence

$$E\Big(\sum_{i=1}^{n}\delta_i\Big)^3 = nE(\delta^3) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} n(n-1)E(\delta_1^2\delta_2) + n(n-1)(n-2)E(\delta_1\delta_2\delta_3)$$
$$= nE(\delta^3) + 3n(n-1)E(\delta_1^2\delta_2) + n(n-1)(n-2)E(\delta_1\delta_2\delta_3).$$

Case $n = 4$ (forth moment). Similarly, we split the sum into equal and not equal indices. To do it, we first fix the number of different indices in the vector $(i, j, k, t) = (i_1, i_2, i_3, i_4)$, say $z = 1, 2, 3, 4$. Then, roughly, consider all combinations of subsets of size z out of $(1, 2, 3, 4)$.

For example, for $z = 2$, by choosing a pair, say $(1, 2)$ we also fix the remaining pair $(3, 4)$. So we take all vectors

$(1, 2), (3, 4)$ corresponding to the case that $i_1 = i_2$ or $i = j$ (for $(1, 2)$) and $i_3 = i_4$ or $k = t$ (for $(3, 4)$) and $i = j \neq k = t$;

$(1, 3), (2, 4)$ corresponding to the case that $i_1 = i_3$ or $i = k$ and $i_2 = i_4$ or $j = t$ and $i = k \neq j = t$;

$(1, 4), (2, 3)$ corresponding to the case that $i_1 = i_4$ or $i = t$ and $i_2 = i_3$ or $j = k$ and $i = t \neq j = k$.

For $z = 3$,

$(1, 2, 3), 4$ corresponding to the case that $i_1 = i_2 = i_3$ or $i = j = k$ and $i = j = k \neq t$;

$(1, 3, 4), 2$ corresponding to the case that $i_1 = i_3 = i_4$ or $i = k = t$ and $i = k = t \neq j$;

$(1, 2, 4), 3$ corresponding to the case that $i_1 = i_2 = i_4$ or $i = j = t$ and $i = j = t \neq k$;

$(2, 3, 4), 1$ corresponding to the case that $i_2 = i_3 = i_4$ or $k = j = t$ and $k = j = t \neq i$.

Overall the sum will be

$$
\begin{aligned}
E\Big(\sum_{i=1}^{n}\delta_i\Big)^4 &= E\Big(\sum_{1\le i,j,k,t\le n}\delta_i\delta_j\delta_k\delta_t\Big)\\
&= \sum_{i=1}^{n}E(\delta_i^4)+\sum_{r\neq m}\sum_{\substack{r=i=j=k,m=t;\\ r=i=k=t,m=j;\\ r=i=j=t,m=k\\ r=j=k=t,m=i}}E(\delta_r^3\delta_m)+\sum_{r\neq m}\sum_{\substack{r=i=j,m=k=t;\\ r=i=k,m=j=t;\\ r=i=t,m=j=k}}E(\delta_r^2\delta_m^2)\\
&\quad+\sum_{r\neq m\neq q}\sum_{\substack{r=i=j,m=k,q=t;\\ r=i=k,m=j,q=t\\ r=i=t,m=j,q=k\\ r=j=k,m=i,q=t\\ r=j=t,m=i,q=k\\ r=k=t,m=i,q=j}}E(\delta_r^2)E(\delta_m)E(\delta_q)+\sum_{i\neq j\neq k\neq t}E(\delta_i)E(\delta_j)E(\delta_k)E(\delta_t).
\end{aligned}
$$

Therefore, the sum can be computed as

$$
\begin{aligned}
E\Big(\sum_{i=1}^{n}\delta_i\Big)^4 &= nE(\delta^4)+4n(n-1)E(\delta_1^3)\delta_2+3n(n-1)E(\delta_1^2\delta_2^2)+6n(n-1)(n-2)E(\delta_1^2\delta_2\delta_3)\\
&\quad+n(n-1)(n-2)(n-3)E(\delta_1\delta_2\delta_3\delta_4).
\end{aligned}
$$