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Teachers' Orchestration of Mathematics for Different Groups of Students

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Fay Baldry

School of Education

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Thesis Abstract

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Fay Baldry

Class composition for secondary mathematics lessons in England is often decided by measures related to prior attainment, with students grouped with others of similar 'ability' and referred to as 'setting'. Research suggests setting does not raise overall attainment and disadvantages students placed in lower attaining sets, with those students experiencing an impoverished curriculum, such as more 'drill and practice' (Wiliam and Bartholomew, 2004). This classroom-based video study explores mathematics teachers' practice in 'typical' classrooms and changes that come about when they teach sets with different attainment profiles. The Orchestration of Mathematics Framework (OMF) was developed to interpret classroom activities; this framework integrates a range of theoretical perspectives, including variation theory (Marton and Pang, 2006) and classroom norms (Cobb et al., 2009). The application of the OMF in different settings demonstrated it facilitated cross-class comparisons, and while further research is needed, this evidenced the potential of the OMF as an analytical tool. Three teachers participated in this study; for each teacher two sets with different attainment profiles were studied. When the findings were analysed, a complex picture of teachers' practice emerged. There were differences between sets, but many facets of teachers' practice were relatively stable across classes. Some differences reflected previously reported characteristics associated with low attaining sets, such as tighter control over classroom talk (Kutnick et al., 2006), but others were absent; for example, there were no discernible shifts to 'drill and practice'. One common thread was the low frequency of attention being drawn to mathematical concepts beyond that provided by examples. One question raised is whether students in higher attaining sets are those better placed to read the implicit mathematical meanings available in the act of 'doing' tasks.

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1. Introduction

1.1 Overview of the Study

Mathematics holds a singular position in English schools, having a privileged status as one of the core subjects studied by all students, whilst being repositioned from one of the most popular subjects in primary schools to one of the most disliked by the end of compulsory education (e.g. Brown et al., 2008; Noyes, 2013). Whilst international comparisons of attainment place England above the international mean, performance does not match East Asian jurisdictions (Greany et al., 2016) and England has a higher proportion of students in the lowest attainment band (Andrews et al., 2017). Moreover, concerns persist about low levels of numeracy in adults (Evans et al., 2017), and the impact this has on individuals and the wider economy (Kuczera et al., 2016). Consequently, there is ongoing interest from a wide range of stakeholders in the teaching of mathematics.

Discussions about the teaching and learning of mathematics are often framed in terms of dichotomies (Fan and Bokhove, 2014). For example, Skemp's (1976) seminal work on relational and instrumental understanding has been frequently cited in support of the argument that both learners and teachers can hold very different views of what constitutes mathematical activity. For some, mathematics is about the efficient application of algorithms, relying on recall and reproduction, whilst for others it is about flexible thinking, based on an understanding of the structures within mathematics and links between concepts (Remillard and Bryans, 2004; Zan and Di Martino, 2007). Similarly, a 'traditional' style of teaching is frequently contrasted with inquiry-oriented approaches; with the former characterised as teacher-centred, using a transmission style to impart rules and procedures, and the latter as student-centred, focussed on developing conceptual understanding through the use of rich tasks (Stein, 2000). Whilst these dichotomies represent an oversimplification of the complexities in the teaching and learning of mathematics, questions are raised as to what prompts these dichotomous perspectives and whether different groups of students experience mathematics in distinctive ways.

There does appear to be a disconnect between educational research and school practice (Boyd, 2007). For instance, mathematics educational research, often underpinned by a constructivist perspective (Lerman, 2013), tends to advocate more open problem-solving approaches to teaching (e.g. Mason, 2000; Boaler, 2010), but traditional ‘chalk and talk’ methods hold sway in many UK classrooms (Watson and Evans, 2012). Of particular interest here are issues related to ‘setting’, which is the practice of placing students of similar ‘ability’ in classes together, referred to as sets. Schools report they use measures of prior attainment to decide on the composition of sets (Francis et al., 2017). In England, setting happens more often and earlier in mathematics than in other subjects (Ireson et al., 2002; Taylor et al., 2017).

The motivation for this research originates in the disconnect between research and practice in relation to setting and the implications for social justice. In English secondary schools, the practice of setting endures in spite of evidence that overall attainment is, at best, not improved, and students with low prior attainment and those from low socioeconomic backgrounds are disproportionally affected (e.g. Ireson et al., 2005; Forgasz, 2010). Whilst researchers have reported the detrimental effects of grouping students in this manner for many years (e.g. Boaler, 1997; Ireson et al., 2002; Noyes, 2012), setting has been recommended by previous UK government departments responsible for education (DfEE, 2001; DfES, 2005) and is currently the dominant form of class composition for mathematics provision in English secondary schools (Taylor et al., 2017). Recently, the Education Endowment Foundation (EEF), a government funded body with a remit to embed evidence in policy decision-making (Gough et al., 2018), has reported that setting has a slight negative effect on student progress (EEF, 2018). Schools, however, appear reluctant to consider alternatives, such as mixed-attainment teaching for all subjects (Taylor et al., 2017), and struggle to adapt their setting practices, say to a more equitable deployment of teachers (Francis et al., 2019).

In recent years, the UK government has focussed attention on the attainment of ‘disadvantaged’ students through funding and accountability measures. Additional funds are provided for students identified as eligible for ‘Pupil Premium’, where the main criteria is eligibility for free school meals, a proxy for low socioeconomic background. Schools also have to publish the gap in attainment between students eligible for Pupil Premium funds and others, with implications for school inspection outcomes if the gap is not narrowed (DfE, 2018). Research has demonstrated that students from low socioeconomic backgrounds are over represented in low sets, even when prior attainment is taken into account, which impacts negatively on their attainment (Dunne et al., 2011). However, in spite of the drivers to raise attainment of 'disadvantaged' students, schools remain reluctant to challenge setting as a method of grouping students (Taylor et al., 2017).

Published research about student attainment has included the analysis of large attainment datasets and international comparison studies, and have considered factors such as the impact of different class compositions and how teaching approaches vary (e.g. Hiebert et al., 2003b; Wiliam and Bartholomew, 2004; Francis et al., 2019). Research has identified a range of factors that influence students' experience of mathematics. For example, classes with lower attainment profiles tend to experience a more restricted curriculum and be taught by less qualified and less experienced teachers (Wiliam and Bartholomew, 2004; Francis et al., 2019). There is also evidence that pedagogical approaches seen in sets with lower attainment profiles tend to have particular characteristics, such as more 'drill and practice' and less peer interaction, which results in the students experiencing an impoverished version of mathematics (Boaler, 1997; Watson, 2001; Hallam and Ireson, 2003).

Whilst these general pedagogical characteristics have been identified, it is not clear the extent to which differences relate to any variation in the demographic of teachers allocated to lower or higher attaining sets. Drawing on self-reporting questionnaires, researchers have published findings about teachers' perceptions of how they change their practice as they move between mixed-attainment classes and sets (Hallam and Ireson, 2005), and more recently between sets with different attainment profiles (Mazenod et al., 2019). However, there is less evidence as to how these changes are enacted in classrooms and the extent to which reported pedagogical differences between sets are manifest in individual teachers' practice.

This study seeks to contribute to our understanding of how students in different sets experience mathematics by exploring three secondary teachers' practice as they teach mathematics classes with different attainment profiles. The argument made is that there is little published research that directly compares individual teachers' enacted classroom practice in contrasting sets, so a detailed analysis adds to our understanding of possible antecedents for the different pedagogical characteristics reported in sets with different attainment profiles. Moreover, insights provided by this study extend what is known about approaches that have the potential to ameliorate the effects of setting, and in particular how individual teachers might adapt their practice.

In order to explicate teachers' practice, a framework for systematically describing and interpreting teachers' pedagogical moves was developed in this study, namely the Orchestration of Mathematics Framework (OMF). As the focus was on teachers' current practice, independent of topic taught or pedagogical orientation, a mechanism for interpreting 'typical' lessons was required. One key contribution to knowledge of this study is the power of the OMF to integrate the myriad of interconnected facets of teachers' pedagogical practice into meaningful narratives about mathematical practices in 'typical' lessons.

1.2 Background of the Author

I taught mathematics in English secondary schools for seventeen years. I then moved into the higher education sector, where I have worked on Initial Teacher Education (ITE) and education master's courses for the past eight years. In my transition to the higher education sector, my engagement with educational research became much more extensive, which led to a growing awareness that many of my school practices were based on tacit knowledge. I also noticed a disconnect between my previous school practices and what was 'known' in research. One particularly stark contrast was in relation to setting; I had not only taught in sets, as head of department I had been responsible for their organisation, but I was unaware of the body of research on the topic.

In my current role, I visit over twenty schools each year and observe student-teachers' lessons. Whilst student learning is an incremental process over time, all parties involved, the student-teacher, the school-based mentor and I, are expected to have a sufficient understanding of the single observed lesson to hold discussions about student learning and the student-teacher's capabilities. Indeed, in schools, most observations of teachers that occur as part of professional development practices or performance management processes are similarly single lessons. Therefore, while 'the lesson' might not be an idea unit of study in relation to learning, it does reflect practice in schools.

What I noticed during joint observations with school-based staff was that we appeared to read the classroom in very different ways. Work I have undertaken on lesson study (Larssen et al., 2018) drew my attention to the complexities of building a shared understanding of what may have happened in the lesson, and in particular, what might have been mathematically significant events. This led me to the challenging task of looking at typical lessons, with the aim to generate an understanding of the mathematics the students experienced and how the teachers' actions might have brought that about. But to be clear, I am interested in how we can interpret the mathematics classroom; making evaluative judgments of teaching is not my purpose.

These two strands, the inequities inherent in setting and the complexities in interpreting mathematics classrooms, led me to undertake a classroom-based study in the contexts of setting. Many professional development initiatives that aim to develop teachers' practice are reported as being ineffective (Pedder and Opfer, 2011). If I can articulate how individual teachers shift their practice when teaching different sets, then ameliorating some of the effects of setting may be achievable through teachers implementing some of their current practices in different settings, rather than having to develop new ways of working; potentially a more achievable strategy.

1.3 Outline of the Thesis

This first chapter has provided an overview of the study and has outlined the motivational drivers behind some of the choices made. The second chapter offers an overview of current research about setting and the impact on attainment. The third chapter discusses the development of the Orchestration of Mathematics Framework (OMF) as a framework for interpreting classrooms. The first section draws on the didactic triangle (Straesser, 2007) and the notion of classroom norms (Cobb et al., 2001). The second section addresses teachers' pedagogical moves and focusses on task features, classroom discourse and the teachers' management of the lesson trajectory. This includes extending the framework of a hypothetical learning trajectory (Simon, 1995) to incorporate different types of lessons. In section three, the Orchestration of Mathematics Framework (OMF) is presented as a mechanism to draw together the myriad of strands that form teachers' pedagogical moves. This instrument was subsequently used throughout the data collection and analysis undertaken in this study. The fourth section discusses the further developments of the OMF during its application in the pilot study and the transition to the main study.

Chapter four, the methodology, outlines the research questions that relate to how teachers' pedagogical practice shifts when they teach different groups of students. The chapter details the opportunistic recruitment of three teachers from two schools and the associated data collection and analysis. For each teacher, two classes with different attainment profiles were observed and video recorded. Chapter five presents the findings, which form the evidence for the claims made in chapter six. After an overview, the substantive part of the chapter presents the lesson narratives. For each of the three teachers, one lesson from a set with a higher attainment profile and one lesson from a set with a lower attainment profile are presented in detail. When other lessons were also recorded, a summary of the analysis is provided.

In chapter six, after an introduction the second section offers a response to the research questions. The remaining sections discuss the detailed claims made and provides the warrants (Toulmin, 2003) for those claims. Section three reviews the efficacy of the OMF, after which the fourth section discusses each element of the OMF in detail and concludes with an overview of the shifts in practice. Chapter seven is the conclusion; a summary of the responses to the research questions is presented along with the limitations of the study. Implications and further areas of research are considered.

2. Literature Review: Setting

In order to explore teachers' practice with classes that have different attainment profiles, an understanding of the implications of linking students' class allocation with prior attainment is required. This chapter addresses this issue and establishes the context within which this study is situated.

A limited range of schools and teachers participated in this study. In order to site this work in a wider context, national and international research related to setting is discussed in the following sections. First, the position of setting in English schools is discussed, and then evidence about the relationships between setting practices and student outcomes is explored. Pedagogical characteristics that have been associated with particular sets are then discussed, after which the implications of these issues are outlined.

2.1 Setting in English Secondary Schools

In England, classes in secondary schools are almost exclusively composed of students from the same year group. A common feature in most secondary schools is that students are grouped by 'ability' for at least some subjects; in England this is usually referred to as 'setting'. Mathematics is one of the subjects where setting is most common, especially for older pupils (Ireson et al., 2005; Francis et al., 2019). There are reported issues of equity when this system is used, such as placement in lower attaining sets being more likely for students from low socioeconomic backgrounds (Wiliam and Bartholomew, 2004; Dunne et al., 2011), with this placement being detrimental to students' progress (Boaler et al., 2000; Hallam and Ireson, 2003; Noyes, 2012). In spite of the longstanding and ongoing debate in educational research about the effects of grouping students by notions of ability, setting remains predominantly unchallenged in English schools (Taylor et al., 2017); it is often seen as the only practical way to teach students who have a range of 'abilities' (Wiliam and Bartholomew, 2004; Hallam and Ireson, 2007). Boyd (2007) argued that it would take 'a paradigm shift in thinking about learning to dislodge [setting] from its pre-eminent position' (p.293). More recently, Taylor et al. (2017) reported a commonly held view by English secondary school teachers was that mixed attainment classes were problematic. So, it appears that this shift has not yet occurred, and setting is likely to remain a feature of the English education system for the time being at least. Therefore, research that adds to the understanding of the relationship between setting and the learning of mathematics should be of value.

As will be discussed in more detail later (2.3), schools often present the mechanism for set allocation as measures of prior attainment (Muijs and Dunne, 2010). However, there is evidence that these processes are influenced by other factors and are often far from transparent (e.g. Dunne et al., 2011; Francis et al., 2017). In contrast, the language commonly used in English schools relates to 'ability', where the phrase 'setting by ability' is the norm, with 'tracking' the US equivalent; as a consequence this language appears in research papers (Wiliam and Bartholomew, 2004). Therefore, the term 'ability' may be used in this study when referring to the setting process, as it forms an expedient label, but I would wish to echo the sentiments of Wiliam and Bartholomew (2004) when they stated 'we believe that such notions of ability are not in any way well founded and are of dubious validity as predictors of potential' (p.281). To reflect this stance, students will be referred to as low or high attaining when this will not cause confusion. The phrase 'low attaining sets' will be used to describe what schools often refer to as 'bottom' sets, and similarly 'high' in reference to 'top' sets, although it is acknowledged that there is considerable variety in how schools arrange the composition of sets (Roy et al., 2018). For example, some schools state they employ a strict hierarchy based on attainment data, whilst others will set using broader bands of attainment or will include parallel groups.

2.2 Relationships between Setting and Cohort Attainment

A large number of studies have been undertaken into the impact of setting on attainment (Kutnick et al., 2005b). In order to gain an understanding of the research evidence available, a search was made of the British Educational Index and ERIC for peer-reviewed articles published after 2000, using the key terms 'ability grouping', 'setting', 'class' and 'school'. This was supplemented with a further search with 'setting' replaced with 'tracking', as the latter is more commonly used in the US for grouping students by attainment. In addition, a search was made on Google Scholar with the inclusion of 'mathematics'. These were filtered and considered in more detail if they included first-hand analysis of class-level attainment data for mathematics or were a meta-analysis of studies related to setting.

However, there are considerable difficulties in separating class composition from other influences on attainment. National and international data on student attainment has become increasingly available over recent years. In England, the National Pupil Database (NPD) collates attainment data for all students in state education, recorded at different stages in their schooling. This includes results from standardised Key Stage assessments taken in primary schools, and GCSEs taken at the end of compulsory education in secondary schools. Contextual information, such as gender, ethnicity, eligibility for free school meals and school details are also included. Researchers have interrogated these large datasets; contextual information and multiple assessment points has allowed the attainment for different groups of students to be analysed over time (e.g. Strand, 2014).

Analysis has highlighted variation between schools. For example, at least 'one quarter of schools in England are differentially effective for students of differing prior ability levels' (Dearden et al., 2011, p.1). However, as Strand (2014) argued, differential attainment is complex, being influenced by the interaction of many variables: school-level factors, such as school ethos and quality of teaching; class-level factors, such as teacher expectations; and students' backgrounds, such as ethnicity and class. Unfortunately, understanding the role of setting is an even more complex undertaking, as while there is a large amount of school-level and student-level data, class-level data is not routinely available. For those interested in the impact of class composition, individual schools need to be approached, which has resulted in smaller scale studies.

The difficulties in obtaining class-level data were reflected in the number and scale of research studies found in the literature search. Due to the complexities of taking into account cultural differences (Andrews, 2007), empirical studies based in England were the primary focus of the literature search, with international studies being drawn on when they offered supporting or contrasting evidence. Two studies were identified that included the analysis of class-level attainment data from national tests to explore the effects of set allocation on attainment; namely, those by Ireson et al. (2005) and Wiliam and Bartholomew (2004), which are considered in more detail below. The Education Endowment Foundation (EEF), a UK government funded What Works Network centre with a remit to generate evidence to improve teaching (Gough et al., 2018), recently published their synthesis of research into the impact of setting or streaming. Their report mirrored the literature search undertaken for this study, stating there were relatively few UK based research studies in this field (EEF, 2018).

Drawing on five meta-analyses, Hattie (2002) acknowledged that there were variations in findings from different studies, but concluded that the 'overwhelming message is that tracking [setting] has minimal effects on learning' (p.460). These conclusions were mirrored by Ireson et al. (2005) in their study of GCSE results in forty-five secondary schools in England. They found when prior attainment was taken into account, setting 'had little overall impact on GCSE attainment in English, mathematics or science' (p.454). The EEF report concluded that setting had a slight negative impact on attainment as compared to mixed attainment teaching (EEF, 2018). As these studies indicate, research evidence about the impact of setting on overall attainment is not unequivocal but the prevailing view appears to be that any impact on overall attainment is marginal. However, as discussed in more detail below, there is more evidence about the differential impact of setting on different groups of students.

A complex picture emerges when the details of research studies are examined. For example, the study by Ireson et al. (2005) involving forty-five schools, reported setting had little impact on overall attainment when factors such as prior attainment, gender, free school meals and attendance were controlled. They also reported that setting did not have a differential impact on high or low attaining students in mathematics or English, but slight differences were seen in science, with exam results from higher attaining sets depressed and lower attaining sets increased. However, they acknowledged previous studies, including their own, had conflicting findings, some of which indicated that mixed-ability classes benefited students who would otherwise be placed in lower attaining sets, with setting benefiting those in higher sets. Hanushek and Woessmann (2006) analysed international attainment data for twenty-six countries in relation to setting policies. By comparing progress across secondary schools with different systems, they concluded that setting increased the spread of attainment and educational inequality. While the evidence was less conclusive, they also found there was a tendency for early setting to lower overall attainment. However, a second-order meta-analysis by Steenbergen-Hu et al. (2016) reported that setting had no impact on overall student outcomes.

Whilst findings are not completely consistent, any differences reported were small (Steenbergen-Hu et al., 2016; EEF, 2018). The prevailing view appears to be that setting does not increase overall attainment, but has a differential effect on different attainment groups, with setting increasing the spread of attainment, depressing the attainment of students placed in lower attaining sets (Hattie, 2002; Ireson et al., 2005; Kutnick et al., 2006; Dunne et al., 2011; Yu et al., 2014; EEF, 2018). There appears to be more debate as to whether setting has a slight positive impact on students placed in higher attaining sets or has no effect (e.g. Hattie, 2002; EEF, 2018).

2.3 (Mis)allocation of Students to Sets: Implications for Students

Researchers have considered the processes used to allocate students to different sets and the apparent differential impact set placement has on student attainment. Teachers generally describe the mechanism for allocating students to sets as being based on prior attainment (Hallam and Ireson, 2003; Dunne et al., 2011; Francis et al., 2017). However, whilst there is variation between schools, there is usually a less transparent element of teacher recommendation within this process, with evidence that other criteria such as behaviour and perceptions of 'educability' are taken into account (Araujo, 2007; Taylor et al., 2017). The evidence presented in the previous section indicates setting depresses the attainment of lower attaining students. As discussed in detail below, there are further layers of potential inequity that arise from the use of teacher judgment in set allocation. First, for students with the same prior attainment, exam results are higher for students placed in a higher set compared to students placed in a lower set (Wiliam and Bartholomew, 2004). Second, students from low socioeconomic backgrounds and students with special educational needs (SEN) are overrepresented in low sets, even when prior attainment is taken into account (Dunne et al., 2011).

This is of particular concern in mathematics, as setting is more common and starts earlier than in other subjects. Moreover, any negative effects are compounded by the fact that students rarely move between sets once established (Kutnick et al., 2006; Dunne et al., 2011). However, drawing conclusions about relationships between setting and attainment is not straightforward. In order to consider the complexities, the two UK studies identified in the literature review as drawing on class-level data are discussed in detail below.

Wiliam and Bartholomew (2004) followed one cohort of students as they moved from year 8 to year 11. Starting in 1996, they tracked over 900 students in 42 classes from 6 London secondary schools. Student questionnaires and interviews, lesson observations and attainment data from Key Stage 3 tests, and GCSE examinations in year 11 were analysed. There were variations in levels of attainment between the schools, but the progress made from Key Stage 3 to GCSE was similar. One key finding common to all the schools, though to different degrees, was that students with the same Key Stage 3 score did better if they were placed in a higher set and worse if placed in a lower set. In other words, a dividend accrued in terms of higher GCSE results for those students placed in higher sets compared to other students with the same Key Stage 3 scores.

Wiliam and Bartholomew (2004) presented a nuanced argument regarding the role of set allocation on attainment. In all the schools, prior attainment in the form of Key Stage 3 test scores were used as part of the set allocation process, but all had some element of teacher recommendation; consequently, students with the same scores could be found in a range of sets. If teachers were able to accurately spot 'potential' or otherwise in students, which led to them being placed in sets other than their scores would dictate, then this would offer an explanation for differential set allocation that would not raise issues of equity. Wiliam and Bartholomew (2004) argued that if this was the case, then larger dividends would accrue when teachers overrode Key Stage 3 scores more often; however, the reverse occurred. They claimed this, combined with their previous research, provided evidence that the differences in attainment by students of similar prior attainment when placed in different sets 'are attributable to the process of setting, and the kinds of teaching that result' (p.90). The implication being that students placed in lower sets would probably have gained higher exam results if allocated a place in a higher set and were thereby disadvantaged by that decision.

Hallam, Ireson and colleagues published a number of papers based on a longitudinal study of 45 secondary schools, which followed students from a single cohort from 1998 to 2000 and encompassed year 9 through to year 11 (e.g. Ireson et al., 2002; Hallam and Ireson, 2003; Ireson et al., 2005). Student and teacher questionnaires were used and Key Stage 3 test results and GCSE grades in English, mathematics and science were obtained, with Key Stage 2 results retrieved retrospectively. They used multilevel modelling to explore the effect of setting, taking into account prior attainment, gender, social disadvantage and attendance. However, this was only undertaken at two levels, student and school; the clustering based on class composition was not included. Set lists for mathematics were provided by 27 schools and a cross tabulation of Key Stage 3 levels with mean GCSE scores for top, middle and low sets was used to explore the impact of set placement. Their results mirrored those of Wiliam and

Bartholomew (2004); students of similar prior attainment were placed in different sets and attained higher GCSE grades when placed in higher sets, with the additional gains largest in mathematics. Some of this difference could be explained by the use of Key Stage levels by Ireson et al. (2005), as the schools may have placed students on the finer grained measure afforded by raw Key Stage scores. However, this could not explain all the disparity, as some students were placed out of rank order when levels were considered.

These studies offered reminders that the analysis of data is complex and open to interpretation. For example, Wiliam and Bartholomew (2004) discussed the study by Ireson et al. (2002) and commented ‘the fragility of these effects suggests that between-class ability grouping cannot be understood as a simple phenomenon with predictable results’ (p.282). Moreover, they acknowledged the sample of six schools used in their study was too small for the findings to be statistically significant. In addition, both studies looked at set allocation in relation to progression across Key Stage 4. With many, but not all schools setting across Key Stage 3, the potential influences of earlier setting practices were not taken into account. In spite of these limitations, the argument that there is a differential effect on attainment based on set allocation, above and beyond that expected by prior attainment, is echoed by other researchers that have drawn on class-level data (Linchevski and Kutscher, 1998; Boaler et al., 2000).

Dunne et al. (2011) surveyed over a hundred secondary schools about their setting policies and cross-referenced this information with attainment data from the NPD. Echoing the findings of Wiliam and Bartholomew (2004), they found while schools stated their setting policies were based on measures of attainment, other factors were taken into account, which resulted in disparities between set allocation and prior attainment. A significant feature of this analysis was that particular groups of students were disproportionately affected by this ‘misallocation’ to sets. They demonstrated that students from low socioeconomic backgrounds and students with SEN were overrepresented in low attaining sets, even when prior attainment was taken into account. It appears that setting may contribute to the perpetuation of social selection (Reay, 2010).

Comparable evidence was found in the Wiliam and Bartholomew (2004) study, albeit from a smaller sample of schools. Students with low socioeconomic backgrounds were overrepresented in the low attaining sets in all six schools, but the levels varied. In two schools, the placement of students with low socioeconomic backgrounds in sets lower than their Key Stage 3 results justified was large enough to be statistically significant. Indeed, in one school this represented nearly half of students with low socioeconomic backgrounds.

The notion that there is ‘misallocation’ of students to sets has been reported in recent publications, and appears to have particular traction because particular groups are disproportionally affected (EEF, 2018; Roy et al., 2018). Suggested changes to practise, however, follow two different paths; those who look to reform setting practices and those who advocate mixed-attainment teaching. The EEF commissioned research into ‘best-practice setting’ involving over a hundred schools. The ‘best-practice’ principles were specified as set allocation to be based purely on attainment, to at most four sets, with frequent reassignment of students and random allocation of teachers who are provided with pedagogical training (Roy et al., 2018). The notion appears to be that a ‘pure’ version of setting could ameliorate any differential effects. However, the EEF reported that over half the schools dropped out of the study, and for those that remained there was low fidelity to the study protocols. Roy et al. (2018) concluded that schools found it difficult to alter their setting practices. There is a longer history of mathematics education researchers advocating mixed-attainment teaching (Boaler, 1997; Linchevski and Kutscher, 1998; Hodgen, 2007; Boaler, 2008). Taylor et al. (2017) reported that schools are also reluctant to move towards mixed-attainment classes. It appears, therefore, that setting in its current form is likely to remain a feature of the English education system for some time.

2.4 Relationships between Teachers, Setting and Pedagogy

Studies have considered a wide range of issues connected with setting, from curriculum access to the role of teachers' expectations and beliefs (e.g. Zohar et al., 2001; Watson and De Geest, 2005). The following section discusses the relationships between some of these key issues and pedagogy.

2.4.1 Organisational Factors

2.4.1.1 Classroom Organisation

There are a number of ways that classroom organisation in lower attaining sets tends to differ from higher attaining sets or indeed mixed-attainment classes. They are usually smaller in size and more frequently have a teaching assistant working alongside the teacher (Blatchford et al., 2011). The situation is complicated by the fact that these same differences can be seen as offering support for lower attaining students or as mechanisms that exacerbate negative differential effects. For example, findings have indicated that teaching assistants can have a positive or negative impact on achievement (Farrell et al., 2010; Radford et al., 2011).

Class size has similarly contrasting findings. In lower attaining sets, class sizes tend to be smaller, with students seated on their own more often, with less group work and peer interaction, which provides evidence of greater levels of students working in isolation (Kutnick et al., 2006; Mazenod et al., 2019). For those who perceive collaborative working as beneficial for developing mathematical reasoning, this increase in lone working would have negative implications. On the other hand, the higher allocation of staff resources could allow greater teacher-student interaction (Blatchford et al., 2011), which could allow support to be more easily tailored to individual needs (Dunne et al., 2011).

2.4.1.2 Teachers

A number of studies report that low attaining sets tend to be taught by less experienced and less qualified teachers (Hattie, 2002; Wiliam and Bartholomew, 2004; Dunne et al., 2011). Francis et al. (2019), in their study of over a hundred secondary schools, also reported it was less likely for highly qualified teachers to teach low sets, findings that replicated the smaller Boaler et al. (2000) study of six schools.

2.4.1.3 Curriculum Access

At the end of Key Stage 4, the majority of students in England sit mathematics GCSE exams, with schools following curricula specified by the respective exam specifications. Due to the Office of Qualifications and Examinations Regulation (Ofqual) requirements, the content specified by the different exam boards are very similar; all boards offer tiered exams, higher and foundation, with additional 'harder' content specified for the higher tier. The vast majority of schools set for mathematics in Key Stage 4 and the designation of sets as following either a higher or foundation track is common practice (Taylor et al., 2017). In 2018, 44% of students took the foundation tier (Ofqual, 2018) which places an upper limit on the grade that can be awarded. Whilst some middle attaining students study the higher tier curriculum but take the foundation papers, a significant minority of students only study the content found in the foundation tier (Boaler et al., 2000). For students in lower attaining sets, the restricted access to the curriculum often starts earlier than Key Stage 4, with different routes specified for different sets (Kutnick et al., 2005a). In addition, this stratification makes transitions between sets problematic, as students in lower attaining sets have often not experienced material and ideas met in higher attaining sets (Boaler et al., 2000).

2.4.2 Pedagogical Characteristics of Sets

In addition to differential access to teachers and the curriculum, there are also studies that explore links between setting and pedagogical approaches. A number of researchers have argued that low sets tend to have distinctive pedagogical characteristics that limit students' access to mathematical ideas, thereby perpetuating a cycle of low attainment (Boaler, 1997; Kutnick et al., 2005a; Wilkinson and Penney, 2014).

Watson (2001) argued that secondary mathematics teachers often restrict low attaining students to activities requiring low-level recall, rather than allowing them to engage in sense-making activities. She found teachers focused on procedures and broke tasks down into small simple steps, leading students through the mathematical reasoning they themselves structured. These findings are replicated by other researchers, with 'drill and practice' and low cognitively demanding tasks (CDTs) commonly reported features for low attaining sets (Boaler et al., 2000; Hallam and Ireson, 2005; Kutnick et al., 2006; Mazenod et al., 2019). This oversimplification of mathematics, combined with a more limited and conceptually fragmented curriculum, often results in low attaining students having limited access to mathematically significant ideas (Linchevski and Kutscher, 1998; Jorgensen et al., 2013).

2.4.2.1 The Influence of Teachers' Beliefs and Expectations

As previously discussed, students from low socioeconomic backgrounds are overrepresented in low sets, even when prior attainment is taken into account (2.3). This appears to be related to teachers' 'interference' in the setting process when they override attainment data when placing students in sets. A number of studies have made this connection through the analysis of results and have suggested teachers' perceptions of 'educability' are a factor (e.g. Wiliam and Bartholomew, 2004), but there appears to be less evidence about how this social bias occurs in individual decisions.

There is a long history of studies that have linked teacher expectations to student attainment (e.g. Brophy and Good, 1970; Slavin, 1990; de Boer et al., 2010). These are often described as self-fulfilling prophecies, as higher teacher expectations are associated with higher attainment, with the reverse for low expectations. There appears to be two intertwined aspects of teachers' thinking in relation to students with low prior attainment; their beliefs about what pedagogical approaches are most suitable and expectations in relation to attainment and engagement (Knipping et al., 2008).

Above and beyond the content of mathematics lessons, there is evidence that teachers vary their pedagogical approaches in response to attainment profiles. For example, Zohar et al. (2001) interviewed 40 Israeli teachers and reported that nearly half of these teachers thought it was inappropriate to give low attaining students tasks that required higher-order thinking. Mazenod et al. (2019), from a survey of about 600 teachers, reported that three-quarters agreed with the item: 'I expect more independent work from high-attaining students' (p.60). Kutnick et al. (2006) argued that some teachers perceive student behaviour as being worse in lower sets and respond by employing tightly controlled whole-class explanations and individual seatwork, thus limiting peer interaction. These differences have been associated with low sets being exposed to an impoverished curriculum, with a negative impact on attainment (Boaler, 1997; Mazenod et al., 2019).

Whilst the preceding discussions indicate that many studies link teacher beliefs and expectations with lowering attainment for students in lower attaining sets, there are also a few that have more positive outcomes. For example, in an action research project with ten secondary teachers, Watson and De Geest (2005) sought to develop innovative practice in the teaching of low attaining students. They argued that improvements in learning occurred but 'the methods and strategies the teachers used were not always generalisable across the project, indeed some were contradictory' (p.209). Instead, they identified a set of beliefs, common to all the teachers in the study, namely, all students could learn, and get better at learning, mathematics. Watson and De Geest (2005) argued this was a key factor in improving learning when combined with a commitment to the long-term development of students as learners of mathematics. However, this study was based on case studies of teachers with track records of success with low attaining students and were reported as a contrast to the norm.

Stereotypical pedagogical characteristics are not, however, limited to low attaining sets. Teachers seem predisposed to teach sets as if they were one homogenous group, with little attention paid to differentiation (Wiliam and Bartholomew, 2004). For example, Kutnick et al. (2006) reported a greater use of whole-class teaching when classes were organised into sets rather than mixed-attainment classes, and Boaler (1997) reported that some students in top sets found the pace too fast for them. The Wiliam and Bartholomew (2004) study offers some evidence, albeit limited, that pedagogical characteristics of classes can influence attainment. In their study, two schools had 'mixed ability' classes in the lower years, where individual and small group working was established practice. These atypical pedagogical practices were carried over into Key Stage 4 when the students were taught in sets. In comparison to the other four schools in the study, these two schools had the smallest difference in exam results between students with the same prior attainment who were placed in different sets.

2.4.2.2 Individual Teachers' Shifts in Practice

As previously indicated, there is evidence that lower attaining sets tend to be taught by less experienced and less qualified teachers (Hattie, 2002; Wiliam and Bartholomew, 2004; Dunne et al., 2011; Francis et al., 2019). This could explain some of the differences between how sets are taught, and raises the question as to how far differences permeate to individual teachers' practice. However, as discussed below, there is also evidence that individual teachers change their pedagogical practice when they move between different sets.

For example, Hallam and Ireson (2005) surveyed English, mathematics and science teachers from forty-five UK secondary schools. Those teachers reported that for low attaining sets they structured activities more tightly, with greater levels of repetition and with a focus on basic skills. Forgasz (2010) surveyed thirty-six secondary mathematics teachers in Australia; thirty-three of whom indicated that they did change their pedagogy in response to teaching groups of different ability, but few commented on what changes were made. In the few reported explanations, reference was made to using more concrete, practical and real-life applications in low sets, whilst concept development and problem solving were mentioned in relation to high attaining sets. More recently, Mazenod et al. (2019) surveyed about six hundred teachers, the majority of whom reported they modified their teaching to meet the needs of students based on prior attainment. Whilst there was variation, teachers tended to say they used more repetition and practice, more structured work, more practical activities and less independent work with low attaining students. As the survey questions asked about high or low attaining students, it was less clear as to how these self-reported modifications translated into classroom practice when they taught in sets.

These self-reported changes in practice do reflect the different pedagogical characteristics associated with low sets, which many researchers have argued contribute to low attaining sets having restricted access to significant mathematical concepts (Boaler et al., 2000; Watson, 2001; Hallam and Ireson, 2005; Kutnick et al., 2005a). This suggests that some of the distinctive pedagogical characteristics of low attaining sets are likely to be reflected in shifts in individual teachers' practice when they teach different sets. However, the nature and extent of any shifts in practice are less clear as teachers' enacted classroom practices have been shown to differ from their espoused beliefs (Ernest, 1989), and few studies have included classroom observations of individual teachers with different sets.

2.5 Implications for this Study

Whilst there are variations in reported findings, there appears to be shifts in pedagogical practices when high and low attaining sets are considered (Boaler et al., 2000; Wiliam and Bartholomew, 2004; Ireson et al., 2005). The prevailing view appears to be that it is not the setting process *per se* that causes a lack of student engagement with mathematically significant ideas in lower attaining sets, but it is how pedagogical practices are enacted (Boaler, 1997; Hallam and Ireson, 2005; Kutnick et al., 2005b; Dunne et al., 2011). Hallam and Ireson (2005) are amongst many that argue that it is the adoption of the pedagogical practices described above that drives the depression of attainment in lower attaining sets (Wiliam and Bartholomew, 2004; Watson and De Geest, 2005; Boaler, 2010).

The implication being that changing pedagogical practices could bring benefits to low attaining students. Watson and De Geest (2005), drawing on case studies of ten teachers, articulated principles that contributed to improved attainment of low attaining students, and Dunne et al. (2011) argued they had found examples of innovative pedagogical practices that avoided the creation of stereotypical 'bottom' sets. However, in both studies, the cases were chosen to represent best practice as a contrast to the norm. Whilst acknowledging it may be possible to ameliorate negative aspects of setting, Boaler et al. (2000) cautioned 'many of the disadvantages of setting… are contingent rather than necessary features of ability-grouping but we believe that they are widespread, pervasive and difficult to avoid' (p.644).

When pedagogical approaches are considered, it has been argued that conclusions cannot be drawn about students' learning of mathematics from the mere presence or absence of particular features (Hiebert et al., 2003b; Watson and De Geest, 2005). For example, in the influential Trends in International Mathematics and Science (TIMSS) video study, Hiebert et al. (2003b) demonstrated that high achieving countries had different pedagogical practices, and argued that it was the nuances of how particular features were implemented that distinguished effective teaching. By implication, any study that aims to interpret a learning environment needs an instrument that moves beyond the categorisation of pedagogical approaches used.

The tendency for different approaches to be taken when different sets are taught does appear to be embedded in practice (Solomon, 2007; Dunne et al., 2011), but Wiliam and Bartholomew (2004, p.280) argued ‘what teachers actually do in classrooms is so weakly theorized’ that practice is not fully understood. The previous discussion has outlined key differences that are known; by studying the same teachers as they teach different groups of students, this study seeks to contribute to the understanding of any shifts in pedagogy that are not explained by a change in the demographic of teachers assigned to different sets. It is hoped that this focus on pedagogical moves could illuminate differences in learning environments experienced by different groups of students.

3. The Development of the Orchestration of Mathematics Framework (OMF)

3.1 Overview

Exploring teachers' practice necessitates the interpretation of classroom behaviours. This chapter discusses theoretical perspectives drawn on in this study and articulates how a conceptual framework for interpreting classrooms was developed. Consequently, this chapter serves two purposes. First, the following includes a review of previous studies that have offered models for interpreting mathematics classrooms and those studies that have identified important pedagogical features in the teaching and learning of mathematics. Second, the chapter outlines how the process of undertaking the literature review led to the development of a conceptual framework that coordinated theoretical perspectives of classroom practice (figure 3.4: model A). Progressively, as a wider range of research was considered, this conceptual framework was developed into an overarching model of teachers' pedagogical practice, which I have called the Orchestration of Mathematics Framework (OMF) (figure 3.8: model B). This process drew together features of classroom practice, identified as important in the literature, and integrated these from the perspective of the teacher. This framework was revised during the pilot study and in the transition to the main study. The resulting OMF (figure 3.14: model D) provided the model for interpreting 'typical' lessons that was used in the data collection and analysis in this study.

Section 3.2 draws on the notion of the didactic triangle (Straesser, 2007) and classroom norms (Yackel and Cobb, 1996) to discuss models for interpreting mathematics classrooms. After defining orchestration, section 3.3 focusses on pedagogical features that prior research has identified as important in the learning of mathematics. The three key areas discussed are task features, such as multiple representations, the management of discourse, such as initiate-response-evaluate interactional patterns, and the management of the lesson trajectory. Section 3.4 discusses the development of the OMF framework, from its inception in the literature review through to its iterative development during application in the pilot and main study.

3.2 Interpreting Classrooms

In this section, the didactic triangle (Straesser, 2007) and the notion of classroom norms (Yackel and Cobb, 1996) are drawn on to consider how interactions in mathematics classrooms can be interpreted.

3.2.1 The Didactic Triangle

It is widely acknowledged that classrooms are complex, dynamic environments, and the multifaceted nature of classroom interactions means that there are no simple ways to understand pedagogical practices (Potari and Jaworski, 2002; Hiebert et al., 2005; Derry et al., 2010). In order to develop an understanding of teachers' practice, a theoretical framework for interpreting classroom activities is required. One common conceptualisation of the teaching and learning of mathematics is the 'didactic triangle' (figure 3.1), in which the teacher, students and mathematical content form the nodes of a triangle (Herbst and Chazan, 2012; Schoenfeld, 2012; Lerman, 2013). Whilst acknowledging that the complexity of the classroom cannot be reduced to a simple model, Goodchild and Sriraman (2012) argued that the triangle 'serves as a starting point to theorize the dynamics of teaching–learning' (p.581).

The connections between nodes can be considered in terms of social interactions (Straesser, 2007), but equally the connections could be considered in terms of internal psychological processes. The interpretative framework offered by Cobb et al. (2001), in which cognition is considered as integrating interactional elements and psychological processes, allows each perspective to be foregrounded as appropriate. Taking their lead, both social constructivist and constructivist perspectives will be drawn on in this study.

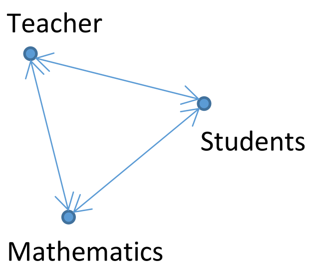


Figure 3.1: Didactic Triangle (Straesser, 2007, p.165)

This study focusses on how the teacher’s actions shape the mathematics made available in the classroom, and how this might change for different groups of students. At first sight, this could be interpreted as ignoring the fundamental role of individual student participation. However, as discussed in detail in the following section, this study draws on the concepts of classroom norms and normative identities developed by Cobb and his colleagues (Yackel and Cobb, 1996; Yackel, 2001; Cobb et al., 2009; Cobb et al., 2011). The argument made below is that these constructs allow individual student participation to be taken into account through consideration of how their actions relate to those classroom norms. Moreover, classroom norms offer a mechanism to interpret specific instances of teachers’ pedagogical moves and classroom interactions more widely. The following section offers an outline of the research in this field and its relevance for this study. Limitations will also be discussed.

3.2.2 Classroom Norms

Classroom norms are considered to be recurring patterns of behaviour that fulfil the expectations the teacher and students have for the actions of others (Cobb et al., 2009). In a yearlong teaching experiment, Yackel and Cobb (1996) sought to account for how students come to develop a mathematical disposition in a classroom environment. Building on the concept of classroom social norms, they developed the concept of sociomathematical norms. Yackel and Cobb (1996) defined sociomathematical norms as ‘normative aspects of mathematical discussions that are specific to students' mathematical activity’ (p.458); a notion that has been widely regarded as making a significant contribution to how mathematics classrooms are interpreted (e.g. Lopez and Allal, 2007; Staub, 2007; Levenson et al., 2009). In their analyses of classroom interactions, Yackel and Cobb (1996) associated a range of recurring patterns of behaviour with different norms. For example, a teacher accepting ‘correct’ answers whilst following-up on ‘errors’ was taken as an indication that the teacher was the arbiter of correctness. One indication of this norm was students changing their answers when their response prompted further questioning by the teacher.

Yackel and Cobb (1996) documented the way a teacher challenged and modified these norms over a year. Their analysis led them to classify the recognition of what constituted a different, efficient or sophisticated explanation as a sociomathematical norm, along with the recognition of what counts as a mathematical explanation or justification. In contrast, an example of a classroom social norm would be the expectation that explanations would be given, as this could apply to any subject. They argued this level of discrimination built on previous work on classroom microcultures and social norms.

The modification of norms over time reflects their reflexive nature, insofar as actions both influence, and are influenced by, classroom norms. Of significance to Yackel and Cobb (1996) was the finding that the presence of sociomathematical norms, as established in their classroom experiment, were associated with an effective inquiry-oriented learning environment. For example, a discussion as to what counted as a different mathematical explanation required comparison and evaluation, which provided opportunities for higher-level thinking. In their study, they reported the development of an inquiry-oriented classroom occurred with a corresponding modification of classroom norms over time. Whilst this enhanced the importance of sociomathematical norms for Yackel and Cobb (1996), it could be more problematic to apply this analytical approach to non-inquiry orientated settings. Cobb et al. (2001) saw no reason, in principle, as to why this analytical approach could not be applied to classrooms with a more traditional style, but they did foresee difficulties in so far as student reasoning would be less visible.

Cobb et al. (2001) continued with a design research approach, which was an iterative cycle that combined classroom teaching experiments with theory informed instructional design. In describing the theoretical framework for their study, they situated sociomathematical norms in a wider interpretative framework, in which the social and psychological perspectives are in a reflexive relationship. In this framework, the social perspective encompasses normative activities about ways of working and reasoning. Three categories ascribe the level of association with mathematics: social norms relate to activities that are not mathematics specific; sociomathematical norms relate to ways of working mathematically; mathematical practices are those related to a specific mathematical idea. The psychological perspective concerns individual cognition and there are three corresponding categories: beliefs about roles and school activities; mathematical beliefs and values; mathematical interpretation and reasoning (p.119). Cobb et al. (2001) argued that these two perspectives are so interdependent that each exists as 'the background against which mathematical activity is interpreted from the other perspective' (p.122). The social perspective foregrounds the 'taken-as-shared' activities, where 'an individual student's reasoning is framed as an act of participation in these normative activities' (Cobb et al., 2001, p.119). The psychological perspective foregrounds individual students' reasoning and their diversity of participation within these established norms.

The interpretative framework introduced by Cobb et al. (2001) has been used in a number of other studies, especially the notion of sociomathematical norms. In their study of four primary classrooms, Kazemi and Stipek (2001) found that although the classes had similar social norms, such as collaborative group work and the sharing of strategies, subtle differences in sociomathematical norms limited opportunities for students to think conceptually. For example, the social norm of presenting different strategies was present in all the classes, but not the sociomathematical norm of understanding relationships amongst the different strategies.

In an Australian study by Makar et al. (2015), a teacher's explicit aim was 'the development of argumentation-based inquiry norms and practices in a mathematics classroom' (p.1107). The study documented the students' adoption of these norms in response to the teacher's pedagogical moves across a year. This did offer evidence that the interpretative framework allowed the dynamic nature of classroom microcultures to be interpreted, where actions by the teacher and students both shaped and were shaped by classroom norms. Makar et al. (2015) argued the teacher used questioning to indicate that explanations needed to go beyond the recital of procedural steps, which then contributed to a shift in students' beliefs about what it meant to do mathematics. This in turn influenced students' future actions when explanations were given, which contributed to a shift in a sociomathematical norm, demonstrated by peers holding each other to account if procedural explanations were given.

Cobb et al. (2001) argued that it would not be feasible for a teacher to consider each individual student's anticipated reasoning when planning or delivering lessons. A key strength of this interpretative framework is that an understanding of the overall learning trajectory of a class can be developed, with 'students' participation in collective practices as constituting the conditions for the possibility of their mathematical learning' (Cobb et al., 2011, p.110). Rasmussen et al. (2015) argued that particular ideas and ways of reasoning become normative in classroom discourse, and these norms function as a shared understanding of what constitutes mathematics in that setting; however, differences in students' understanding remain.

The argument made here is that the identification of sociomathematical norms and mathematical practices foregrounds what is mathematically available for students to learn, if they choose to participate. Moreover, it should be possible to analyse the teacher's role in establishing and maintaining these norms without analysing individual student reasoning and engagement at each stage of a lesson. However, while sociomathematical norms and mathematical practices provide the taken-as-shared view of what is acknowledged as mathematics in particular settings, 'they do not inform us whether the constructed knowledge is or is not mathematical in character' (Kaldrimidou et al., 2008, p.237). Norms, therefore, appear to offer a way to foreground teachers' actions in relation to students' learning, but with the caveat that evidence about the mathematical nature of what is made available to students resides elsewhere.

3.2.3 Identities

3.2.3.1 Student Identities

Cobb et al. (2009) continued to develop their analytical framework with work on identities. They posited moving the focus from individual to normative identities could inform pedagogical design and teaching. One of their central constructs was a normative identity as a doer of mathematics, which was defined as 'the general and specifically mathematical obligations that delineate the role of an effective mathematics student in that classroom' (p.43) (figure 3.2). This construct is closely associated with classroom norms, as sociomathematical norms provide the framework within which students develop their understanding of what it means to do mathematics, and a normative identity is the taken-as-shared view of what is recognised as an 'ideal' student in that classroom.

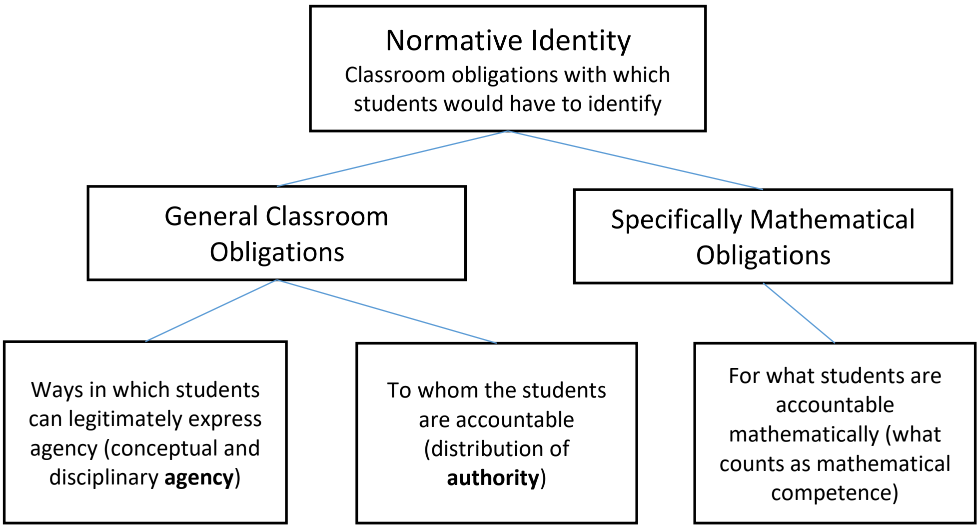


Figure 3.2 Facets of the normative identity as a doer of mathematics
(Cobb et al., 2009, p.46).

However, Cobb et al. (2009) also considered the relationship between normative identities and individuals. A second central construct was that of personal identities of individual students. This encompassed a student's view of their own competence, their perception of how peers viewed their competence and how closely they identified with the normative identity. Drawing on previous work, such as Boaler and Greeno (2000), Cobb et al. (2009) argued students could be classified as to whether they identified with, merely complied or resisted their classroom obligations in relation to engagement in mathematical activity. In other words, how closely they identified with the normative identity of an 'effective student of mathematics'. There is some evidence that students in low sets engage less with mathematics (Boaler, 1997; Horn, 2008; Noyes, 2012), which could be manifest in higher levels of resistance. Certainly, the levels of identification with normative identities have been shown to vary between classes (Boaler and Greeno, 2000). With the acknowledgement that factors from outside the classroom influence classroom behaviours (e.g. Boaler, 2005; Strand, 2014), the separation of normative and personal identities is of interest to this study, as it has potential to separate individual student engagement from the mathematics made available to them.

3.2.3.2 Teacher Identity

As with normative identities for an effective student, there is an equivalent construct for teachers, defined by Gresalfi and Cobb (2011) as 'a set of obligations that a teacher would have to fulfil to be recognised as a competent mathematics teacher in that setting' (p.275). It could therefore be possible to consider if or how normative identities are different for different classes, but as Levenson et al. (2009) cautioned, how these vary may depend on who is constructing the identity. In the Gresalfi and Cobb (2011) interpretative framework, the teacher's personal identity is conceived as developing as the teacher interacts in that environment. This offers a mechanism through which teachers' perceptions of expectations and obligations for the teaching of mathematics could be considered. Moreover, this framework could allow expectations from outside the classroom to be taken into account.

In a similar manner to student identities, the relationship between a teacher's personal and normative identity could be explored. When focussing on the teacher's pedagogical practice, it could be productive to consider the two interrelated facets of classroom norms and teacher identity. As discussed, a number of studies have demonstrated that different teachers establish different classroom norms (Lopez and Allal, 2007; Straehler-Pohl et al., 2013), whilst others have discussed how teachers modify classroom norms over time (Cobb et al., 2009; Makar et al., 2015). Taken together, normative identities and classroom norms offer a window through which to consider the relationship between the teacher and the mathematics in different settings.

3.2.3.3 Mathematical Profiles of Classes

Of key interest to this study is the mathematics that is made available to students. One thread that runs through educational research is that the mathematics experienced by learners varies in fundamental ways. For example, the relational versus instrumental divide or mathematics being seen as the production of fast and accurate solutions versus a problem solving process (e.g. Skemp, 1976; Di Martino and Zan, 2010). The notions of classroom norms and identities proposed by Cobb et al. (2011) could allow a complex image of what constitutes mathematics in particular classrooms to be built. The argument made is that, in addition to being able to describe 'an effective student of mathematics' or a 'competent teacher', it is possible and useful to describe the normative mathematical profile of particular classrooms.

Teachers and students construct significant narratives about mathematics, though the narratives may reside in intuitive behaviours rather than being well-articulated and shared explicitly (Sfard and Prusak, 2005). The affordance of the notion of a normative mathematical profile lies in its status as an object, albeit one that evolves over time. In particular, it allows the nature of mathematics made available to students to be brought into focus. For example, low attaining classes have a reputation for having students who are less engaged (Boaler and Greeno, 2000); the separation of the mathematical profile of the classroom from the students levels of participation could offer a window into teachers' pedagogical practices that may otherwise be obscured by non-mathematical activity. It may also answer the question as to whether, from a learning of mathematics perspective, it would be worth the students engaging in the tasks offered.

The question to be answered here is whether different classes for the same teacher have different normative mathematical profiles, and if so, how does this relate to classroom norms and different normative identities for both the teacher and students. In addition, how does a teacher contribute and respond to those norms, and how do they negotiate their personal identity within this complex environment.

3.2.4 Authority and Agency

Authority and student agency in classrooms have been identified by a number of researchers as critical features of mathematics lessons, as they can be determining factors in the nature of mathematics experienced by students (Boaler and Greeno, 2000; Martin, 2000; Watson and Mason, 2007; Schoenfeld, 2013a). For example, Schoenfeld (2014) identified 'agency, authority and identity' as one of his five dimensions of mathematically powerful classrooms, which also included cognitive demand, the mathematics, use of mathematics and use of assessment (p. 407) (see 3.4.3.2 for details). Cobb et al. (2009) similarly argued that authority and agency were important features of what it means to be an 'effective student of mathematics'. For example, in the cases used to illustrate their interpretative framework, they demonstrated that in one class the authority to evaluate responses resided with the teacher, whilst in the other both the teacher and students were involved. However, Cobb et al. (2009) acknowledged the context of the second class, where attendance was voluntary, could have contributed to the different dynamics.

A number of studies have drawn on Cobb and colleagues' model of classroom microcultures involving the notion of authority. Lopez and Allal (2007), in their study of two primary classes, also found the distribution of authority varied with different teachers, along with other aspects of sociomathematical norms. One teacher retained authority whilst developing an expectation that different approaches to solving problems would be considered. The other teacher distributed authority more widely but with a norm established of focusing on the most effective procedure. However, Lopez and Allal (2007) did offer a note of caution when they argued that the interpretation of sociomathematical norms was complex as they could only be interpreted in relation to each other.

3.2.5 Affordances and Limitations of Classroom Norms

The majority of studies conducted by Cobb and his colleagues (e.g. Yackel and Cobb, 1996; Cobb et al., 2001; Gravemeijer and Cobb, 2006; Cobb et al., 2009) were reform-orientated design research, involving classroom experiments undertaken in the elementary sector in the US. One consequence was an underpinning ethos of improving teaching and learning through the development of effective inquiry-orientated classrooms. It does appear that this type of analysis, which utilises classroom norms and identities to investigate classroom microcultures, offers a mechanism through which behaviours valued in inquiry-oriented settings are captured. For example, a sustained press for explanations (Stein, 2000) and students holding peers to account (Goos, 2004) are two of those behaviours.

A number of subsequent studies have illustrated the use of these constructs in the analysis of classroom interactions and the teachers' role within those dynamic environments (Kazemi and Stipek, 2001; Lopez and Allal, 2007; Makar et al., 2015). Although many of those studies were also undertaken in primary inquiry-oriented settings, they did identify a wider range of norms, which went beyond those utilised by

Yackel and Cobb (1996). Some of these norms, such as authority residing with the teacher, are more closely associated with traditional classrooms. As such, these offered some evidence of the potential of analysis from the perspective of classroom norms to be viable beyond inquiry-oriented settings.

A search of the British Educational Index and ERIC (terms: sociomathematical, microculture and norms) indicated there are few studies that utilised these constructs in more traditional settings or in secondary schools in England. Consequently, as yet, there is less evidence of the efficacy of these constructs in the analysis of typical English secondary classrooms. In spite of these limitations, the affordances offered by these constructs to interpret constantly varying situations and to identify patterns of participation have the potential to contribute significantly to this study. Specifically, classroom norms, and in particular sociomathematical norms and mathematical practices, allow specific interactions to be interpreted as to whether they represent typical or atypical occurrences. Consequently, the analysis of a relatively small number of lessons could offer a sufficiently representative picture of the mathematics made available to students more generally.

3.2.6 The Relationships between Identities and the Didactic Triangle

In this study, the affordances of classroom norms and normative identities, as co-constructed in particular classrooms (Cobb et al., 2009), for interpreting classrooms lies in how different aspects of the didactic triangle can be brought into focus. In particular, individual student participation can be taken into account through how students' actions relate to the normative identity of an 'effective student of mathematics'.

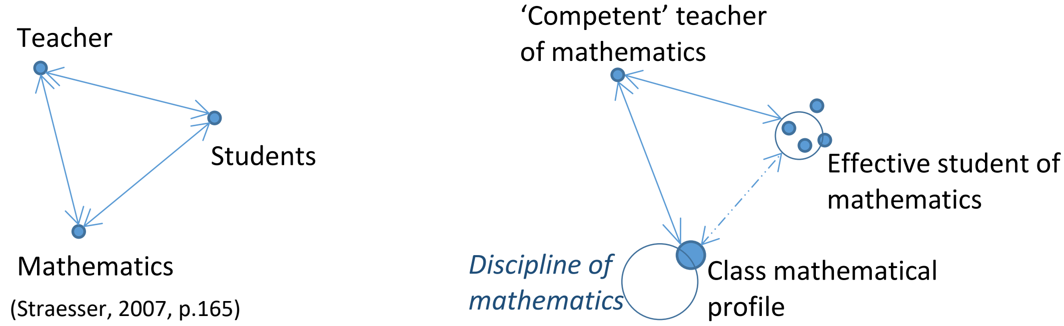


Figure 3.3: Identities in relation to the didactic triangle

Viewed from this perspective, the student node in the triangle embodies an 'effective student of mathematics', to which individual students will have varying levels of identification (figure 3.3) (3.2.3.1). Similarly, the teacher node embodies the normative identity of a 'competent' teacher, taking into account that different settings can create different sets of obligations. This model allows a teacher's actions to be foregrounded through consideration of their relationship to these norms, without interrogating the actions of all the individual students at each stage of the lesson.

As discussed in 3.2.2, the argument made is that a shared understanding of what it is to 'do mathematics' is built in individual classrooms. Within the didactic triangle, the mathematics node embodies the class mathematical profile. It should be noted, however, that activities undertaken under the label of 'mathematics' carry different meanings when applied to a mathematics classroom as compared to academics working in the field (Moreira and David, 2008). Ernest (1998) identified the differences between school and research mathematics as:

- participants: all students - small groups of adults;
- knowledge: learning existing knowledge that matters to the student - creating new knowledge for the public domain;
- tasks and assessment criteria: imposed by the teacher - self-selected and shared;
- tasks: short with short written answers - long-term problems with longer written outcomes (p.247).

Burton (2002) argued learners would benefit if more elements of research mathematics were incorporated into school mathematics, such as a greater use of inquiry-based collaborative tasks. However, the characteristics of school mathematics identified by Ernest (1998) appear to have remained common practice in English secondary schools, although potentially more entrenched with low attaining sets (Noyes, 2012). Consequently, the intersection between school mathematics and the wider discipline may vary between classes. As such, norms offer a profile of what is accepted as 'doing mathematics' in particular settings, rather than the character of that knowledge in relation to mathematics as may be described by academics (Kaldrimidou et al., 2008)

3.3 Teachers' Orchestration of Mathematics

In this section, orchestration is defined and then research that explores teachers' pedagogical practices is discussed. In particular, those studies that have sought to relate particular pedagogical approaches with student learning are considered in more detail.

3.3.1 Introduction to Orchestration

Classrooms are widely acknowledged as complex, dynamic environments that are influenced by a wide range of factors. As discussed in section 3.2, normative constructs offer a lens through which to interpret classroom interactions. Developing an understanding of how teachers teach mathematics requires consideration of a wide range of factors that are both brought to the classroom and evolve as the lesson unfolds. Numerous and diverse theoretical perspectives have been developed to consider different facets of teaching and learning. For example, Pedagogical Content Knowledge (PCK) for teaching was a theoretical framework introduced by Shulman (1986), which has been widely cited and developed since (e.g. Ball et al., 2008). Teachers' knowledge, along with their beliefs and values, has been widely acknowledged as influencing teachers' practice (e.g. Stipek et al., 2001; Hannula, 2002; Leatham, 2006). Schoenfeld (2011) drew together a number of these strands in his theoretical framework for teachers' decision-making, with the model including teachers' knowledge, goals, beliefs and values, alongside mechanisms for routine and non-routine decision-making. Whilst acknowledging this complex interplay of factors that influence teachers' practice, the aims of this study are more modest. Specifically, the objective is to identify the mathematics the teachers make available to different groups of students and the actions taken by them to bring this about. In other words, how teachers orchestrate mathematics for different groups of students.

3.3.2 Defining Orchestration

The term 'orchestration' is often used in relation to teachers' management of classroom discourse, although the metaphor is rarely explained (e.g. Henningsen and Stein, 1997; O'Connor, 2001; Goos, 2004; Leinhardt and Steele, 2005). For example, in their review of research relating to mathematics classrooms, Walshaw and Anthony

(2008) stated ‘the teacher orchestrated mathematical events by securing student attention and participation in the classroom discussion’ (p.527). In this study, the focus is the teachers’ orchestration of mathematics, where ‘orchestrate’ is taken to encapsulate the actions a teacher takes to select, organise and make available to students the mathematical tasks used in class, alongside their management of students’ mathematical contributions. The term ‘discourse’ is also used in different ways (Gee and Handford, 2013); see section 3.3.6.1 regarding use in this study.

The power of any metaphor does not lie in any literal translation or in the detail it provides, but in the image evoked. The following discussion considers possible interpretations of ‘orchestration’ and hence offers a rationale for its use. The Oxford English Dictionary (2004) defines orchestration as: ‘To combine harmoniously, like instruments in an orchestra; to arrange or direct (now often surreptitiously) to produce a desired effect; To compose or arrange for an orchestra; to score for orchestral performance’ (entry 156048). Key differences between these literal meanings and their use in relation to discourse appear to be the in-the-moment decision making and the level of contingent responses a teacher has to make.

When ‘orchestrate’ is used as a descriptor for classroom discussions, this could shift the image towards the class as an orchestra with the teacher conducting. In this context, the notion of leadership by the teacher, with authority to select, control and sequence activities could be brought to mind. Writing from the field of technology use in the classroom, Dillenbourg and Jermann (2010) explicitly discussed this metaphor. Whilst acknowledging that the teachers’ actions may be more reminiscent of a conductor, they highlighted orchestration’s intuitive appeal based on a shared purpose of the harmonisation of multiple voices. A teacher draws on their prior knowledge and planning to manage the complexities of integrating multiple contributions to a coherent whole, which resonates with an image of a composer come conductor. In this study, the phrase ‘orchestrate the mathematics’ is intended to foreground the mathematics in a manner akin to music that is produced by an orchestra, as something that is created in the shared space of the classroom and to be perceived by all, albeit in different ways.

3.3.3 Cognitive Demand

In order to discuss how teachers orchestrate mathematics, aspects of their activities that are significant in relation to the learning of mathematics need to be identified. The notion of cognitive demand has been drawn on by a number of studies, especially those looking to explicate the impact of particular pedagogical practices on student learning (e.g. Charalambous, 2008; Boston and Smith, 2009; Schoenfeld, 2014; Leatham et al., 2015). As such, cognitive demand seems to offer a mechanism through which to evaluate the significance of particular practices in relation to the learning of mathematics. Therefore, this was of relevance to this study and is discussed below.

One of the more commonly cited studies on cognitive demand is the work by Stein et al. (1996), where they introduced the Mathematical Task Framework (MTF) that modelled the relationship between task related variables and student learning. As part of this wider conceptual framework, they offered a rubric for categorising tasks as having low-level or high-level cognitive demand, where the purpose was to capture 'the level and type of thinking that a task has the potential to elicit' (Boston and Smith, 2009, p.122). 'Memorisation' and 'procedures without connections to concepts' were indicative of low-level tasks, which included following narrow algorithms without any requirement for explanations to go beyond describing the procedure (Stein et al., 1996, p.455) (appendix 5). 'Procedures with connections to concepts' and 'doing mathematics' were classified as high-level. This included tasks that required connections to be made between multiple representations and cognitive engagement with concepts, such as justifications that go beyond describing procedures. The latter category encompassed self-regulatory behaviours as well as non-routine problem solving that required students to develop an understanding of the underlying mathematical structures.

The term 'rich task' is often used to describe tasks that provide students with the opportunity to 'do mathematics' (e.g. Henningsen and Stein, 1997; Anderson, 2003; Watson and De Geest, 2005). However, the term is used in different ways. For example, Aubusson et al. (2014) include 'authenticity in their relationship to real-world application and context' (p.220). Here, 'rich task' is used as a label for those meeting the criteria for high-level cognitively demanding tasks.

A number of subsequent studies have drawn on the work by Stein and her colleagues on cognitive demand. For example, Schoenfeld (2014) identified cognitive demand as one of the five features of a powerful mathematics classroom, and Imm and Stylianou (2012) explored the relationships between cognitively demanding tasks and discourse. However, a note of caution was made by Charalambous et al. (2010) when they stated that they were not able to distinguish between 'procedures with connections' and 'procedures without connections' with acceptable reliability. Furthermore, the development of the MTF was firmly rooted in the US reform agenda, underpinned by a constructivist stance, and consequently may not be as relevant in other contexts. In spite of these limitations, it appears that the construct of cognitive demand describes an important aspect of mathematical classrooms.

The MTF modelled how tasks evolved from the way they appeared in curriculum materials through to how they were enacted in the classroom. One of the key affordances of the framework was that it allowed Stein et al. (1996) to track the maintenance or decline of cognitive demand as lessons unfolded. Of importance here is that they identified factors associated with this maintenance or decline, all of which fell within the remit of the teacher. For example, maintenance was associated with the presence of: scaffolding for thinking; modelling of high-level performance; support for self-assessment; tasks that build on prior knowledge; sufficient time; the teacher pressing for justifications, explanations and meaning; and frequent drawing of conceptual connections by the teacher (Smith and Stein, 1998, p.15) (appendix 5). Similarly, decline was also associated with teachers' actions, with identified factors: a shift to a focus on the correctness of answers; routinisation of tasks; inappropriate amounts of time; unsuitable tasks; and students not being held accountable for high level reasoning.

The relationship between the presence or absence of cognitively demanding tasks (CDTs) and the development of students' mathematical thinking is not straightforward. Whilst many may agree that 'high cognitive demand tasks create particularly fertile ground for student thinking to occur' (Leatham et al., 2015, p.90), studies have shown decline to low level activity is common, especially in classrooms without an inquiry-oriented tradition (Henningsen and Stein, 1997; Boston and Smith, 2009; Tekkumru

Kisa and Stein, 2015). Conversely, there is also evidence that high level thinking can occur in the absence of rich tasks. For example, Leatham et al. (2015) identified elements of lessons where opportunities for higher level thinking arose as teachers responded to student contributions, and which were not by necessity associated with CDTs. In their influential analysis of lessons from the TIMSS video studies, Hiebert et al. (2003b) concluded that the level of cognitive demand, as experienced by students, resided in the nuances of how classroom features were coordinated and classroom approaches were implemented, rather than in particular features of teaching *per se*.

What appears undisputed, however, is that the level of cognitive demand is a key characteristic of mathematics classrooms. Although it is important to remember that the task difficulty as experienced by the student is related to their prior knowledge and their familiarity with problems of a similar nature (Burkhardt and Swan, 2013). So, whilst there is a complex relationship between tasks used and enacted classroom activity, the teacher's classroom practice is the key component in determining the level of cognitive demand available to students.

3.3.4 The Initial Development of the Teacher's Orchestration of Mathematics Framework

There is an extensive range of published research that explores teachers' pedagogical activities and the implications for learning. As my review of literature continued, I found myself searching for ways to structure my understanding of such a wide range of perspectives. The previous discussions highlight that teachers have a pivotal role in establishing the level of cognitive demand available to students, and hence the type of mathematical thinking that might occur. There appears to be two interconnected elements of teachers' practice that influence cognitive demand, namely their selection of tasks and their management of classroom interactions. Moreover, this demarcation aligns with wider research, with task design and teachers' management of classroom discourse forming substantial bodies of work (e.g. Ainley et al., 2006; Liljedahl et al., 2007; Stein and Smith, 2011). As such, I found this categorisation useful in my synthesis of literature. This prompted me to look for a more systematic way to integrate other substantive issues met in the literature, which led to the development of a conceptual framework as outline below.

In order to develop a conceptual framework for understanding how teachers orchestrate mathematics, both the mathematical tasks and the teachers' activities in relation to mathematics would need to be categorised in meaningful ways. Moreover, how those actions would be recognised in classrooms would need to be articulated. The didactic triangle (3.2.1) had provided a model for viewing classroom activities from the perspective of the teacher, whilst still taking account of the students and the mathematics. It appeared that consideration of classroom norms (3.2.2), cognitive demand (3.3.3), classroom discourse and task features all had the potential to offer key insights into the mathematics made available to students through teachers' pedagogical activities. This led to the development of the initial version of the Teacher's Orchestration of Mathematics framework (figure 3.4: model A). This conceptual framework provided a mechanism through which to view the coordination of theoretical perspectives from the point of view of the teacher's pedagogical moves.

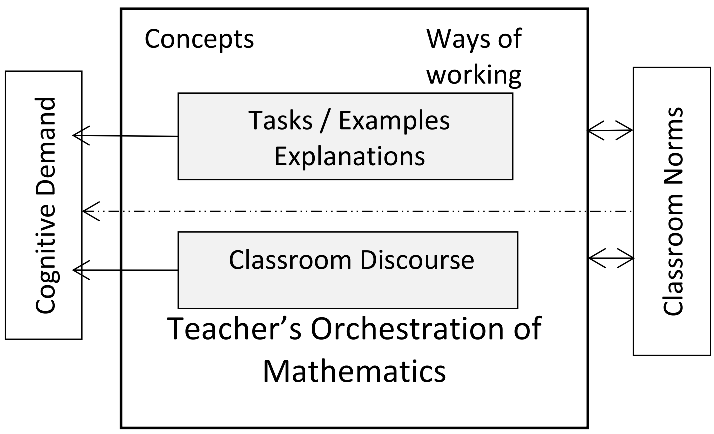


Figure 3.4: Model A - The Teacher's Orchestration of Mathematics (TOM) Framework

The shaded blocks represent the teacher's task selection and management of classroom discourse, which classify pedagogical moves that have a critical effect on the mathematics made available to students. The two vertical blocks represent cognitive demand and classroom norms, features that run throughout each lesson. Cognitive demand offers a way of evaluating the type of thinking made available to students. This could be thought of as the potential of the teacher's pedagogical moves to influence individual student's cognition, as set up in the tasks and maintained throughout the lesson. Classroom norms feature here as a mechanism through which classroom interactions can be interpreted, and activities acknowledged as mathematically legitimate can be recognised. Classroom norms both frame and are framed by classroom interactions; the arrows represent the significant and reflexive relationship between the teacher's pedagogical moves and classroom norms, whilst the separation represents the role others have in shaping these norms.

As the literature review continued, refinements were made to the framework. The following sections discuss the key issues that informed this ongoing development. First examples and tasks will be discussed in more detail. This will be followed by a discussion of the management of discourse in mathematics classrooms. Finally, the management of the lesson trajectory will be considered. Section 3.4 will then discuss how these theoretical perspectives were synthesised into a broader conceptual framework, the Orchestration of Mathematics Framework (OMF).

3.3.5 Tasks: Activities, Explanations and Examples

Whilst mathematics education research often focuses on the use of rich tasks, which encourage higher levels of cognitive demand and the development of inquiry-orientated approaches, there is evidence that many English classrooms follow a more traditional structure (Boaler, 2000; Watson and Evans, 2012). A common structure sees lessons starting with teacher exposition, used to introduce topics where examples play a key role, which is then followed by students completing practice exercises (Bills et al., 2006). However, establishing absolute demarcations between classroom activities would be difficult. For example, teacher exposition may be interwoven with whole-class question-and-answer sequences. In order to take account of different classroom practices, here tasks are considered as anything given to students to do that is related to mathematics, encompassing anything from rich tasks through to completion of routine exercises.

The relationship between task features and the development of students' mathematical reasoning has been subject to substantial study. Tasks that lend themselves to use of multiple representations, tasks that have multiple solutions strategies, and tasks that have a requirement for explanations and justification have all been identified as important (Stein et al., 1996; Duval, 2006; Guberman and Leikin, 2013; Schoenfeld, 2013a). Furthermore, the use of real-life examples and the role of context have been debated (Cooper and Harries, 2002; Stylianides and Stylianides, 2008). Identifying the presence or absence of these features may be relatively straightforward, but the implications for students' learning are necessarily complex. As previously argued (3.3.3), it is not the presence or absence of particular features *per se* that influences the mathematics made available to learners, but rather it is in the nuances of how those features are coordinated and enacted (Hiebert et al., 2003a).

At first sight, it might appear that there is a relatively direct route from tasks requiring explanations to high cognitive demand, but differing expectations for a process versus conceptually orientated contribution, or variation in accountability norms, could fundamentally change the nature of the task. To illustrate, responses to a teacher's request to explain why $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$ could range from "you add the numerators but not the denominators", through to a contribution that drew on diagrammatical representations and number lines to illustrate the structure of the problem. The former outlines a procedure without reference to concepts, whereas the latter could offer insights into the multifaceted nature of fractions. Moreover, a contribution from a single student has wider relevance if the remaining students take responsibility for understanding explanations made by others, and take action if their perception is different (Cobb et al., 2009). However, without this wider responsibility, any potential higher-level thinking could remain in the purview of one student.

Of interest to this study is whether it is possible to identify mathematically salient features of tasks chosen by teachers for use in their 'typical' lessons. The purpose here would be to explore whether teachers offer tasks with different features to different groups of students. The following discussion seeks to articulate in more detail how mathematically salient features could be identified.

3.3.5.1 Salient Mathematical Features

The abstract nature of mathematical concepts means that they are only accessible through the use of representations (Duval, 2006), where different representations draw different facets of a concept into focus. Research indicates that multiple representations of a concept have a paradoxical role; simultaneously being both a key mechanism for developing understanding, whilst also constituting an area of profound difficulty for many learners (Ainsworth, 2006; Duval, 2006; Dreher and Kuntze, 2015). Duval (2006) discussed two types of transformations of representations, those within and between mathematical registers. Transformations within one register, such as manipulation of equivalent algebraic expressions, are more common. Whereas conversion between registers, such as between algebraic and graphical representation of a function, are more important for learning but harder to achieve. Duval (2006) argued that learners could only develop a limited and compartmentalised view of mathematics if they were not able to switch between different representations in different mathematical registers. However, this flexibility requires the recognition and interpretation of the links between the manifestations of a concept in different representations (Dreher and Kuntze, 2015). A challenge as this requires a more complete understanding of the concept, which is often derived from being able to access the concept from this very same range of different representations (Ainsworth, 2006).

Rau and colleagues published a number of papers that explored links between student learning and the use of multiple representations with fractions (e.g. Rau et al., 2013; Rau et al., 2017; Rau and Matthews, 2017). Drawing on their work with over five hundred students aged between 9 and 12 years, Rau and Matthews (2017) postulated learning principles that enhance student understanding and fluency with visual representations. In relation to understanding, they argued students should self-explain connections between visual representations and concepts, and between visual features of different representations. For example, ‘number lines can more easily depict fractions larger than 1, compared to circles and rectangles’ (p.538). However, they cautioned students needed support to identify mathematically salient features, as otherwise the tendency would be to describe superficial elements of the particular example. In relation to fluency, they argued that the order in which students met different examples of the same representation, different representations and the switching between representations was critical. The strategies described align with aspects of variation theory and levels of explicitness that are discussed in more detail below.

The Rau and Matthews (2017) research is of relevance as it highlights that the presence of multiple representations is not sufficient *per se* to provide effective learning opportunities for students. Instead, the activities that students are prompted to undertake need to include attention being drawn to the underlying mathematical structure or concepts. To illustrate, an activity where a student relates $^4/_5$ to a circle split into five equal parts with four shaded by saying "there are five equal parts and you shade four" would not meet the learning principles outlined by Rau and Matthews (2017). However, activities that would meet their criteria for linking visual representations to concepts include students making reference to the shape being designated as the unit whole where its size and shape are arbitrary, or students explaining the denominator indicates how many equal parts but those could be any shape with equal area. Analysis of tasks should allow the intended role of multiple representations to be traced through the activity, to determine the potential to make links accessible to learners, although actual use would only become apparent when enacted in the classroom.

3.3.5.2 Examples and Variation Theory: Links to Generalisation

Bills et al. (2006) argued that examples are a fundamental feature in the teaching of mathematics, regardless of the pedagogical orientation of the classroom. Concepts are often introduced to students inductively, through a sequence of examples. In this context learning is a process of seeing the general through the particular, leading to understanding that in turn could be applied to previously unseen examples (Mason and Pimm, 1984; Park and Leung, 2006). Examples can have a range of purposes: typical examples used to introduce topics; paradigmatic, intended to illustrate archetypal features; atypical, peculiar, boundary or non-examples to broaden an understanding of a concept (Watson and Mason, 2002).

Mason (2015a) argued 'without generality there is no mathematics'; one fundamental issue is whether learners make the step from particular examples to a generalisation, and if they do whether they perceive the same generalities as the teacher (Bills et al., 2006). To illustrate, Watson and Mason (2002) offered the example of multiplying by 0.3. They argued a teacher may see this as exemplifying multipliers in the 0 to 1 range make products smaller, but they questioned whether students would see beyond the specific to perceive a more general outcome (smaller product) would occur for a range of values (0 to 1) (p.2). Furthermore, if the 0.3 did prompt a student to bring to mind the 0 to 1 range, would their numbers be restricted in some way that the teacher did not foresee, say to terminating decimals, and to the exclusion of others, such as fractions or irrationals.

In recent years, a number of researchers have explored how variation theory could provide a framework for how learners might develop an understanding of mathematical concepts (e.g. Lam, 2013). Variation theory is based on the premise that learning is discernment, which requires variation set against a background of invariance (Lo, 2012); Runesson (2005) argued it was powerful enough 'to reveal constraints on what it is possible to learn in mathematics classrooms' (p.69). Inspired by Japanese and Chinese approaches to lesson and teaching study, Marton and Pang (2006) developed a framework for identifying conditions for learning. They explicated how discernment of critical features, from a structured sequence of particular instances of a concept, could lead to generalisations and hence meaning.

With examples forming many of the 'particular instances of a concept', a simple illustration of the principles would be to consider how an understanding of 'triangle' could be constructed. Contrasting triangles from 'not triangles', by highlighting triangles from a range of polygons, could develop awareness of the critical 'three-ness' aspect (Marton and Pang, 2006). However, this is not sufficient to understand triangles. Mason (2011a) stated learning 'is to become aware of aspects that can vary in examples while remaining as examples' (p.108). In this case, keeping the triangle aspect invariant while systematically varying size, angles, orientation and position could allow the defining characteristics of triangles to be foregrounded, with non-essential features 'put away' (Gu et al., 2004), but this can only occur once the critical aspects of a triangle are recognised (Lo and Marton, 2011). From this perspective, an understanding of the general notion of 'triangle' is the recognition of all possible variation.

However, in practice it is far more complex, as there are a large number of interrelated features and what is attended to can vary. For example, in comparing triangles to other polygons, a student could focus on how polygons vary in the number of sides, rather than ‘three-ness’ that makes this critical feature of triangles more visible. Moreover, Lam (2014) argued it was also possible for learners to discern the invariant aspect rather than the variant. Prior understanding could also change the nature of the discernment (Lo, 2012). For instance, in the above example, polygons would already have had to be separated from shapes with non-straight sides in order to attend to the number of sides.

Drawing on work by Marton and colleagues, Watson and Mason (2005) offered their notions of ‘dimensions of possible variation’ and ‘range of permissible change’ as a way of exposing mathematical structure, and hence the mathematics made available for students to discern. The dimensions of possible variation are the aspects that can be varied. For example, if the object of learning was understanding straight-line graphs, the dimensions of possible variation would include gradient and position (e.g. the y intercept). Discernment is possible if one aspect is varied against a backdrop of relative invariance of other aspects (Watson and Mason, 2006). The argument made is that discerning gradient would be more likely if position was relatively invariant, say by having a common point. Whereas, the range of permissible change is how that variation could happen; for straight-line graphs, gradient could vary between $-\infty$ and ∞ and position could be anywhere on the Cartesian plane.

However, the situation with straight-line graphs is far more complex than two dimensions of variation. Different representations, and their relationships, and notions of infinity are just two of the multiple facets of a more complete understanding of linear functions. A known issue is that students can fail to recognise representations in atypical formats (Watson and Mason, 2006). In this example, vertical and horizontal lines are known to cause issues for some learners. Exposure to a wide range of permissible change within each dimension of possible variation could deepen students understanding of a concept, and the recognition of all possible variation would be generalisation (Watson and Mason, 2002).

Watson and Mason (2006) contended that carefully controlled variation could foreground mathematical properties, structures and relationships, which in turn is ‘the raw material for mathematical conceptualization’ (p.95). They argued that any exercise given to students could be considered as a set of examples, thereby could be treated as a single mathematical object analysable from a perspective of variation. To illustrate their argument, Watson and Mason (2006) offered a number of contrasting exercises, including:

A. $Reduce\ to\ simplest\ terms$:

$(a)\,\frac{4}{12}\quad (b)\,\frac{36}{12}\quad (c)\,\frac{240}{300}\quad (d)\ 5{:}\,5\quad (e)\ ab{:}\,ab\quad (g)\ 2\frac{1}{4}{:}\,1\quad (h)\ \frac{6ab}{3b}$

(Lerman, 2001, cited in Watson and Mason, 2006, p.105)

B. $Simplify\ these$: $6/10 \quad 18/20 \quad 6/8 \quad 14/16$

$Now\ simplify\ these$: $15/25 \quad 45/50 \quad 15/20 \quad 35/40$

$Compare\ the\ answers$

(Watson and Mason, 2006, p.107)

Example A was considered unsystematic variation, as there were no discernible connections between questions, and they could be completed without connections being made to the concept of ratio. Whereas, example B was considered to have some controlled variation as ‘correctly-performed techniques were only the starting point for mathematization’, as connections could be made between questions (Watson and Mason, 2006, p.107). It appears that textbooks used in England often offer exercises with similar questions, but with arbitrary rather than systematic changes (Haggarty and Pepin, 2002; Watson and Mason, 2006). Therefore, a question remains as to how much systematic variation occurs in English mathematics classrooms and the impact this has on students’ access to mathematical concepts and generalisation through that variation.

Moreover, there is some evidence to suggest that teachers in England do not readily consider the implications of variation theory when selecting their own examples (Watson and Mason, 2006; Zodik and Zaslavsky, 2008). Even if we are cognisant of issues relating to variation, understanding where our own attention lies is complex; interpreting this in others is even more difficult (Mason, 2015b). Consequently, analysing examples from a variation perspective could indicate what mathematics is potentially made visible, but interpreting how this may be seen by students would be non trivial. In spite of these difficulties, the apparent prevalence of ‘drill and practice’ exercises in low attaining classrooms means the analysis of tasks from a variation perspective could be particularly relevant.

Students' mathematical errors can also be considered through the lens of variation theory. Overgeneralisation has been identified as one route for students to make rational decisions based on the information available to them, but which do not align with accepted mathematical reasoning (Swan, 2001). For example, a common recited misconception seen in primary students is that multiplication makes things bigger, which has been traced back to students only being exposed to positive integer examples. In other words, exposure to a limited range of permissible change.

Variation theory can also suggest a mechanism by which the broader range of previously identified beneficial features could support learning. Specifically, this perspective can provide a rationale for the affordances of multiple representations, multiple solution strategies and rich tasks. The use of multiple representations has been associated with developing an understanding of mathematics (Duval, 2006). Rau and Matthews (2017) outlined their preferred order of student exposure to visual representations, arguing that they should first experience examples of the same representation within one type of activity, followed by the same representation in different activities, before switching between representations. In these situations, the concept is being held invariant whilst the context and then representation is varied; the term 'conceptual variation' has been used to describe this type of use of multiple representations (Gu et al., 2004).

Many mathematical problems have a range of possible solutions strategies. If students experience multiple solution strategies, this would keep the problem invariant. Comparison of different strategies could shift the focus from a reproduction of a single procedure and allow the underlying structure of the problem to become more visible. Lai and Murray (2012) categorised multiple solution strategies for the same problem as one type of procedural variation. They included two further types in this category: the first being the same method applied to similar problems, which is common in English textbooks (Haggarty and Pepin, 2002); and the second where aspects of a problem are varied, such as changing conditions, which aligns with the notion of controlled variation discussed previously (example B) (Watson and Mason, 2006). If teachers employ rich tasks, then these would include some or all of multiple representations, multiple strategies, the press for explanation and justification, and the search for patterns. As such, engagement with rich tasks could be construed as the student's own exploration of variant/invariant relationships.

3.3.5.3 Making 'Visible'

A common thread that runs through this work is the notion of identifying which mathematical concepts are made more 'visible' to students and how this relates to teachers' pedagogical moves. Adler and Ronda (2015) posited that variation theory is one way to analyse 'what is mathematically available to learn' (p.1). The argument made is that knowing which items are juxtaposed, and with what type of variation, allows the identification of which mathematical features are discernible (Bills et al., 2006; Marton and Pang, 2006; Adler and Ronda, 2015). Mason (2011a), however, raised some interrelated issues; variation may be present but not experienced by learners, and while some level of explicitness may be helpful, understanding the role of implicit and explicit attention is complex:

> The conjecture is proposed that tension between explicitness and implicitness is present in all attempts both to implement theories in practice and to justify or analyse pedagogical choices using theories, of whatever kind. (p.107)

For example, if students were exposed to different types of triangles, questions might be how much of the 'three-ness' of triangles is understood, and what overt action by the teacher might draw attention to the critical features. Mason (2011a) argued 'it is part of the art of teaching to make choices about appropriate degrees of explicitness' (p.111), including when and how to shift students' attention from 'doing' tasks to the interrogation of those tasks. As will be discussed in section 3.3.7, this highlights the impact teachers' 'in-the-moment' actions taken in the classroom have on the learning opportunities for students.

Of interest here is the argument from Lo (2012) that students identified as high attaining are those more able to discern critical features independently, whereas students who do not identify these key features make less progress and are thereby identified as lower attaining. In a broader context of classroom interactions, Knipping et al. (2015) highlighted how some teachers' intentions are communicated more implicitly than others, and students vary in their understanding of these ‘unwritten rules’.

The notion that raising levels of explicitness supports lower attaining students has some resonance with the formative assessment literature. Black and Wiliam (1998), drawing on over 250 published works, stated many studies had concluded ‘improved formative assessment helps low achievers more than other students’ (p.3). Black and Wiliam (2009) later argued that formative assessment, which entails providing students with feedback about where they are, where they are going and how they are going to get there, enhances learning by making more explicit what needs to be done. Likewise, in articulating their principles for supporting low-attaining students, Watson and De Geest (2005) listed ‘be explicit about connections and differences in mathematics’ (p.228). It appears, therefore, that there are varying levels explicitness in classrooms in relation to teachers’ intentions, task requirements and in how connections are made to mathematical principles. Moreover, there is the possibility that higher attaining students are more proficient at interpreting the implicit; with the corollary being that, whilst it would not be a straightforward process, increasing levels of explicitness may benefit learners, especially low attainers.

3.3.5.4 Use of Context

The use of context in mathematics lessons has been the subject of debate (Cooper, 2001; Ainley et al., 2006). In particular, the potential tension between the use of context to support the understanding of mathematics and the role of mathematics in modelling and explicating real situations has been highlighted (Van Den Heuvel-Panhuizen, 2003); the analysis of the use of contextualisation in the classroom is complicated by this duality. Moreover, what is considered as contextualisation varies. For example, proponents of Realistic Mathematics Education (RME), where realistic has a meaning of ‘what can be imagined’, argue that context relates to ‘working from contexts that make sense to them’ (Dickinson and Hough, 2012, p.1), whilst for others, the term is used when drawing on real-life experiences (Lowrie, 2011).

One of the key debates for this study relates to the reported use of pseudo-contexts. By which is meant problems that use objects that do exist in the real world, such as cars or apples, but are used in contrived and unrealistic ways. One example, taken from an English national assessment, and discussed by Cooper and Harries (2002), relates to asking students to work out how many lift journeys, with a given capacity, would be needed for a set number of people. The expected solution was the number of people divided by the lift capacity, rounded up to the nearest integer. Students were not expected to draw on other contextual information, such as the possibility of people using the stairs or not using the lift to capacity. Of relevance here is the finding that students with low socioeconomic backgrounds more frequently drew on 'inappropriate' contextual information, as they failed to understand the implicit 'rules of the game' about how much context to use (Lubienski, 2000; Cooper and Harries, 2002).

Student motivation is often reported as a rationale for using real-life context rather than for direct pedagogical reasons *per se*, but some detrimental effects have been reported (Stylianides and Stylianides, 2008; Lowrie, 2011). For instance, Stylianides and Stylianides (2008) reported a shift towards trial and check processes when the mathematical tasks were contextualised. Also, Lowrie (2011) argued that students use of genuine artefacts resulted in over-personalisation of the problems, which subsequently inhibited the establishment of a shared understanding of the situation. There is some evidence from international studies that the use of context varies between classes. For example, Gainsburg (2008) reported that teachers from four schools in her US study tended to use more connections to real-life contexts with higher attaining and well-behaved classes. In a small study of three teachers in Greece, Straehler-Pohl et al. (2013) presented a detailed description of how teachers shifted attention from the contextualised problem to an abstracted mathematical process. They contrasted the high and low attaining sets; for the former this was gradual and exposed the relationships between the context and the mathematics, and with the latter, this was abrupt and 'the teacher created an artificial relation to the students' supposed lives' (p.195). Due to the differing cultural contexts, the relevance of these last two studies to an English setting could be questioned. However, they do suggest that the use of context could be of interest when shifts in teachers' pedagogy between sets are under consideration.

3.3.6 Management of Discourse

3.3.6.1 Classroom Interactions: Discourse

The previous sections discussed task features and how these could be related to the mathematics made available to students. If task selection provides the initial content for any lesson, then it is classroom interactions that govern how these tasks are enacted and hence shape how the mathematics unfolds as the lesson progresses. In this study, the term 'discourse' is used in the sense of 'discourse as social interaction' (Ryve, 2011, p.171), as the focus is on how teachers coordinate local face-to-face interactions and influence classroom norms. Gee (2004) included broader issues in his definition of 'Discourse', such as 'ways of thinking, believing, valuing, and using various symbols, tools, and objects to enact a particular sort of socially recognizable identity' (p.29). Some of these issues are of interest to this study but are signalled using terminology other than 'discourse' to avoid confusion.

Classroom talk is integral to many classroom activities and plays a fundamental role in most classroom interactions (Lefstein and Snell, 2011). The amount and nature of teacher talk has been shown to vary between teachers, with consequential differences in student participation. For example, Truxaw and DeFranco (2008) analysed classroom interactions, categorising teachers talk as either univocal, where the purpose was transmission of meaning, or dialogic 'that uses dialogue as a process for thinking' (p.489); with the latter being more closely associated with developing students' conceptual understanding (Imm and Stylianou, 2012).

Lefstein and Snell (2011) stated that teachers typically spoke more than students, and they controlled classroom talk, both in terms of content and who could talk when. Teachers' talk tends to include a mixture of exposition and teacher-student exchanges. In this study, teachers' explanations in the form of monologues are considered to form part of mathematical tasks (3.3.5). The next section focusses on whole-class patterns of interaction and consideration will be given to the possible impact on classroom norms, especially authority and agency. Section 3.3.7 focusses on the function of classroom discourse in terms of how this shapes the mathematical trajectory of the lesson.

3.3.6.2 Patterns of Classroom Talk

The most commonly found interactional pattern in whole-class discussions is an Initiate-Response-Evaluate (IRE) cycle, where the teacher asks a relatively closed question, for which they know the answer, a student responds and the teacher evaluates (Drageset, 2014; Lefstein et al., 2015). This IRE pattern is more closely associated with traditional approaches, where the teacher not only controls the questions asked but also acts as the arbiter of right and wrong (Imm and Stylianou, 2012). This use can play a significant role in establishing norms relating to authority in the classroom (Cobb et al., 2011). However, questions form an integral part of mathematics lessons, both in written and verbal forms, where they can perform a variety of roles, from assessment of understanding to a tool for highlighting what is mathematically important (Mason, 2000). It is, therefore, important to understand what questions are asked, by whom, and how these are responded to, as this forms an important part of understanding mathematics classrooms.

Classroom norms can provide insights into how interactions build up notions of accountability and agency. For example, one marker for accountability is a class's response to being asked to evaluate a particular contribution. In a teacher-authority context, the classroom norm is often for 'correct' answers to be immediately acknowledged as such by the teacher, so any further scrutiny of the response cues the students to treat the original answer as incorrect, regardless of its mathematical validity. Whereas, in a class-authority context, the original response is considered on its merits (Brodie, 2014).

Even within the apparently restrictive pattern of IRE interactions, there can be significant differences relating to who or what is asked. For example, even when the teacher decides what counts as a mathematically acceptable solution, the influences on the sociomathematical norms are likely to vary between responses that state an answer, outline a procedure or engage with concepts. Many researchers advocate the use of a broader range of questioning strategies to encourage discussion, where the type of questions asked, by whom and how solutions are evaluated are all opened up (e.g. Mason, 2000; Krussel et al., 2004; Cobb et al., 2011). For example, a strategy that extends the IRE sequence so that multiple students are expected to respond, where they offer alternatives and evaluate or rephrase previous contributions, have been advocated as feasible ways to build on a traditional approach (Watson et al., 1998).

In addition to issues of authority, the IRE pattern of interaction has drawn criticism, due to an association with low-level cognitive demand that can occur when recall questions are asked or if ‘funnelling’ occurs. The latter is where the teacher leads the students to the correct answer by progressively providing more and more information (Lefstein and Snell, 2011), thereby removing the requirement for the students to undertake any significant mathematical work. Though it should be noted that the line between funnelling and focussing students’ attention on what is mathematically significant is not always easy to determine (Wood, 1994). Moreover, Lefstein et al. (2015) argued the significance of individual questions cannot be considered in isolation, but rather its contribution to the sequence of events must be taken into account. Questioning that press for explanations and justifications have been associated with higher cognitive demand, although it is necessary to distinguish between contributions that focus on replaying procedures from those that discuss concepts (Cobb et al., 2001; Rau and Matthews, 2017).

Of particular relevance here is the evidence suggesting that low attaining students are more likely to be asked low-level questions (Watson, 2001; Zohar et al., 2001). Moreover, students with low socioeconomic backgrounds, who are overrepresented in low sets, tend to be less successful in accessing the mathematics through an IRE interactional pattern, as their home experience is more likely to have a declarative structure (Jorgensen et al., 2013).

The nature of teachers’ reactions to student contributions varies, even in IRE dominated classrooms. After a student contribution, the next turn is usually taken by the teacher. Whilst simple acknowledgements do occur, teachers also ‘revoice’ student contributions in different ways (Cazden and Beck, 2003). For example, teachers might repeat, rephrase or extend the contribution. These types of teacher actions can have a complex relationship to classroom norms, as they can indicate the student’s contribution is valued whilst simultaneously taking control. Moreover, studies using discourse analysis have demonstrated that understanding classroom interactions is complex, as pauses, intonations and gestures also have nuanced meanings (Forman and Larreamendy-Joerns, 1998).

One particular area where complex interactional patterns can occur is the treatment of ‘errors’. A common teacher response to a student ‘error’ is to highlight the problem in an indirect manner, say by pausing or repeating the problematic element of the response (Ingram et al., 2015). These actions allow further consideration of the source of the error. However, from a conversational analysis perspective, avoiding direct negative evaluation conveys the message that what has been said, namely the ‘error’, should be avoided (Sacks et al., 1974). As such, there is a potential tension between the learning potential of exploring mathematical errors and the message initiating this exploration could convey.

Errors in mathematics can be viewed from different perspectives. The mathematical validity of some statements could be determined, such as $2 + 1 = 3$ being considered valid and $2 + 1 = 2$ invalid, whereas others are more ambiguous and open to interpretation. In classrooms, teachers may treat responses as ‘correct’ or as an ‘error’, but these treatments may not perfectly align with the mathematical validity of the statements as viewed from the position of an expert mathematician. When a distinction needs to be made, quotation marks will be used when referring to the teachers’ treatment of student contributions as ‘correct’ or an ‘error’, and the phrase mathematical error will be used when referring to validity with respect to accepted mathematical reasoning (Swan, 2001). In the lesson analysis in this study, the terms satisfactory and unsatisfactory will be used when referring to the teachers’ treatment of student responses as either ‘correct’ or an ‘error’, respectively.

Even when student responses do not contain mathematically invalid statements *per se*, teachers often filter and rephrase student contributions (Imm and Stylianou, 2012). This often involves the teachers ‘tidying up’ responses to make them conform to mathematical conventions, which could reinforce the teacher’s authority to decide on the legitimacy of solutions and appropriateness of terminology (Forman and Larreamendy-Joerns, 1998). Teachers ‘rebroadcasting’ of a student’s response through repetition could indicate the response is valued and accepted (Cazden and Beck, 2003). However, it could also reinforce the notion that it is the teachers’ responsibility to ensure that all the students heard and understood. Similarly, teachers may repeat with emphasis. This allows the teacher to use a student's response to draw attention to a mathematically significant feature, simultaneously valuing the response whilst shifting ownership of the contribution from the student to the teacher.

Whilst it remains the case that in many classrooms the teacher evaluates contributions (Krussel et al., 2004; Walshaw and Anthony, 2008; Lefstein and Snell, 2011; Imm and Stylianou, 2012), not all teachers conform to these dominant discourse patterns. In some classrooms students hold each other to account, although the evidence is often based in US reform contexts where the teachers are involved in professional development focussed on developing student participation (Larsson and Ryve, 2012; Wachira et al., 2013).

The use of praise as part of a teachers' evaluative move is also a complex issue. There are some who advocate the use of contingent praise, although this appears to be related to social behaviours (Simonsen et al., 2013). Dweck (2007), making links to her mindset theory, argued that praise had positive effects if directed at effort but negative if directed at notions of ability. However, difficulties could emerge if praise was given for effort when actually little effort was made (Hattie and Timperley, 2007). Meyer (1982) made a nuance argument that a recipient of praise may interpret this as the speaker believing their ability was low. This could have a negative impact on the student's academic self-concept (Ireson and Hallam, 2009). Brophy (1981) argued the principal driver for the use of praise was the teachers reading of student needs, rather than any judgment of the quality of student contributions. As such, this appears to be linked to teachers' expectations of students. Consequently, the use of praise could be of relevance to this study, especially if there is a differential use of praise by teachers with different groups of students (2.4.2).

A number of researchers have argued for the value of classrooms where students are expected to make longer contributions and offer their own opinions, but studies have indicated teachers experience difficulties in establishing and managing these broader classroom discussions (e.g. Stein et al., 2008; Larsson and Ryve, 2012; Adler and Ronda, 2015). Difficulties appear to arise from the complexities involved in the interpretation of student contributions and the management of the tension between exploring student reasoning and drawing attention to the logic of the discipline (Ball, 1993; Sherin, 2002; Scherrer and Stein, 2013). Maintaining a productive discussion appears to be especially problematic when student responses are not easily recognised as part of a standard solution strategy. The evidence shows there is a tendency to reduce the level of cognitive demand by prompting specific procedural strategies (Henningsen and Stein, 1997; Krussel et al., 2004; Wilhelm, 2014).

Of interest to this study is whether the same teacher exhibits different patterns of interaction with different groups of students, leading to the establishment of different classroom norms.

3.3.6.3 Registers

Documented differences between low and high attaining sets include how supposed real-life experiences are drawn on and how registers are used. Specifically, variations in the relationships between colloquial language and more formal mathematical terminology have been found. Drawing on his previous studies, Dowling (2010) argued that students in low sets encounter 'mythologised versions of the students' own lives' (p.7), whilst the higher attaining groups focused on more formal mathematics.

Gellert and Straehler-Pohl (2011) described horizontal discourse as relating to contextualised language and vertical discourse as more formal decontextualised mathematical discourse. They argued access to both registers is needed in order to develop mathematical understanding, but teachers varied in how they inducted students into the use of vertical discourse, tending to restrict access to those they considered more able. Similarly, Adler and Ronda (2015) argued that the use of colloquial discourse is crucial in allowing students to access mathematical concepts, but that the transition to a formal register is necessary to fully participate in the learning of mathematics. They contended there was a disinclination on the part of some teachers to use a more formal register, especially with low attaining students.

As with context, it appears how language is used requires a nuanced interpretation, with informal discourse being necessary to support students' access to mathematical concepts but would restrict understanding if relied upon at the expense of induction to formal discourse. Of interest to this study is that it appears that differences in the use of more formal language might be related to different student attainment.

3.3.7 Lesson Trajectories: Planning and In-lesson Sequencing

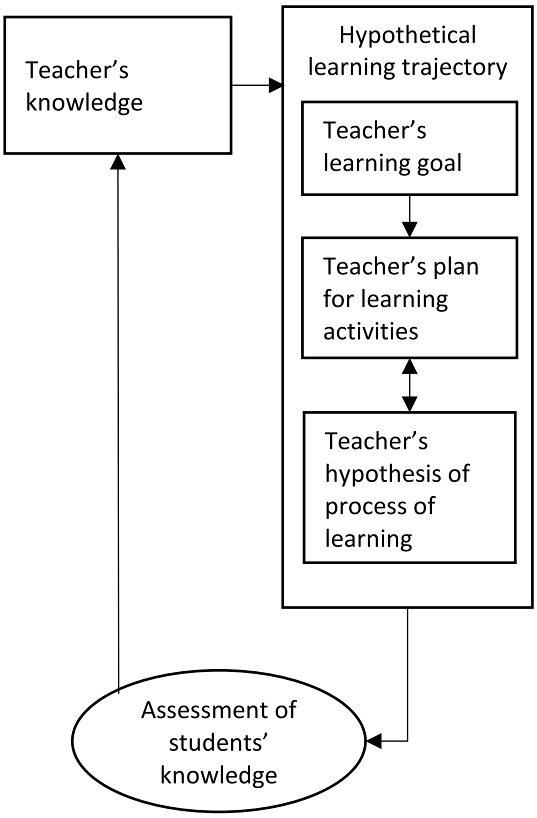
3.3.7.1 Introduction

Mathematics lessons are a sequence of events contingent on a wide range of factors. However, it is usual for teachers to plan classroom activities before the lesson. These evolve into enacted activities during the lesson, shaped by teachers' in-the-moment decision making, students' actions and classroom interactions. This section considers the relationship between these different stages of teachers' thinking about the lesson and the mathematical direction of travel that occurs as the lesson unfolds. The term 'lesson trajectory' is used to describe the course of the mathematical focus of classroom activities as the lesson progresses. First, the notion of learning trajectories will be discussed (Simon, 1995) in order to consider the relationship between pre-lesson planning and the teachers' in-class activities. Then teachers' in-the-moment decision-making and the role of attention will be considered in more detail. Finally, the mathematical focus and actions taken by the teacher to 'steer' the lesson trajectory will be discussed.

3.3.7.2 Learning Trajectories

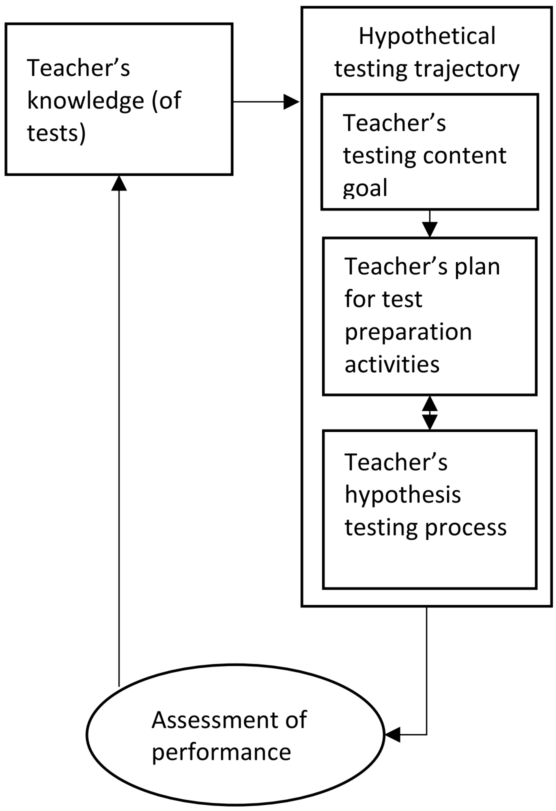
Over recent years, the term 'learning trajectories' has been used as a label for theoretical frameworks that describe the course of students' mathematical reasoning through particular topics (Simon, 1995; Clements and Sarama, 2004; Wilson et al., 2014). Whilst these discourses are rooted in a constructivist perspective, the meaning ascribed to the term learning trajectory has developed as the research focus shifted in grain size from classrooms to large-scale curriculum development. Of interest here is the analysis of classroom practice, so the origins of 'hypothetical learning trajectories' (HLT) introduced by Simon (1995) are discussed. Attention will be drawn to core features of the framework through consideration of how other researches have drawn on Simon's construct to analyse classroom practice. Finally, consideration will be given to how the HLT framework was adapted to inform this study.

Simon (1995) argued that constructivism provides a valuable way to think about mathematical learning, but that there are no simple ways to translate this understanding into particular teaching approaches. In order to contribute to a dialogue about pedagogy, he conducted classroom teaching experiments with his own class of student-teachers. The analysis of which led to the development of his 'Mathematical Teaching Cycle', a theoretical framework for pedagogy (p.131). One of the key issues a constructivist perspective brings to the classroom is the tension arising from responding to students' thinking whilst also keeping 'an eye on the mathematical horizon' (Ball, 1993, p.373). Simon (1995) coined the term 'hypothetical learning trajectories' (HLT) to describe 'the teacher's prediction as to the path by which learning might proceed' (p.135); as such his framework articulated how a teacher might manage that tension.



Simon (1995, p.136)

Figure 3.5: Mathematical Teaching Cycle



Adapted from Amador and Lamberg (2013, p.161)

Figure 3.6: Testing Cycle

A trajectory starts with the teacher's learning goals, which indicates the planned direction of travel. This informs the teacher's plan for learning activities, which in turn is in a reflexive relationship with their hypotheses of how students' conceptual development evolves during participation in those mathematical activities. This formed part of his wider theoretical framework of the Mathematical Teaching Cycle, where HLTs are continually revised in light of the teacher's assessment of student reasoning and is informed by the teacher's own knowledge (Simon, 1995) (figure 3.5).

The narrative Simon (1995) provided was a detailed account of his assessment of the students' understanding of concepts related to area, and how he developed mathematical activities based on student responses. Whilst this appeared to demonstrate a coherent way to model the iterative processes involved in his own pedagogical thinking and actions, it was strongly tied to the particular mathematical concepts and the individual student responses. Moreover, it was inextricably linked to Simon's mathematical knowledge for teaching and his constructivist stance. Simon (1995) himself posited that it was unlikely that practicing teachers would have the time or resources to replicate this process in their own classrooms.

The HLT framework is predicated on students' individual meaning making, with the result that there could be as many different actual pathways of concept development as there are students. As such, it would be impractical for a teacher to generate HLTs on this scale. Cobb et al. (2001) argued their interpretative framework that utilises classroom norms, overcomes the difficulties in dealing with multiple, qualitatively different, student reasoning, and consequently allows HLTs to be utilised as an analytic tool at a class level. In the classroom experiments reported by Cobb et al. (2001), the research staff recorded their interpretation of diverse student thinking and then discussed this information with the teacher in 'real-time', which he then used to inform the latter stages of the lesson. So, whilst Cobb et al. (2001) demonstrated how they used HLTs to inform their instructional design, the use of staff beyond that normally available to teachers reinforces Simon's (1995) assertion that HLTs in this form are unlikely to be accessible to teaching staff on a routine basis.

3.3.7.3 Generic Trajectories

In a US based study, Amador and Lamberg (2013) investigated four teachers' planning and teaching utilising the HLT methodology. They were only able to analyse one teacher's practice using this framework, suggesting, for some teachers at least, the fundamental features of the HLTs are not a regular part of practice. The one teacher's practice they could analyse utilising HLTs was aligned with an inquiry approach, with attention being paid to developing students' conceptual understanding. Even though this teacher did not explicitly refer to HLTs, the goals she articulated, and the planning-assessment cycle undertaken, were captured by the model, providing some evidence of the potential explanatory power of the HLT framework in inquiry-oriented contexts. The remaining three teachers did not articulate or enact learning goals as construed by Simon (1995). That is to say, goals related to the development of a conceptual understanding of significant mathematical ideas were not present. Instead, Amador and Lamberg (2013) developed an alternative trajectory to model these teachers' actions, arguing that they followed a testing trajectory where the key focus was on student performance in high-stakes tests (figure 3.6).

Whilst this study highlighted the difficulties in drawing on HLTs when exploring teachers' typical classroom practice, it also offered some evidence that the underlying structure could be applied to a wider range of contexts. Amador and Lamberg (2013) were able to construct an alternative model that linked the teachers' plans, predictions and assessment of classroom outcomes together. In doing so, the model still encapsulated the tension the teachers needed to manage between their intended lesson goals and being responsive to student activity, albeit in terms of test performance. With the English school system often characterised as highly performative in nature (Sealey and Noyes, 2011), this adaptability may make this model more relevant to this study. Moreover, it may offer a way to identify different trajectories for different classes, which may be of particular relevance as engagement rather than conceptual understanding is seen by some as an appropriate goal for low attaining students (Boaler et al., 2000).

The term 'lesson image' has been used to describe 'a broad vision of what the teacher expects to happen' (Rowland et al., 2015, p.75). The previous discussion indicates that a generic trajectory, which encompasses a goal for the lesson, a plan for activities and anticipated student responses, has the potential to provide a framework for analysing a range of different styles of lessons. So here the term 'lesson image' has been adopted to describe a generic trajectory that encapsulates learning, performance and engagement orientations (figure 3.7). The original teaching cycle by Simon (1995) incorporated a feedback loop as the trajectory was considered to be under constant review. In the model developed for this study, the teachers' interpretation of the unfolding lesson trajectory is reviewed against their lesson image (hypothetical generic trajectory), with incremental or substantial adjustments possible as a result.

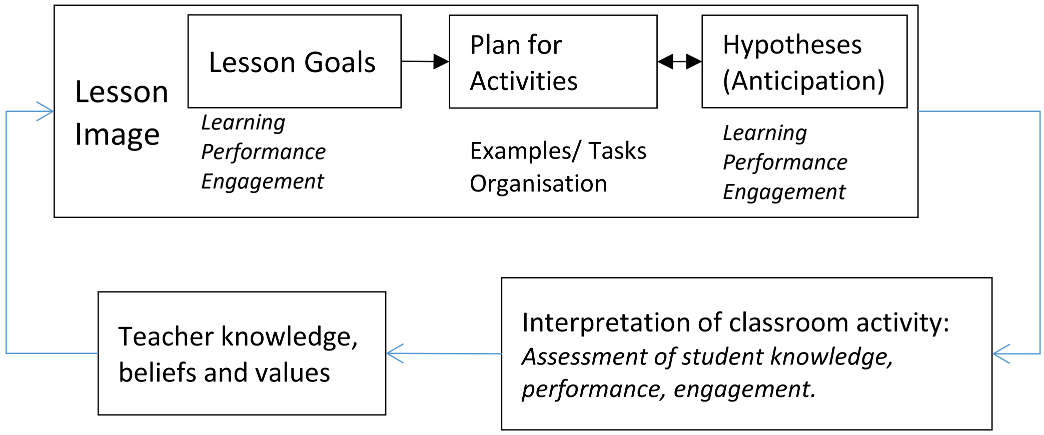


Figure 3.7: Teaching Cycle including Generic Trajectory (Lesson Image)

3.3.7.4 Algorithmic Reasoning

The processes by which teachers make decisions have been considered from a range of perspectives. For example, Schoenfeld (2011) argued that teachers' decision making could be modelled through a consideration of teachers' goals, orientations and available resources, which included beliefs, values and teachers' knowledge; this aligns well with the teaching cycle described above (figure 3.7). However, as suggested in the previous section, teachers' thinking about lessons occurs in two distinct phases. First, the out-of-class pre- and post-thinking related to the planning before the lesson and the reflections that continue afterwards. Second, the in-the-moment thinking that occurs as the teacher interacts in the lesson (Clark and Peterson, 1986). Watson (2019) argued there are qualitative differences between out-of-class and in-the-moment decisions due to the dynamic and complex nature of classroom interactions.

Drawing on cognitive psychology, Watson (2019) first differentiated between type 1 and type 2 processes, with the former 'unconscious or intuitive' and the latter 'rational and conscious deliberative' (p.2). Acknowledging more recent research, he drew on the notion of algorithmic reasoning, which is an intermediate form of reasoning based on 'rehearsed actions and pedagogic routines' (p.3). Specifically, he concluded algorithmic reasoning was the dominant form of teachers' reasoning in classrooms. The argument being that once pedagogical routines are established, such as IRE patterns, these are easily accessible to teachers in-the-moment. Due to the amount and dynamic nature of information available in the classroom, algorithmic reasoning often occurs rather than more conscious deliberations. Watson (2019) argued this could at least partially explain the differences between teachers' espoused and enacted beliefs. For example, a teacher might state they value student explanations but in practice limit students to short responses. An explanation suggested by this model is that if IRE patterns are normative for the teacher, there may be insufficient triggers to shift them to more deliberate, reflective processes needed for them to notice, and therefore be able to modify, the type of responses they accept.

3.3.7.5 Steer: Focus of Attention

Mason (2011b) stated that 'noticing is a movement or shift of attention' (p.45), where a person's attention can be spontaneous or intentional, with varying levels of awareness about what they are attending to. For example, a mathematics teacher could interpret a diagram as representing an equilateral triangle without consciously bringing to mind that the three marks on the edges signify the sides were the same length. Mason (2011b) argued that mathematics necessitates shifting attention, which is often an intuitive action by experts but more difficult for novices. For example, when asked to graph a quadratic, an expert may shift attention between a few values, the roots, a translated mental image of $y = x^2$ and the associated symmetry in order to draw a graph efficiently and accurately. Whereas, a novice might focus solely on a table of values; without recourse to the relationships between the particular example and properties of quadratics more generally, mathematical errors could go unnoticed.

One significant implication for the classroom is the role attention has in communication:

> When teacher and students are attending to different things, communication is unlikely to be efficient. Even when teacher and students are attending to the same things, they may be attending differently, and so communication may be, at best, restricted and incomplete, if it does not break down altogether.
> (Mason, 2011b, p.47)

The situation is complicated by the fact that individuals have varying levels of awareness of where their own attention lies (Mason, 2001); at best, inferences could be drawn about where other peoples' attention lies based on their behaviours (Mason, 2011b). From this perspective, the expertise of a teacher could be linked to the level to which they understand where students' attention might lie.

Jacobs et al. (2010) coined the phrase 'professional noticing' to describe the in-the-moment processes of teachers attending to students' strategies, interpreting students' understanding and making decisions about their own responses (p.169). With the focus on student reasoning, this appears to align with an in-the-moment enactment of the mathematical teaching cycle (3.3.7.2 & figure 3.5), but with the role of attention in the assessment of students' activities acknowledged more explicitly. Jacobs et al. (2010) found that teachers had different levels of expertise in terms of professional noticing, but this could be developed. However, they also acknowledged that their research was set in a context of reform orientated CPD. It appears likely, therefore, that as with hypothetical learning trajectories, this type of noticing is less likely to occur in all classrooms, and performance and engagement orientations would need to be taken into account.

The tension a teacher has to manage between attending to students' thinking and maintaining their 'eye on the mathematical horizon' (Ball, 1993, p.373) was discussed in section 3.3.7.2. However, the lesson image that teachers hold may have performance and/or engagement orientations (Amador and Lamberg, 2013). As such, the teacher may attend to aspects of students' activities other than reasoning, and their mathematical horizon might relate to the efficient application of algorithms. So, while the tensions between being responsive to student activities and managing the lesson trajectory remain for all teachers, a teacher's orientation could lead to resolutions being a long way from the vision Ball (1993) held 'that both honors children and is honest to mathematics' (p.31).

Mason (2011b) argued that inferences could be made about what teachers and students may be attending to and that different levels of attention occur. For example, at the micro-level attention can shift between the whole and details, or between relationships and properties, whereas at a macro-level attention can vary in focus, strength, scope and source (p.46-7). In terms of observable behaviours that could be interpreted in classrooms, identifying macro-level attention appears to be more achievable, especially in terms of focus and source. In order to have productive discussions in the shared space of whole-class interactions, a shared focus of attention would be needed at some level (Mason, 2011b). In a classroom context, teachers' actions that direct students' focus of attention might be inferred through the analysis of classroom discourse. Whilst far less would be known about the attention of students not directly participating in whole-class discussions, if a class level focus could be established then it would be reasonable to consider that the attention of an 'effective student of mathematics' would be drawn to that focus.

With the nature of the foci forming an integral part of the lesson trajectory, who determines the foci and controls any shifts provides the mathematical steer for the lesson. Teachers tend to control classroom talk (3.3.6.1) and are therefore likely to play a significant part in managing the lesson trajectory, but some of this steer may come from algorithmic reasoning (3.3.7.4). In terms of the mathematics made available to students over the course of the lesson, how and when attention is drawn to mathematically significant features is important. For example, from a perspective of variation theory, the sequencing of examples is important, as features need to be varied and juxtaposed in relatively close time frames (3.3.5.2). However, whilst variation might make aspects of mathematics more visible, its presence alone does not guarantee learning (Mason, 2011a). Whilst making explicit reference to mathematically significant features may make learning more likely, it is a complex decision as to when and how to shift students' attention from 'doing' tasks to the scrutiny of the mathematically significant features embedded in those tasks. Moreover, the level of explicitness that students might find useful is contingent on a range of factors and is likely to vary between students (Mason, 2011a).

3.3.7.6 Summary: Lesson Trajectories

The previous sections have outlined different aspects that regulate the mathematical direction of travel, from teachers' pre-planning through to in-the-moment decision making. Key elements include the teachers' orientation in terms of learning, performance or engagement, and how this influences their attention and where they direct the attention of students. All teachers have to balance their attention to student contributions with how they focus on their mathematical horizon, but again the nature of their engagement with student contributions and the nature of their mathematical horizon are shaped by their orientations and beliefs. What the teacher draws attention to shapes the lesson trajectory and the mathematics made available to students.

3.4 The Orchestration of Mathematics Framework (OMF)

Early in the literature review process, the Teacher's Orchestration of Mathematics (TOM) conceptual framework (figure 3.4: model A) was developed to facilitate the synthesis of prior research (3.3.4). As the study progressed, this early model continued to be refined through the systematic integration of a range of theoretical perspectives that can be brought to bear on the interpretation of the mathematics classroom. In these developments, as discussed in detail below, the model was expanded and refined. A revised version of TOM was retained as the central element of the framework as this focussed on the teacher's classroom activities. This was integrated with a teaching cycle (3.3.7.3) that could take into account pre-lesson planning and in-lesson assessment, which resulted in the first iteration of the Orchestration of Mathematics Framework (OMF) (figure 3.8: model B). As the study moved into the data collection and analysis stages, the OMF was repositioned from a synthesis tool to an analytical framework, with further revisions made as a result. In the pilot stage a revised version of the OMF was used (figure 3.11: model C) and the transition to the main study brought the final iteration of the OMF (figure 3.14: model D).

In designing the study, consideration was given to the use of classroom observation tools. A number of studies have developed observation frameworks, but many held an evaluative orientation (e.g. Learning Mathematics for Teaching Project, 2011). Also, they were often underpinned by particular initiatives, such as the reform agenda in the US (e.g. Remillard and Bryans, 2004; Schoenfeld, 2014) (3.4.3), or focussed on specific aspects of lessons, such as cognitive demand (Stein and Smith, 1998) (3.3.3). The aim of this study was to describe and analyse 'typical' lessons rather than offer an evaluation of teaching. As such, none of the frameworks reviewed appeared to have the breadth of coverage to capture the range of mathematical practices that prior research indicated were likely to occur in English classrooms, especially practices reported as more prevalent in lower attaining sets. Consequently, as the study moved towards the pilot stage, the decision was made to develop the OMF so that it could be used as an observation and analytical framework.

This transition, from a conceptual framework as a distillation of relevant theoretical perspectives to one for the observation and interpretation of teachers' pedagogical practices, brought additional challenges. The framework would need categories that were both necessary and sufficient to capture mathematically significant events, with internal coherence and acceptable levels of separation (Schoenfeld, 2013a). The interconnectedness of classroom activities means that a completely unambiguous categorisation of teachers' pedagogical moves is unlikely to be found. Moreover, it is recognised that the connections between different elements of a framework are as important as the elements themselves. Therefore, developing a coherent framework was a complex undertaking.

Throughout the study, the development of the OMF remained an iterative process. Initially the moves were between the literature and the conceptual model, starting with the Teacher's Orchestration of Mathematics framework, the first model of teachers' pedagogical moves (figure 3.4: model A), and then with the broader OMF (figure 3.8: model B). Sections 3.2 and 3.3 outlined the relevant research that informed the developments from a theoretical perspective. As the study progressed, data from the pilot study were mapped against the framework, with initial revisions leading to the pilot version of the OMF (figure 3.11: model C). In the transition to the main study, the relationships between the coding protocols and the OMF led to further refinements, with amendments made and further literature sought as gaps and potential conflicts in the framework arose. This led to the final iteration of the OMF, as presented in figure 3.14 (model D).

The following provides an overview of the influences of the literature and empirical data on the developments of the OMF. Section 3.4.1 outlines how the separate elements identified in the literature review were synthesised into the Orchestration of Mathematics Framework (OMF) (figure 3.8: model B). Section 3.4.2 discusses how further adjustments were made to the OMF during the application of the framework to the pilot study (figure 3.11: model C) and the transition to the main study (figure 3.14: model D). In order to site the OMF in the wider field, section 3.4.3 undertakes a comparison with two other observation frameworks, and section 3.4.4 considers the affordances and limitations of the OMF.

3.4.1 OMF: Summary from the Review of Literature

The initial model for teachers' pedagogical moves was the Teacher's Orchestration of Mathematics framework (figure 3.4: model A). This was structured around cognitive demand, classroom norms and the teachers' activities related to their management of discourse and task design. The model was then expanded to include the teaching cycle, which captured the lesson image, the interpretation of classroom activity and teacher knowledge, beliefs and values. Additional dimensions were also added to the central TOM section. These features were integrated to form the first iteration of the OMF (figure 3.8: model B), as discussed in detail below.

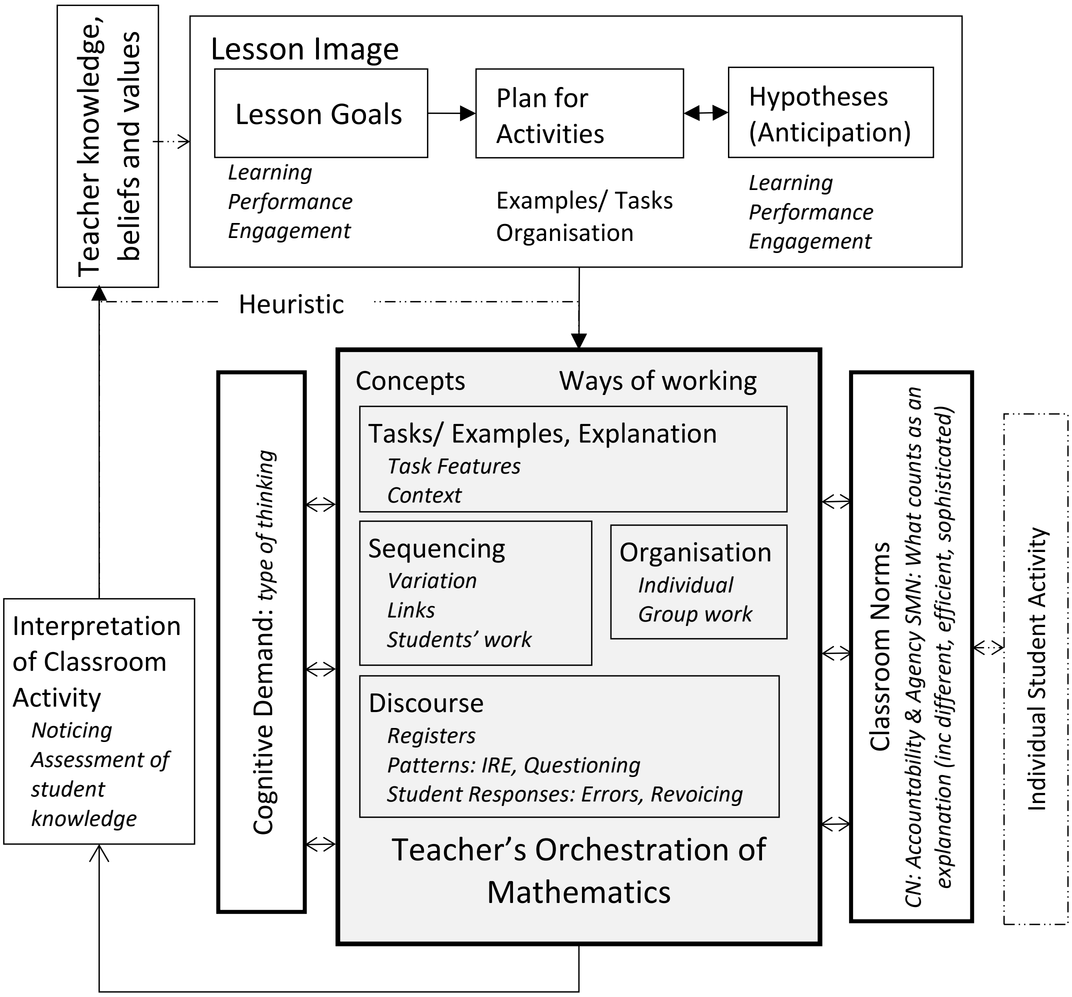


Figure 3.8: Model B – Orchestration of Mathematics Framework (OMF) – Literature Review

The teaching cycle (figure 3.7) was incorporated due to its power to model teachers' planning and assessment both prior to and during the lesson. TOM was included in the cycle between the lesson image and the interpretation of classroom activities as this represented the stage where the teacher enacted their plans prior to review. The heuristic label was included to acknowledge some decision making is more intuitive and based on established pedagogical routines. In the central TOM section, classroom organisation (2.4.1.1) and sequencing (3.3.7) dimensions were added. These were included when the literature signalled their relevance to setting and their importance in relation to teachers' pedagogical moves but did not fit within the developing descriptions of the task or discourse dimensions.

Figure 3.8 (model B) represents the last iteration of the OMF from the substantive literature review undertaken before the pilot study commenced. However, further literature was drawn on after this point in the study. For example, the heuristic label was always an integral part of the OMF, but the work by Watson (2019) on algorithmic reasoning was subsequently drawn on to offer a more detailed theoretical perspective on that aspect of the OMF (3.3.7.4). Consequently, the main concepts discussed in sections 3.3.3 to 3.3.7 are to be found in figure 3.8 (model B), but there are some elements, such as algorithmic reasoning and the role of attention (3.3.7.5), that informed the last iteration of the OMF (figure 3.14: model D).

3.4.1.1 OMF: Summary of the Theoretical Perspectives

The central core of the OMF has the same overall structure as the original framework, and consists of the Teacher's Orchestration of Mathematics (TOM), cognitive demand and classroom norms. Although TOM now only refers to the shaded section and has the added dimensions of sequencing and organisation. TOM encapsulates the dimensions related to the teachers' in-class mathematically related activities. Specifically, the organisation (2.4.1), tasks (3.3.5), discourse (3.3.6), and sequencing (3.3.7) dimensions of TOM captures the teachers' pedagogical moves that can have a critical effect on the mathematics made available to students.

As previously discussed, cognitive demand offers a way to capture the potential of the teacher's pedagogical moves to generate different types of student thinking, as set up in the tasks and managed throughout the lesson (3.3.3). Classroom norms are co-constructed by all members of a class, where actions both shaped and are shaped by these norms. They provide a mechanism to identify what is recognised as mathematically legitimate in particular settings (3.2.2). In order not to imply that teachers are the only influence on these norms, individual student activity is included in the framework, though the hashes signal that the student perspective is not the focus of this study.

The extended teaching cycle is composed of the Lesson Image, TOM, the Interpretation of Classroom Activities and Teacher Knowledge, Beliefs and Values. This was included as it captures the teacher's planning that begins prior to the lesson and the ongoing cycle of classroom activity, interpretation and revision of plans that continues as the lesson unfolds, all of which are influenced by the teacher's knowledge, beliefs and values (3.3.7). The algorithmic reasoning is indicated by the 'heuristic' label, and signifies the significant influence pedagogical scripts that have been built up over time can have on teachers' decision making (Watson, 2019). This was included to acknowledge that many of the teacher's in-class actions might be based on reasoning of a more autonomous nature (3.3.7.4).

Teachers' practice is a mixture of planned and contingent activities, based on conscious decisions and intuitive responses (Schoenfeld, 2011), and teachers respond to what is noticed, which varies (Mason, 2015b). Moreover, from a constructivist perspective, meaning making resides with the learner, so student responses will never be entirely predictable. Consequently, no two lessons are the same, even if the same tasks are used by the same teacher. A valuable analytic tool in these circumstances is the use of classroom norms, as this allows regularities in these constantly varying situations to be identified (Cobb et al., 2011).

This study sought to build an understanding of the nature of mathematics made available to students resulting from the teacher's pedagogical moves. This should be achievable through the identification of activities that are legitimised and accepted in particular classes as part of established classroom norms (Cobb et al., 2011). When considering mathematical activity, a common demarcation is distinguishing between particular mathematical concepts and ways of working (Schoenfeld, 2013a). A corresponding demarcation is made by Cobb et al. (2001) when they distinguish between mathematical practices and sociomathematical norms. As mathematical topics vary from lesson to lesson, and this study will not be imposing particular tasks, it was anticipated that the focus would be orientated towards sociomathematical norms.

3.4.2 OMF: Developments during the Pilot and Main Study.

The OMF continued to be developed during its use in the pilot study and in the transition to the main study, both in terms of the refinement of the descriptors and my understanding of the OMF as a conceptual model. The analysis of lessons was an iterative process, moving between the data sources, the OMF and the emerging pedagogical profiles. In particular, audio transcripts formed a substantive part of the data, and their coding was conceptually and procedurally related to the OMF. Discussions of these issues would usually be considered in a methodology chapter. However, in order to provide a coherent narrative of the development of the OMF, the following sections discuss aspects of this study that were instrumental in that development, whilst chapter 4 addresses the wider methodological issues. The following sections include the formalisation of the coding strategy for the lesson transcripts and the transition to the main study. Figure 3.9 outlines the key OMF transition points and signposts where each version of the OMF is located.

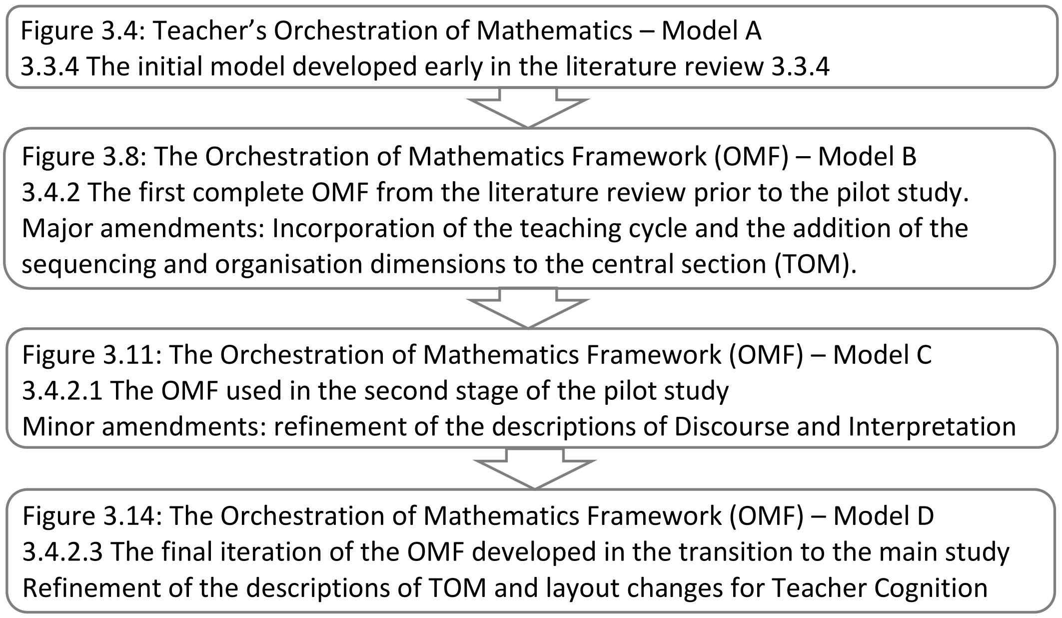


Figure 3.9: Summary of the Development of the Orchestration of Mathematics Framework (OMF)

3.4.2.1 Pilot Study

The pilot study consisted of the analysis of three publicly available videos from the TIMSS video study (Hiebert et al., 2003a) and two lessons recorded for this study.

(a) Stage One: TIMSS Video Study

The TIMSS videos were used as they were complete, unedited lessons, selected as representative samples rather than for the presence of specific features. In addition, a range of supplementary data was available (appendix 2), and the recordings took 'the perspective of an ideal student' (Hiebert et al., 2003a, p.15) that aligned well with this study. In this stage of the pilot, the OMF represented by figure 3.8 (model B) was used.

Included in the TIMSS material were brief commentaries from the teachers and researchers. This was useful in gaining an understanding of how cognitive demand was interpreted by the researchers and offered a mechanism for comparing my analysis with that of other researchers, albeit in a very limited fashion (appendices 2.1.3, 2.2.3 & 2.3.3). There was less information about the teachers' planning. Therefore, only the central part of the OMF was the focus of the first stage of the pilot study (figure 3.10). The purpose of this stage was to explore the viability of this restricted framework as an analytical tool when the lessons were considered in isolation, thus providing evidence to shape the wider study.

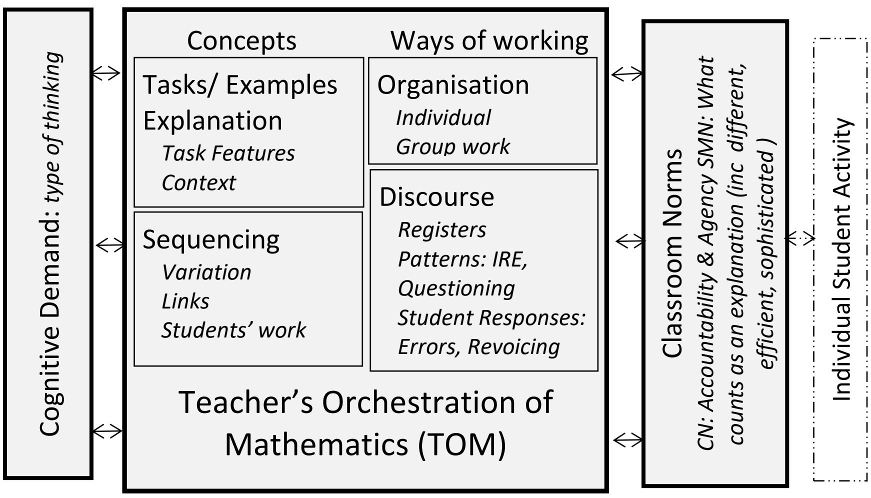


Figure 3.10: Central Section (TOM) from Model B (OMF)

After an initial viewing of the videos and the accompanying material, each section on the lesson transcripts was coded for mathematical activity (appendices 2.1.1, 2.2.1 &

2.3.1). Classroom activities that were unrelated to mathematics, such as social interactions or classroom management, were coded as not mathematically related and were not subject to further analysis. The remaining sections were coded as mathematically related and were subject to further analysis. Those activities related to the organisation of the mathematical tasks, such as arranging students into groups, were coded and recorded on the organisation dimension of a lesson specific OMF. Other mathematically related episodes were subject to a more detailed analysis. Initial impressions for each section of the OMF were also recorded at the end of the first viewing. These interim representations consisted of annotated transcripts and a partially completed lesson-specific OMF (e.g. appendices 2.1.1 & 2.1.2).

Mathematically related sections of the transcripts were revisited and considered in relation to the dimensions of TOM, cognitive demand and classroom norms. In effect, the OMF categories were used as an initial coding protocol (appendices 2.1.1, 2.2.1 & 2.3.1). In addition, the tasks published in the supplementary information were analysed and compared to the enacted classroom tasks. During the analysis, I found myself moving between the lesson videos, transcripts and classroom artefacts in order to develop an understanding of the individual episodes. As characteristic themes in each dimension emerged, these were compared with other sections of the lesson. Features seen only when students were working individually or in small groups were noted and only included in the overall annotated OMF if they were seen more than once and included more than one student. This resulted in an annotated OMF for each lesson, with themes being linked with specific passages in the transcripts (appendices 2.1.4, 2.2.4 & 2.3.4).

Across the elements of the OMF different lesson characteristics did appear. For example, in relation to classroom norms, two teachers were the sole arbiter of what constituted an acceptable mathematical solution, and explanations were not expected when students answered questions. For the third lesson, there did appear to be a broader range of accountability (appendix 2.3). Whilst the teacher did evaluate the responses of many questions asked, there was a press for explanations in some contexts and some students held each other to account in group work. When TOM was considered, the interrelationship between the dimensions became apparent. For example, in the third lesson the task dimension highlighted that the multiple solution strategies and multiple representations inherent in the task had the potential to expose critical features of triangles. However, the sequencing dimension highlighted the restricted nature of the examples used throughout the lesson, which then limited exposure to the range of permissible change, and hence generalisation.

This analysis was compared to the publicly available material and none of the analysis above appeared to contradict the published commentary. For example, classroom organisation and the identification of multiple solution strategies were common to both my analysis and the published material for the ratio lesson (appendix 2.2.3). However, the detail provided in the published material was limited and focussed on describing the sequence of events (e.g. appendix 2.2.2). The analytical framework provided by the restricted OMF did provide a framework for summarising the lesson observations that allowed characteristics such as student accountability to be compared. As such, this provided some evidence that pedagogically noteworthy features were captured by this process.

One amendment made to the OMF during this stage of the study was the inclusion of procedure/concept as one of the descriptions in the discourse dimension. This arose out of cognitive demand being categorised as low due to a focus on procedures (e.g. appendix 2.1.4). Tracing this back to the teachers' pedagogical moves, these procedural foci were found to be embedded in the teachers' questions and the type of responses they treated as satisfactory. However, as will be discussed in section 3.4.2.2, this classification was subject to further revisions later in the study.

(b) Stage Two: Two Lessons

Sam, one of the participating teachers, volunteered to have two lessons with the same high attaining class videoed. In addition to videoing the lessons, pre- and post-lesson semi-structured interviews were undertaken, and classroom artefacts were collected. Consequently, the complete version of the pilot OMF was used (figure 3.11: model C).

This had a more compact layout, used for ease of publication. There were minor revisions to the discourse dimension and the interpretation of classroom activity in comparison to figure 3.8 (model B), the Literature Review version of the OMF (the revisions are indicated by double underlining).

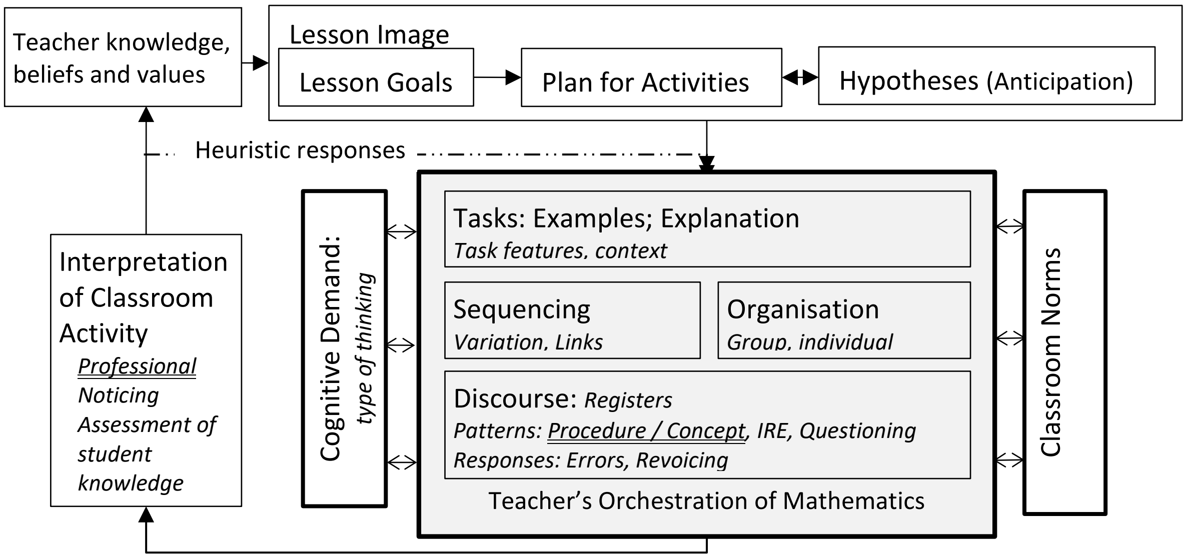


Figure 3.11: Model C – Orchestration of Mathematics Framework (OMF) – Pilot Version

The audio data from the lessons were transcribed and initially coded as mathematically related or not. Those episodes coded as mathematically related were then mapped against the OMF. As with the pilot study, the OMF dimensions were populated and the transcripts were subject to initial coding based on the OMF categories (an example is given below: extract 3.1 and table 3.1).

The following is an extract of the transcription from the first stage 2 pilot lesson. Links made to the OMF are given in square brackets [ ]. The students were asked to answer six area questions and, after nine minutes, solutions were discussed as a whole class.

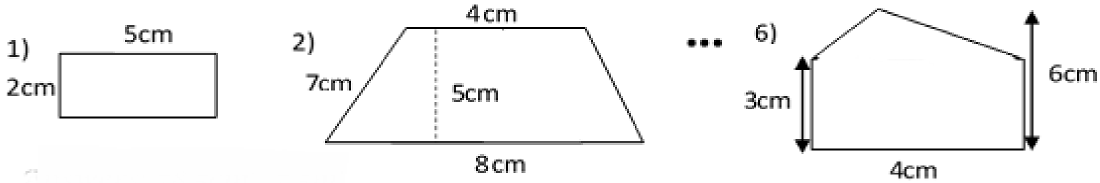


Figure 3.12 Questions presented on PowerPoint Slide

78 T: OK number one, what did you do and what answer did you get? [*Discourse: I-IRE*]

79 Azar: I got five times two (.) ten meters (..) centimetres meters squared

80 T: Ten centimetres squared, perfect, units are really important OK

Fin second one then

[*IRE-E immediate acknowledgment (evaluation) and initiation of new IRE*]

81 Fin: I did four add eight (.) twelve divided by two, six (..) times five makes thirty centimetres squared.

[*Norms: procedure accepted as explanation*]

⋮

113 T: Sixteen centimetres squared, what did you do to get it?

114 Shaz: It was just the last (…)

115 T: Just the last answer (.) did anyone manage to do it with the maths? Raj

[*'error' – not mathematical procedure – moved on to another student IRE reinstated*]

116 Raj: Three times (.) three times four so it's the bottom rectangle

117 T: Yep

118 Raj: and that's 12 meters squared (.) but then you triangle the top bit (.) which is three times four meters and divide by two

119 T: So what Raj did, and I'm guessing what most people did who managed to get that was to draw a little line there to do our 3 times 4 which is 12 meters squared and work out the triangle on top which was 4 meters squared OK good.

[E-IRE including revoicing explanation. Sequencing: Unsystematic variation – no links between questions]

Extract 3.1: Pilot Lesson
(adapted from Baldry, 2017, p.3045)

The pre- and post-lesson interviews were transcribed and were similarly mapped against the OMF. These predominantly contributed to the lesson image and the interpretation of classroom activity sections.

The data was then summarised in relation to the OMF (table 3.1) and cross-referenced with the lesson observation notes. Due to the nature of what can be more easily observed, the task, organisation and discourse dimensions of TOM were more straightforward to populate, with lower levels of inference required.

<table>
<tr><td></td><td colspan="2">Lesson Image</td><td></td></tr>
<tr><td rowspan="4">Interpretation of classroom activity: errors, explanations not explored</td><td>Goal: Performance</td><td>Plan: Exam style questions</td><td>Hypotheses:
Familiarisation</td></tr>
<tr><td colspan="2">Tasks: Example; Explanations
Multiple solution strategies were possible but no evidence of acknowledgement. All questions of standard format.</td><td rowspan="4">Classroom Norms:
Social Norms: Agency and accountability: resides predominantly with the teacher.
SM norms: procedural explanation counts as explanation
Mathematical practices: units important; area equates to multiplication</td></tr>
<tr><td colspan="2">Sequencing: Unsystematic variation and links were not explored.</td></tr>
<tr><td colspan="2">Organisation: Individual working</td></tr>
<tr><td>Cognitive Demand: Potential-high; As enacted- low</td><td colspan="2">Discourse: Teacher led; Teacher requested (procedural) explanation, followed up when not provided
IRE with teacher evaluation
Teacher re-voiced contributions; repeating correct answer, rewording more complex explanations</td></tr>
</table>

Table 3.1: OMF summary for lesson 1
(adapted from Baldry, 2017, p.3046)

In order to explore the potential of the OMF as an analytical tool to capture mathematically significant pedagogical characteristics, with resonance beyond the individual episodes, the two lesson OMF profiles from the pilot study were compared. Drawing on the similarities and differences, an integrated OMF was compiled (table 3.2).

<table>
<tr><td>OMF</td><td colspan="3">Lesson Image</td></tr>
<tr><td rowspan="3">Interpretation of classroom activity: Professional noticing -….</td><td>Goal: Performance</td><td>Plan: Exam style questions</td><td>Hypotheses: Familiarisation</td></tr>
<tr><td colspan="2">Tasks: Multiple solution strategies were possible, but …</td><td rowspan="4">Classroom Norms:
Social Norms: Agency and accountability: resides predominantly with the teacher. Sociomathematical norms: procedural explanation counts as explanation. Mathematical competence equates to obtaining correct answers efficiently (errors to be avoided).
Mathematical practices: Lesson 1: Area equates to multiplication. Lesson 2: Proportional reasoning equates to multiplication</td></tr>
<tr><td colspan="2">Sequencing: Questions sets unsystematic variation. [Dimensions of variation and range of permissible change not made explicit]</td></tr>
<tr><td rowspan="2">Cognitive demand: Potential - high
As enacted - …</td><td colspan="2">Organisation: Individual working- tables in groups of four; peer-to-peer discussions were had.</td></tr>
<tr><td colspan="2">Discourse: Registers: teacher used colloquial language …
Patterns: IRE dominant form of interaction. Correct answers acknowledged, often repeated or extended. Errors often ignored; when acknowledged focused on moving to standard solution, reverting to direct explanation if initial follow-up failed. Extended student explanations ….</td></tr>
</table>

Table 3.2: Compilation OMF
(extracts from Baldry, 2017, p.3047)

The two lessons were on different topics, area and proportional reasoning, so as anticipated the mathematical practices within the classroom norms were different, but across the other dimensions of the OMF the profiles were similar. For example, the sequencing of questions was categorised as unsystematic variation (Baldry, 2017). The analysis also identified a range of classroom norms, such as mathematical competence being equated to the efficient production of ‘correct’ answers. These were drawn from the discourse patterns in relation to the teacher’s reactions to student contributions, including errors.

The interaction between the categories raised the question as to whether the OMF was sufficiently well-defined. Schoenfeld (2013b, p.614) argued we should ‘aim for a “nearly decomposable system” … in which the parts cohere internally and have minimal overlap’. Here, whilst features such as multiple solution strategies could have aspects that appear in the tasks, sequencing and norms sections of the OMF, the argument made is that the purposes were different. In tasks, multiple solution strategies related to the embeddedness in the planned tasks; in sequencing, it related to how the teacher drew attention to alternatives strategies in the management of the lesson trajectory; and in norms, it contributed to what counted as an explanation.

Errors also appeared in different sections. Within the discourse dimension of TOM, errors related to the teacher’s actions, such as offering a partial evaluation of “not quite” and moving onto another student, whereas within classroom norms the teachers’ and students’ responses to errors provided information about accountability. Errors also featured in the interpretation of classroom activity, as this was one of the more overt means of interpreting evidence related to the teacher’s professional noticing (Jacobs et al., 2010). As such, the pilot study indicated that the orientation provided by the OMF was sufficient to capture mathematically significant pedagogical moves with adequate separation of categories.

3.4.2.2 Transition to the Main Study: Coding of Lesson Transcripts

In the pilot study, the pedagogical profiles for the lessons were built up by populating the dimensions of the OMF with lesson specific analysis. However, the analysis of the lesson transcripts formed a substantial part of this work. Whilst the OMF categories were the reference point during the initial coding, to ensure substantive parts of teachers' practice were not excluded by this process it was important that codes were also allowed to emerge from the data (4.6.2). Consequently, in the transition to the main study, coding protocols were revisited. Whilst the OMF categories were inevitably 'kept in mind', the lesson transcripts were scrutinised to compare incidents so categories and their properties could emerge, which were then mapped to the OMF.

As previously discussed, the initial coding categorised talk as either mathematically related episodes or not related to mathematics. The latter category was not subject to further scrutiny. Whilst coding was an iterative process, for clarity an overview is provided first, after which the relationship to the OMF is discussed. Appendix 3 provides a timeline of activities, an example of the final transcription protocols and further illustrations of how the codes were developed.

(a) Coding Classifications

Mathematically related episodes were reviewed sequentially; three levels of interaction were noted:

- Whole class:
    - Everybody expected to focus on the same shared talk/activity
    - An 'effective student of mathematics' participated by listening and cognitive engagement (when not directly involved in exchanges).
- Semi-public:
    - Talk/activity involved a limited number of students but heard/observable by other students (e.g. an exchange across the classroom, so it could be heard by others but without signals that listening was expected).
- Local:
    - Talk/activity between individuals with no expectation of a wider audience.

When students were seated at their desks, working individually or in small groups, this was collectively referred to as seatwork.

The focus of the study led to teacher-talk being analysed in detail. Types of interaction were encompassed by the following categories:

- Turn-taking
    - teacher-initiated question-and-answer sequences (IRE)
    - facilitating peer-to-peer turn-taking
    - responding to student-initiated questions/comments
- Monologues
    - exposition related to mathematical ideas (teacher explanations)
    - instructions to students about required actions

Much of the talk fell into 'conversational' patterns of turn-taking, where alternate turns were taken by the teacher and individual students. Initiate-Response-Evaluate (IRE) (3.3.6.2) was the dominant form of turn-taking for all the teachers. Individual IRE exchanges were often followed by others to form extended whole-class question-and-answer sequences. Another common feature was the teacher extending their evaluative IRE turn by the inclusion of an explanation, summary or more extensive revoicing of the student's contribution. This occurred so regularly with all teachers that this variant was included within the IRE classification. In addition, there were occasions where more than one student responded to the initial questions in an IRE sequence before the teacher responded. There were also some teacher/student turn-taking exchanges, prompted by student-initiated questions or comments. In that context self-initiated indicated the student was not responding to a question or direct invitation to contribute. There was also some student-to-student turn-taking, prompted by a teacher's question but followed by multiple student contributions where they were responding to each other before a further teacher turn.

Whilst many whole-class episodes contained sequences of alternating voices, there were also episodes where comments by individuals were more self-contained and without the structural mechanisms of turn-taking. Classified as monologues, these were usually extended mathematical explanations or instructions given by the teacher, where meaning was less dependent on preceding or following utterances. However, this categorisation was not unproblematic, as the regular inclusion of explanations in teachers' evaluative IRE turns raised the question as to the boundary between IRE exchanges and monologues. Consequently, determination was based on the level of self-containment and duration; typically, monologues reported on multiple stages of a procedure and lasted over twenty seconds. Occasionally, there were extended student explanations classified as monologues as they met the self-containment and duration criteria.

Another aspect of talk related to the use of contextualised or colloquial language, as compared to inducting students into a more formal decontextualised mathematical discourse. The terms horizontal and vertical discourse were used to differentiate between these usages (Gellert and Straehler-Pohl, 2011).

Another overarching theme was the role of talk in the regulation of the lesson trajectory. Almost always, it was the teacher who determined the mathematical focus, providing steer for the mathematical direction of travel. The apparent mathematical foci were traced, and the teachers’ talk was categorised based on the different regulatory functions that over time shaped the lesson trajectory:

- Launch - when a new topic was introduced
- Direction - when the talk maintained the focus on the same mathematical feature
- Redirection – when the speakers’ foci diverged, and the teacher controlled the mathematical focus of the talk
- Student reasoning - when student reasoning was explored

Occasionally, student talk provided the mathematical focus, through peer-to-peer interactions at a whole-class level, student-initiated approaches or student explanations.

For talk where the teacher controlled the mathematical focus, there were variations in how the complexity of mathematics was regulated. Specifically, three categories were identified, namely:

- Simplifying - when the mathematics the students were asked to undertake was reduced in complexity.
- Processing - when the focus was implementation.
- Conceptualising - when attention was drawn to mathematically significant features or concepts.

In the simplifying category, the term funnelling was used when the reduction in complexity was to such an extent that students no longer had any meaningful mathematics to undertake (Wood, 1998).

In mathematically related episodes, the type of classroom talk varied. Some was ‘talk as mathematics’, in so far as the comments contained mathematical statements that could be considered part of the verbal register of mathematics (Duval, 2006). There were two aspects to ‘talk as mathematics’: the particular mathematical features attended to and the ‘level’ of mathematics. The levels were classified as:

- Recall - when mathematical facts were stated.
- Computation - when results of calculations or procedures were stated.
- Procedures - when what was done was described.
- Process - when the talk remained focussed on how particular mathematical problems were completed but comments had some level of meaning outside the specific example.
- Mathematical concepts - when links to underlying concepts or structures were made.

These classifications of classroom talk have a structure analogous to the notion of levels of cognitive demand. Stein et al. (1996, p.455) defined low-level cognitively demanding tasks as those with the potential to educe thinking classified as ‘memorisation’ and ‘procedures without connections to concepts’, with high-level tasks as those with the potential to educe thinking classified as ‘procedures with connections to concepts’ and ‘doing mathematics’ (see 3.3.3 for a detailed discussion). In the shared space of the classroom, the level of ‘talk as mathematics’ was considered the verbalisation of different types of mathematical thinking, with ‘recall’, computation’ and ‘procedures’ aligning with low-level cognitive demand, with ‘process’ and ‘mathematical concepts’ aligning with high-level demand.

On other occasions, there was ‘talk about mathematics’, where perceptions about the nature of mathematics and the learning of mathematics were implied. For example, Rowan, one of the participating teachers, commented “… ready to move straight onto the algebra… some of you practise doing it with the sums”. The implied hierarchy and the reference to practise was taken as communicating embedded ideas about the nature of mathematics and the nature of the learning of mathematics, respectively.

There was also some ‘talk about students’ that focussed on the student as a learner. This classification included notions of motivation and the expectations of an effective student of mathematics, where examples included awarding reward points for the accurate completion of work.

(b) Coding Summary

A summary of themes that emerged from the coding:

- Structure of Talk
    - Level of interaction:
        - Whole-class; semi-public; local
    - Type of interaction:
        - Turn-taking
            - Teacher: **IRE**; IRE variant (extended teacher turn); multiple R
                - I: Simple; Self-contained; Single/Multiple Solutions
            - Student-initiated: peer-to-peer; questions/comments
        - Monologue Exposition:
            - Teacher: explains; instructs
            - Student: explains
    - Register (vertical ⇔ horizontal)
- Steer: Regulation of Lesson Trajectory
    - Teacher led:
        - Launch, direction, redirection (simplifying, processing, conceptualising)
        - Focus: Mathematical horizon or student reasoning
        - Feedback: sharing solutions
    - Student led:
        - Student-initiated approaches
- Type of Talk
    - Talk as mathematics:
        - Mathematical focus
        - Level (recall; computation; procedure; process; mathematical concepts)
    - Talk about mathematics:

- The nature of mathematics
- The learning of mathematics

- Talk about students:
    - Effective student of mathematics
    - Motivation and engagement

(c) OMF: Lesson Transcripts, Interviews and Classroom Artefacts

The coding derived from the lesson transcripts was cross-referenced with the OMF, as summarised below (figure 3.13). In general, the coding was consistent with the framework, by which is meant the themes identified above contributed to different sections of the OMF, albeit with some overlap.

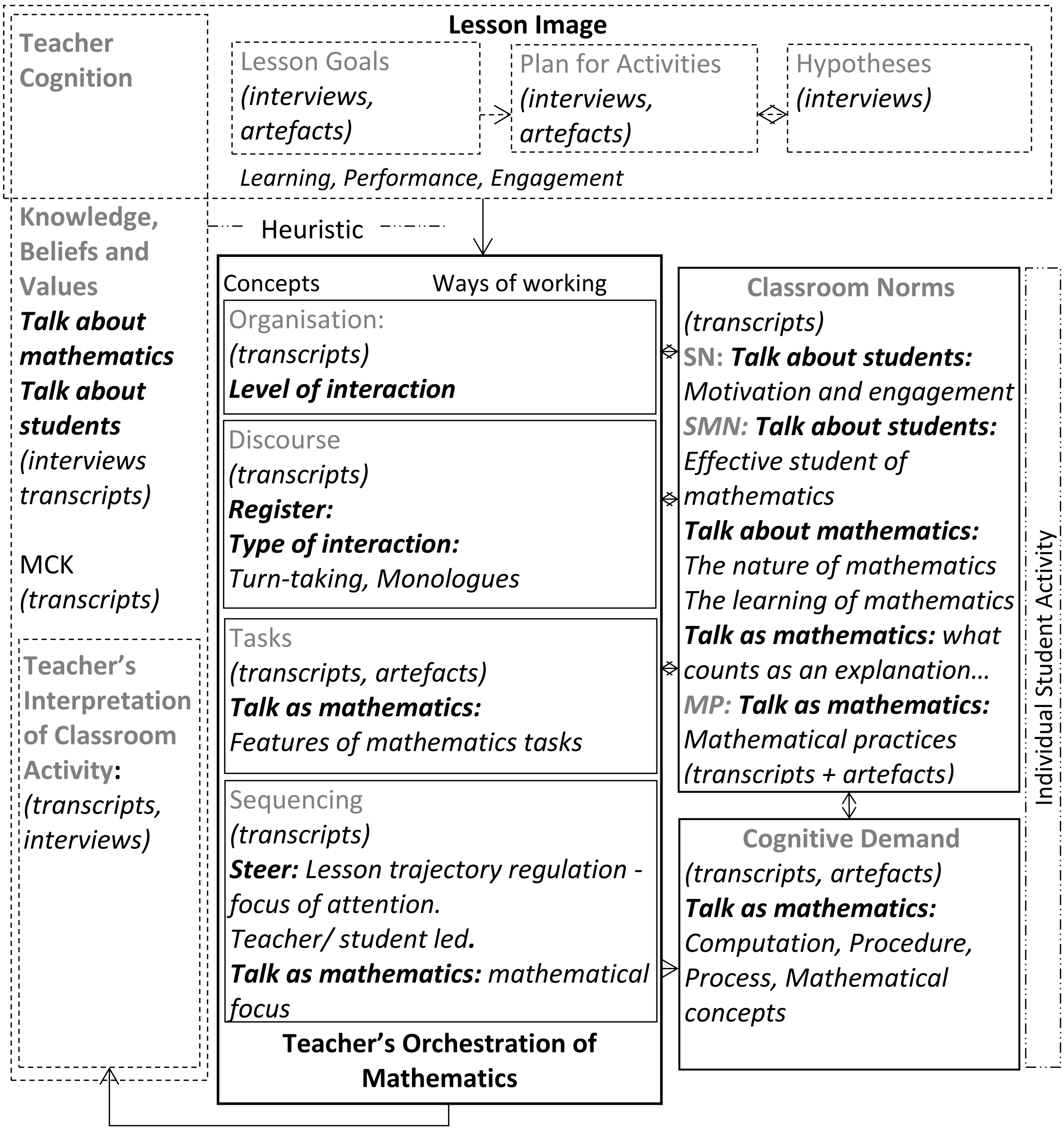


Figure 3.13: Mapping of coding themes to OMF

The pre- and post-lesson interviews and other data derived from the classroom artefacts were also analysed. The pre and post-lesson interviews, in particular, provided information relating to the lesson image and some insights into the teachers' beliefs.

The separate sections of the OMF have qualitatively different natures. For example, the central section, the Teacher's Orchestration of Mathematics (TOM), comprises the more visible actions the teacher takes, whereas cognitive demand involves a judgement of the mathematical potential of an activity. These differences were mirrored in the coding. For example, the structure of teacher talk was drawn from the structures apparent in the transcripts, whereas talk about mathematics involved reading meaning into comments made.

In terms of TOM, the level of interaction, types of interaction, and steer were mapped to the organisation, discourse, and sequencing dimensions respectively (figure 3.13). 'Talk as mathematics' was the most common type of talk and contributed to both the task and sequencing dimensions. Whilst the main themes from the lesson transcripts aligned with dimensions of TOM, features did emerge that also related to the wider framework. For example, the levels of 'talk as mathematics' contributed to cognitive demand, as this verbalisation was considered to have the potential to prompt that type of thinking in cognitively engaged listeners. Whereas all types of talk were considered as contributing to classroom norms. For example, Sam's statement "I should be hearing discussions" was coded as 'talk about mathematics' as this commented on the nature of learning mathematics. When considered in isolation, this talk was not a recurring pattern of behaviour that fulfils the expectations teachers and students have for the actions of others, and thereby, in of itself, is not a classroom norm. However, this talk was considered as having the potential to contribute to the building of a taken-as-shared view of what is expected in a mathematics classroom.

(d) The Influence of the Coding Process on the OMF

The coding process contributed to the refining of the properties of the dimensions of TOM and the wider framework. For example, the type of interaction and the regulation of the lesson trajectory categories influenced a clearer demarcation between the form of an interaction and its function, resulting in revisions of the discourse and sequencing dimensions, as discussed in detail below. Also, additional descriptors were added. For example, to capture the variations in turn-taking patterns the coding identified, IRE was expanded to include variants with extended teacher turns and student-initiated interactions. In addition, subcategories were added to the questioning description.

At the pilot stage, the discourse dimension was adapted when the procedure/concept description was added (3.4.2.1 & figure 3.11: model C). This was the first attempt to capture the dual aspects of form and function that are inherent in any interaction. During the coding process, the demarcations between the discourse and sequencing dimensions became more clearly defined. The discourse dimension became more tightly focussed on the form of interactions, and was renamed as discourse patterns to reflect this, whereas the function of those interactions was captured in the sequencing dimension. Consequently, the procedure/concept label was removed as this was considered a function, and instead was captured in the more clearly defined sequencing dimension.

In his detailed discourse analysis, Drageset (2014) categorised teachers' use of students' contributions in terms of their function, with labels of redirecting, progressing or focusing. In this study, the coding categorised similar types of functions of the discourse in relation to the regulation of the lesson trajectory, using the labels launch, direction/redirection (simplifying, processing, conceptualising). This led to the refinement of the sequencing dimension. The key change was the inclusion of descriptors to capture the management of the lesson trajectory in relation to the focus of attention. In particular, the launch, direction and redirection descriptions captured how the lesson trajectory was managed. Student reasoning or the teacher's mathematical horizon captured whose reasoning was the focus of that attention. The 'talk as mathematics' category led to the mathematical focus being included in the sequencing dimension, in terms of how the focus on mathematically significant features evolved over time.

Categories that were retained were also scrutinised, and the process highlighted the complexities involved in achieving a structure 'in which the parts cohere internally and have minimal overlap' (Schoenfeld, 2013b, p.614). For example, the register classification relates to the use of vertical or horizontal discourse (3.3.6.3) and could be viewed as simply the form of language used. However, there is also the perspective of using language as a means to induct students into vertical discourse, and thereby be considered a function of that discourse. Here the register classification was retained in the discourse dimension but could be seen to be at variance with the tightening of the focus to discourse patterns discussed above.

Similarly, the separation of the task and sequencing dimensions was not straightforward, as task features changed as they were enacted in the classroom. The form and function approach for discourse cued the clarification of these properties. The task dimension related to the more static mathematical features inherent in a task, with the sequencing dimension focussed on the mathematical function of those tasks under the teachers' in-class deployment. However, the dynamic, interconnected nature of classrooms meant that the challenge of classifying teachers' pedagogical activities in meaningful ways remained throughout the study.

3.4.2.3 OMF: Final Iteration

As discussed above, the main categories in the OMF were retained during the analysis of lessons in the pilot study and the coding processes, but the detailed descriptors were refined. During the transition to the main study, the final adjustments to the OMF were made as I reflected on the data collection and analysis, with the final iteration shown in figure 3.14 (model D).

The main change related to the inverted L of Teacher Cognition. This change originated in the connectedness between the teachers' talk about the individual lessons and their more general comments. The inverted L was introduced to connect the teacher's knowledge and lesson image elements of the teaching cycle to emphasise the interrelated nature of these aspects of teachers' cognition (figures 3.5, 3.6 & 3.7). This included embedding the teacher's interpretation of classroom activity within teacher cognition as a means of acknowledging the influence of psychological processes on what is noticed and how noticed activities are interpreted (Mason, 2015b). The hashed lines for 'Teacher's Cognition' was intended to make a clearer distinction between the Teacher's Orchestration of Mathematics (TOM) section, which represents the more visible in-class activities, from the less visible but highly influential aspects of teachers' internal psychological processes.

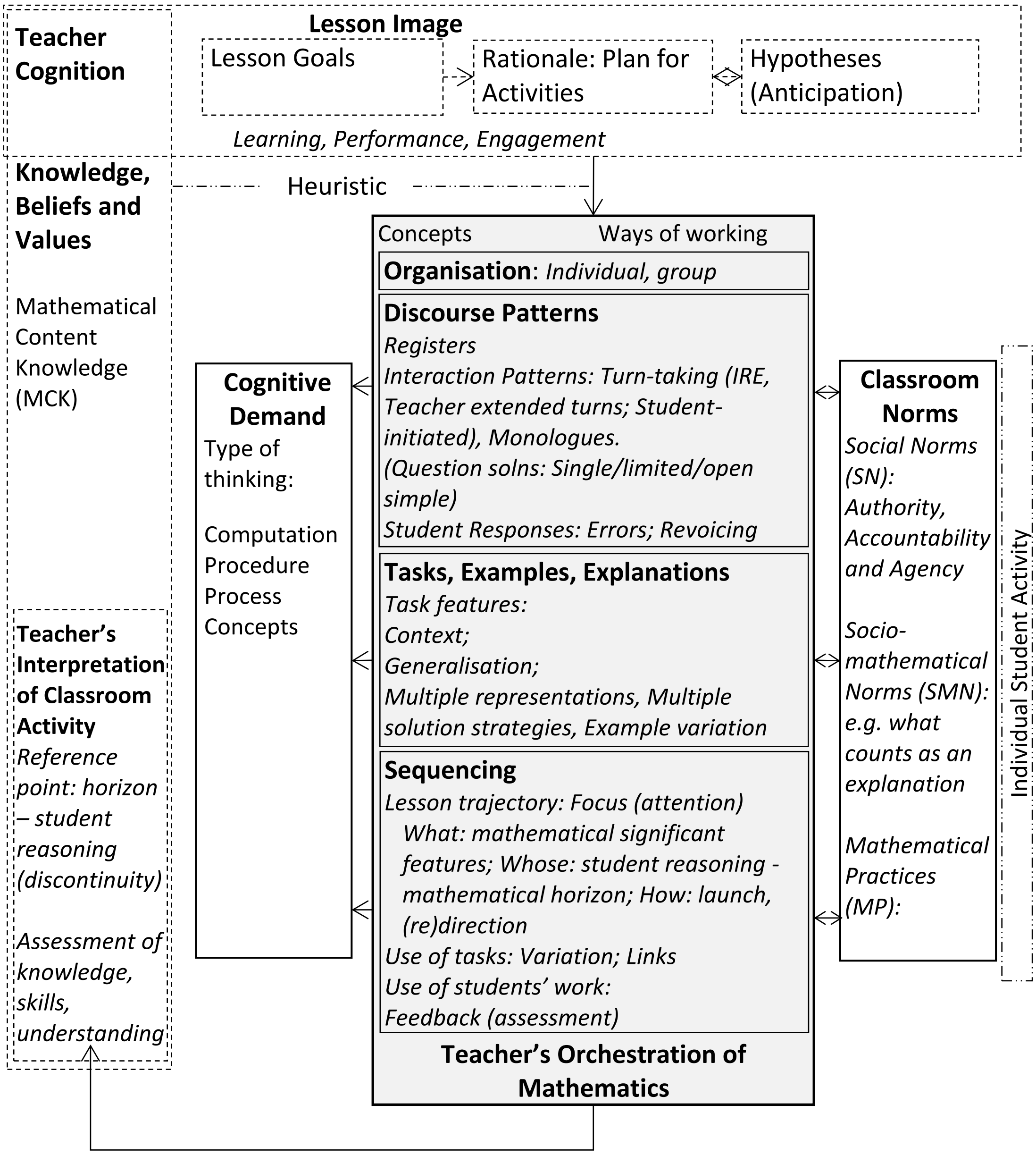


Figure 3.14: Model D – Orchestration of Mathematics Framework (OMF) – Final Iteration

During the main study, the layout was sometimes adapted for practical reasons, such as space for writing when presenting the summary OMFs (e.g. figures 4.3 & 5.21). There were also some additional details added to the task feature descriptions to differentiate between the potential features of tasks and those planned for use by the teacher (see 4.4.1). Whilst details were added, these were viewed as being conceptually congruent to the OMF above.

3.4.3 OMF: Relationships to Other Observation Frameworks

As previously discussed, the development of the OMF was an iterative process, with figure 3.14 (model D) the final version. Whilst the previous sections have sought to outline the development process, the very nature of a linear narrative could mask the ongoing interplay between my analysis of the data and my understanding of the literature as embodied in the OMF as a conceptual framework. Moreover, whilst the OMF might represent a distillation of my understanding, the model remains untested outside of this study, and this includes how others may interpret this framework. The following section compares the OMF to two published observation schedules, in anticipation that this comparison will provide alternative ways to understand the OMF, whilst also providing the opportunity to consider the OMF's affordances and limitations.

In recent years there have been efforts to compare classroom observation frameworks, many of which are focussed on the evaluation of teaching through studying the quality of instruction (e.g. Ingram et al., 2018). For example, in 2018, a ZDM issue published papers that analysed the same three videos of mathematics lessons. Twelve papers reported their analysis using different frameworks; these approaches were then compared by the editors (Charalambous and Praetorius, 2018). Praetorius and Charalambous (2018) argued this comparison allowed the frameworks to be viewed synergistically, developing a shared understanding that would allow future researchers to build on prior work more effectively. Whilst it is beyond the scope of this study to undertake comparisons with other frameworks drawing on the same data sets, other than that undertaken in the pilot study with the TIMSS material (3.4.2.1), the following section reflects on the similarities and differences between the OMF and two of the mathematics specific frameworks, namely the TRU (Teaching for Robust Understanding) framework (Schoenfeld, 2013a) and MQI (Mathematical Quality of Instruction) (Learning Mathematics for Teaching Project, 2011). Both TRU and MQI are mathematics specific lesson observation frameworks that intend to offer a holistic view of a whole lesson. Whilst both provide measures that report on the quality of instruction, the MQI focusses on the teacher and TRU is more student oriented (Praetorius and Charalambous, 2018).

3.4.3.1 Mathematical Quality of Instruction (MQI)

As the design team for MQI, Learning Mathematics for Teaching Project (2011) outlined the development of the framework and offered examples of how codes were applied. They described six dimensions: the richness and the development of mathematics; responding to students; connection to mathematics; language; equity; and the presence of unmitigated mathematical errors. For each dimension, there were scales that contained codes in the form of scoring rubrics. The full MQI coding rubric was only accessible to those that undertook specific training. As both time and cost prohibited access, comparisons here are made in reference to their descriptions of the scales in the published paper. For all of their dimensions, the majority of the descriptions appeared to be represented in the OMF. For example, the discussion of the richness dimension contained the following:

> Our instrument contains seven codes to capture the ‘‘richness’’ of the mathematics in a lesson. These include the presence of multiple mathematical models in classrooms (e.g. symbols and visual representations); links made between multiple models; mathematical explanations; mathematical justifications; and explicit talk about mathematical language, reasoning, and practices.
>
> (Learning Mathematics for Teaching Project, 2011, p.34)

The multiple mathematical models could be captured by multiple representations in the OMF task dimension, with the links between models captured by the sequencing dimension. Whereas mathematical explanations and justifications would depend on who is undertaking the talk. These could contribute to classroom norms in the OMF, in relation to the expectation for explanations and justifications, whilst also contributing to cognitive demand, in terms of the level of mathematics. The level of explicitness could be captured in the task dimension, if those features were embedded in the activity, or in the sequencing dimension, if the teacher drew attention to those features as the lesson unfolded.

A similar pattern occurred when the other dimensions were considered. The majority of descriptions could be matched to elements of the OMF, such as language to the register element of the OMF discourse pattern dimension. The MQI equity dimension was described as having codes related to teacher explicitness, which could be captured in the sequencing dimension, although it was not fully transparent how the explicitness in the MQI equity dimension was distinct from the richness dimension discussed above. However, another strand of the MQI equity dimension, namely the opportunities for students to participate, does appear to be absent from the OMF. The MQI approached parsed the lessons into five-minute sections and observers recorded if students were engaged in mathematics. In this study, the breakdown of whole class activities (e.g. figure 5.6) indicated the relative amounts of mathematically related activities to non-mathematically related, so engagement was partially captured, though indirectly.

When the comparison is considered from the perspective of the OMF, the details of discourse patterns, the explicit identification of classroom norms and organisational approaches did not appear in the commentary offered by Learning Mathematics for Teaching Project (2011). However, without access to the complete rubrics, firm conclusions cannot be drawn about equivalent notions being present or not in the MQI.

Whilst the majority of MQI descriptions for each dimension mapped to the OMF, they mapped to many different parts. In other words, there were no clear links between the categories established for the MQI and the OMF. Whilst the non-alignment of dimensions raises challenging questions about the theoretical basis for the structure of the OMF, especially considering the scale of the MQI project, this issue is not unique to the OMF. This lack of consistency regarding the conceptualisation of dimensions has been found when a range of mathematics specific observation frameworks have been compared, including the MQI when this has been compared to other well-established frameworks (Schlesinger and Jentsch, 2016; Praetorius and Charalambous, 2018).

In the early stages of this study, the MQI was considered but the evaluative perspective did not align with the research questions. In particular, the codes were intended to capture if particular features were present or not, and whether this was appropriate or not appropriate. This latter element held a presumption that the observer could judge what should have been happening. In this study, the purpose was to describe and interpret so that level of judgment was deemed inappropriate. However, the Learning Mathematics for Teaching Project (2011, p.30) stated ‘the word quality can refer both to the distinctive character of something or to its rank, level, or grade. We use it primarily in the former sense’. In other words, it may have been possible to use the codes in a less evaluative manner. However, the orientation of the MQI to the reform agenda in the US suggested that certain classroom activities that may feature in a UK setting, such as ‘drill and practice’, may not have been captured by the codes (Praetorius and Charalambous, 2018). Consequently, with the full rubrics not readily available, the possibilities of using the MQI was not pursued in this study.

After the conclusion of the funded research project, which included the publication by Learning Mathematics for Teaching Project (2011), the MQI appears to have shifted in focus. The Center for Educational Policy Research, at Harvard University, currently publishes the MQI framework on their website (CfEOR, 2019), where they offer MQI training and access to a video library. This appears to be orientated towards practicing teachers and there have been some changes to the MQI dimensions. The information provided does not indicate whether the changes were due to developments in the conceptual framework from a theoretical perspective or were in response to the different context, or some combination of the two. However, this served as a reminder that any shifts in use or user of the OMF would require a re-examination of its structure. For example, whilst recording of lessons is becoming more common in schools, its use is not ubiquitous, and even when used teachers are unlikely to have the time to commit to lengthy analysis.

3.4.3.2. Teaching for Robust Understanding (TRU)

The TRU framework was developed by a research team led by Alan Shoenfeld over a three-year period (Schoenfeld, 2013a). The overarching structure has five dimensions that Schoenfeld (2013a, p.607) argued ‘may have the potential to be a necessary and sufficient set of dimensions for the analysis of effective classroom instruction’. The dimensions and associated descriptors are summarised below:

The Mathematics

How accurate, coherent, and well justified is the mathematical content?

Cognitive Demand

To what extent are students supported in grappling with and making sense of mathematical concepts?

Access to Mathematical Content

To what extent does the teacher support access to the content of the lesson for all students?

Agency, Authority, and Identity

To what extent are students the source of ideas and discussion of them? How are student contributions framed?

Uses of Assessment

To what extent is students' mathematical thinking surfaced; to what extent does instruction build on student ideas when potentially valuable or address misunderstandings when they arise?

(adapted from Schoenfeld, 2014, p.408)

[There is some variation in the descriptions used in different publications.]

From inception, one articulated goal was for the TRU framework to be usable in teachers' professional development and there are a number of professionally orientated publications (e.g. Schoenfeld et al., 2014b). For each dimension there is a scoring rubric with three levels, and further more specialised rubrics have been written for whole class activities, small group work, student presentations and individual work, as well as topic specific rubrics for algebra (Schoenfeld et al., 2014a).

On an initial inspection, some commonalties and differences between TRU and the OMF could be posited. For example, cognitive demand is named as a key dimension in each framework. However, the different orientations, namely the student versus the teacher perspectives, are apparent. In TRU, the observer is asked to make an inference about the students' type of engagement, from memorised procedures to productive struggle (Schoenfeld et al., 2014a). With the OMF, cognitive demand is conceptualised in terms of potential. Whilst inferences could be made about individual students based on their verbal contributions to whole-class interactions, the coding relates to the mathematical potential if the students chose to engage with the activity.

The agency, authority and identity dimension in TRU appears to align with the social norms element of the OMF, at least in terms of the labels used. As before, the TRU framework specifically refers to students, whereas the OMF makes reference to all classroom norms. However, when the scoring rubrics for TRU are inspected, other structural differences of the two frameworks become apparent. For example, the lowest descriptor in TRU states ‘The teacher initiates conversations. Students’ speech turns are short (one sentence or less), and constrained by what the teacher says or does.’ (Schoenfeld et al., 2014b, p.2). These types of interactions would be captured in the discourse patterns dimension of TOM. As previously stated, the purpose of the OMF is to describe and interpret teachers’ pedagogical moves, where the TOM section captures the more visible in-class activities. As such, these descriptions themselves are fundamental to the analysis. It appears that in the TRU framework, these types of descriptions are an interim stage for the generation of scores for evaluation. This student orientated evaluative stance influenced the decision not to pursue the use of TRU in this study.

The mathematics and use of assessment dimensions of TRU appear to be most closely aligned with the tasks and sequencing dimensions of TOM, respectively. For example the middle rubric descriptor for the mathematics states ‘Activities are primarily skills oriented, with cursory connections between procedures, concepts and contexts (where appropriate)’ (Schoenfeld et al., 2014b, p.2). This appears to align with the task dimension in the OMF, but the making of connections could be captured by the sequencing dimension and the resulting potential of any activities would contribute to cognitive demand. This does reflect the challenge found of clearly delimiting between dimensions of the OMF (3.4.2.2). Moreover, whilst there appears to be more links between the OMF categorisations and TRU than with MQI, there is still significant non-alignment of dimensions. Consequently, the questions about the theoretical basis for the structure of the OMF are not resolved through this comparison. Although, as with MQI, similar issues have been raised when TRU was compared to other frameworks (Praetorius and Charalambous, 2018).

The most apparent difference between the OMF and TRU relates to the Access dimension of TRU; a comparable difference was also found with the MQI in relation to equity of access. With the OMF, the use of the notion of an ‘effective student of mathematics’ structurally removed differential access by different students. The argument made was that this allowed the OMF to focus on the teacher, which would allow the research questions to be investigated. However, student engagement has been reported as a differential feature of different sets (2.4.2), so the fact the OMF does not capture teachers’ moves in relation to equity of access for all students must be considered a limitation. That said, as Praetorius and Charalambous (2018) argued, there is considerable variation in the pedagogical features captured by different observational frameworks. As such, the hope is that the OMF can add a complimentary perspective to the interpretation of classrooms, albeit with acknowledged limitations.

3.4.4 OMF: Affordances and Limitations

The focus of this study was on describing and interpreting teachers’ pedagogical moves, so that any shifts in practice between sets could be explicated. An instrument that could build a holistic view of a typical mathematics lesson was sought. Most of the previously published mathematics-specific observation frameworks focussed on the quality of instruction (Praetorius and Charalambous, 2018) and those evaluative orientations did not align with the research questions in this study. As outlined in this chapter, this led to the development of the OMF.

3.4.4.1 Research Perspective

The attempt to capture the complexity of a mathematics lesson and provide a coherent picture of a teacher’s pedagogical activities, frames both the affordances and limitations of this study. In many respects, ‘the lesson’ is not an ideal unit of analysis. When any of the elements of the OMF are considered, more in-depth research has been undertaken in those fields and more comprehensive descriptors should therefore be possible. Moreover, finer grained analysis of classroom data could have been undertaken, such as conversation analysis of interactions. On the other hand, classrooms are dynamic environments and the relationship between features is often as important as individual elements. Therefore, there appears to be an inherent tension between the level of detail about particular aspects of the lesson under study, and a more holistic view that allows the interrelated nature of classroom activities to be considered.

Moreover, a lesson is only a snapshot of the teacher's pedagogical practice and the students' experience of mathematics is built up over years. However, 'the lesson' has been used in other observation-based studies (e.g. Hiebert et al., 2003b) and teachers are expected to make judgements about learning from single live lesson observations. Therefore, providing a coherent picture of a teacher's pedagogical activities for a lesson appears to be a worthwhile goal. The question posed is whether the OMF framework balances the tension between detail and interconnectedness with sufficient coverage of mathematically significant events. It is believed that the findings of this study provide evidence of the affordances of the OMF in this regard.

It has been acknowledged that no single model captures the complex and dynamic nature of the classroom (e.g. Derry et al., 2010). For example, Herbel-Eisenmann and Otten (2011) argued that 'studies of mathematics classroom discourse need to attend further to the mathematics being construed in the discourse' (p.452). Likewise, as powerful a construct as sociomathematical norms have become, they capture what is acknowledged as legitimate rather than the mathematical nature of any reasoning (Kaldrimidou et al., 2008). The intent underpinning the OMF is to allow different lenses to be made available as typical lessons unfold. So, for example, the analysis of the mathematics made available to students by drawing on variation theory is integrated with social perspectives, such as sociomathematical norms. It is not argued that the OMF captures all, but rather it provides a holistic pedagogical profile of a mathematics lesson that allows different perspectives to be interpreted in relation to each other. Thereby the OMF contributes to the debate, and indeed offers a model, as to how different theoretical perspectives can be brought to bear on the interpretation of pedagogical activities.

A range of lesson observation frameworks have been developed and the comparison of these frameworks has been the focus of a number of papers in recent years (Schlesinger and Jentsch, 2016; Ingram et al., 2018; Praetorius and Charalambous, 2018). On a smaller scale, section 3.4.3 compares the OMF to two established frameworks, namely MQI and TRU. The comparison demonstrated some areas of commonality. In particular, a number of classroom activities, such as the use of multiple representations, were included in the OMF and MQI and/or TRU. However, in addition to having different purposes, the categorisations that operationalised the conceptual frameworks differed considerably. Whilst this was also found to be the case when other published frameworks were compared (Schlesinger and Jentsch, 2016), this raises questions about the theoretical and empirical justification for the structure and categorisations within this or any other bespoke framework.

3.4.4.2 Teacher Professional Development Perspective

Many of the lesson observation frameworks have been designed to measure the quality of instruction (Charalambous and Praetorius, 2018). These evaluative frameworks often include a professional development element, where it is posited that teachers can use the frameworks to improve practice (e.g. Learning Mathematics for Teaching Project, 2011; Schoenfeld, 2014). For example, it was intended from the outset that the TRU framework would be used in professional development, and there are both professional and research orientated publications (e.g. Schoenfeld et al., 2014b). The descriptions of the dimensions did vary between publications. For example, the mathematics dimension descriptor was given as:

> 'How accurate, coherent, and well justified is the mathematical content?' (Schoenfeld et al., 2014b, p.2)
>
> 'To what extent is the mathematics discussed clear, correct, and well justified (tied to conceptual underpinnings)? (Schoenfeld, 2014, p.616).

Whilst the origins of these changes were not made explicit, it appeared that the language was adapted for use by teachers. Moreover, with the MQI framework, there have been changes to the dimension and wording from the research orientated paper by Learning Mathematics for Teaching Project (2011) to the current professionally orientated MQI website (CfEOR, 2019).

These changes highlight the distinction made between using frameworks for informing practice versus use in research. The argument made for the OMF is that its descriptive power has the potential to contribute to teachers' understanding of their practice, an essential prerequisite for any professional development. However, the adaptions made to both TRU and MQI signal that adaptions would need to be made for the OMF to be accessible and useful for teachers. For example, in terms of accessing appropriate data, the more static nature of the task dimension of TOM would make this dimension relatively accessible to teachers. Whereas recognising discourse patterns, that are normalised and happen in-the-moment, would be harder without access to recordings and time for analysis.

Recently, Watson (2019) drew attention to the role algorithmic reasoning plays in teachers' in-class decision making. By the very nature of this more intuitive reasoning, it is less available for teachers to review. Similarly, teachers develop normative patterns of participation, which they are less likely to notice due to their more autonomous nature. Consequently, it is more difficult for teachers to consider the pedagogical implications of those actions. In the OMF, the inclusion of 'heuristic' in the teaching cycle and the explicit reference to classroom norms (figure 3.14: model D) allows attention to be drawn to these less accessible aspects of teachers' pedagogically related activities. Moreover, stepping back from evaluation provides the space for building a shared understanding of classroom activities and a closer interrogation of mathematically significant events.

The wider value of the OMF can only become apparent with future use, but with the framework being structured from the perspective of the teacher, it could contribute to teachers' professional development as a planning and review instrument. One starting point could be for a teacher to map an existing lesson plan to the OMF, which would have the potential to draw attention to alternative perspectives. For example, typical planning documents refer to the mathematical tasks to be completed, which could be used to populate the lesson image and the task dimension of TOM. The findings in this study indicate teachers rarely make explicit links between examples and general concepts; the descriptions in TOM could draw attention to the potential of increasing levels of explicitness. On a broader scale, the OMF draws on a range of theoretical perspectives, so the framework could provide a structure for teachers to relate theory with practice. For example, a teacher may be aware of the IRE pattern of interaction, but the OMF could prompt them to consider the impact this has on classroom norms and the taken as shared view of what constitutes mathematics in their classroom.

In this study, the decision to focus on description and interpretation, rather than evaluation, was made in response to the nature of the research questions. In relation to teachers' professional development, this decision might be seen as a counter to a performativity culture. However, that might obscure the value of finding more effective means of building a shared understanding of what might be happening in a mathematics classroom. Fifteen years ago, Wiliam and Bartholomew (2004, p.280) argued that teachers' actual classroom practice was 'weakly theorized'. Whilst there has been considerable research undertaken since then, I would argue there is still work to be done to understand how a range of theoretical perspectives can be brought to bear on the interpretation of classroom activities, and in a manner that is accessible and useful to teachers; I would hope the OMF contributes to that debate.

4. Methodology

4.1 Introduction

The motivation for this research originates in the retention of setting for mathematics in many secondary schools, and in spite of evidence of the negative effects of this policy (e.g. Ireson et al., 2002; Forgasz, 2010). As there tends to be variation in both the curricula and teacher demographics when different sets are considered (e.g. Hiebert et al., 2003b; Wiliam and Bartholomew, 2004) (2.2), this research seeks to contribute to the field by focusing on how individual teachers adapt their practice when they teach different groups of students. This led to the development of the research questions discussed in the next section and the associated research design.

4.2 Research Design

This section outlines the rationale underpinning the development of this study and the approaches taken. The following sections outline in more detail the methods employed and the affordances and limitations of approaches taken.

4.2.1. Research Questions

RQ: How does a teacher orchestrate mathematics for different groups of students?

> RQa: How does a teacher shift their pedagogical approaches when teaching different groups of students?
>
> RQb: How does the character of the mathematics made available to students vary when a teacher teaches different groups of students?

As previously defined, ‘orchestrate’ means all the actions a teacher takes to select, organise and make available to students the mathematical tasks used in class, and the management of the classroom discourse, including student contributions (3.3.2).

4.2.2 Theoretical Background

In seeking to answer these research questions, teachers’ activities in relation to mathematics and students need to be interpreted. In common with many mathematics education studies, here classroom interactions are considered to be complex and dynamic, with the cognition related to the learning of mathematics equally complex and multifaceted (e.g. Cobb et al., 2009; Jaworski, 2012). An interpretivist paradigm underpins this work; the goal is to understand and interpret classroom behaviours that offer insights to compliment other perspectives, rather than to provide a single generalisable truth.

Research has offered a wide range of models to help explicate the learning of mathematics, from more global psychological processes, such as Piaget's logico-mathematical thinking, to more local frameworks, such as the duality of process-object approaches (Pegg and Tall, 2005). Furthermore, the analysis of classroom norms is an example where a social perspective can contribute to the understanding of learning. Taking the lead from researchers such as Cobb et al. (2009) and Illeris (2003), here both internal cognitive processes and social interaction are taken to be integral and complimentary elements of learning. In other words, a constructivist perspective is considered not only to be compatible with a social constructivist perspective, but interdependent to the extent that each perspective offers the background against which the other is interpreted (Cobb et al., 2001).

4.2.3 Development of the Research Design: Case Study

In order to explore teachers' pedagogical practices, data from classrooms are required. As classroom interactions are an integral part of that process, then lesson observations need to be part of that data. Whilst there is some limited open source material that includes video recordings of lessons, no publicly available data was found that included multiple classes for the same teacher in England. Consequently, primary data collection from lesson observations forms a substantial part of this study. The nature of the data and subsequent analysis determined that this would be a qualitative study. A case study approach has been undertaken, as this is an empirical study where the complexity of classrooms and the interrelated nature of activities necessitates an in-depth exploration of each classroom. Moreover, the purpose is to explore a complex situation in a context that is bounded by time, with lessons being of a fixed duration, and by space, that of the classroom (Thomas, 2015). In addition, there is no control over the events, which cannot be separated from the context, and the research questions (RQ) are of a 'how' form that focus on contemporary events (Yin, 2018).

As discussed in detail in chapter 3, a conceptual framework for the interpretation of teachers' pedagogical practice was developed. The result was the Orchestration of Mathematics Framework (OMF), designed to capture teacher activity in meaningful ways. Whilst a number of published studies have developed observational frameworks, some focussed on measuring the impact of particular initiatives (e.g. Remillard and Bryans, 2004) and others on specific aspects of lessons, such as cognitive demand (Stein and Smith, 1998). At the outset of this study, it appeared these did not have the breadth of coverage to capture and describe the range of practices that literature indicated would be found in 'typical' secondary mathematics lessons, particularly those activities thought to be prevalent in lower attaining sets. For example, the mathematical quality of instruction (MQI) instrument (Learning Mathematics for Teaching Project, 2011), developed as part of a large-scale study, was one of the instruments considered for use in this study. Whilst the associated reliability and validity measures would have been of benefit, it was decided that the focus on quality, as distinct from pedagogical processes, might preclude key elements of teachers' practice from being captured.

Consideration was given to using other mathematics specific frameworks, such as the Knowledge Quartet (Rowland et al., 2005). In that case, whilst acknowledging the framework has been employed in a wider range of contexts in more recent years, the original focus was on primary student-teachers and mathematical content knowledge. As such, this did not appear to be an appropriate instrument at the outset of this study. Recently, there have been some comparisons of observation frameworks and protocols (e.g. Boston et al., 2015; Ingram et al., 2018), which have demonstrated the range of instruments available and discussed their purpose and use in a range of contexts. Whilst it needs to be acknowledged that use of a pre-existing framework could have brought benefits of tested validity and reliability, a different context with a different researcher meant there would be no way to guarantee adoption with sufficient fidelity to automatically 'transfer' validity and reliability of previous measures. As this author was not aware of an approach that captured the sequences of planning and management of the lesson trajectory from an interpretative rather than evaluative perspective, the decision was made to develop the Orchestration of

Mathematics Framework (OMF). This added another dimension to the study, as the efficacy of the OMF would need to be considered.

4.2.4 Scale

This was a solo research project, which did present some natural limitations in terms of the scale of the research. In particular, the number of lessons observed and analysed, and the number of participating teachers were restricted due to time requirements and difficulties in recruitment. More significantly, only one person was substantially involved in the observation and analysis of lessons.

4.3 Methods

4.3.1 Case Study

In order to explore the complexities of a mathematics classroom, in-depth research into a small number of cases was undertaken (Thomas, 2015). A nested model was used, with two different classes for the same teacher studied in parallel, with three teachers recruited in total. The subject of the cases was the mathematics class taught by participating teachers, with the analytical frame the influence the teacher's pedagogical approaches had over the nature of mathematics made available. As mathematics classes convene for short periods of time at regular intervals, a number of separate lessons were studied.

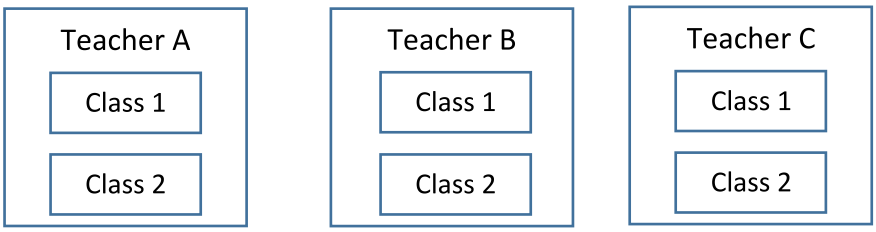


Figure 4.1: Nested Parallel Case Studies

The ultimate goal of this study is to contribute to the wider debate about the nature of mathematics made available to different groups of students and the pedagogical actions taken by teachers that contribute to any differences. As the intention was to offer insights into issues that go beyond the individual cases, with data drawn from a number of sources, this could be described as a collective case study (Stake, 1994). However, an in-depth understanding of each class was needed, both to consider the efficacy of the framework and to analyse any shifts in pedagogy when the two classes for each teacher were compared. Therefore, the first stage was to develop an understanding of the mathematical characteristics of each class before comparisons were made. The latter stage was to consider all three teachers; whilst the intention was to offer wider insights, the variability in teaching approaches meant that the identification of similar shifts in pedagogy for all three teachers was not anticipated. Rather, it was hoped that the detailed description and analysis would allow the reader to understand how potentially subtle changes in teaching approaches could occur and the impact this might have.

4.3.2 Participating Teachers and Classes

Due to the scale of the research and the in-depth study of particular classrooms, only a limited number of teachers and classes were involved. One decision was whether teachers from a range of schools should be approached or whether two or three teachers from the same school should be sought, as both approaches have advantages and disadvantages relating to the influence of the school context. However, finding teachers willing to commit the time, in schools prepared to support classroom-based research, was problematic, especially when consent from all parents and students in the participating classes also had to be obtained. Consequently, the decision was made pragmatically, based on the teachers I was able to recruit. I drew on personal contacts, developed through my role as a university-based Post Graduate Certificate of Education (PGCE) tutor, to recruit three mathematics teachers from two local schools where mathematics was taught in sets.

A number of limitations arose from this opportunistic recruitment. A regular feature of my tutor role is the observation of student-teachers on school placements, which are usually undertaken jointly with the student-teachers' mentor. This may have influenced the expectations of the participating teachers, especially as they had all been mentors of student-teachers and two had also been student-teachers on the PGCE course where I was a tutor. Consequently, previous discussions could have left them with beliefs about what I consider a 'good' lesson and they may have selected classes or planned lessons with those features in mind. In addition, they may have held an expectation that I would make judgments about their teaching. Teachers who volunteer for research projects may not have the most typical profile. Indeed, my personal knowledge of the participating teachers indicated they were more proactive than most in terms of engagement with professional development opportunities.

The teachers were asked to select classes of a similar age but different attainment profiles. After gaining agreement from the head teachers, who act as the schools' gatekeepers, parents and students from two classes for each teacher were approached. The final decision about which classes were selected and when observations could be undertaken resided with the teachers. Practical considerations, such as my availability, also influenced which lessons were observed. This did lead to a variation in the number of lessons observed with each teacher (figure 4.2). One teacher, Joe, was observed once with each class. The second, Rowan was observed twice with each class two weeks apart. In addition to two lessons observed as part of the pilot study, the third teacher, Sam, was observed five times with each class, three of which were sequential lessons taught in the same week. Rowan and Sam were at the same school.

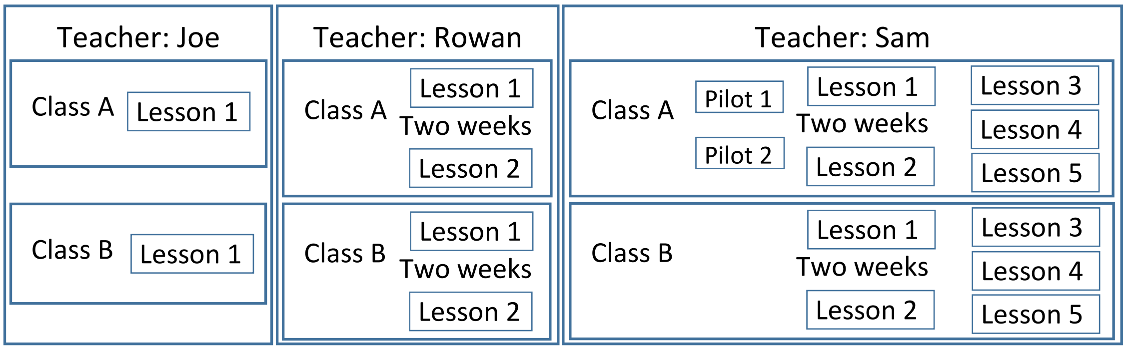


Figure 4.2: Nested Parallel Case Studies with Lessons

4.3.3 Ethics

Ethical approval was obtained from the University of Leicester (appendix 1.1) and BERA ethical guidelines have been followed.

The majority of students in the participating classes were under 16 years of age and were therefore considered vulnerable. However, the research was undertaken in school settings, where safeguarding policies and practices are established and are designed to protect the welfare of students. The research was approved by the schools' head teachers, who had overall responsibility for the welfare and education of the students, and acted as gatekeepers, overseeing the recruitment of all participants. Informed consent was obtained from all participants, with approved standardised letters of consent used for parents/students (appendix 1.2).

The research involved the exploration of normal school practice, so there was minimal change to the classroom experience for students. Within schools' normal practice, lessons are regularly observed, and even though recording of lessons is less common, its use is increasing and happens in both schools on occasions. I regularly work in schools, hold a current Disclosure and Barring Service (DBS) certificate and have a good understanding of how to minimise the impact of observations on student learning.

The small number of participating teachers and their unique nature of participation, from the number of lessons recorded to the nature of the topics taught, would mean that teachers could recognise their part in the study. Consequently, care was needed to ensure that all the teachers gave informed consent. The analysis of the lessons in the pilot study was reported at a mathematics education conference, and I was very conscious that I was discussing an individual teacher. This prompted me to recheck the ethical issues relating to the participating teachers. In particular, this reinforced the need for clarity in the discussions I held with each participant, especially in relation to their expectations about the reporting of the analysed lessons. They had all undertaken small-scale classroom-based research projects as part of their own PGCE courses, and thereby had some prior knowledge of ethical considerations relating to research and the reporting of qualitative data. I ensured they were aware of the level of detail that would be reported in a doctoral thesis and the type of discussions that would be undertaken. In addition, I ensured that they understood that the unique nature of the lessons would mean that they could recognise their participation.

The making of evaluative judgments of teaching is not part of the study, but when notions such as cognitive demand are discussed, phrases such as ‘low cognitive demand’ could be read as a negative judgement rather than an analysis of the data. Moreover, many pedagogical practices could be considered as being rooted in wider curriculum expectations that characterise mathematics teaching in England, rather than being a proactive choice by the individual teacher. For example, many of the textbooks and resources in the UK appear to have examples and question sets with unsystematic variation (Park and Leung, 2006; Rowland, 2008), with limited exposure to the full range of permissible change (Watson and Mason, 2006). Therefore, ‘unsystematic variation’ could be seen as a cultural norm in English classrooms. Moreover, reported summaries do not provide all the contextual information. For example, a performance goal orientation could be seen differently if it is known the lessons occurred towards the end of the academic year just prior to exams. Consequently, to ensure the participants were fairly represented, care was needed in both the analysis and the reporting of findings to ensure the context and the wider educational landscape were taken into account and communicated in published work.

In summary, in order to ameliorate any issues related to the participating teachers being able to recognise themselves in published work a number of steps were taken. First, discussions were held with each participant. All had some prior understanding of research ethics and the reporting of qualitative data, but I ensured they understood the level of detail that would be reported and the likely nature of the discussions. They were made aware that they would be able to recognise their participation due to the unique nature of the data. Second, settings were anonymised as far as possible. For example, the key stage but not the year groups of the classes were reported. Third, reporting adheres to the aim of describing and interpreting classroom activities and seeks to clarify the use of terms, such as cognitive demand, that could otherwise imply evaluation. Fourth, the study outlines the context in which the lessons are set and highlights the influence of schools, educational traditions and curricula have on the teaching of mathematics.

4.4 Data

4.4.1 The Orchestration of Mathematics Framework

In order for the OMF to have explanatory power, the framework needed to orientate the data collection and analysis in a manner that offered insights above and beyond those obtainable from individual elements. To develop an understanding of how the OMF could be used as an analytical tool to investigate teachers' orchestration of mathematics, data relevant to the different dimensions of the OMF were gathered.

Figure 4.3 is equivalent to the final iteration of the OMF (figure 3.14: model D) but with the layout adapted for practical reasons. Specifically, the cognitive demand box was moved to allow more room for collating typed data (figure 4.3). Also, additional details were added to the task feature descriptions to differentiate between the potential features of tasks and those chosen for use by the teacher.

The central shaded boxes relate to the in-class mathematical activity and are at the centre of this study. As a consequence, observations of classrooms and classroom artefacts formed key sources of data. Inferences about aspects of the broader teaching cycle were made from lesson observations, supplemented by information from participating teachers. Nonetheless, it is acknowledged that differing levels of inference were required for different elements of the framework. For example, establishing the organisational structures of lessons was relatively straightforward from direct observations, whereas determining what teachers attended to required a higher level of inference.

Classroom norms had two important purposes in the analysis. First, identifying classroom norms, including sociomathematical norms and mathematical practices, provided a characterisation of the nature of mathematics in particular classrooms. It was this characterisation that had the potential to allow for comparisons between classes, even when the topics taught and tasks were different. Second, once established, classroom norms lowered the level of inference required when interpreting local interactions and behaviours.

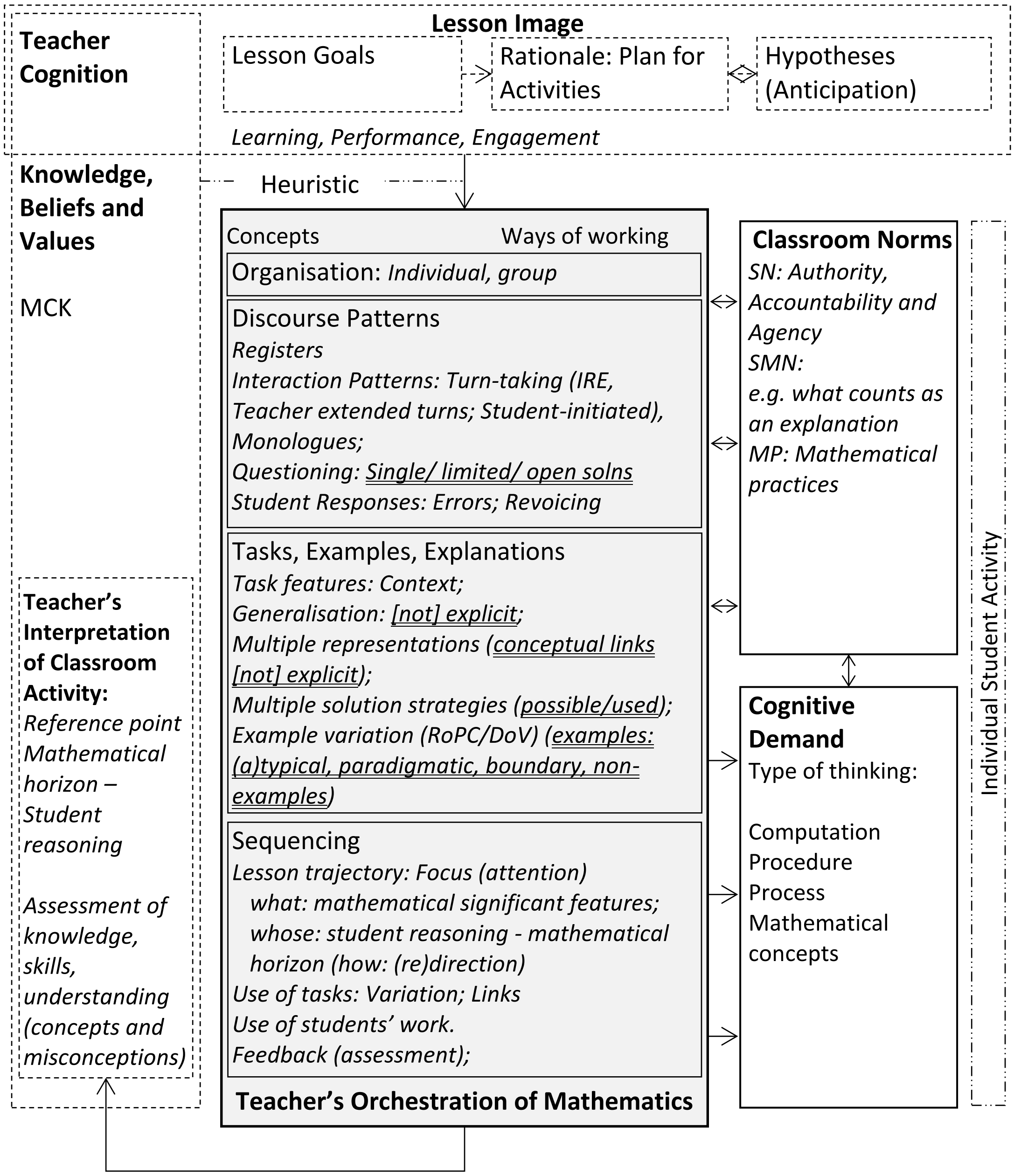


Figure 4.3: Orchestration of Mathematics Framework (OMF): Lesson Summary Layout

4.4.2 Data Collection

In line with similar classroom-based studies, the complexity of interpreting classroom interactions meant that videoing of lessons in conjunction with real-time observation was the preferred option (e.g. Hiebert et al., 2003a; Staub, 2007; Even and Kvatinsky, 2009). Jacobs et al. (2007) found that teacher questionnaires and copies of text-based classroom resources were the most useful forms of additional data. In this study, contextual information was gathered from the teachers, obtained through semi-structured interviews (appendix 6) and sight of planning documentation and classroom artefacts. The full range of classroom resources used in the lessons informed the task and sequencing dimensions of the OMF. Therefore, copies of resources in the form of textbooks, worksheets or material presented to the whole class via a data projector were gathered. Boston and Smith (2009) concluded that students' work could be analysed in relation to cognitive demand, and that this analysis was highly correlated with judgments based on lesson observations; therefore, copies of students' work were collected.

4.4.2.1 Observations

Lesson videos form a rich data source but, as with any observation, it is not possible to capture everything that happens and analysis starts immediately; observers attend to different aspects of the lesson and decisions are made as to which elements of the lessons are recorded (Derry et al., 2010; Cohen et al., 2011). I was present in recorded lessons, acting as a non-participant observer (Ritchie and Lewis, 2013) taking field-notes based on copies of the OMF. Throughout all the data collection and analysis, I needed to be cognisant of how my experience as a teacher and PGCE tutor affected what I attended to in classrooms (Mason, 2002). The presence of an observer and recording equipment could, by its very nature, be disruptive. However, Jacobs et al. (2007) argued that teaching is such an ingrained cultural activity that it changes little when observed and recorded.

Jacobs et al. (2007) advocated that decisions about what to film should be based on the notion of where the attention of an 'ideal student' is likely to be. It seemed reasonable, therefore, to aim for the video to capture the teacher's actions, including interactions with students at a whole-class level, and how they managed tasks presented to the class, including writing and gestures. The capturing of student contributions at a whole-class level was also desirable. Consequently, the decision was taken to use two static video cameras, one with a clear view of the board and the other a wider view of the class.

The systematic collection of individual teacher-student and student-student interactions was practically more difficult and would have been more intrusive (Derry et al., 2010), so data of this type was limited to that captured by proximity to the static cameras. Whilst individual-level data could have been informative, the fact that an 'ideal student' would only be party to those interactions if they were directly involved, or in close proximity, should limit the impact of not systematically collecting that data. In addition, individual students' seatwork can form a substantial part of student activity (Watson and Evans, 2012), but no attempts were made to capture data in relation to this activity during the recording or observation of lessons. Whilst this might be partially ameliorated by the retrospective examination of students work (Boston and Smith, 2009), this was another limiting factor in this research.

Videos of lessons provide rich data and can be analysed in a variety of ways (Derry et al., 2010). In this study, audio data was transcribed as heard, rather than written as grammatically correct sentences. Based on a protocol developed from the TIMSS video study (Hiebert et al., 2003a), classroom episodes related to mathematics were identified. In addition, live observation notes were abridged, and classroom artefacts were scrutinised, to contribute to a more complete picture of the publicly shared mathematics. These intermediate representations were then used to select episodes for more in-depth analysis (Derry et al., 2010). The coding of the transcripts was an iterative process (see 3.4.2) and established classifications were mapped to the OMF.

4.4.2.2 Lesson Structures

Lessons have a sequential structure, with different types of activities being undertaken at different stages in the lesson. In this study, the term 'phase' was used to describe time periods where the focus was on a common mathematical theme, such as the completion of a mathematics exercise or the discussion of learning objectives. Within these phases, different types of activities could occur, such as whole-class talk, small group work or individual seatwork, as long as the mathematical focus was retained. The term 'episode' was used to describe sub-divisions of phases that had a level of mathematical coherence. For example, a sequence of turn-taking between a teacher and student as one question was discussed, or a monologue by the teacher as they modelled a solution, would be classed as an episode. The smallest sub-division of an episode was an 'event', which contain a single occurrence or linguistic feature.

4.4.2.3 Complementary Data

Whilst the focus was on the mathematics in particular classrooms, the influence of the wider school context was taken into account. In particular, information about how class composition was decided and how the curricula were designed was sought. However, consideration was given to the research that indicates that setting practices are often far from transparent, with the implication that reported policies might not match practice (Hallam and Ireson, 2003).

In order to gain a better understanding of the participating teachers' lesson image, information was sought about the class and the observed lessons. This was in the form of semi-structured interviews with the teachers, conducted before and after the lessons (appendix 6), which drew on the protocols from the TIMSS video study (TIMSS, 1999). This included how planning decisions were made, as well as lesson specific information such as how particular lesson content related to previous work covered and the school curriculum. After the lesson, participating teachers were asked if they thought the lesson progressed as anticipated or whether there were any notable incidents. However, due to other teaching commitments, the amount of time the teachers could spend before and after the lesson answering questions did vary, so the level of detail obtained fluctuated. Copies of any classroom resources and any content presented to the class by the teacher was collected, which supported the analysis in relation to task features and the use of context, as well as facilitating analysis from a perspective of variation theory. In addition, examples of student work were collected to support the interpretation of the video analysis and the evaluation of the level of cognitive demand.

4.4.2.4 Summary of Data Sets

The analysis was an iterative process, moving between the transcriptions, sections of videos and the supporting data, such as classroom artefacts and teacher interviews, as the developing interpretations were constructed.

data sources	data sets	research focus
video of lessons	transcripts: • audio transcribed • gestures indicated • mathematics displayed on board described	Viability of the Orchestration of Mathematics Framework (OMF) as an analytical tool
	abridged OMFs	RQ Teacher's orchestration of mathematics section of OMF
live observations	annotated OMF	
classroom artefacts	copies of mathematical tasks	
	copies of students' written work	
teacher questionnaires pre and post	transcripts	RQ Teaching cycle elements of OMF (Lesson Image)

Table 4.1: Summary of Data Sets

4.4.3 Validity and Reliability

Generalisation in qualitative studies has been widely discussed (e.g. Hartas, 2010; Cohen et al., 2011), where one key argument is that the statistical perspective of generalising from measurably representative samples to a population is inappropriate. Instead, notions such as 'transferability' and 'trustworthiness' are offered as means to judge whether the study has meaning 'beyond the context in which it was derived' (Ritchie and Lewis, 2013, p.263). Generalisation in qualitative studies can be considered from a number of different perspectives. For example, it is hoped that this study could contribute to the debate about equity issues relating to setting, whilst also allowing the reader to relate findings to other settings, which Ritchie and Lewis (2013) would characterise as theoretical and empirical generalisation, respectively.

Similarly, issues of reliability and validity in qualitative studies are debated, including how these notions relate to generalisability (Hartas, 2010). The key tenet of reliability is often considered to be replicability (e.g. Cohen et al., 2011), but due to the complex, dynamic and unique nature of each lesson, a narrow interpretation of replicability is inappropriate in education contexts (Schoenfeld, 2002). Here it is understood that no two classrooms would be identical, and that teachers would not act in exactly the same ways as the teachers in this study. Rather, reliability in education contexts can be thought of as the 'soundness' of the study in relation to both the generation and analysis of the data (Ritchie and Lewis, 2013). Whilst ‘duplication’ of context is not possible, the structural features of the situations that generated the data need to occur outside of the study in a recognisable form. Here it is taken that setting and the dimensions of teachers’ classroom activities, encapsulated in the OMF, would be recognisable in other classrooms.

There are a number of limitations to this study. First, there was selective coverage of classroom activity due to the focus on whole-class interaction, which did limit the range of perspectives captured. Also, in terms of analysis in qualitative studies, reliability as replicability could be thought of as the level of agreement about judgements by others, and with the robustness of theoretical constructs established by wider use (Schoenfeld, 2002). The scale of this study does limit the level of demonstrable reliability (Hollingsworth and Clarke, 2017); as a solo research project the analysis has been undertaken by one person and the theoretical constructs have not yet been tested in other studies.

Validity is frequently considered in terms of internal and external subconstructs. Whilst there are many subconstructs within validity, in broad terms the former considers the extent to which researchers research what they intended to research, and the latter the extent to which findings can be applied to other contexts or settings, often considered synonymous to generalisation (Hartas, 2010). Here, internal validity is focused on capturing of the phenomena, namely teachers’ orchestration of mathematics. In other words, validity is the viability of OMF as an analytical tool that can be brought to bear on the research questions. The level of external validity is dependent on the quality of ‘thick descriptions’ provided in this study. The depth and breadth of descriptions governs how far the reader can understand the findings in relation to this context, and thereby the level of ‘transferability’ to other settings with which they are familiar (Ritchie and Lewis, 2013).

4.5 Pilot Study

As the OMF was integral to this study, a two-stage pilot study was undertaken to test its efficacy. As obtaining video recordings of lessons is difficult, the decision was made to use publicly available TIMSS videos for the first stage of the pilot. The second stage moved onto primary data collection. One teacher, Sam, requested trial recordings, so he could decide if he wanted to participate in the full study, and these two lessons formed the second stage of the pilot study.

Section 3.4.2.1 discussed the pilot study in relation to the development of the OMF. Consequently, there is some overlap between sections, but here the focus is on the wider methodological issues.

4.5.1 Stage One

Whilst the TIMSS videos are over fifteen years old, and examples are not available from England, these videos had the advantage of being complete, unedited lessons, which were selected as representative samples, rather than for the presence of particular features or undertaken in response to professional development projects. In addition, they were accompanied by a range of supplementary data that is not routinely made available. This included the text-based resources used in lessons, lesson transcripts and detailed research protocols. The latter outlined the filming protocols, where the camera operator was asked to take 'the perspective of an ideal student' (Hiebert et al., 2003a, p.15), which aligned well with this study. The four Australian videos were selected, as research indicated these were likely to be the closest to practices in England (Vincent and Stacey, 2008). One was subsequently dropped as the majority of the lesson involved individual computer-based work.

Included with the TIMSS videos were brief commentaries from the teachers and researchers (appendix 2). In a limited fashion, this provided a mechanism for comparing my analysis with other researchers. The central part of the OMF was the focus of the first stage of the pilot study (figure 3.10). The videos were viewed, lesson transcripts were annotated and lesson specific OMFs were populated (see 3.4.2.1 for details). This was an iterative process, moving between the data sources, primarily the transcripts and videos, and the restricted OMFs. This resulted in an annotated OMF for each lesson, with themes being linked with specific passages in the transcripts (appendix 2) and reinforced the iterative nature of the analysis and the links between the OMF structure and the interpretation of the transcripts.

4.5.2 Stage Two

Early in the study, Sam volunteered to have two lessons of a high attaining set videoed as a trial. He wanted to find out how the videoing would work in practice before committing to the full study. This presented an ideal opportunity for the second stage of the pilot study, in which I could test the OMF in a real context and develop an understanding of the data capture processes. The complete version of the pilot OMF was used (figure 3.11: model C) as pre- and post-lesson interviews could be undertaken and classroom artefacts could be collected.

I was present as a non-participant observer in both lessons, taking observation notes on a pro forma based on the OMF. Two static digital video recorders were placed so that the board was captured by one and the class perspective from the back of the classroom was captured by the other. Whilst this limited the level of individual teacher-student or student-student interaction that was captured, this also minimised the impact on the lesson. A couple of students did look at the viewfinder display and appeared to lose interest when they realised that, at most, students' backs were visible. Whilst in the first ten minutes of the lesson there were a couple of comments, such as "you're on camera", after that there was no other overt evidence of students responding to the presence of the cameras or myself.

The audio data were transcribed and coded as mathematically related or not, with the former mapped to the OMF. In attempting to understand the mathematics of the classroom, I found that the video recordings needed to be reviewed alongside the transcripts. In particular, views of the board that displayed the written mathematics and the teacher's gestures were an integral part of the analytical process. Drawing on work by Derry (2007), I increased the level of detail in the transcriptions to include pauses and descriptions of visual features (appendix 3.3.1). Rising intonations at the end of statements were used as an indication of a question being asked, and appeared useful when interpreting student contributions. For example, if asked about fractions by their teacher, a student reply of “part of a whole” has subtle shifts in interactional meaning if interpreted as a statement as compared to a question. The latter potentially indicating a level of uncertainty and could act as a self-mitigating tool for the student in the event that the response was treated as an error.

In addition to the lesson transcripts, the pre- and post-lesson interviews were also transcribed and incorporated into the summarised OMF (table 3.1). The two lesson OMF profiles were then compared and an integrated OMF was compiled (table 3.2) as overall profiles were similar. For example, the sociomathematical norm of mathematical competence being equated to the efficient production of ‘correct’ answers was common to both. The question of whether the OMF was sufficiently well defined was considered (3.4.2.2). From my perspective, the ongoing process of mapping events to categories refined my understanding of the dimension descriptors, as evidenced by more rapid coding with less contradictory examples. As such, the pilot study contributed to the coherence of the dimensions of the OMF and a reduction in overlap, albeit from the perspective of a lone researcher. The populated OMFs indicated that mathematically significant pedagogical moves were captured by the analysis orientated by the OMF with adequate separation of categories.

4.6 Main Study

4.6.1 Lessons from the Pilot Study

The pilot study did raise broader questions, and in particular the structure and role of the OMF. The literature review made the argument for the OMF as a conceptual framework for interpreting teachers' pedagogical practice, and as such, it could be construed as a theory of teaching. The complexities of classrooms would make any claim of the completeness of the OMF as a theory of teaching bold indeed, with the requirement that categories offered would be both necessary and sufficient (Schoenfeld, 2013a). In this study, however, the purpose of the OMF is to provide an instrument sensitive enough to reveal differences in a teacher's pedagogical actions as they interact with different groups of students. In other words, completeness is not claimed, but rather the OMF orientates data collection and analysis in a manner that allows comparison of pedagogically significant aspects of teachers' practice. The summary OMFs generated from the pilot study offered a picture of teachers' practice, but as these related to individual lessons rather than two classes with the same teacher, the sensitivity of the OMF for comparisons across classes was not tested.

The pilot study process also allowed some of the limitations of the data to be considered. For example, in the live lesson recordings there were some elements of classroom dialogue that were hard to discern, particularly when multiple voices were heard, and the changing light conditions meant that on a few occasions some of the written material on the board was unclear. As outlined in section 4.4.2, students' seatwork and one-to-one interactions were unlikely to be captured through the use of static cameras, but in the pilot study the level of analysis possible drawing on the captured whole-class activity offered evidence that the research questions could be suitably investigated. Therefore, on reflection, it was decided that the advantages of two static cameras, in terms of minimising the level of intrusion, outweighed the disadvantages of any reduction in the data available.

The semi-structured interviews with the teacher were transcribed, with the data principally informing the wider cycle of the lesson image and interpretation of classroom activity, rather than the dimensions of TOM. Whilst copies of students' written work were collected these were not used in the lesson analysis. Although Boston and Smith (2009) argued that students' work is 'stable within teachers and highly correlated with observed instruction' (p.136), here it was found that students' work was more difficult to interpret. This may have related to this particular teacher, as a large proportion of the lessons involved whole-class activities, with relatively little sustained independent seatwork. Consequently, it was difficult to ascertain if the written work represented the students' thinking or whether it arose from the whole-class interactions that included the sharing of solutions. Whilst copies of students' written work continued to be collected in the main study, as this was a simple process with minimal impact on participants, the pilot study did indicate that this source of data might not play a significant part in the main study.

Generating the lesson transcripts was the early stage of the analysis of the two live lessons. Speech was transcribed as heard, which included pauses and hesitation sounds (appendix 3.3). As discussed in the previous sections, the pilot study saw developments in the analysis of data. Specifically, the type and level of detail in the lesson transcripts was revised and adaptions were made to the OMF. During the transcription and coding of mathematically related episodes, in order to understand the mathematical activity, the videos had to be viewed to ascertain the mathematics presented on the class boards. This led to an increase in the level of descriptions of this shared written material in the transcripts. These annotated transcripts formed the principal data source; the mathematically related episodes were further scrutinised and coded. However, as non-mathematically related episodes were not scrutinised any further, the transcripts formed an abridged version of classroom activities. A further challenge was raised by Drageset (2015), who argued that both coding derived from the literature and coding derived from the data face considerable challenges. As the OMF, and hence the coding, was drawn from literature, his argument that noteworthy features could be missed through working within established categories needs to be acknowledged.

The pilot study process raised some questions regarding the grain size of the analysis. In the transcripts, it became apparent that intonation was relevant when distinguishing between assertions and questions. This led to a wider review as to how detailed the transcripts and how fine grained the analysis should be. One of the first interaction patterns noticed was how the teacher dealt with student errors. Drawing on a conversation analysis term, the teachers' treated student errors as dispreferred. However, the transcripts were not as finely grained as conversation analysis would require. For example, whilst pauses were indicated these were not precisely timed. Nevertheless, the principle of orderly interactions, with turn taking following identifiable patterns that is associated with conversation analysis (Sacks et al., 1974), appeared to offer insights into these interactions.

Within a western culture, one conversation convention is that an invitation has a preferred response that signals acceptance, and is given without hesitation. Whereas if one is going to decline, it is usually done with caveats, pauses and/or justification; a bald "no" is quite rare (Schegloff, 1987). Conversation analysis originated in the analysis of naturally occurring conversations in small groups (Sacks et al., 1974) and classroom interactions have their own customs and practice that can be distinct from everyday conversations. However, McHoul (1990) demonstrated that comparable initiation and response patterns could be identified in classroom exchanges.

The notion of preferred and dispreferred responses had a particular resonance with the teachers handling of 'correct' responses, 'errors' and unwanted student contributions. Some student contributions were accepted immediately; this group was mainly responses to closed questions that were mathematically valid. Other student contributions were followed by hesitations, follow-up questions or redirection. Student contributions that were incomplete or had mathematical errors fell into this category, but there were also some mathematically valid contributions that were treated in this dispreferred manner. This appeared to signal that the teacher was attending to a different aspect of mathematics, related to their mathematical horizon, rather than exploring the students' mathematical reasoning.

Other aspects of a conversation analysis approach were intuitively appealing. For instance, analysis of interactional patterns have shown that people often employ conversational strategies to avoid dispreferred responses, such as asking someone about their plans before initiating an invitation (Schegloff, 1987; Goodwin and Heritage, 1990). Within the stage two pilot study lessons, there were some examples where students appeared to adopt strategies to avoid dispreferred responses related to errors, which included phrasing responses as questions.

Moreover, the reflexivity, whereby context is both shaped by and shapes each action, aligns with notions of the development and influence of classroom norms. However, significant points of departure needed to be considered. For example, in conversation analysis, categories for analysis are derived from participants' interactions, with the influence of context derived from how the participants position themselves within the dialogue, rather than from outside information (Goodwin and Heritage, 1990). In this study, however, a wider range of contextual information was considered. This highlights a wider debate about the structure and role of the OMF and its coordination of a range of theoretical perspectives, with different grain sizes and underpinning theoretical principles. This question needs to be considered in the main study.

4.6.2 Analysis in the Main Study

As previously described, the data collection consisted of video recording of lessons using two static cameras, non-participant observation of lessons, collection of classroom artefacts and semi-structured interviews with the teachers. As I observed the lessons, analysis started immediately, through what and how I observed, noticed and noted. However, due to the transient nature of classroom interactions and the limitations of capturing classroom activity in field notes, the videos were treated as the principal source of data. Consequently, the transcription of these videos provided the starting point for the substantive analysis. It was easier to transcribe when I understood the mathematics being discussed. Consequently, additional descriptive details drawn from stills of the classroom boards or artefacts were added as the transcription progressed.

The completed transcripts were then coded (3.4.2.2) and cross-referenced with the lesson-specific OMFs. The analysis was an iterative process that involved moving between sections of the data and between the data and the theoretical framework outlined in the OMF, and as such could be described as a 'constant comparative method'. However, this term is often associated with grounded theory. As my coding drew on the OMF in addition to emerging from the data, then this study would be considered as falling outside the parameters of a grounded theory study (Strauss and Corbin, 1994; Suddaby, 2006). Whilst the constant comparative method is most closely associated with grounded theory (e.g. Cohen et al., 2011), it is an approach that is also found in a broader range of studies (Fram, 2013). Consequently, the following offers a brief overview of the origins of the constant comparative method, before describing how this method has been applied in the context of this study and the resulting methods of analysis.

Throughout the discussion it is useful to keep in mind that this iterative process was non-linear, with constant movement between the data and the analysis, whereas for clarity, both the description and reporting often takes a summary form that sometimes obscures this very process (Strauss and Corbin, 1994). In his early work, closely associated with grounded theory, Glaser (1965) stated that 'the constant comparative method can be described in four stages: (1) comparing incidents applicable to each category, (2) integrating categories and the properties, (3) delimiting the theory, and (4) writing the theory' (p.439). The first stage is to code incidents, which are initially compared with other incidents in the same category, then as coding continues into the second stage, comparisons between incidents means category properties can begin to emerge. This continual comparison of new data with the emerging constructs, undertaken with a critical eye, is recognised as an integral part of a constant comparative method (Suddaby, 2006). Fram (2013) made the argument that a constant comparative method can be used in conjunction with a theoretical framework drawn from literature, as it could be considered comparable to the knowledge that an experienced researcher would bring to the analysis.

In this study, incidents were coded and compared to other incidents from the sections of the transcripts identified as mathematically related. For example, the management of errors in students' contributions quickly emerged as a category. Glaser (1965) argued that it is out of this constant comparison that properties of categories and relationships with other categories evolve. This is where this method diverged from a process grounded entirely in the data, as the theoretical properties of categories and their relationship to other categories had been postulated in the OMF before the interrogation of the data. As the OMF was 'kept in mind' during the lesson observations, the transcription processes and the coding, the comparison included reference to the theoretical framework encapsulated in the OMF. The analysis did reveal some new categories that led to some developments in the OMF. For example, the management of the lesson trajectory was not highlighted in the early iterations of the OMF but was added with the sequencing dimension as the analysis continued. Whereas other aspects that emerged from the data, such as the role of errors, were already included in the OMF. The OMF postulated theoretical relationships between elements of the framework; the data provided sets of examples to test and map the scope of these relationships.

Glaser (1965) argued that the next stage was 'delimiting the theory', where the boundaries of the theory are determined. This occurs as fewer and fewer adjustments need to be made when new incidents are considered in relation to the properties of established categories. In the last stage, the properties of the categories are summarised and form the themes for the writing of the theory. In this study, the boundaries of the theory were postulated in the OMF and tested as the data from the lessons were categorised within this framework. The overarching theme of this study is the teachers' orchestration of mathematics as they teach different groups of students. As my study consisted of three teachers, each with two classes and a number of different lessons (figure 4.4) the constant comparative method had a number of levels: for each case, there were comparisons within a single lesson and comparisons between lessons with the same class; in the nested cases there were comparisons between classes with the same teacher; and between parallel cases there were comparisons between teachers.

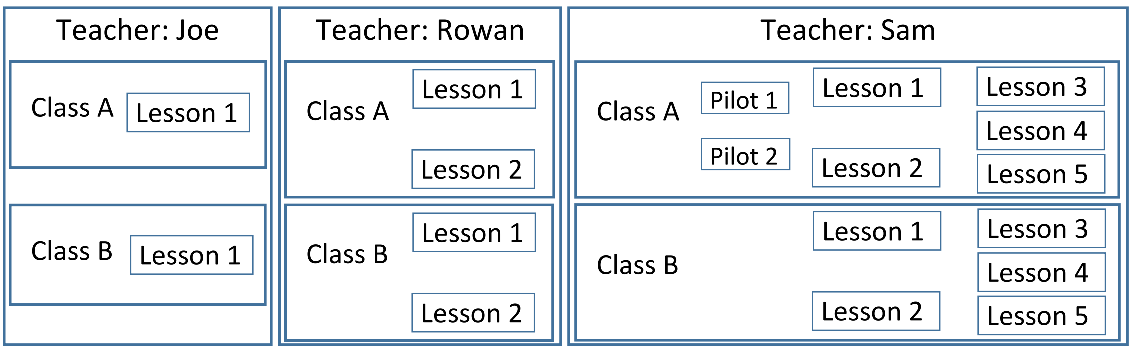


Figure 4.4: Nested Parallel Case Studies with Lessons

At each level of the comparison, the focus shifted. When individual lessons were considered, the constant comparative analysis classified mathematically related episodes to determine whether there were distinguishing features of the teacher's pedagogy, and to consider any contradictions. This also allowed the categories within the OMF to be developed. When multiple lessons with the same teacher and class, were considered, the focus shifted to analysing whether distinguishing features established within single lessons were common to others. In other words, were there elements of the teacher's practice that could be seen as stable characteristic traits of their teaching more generally. Episodes with the same coding were compared, allowing properties of categories to be developed. Consideration was also given to why some features appeared in some lessons but not others. Comparison between classes with the same teacher was the key part of this study. Comparisons between teachers allowed the efficacy of the OMF to be considered in a different context, and in particular, whether the dimensions of the OMF appeared sufficient.

5. Findings

5.1 Introduction

The previous chapter outlined the methodological approaches taken to gather and analyse data that would allow an exploration of how individual teachers adapt their practice when they teach different groups of students. Data has been obtained from two classes for each of three teachers, with lessons observed and videoed, teachers interviewed, and classroom artefacts collected. This chapter analyses that data with the purpose of identifying features of the teachers' pedagogical activities that are mathematically important and the mathematics made available to students through those pedagogical moves.

As previously discussed, classrooms are complex, dynamic environments with many interdependent factors, and the data reflected this level of complexity. The analysis was an iterative process, moving between data sources and different lessons, with pedagogical portraits for each class gradually built and refined over time, which formed the six cases. However, in order to communicate the findings, individual lessons for each teacher are discussed. This establishes a mathematical narrative for each lesson and provides frames of reference for the subsequent more nuanced discussions. All the lessons were coded and analysed, and summaries are provided in the form of summary OMFs, but one lesson for each class is reported on in detail to exemplify the pedagogical moves the teachers made; confirmatory, complimentary or contradictory evidence from other lessons is included in the discussions as appropriate.

5.2 Description and Interpretation

As previously stated, the purpose of this study is to describe and interpret teachers' pedagogical practices in order to explicate any shifts in practice when teachers teach different sets. Evaluation is not the purpose, but phrases such as 'low cognitive demand' or 'unsystematic variation' could be read as judgments. Moreover, one area that is potentially problematic to discuss is the teacher's mathematical knowledge for teaching. Teachers continually draw on subject knowledge and pedagogical knowledge in the planning and execution of lessons, but interesting episodes can arise when it appears that teachers have made errors in interpreting student contributions or in their own explanations. In discussing these issues, it may appear that teaching is being evaluated and judgments made. Whilst description and interpretation of classrooms is the cornerstone of this study, making judgements is not the intention.

Chapters 2 and 3 have outlined the context in which the study is set. For example, textbooks being characterised as having unsystematic variation (3.3.5.2) or the stratified curriculum (2.4.1) frames what happens in many mathematics classrooms in England. Nevertheless, the findings discuss three individual teachers, so before the presentation of the findings I would like to reiterate my opening acknowledgement as a means of conveying my stance:

> I would like to thank the three teachers who participated in this study and for the privilege of spending time in their classrooms. This study has reminded me above all else of the complexity of teaching mathematics, and the skill and dedication teachers demonstrate every day.

5.3 Lesson Narratives

The following sections provide lesson narratives for each teacher. After providing some background information, the OMF is used to structure the description of each lesson in order to provide an overview of the significant mathematical activities (figure 5.1). To report the findings in sufficient detail to make comparisons, a lesson narrative from one lesson for each class is provided to exemplify how pedagogical profiles were constructed.

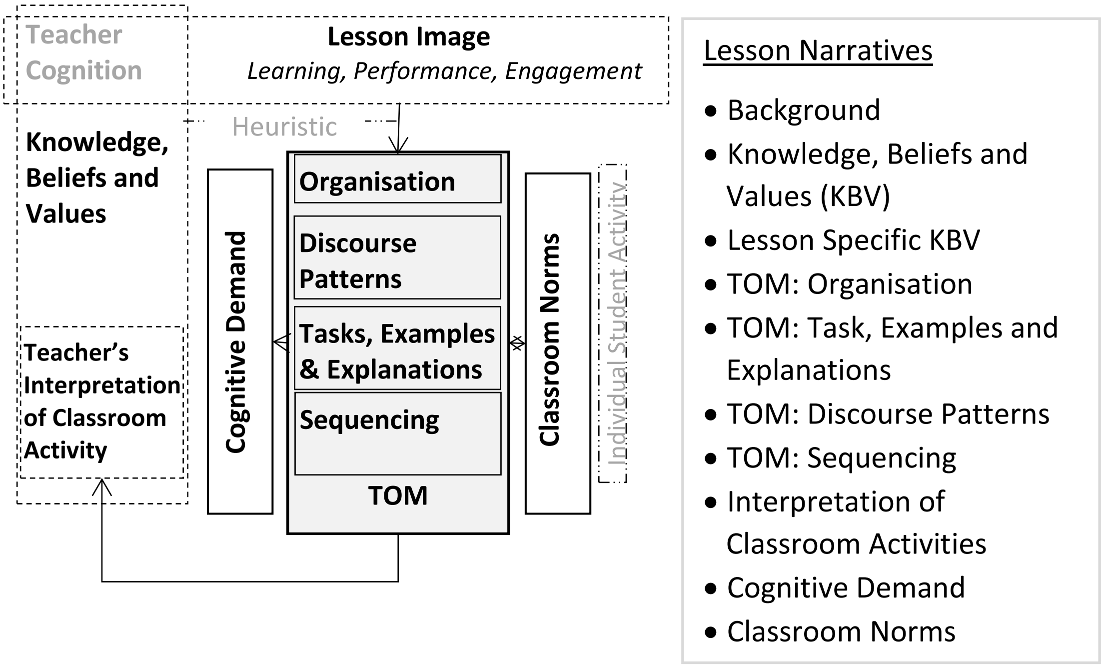


Figure 5.1: Orchestration of Mathematics Framework (OMF) and Lesson Narrative Structure

5.3.1 Teacher: Joe

5.3.1.1 Background

In pre-lesson interviews, Joe explained that in recent years his school leaders had drawn on national and international research to inform their curriculum design. This resulted in a curriculum that sought to develop mathematical fluency in tandem with students tackling 'big questions', intended to offer the students a real-world application for the mathematics they undertake. The school also adopted a textbook, designed overseas, that follows a 'concrete, pictorial, abstract' approach to developing mathematical ideas. Joe engaged in internal CPD focused on the development and use of these curriculum plans.

The school placed students in sets for mathematics; the policy was to base this on targets generated from SATs performance and internal assessments. After the sets were first established, most movement between sets occurred once a year and involved about ten percent of students identified as outliers of each set, rather than an absolute re-ranking of all students. All classes in the same year followed the same curriculum plan in terms of the overall topics taught, but tiered routes were specified so sets did not always cover the same material.

As the observed classes were from the same year group, they were following the same overarching curriculum as laid out in the school's curriculum plans. Whilst both lessons were drawn from the same number topic, different routes were identified. For class A, the starting point for the overarching topic was 'percentage of', which led to 'percentage change', with a final goal of 'inverse percentage change', whereas for class B the starting point was the multiplication of powers of ten, which would be followed by 'percentage of', with a final goal of 'percentage change'. In effect, the initial lesson for class B was treated as presumed knowledge for class A and it was anticipated that class B would not meet inverse percentage changes.

The recorded lessons were an hour long and were one of seven lessons held over a fortnight. Joe followed his departmental scheme of work and taught the recorded lessons as per his existing curriculum plans, with the level of planning typical for his lessons. Consequently, the lessons reported should represent examples of students' everyday experiences.

5.3.1.2 Teacher's Knowledge, Beliefs and Values

Joe's key priority was to develop students' deeper understanding of mathematics, with a particular focus on problem solving. In general, he felt the department's curriculum plans aligned with his approach to teaching mathematics. In particular, he thought discussions were given greater prominence in the plans, which enriched students' understanding. Whilst the 'big questions' had the potential to draw attention to the relevance of mathematics, which in turn could lead to greater student engagement, he felt that finding "genuinely relevant" problems was problematic at times. Whilst different routes through the curriculum were identified for different sets, Joe felt he had the discretion to adapt the routes based on his assessment of the classes. However, Joe reported that in practice, for the majority of the time, he followed the planned routes.

5.3.1.3 Joe: Class A

This was a Key Stage 3 class, composed of about twenty-five students who had attainment profiles in line with the average for their year group in the school.

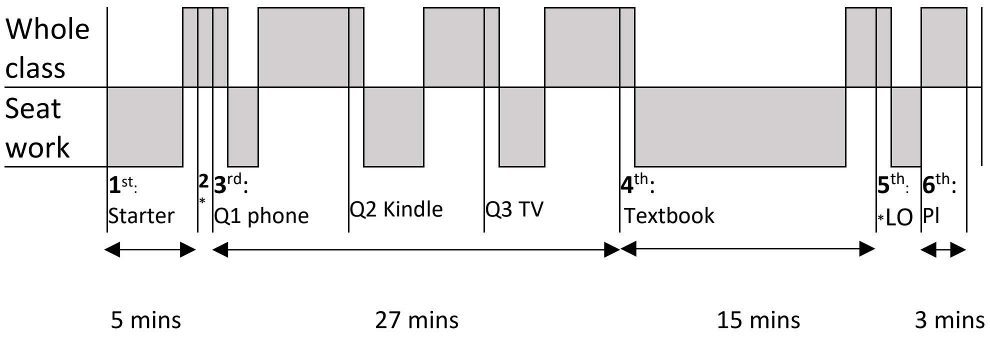
(a) Lesson Specific: Teacher's Knowledge, Beliefs and Values, incorporating Initial Lesson Image

In the pre-lesson interview, Joe's stated lesson goals were for students to be able to work out percentage change, with some students discovering alternative solution strategies; these aligned with learning goals shared with students in the lesson. As such, his articulated lesson goals were considered to be framed as performance for all students, with a learning orientation a possibility for some.

The students had met simple percentage change the previous year and had completed 'percentages of' questions in the preceding lessons. Joe anticipated that many students would be able to "find the percentage and add or subtract" ('percentage first' strategy) for simple problems with relatively little teacher intervention. He anticipated students would have greater difficulties when percentage change was applied to real-life contexts and when non-integer solutions meant rounding was involved. Joe was uncertain if any of the students would discover alternative approaches themselves.

(b) TOM: Organisation

The lesson was timetabled for one hour and lasted fifty-seven minutes due to lesson transitions. Twenty-seven minutes were spent at a whole-class level, with the remaining time spent on seatwork. Desks, large enough to seat two students, were arranged in groups of three, which resulted in students seated in groups of five or six. Tasks set could have been completed independently, but peer discussion was encouraged. For example, in phase 3, students were asked to discuss possible solutions with peers, although solutions were taken from individual students, and during seatwork, when a student indicated she was stuck, Joe asked if she had spoken to her peers.



*Learning Objectives: 1 minute in phase 2 and 3 minutes in phase 5.

Figure 5.2: Joe Class A Organisation

(c) TOM: Tasks, Examples and Explanations – Overview

The lesson was entitled ‘Percentage Change’. It was from a sequence of lessons that started with calculating percentages and the intention was to move onto reverse percentage calculations.

Phase 1: Starter

The departmental scheme of work specified the start of the lesson should focus on developing fluency. Joe displayed a PowerPoint slide (figure 5.3) and students selected which set(s) of questions to attempt in four minutes. Students were familiar with the form of the task and no specific instructions were given.

Start **33** use your answer in the next sum

(green)	(orange)	(red)
$+17$	$\times 2$	$\times 2$
$+50$	$+34$	$+24$
-51	-22	$10\%\ of$
$+11$	$\div 2$	$square\ root$
-22	$+21$	$\times 25$
$+16$	$\div 12$	$+45$
-27	$\times 15$	$5\%\ of$
$+43$	$+25$	$\times 30$
-40	$\div 5$	$\div 9$
$+24$	$+22$	$\times 5$

Figure 5.3: Joe Class A Tasks

The students were expected to complete this using any mental or written methods. After the individual seatwork, answers were displayed on a PowerPoint slide. Students self-marked and points were awarded for correct answers, with graded tariffs from green to red.

Phase 2: Sharing Learning Outcomes

There was a departmental expectation that each lesson had a ‘big question’, which could be an overarching theme for a sequence of lessons, and learning outcomes that were shared with students. These were displayed on a PowerPoint slide and Joe read them out.

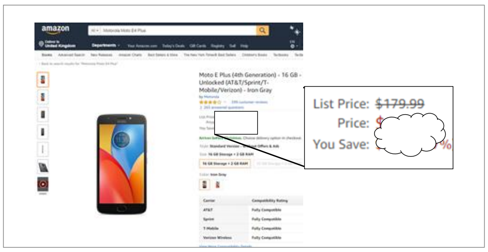
> The ‘big question’: What is a pay-day loan and why are they called loan sharks
> Learning outcome: To apply what we have learnt about percentages to work out percentage change.

The learning outcome was split into tiered learning goals:

- Bronze: Calculate integer percentages of a number
- Silver: Calculate percentage change
- Gold: Create alternative methods to calculate percentage change

Phase 3: Percentage Change Problems

Percentage reduction problems, based on online offers, were presented. Students were given a few minutes of seatwork to find a product’s cost (figure 5.4), after which there was a whole-class discussion. This pattern was repeated for three questions.



Q1. Phone: list price of \$179.99 with 43% off.

Q2. Kindle: list price of £79.99 with 38% off.

Q3. OLED TV bundle: list price of £3499.99 with 18% off.

Figure 5.4: Joe Class A Tasks

As part of extended question-and-answer sessions, Joe structured a solution strategy that he wrote on the class whiteboard. For each question the same ‘percentage first’ strategy was used, namely dividing the original amount by 100, multiplying by the percentage reduction and subtracting this amount from the original. For example:

$$
\begin{aligned}
&179.99 \div 100 = 1.7999 && (1\%) \\
&1.7999 \times 43 = 77.3957 && (43\%) \\
&179.99 - 77.3957 = 102.5943 &&
\end{aligned}
$$

Figure 5.5: Joe Class A Tasks

The percentages written in brackets were linked to the calculation by Joe asking what the amount represented. For example, after writing 77.3957 Joe asked:

66 T: what does that give you (.) what does that represent Tom (.)

67 Tom: erm (.) it would (.) be forty-three percent

[T writing: (43%)]

Extract 5.1: Joe Class A

The calculations for all three questions resulted in answers with four decimal places. For each question, rounding in the context of money was discussed, and then Joe revealed the offer price from the websites, which were compared to their solutions.

Phase 4: Textbook Exercises

Students worked from an online textbook and were given the choice of which exercise to complete; as with the learning goals these were tiered. (A show of hands later in the lesson indicated all students started at exercise B).

Bronze - Ex A: Q1. Find 12% of 120 … Q6. Find 15% of $50 …

Silver - Ex B: Q1. A bag valued at $240 has 7% GST added before it goes on sale.
How much did it cost?
Q2. An investment of $8000 attracted 5.5% interest per year.
How much was it worth at the end of one year? …
⋮
Q6. A car originally costing $82500 lost 40% of its value.
How much is it now worth?

Gold – Ex B: Create new methods and see if questions work both ways.

All the percentages and solutions were either integers or terminating decimals, with all percentages less than one hundred. In contrast to problems met in phase A, there were no cases that required rounding because of the context.

Phase 5: Review of Learning Objectives

The students reviewed the learning objectives by writing about WWW (what went well) and EBII (even better if I…); this activity was undertaken in most lessons.

Phase 6: Plenary

It was usual for Joe to undertake a whole-class discussion with the aim to review a key mathematical feature and to make links to future lessons. In this case, an alternative strategy for calculating percentage change was discussed.

General

Whilst the 'big question' was not addressed in this lesson, Joe's intention was that future lessons would consider percentage change in the context of loan sharks. The majority of the percentage problems were set in contexts with some potential links to the 'real world', but the problems were of a form students were unlikely to meet outside the classroom.

The examples in phase 3 were drawn from current online offers from well-known websites. Joe drew attention to his personal interest in the products. For example, to explain how he found one offer he stated "my wife has a Kindle... she really loves it (.) erm so I was having a look…", and another was a product he had bought. One student stated, "sir I know how much you spent on it (.) too much", indicating they were making personal judgments about the value of the products. However, the problems were constructed for the classroom, in so far as Joe had hidden the offer prices that were visible on the websites. So, while the websites offered a potential link to students' personal experiences, the students were unlikely to meet these calculations outside of the classroom.

In phase 4, the problems drew on different contexts, but were from an overseas textbook; students' questions indicated that many were unfamiliar with local terminology, such as GST. Moreover, 'just sufficient' mathematical information was provided to undertake the required calculations, and were of a form usually only seen in classrooms; a label of 'pseudocontext' was considered appropriate.

Multiple solution strategies and multiple representations were possible with these percentage change problems. However, a numerical calculation based on the 'percentage first' approach was used at a whole-class level for most of the lesson, and was the only one presented in written form on the whiteboard.

When the overall sequence of questions was considered, the early focus on percentage reduction was extended to include percentage increases, but the problems remained similar whilst the 'contexts' changed. In phase 3, questions were percentage reductions with non-integer answers that required rounding to have meaning in the context. In phase 4, exercise B contained decimal percentages and introduced percentage increases alongside decreases, although rounding was not required. Apart from these differences, the main changes to the questions were the context in which the problems were set. For each question, there was no requirement to go beyond the correct application of the procedure and there were few discernible connections between questions. As such there did not appear to be systematic variation in the exercises and examples (Watson and Mason, 2006) (3.3.5.2). That is to say, the same approach could have been undertaken in each question and there were no prompts to cue further interrogation of the problems or the approaches taken. In addition, all percentages remained less than one hundred and, as students started at exercise B, they did not have to attempt any 'not' percentage change questions.

(d) TOM: Discourse

In terms of the mathematical register, mathematical operations were the main examples of mathematical terminology. 'Times' was commonly used to indicate multiplication by both Joe and the students, with Joe occasionally using 'multiply'. Some terms related to unfamiliar contexts were explained, such as GST, but 'integer' was the only mathematical term where meaning was discussed.

In whole-class episodes, approximately 90% of talk was classified as mathematically related (figure 5.6: subdivision 1). There were two sub-categories of mathematically related whole-class episodes; turn-taking was the most common form of talk, with the remaining time classified as monologues (subdivision 2). The monologues consisted of explanations or instructions given by Joe (subdivision 3 T:E). Within turn-taking, the initiate-respond-evaluate (IRE) pattern, or a variant thereof, was the most common form of talk (subdivision 3 T:IRE); the variant was the extension of the evaluative turn by the inclusion of a short explanation or summary by Joe. It was common for IRE exchanges to be followed by additional IRE turns, forming extended whole-class question-and-answer sequences. Periodically, Joe shifted away from the IRE pattern by collecting a range of final answers from a number of students (subdivision 3 T:M), postponing any evaluation until they could be compared. Occasionally, students initiated a turn-taking exchange by asking a question or making a comment (subdivision 3 S:L).

<table>
<tr><td colspan="5">Whole-class talk</td><td></td></tr>
<tr><td colspan="4">Mathematically Related Episodes</td><td>Other</td><td>Subdivision 1</td></tr>
<tr><td colspan="3">Turn-taking</td><td>Monologues</td><td>Other</td><td>Subdivision 2</td></tr>
<tr><td>T: IRE (inc. IRE variant)</td><td>T:M</td><td>S:L</td><td>T:E Explains/ Instructs</td><td>Other</td><td>Subdivision 3</td></tr>
</table>

Figure 5.6: Joe Class A Breakdown of Whole-Class Episodes

In IRE exchanges, Joe indicated student responses were satisfactory in a number of ways: repetition of the student's comment; utilisation of the response, either by translation to written form on the class whiteboard or incorporation into the next question; use of an affirmative word, such as "OK" or a superlative such as "fantastic" (extracts 5.2 & 5.7); or some combination of these. About three-quarters of IRE responses were treated as satisfactory, in other words as 'correct'. The remaining responses consisted of 'don't know' comments or mathematical statements treated as unsatisfactory, otherwise referred to as 'errors'. To indicate responses were unsatisfactory Joe sometimes replied with an evaluative phrase, such as "not quite", and once challenged a response with "is it?". On other occasions, he used repetition in combination with referral to other students. For example, when the 43% reduction of $\$179.99$ was discussed the following exchange occurred:

48 T: … how did you start this (.)

49 S1: found one percent

50 T: you found one percent (..) how did you find one percent

51 S1: by dividing by a hundred

52 T: so what did you divide by a hundred (..)

53 S1: one seven nine (.) nine nine

54 T: so you did a hundred and seventy-nine ninety-nine right (.) divided that by one hundred

[writing: $179.99 \div 100 = 1.7999$ board]

⋮

72 T: seventy-seven point three nine five seven (..)

[writing: $\times\, 43 \ = 77.3957$ (43%)]

are we done (.) you're saying no (.) I saw a bit of a no but why

73 S2: erh (.) cause you have to (..) I don't know

74 T: OK (.) you don't think we're finished yet (.) OK Mel

[a few students had their hands raised, including Mel]

⋮

82 T: … so because it's forty-three percent off what have I got to do (..)

83 S2: divide it

84 T: divide it (..) do you agree (..) you don't (.) go on then

[Joe scanning then looking at a student shaking their head]

85 S3: you take it away from er (.) that (.) first price (.)

Extract 5.2: Joe Class A

Lines 50, 52 and 54 are examples of Joe treating responses as satisfactory. Line 50 has repetition and a follow-on question and line 52 incorporated the response into another follow-on question. In line 54, Joe rephrased the number in a more standard form and summarised the whole calculation, which he wrote on the whiteboard. Lines 48 to 52 were considered to be two cycles of IRE, whilst the revoicing, writing and summarising in line 54 was coded as the IRE variant. The IRE sequences continued, forming an extended question-and-answer sequence about the same problem. Line 73 contained a 'don't know' response. In this case, Joe asked another student, who had indicated their willingness to respond by their raised hand. Lines 82 to 85 contained an example of the treatment of an unsatisfactory response following the repetition and referral pattern. In this case, after the repetition Joe paused and asked, "do you agree" (line 84); his scanning of the room indicated this question was directed to the class in general. A student who had shaken his head was invited to contribute and offered an alternative response (line 85). Thus, there appeared to be a shared understanding that Joe's actions indicated the response in line 83 was unsatisfactory.

A common feature in the treatment of satisfactory responses was the immediacy of the transition away from the original question, often to the next stage of the procedure, via a follow-on question or explanation, or to a new idea. After 'don't know' responses Joe either redirected the original question to another student, as above, asked the same student a simpler follow-up question or offered his own explanation. The treatment of unsatisfactory responses was either repetition followed by redirection to another student, as above (extract 5.2, line 84), or indirect evaluation used in conjunction with a follow-up question to the same student, usually simplified (e.g. extract 5.6, line 138).

As in extract 5.2 above, the last turn in one IRE exchange often formed the first turn of another, forming a step-by-step approach to an overarching procedure for a multistage problem. The majority of Joe's questions had a limited range of mathematically valid responses. After a particular approach to a multi-step problem had started, Joe's questions often asked for a result of a calculation or one step in the procedure. Joe translated students' verbal contributions into written calculations on the board; he often accepted single phrase responses that he periodically revoiced into a more extensive description of the procedure (e.g. extract 5.2, line 54). He occasionally asked 'why', but these occurred in reference to particular steps in a procedure (e.g. extract 5.2, line 72). Consequently, whilst this language is usually associated with more open questions, in this lesson the range of mathematically valid responses remained limited.

Within IRE sequences, responses treated as satisfactory were mathematically valid statements. That is to say, they could be interpreted as a mathematically appropriate response to a question posed. Whereas responses treated as unsatisfactory contained mathematical errors, meaning they included a mathematically invalid statement or response to a posed question, such as a numerical value not being the result of the requested calculation.

In overall terms, IRE exchanges were predominantly initiated with a question with a limited range of mathematically valid responses. The most common response was a valid mathematical contribution that was treated as satisfactory, which was followed by an immediate transition to a new question. The remaining responses were either 'don't know' or mathematically invalid contributions that were treated as unsatisfactory. In these cases, Joe subsequent actions maintained the focus on the original question, either by asking a follow-up question, redirecting the question to another student or by offering a direct explanation himself. Joe often extended his own turn by including an explanation or summary.

There were a few occasions when Joe departed from the IRE pattern by collecting a range of final answers without any immediate evaluation. For example, after Joe wrote $\$179.99 - 77.3957 = \$102.5943$ on the class whiteboard when calculating 43% off $\$179.99$ the following exchange occurred:

106 T: … so if we rounded would we have (..)
107 S1: erm a hundred and three dollars
108 T: oh all right you'd go straight for a hundred and three dollars (.) OK interesting (.) interesting (.)
[writing: $103.00]
Nat you said rounding as well what d'you think
109 Nat (.) er (.) maybe a hundred and two dollars and fifty no sixty cents
110 T: so you're going for hundred and two dollars and sixty cents (.) OK (.)
[writing: $102.60]
interesting (.) anybody else yeh
111 S2: I'm saying a hundred and (.) two dollars and fifty-nine
[T writing: $102.59]

Extract 5.3: Joe Class A

After Joe had written the different answers on the whiteboard, he revealed the reduced price listed on the website of $\$102.50$ and the students' responses were compared to this price. There was no comparison of the students' responses with each other.

There were a few occasions when students-initiated turn-taking exchanges by making a comment without a direct invitation from Joe. For example, the discussion about the phone continued with:

112 T: … I feel robbed cause that's not thirt- that's not forty-three percent off (.) we've just shown that that's not forty-three percent off (.) Amazon needs to sort out its pricing (.) a hundred and two dollars fifty they're charging which I think is a tiny bit over
113 S: it's cheaper
114 T: (..) it is isn't it (.) to be fair we're saving nine cents we should be fairly happy shouldn't we…

Extract 5.4: Joe Class A

In line 112, Joe was in the process of offering an explanation that implied Amazon was overcharging. In line 113, a student called out, without a direct invitation from either a posed question or a request for a contribution, in which he offered an alternative interpretation. After a short pause, Joe acknowledged and agreed with the student's contribution and goes on to add more detail by quantifying how much money is saved.

Extended periods of talk by Joe were classified as monologues based on their duration and levels of self-containment. These episodes usually occurred when Jo was 'setting up' the examples used in phase 2. There was no student talk long enough to be similarly classified. In almost all IRE exchanges, the students' contributions were shorter than Joe's. Taken in conjunction with monologues, this resulted in over three-quarters of class-level talk being undertaken by Joe.

(e) TOM: Sequencing

Joe controlled the overall trajectory of the lesson, shaped by the prepared resources, and he regulated the mathematical focus through questions asked and explanations given. For the majority of the lesson he directed or redirected student responses towards approaches he introduced. The inference made was that Joe usually attended to his mathematical horizon rather than the interrogation of student reasoning when managing whole-class interactions.

The learning outcomes indicated alternative approaches were encouraged, but Joe maintained the focus on the 'percentage first' method for most of the lesson (figure 5.5). He gave students time to attempt problems with their peers before whole-class discussions were held, which provided an opportunity for students to use methods of their choosing. However, in phases 3 and 4 only the 'percentage first' approach was discussed at a whole-class level; having established this approach in Q1, Joe drew attention to this procedure as being appropriate for the next calculation, stating:

120 T: … so we've just done one previously and the workings here (.) so we divided by a hundred to find one percent (.) we times'd by forty-three to give us our forty-three percent so we'll have to change that a little bit because we've got a different amount this time
[pointing at original amount and percentage]

Extract 5.5: Joe Class A

During the discussion of Q2 Joe curtailed and postponed discussions of alternative approaches:

130 T: you've divided something by a hundred
131 S: divided thirty-eight
132 T: you divided thirty-eight by a hundred
[writing: $\frac{38}{100} = 0.38$]
OK so thirty-eight divided by a hundred equals nought point three eight (.) then what did you do (.)
133 S: erh don't know
134 you don't know (.) ok so you're going to have to listen really really carefully to what other people say aren't you (.) Lex what did you do (.)
[erasing board]
135: Lex: I did (..) I did (.) seventy-nine nine nine times one hundred (.)
136 T: times a hundred
137 Lex: yep
138 T: you've got a couple of people looking at you a bit weirdly there Lex (.) what should you have done instead
139 Lex: divide
⋮
164 T: one final one (.) Mel
165 Mel: the method does work
166 T: argh does it brilliant can I share that with everyone a little bit later…

Extract 5.6: Joe Class A

In line 132, Joe repeated the student's response and wrote on the board, which was a normal indication of a satisfactory response. However, when the student was unable to continue the calculation (line 133) Joe moved onto another student (line 134) and erased the 0.38 calculation without any further discussion. Whilst the 0.38 calculation was originally accepted, and there was no indication as to the source of any mathematical error, the likely message was that this approach was without merit, even though a number of successful solution strategies could follow from this starting point. In line 135, Lex offered an alternative starting point, which contained a mathematically invalid response to the problem. Joe then repeated part of Lex's response; repetition usually indicated a satisfactory response and Lex's next turn indicated that he did not identify any issues with his original response (line 137). In his next turn, Joe's more explicit indication of an issue (line 138) led to Lex correcting the operation. After this point, the IRE exchanges continued and resulted in the 'percentage first' method being written on the board.

After Joe had written the final line, $79.99 - 30.6962 = 49.9538$, he asked students to "put that back into a price". Alternative rounded answers were offered by a few students. As part of those exchanges, rather than offer another rounded answer, Mel responded by indicating he had an alternative approach that worked (line 165). Joe positively acknowledged this contribution including a superlative (line 166) but postponed the discussion until the last three minutes of the lesson:

269	T:	… so does that person want to share (.) yep OK go for it then
270	Mel:	because percentages are out of a hundred (.) er to get to a hundred you do (.) sixty-six percent
271	T:	fantastic (.) thank you so that's the first bit (.) thirty-four percent off (.) so if we start with one hundred percent … if we take away thirty-four percent (.) what that means is that we are left (.) with sixty-six percent left (.) er of our original total…

Extract 5.7: Joe Class A

Joe had previously identified Mel as a student with an alternative strategy and asked if she wanted to share (line 269). After the student's contribution (line 270), Joe used a superlative to positively evaluate the contribution; he used superlatives on four occasions, three of which were in relation to this explanation. He then revoiced the student's response, extending the explanation by outlining the origins of sixty-six percent and completing the calculation.

For all three questions in phase 3, the 'one percentage' approach established the offer price in an unrounded form, after which students offered a range of rounded answers that Joe wrote on the whiteboard. For example, in Q1 $\$103.00$, $\$102.60$ and $\$102.59$ were associated with $\$102.5943$ (e.g. extract 5.3, line 106-111) and for Q2, $£50$, $£49.50$, $£49.60$ and $£49.59$ were associated with 49.9538. There was no explicit discussion about the accuracy or appropriateness of the rounded figures in any of the questions. Instead, Joe drew attention to the accuracy of the online price by revealing the online offer (e.g. extract 5.4, line 112-114).

(f) Interpretation of Classroom Activities

For the majority of the lesson, Joe directed or redirected students' attention to mathematical foci he introduced. As this included his management of student responses, the inference made was that he was often attending to his mathematical horizon when interpreting student contributions; the interrogation of student reasoning was less common. For example, in the discussion of Q2 (extract 5.6, lines 130-134) Joe moved on to another student and the 'one percent' strategy when the student could not offer the next step after $\frac{38}{100} = 0.38$; the reasoning behind this response was not interrogated further.

There were occasions where Joe's revoicing of responses may have gone beyond the student's understanding. For example, when Joe was outlining the learning outcomes the term 'integer' was discussed:

23	T:	… so I want <u>everybody</u> to be able to calculate an integer percentage of a number (.) what's an integer (..) go on Nat
24	Nat:	it's a number without a tenth
25	T:	fantastic it's a whole number so thirty-three twenty-one seventy-four (.) all integer percentages

Extract 5.8: Joe Class A

In line 25, Joe explicitly acknowledged Nat's contribution as satisfactory and then offered his own explanation. He appeared to interpret Nat's comment about tenths as inferring a broader understanding of integers.

In the post-lesson interview, Joe stated that he had chosen to postpone the discussion of alternative strategies until the end of the lesson, as not all students appeared to be confident to apply the first method. He thought it was important that the students were successful with one method before others were introduced. He attributed most issues students encountered in phase 4 to the use of a Singapore based text, with students having to work with unfamiliar contexts, such as dollars and GST.

(g) Cognitive Demand

Multiple solution strategies were possible, which was acknowledged and encouraged in the articulation of the learning goals. Moreover, students were asked to complete percentage change questions and participate in small group discussions before these were discussed at a whole-class level. Consequently, students had the opportunity to structure solutions themselves. However, the whole-class discussions predominantly focussed on the 'percentage first' approach, with this solution strategy structured step-by-step by Joe through his use of IRE sequences. As such, this 'talk as mathematics' could be seen as an articulation of a particular procedure.

Joe also signalled that the 'percentage first' approach outlined for Q1 should be used in subsequent questions (extract 5.5). This had the potential to draw attention to the structure of the problem by highlighting the similarities and differences, but could have limited students' subsequent considerations of alternative strategies. In addition, the requirement for students to articulate what the amount represented in terms of percentages in the early stages of the calculations provided some justification for the procedure. So, whilst the talk remained focussed on particular examples, some discussions had the potential to convey meaning beyond the examples used. Consequently, the majority of 'talk as mathematics' was classified as procedural or process.

When the sequence of questions is considered, the main differences were the context in which the problems were presented. Indeed, all the questions contained the original cost, the percentage change and whether it was an increase or decrease. Once the students had extracted this information, they could have successfully completed all the tasks using the 'percentage first' approach (figure 5.5). Difficulties seemed to arise in interpreting the contexts as presented. For example, interpreting GST and interest as percentage increases. The questions drawn from online products exposed students to 'messier' answers that can arise when 'real' data is considered. This allowed attention to be drawn to the necessity of rounding to give meaning in monetary contexts. However, there was no explicit discussion of the different levels of accuracy when students offered their rounded answers.

Joe signalled that other approaches were possible, and it was desirable to find these methods, albeit with the expectation that only a few students working at 'gold' would engage in this activity. As such, students had the opportunity to work on the tasks in a manner that required higher levels of cognitive demand, but there was no requirement to do so. For the student, Mel, who offered the alternative strategy at the end of the lesson, there was no clear evidence as to whether she had met this approach previously or had 'discovered' it for herself in the lesson. There was a short whole-class discussion of this alternative approach, and as such an 'effective student of mathematics' would have had the opportunity to consider at least one alternative strategy. However, this was a wholly verbal exchange between Joe and Mel held at the end of the lesson and without a written solution being presented on the class whiteboard. This would have made it more difficult for the remaining students to consider this strategy and compare it with the 'percentage first' approach.

Consequently, the potential cognitive demand of tasks was high. However, if students had followed the 'percentage first' approach that dominated the whole-class discussions, they could have successfully completed all tasks without working at these higher levels.

(h) Classroom Norms

Within the lesson, Joe directed student activities and regulated the mathematical direction of travel, determining both the content and mathematical approaches to be taken. As such, he could have been seen as having high levels of agency within his classroom. The learning goals and tiered exercises contributed to the notion that mathematics has a hierarchy, both in terms of the mathematical content and the learning of mathematics. In particular, demonstration of competence in terms of performance was the expectation for all students and was a prerequisite for moving onto tasks that involved more student-led thinking. Moreover, the latter was appropriate for a minority of students who moved onto the highest learning goal.

The most common form of interaction was IRE turn-taking, and Joe tended to revoice student contributions, often summarising or extending explanations. These actions reinforced the norm that Joe was the arbiter of correctness and that he was responsible for explanations. Although, towards the end of the discussions of each question in phase 3, Joe collected multiple answers from students and postponed any evaluation. This shifted Joe out of an evaluative role, albeit for short periods of time, and provided exposure to situations where more than one solution could be considered acceptable. There were a few occasions where students took more of an initiative. For example, a student 'corrected' Joe's interpretation of the accuracy of rounding on a website (extract 5.4, line 113), thus demonstrating some aspects of agency. Thus, there were signals that student contributions were valued, but the final judgment about the legitimacy of mathematical contributions resided with Joe.

Most of the questions asked by Joe were related to a particular procedure and had a limited range of valid mathematical solutions. Whilst explanations were part of some student responses, a description of the next step in a procedure was treated as a satisfactory explanation. Within IRE sequences, Joe usually indicated a response was satisfactory with an immediate transition to a new question, whereas unsatisfactory responses led to follow-up questions until the 'correct' response was forthcoming. These contributed to the narrative that the efficient production of the correct answer was the expectation of an 'effective student of mathematics', and if met, errors should be corrected. However, while Joe structured the solutions, the students were regularly asked to provide the steps, with Joe occasionally providing the answer; for example:

199	T:	what's our first step
200	S:	divide (.) erm three four nine nine (point nine nine) by a hundred
201	T:	so three thousand four hundred and ninety-nine divided by a hundred (.) and that gives us thirty-four point nine nine (.) nine nine and what does that represent…

Extract 5.9: Joe Class A

This could contribute to the narrative that an 'effective student of mathematics' can also be responsible for structuring the problem, indicating the process carries importance as well as the numerical answer.

5.3.1.4 Joe: Summary for Class A

Throughout the iterative analysis process, pedagogical features of particular lessons were mapped to lesson-specific OMFs. These OMFs were working documents, gradually populated as different data were analysed. Figure 5.7 was the final working document generated through this process for Joe's class A. This exemplifies how the OMF summarises the pedagogical profile for a lesson. This document was drawn on in the further analysis, forming a key point of comparison when Joe's class A was compared to class B, and later when the three teachers were compared.

However, these lesson-specific summary OMFs were working documents that contained descriptions in note form and bespoke abbreviations. Accordingly, in order to communicate the key themes from the analysed lesson, a written summary has been provided before the presentation of the OMF. This written summary offers an overview of the pedagogical features of the lesson, though it should be noted that the OMF models the relationships between features.

(a) Joe Class A: Written Summary – Percentages Lesson

The following outline draws together the key themes from the analysed lesson discussed in the preceding section (5.3.1.3). As such, this is a summary of the pedagogical features for the observed lesson.

A) Curriculum
   a) From a 'middle' curriculum route
B) Organisation
   a) Seatwork: individual, peer discussion encouraged but not required; self-selected tiered textbook work
   b) Interleaved seatwork with whole class
C) Discourse patterns: aligned with patterns previously reported (e.g. Drageset, 2015)
   a) IRE dominant, limited solution questions in linked sequences
   b) Typical satisfactory/unsatisfactory norms: 'Correct' responses ⇨ follow-on questions; 'errors' ⇨ follow-up questions
   c) Revoicing; rephrasing (increasing precision) and explanations extended
D) Tasks

a) Links to ‘real-world’ contexts but psuedo-contexts for textbook work

b) Focus on one solution strategy and limited use of multiple representations.

c) Model – exercises; limited range of permissible change (Bills et al., 2006)

E) Sequencing

a) Focus on mathematical horizon; (re)direction to solution strategy introduced

b) Some attention drawn to mathematical structure through questioning

F) Teacher Cognition

a) Espoused priority was to develop student understanding; articulated lesson goals had performance orientation

b) Privileged his mathematical horizon when interpreting student responses

G) Classroom norms

a) Teacher arbiter of correctness

b) Procedure counts as explanation; mathematics as a hierarchy

H) Cognitive Demand

a) Potential high but range low to high (limited press to move beyond procedural)

(b) Joe Class A: Summary OMF – Percentages Lesson

As previously indicated, the lesson-specific summary OMF was a working document. Descriptions were in note form and abbreviations were used, which are summarised in table 5.1.

Standard subheadings		Text abbreviations	
MSS	Multiple solution strategies	Soln	Solution
MR	Multiple representations	Sat	Satisfactory
General^n	Generalisation	Unsat	Unsatisfactory
SN	Social norms	SS	Solution strategy
SMN	Sociomathematical norms	RoPC	Range of permissible change
MP	Mathematical practices	DoV	Dimensions of variation
ESM	Effective student of mathematics	T	Teacher
		SW	Seat work
		ISW	Individual seat work
		WC	Whole class

Table 5.1: Abbreviations for Summary OMF

Joe Class A Summary OMF

Lesson Image: Teacher Class

Teacher Cognition

Lesson Goals: *Performance, learning for some (silver: find % change; gold: find alternative methods)*

Plan Rationale: *Online examples offered relevance and rounding opportunities.*

Hypotheses: *Find % then ± would be students' default approach. Contexts may cause issues.*

Heuristic

Knowledge, Beliefs and Values

MCK

Priority: to develop student's understanding of mathematics (particularly problem solving). Discussion beneficial to enrich understanding.

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon (over-interpretation of student reasoning?)

Organisation: *Individual (peer encouraged); interleaved SW-WC (ISW first); self-selecting tiered tasks*

Discourse Patterns

Registers: *term explained (integer) but no press for students' use*

Interaction: *IRE dominant, extended teacher turns, linked longer sequence. Some multiple responses pre evaluation*

Questions: *Most limited soln – part of step-by-step procedure (T summarised)*

Responses: *Typical satisfactory & unsat*

Revoicing: *repeat, rephrase-formal, extend*

Tasks, Examples, Explanations: *(model-exercises)* Features Context: *some 'real-world' questions constructed for classroom. Textbook psuedocontexts.* General^{n}: *not explicit* MSS: *possible, acknowledged and shown, but not required. One method in teacher explanations (% +/-)* MR: *possible-not used* Example variation: *limited RoPC DoV (% change with all information provided), 'contexts' varied but similar form/ format, no 'not'*

Sequencing: Lesson trajectory

Focus on mathematical horizon, redirection to single procedure (% change, finding % then +/-). Drew attention to the relationship between value and % represented (structure). Focus on comparison with online reductions ➪ excluded exploration of rounding. Variation: *Alternative SS acknowledged (used at end of lesson*):* Students' work: *alternative SS shared**

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act). Errors: avoid/ correct.*

SMN: *Procedure as explanation, (one procedure to soln); students articulate next step a few times (prompted). Mathematics as a hierarchy (content, ways of working). ESM: efficient & correct answers*

MP: *'one percent' approach for %*

Cognitive Demand

Type of thinking:

Potential: *High (discovering alternative strategies 'gold goal)*

Range: *Low to high Procedural completion of tasks following modelled examples was possible (no press for higher).*

Individual Student Activity

Figure 5.7: Joe Class A Summary OMF

5.3.1.5 Joe: Class B

This was a Key Stage 3 class, composed of about fifteen students who had attainment profiles below average for their year group in the school.

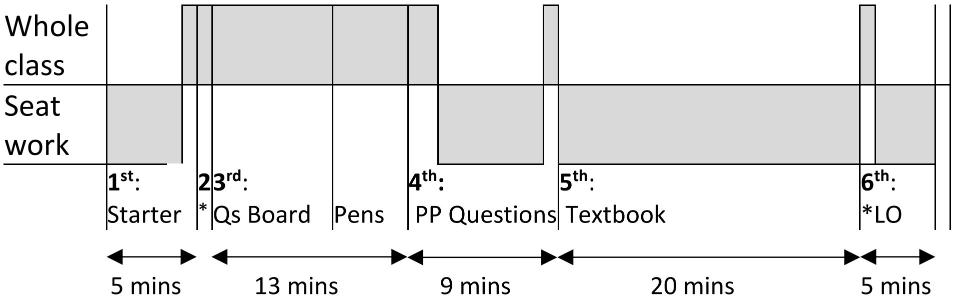
(a) Lesson Specific: Teacher's Knowledge, Beliefs and Values, incorporating Initial Lesson Image

Joe's stated lesson goal was for students to become more fluent with multiplying by tens, hundreds and thousands when decimals were involved, which aligned with the learning goals shared with students. As such, Joe's articulated lesson goal was considered performance orientated. Joe added that a few students might progress onto division. This was set up as a student led activity, where they would attempt problems without seeing modelled solution strategies. As such, a learning orientation was considered a possibility for some.

In his assessment, a number of the students had weak mental arithmetic skills that could hinder progress. For this reason, he felt the 'fluency starter' was particularly beneficial for this class. Preceding lessons focussed on multiplication by powers of ten; Joe's assessment was the majority of students could accurately multiply integers by ten, a hundred and a thousand. He anticipated errors would occur with decimals, with a variety of origins. For instance, he thought some might "add zeros", rather than use a place value approach, such as headed columns. In addition, he thought difficulties would arise in managing the decimal point if calculations required decimals to be multiplied by integers. Consequently, he planned a $\times 10^{n}$ first approach to reduce the requirement to use decimals in latter stages. Joe did plan for the class to move onto percentage calculations and percentage change in subsequent lessons, but anticipated more time would be needed. Specifically, he thought it would take two lessons to cover the percentage change content delivered to class A in one lesson, and that he would need to take out the rounding aspect.

(b) TOM: Organisation

The lesson was timetabled for one hour and lasted fifty-six minutes due to lesson transitions. Nineteen minutes were spent at a whole-class level with the remaining time spent on individual seatwork. Desks were arranged in groups; most students sat in pairs or threes, with a couple of students sitting on their own. Students were encouraged to work with their peers. For example, when a couple of students were self-marking, Joe suggested they worked together to correct mistakes. During the second half of the lesson, Joe circulated and interacted with all students at least once, either individually or in small groups.



*Learning Objectives: 1 minute in phase 2 and 5 minutes in phase 6.

Figure 5.8: Joe Class B Organisation

(c) TOM: Tasks, Examples and Explanations – Overview

The lesson was entitled ‘Multiplying by 10s, 100s and 1000s’. It was part of a sequence of lessons that started with multiplying by powers of ten. The intention was to move onto percentages, including percentage change.

Phase 1: Starter

Joe used the same fluency starter as for class A (figure 5.3); these students were also familiar with this type of task. After individual seatwork, students self-marked their work and points were awarded for correct answers.

Phase 2: Sharing Learning Outcomes

The ‘big question’ and learning outcomes were displayed on a PowerPoint slide and there was a short discussion.

> The ‘big question’: What is the relevance of these
> [a picture of a pile of hundred and thousand sprinkles]
> Learning outcome: To be able to multiply decimals by 10s, 100s and 1000s.

The learning outcome was split into tiered learning goals:

- Bronze: Multiply decimals by ten, hundred and a thousand
- Silver: Multiply decimals by tens, hundreds and thousands
- Gold: Divide decimals by ten, hundred and a thousand

Phase 3: Multiplication Problems

Questions related to the bronze and silver learning goals were discussed at a whole-class level. Joe initially asked what the difference was between multiplying by ten and by tens. When students were initially unable to recall, Joe hand wrote a set of questions on the board:

Q1 3.14×10 Q2 3.14×20 Q3 3.14×40

The approach taken was to write the tens, hundreds or thousands as factors of powers of ten and a single digit; for example:

$$\begin{aligned} 3.14 \times 20 &= 3.14 \times 10 \times 2 \\ &= 31.4 \times 2 \\ &= 62.8 \end{aligned}$$

Figure 5.9: Joe Class B Tasks

During the discussions about Q1 Joe used column headings to demonstrate how the digits 'moved' when multiplying by 10: $H \mid T \mid ones \mid tenths \mid hundredths$

Two questions were then presented on PowerPoint slides and discussed.

For the second slide, '100 packs' was replaced with '30 packs'.

Figure 5.10: Joe Class B Tasks

Phase 4: Practice Questions

With the next PowerPoint slide (figure 5.11), Joe talked through the example, making reference to the different order of the first line in comparison to his previous examples (figure 5.9). Students were then asked to complete the questions.

PowerPoint

Example: Multiply 0.8 by 200

$$\begin{aligned} 0.8 \times 200 &= 0.8 \times 2 \times 100 \\ &= 1.6 \times 100 \\ &= 160 \end{aligned}$$

Q What are the missing numbers?

$$a)\ 0.7 \times 400 = 0.7 \times \sim \times 100$$
$$= \sim \times 100$$
$$= \sim$$

$$b)\ 0.19 \times 4000 = 0.19 \times 4 \times \sim$$
$$= \sim \times 1000$$
$$= \sim$$

$$c)\ 0.143 \times 3000 = 0.143 \times \sim \times 1000$$
$$= \sim \times 1000$$

Figure 5.11: Joe Class B Tasks

Phase 5: Textbook Exercises

Students were given a choice of which exercise to complete from an online textbook; as with the learning goals these were tiered.

Bronze:	Q1 0.3×100 …	Q6 1.43×1000 …
Silver:	Q1 0.7×300 …	Q6 3.24×30 …
Gold:	Q1 $3.2 \div 10$ …	Q6 $0.3 \div 10$ …

In all the multiplication sums, the multiplier was a decimal and the multiplicand was a multiple of a power of ten. The decimal multipliers had one, two or three digits and were less than one in early questions, with decimals greater than one introduced later. Some questions needed the use of zero placeholders. Early answers were integers, with later answers a mixture of decimals and integers but there appeared to be no connections between questions. The division questions started with dividends greater than one, moving onto decimals less than one in later questions. Students were asked to check their own work using a calculator.

Phase 6: Review of Learning Objectives

The students reviewed the learning objectives by writing about WWW (what went well) and EBII (even better if I…); this activity was undertaken in most lessons. This included time to self-mark their work.

General

Whilst the intention behind the ‘big question’ was to offer a real-world application, Joe stated this was one of those occasions where he had not found anything suitable. This led him to focus on the learning outcomes in phase 2. The first set of questions in phase 3 was context free. The PowerPoint questions were based on a real website and Joe drew attention to his personal interest in the product, stating "these are my favourite type of pens … they take a little while to run in but they're good when they're run in". However, while the students were likely to be familiar with online ordering, it was far less likely there would be a need to perform calculations of this type. As such, students could have perceived the activity as classroom mathematics rather than having relevance outside the school context.

Multiple solution strategies were possible, including formal and informal written methods and mental strategies. A range of strategies was seen, although these were usually applied separately to different questions rather than in parallel and applied to the same question. For example, the shared strategy in phase 3 was factorising the multiplicand into a power of ten and a single digit (figure 5.9), with the product of the initial multiplier and power of ten calculated first. The phase 4 PowerPoint (figure 5.11) introduced a different order; Joe stated this was also "perfectly reasonable" but there was no further discussion about why this was the case. In the whole-class discussions, multiplying by a power of ten was done mentally, but having modelled the use of columns early in the lesson, Joe advocated their use to check if students were unsure.

When the overall sequence is considered, there was a transition from multiplying by ten, hundred and a thousand to tens, hundreds and thousands. In addition, multipliers and multiplicands varied such that zero placeholders were needed in phase 4. However, all the multiplication questions were of the same format, and could have been completed using the strategy modelled in phase 3. For each question, there was no requirement to go beyond the correct application of a procedure and there were few discernible connections between questions. As such, there did not appear to be systematic variation in the exercises (3.3.5.2).

(d) TOM: Discourse

In terms of the mathematical register, the main use of mathematical terminology in whole-class talk related to multiplication. Whilst Joe used 'multiply' and 'times' in about equal proportion, students only used the term 'times'. Early in the lesson, Joe asked the students about the difference between multiplying by 'ten' and 'tens'. A student responded, “would it be like like (.) times tweny and times thirty”. After Joe offered an explicit positive evaluation of this response, there was no further reference to these terms.

In whole-class episodes, approximately 90% of talk was classified as mathematically related (figure 5.12: subdivision 1). There were two sub-categories of mathematically related whole-class episodes; turn-taking was again the most common form of talk, although monologues by Joe formed a higher proportion of talk compared to class A (subdivision 2). The monologues consisted of explanations or instructions given by Joe (subdivision 3 T:E). Within turn-taking, taken together, IRE exchanges and the variant with an extended evaluative turn were the most common form of talk, which were often linked to form extended question-and-answer sequences (subdivision 3 T:IRE). On a couple of occasions Joe shifted away from an IRE pattern by not immediately evaluating a student’s response (subdivision 3 T:M); this allowed different answers to be collected and compared. Occasionally students initiated a turn-taking exchange by asking a question or making a comment (subdivision 3 S:L). These general discourse patterns were similar to his other class.

<table>
<tr><td colspan="5">Whole-class talk</td><td></td></tr>
<tr><td colspan="4">Mathematically Related Episodes</td><td>Other</td><td>Subdivision 1</td></tr>
<tr><td colspan="3">Turn-taking</td><td>Monologues</td><td>Other</td><td>Subdivision 2</td></tr>
<tr><td>T: IRE (inc. IRE variant)</td><td>T
M</td><td>S
L</td><td>T:E Explains/ Instructs</td><td>Other</td><td>Subdivision 3</td></tr>
</table>

Figure 5.12: Joe Class B Breakdown of Whole-Class Episodes

Joe indicated whether student responses in IRE exchanges were satisfactory or unsatisfactory in similar ways as with class A. Namely, satisfactory responses were acknowledged immediately with affirmative words, by repetition, the utilisation of the response, or some combination of these. About three-quarters of IRE responses were treated as satisfactory, with about one fifth of the evaluative turn containing a superlative. Most of the remaining responses consisted of ‘don’t know’ comments or mathematical statements that were treated as unsatisfactory. Joe used indirect indicators that responses were unsatisfactory, such as repeating the question, asking a follow-up question or offering an explanation; on a couple of occasions, he offered his interpretation of student reasoning. Occasionally, Joe's evaluative turn was more neutral as he sought clarification or redirected the question. For example, having rephrased the first 'pens' question (figure 5.10) as "If I want a hundred packs of pens, how much is that going to cost me", Joe continued:

109	T:	… how many do I want (.)
110	S1:	you need ten packs
111	T:	ooh do I need ten packs (.) arh no I want a hundred packs of pens not a hundred pens OK (.) I want a hundred packs (.) so what am going to need to do (.)
112	S2:	you (.) times it by a hundred
113	T:	brill so you're going to do three pounds forty-nine (.) multiplied by a hundred [writing: 3.49×100]

Extract 5.10: Joe Class B

When line 111 is taken into account, in line 109 Joe appeared to be asking students to state the first part of the 'pens' question, namely how many packs were wanted, although the phrasing was ambiguous. The student's response (line 110) was not valid with respect to that question. In line 111, Joe reworded the student's response as a question, but before anybody responded he appeared to interpret the student's reasoning as misreading the question as 'how many packs for a hundred pens', for which ten is correct. Joe then answered the question himself and concluded his turn with a new question. So, while Joe did not treat the response as satisfactory, he identified the source of the trouble as the reading of the question rather than a 'faulty' mathematical calculation. In line 113, Joe used a superlative as an explicit positive evaluation. This was a typical example of how he used superlatives, namely in the evaluation of a correct calculation. He then extended the student's response by including the complete sum.

As with class A, Joe regularly revoiced the students' satisfactory contributions, often extending the response by including more of the mathematical calculation (e.g. extract

5.10, line 113). Another common feature of Joe's treatment of satisfactory responses was the use of his turn to transition to a new question or idea, whereas, his actions associated with unsatisfactory responses kept the focus on the original question.

As with class A, IRE exchanges were often linked to form a step-by-step approach to a multistage procedure. The majority of questions had a limited range of mathematically valid responses, with the questions posed often a request for one step in a procedure (e.g. extract 5.10, line 112) or the result of a calculation. Within IRE sequences, responses treated as satisfactory could be considered to be mathematically valid statements, albeit with varying levels of mathematical precision. Responses treated as unsatisfactory either did not address the question posed (e.g. extract 5.10, line 110) or contained a mathematical error, such as "three hundred and ten" in response to 3.14×10.

In overall terms, most IRE exchanges were initiated with a question with a limited range of mathematically valid responses. The most common pattern was a valid mathematical response, treated as satisfactory and with an immediate transition to a new question. In the remaining cases, Joe's subsequent actions maintained the focus on the original question, either by seeking clarification, asking a follow-up question or by offering a direct explanation himself. Joe often extended his own turn, by including an explanation or summary.

There were a few occasions when students-initiated turn-taking exchanges by making a comment without a direct invitation from Sam. For example, the discussions about multiplying by a hundred (figure 5.10) continued with:

25	T:	we would move the numbers over two
26	S:	basically er add a zero on them
27	T:	mmm sometimes (.) sometimes (.) how far do we move the numbers if we are multiplying by ten

Extract 5.11: Joe Class B

As Joe did not ask a question in his turn (line 25), the student's comment was classified as student-initiated. In line 27, Joe partially acknowledged the student's contribution before redirecting attention back to using place value and column headings.

Some periods of talk by Joe were classified as monologues, which were mostly extended explanations; there was no student talk long enough to be similarly classified. In all bar one IRE exchange, the students' contributions were shorter than Joe's. Taken in conjunction with monologues, this resulted in about four fifths of class-level talk being undertaken by Joe.

(e) TOM: Sequencing

Joe controlled the overall trajectory of the lesson, shaped by the prepared resources, and he regulated the mathematical focus through questions asked and explanations given. For much of the lesson he directed or redirected attention towards approaches he introduced, with the inference made that he attended to his mathematical horizon more often than student reasoning when managing classroom interactions.

There was one occasion when Joe maintained the focus on the original response after treating it as satisfactory. This occurred after a student had offered "one hundred and four point seven pounds", which Joe had accepted but went on to ask "can I have point seven pounds". The following exchange then occurred:

135	S1	no it's one hundred and four pounds point er (.) erm and seven pence
136	T:	is it seven pence
137	S2:	yes
138	S3:	seventy
139	S4:	no seven
140	T:	this is an interesting one isn't it (.) how would we write (..)

Extract 5.12: Joe Class B

It appeared that Joe had used the opportunity to explore students' understanding of decimals in a monetary context. However, he structured the explanation and drew to a conclusion by writing £104.07 for seven pence and £104.70 for seventy pence and used columns to demonstrate £104.7 was the same as £104.70.

Multiple solution strategies are possible for multiplying by tens, hundreds and thousands. For the majority of the lesson Joe maintained the focus on factorising as a power of ten and a single digit, multiplying by the power of ten first. This included redirecting students back to this approach when they were offering an alternative starting point that could have led to a correct solution. For example, when discussing the cost of thirty packs of pens, the following exchange occurred:

124 T: … which as you quite rightly said is three point four nine times ten times three (.) go on

[on board: $3.49 \times 30 = 3.49 \times 10 \times 3$]

125 S1: so the nine would be um er (.) twenty-seven

126 T: oh hang on can we times by ten first

Extract 5.13: Joe Class B

The student's comment on line 125 could have been part of a mathematically valid approach to calculating the required cost. However, in line 126, Joe redirected the student's attention to multiplying by ten, without ascertaining how the student was going to use the partial calculation.

When the example on the PowerPoint used a different order, multiplying by the power of ten last, Joe explained this was also "perfectly reasonable", but there was no further whole-class discussion or comparison of methods. For the stage when students needed to multiply by a single digit, Joe accepted a range of strategies, such as doubling twice for multiplying by four, but stated, "I would recommend maybe doing it like that" whilst writing out a 'long multiplication' sum. So, whilst acknowledging alternative strategies, it appeared that Joe endorsed particular approaches.

(f) Interpretation of Classroom Activities

Joe regulated whole-class interactions through his use of questions and explanations and he usually directed or redirected student responses towards approaches he introduced (e.g. extract 5.13, line 126). Whilst there were a couple of cases where Joe appeared to be interpreting students' reasoning (e.g. extract 5.10, line 111), on many occasions Joe revoiced student contributions, rephrasing in more formal terms. The inference made was that Joe often attended to his mathematical horizon when interpreting whole-class interactions.

Some of Joe's revoicing into more precise language may or may not have represented the student's understanding. For example, when Joe was introducing the lesson goals the following exchange occurred:

19 T: … what do we do when we multiply by a hundred
20 S: would you (.) er (.) take it up twice
21 T: what do you mean by take it up
22 S: so when you have the grid
23 T: er the number columns
24 S: yeh you would take the decimals up twice
25 T: we would move the numbers over two

Extract 5.14: Joe Class B

In line 21 Joe seeks clarification from the student; as such this was classified as a neutral evaluative turn. In Joe's next turn (line 23), he reworded the student's comment, which the student accepted in their following turn (line 24). In line 25, Joe reworded the student's contribution into more precise terms. As there was no example involved, it was unclear how the student would enact her manipulation of decimals.

In the post-lesson interview, Joe stated that he had added the additional examples of $3.14 \times 10, 3.14 \times 20$ and 3.14×40 after the discussion of the learning goals, as he felt the students had not recalled the difference between multiplying by ten and tens. However, he said in retrospect that 20 and 40 were not the best choice for examples, as the students could use doubling strategies that would not work on other decimal multiplications.

(g) Cognitive Demand

Multiple solution strategies were possible and were acknowledged at some points in the lesson. However, the whole-class discussions predominantly focussed on the one approach and this solution strategy was structured step-by-step by Joe through his use of IRE sequences. Consequently, such talk could be seen as an articulation of a particular procedure. The inclusion of column headings offered a more explicit representation of the decimal number system and the factorisation/product approach allowed students to manipulate numbers in different ways, both of which provided an opportunity for students to make links to the underlying mathematical structure. For example, the requirement for and the placement of zero as a placeholder was illustrated through this modelling. So, whilst the talk remained focussed on particular examples, some discussions had the potential to convey meaning beyond the particular examples used. Consequently, the majority of this 'talk as mathematics' was classified as procedural or process.

In phase 3, there was a whole-class discussion about a question that required the modification of a decimal answer for a monetary context (extract 5.12), but this type of question was not met again, so the students had no further opportunities to engage with this issue. Different types of decimal manipulation were required in different questions. For example, later questions had calculations that most students would not have been able to do with mental methods and some required the use of zero placeholders. So, the efficiency and ease of different approaches varied between questions. As students could choose their own strategies, there was the potential for them to develop a more flexible approach to calculations, but there was no whole-class comparison of different approaches. Consequently, whilst it was possible for students to work on the tasks in a manner that required high levels of cognitive demand, this would have been at their own instigation; the students could have successfully completed the tasks by applying one approach with which they were familiar, thereby working at lower levels of cognitive demand.

(h) Classroom Norms

As with class A, Joe directed student activities, determined the mathematical content and regulated many of the mathematical approaches taken. As such, he could have been seen as having high levels of agency within his classroom. The learning goals and tiered exercises contributed to the notion that mathematics has a hierarchy, both in terms of the mathematical content and the learning of mathematics.

IRE was the most common interaction, and Joe tended to revoice student contributions, often summarising or extending in his turn. These actions reinforced the norm that judgments about the legitimacy of mathematical contributions resided with Joe. There were a few occasions where students made self-initiated contributions, demonstrating some agency (e.g. extract 5.11). There were a couple of occasions where Joe postponed an evaluation (e.g. extract 5.12, lines 137-139), which shifted him out of the evaluator role for short periods of time, but he drew the discussions to a conclusion by offering an explanation, reinforcing the notion that he was the arbiter of correctness and was responsible for explanations.

Most of the questions asked by Joe related to a particular procedure and had a limited range of valid mathematical responses. Most student responses were short and were often only part of a calculation; a description of a partial next step in a procedure was treated as satisfactory, with Joe often adding greater mathematical detail in his turn. Within IRE sequences, satisfactory responses were usually followed by an immediate transition to a new question, whereas unsatisfactory responses led to follow-up questions until the 'correct' response was forthcoming. These contributed to the narrative that the efficient production of the correct answer was the expectation of an 'effective student of mathematics' and that if errors were met, they should be corrected. The fact that students were asked to check their answers using a calculator, and to move on if they were correct, could have reinforced the notion that finding the 'correct' answer was the goal, with the process used of less importance.

5.3.1.6 Joe: Summary for Class B

As with class A, analysis was an iterative process and an OMF for class B was populated during this process. The lesson-specific summary OMF (figure 5.13) was the final working document that summarised the pedagogical profile for the lesson. Again, it was this lesson-specific summary OMF that was used in the further analysis when lessons and teachers were compared. As before, a written summary is provided first to communicate the key themes from the analysed lesson.

(a) Joe Class B: Written Summary – Multiplying Lesson

As with class A, the following outlines the key themes from the analysed lesson discussed in the preceding section (5.3.1.5).

A) Curriculum
   a) From a 'lower' curriculum route
B) Organisation
   a) Seatwork: individual, peer discussion encouraged but not required; self-selected tiered textbook work
   b) Block of whole-class work followed by block of seatwork
C) Discourse patterns: aligned with patterns previously reported (e.g. Drageset, 2015)
   a) IRE dominant, limited solution questions in linked sequences
   b) Typical satisfactory/unsatisfactory norms: 'Correct' responses ⇨ follow-on questions (superlatives used in one fifth); 'errors' ⇨ follow-up questions
   c) Revoicing; rephrasing (increasing precision) and explanations extended
D) Tasks
   a) Initial questions: tenuous links to 'real-world' (psuedo-contexts)
   b) Focus on one solution strategy and limited use of multiple representations.
   c) Model – exercises; limited range of permissible change (Bills et al., 2006)
E) Sequencing
   a) Focus on mathematical horizon, (re)direction to solution strategy introduced
   b) Some attention drawn to mathematical structure through questioning
F) Teacher Cognition
   a) Espoused priority was to develop student understanding; articulated lesson goals had performance orientation

b) Interpretation of student responses usually privileged his mathematical horizon; occasional attention to student reasoning

G) Classroom norms

a) Teacher arbiter of correctness

b) Procedure counts as explanation; mathematics as a hierarchy including ways of working

H) Cognitive Demand

a) Potential high but range low to high (limited press to move beyond procedural)

(b) Joe Class B: Summary OMF – Multiplying Lesson

This lesson-specific summary OMF was a working document, with descriptions in note form and bespoke abbreviations used (see table 5.1).

Joe Class B Summary OMF

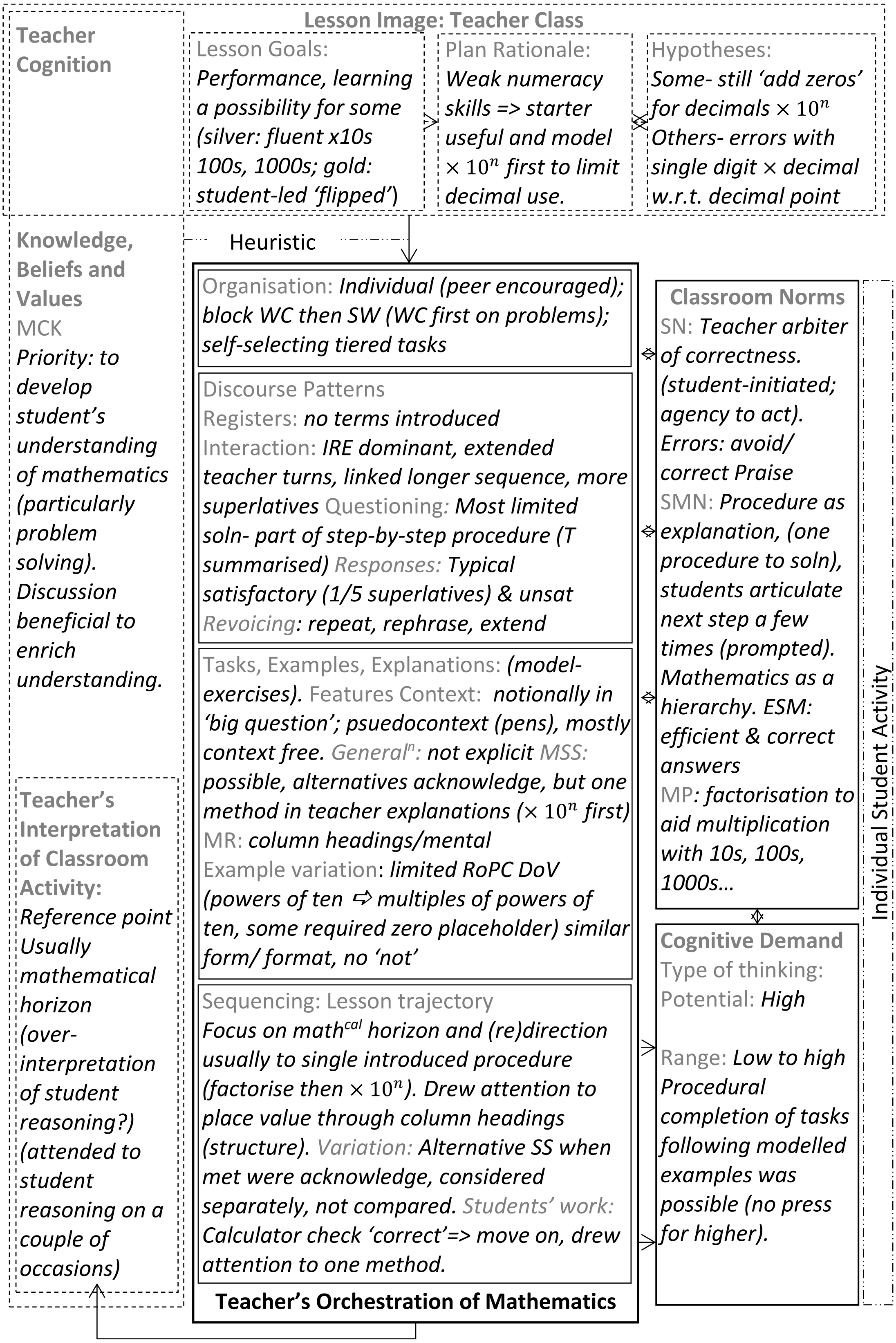


Figure 5.13: Joe Class B Summary OMF

5.3.1.7 Joe: Class Comparisons

One lesson from each class was recorded and analysed. Joe's two lessons were compared, which used the lesson-specific summary OMFs as a key point of comparison. These allowed common and differential themes to be identified and facilitated the interrogation of the lesson narratives to identify episodes that exemplified these features. When these two lessons were compared, there were many similarities in Joe's pedagogical approaches, although differences were also noted

The comparison of lessons from the perspective of the summary OMFs can be found in appendices 3.5.1 & 3.5.2, where underlining was used to highlight common and differential features. These comparisons, cross-referenced with the more detailed lesson narratives, informed the written summary given below. For ease of comparison, this follows the same structure as the written summaries for the individual lessons. Similarities between the classes are indicated with normal type and *differences* are indicated with *italics.*

A) Curriculum
    a) Class A: '*Middle*' route (sequence of lessons: 'percentage of', percentage change, *inverse percentages*). Class B: '*Lower*' route (sequence of lessons: *powers of ten(s)*, 'percentage of', percentage change)
B) Organisation
    a) Seatwork: Individual, peer discussion encouraged but not required; self-selected tiered textbook work
    b) Class A*: Interleaved seatwork with whole-class*. Class B: *Block of whole-class work followed by a block of seatwork*
    c) *Class B contained fewer students*
C) Discourse patterns: aligned with patterns previously reported (e.g. Drageset, 2015)
    a) IRE dominant, limited solution questions in linked sequences
    b) Typical satisfactory/unsatisfactory norms: 'Correct' responses ⇨ follow-on questions (Class B: *superlatives used more frequently*); 'errors' ⇨ follow-up questions
    c) Revoicing; rephrasing (increasing precision) and explanations extended
D) Tasks

a) Class A: *some links to 'real-world' contexts*. Both: elements of psuedo-contexts
b) Focus on one solution strategy and limited use of multiple representations.
c) Model – exercise; limited range of permissible change (Bills et al., 2006)

E) Sequencing
a) Focus on mathematical horizon, (re)direction to solution strategy introduced
b) Some attention drawn to mathematical structure through questioning

F) Teacher Cognition
a) Espoused priority was to develop student understanding; articulated lesson goals had performance orientation
b) Interpretation of student responses usually privileged their mathematical horizon. Class B: *occasional attention to student reasoning*

G) Classroom norms
a) Teacher arbiter of correctness
b) Procedure counts as explanation; mathematics as a hierarchy including ways of working

H) Cognitive Demand
a) Potential high but range low to high (limited press to move beyond procedural)

5.3.2 Teacher: Sam

5.3.2.1 Background

Sam engaged in a wide range of CPD opportunities that included both school-initiated and self-directed courses. This resulted in Sam being involved in more CPD than the majority of his colleagues.

The school placed students in sets for mathematics. Their policy was to use targets generated by SATs performance and internal assessments to make judgments, but after the sets were first established, this also involved teacher recommendation. Most movement occurred once a year and, rather than use a re-ranking of all students based on attainment, the two or three students with highest and lowest attainment in each set was considered and moved based on teacher recommendation and set sizes. However, the school leadership also placed a few students in particular sets based on issues related to behaviour. Consequently, about ten percent of students moved classes each year. There was a single curriculum plan for each Key Stage 3 year group, but alternative routes were identified for different sets. In each topic area, content was tiered, with different content listed as 'core' or 'optional' for different sets.

The pre-lesson interviews indicated that the level of planning and resource development for the recorded lesson was typical of about one in five of his lessons. He followed his departmental scheme of work and the specific topic chosen for the recorded lessons fell within the then current overarching theme. However, the recording did prompt some adjustments. First, Sam chose to teach lessons on indices out of sequence from the order specified in the scheme of work, as he felt that this was a "rich topic full of potential". Second, he used very similar resources for both classes and planned the two lessons together, as he thought this would "aid comparison"; this was not his usual practice. As will be discussed in more detail in subsequent sections, this resulted in a typical lesson for class A, but a more atypical lesson for class B. The recorded lessons were one hour long and were one of four mathematics lessons held each week.

5.3.2.2 Teacher's Knowledge, Beliefs and Values

In the pre- and post-interviews, Sam outlined his priority was to enable students to engage in tasks where they could develop an understanding of the mathematics. In particular, he wanted students to have the opportunity to discuss mathematical ideas and consider the 'why'. He felt student engagement in "rich tasks" that allowed discussions was important, as this was an effective way to expose and challenge misconceptions whilst also leading students to become more invested in the mathematics. Whilst different routes through the curriculum were identified, Sam felt he had the discretion to develop bespoke routes based on his assessment of the classes. However, in practice, Sam reported that more often than not his lessons aligned with the suggested content for each set; the recorded lessons for class B were some of the exceptions.

A key strategy Sam identified for developing students' understanding was planning lessons around "what about the mathematics". He explained this as having a clear idea as to what mathematical concepts students would meet when engaging in particular activities. Referencing variation theory, Sam talked about how, both individually and in collaboration with departmental colleagues, he considered the specifics of the examples being used. He used an example of algebra, where from a starting point of $x + x$ he explained he would consider what difference it would make moving onto, say, $y + y$ or $2x + x$. Sam stated he had paid particular attention to the impact of changing exponents when planning the PowerPoint questions.

5.3.2.3 Sam: Class A

This was a Key Stage 3 class, composed of about thirty students who had attainment profiles higher than average for that year group in the school.

(a) Lesson Specific: Teacher's Knowledge, Beliefs and Values, incorporating Initial Lesson Image

In the pre-lesson interview, Sam's stated lesson goal was to enrich the students' understanding of powers. He linked this to the demonstration of particular skills, namely the recall and application of the rules of indices. As such, his articulated learning goal was considered to be learning orientated, albeit with a potentially narrow interpretation of learning.

The students had done some work on indices in the previous year but had not met formally written rules of indices. Sam planned to present completed examples of how two powers could be combined and ask “why?”; he thought this would draw attention to the reasoning behind the ‘rules’. A questions-rules-questions pattern was planned as he thought this would help students bridge the gap between specific examples and the general case. He anticipated students would be confident using individual rules with integers but thought algebraic powers and the inclusion of coefficients might bring errors.

(b) TOM: Organisation

The lesson was timetabled for one hour and lasted fifty-six minutes due to lesson transitions. Twenty-four minutes were spent at a whole-class level, with the remaining time spent on individual seatwork. The desks, large enough to seat two students, were arranged in two concentric horseshoes. Whilst Sam encouraged students to talk to their peers, the tasks set could have been completed independently.

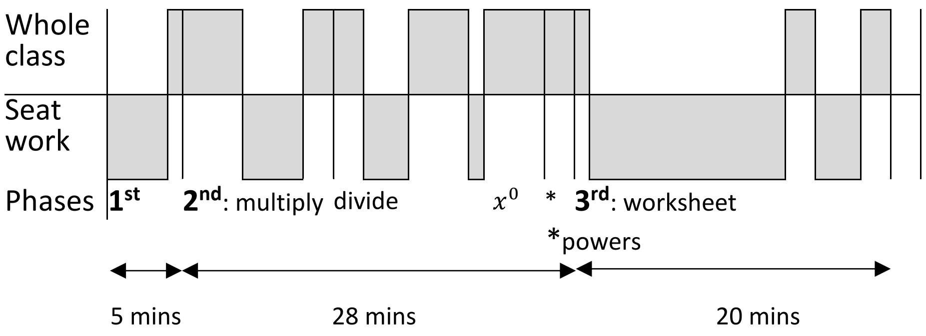


Figure 5.14: Sam Class A Organisation

(c) TOM: Tasks, Examples and Explanations - Overview

The lesson was entitled ‘The rules of indices’. This was one of two planned lessons on indices drawn from the department’s three-week topic plan on algebra.

Phase 1: Starter

Sam displayed a PowerPoint slide (figure 5.15) and students were expected to start once seated.

Starter – no calculators! Write down the first 12 square numbers Write down the first 6 cube numbers What is the value of 2^4

Figure 5.15: Sam Class A Tasks

The first two parts required the recall of the meaning of square and cube numbers and the capability to calculate or recall their values, whilst the last part required some knowledge of indices.

Phase 2: Rules of Indices

The mathematics focussed on the ‘rules of indices’ relating to multiplication, division and powers. Each operation was introduced with a completed example, accompanied with “why”. There was a mixture of seatwork and whole-class discussions for each rule and students had the opportunity to complete questions displayed on the PowerPoint.

Expanding powers, writing them as a repeated multiplication followed by simplification, was a recurring part of Sam’s explanations. This approach was included in some of the prepared PowerPoint slides.

Prepared Resources

Multiplication introduction: Sam stated, “two squared times two cubed is two to the power five; why?”, accompanied by $2^2 \times 2^3 = 2^5$ written on the whiteboard.

<u>Multiplying – spot the rule</u> What is $2^5 \times 2^4$ What is $3^4 \times 3^2$ What is $8^2 \times 8^3$ Pattern? Note: same base! What is $x^2 \times x^6$ What is $x^a \times x^b$	<u>Write down the rule!</u> Index Law 1: Multiplication Rule $a^m \times a^n = a^{m+n}$ Example: $5^4 \times 5^9 = 5^{13}$	<u>Three quick questions</u> Find the value of a $y^5 \times y^8 = y^a$ $3^6 \times 3^a = 3^{17}$ $2k^3 \times 4k^2 = ak^5$

Division introduction: “two the power five divided by two cubed”, accompanied by $2^5 \div 2^3 = 2^2$ written on the whiteboard.

<u>Write down the rule!</u> Index Law 2: Division Rule $a^m \div a^n = a^{m-n}$ Example: $5^9 \times 5^5 = 5^4$	<u>Three quick questions</u> Find the value of a $y^9 \div y^2 = y^a$ $14t^6 \div 7t^4 = at^2$ $k^7 \div k^7 = a$	Anything to the power of zero is 1 $a^0 = 1$

Powers introduction: The ‘Powers – spot the rule’ slide.

<u>Powers – spot the rule</u>

What is $(5^2)^3$

$$(5 \times 5)^3$$
$$= (5 \times 5) \times (5 \times 5) \times (5 \times 5)$$
$$= 5^6$$

Figure 5.16: Sam Class A Tasks

Some further prepared slides were not used and Sam inserted three additional examples into the sequence of planned questions. First, after the initial example of $2^5 \times 2^4$, Sam added $3^4 \times 2^3$. Second, after the powers to powers’ introduction, he added $(2^3)^2$. Finally, after a whole-class discussion about “anything to the power zero is one”, Sam introduced a pattern of:

$$10^3 = 1000 \qquad 10^2 = 100 \qquad 10^1 = 10 \qquad 10^0 = \_\_ \qquad 10^{-1} = \_\_$$

Figure 5.17: Sam Class A Tasks

For all the prepared questions, there was a common base, either a positive integer or a letter. The requirement for ‘the same base’ for the ‘rules of indices’ to apply was highlighted by “note: the same base!” on the first slide and by the inserted question, namely $3^4 \times 2^3$.

Phase 3: Worksheet

A worksheet was completed by the students (figure 5.18).

As with all bar one question in phase 2, all the questions had a common base and focussed on multiplication, division and powers. There was some variation, in so far as addition of like terms and three power terms were introduced.

1. $y^3 \times y^2$
2. $y^5 \times y^6$
3. $y^7 \times y^2$
4. $y^3 \times y^{16}$
5. $(y^2)^2$
6. $(y^3)^5$
7. $(y^4)^5$
8. $(y^7)^2$
9. $(y^5)^{15}$
10. $y^8 \div y^2$
11. $y^{16} \div y^4$
12. $y^7 \div y^6$
13. $y^{16} \div y^8$
14. $y^{15} \times y^9$
15. $(y^{16})^2$
16. $y^8 \times y^9 \times y^{12}$
17. $y^0 \times y^0$
18. $y \times y^{10} \times y^{20}$
19. $y^0 \times y^2$
20. $y \times (y^7)^9$
21. $y^2 + y^2$
22. $y^{28} \div y^2$
23. $y^4(y^{28} \div y^2)$
24. $y^3 + 3y^3$
25. $y^3(y^{25} \div y^5)$
26. $y \times y^{29} \times y^3$
27. $2(y^2)^4$
28. $3(y^4)^3$
29. $y^0 \times (y^{10})^{10}$
30. $y^0(y^{32} \div y^{16})$
31. $y^4 + (y^2)^2$

Simplify the calculations, cross out the corresponding squares in the grid. When finished the remaining squares will reveal a message.

y^6	C	y^{13}	Y	y^{100}	A	y^7	0	$3y^{12}$	R	y^{75}	B
$2y^4$	T	y^{20}	Y	1	A	y^{22}	U	y^{29}	H	y^{11}	S
y^{15}	I	y^4	E	y^{18}	A	y^9	P	y^{33}	T	y^{19}	U
y^{21}	R	y^{16}	Q	$2y^2$	C	y^{28}	E	y^5	X	y^{26}	I
y^{64}	N	y^3	R	y^{24}	K	y^2	Y	y^{32}	B	y^{23}	H
y	D	y^8	M	y^{17}	I	$2y^8$	A	y^{30}	G	y^{27}	G
y^{10}	H	y^{12}	J	$4y^3$	W	y^{14}	F	y^{36}	T	y^{31}	S

Figure 5.18: Sam Class A Tasks

General

In phase 2, Sam introduced the rules of indices sequentially. The expansion of power terms as repeated multiplication followed by simplification had potential to provide insight into the mathematical structure underpinning ‘the rules of indices’, but justification was based on single examples. In the ‘spot the rules’ and ‘quick questions’ there were multiple changes between each question; bases, exponents and coefficients were changed. Varying more dimensions was likely to have made it harder for students to discern the variation, making relationships less visible, especially with the small number of examples. Moreover, there was no requirement to go beyond finding individual answers, and with few discernible connections between questions, variation did not appear to be controlled systematically (3.3.5.2)

Some task elements had the potential to draw attention to aspects of the ‘rules of indices’. Specifically, the inserted ‘not the same base’ question could have highlighted the requirement for the ‘same base’, and the inclusion of non-unitary coefficients to the requirement for correct manipulation of the non-power elements. In addition, the inclusion of ‘not multiply’ (addition) could have highlighted the relationship of the ‘rules of indices’ to multiplication. However, other boundaries, such as bases not being zero, were not part of the tasks. The range of permissible change (Bills et al., 2006) of bases and exponents for the 'rules of indices' to hold are not straightforward. The rules hold for all integer exponents provided the base is non-zero and for any positive base with rational exponents. In this lesson, in pre-prepared questions all powers had exponents that were positive integers or zero, which included the answers to division questions as all the exponents of the dividend were set to be greater or equal to the exponent of the divisor. Whilst the inserted powers of ten example (figure 5.17) offered a window into how integer powers could vary, drawing attention to the range of permissible change of bases and exponents was not an explicit part of the prepared tasks.

(d) TOM: Discourse

In terms of the mathematical register, students were exposed to topic specific language, but it appeared they were not expected, as a matter of routine, to develop their use of formal language in their own public talk. For example, the term 'power' was commonly used by all in phrases such as "two to the power (of) four", with the abbreviated form "two to the four" also heard and accepted. Normally, bases and exponents were referred to by their value or representation, namely "two" and "four" in the example given above, but there were a few occasions where these component parts of powers were indicated in more general terms. For example, the context implied "just add the powers" referred to the exponents rather than the whole term, and there were other similar examples, from both students and Sam. The term 'base' was first introduced by a student when discussing $3^4 \times 2^3$:

66	T:	oh did someone just say what dif- what is the difference (.) it's a different (..)
67	Ss:	denominator base base
		[answers called out in quick succession by multiple students]
68	T:	excellent word Jade (.) base well done (…)

Extract 5.15: Sam Class A

In line 68, Sam made no overt response to "denominator" and praised the use of "base". Subsequently, there were no further examples of students using the term, although Sam used the term twice; for example:

219	S:	… do you just keep the number each time

220 T: (.) d- yes you keep the base the whole time

Extract 5.16: Sam Class A

In line 220, Sam accepted the student’s ambiguous response, and inferred they had been attending to the base as he revoiced the contribution. However, there was no press for the student to use more formal language themselves.

In whole-class episodes, approximately 90% of talk was classified as mathematically related episodes (figure 5.19: subdivision 1). There were two sub-categories of mathematically related whole-class episodes; turn-taking was the most common form of talk, with the remaining time classified as monologues (subdivision 2). Taken together, IRE exchanges and the variant with an extended evaluative turn were the most common form of turn-taking, which were often linked to form extended question-and-answer sequences (subdivision 3 T:IRE). The remaining turn-taking episodes consisted of questions or comments initiated by students (subdivision 3 S:I) and questioning sequences initiated by Sam but with multiple student contributions between each of his turns (subdivision 3 S:M). The monologues consisted of mathematical explanations and instructions given by Sam (subdivision 3 T:E) and a small number of student explanations (subdivision 3 S:E).

<table>
<tr><td colspan="6">Whole-class Talk</td><td></td></tr>
<tr><td colspan="5">Mathematically Related Talk</td><td>Other</td><td>Subdivision 1</td></tr>
<tr><td colspan="3">Turn-taking</td><td colspan="2">Monologues</td><td>Other</td><td>Subdivision 2</td></tr>
<tr><td>T: IRE (inc. variant)</td><td>S: M (Peer to Peer)</td><td>S: I</td><td>T:E Explains/ Instructs</td><td>S: E</td><td>Other</td><td>Subdivision 3</td></tr>
</table>

Figure 5.19: Sam Class A Breakdown of Whole-Class Episodes

In IRE exchanges, Sam indicated whether the student response was satisfactory or unsatisfactory in similar ways to Joe. Usually, Sam indicated responses were satisfactory immediately and directly by using affirmative words such as “yep”, repetition, the utilisation of the response, or some combination of these; superlatives, such as “wonderful”, were included in about one third of cases. On occasions, Sam moved on and asked a new question without any comment. Common to all positive evaluations was the immediate transition to a new question, task or idea. On the other hand, unsatisfactory responses were indicated by pauses, follow-up questions or by redirection of the question to another student, thus maintaining the focus on the original question. For example:

46 T: two to the power four
47 Ss: sixteen
48 T: sixteen loverly OK (.) Ben (.) what does two to the power four mean
49 Ben: sixteen
50 T: what does it mean
51 Ben: oh it means two times two times two times two
52 T: excellent (.) good (.) so we sort have got a rough understanding of that (.) erm (..) I am going to tell you first that (.) two squared times two cubed is two to the power five (.) ok um why (..) Mel
[some students' hands were raised]
53 Mel: you add the powers
54 T: yes so that's what we can do (…) Fay
[some students' hands were raised]

Extract 5.17: Sam Class A

In line 48, Sam's use of repetition and a superlative indicated a satisfactory response, and he concluded with a new question, which led to an extended question-and-answer sequence. After Ben responded (line 49), Sam's partial repetition of the question, with an emphasis on 'mean', indicated a deficiency and the location of the issue (line 50). Ben treated Sam's evaluation as an opportunity to take a further turn (line 51), which was accepted by Sam immediately and directly with the use of a superlative (line 52). Sam extended his turn and transitioned to the first operation by providing a completed example. In Sam's next evaluative turn, he offered agreement (line 54), but the emphasis on "do", followed by the nomination of another student, indicated Mel's response, whilst not incorrect, was not sufficient. As such, this had features of both positive and negative evaluations and was coded as both.

One indication of an unsatisfactory response was Sam asking a follow-up question (e.g. extract 5.17, line 50). A second indication was Sam redirecting the question to another student. In these circumstances, he continued until either a satisfactory answer was heard, or he offered an explanation himself. For example, Sam asked what $2k^3$ meant and the following exchange occurred:

173 S: erm (.) is it (.) six (like) k four times (.)
174 T: Mel
175 Mel: is it two k times two k times two k
176 T: (.) any other ideas Jan
177 Jan: is it k to the power three times two
178 T: (.) how I'm going to write it out for now (.) er you do two times k times k times k...
[writing: $2 \times k \times k \times k$]

Extract 5.18: Sam Class A

In this exchange, Sam redirects the question to three students before offering his own explanation after a short pause (line 178). Whilst Jan's response was mathematically valid (line 177) it was not acknowledged by Sam (line 178).

A third indication of an unsatisfactory response was Sam not responding verbally, which resulted in an extended pause; in some cases, the interaction shifted from IRE to a peer-to-peer exchange. These had characteristics of an informal debate, in so far as opposing points of view were put forward with multiple student contributions. For example, when Sam asked for the value of a in $k^7 \div k^7 = a$ the following exchange was heard:

243 Ss: k
[chorused by multiple students]
244 T: (...)
245 S1: yes its k isn't it
246 S2: one
247 S3: cause seven divided by seven is one
248 S4: seven minus seven is zero
249 S5: no isn't it zero
250 S6: k divided by k is k
251 S7: one
252 S8: I think it's zero
253 S9: no it isn't its one guys OK its one

254 T: er vote then (.) vote for zero

Extract 5.19: Sam Class A

The opening turns of this exchange was coded as IRE, with line 243 coded as R and the pause by Sam treated as a turn and coded E. Line 245 onwards was coded as peer-to-peer turn-taking. In this exchange, there appeared to be a combination of students offering opinions and responding to others, though not necessarily to the directly preceding turn. Sam drew the peer-to-peer exchanges to a close by asking for votes for zero then one, after which he offered his own explanation.

Within IRE sequences, there were also differences in the precursors to responses being treated as satisfactory or unsatisfactory. In all bar one case, responses treated as satisfactory were mathematically valid statements. In the exception, a student offered “five k”, which Sam initially acknowledged before correcting himself to “k to the power of five not five k” after a short pause. Whereas responses treated as unsatisfactory occurred with two different precursors. First were mathematically valid statements that did not meet Sam’s requirements. Whilst the student responses did not contain mathematical errors *per se*, Sam usually indicated where the response did not match his expectations (e.g. extract 5.17, line 50). Second were mathematical errors, with mathematically invalid answers or statements (e.g. extract 5.19, line 243). There was another category of ‘don’t know’ responses, which Sam treated in a similar manner as unsatisfactory responses.

IRE exchanges were often linked to form a step-by-step approach to a multistage procedure and regularly contained some level of explanation by Sam. The majority of questions in these exchanges had a limited range of mathematically valid responses. In addition to the recall of multiplication facts in phase 1, about one quarter of questions were considered simple and self-contained, in so far as they were considered easily answerable by students from the information contained in the question. For example:

235 T: … that is my representation of fourteen t to the power six (..) and that’s my representation of seven times t to the power four and I’m going to do the same as I did before there are t’s divided by t’s (.) and one times by one it doesn’t change (.) what is this

[written: $\frac{14\times\cancel{t}\times\cancel{t}\times\cancel{t}\times\cancel{t}\times t\times t}{7\times\cancel{t}\times\cancel{t}\times\cancel{t}\times\cancel{t}}$ and pointing at the uncancelled ts]

236 S: t squared

237 T: and fourteen divided by seven (.)

238 S: two

Extract 5.20: Sam Class A

In line 235, Sam's explanation included the expansion of power terms and a structure for division. The prompts given by Sam as he asked the question at the end of his turn meant students could have answered without a full understanding of the division of powers. In line 237, Sam used a simple and self-contained question to complete the process. Considered as a sequence, Sam's explanation and questions offered a structure for the division of powers.

In overall terms, the most common IRE pattern was a limited solution question, followed by a valid student response treated as satisfactory, and concluded with an immediate transition to a new question or idea. A regular but less common pattern was a student response containing a mathematically invalid element, which was treated as unsatisfactory. On the few occasions when more open questions were posed, such as when meaning was asked for, all types of student responses and teacher evaluative turns occurred, which included responses that did not contain mathematical errors *per se* but were treated as unsatisfactory.

On a few occasions, students initiated a turn-taking exchange by asking a question or making a comment without a direct invitation from Sam. In all bar one case, Sam acknowledged the contribution and treated it in the same manner as a satisfactory response to a question. For example, in extract 5.16, line 219, a student asked, "do you just keep the number each time", which Sam positively evaluated. The exception occurred when negative exponents were discussed:

369 S: sir you know if you got to like nought point nought nought nought whatever (.) then we are still not ever going to minus like

370 T: don't worry about minuses they'll look after themselves…

Extract 5.21: Sam Class A

In line 369, Sam appeared not to engage with the student's comment, and he moved the talk on by asking a new question at the end of his turn.

Some periods of talk by Sam were classified as monologues, most of which were extended explanations. There were three student explanations long enough to be similarly classified. These occurred after Sam stated, "anything to the power zero is one (.) sixty seconds talk between yourselves to tell me why that is". After a period of seatwork, the students offered extended explanations that contained multiple stages.

(e) TOM: Sequencing

Sam controlled the overall trajectory of the lesson, shaped by his use of prepared resources and initiation of transitions between activities. He directed or redirected attention to his foci, namely onto strategies he introduced or procedures he was structuring through questioning. The inference made was that Sam attended to his mathematical horizon rather than the interrogation of student reasoning when managing most classroom activities.

A recurrent theme was the expansion of power terms as repeated multiplication, followed by simplification. Sam drew attention to this through direct explanation and the treatment of student responses as satisfactory or unsatisfactory dependent on whether they conformed to this approach (e.g. extract 5.17, lines 46-52). Indeed, Sam indicated that this approach offered meaning beyond a particular example used. For example, when $x^a \times x^b$ was discussed students' initial responses related to x^{ab}, and peer-to-peer debates followed. Sam drew these to a conclusion by referring back to a previous numerical example, stating "so if our base is exactly the same (.) and we have got our powers there we can add them (.) OK", while pointing sequentially at the numerical exponents. Moreover, at the end of a discussion about $2^5 \div 2^3 = 2^2$ Sam stated "so (.) your rule that you have spotted then erm is to take away those top powers". Consequently, this approach, drawing on examples, was linked to meaning and justification of the 'rules of indices'.

The key point of departure from drawing on an expanded layout as the principal explanatory approach was when exponents of zero were met. This started with the example $k^7 \div k^7 = a$. The initial response chorused by students was "k" and a peer-to-peer debate followed, focussed on zero and one (extract 5.19). Sam then intervened and asked questions where the dividend and divisor were the same: "nine divided by nine"; "one million four hundred and twenty-three thousand two hundred and sixty eight divided by one million four hundred and twenty-three thousand two hundred and sixty eight"; "smiley face divided by smiley face". After "one" had been chorused for each answer, Sam repeated the initial question "so k to the power seven divided by k to the power seven is …" and students chorused "one" in response.

This was followed by a few overlapping student comments until the following interaction occurred:

282	Lyn:	but seven takeaway seven is zero
283	T:	is zero (.) absolutely (.)
284	S1:	so what's the answer
285	S2:	one
286	T:	so if we are doing our takeaway rule Lyn (.) that would give us k to the power zero
287	Ss:	that's what I put; which is zero; that's zero
288	T:	let's see what it is now (.) erm now I'm going to tell you to add fuel to the fire (..) that anything to the power zero is one [showing PowerPoint slide: $a^0 = 1$]

Extract 5.22: Sam Class A

The "but" in Lyn's comment (line 282) indicated that she might have had an issue with the previously acknowledged answer of "one" and the reference to sevens indicated she was applying a 'rule of indices' to the exponents. After Sam agreed with Lyn's comment another student called out a question (line 284), answered by a third student. In line 286, Sam readdressed Lyn, confirming that based on takeaway rule the answer would be "k to the power zero", which was followed by a number of overlapping student voices (line 287). It was unclear as to whether the students were referring to the base, exponent or the resulting value, namely the quotient, but it appeared there was still some debate about the role of zero. This appeared to be an example of the teacher and students attending to different aspects of powers, resulting in inefficient communication (Mason, 2011b) (3.3.7.5).

When the overall trajectory of the $k^7 \div k^7 = a$ discussion is considered, Sam appeared to be attending to a direct route to the resulting value of one, whilst students who contributed to whole-class discussions appeared to be drawing on the subtraction of exponents used in previous division questions. In particular, in the early part of the discussion Sam drew attention to "one" when the dividend and divisor were identical, whilst some students appeared to have subtracted exponents and were attending to the resulting exponent, namely zero. Sam engaged with the student-initiated focus on the manipulation of exponents and zero, acknowledging that k^0 was the result of following the rule (extract 5.22, line 286), but an explicit link was not made between identical dividends and divisors that resulted in "one" and $k^{7-7} = k^0$. In other words, students would have to have noticed the juxtaposition of $k^7 \div k^7 = 1$ and $k^{7-7} = k^0$ in the discussion, and understood that both these arguments were correct, in order to apprehend the respective roles of zero and one inherent in the relationship $k^0 = 1$.

Sam then stated, "anything to the power zero is one" and gave the students sixty seconds to prepare an explanation through discussions with a peer. Three student explanations followed; for example:

304	S:	so could it be a hundred of that to the power seven divided by a hundred to the power seven yeh (.) then a hundred divided by a hundred will equal one then seven minus seven will equal zero so that that just equal one

Extract 5.23: Sam Class A

The students made reference to cancelling, but the language remained ambiguous, particularly in relation to zero and one. Whilst Sam acknowledged these student explanations, with comments such as "that looked quite good", there was no interrogation of the individual students' reasoning or reference to how the rules of indices related to the dividend/divisor approach.

Sam then introduced patterns related to powers of ten, moving from 10^3 to 10^0 initially, after which 10^{-1} was also considered. This was the second occasion when he moved away from prepared resources. He used a sequence of simple questions to establish the following pattern:

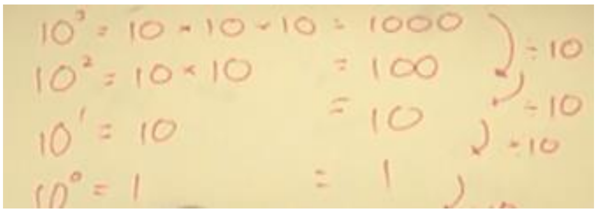


Figure 5.20: Sam Class A Artefact

Sam concluded with:

370	T:	… so can everyone see from this pattern at least that 10 to the power zero could equal 1 (..)
371	Ss:	yeah
372	T:	thank you (.) look have a look at it see what you think I quite like that erm (.) but there you go

Extract 5.24: Sam Class A

The use of "could" in line 370 and "see what you think" framed this as a possibility. Sam indicated he liked the approach, but with the implication that the students could make their own judgment. This was offered as an alternative approach to understanding powers, and particularly when the exponent was zero, but there was no explicit link made between this and the earlier approaches.

Over the course of the lesson the students were exposed to three approaches related to an exponent of zero: the division 'rule of indices'; quotient when the divisor and dividend were identical; descending powers of ten. Consequently, the students had the opportunity to experience and gain insights from different perspectives. However, each approach was treated separately, and language was ambiguous at times, so it was not always clear which aspect of powers were being referred to. Indeed, in the $k^7 \div k^7 = a$ discussion there was evidence that Sam attended to the quotient whilst students attended to the exponent, with some remaining confusion regarding the relationship between zero and one. Taken as a whole, these episodes allowed the exploration of exponents from different perspectives, but discontinuities in attention may have made the mathematical features less apparent.

There were two occasions when additional examples had the potential to draw attention to the range of permissible change. The first was $3^4 \times 2^3$, inserted after multiplication had been introduced. The initial student response “is it six to the power seven” indicated they had misapplied the ‘rule’ for exponents and applied it to bases. The treatment of this response as unsatisfactory and the subsequent discussion had the potential to draw students’ attention to this boundary, namely the requirement for the same base for the ‘rules of indices’ to be applied. Whilst Sam made reference to identical bases when discussing ‘rules of indices’, and written on a PowerPoint slide was ‘Note: same base!’, this was the students only exposure to a ‘not the same base’ question (Watson and Mason, 2006).

The second was when Sam extended his powers of ten example to negative exponents.

361	T:	lets make a prediction (.) erm what’s ten to the power minus one
362	S1:	minus ten (.) or is it minus one
363	Ss:	minus ten (inaudible)
364	T:	who said that (.) was that you Jon
365	Jon:	nought point one
366	T:	listen (.) if I carry on the pattern of keep dividing by ten (.) divide by ten again here [writing: $10^{-1} = 0.1 \quad = {}^{1}/_{10}$]
367	S2:	what s-sir say that was minus 10 (.) minus 10 to (the power) minus two (.) would that be nought point nought one
368	T:	yep
369	S3:	sir you know if you got to like nought point nought nought nought whatever (.) then we are still not ever going to minus like
370	T:	don’t worry about minuses they’ll look after themselves erm so can everyone see from this pattern at least that 10 to the power zero could equal 1 (..)

Extract 5.25: Sam Class A

In line 364, Sam appeared not to react to the mathematically invalid comments and asked Jon to repeat his answer. Sam indicated this response was satisfactory when he wrote on the board. There followed two student-initiated comments. The first, related to 10^{-2} (line 367), was positively evaluated, but without further comment (line 368). In the second, the student appeared to have recognised that powers of ten remain positive with negative exponents. In this case, however, Sam curtailed that line of reasoning and redirected attention back to an exponent of zero and no further references were made to negative powers.

(f) Interpretation of Classroom Activities

Sam regulated whole-class interaction through his questions, explanations and management of student responses. As he usually maintained the focus on strategies he introduced, the inference made was that he usually attended to his mathematical horizon when interpreting classroom activities, with the interrogation of student reasoning far less common.

In the post-lesson interview, Sam did identify elements of the lesson where he adapted his plans. For example, the ‘Note: same base!’ written on the PowerPoint slide had prompted him to include the $3^4 \times 2^3$ example. Sam also identified elements where he reacted to student contributions. For example, he stated he introduced the powers of ten because of the extended debate about $k^7 \div k^7$; he had anticipated the equivalence of k^0 and 1 would have been established quickly, but the variety of answers had been unexpected. He took action by including additional ways of deriving $a^0 = 1$, although his focus returned to his expansion approach.

One other area where students had unexpected difficulties was when x^{ab} was offered instead of the correct mathematical response of x^{a+b}. Sam's interpretation was this was a notation issue rather than a “mathematical misunderstanding”. Sam attributed the issue to the fact that students were “happy” to add the exponents when these were numbers but were confused when they could not combine the a and b into a single result.

(g) Cognitive Demand

During the lesson, the students were asked to complete questions and participate in whole-class discussions related to the manipulation of powers. The approach to each question, as shared at a whole-class level, could be seen as an articulation of a particular procedure. Indeed, with simple, self-contained questions a regular feature, much of ‘talk as mathematics’ was initially classified as procedural or computational. However, the common approach involved expressions with powers being written in an expanded form, then manipulated and simplified. This gave the students the opportunity to view powers in different forms and could have provided an insight into the mathematical structure of the expressions that had meaning beyond the particular example under scrutiny. As such, this ‘talk as mathematics’ was classified as related to process.

Whilst much of the students’ written work was just the final answer, some students had division sums written in an expanded form, providing evidence those students could apply the approach to similar contexts. However, there was no discussion about the underlying numerical properties that made the manipulation possible, or how a general case could be similarly manipulated in order to prove the ‘rules of indices’. Consequently, little of the talk was classified as related to mathematical concepts.

When the sequence of questions is considered, there was some variation in bases, exponents and operations, with the opportunity to notice similarities and differences. Whilst the dimensions of variation and the range of permissible change were not explicitly discussed (Watson and Mason, 2006), and the range of variation was limited, the different forms and operations did provide an opportunity for students to make links between examples. On two occasions student comments offered evidence that the lesson activities prompted thinking about cases that did go beyond the specific example. The first was when negative exponents of powers of ten were discussed (extract 5.25, line 369) and a student seemed to notice the value would remain positive. The second was during a discussion of $(2^3)^2$; the process of writing out in expanded form had been completed when the following exchange occurred:

388	T:	which is (..) 2 to the power 6 (..)
		[writing: $= 2^6$]
		and what’s happened incidentally (…)
389	S1:	times the powers
390	T:	times the powers OK
391:	S2:	does that happen every time
392	T:	yep

Extract 5.26: Sam Class A

Whilst the student's question in line 391 does not capture the full scope and limits of the 'rule of indices', they were prompted to consider the link between the specific example under discussion and a more general context.

There were episodes when students misapplied the 'rules of indices' to inappropriate parts of power terms. The first time, as previously discussed, was in the mixed base question, namely $3^4 \times 2^3$, with the response "six to the power seven". The second was with coefficients other than one; when $2k^3 \times 4k^2 = ak^5$ was discussed some students offered "six" for the value of a. These are common mistakes and the mathematically invalid responses were challenged, in so far as they were identified as incorrect, but student reasoning was not explored. Instead, the 'correct' answers were identified and explained through processes structured by Sam.

In overall terms, the lesson provided questions that could have been answered in a procedural manner. However, the use of expanded forms, patterns of sequential powers of ten and the overall sequence of questions provided an opportunity for students to make links to the underpinning mathematical structures. Moreover, attention was drawn to the justification of the 'rules of indices' through the use of 'why' questions, albeit associated with particular examples rather than explicit proofs. Consequently, the potential cognitive demand was high, but students may have successfully completed tasks without eliciting these high levels.

(h) Classroom Norms

Within the lesson, Sam directed student activities and regulated the mathematical direction of travel, determining both the content and mathematical approach to be taken almost all of the time. As such, Sam could have been seen as having high levels of agency within his classroom.

The IRE interaction pattern was a common feature, and this cycle reinforced the norm that Sam was the arbiter of correctness. However, there were occasions when students took more of the initiative where, for example, they offered their own observations, demonstrating aspects of agency. There were also occasions where peer-to-peer interactions unfolded. Whilst some students stated their position rather than respond directly to a previous student's turn, this shifted Sam out of an evaluative role and placed students more centrally, albeit for short periods of time. However, any peer-to-peer interactions were followed by IRE sequences or teacher explanations that concluded with the identification of the answer found acceptable by Sam. So, there were signals that student contributions were valued, but the final judgment about the legitimacy of mathematical contributions resided with Sam.

Many of the questions asked by Sam had single mathematical solutions, and most of the remaining questions, such as 'why', were treated by Sam as having one acceptable response. As discussed, a particular example in expanded form was accepted as an explanation of how powers were combined. So, whilst explanations were part of some student responses, a description of a procedure was treated as a satisfactory explanation. In particular, this implied that exploration of examples was sufficient justification of a general case.

Within IRE sequences, the established norm was for Sam to immediately signal responses he found acceptable, cued by explicit positive evaluation, repetition or transition to the next question. This contributed to the narrative that mathematics is about finding 'the' answer, which an 'effective student of mathematics' can provide. Sam indicated responses were unsatisfactory by asking follow-up questions or redirecting the question to other students until the 'correct' response was forthcoming, and on occasions through extended pauses that cued peer-to-peer debates. The first two contributed to the narrative that the efficient production of the correct answer was the expectation, whereas the latter broadened the narrative to one where discussions of alternatives have a role to play in determining the answer. Indeed, Sam attached the word "exciting" to the prospect of a debate about two contradictory answers. However, the end goal of a 'correct' answer remained, contributing to the narrative that if errors were met, they should be corrected.

The inclusion of self-mitigating phrases were a regular part of student responses, which often took the form of phrasing responses as a question. This process distanced the students from fully endorsing as correct their own mathematical statements. This may have indicated that students did not want errors to be attributed to them and that an 'effective student of mathematics' would avoid making errors.

5.3.2.4 Sam: Summary for Class A

Five lessons of Sam's class A were recorded and analysed in the main study. As previously discussed, one lesson has been reported in detail in the preceding section (5.3.2.3) and is referred to as the indices lesson. The remaining lessons were similarly analysed and lesson-specific summary OMFs were produced. These summary OMFs structured the comparisons in the analysis, where one output was an all-lesson summary OMF for class A.

As before, these summary OMFs were working documents that used descriptions in note form and bespoke abbreviations. Consequently, written summaries have been provided before the presentation of the OMFs to communicate the key pedagogical features of the lessons under discussion.

First, the written summary for the indices lesson is provided. This is followed by the lesson-specific summary OMF (figure 5.21). After this, a written summary of all recorded lessons for Sam's class A is provided, with additional features not seen in the indices lesson indicated by italics. Finally, an all-lesson summary OMF is presented (figure 5.22).

(a) Sam Class A: Written Summary – Indices Lesson

The following outline draws together the key themes from the indices lesson that was reported in detail in the preceding section (5.3.2.3).

A) Content
   a) Content was tiered: formal rules considered 'core' content for this class
B) Organisation
   a) Seatwork: individual, peer discussion encouraged but not required
   b) Interleaved seatwork with whole class
C) Discourse patterns: mostly aligned with patterns previously reported
   a) Power terms used but no press for student use
   b) IRE dominant, single/limited solutions questions in linked sequences, but some peer-to-peer 'debates'

c) Typical satisfactory norms, some variation in unsatisfactory responses: ‘Correct’ responses ⇨ follow-on questions (1/3 superlatives); IRE ‘errors’ ⇨ often follow-up questions, some debates (resolved by IRE/teacher explanation)

d) Revoicing; repeating (occasional rephrasing into more formal/complete phrase)

D) Tasks

a) Focus on rewriting as multiplication (same register different representations); examples used to justify.

b) Model – exercises; limited range of permissible change (Bills et al., 2006)

E) Sequencing

a) Focus on mathematical horizon, (re)direction to ‘rewriting’ solution strategy

b) Some attention drawn to ‘not’ rules of indices

c) Alternative ways to consider zero exponents introduced but separately

F) Teacher Cognition

a) Discussion considered important in developing student reasoning.

G) Classroom norms

a) Teacher arbiter of correctness

b) Procedure counts as explanation; examples justify rules

c) ESM errors: discussion exciting

H) Cognitive Demand

a) Potential high but range low to high (limited press to move beyond procedural)

(b) Sam Class A: Summary OMF – Indices Lesson

As before, analysis was an iterative process and a lesson-specific summary OMF was populated, with figure 5.21 the final working document. Descriptions in note form and abbreviations were again used (table 5.1).

Sam Class A Summary OMF – Indices Lesson

Lesson Image: Sam Class A Lesson Indices

Teacher Cognition

Lesson Goals: *Learning & Performance*

Plan Rationale: *Expanded form + "why" => reasons for rules. Question-rule- question => support generalisation*

Hypotheses: Issues would be with algebraic exponents and coefficients

Knowledge, Beliefs and Values

MCK

Discussion aids engagement and conceptual reasoning

Examples: Careful choice important in developing mathematical reasoning CPD: variation theory

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon Some interpretation of student reasoning (x^{ab} in post-lesson)

Heuristic

Organisation: *Individual (peer-to-peer discussion promoted, written work individual); interleaved WC-SW*

Discourse Patterns *Registers:* *Power terms used – no press for student use.* *Interaction:* *IRE dominant, extended teacher turns, linked longer sequence; some peer-to peer turn-taking (debates).* *Questioning:* *Mostly limited solution in step-by-step procedure (1/4 simple, self-contained) some funnelling* *Responses:* *Typical satisfactory (1/3 superlatives)* *Errors* *often typical unsat-sometimes debate prompted* *Revoicing*: *repeat, rephrase, extend*

Tasks, Examples, Explanations: *(model-exercises).* *Features* Context: *No Context.* *General^{n}:* *math^{cal} structure exposed (expanded form), examples used to justify.* *MSS:* *mostly single, alternatives separated.* *MR:* *same register - different forms (manipulation principles not explicit part of tasks).* Example variation: *Some variation in examples (operations, coeffs); not explicit, limited RoPC DoV (e.g. integer exponents)*

Sequencing *Focussed on math^{cal} horizon, (re)direction to introduced processes (rearrangement). Plans adapted (zero exponents) – different approaches introduced but separately - links not explicit.* *Variation:* *Example set – 'not' the same base example introduced. Other boundary conditions not explicitly met.* *Students' work:* *Errors used as starting point for peer-to-peer debates*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. Errors discussion exciting but to be corrected. (student-initiated; agency to act, debates but IRE/T explanation resolution). ESM: Efficient production of correct answer & ideas/debate;* *SMN:* *Procedure as explanation, (one at a time procedure to soln). Examples used to justify rules.* *MP*: *Expanded form for manipulation of powers. 'Power' used w.r.t exponent and whole term*

Cognitive Demand

Type of thinking: Potential: *High (MR math^{cal} structure; zero exponents)* Range: *Low to high Procedural approach possible (no sustained press for higher)*

Individual Student Activity

Figure 5.21: Sam Class A Summary OMF – Indices Lesson

(c) Sam Class A: Written Summary of All Recorded Lessons

In addition to the indices lesson, which was reported in detail in the preceding section (5.3.2.3), and the pilot study, four other lessons were recorded and analysed with Sam's class A. Three were consecutive lessons on the topic of fractions and one lesson focussed on the manipulation of algebraic expressions using an area model for multiplication. These lessons were analysed in the same manner, with lesson-specific summary OMFs equivalent to figure 5.21 compiled for each lesson.

These lessons for class A were compared, with the lesson-specific summary OMFs being the key points of comparison. Other than the mathematical practices that were particular to the lesson topic, most dimensions of the OMFs were similar. An amalgamated OMF was used to capture the pedagogical profile for all class A's lessons (figure 5.22).

As with the other summary OMFs, descriptions were in note form and abbreviations have been used (table 5.1). However, in order to provide an overview of the common and differential features for different lessons for class A, a written summary is provided below. For ease of comparison, this follows the same structure as the written summary for the individual lessons. The following outlines the key themes from all Sam's recorded lessons with class A. Additional features not seen in the indices lesson are indicated by *italics.*

I) Content
  a) Content was tiered: topics listed in the top route in the departmental scheme of work were considered 'core' content.

J) Organisation
  a) Seatwork: mostly individual - peer discussion encouraged but not required; *some groups tasks - sharing of resources*.
  b) Interleaved seatwork with whole class.

K) Discourse patterns: mostly aligned with patterns previously reported
  a) Some mathematical terms introduced but no press for use – colloquial terms used and accepted
  b) IRE dominant, single/limited solutions questions in linked sequences, but some peer-to-peer 'debates'

c) Typical satisfactory norms, some variation in unsatisfactory responses: ‘Correct’ responses ⇨ follow-on questions (superlatives); IRE ‘errors’ ⇨ often follow-up questions, some debates (resolved by IRE/teacher explanation)

d) Revoicing; repeating, some rephrasing (more formal/complete), extending

L) Tasks

a) Multiple representations: same register *and different registers*; tasks focussed on specific examples, and examples used to justify.

b) Model – exercises; limited range of permissible change – not explicit

c) Either no context or *context with pseudocontext*

M) Sequencing

a) Focus on mathematical horizon, (re)direction to standard strategies introduced

b) Links, when made, focussed on specific examples rather than on links between mathematically significant features of the representations

N) Teacher Cognition

a) Discussion considered important in developing student reasoning.

O) Classroom norms

a) Teacher arbiter of correctness

b) Procedure counts as explanation; examples justify rules

c) ESM errors: usual - errors to be avoided or corrected, but some debates.

P) Cognitive Demand

a) Potential high but range low to high (limited press to move beyond procedural)

(d) Sam Class A: Summary OMF of All Recorded Lessons

As an integral part of the analysis, lesson-specific summary OMFs were produced, which were then compared, leading to the all-lesson summary OMF (figure 5.22). The same note form and abbreviations were used as in previous summary OMFs (table 5.1).

Sam Class A Summary OMF – All lessons

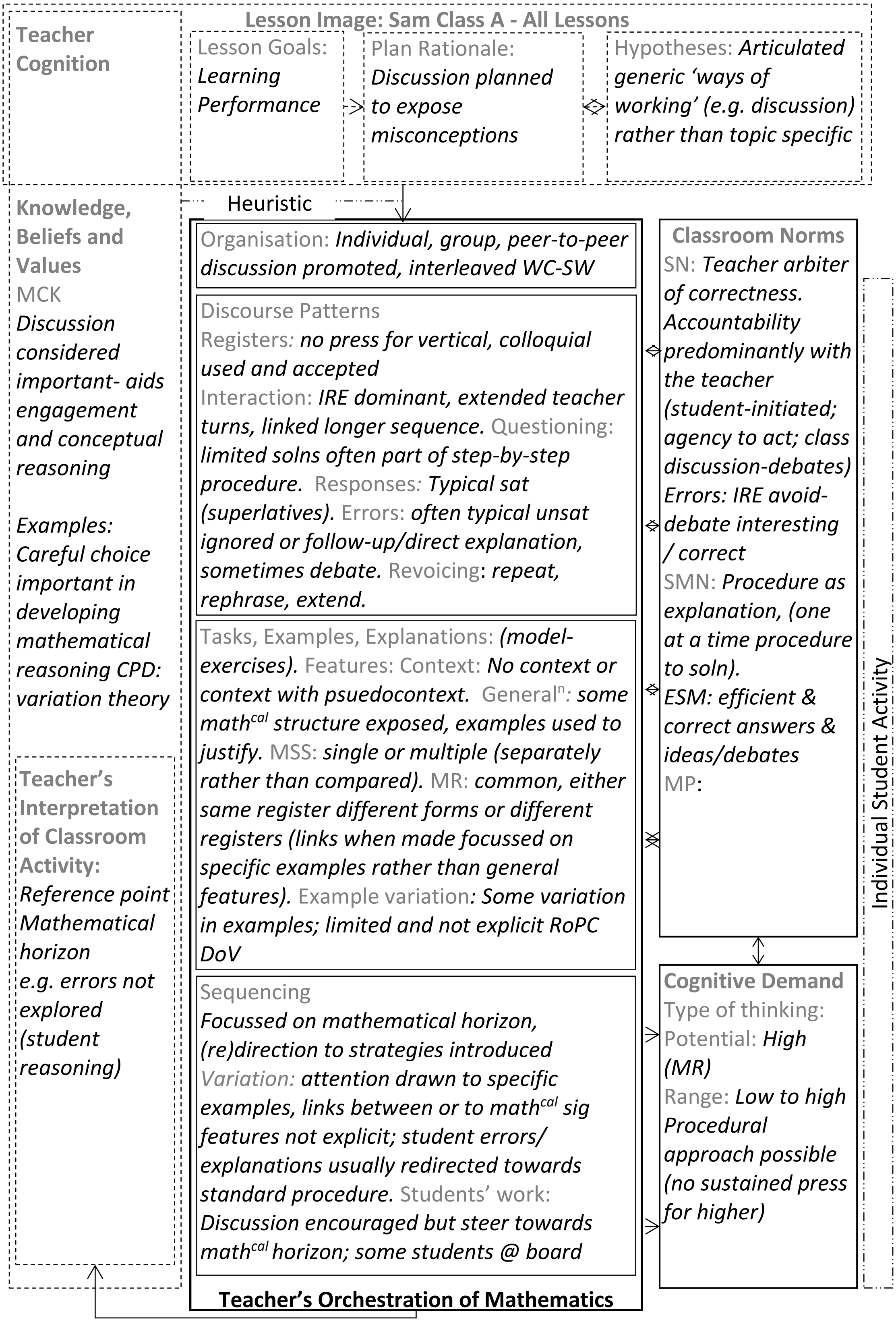


Figure 5.22: Sam Class A Summary OMF – All lessons

5.3.2.5 Sam: Class B

This was a Key Stage 3 class, composed of about twenty-five students who had attainment profiles lower than average for that year group in the school.

(a) Lesson Specific: Teacher's Knowledge, Beliefs and Values, Incorporating Initial Lesson Image

Sam's stated lesson goal was for students to be able to recall and apply the rules of indices. As this was related to the demonstration of particular skills, this was considered to be performance orientated, although his broader discussions also indicated a learning orientation.

The students would have met multiplication of indices in the previous year but had limited experience of division or powers and had not formally met the 'rules of indices'. He decided to base the lesson on the same resources as class A, though anticipated they would spend longer on each operation and he would focus on different PowerPoint slides. Again, Sam thought the questions-rules-questions pattern would help students to bridge the gap between specific examples and the general case. However, Sam thought it was important to use simple numbers, so the students could focus on the patterns, as in his assessment the students had weak number skills. Drawing on his experience of teaching class A, Sam planned to include the question with different bases and the powers of ten example he added to the previous lesson.

Sam acknowledged he had undertaken relatively few discussion-based activities with this class, so elements of the lesson would be atypical. Under normal circumstances, he would have demonstrated each rule with a couple of examples, followed by students practicing similar questions in individual seatwork. Instead, as before, Sam intended to introduce each operation with a completed example and ask "why" but was unsure if these students would engage in discussions in a productive manner, stating he thought off-task behaviour could adversely affect the quality of any discussions. In general, Sam felt that this class needed more encouragement to stay on task.

(b) TOM: Organisation

The lesson was timetabled for one hour and lasted fifty-five minutes due to lesson transitions. Twenty-four minutes were spent at a whole-class level, with the remaining time spent on individual seatwork. The desks were arranged in two concentric horseshoes. Whilst Sam encouraged students to talk to their peers, the tasks set could have been completed independently.

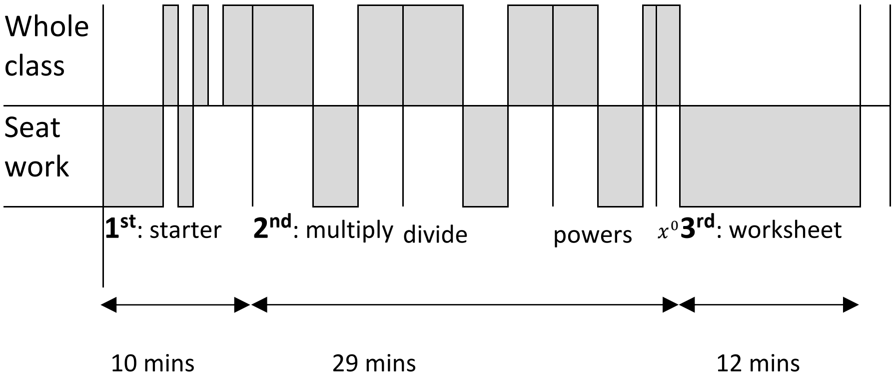


Figure 5.23: Sam Class B Organisation

(c) TOM: Tasks, Examples, Explanations - Overview

As with class A, the lesson was entitled ‘The rules of indices’. This was one of two lessons on indices drawn from the department’s topic plan.

Phase 1: Starter

The same starter was used as for class A (12 square numbers, 6 cube numbers and the value of 2^4) (figure 5.15). Students were expected to begin once seated.

Phase 2: Rules of Indices

The topic was the ‘rules of indices’ relating to multiplication, division and powers. The same overall approach was used as with class A, with operations introduced with a completed example and powers written in an expanded form followed by simplification.

The same PowerPoint presentation was used, although Sam used more of the prepared slides in whole-class discussions.

Prepared Resources

Multiplication: As class A (except $x^a \times x^b$ not discussed).

Division: As class A (except x^0 was discussed after powers) plus:

Division – spot the rule	Division – spot the rule
$2^5 \div 2^3$ $= \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2}$ $= 2^2$	What is $5^7 \div 5^4$ What is $8^3 \div 8^2$ Pattern? Note: same base! What is $x^9 \div x^6$ What is $x^a \div x^b$

Powers: As with class A with an additional introductory question of "five squared to the power two" $(5^2)^2$, plus:

Three quick questions		Anything to the power of zero is 1
Find the value of a $(7^9)^2 = 7^{18}$ $(p^a)^8 = p^{40}$ $(7k^4)^2 = ak^8$	followed by	$a^0 = 1$

Figure 5.24: Sam Class B Tasks

As before, Sam inserted $3^4 \times 2^2$ and the powers of ten pattern, although negative exponents were not included.

Phase 3: Worksheet

The same worksheet was completed by students in individual seatwork (figure 5.18).

General

Apart from negative exponents, the range of examples the students were exposed to was very similar to class A. Consequently, the task features were very similar.

(d) TOM: Discourse

In terms of the mathematical register, students were exposed to some topic specific language, but informal language was frequently used by both Sam and the students; there was no press for students to use formal language in their public talk. 'Power' was heard at a whole-class level and was regularly used by both students and Sam in phrases such as "two to the power (of) four". In addition, 'power' was used to indicate exponents. For example, when $2^3 \times 2^2$ and $2^5 \times 2^4$ were discussed the following whole-class interaction occurred:

125 T: yes there are two times two times two times two times two (.) so noticing (.) what we have written here and what we have done how do you think we could write that

[pointing at $2 \times 2 \times 2 \times 2 \times 2$ written on the board]

126 S1: two five

127 S2: two times five

128 T: good

129 S3: two little five

130 T: two little five (.) two five (.) I'm going to say two to the power of five ok I'm going to say that's a power

[gesturing at the 5 in 2^5]

131 S3: two to the power of five

132 T: that's it well done brilliant OK (..) so (.) let's have a look at this one then this top one here two to the power of five times two to the power of four (.) having a quick look at what happened here

[pointing at exponents in $2^5 \times 2^4 = 2^9$ written on the board]

can we spot anything that might make our lives easier

133 S3: yes cause you could just do two nine

134 S4: two to the power of nine

135 T: you could you could just do two nine

136 S4: two to the power of nine

Extract 5.27 Sam Class B

In line 126, the first student responded with an ambiguous "two five" that required drawing meaning from the context to interpret this as 2^5. In line 127, a second student made a mathematically invalid comment, in so far as a literal translation of the verbal comment as 2×5 is not mathematically equivalent to 2^5. This student's speech overlapped both the first student's talk (line 126) and Sam's subsequent turn (line 128). With no overt interrogation of this comment, there was little information as to the student's intended meaning or whether Sam had noticed this comment. Sam's positive evaluation (line 128) was followed by a third student's self-initiated contribution of "two little five" (line 129), after which Sam repeated both the first and the third student contributions and revoiced by introducing the term "power" (line 130). At the end of this comment, Sam's gestures indicated 'power' referred to the exponent, but it was not completely clear and may have been referring to the whole term. In the following turn (line 131), the third student, who had previously used "little", appeared to rehearse the language of powers introduced by Sam. In line 133, the third student made a further contribution; having previously rehearsed the full version, here they reverted to a more ambiguous form. In this case, Sam mirrored the language of the student's contribution (line 135) and it is another student who revoiced this in a more mathematically explicit form (lines 134 and 136).

Bases and exponents were usually referred to by their value or representation, such as "two to the power four". On occasions, exponents were referred to in more general terms, although informal language was the norm. The most common phrase used for exponents was 'top number'; Sam had referred to exponents as 'powers' a couple of times and 'little' was occasionally used, but after Sam introduced 'top number' this was usually used by all. There were a few occasions when more general reference was made to bases. Sam combined "these things" with pointing at the board and the phrase 'bottom number' was heard from both Sam and students, but 'base' was not heard at a whole-class level.

In whole-class episodes, approximately 85% of talk was classified as mathematically related episodes (figure 5.25: subdivision 1). Turn-taking was the most common form of whole-class talk, with the remaining time classified as monologues (subdivision 2). The monologues consisted of explanations or instructions given by Sam (subdivision 3 T:E). Taken together, IRE exchanges and the variants were the most common form of turn-taking, and were often linked to form extended question-and-answer sequences (subdivision 3 T:IRE). The remaining turn-taking episodes were instigated by student-initiated comments or questions (subdivision 3 S:I); in most of these cases Sam took the following turn and evaluated the students' contributions. The general discourse patterns were similar to his other class, although there were differences in subdivision 3 in relation to the student-initiated or student-led interactions.

<table>
<tr><td colspan="4">Whole-class Talk</td><td></td></tr>
<tr><td colspan="3">Mathematically Related Talk</td><td>Other</td><td>*Subdivision 1*</td></tr>
<tr><td colspan="2">Turn-taking</td><td>Monologues</td><td>Other</td><td>*Subdivision 2*</td></tr>
<tr><td>T: IRE (inc. variants)</td><td>S:I Initiated</td><td>T:E Explains/ Instructs</td><td>Other</td><td>*Subdivision 3*</td></tr>
</table>

Figure 5.25: Sam Class B Breakdown of Whole-Class Episodes

There were two IRE variants. First, in common with class A, an extended evaluative turn was common. The second occurred when more than one student responded before Sam's turn. Often Sam did not directly nominate a student to respond to questions, and students self-nominated by calling out answers. Most of the multiple student responses were characterised by a lack of a pause between responses and/or overlapping speech (e.g. extract 5.27, lines 126-7 & 133-134). These cases were classified as an IRE variant, as the second or third student appeared to be responding to Sam's initial question rather than the previous student's turn.

Within IRE sequences, Sam indicated responses were satisfactory in the same ways as with class A. That is to say, he signalled with affirmative words, repetition, the utilisation of the response or an immediate transition; superlatives were included in about one third of cases. Unsatisfactory responses were usually indicated by a restatement of the question, often with a level of simplification; on a few occasions, a label of funnelling was applied when the level of mathematics was significantly reduced. Occasionally Sam used gestures in his evaluative turn and once used a bald "no". For example, when $(5^2)^3 = 5^6$ was discussed the following exchange occurred:

431 T: … now we get to our final answer five to the power six when we've got these brackets (.) can you think using these numbers here how we might get a shortcut to that (..)

[gesturing at 6]

432 S1: add them

433 T: (…)

[pointing sequentially at 2, 3 and 6]

434 S1: two plus three

435 T: no

436 S2: no it's two times three

437 T: exactly

Extract 5.28: Sam Class B

After the initial response (line 432) and Sam had pointed at the exponents (line 433), the first student changed their response (line 434). This student's behaviour was consistent with her interpreting Sam's actions as communicating her response was deficient. In line 435, Sam gave a bald negative evaluation of this second response, after which another student offered an alternative that Sam immediately positively evaluated. As it was the norm for unsatisfactory responses to be followed by a repetition of the question and for students to self-nominate, additional student turns after unsatisfactory treatment were considered to form IRE sequences (e.g. lines 434 & 436), where the initiating turn was the original question.

About a quarter of turn-taking episodes were not initiated by Sam and instead arose when students made comments or asked questions without a direct invitation. Classified as student-initiated, these occurred after Sam had offered an explanation or positively evaluated another student's response. For example, in extract 5.27 line 129, a student offered "two little five" after Sam had just positively evaluated the previous contribution. As the norm was for Sam to transition immediately to a new question, stage or topic after treating a response as satisfactory, this was deemed to be a student-initiated action. However, whilst these types of instances were coded as student-initiated, they were closely linked to Sam's activities. For example, when Sam asked what $(5^2)^2$ meant the following exchange occurred:

420 T: … so what this actually means is that I need to do five times five (.) and I need to times it by five times five (..)
[writing $5 \times 5 \times 5 \times 5$ on the board]

421 S: so would you do five times five (.) double the answer (.) twenty-five times two

422 T: no you would do five times five5 (.) times (.) five times five (..)

Extract 5.29: Sam Class B

In line 420, Sam concluded his turn by writing the power term as repeated multiplication. As Sam did not ask a question whilst occupied writing on the board, Lee's contribution in line 421 was coded as self-initiated. However, whilst Sam had regularly associated meaning with writing powers in an expanded form in this lesson, the action of writing a sum on the board would normally be associated with a request for the calculation to be undertaken. As such, the student's action in line 421 could be seen as responding to that norm, especially as self-nomination responses were common. In line 422, Sam started his turn with a bald negative evaluation, followed by his own explanation. As above, in the majority of cases, Sam took the turn after the student-initiated comment and he evaluated the contribution.

As with class A, the majority of questions posed by Sam had a limited range of mathematically valid responses. A pattern of step-by-steps solutions, structured by Sam, formed most question-and-answer sequences; about a quarter of questions were considered to be simple and self-contained. For example, after the division question $2^5 \div 2^3 = 2^2$ had progressed to Sam writing $\frac{2\times2\times2\times2\times2}{2\times2\times2}$, the exchange continued:

269 T: … so here I've got all these twos and here I've got two divided by two (.) what's two divided by two

270 Ss: one

271 T: so what I do is cancel these twos out like that (..)
[crossing out pairs of twos]
and then cancel those 2s out like that (.) and those two like that (.)

Extract 5.30: Sam Class B

In line 269, Sam concluded with a simple question and appeared to use this to support his subsequent explanation of how the expression simplified through cancelling (line 271).

Sam treated mathematically valid responses as satisfactory, and mathematically invalid responses as unsatisfactory (e.g. extracts 5.27 & 5.28). There were occasions, however, when more than one student called out answers that contained contradictory answers; for example:

308 T: … what is eight to the power one

309 S1 one

310 S2 eight

311 T: eight (.) well done folks

Extract 5.31: Sam Class B

In line 311, Sam repeats the mathematical response given in line 310 and offers an explicit positive evaluation. There is no apparent reaction by Sam to the mathematically invalid response given in line 309. In general, when mathematically invalid responses occurred alongside valid responses, the former were 'ignored' whilst the latter were positively evaluated.

When Sam asked questions, responses arose from a mixture of teacher-nomination and student self-nomination. The former was less common and occurred when Sam's gestures nominated a student or when he named them. Pauses at the end of a question were an indication that students could self-nominate and call out an answer. For example:

194 T: y to the power five times y to the power eight (…) what is that (…)

[pointing at the exponents in turn]

y to the power of (.)

195 Ss: thirteen

196 T: so y to the power thirteen (.)

Extract 5.32: Sam Class B

At the end of his turn in line 194, Sam stated "y to the power of" and after a short pause a few students chorused "thirteen" (line 195). Sam accepted the response and revoiced in the more complete base-to the power-exponent form. However, earlier in Sam's turn, there were two extended pauses; it would have been an accepted classroom norm to call out responses at these earlier points. In effect, the students had an opportunity to respond but declined to answer. Sam responded by pointing at the exponents and then simplified the question by providing a prompt. Sam's follow-up action after the pauses, namely simplification, was comparable to his treatment of many unsatisfactory responses.

There was a difference in evaluation patterns between IRE sequences and those occasions when Sam evaluated student-initiated contributions (table 5.2). Within IRE sequences Sam treated far more responses as satisfactory compared to unsatisfactory, whilst within student-led interactions the frequencies were similar. When treatment of responses as unsatisfactory were considered, direct acknowledgement through bald negative evaluations, such as “no”, were almost exclusively heard in response to student-initiated comments (e.g. extract 5.29), whereas all bar one of the negative evaluations within IRE sequences was signalled more indirectly (e.g. extract 5.28).

Treated as satisfactory - positive evaluations	IRE	Student-led
Acknowledged indirectly (e.g. limited to repetition, moving on)	8	0
Acknowledged directly (e.g. “yep”)	22	4
Acknowledged directly with judgment of quality (e.g. “excellent”)	13	3

Treated as unsatisfactory – negative evaluations	IRE	Student-led
Acknowledged indirectly (e.g. simplifying the question)	9	0
Acknowledged directly (e.g. “no”)	1	9

Table 5.2: Classification of Evaluations

In overall terms, as with class A, the most common IRE pattern was a single solution question, followed by a valid student response treated as satisfactory, and concluded with an immediate transition. A regular but less common pattern was a student response that contained a mathematically invalid element, which was treated as unsatisfactory. However, there were also student-initiated exchanges where Sam treated mathematically valid contributions as satisfactory and invalid contributions as unsatisfactory, the difference being in the nature of that acknowledgment for unsatisfactory responses (table 5.2).

In addition to offering qualitative feedback on mathematical solutions or comments, Sam also praised students for engagement and task completion. For example, two minutes into the lesson Sam stated, “well done those who have made a start” and later on stated “amazing” when a student reported that they had already copied a rule. In total, in addition to the sixteen occasions when students were praised for offering a satisfactory mathematical contribution, there were thirteen occasions when Sam offered praise related to engagement.

(e) TOM: Sequencing

As for class A, Sam controlled the overall lesson trajectory and regulated the mathematical focus through questions asked and explanations given. Writing powers in an expanded form followed by simplification was similarly central to explanations, and Sam retained the association of this approach with meaning. This included the use of particular examples to justify the general ‘rules of indices’, although there was no explicit discussion as to why examples could be extended to a more general case. In student-initiated exchanges, Sam took the next turn in the majority of cases; he always included some level of explanation that redirected talk back to procedures he had previously introduced.

Again, a key point of departure from the expanded layout was when questions related to zero exponents were discussed. The first instance was $k^7 \div k^7 = a$. Sam pre-empted the discussion of this particular question with nine simpler examples where the dividend and divisor were the same. After ‘one’ was heard as the answer for all, when the $k^7 \div k^7 = a$ question was posed students chorused “one”. As such, this was coded as a funnelling sequence. Sam moved onto the next operation without an explicit discussion of how this linked to processes used in previous division questions; no reference was made to the subtraction of exponents, specifically $k^{7-7} = k^0 = 1$. ‘Anything to the power zero is 1’ was returned to after powers raise to powers had been discussed. As before, Sam used a power of ten example and the following concluded the discussion:

482 T: … ten to the power one is just ten we said earlier (.) so what do we have to do there

[pointing at 100 and then 10]

483 S: divide by ten

484 T: so following the same pattern (.)

485 S: divide by ten

486 T: what is ten to the zero

487 Ss: zero one one

488 T: one (..) dividing by ten each time OK (.) we can do that (.) we can do that (.) with any number we like (.) anything to the power zero is one

Extract 5.33: Sam Class B

As elsewhere in the lesson, Sam structured a step-by-step procedure in which students answered self-contained questions. In line 487, a number of students called out answers simultaneously. In line 488, Sam's repetition indicated acceptance of 'one' and made reference to a general case, but there was no explicit response to zero.

As with class A, the students experienced three approaches to an exponent of zero: the division 'rule of indices'; identical divisor and dividend; descending powers of ten. However, the application of the rule of indices to division with equal exponents was not discussed at a whole-class level and links between the other two approaches were not explicitly made. Taken as a whole, these episodes allowed the exploration of zero exponents from different perspectives, but the separation of approaches may have made the mathematical features less apparent.

In terms of meeting examples of 'not rules of indices', Sam introduced $3^4 \times 2^2$. Once more, a student initially tried to combine both the exponents and bases, stating in a student-initiated comment "three add two equals five and you do five to the power whatever". Sam replied with a bald "no" before offering his explanation. There were further references to 'the same base', both verbally by Sam and on the PowerPoint slides, but this was the only direct exposure to a 'not' example. After $2^5 \div 2^3$ had been discussed, a student asked as self-initiated question:

292 S: but what if the numbers are different

293 T: if the numbers are different it does not work the rule does not work

Extract 5.34: Sam Class B

Whilst it had to be inferred from 'different' that the student was referring to bases, her question drew attention back to the issue of common bases (line 292) and indicated this student was attending to some level of generality beyond the particular question. Sam redirected attention back to the rule without exploring this further (line 293).

(f) Interpretation of Classroom Activities

Sam regulated whole-class interaction through his questions, explanations and management of student responses. He directed or redirected the focus of attention onto strategies he introduced or was structuring through questioning. The inference made was that Sam usually attended to his mathematical horizon when managing classroom interactions, with the interrogation of student reasoning far less common.

In the post-lesson interview, Sam explained he thought the students had grasped division of powers but $k^7 \div k^7 = a$ could have caused confusion if a zero exponent was introduced at that point of the lesson. As such, he pre-empted the discussion of this question by following the approach he felt had worked with class A, namely examples with identical devisors and dividends. However, he also expressed pleasure and some surprise as to how engaged the students had become in the mathematical discussions. He also felt that the students had been able to understand the main concepts in a manner more similar to class A than he had anticipated.

(g) Cognitive Demand

As the resources and approaches taken were similar to class A, the level of cognitive demand was also similar. In particular, the students engaged in activities related to the manipulation of powers, and the approaches taken at a whole-class level were the enactment of particular procedures. However, the expanded form gave students the opportunity to engage with mathematical terms expressed in different forms, with the potential to provide insights into mathematical structures. As such, these elements of 'talk as mathematics' were classified as process.

The sequence of questions introduced some variation in bases, exponents and operations. Whilst the dimensions of variation and range of permissible change were not an explicit part of tasks, and range of variation was limited, the students did have opportunities to make links. For example, when division problems were under discussion, one student asked "but what if the numbers are different"; it appeared, therefore, one student had drawn on the multiplication discussions and had recognised common bases was also an issue for other operations. On occasions, students misapplied the rules of indices. For example, when questions with coefficients were completed some students applied the 'rule' to the coefficients as well as the exponents. As with class A, these mathematical errors were identified and corrected, either directly by an explanation by Sam or through a sequence of simple, self-contained questions. As students' contributions were not interrogated, it remained unclear as to whether reasoning inconsistent with mathematical principles had been challenged.

In overall terms, tasks could have been completed in a procedural manner, but expanded forms, sequential powers of ten and the sequence of questions provided an opportunity for links to be made to the underpinning mathematical structures. Although the notion that consideration of a limited number of examples could justify a more general case could have been reinforced. Consequently, the potential cognitive demand was high, but students may have successfully completed tasks without eliciting these high levels.

(h) Classroom Norms

As with class A, Sam directed student activities and regulated the mathematical direction of travel, and as such, Sam had high levels of agency within his classroom. The dominance of IRE exchanges reinforced the notion that he was the arbiter of correctness. Student-initiated contributions did occur throughout the lesson, demonstrating aspects of student agency. However, these were also evaluated by Sam, who retained the judgment about the legitimacy of mathematical contributions.

Many of the questions asked by Sam had single mathematical solutions, often asked as one step in a sequence of questions related to a larger problem. As discussed, Sam structured the step-by-step approach and it was sufficient for an 'effective student of mathematics' to be able to provide the individual responses. After Sam introduced one 'rules of indices' through the use of a particular example, he announced the result applied to "anything". As with other instances, this carried a message that manipulation of expanded forms of particular examples contained a level of justification of a more general case.

After satisfactory responses were offered Sam moved on, whereas after unsatisfactory responses he took follow-up action that resulted in the 'correct' response being identified in subsequent turns. Moreover, if both satisfactory and unsatisfactory responses were offered, Sam acknowledge the ‘correct’ response and ‘errors’ appeared to be ignored. Within IRE sequences, praise, in the form of superlatives, was only offered in relation to satisfactory responses. These contributed to the narrative that an ‘effective student of mathematics’ would efficiently produce correct answers and if errors were met, they should be corrected. The inclusion of self-mitigating phrases was part of some student responses, which may have indicated that students did not want errors to be attributed to them.

5.3.2.6 Sam: Summary for Class B

As with Sam's class A, five lessons were recorded and analysed, with lesson-specific summary OMFs produced. These were used for comparisons across the different lessons for class B and the cross-class comparisons. As before, written summaries are provided first in order to communicate the key themes from the analysed lessons.

First, the written summary for the indices lesson is provided, which is followed by the lesson-specific summary OMF (figure 5.26). Then a written summary of all the recorded lessons for Sam's class B is provided, followed by the all-lesson summary OMF (figure 5.27).

(a) Sam Class B: Written Summary – Indices Lesson

The following outline draws together the key themes from the indices lesson reported in detail in the preceding section (5.3.2.5).

A) Curriculum (atypical – same resources used as for class A)
   a) Content was tiered: formal rules considered 'optional' content for this class
B) Organisation
   a) Seatwork: individual, peer discussion encouraged but not required;
   b) Interleaved seatwork with whole class
C) Discourse patterns: mostly aligned with patterns previously reported
   a) Informal language the norm
   b) IRE dominant, single/limited solutions questions in linked sequences, some multiple student turns from the same initial I; some student-initiated turns
   c) Typical satisfactory/unsatisfactory norms for IRE: 'Correct' responses ⇨ follow-on questions (1/3 superlatives); IRE 'errors' ⇨ follow-up questions, some funnelling; multiple responses ⇨ correct solution acknowledged/ error ignored; Atypical response for student-initiated 'errors' ⇨ bald 'no'
   d) Revoicing; repeating (occasional rephrasing into more formal/complete phrase)
D) Tasks
   a) Focus on rewriting as multiplication (same register different representations); examples used to justify.
   b) Model – exercises; limited range of permissible change (Bills et al., 2006)

E) Sequencing

    a) Focus on mathematical horizon, (re)direction to ‘rewriting’ solution strategy

    b) Some attention drawn to ‘not’ rules of indices

    c) Alternative ways to consider zero exponents introduced but separately

F) Teacher Cognition

    a) Discussion considered important in developing student reasoning.

        i) For this class: belief discussions may be unproductive – off-task behaviour, praise for engagement

        ii) Interpretation of classroom activity: students engaged in productive discussions and understood main concepts in a similar manner as class A

G) Classroom norms

    a) Teacher arbiter of correctness

    b) Procedure counts as explanation; examples justify rules

H) Cognitive Demand

    a) Potential high but range low to high (limited press to move beyond procedural)

(b) Sam Class B: Summary OMF – Indices Lesson

As before, analysis was an iterative process and an OMF was populated, with the lesson-specific summary OMF (figure 5.26) the final working document. Descriptions in note form and bespoke abbreviations were used (table 5.1).

Sam Class B Summary OMF – Indices Lesson

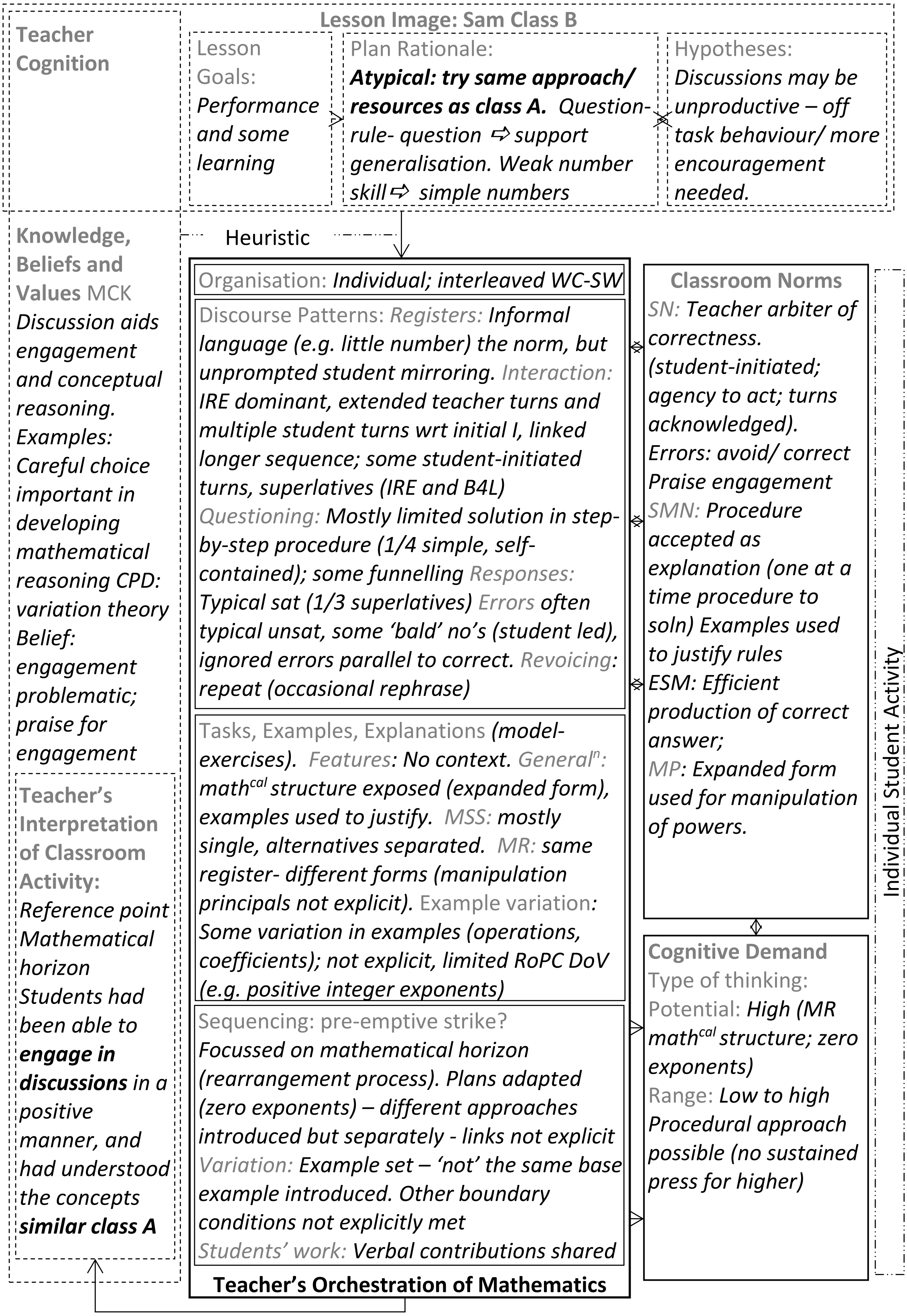


Figure 5.26: Sam Class B Summary OMF – Indices Lesson

(c) Sam Class B: Written Summary of All Recorded Lessons

In addition to the indices lesson, which was reported in detail in the preceding section (5.3.2.5), four other lessons were recorded with class B. As with class A, these were three lessons on fractions and one on manipulation of algebraic expressions. Lesson-specific summary OMFs compiled for each lesson were compared and an all-lesson summary OMF was produced (figure 5.27).

Similar to class A, many of the dimensions of the OMF were stable across class B's lessons, although there was more variation in the use of whole-class discussions and the treatment of errors. The following outlines the key themes from all the recorded lessons. Additional features not seen in the indices lesson are indicated by *italics* and features seen predominantly in the indices lesson are indicated by **bold** type.

A) Content (atypical – same resources used as for class A – less material covered)
   a) Content was tiered: topics listed in the higher route were considered 'optional'
B) Organisation
   a) Seatwork: mostly individual - peer discussion encouraged but not required
   b) Interleaved seatwork with whole class.
C) Discourse patterns: mostly aligned with patterns previously reported
   a) Some mathematical terms introduced but no press for use – colloquial terms often used and accepted
   b) IRE dominant, single/limited solutions questions in linked sequences, some multiple student turns from the same initial I (Initiate-Response-Evaluate); some student-initiated turns.
   c) Typical satisfactory /unsatisfactory norms for IRE: 'Correct' responses ⇨ follow-on questions (superlatives); IRE 'errors' ⇨ follow-up questions, some funnelling; multiple responses ⇨ correct solution acknowledged, error ignored. (the **bald 'no'** predominantly seen in the reported lesson)
   d) Revoicing: repeating, some rephrasing into more formal/complete phrase, extending explanations.
D) Tasks
   a) Multiple representations: same register *and different registers*; tasks focussed on specific examples, and examples used to justify.

b) Model – exercises; limited range of permissible change – not explicit

c) Either no context or *context with pseudocontext*

E) Sequencing

a) Focus on mathematical horizon, (re)direction to standard strategies introduced

b) Links, when made, focussed on specific examples rather than on links between mathematically significant features of the representations

F) Teacher Cognition

a) Discussion considered important in developing student reasoning – conscious decision to try to use more whole-class discussion (structure through IRE) with class B (aware he relied on model-individual practice approach more often with class B). Considered whole-class discussion **'successful'** in reported lesson, *but more unpredictable in the other lessons – some success but some curtailed due to a perceived lack of engagement.*

G) Classroom norms

a) Teacher arbiter of correctness

b) Procedure counts as explanation; examples justify rules

c) ESM errors: usual - errors to be avoided or corrected.

H) Cognitive Demand

a) Potential high but range low to high (limited press to move beyond procedural)

(d) Sam Class B: Summary OMF of All lessons

Lesson-specific summary OMFs were produced as part of the analysis of each lesson. One output from comparing all class B's lessons was an all-lesson summary OMF (figure 5.27). Descriptions were in note form and bespoke abbreviations were used (table 5.1).

Sam Class B Summary OMF – All lessons

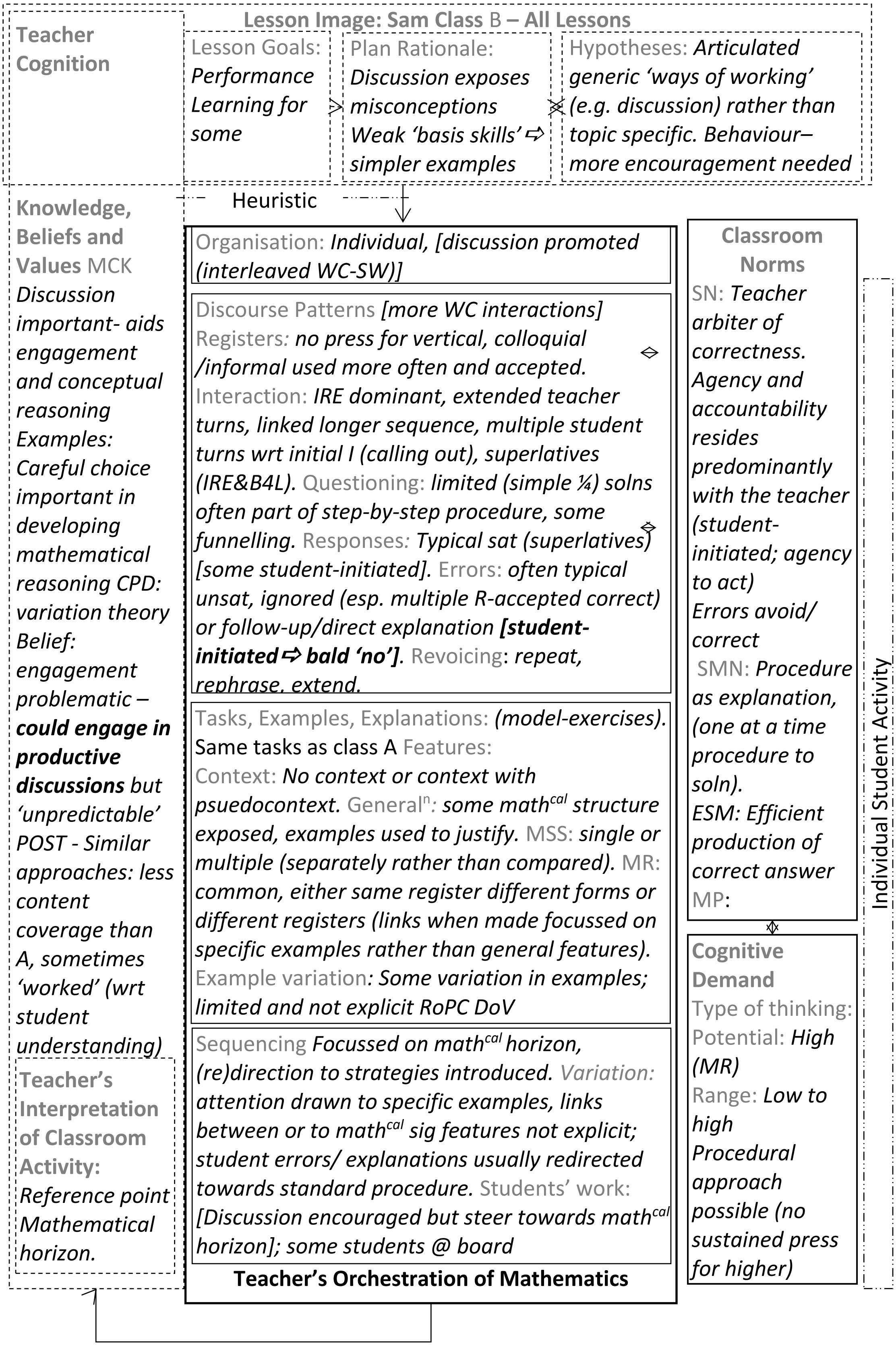


Figure 5.27: Sam Class B Summary OMF – All lessons

5.3.2.7 Sam: Class Comparisons

In the main study, five lessons from each class were recorded and analysed. Lesson-specific summary OMFs were an integral part of the analysis and these structured the comparison processes. When lessons for the same classes were compared, many of the dimensions of the OMF were similar, which allowed all-lesson summary OMFs for each class to be produced (figures 5.22 & 5.27). These in turn structured the cross-class comparison. However, as previously discussed, Sam's decision to base the lessons on the same materials also facilitated cross-class comparisons of individual lessons down to the level of mathematical practices. When the two classes were compared, there were many similarities in Sam's pedagogical approaches, but differences were also noted.

The comparison process from the perspective of the summary OMFs can be found in appendices 3.5.3, 3.5.4, 3.5.5 & 3.5.6, where underlining has been used to highlight common and differential features between the two classes. Those comparisons, cross-referenced with the more detailed lesson narratives, informed the written summary given below. This provides an overview of the similarities and *differences* between class A and class B, with the latter indicated by *italics*. The written summary focusses on the indices lessons so the comparisons can be related to the detailed lesson narratives provided in the previous sections. However, complementary or contradictory features from the other recorded lessons are indicated with square brackets [ ]. The bold indicates where Sam changed his practice for the recorded lessons.

A) Curriculum
   a) Content tiered: formal rules '*core*' for class A and *'optional'* for class B
   b) **Atypical** – used same resources for both classes
B) Organisation
   a) Seatwork: individual, peer discussion encouraged but not required;
   b) Interleaved seatwork with whole class
   c) *Class B contained fewer students*
C) Discourse patterns: mostly aligned with patterns previously reported
   a) *Power terms used* but no press for student use, *class B informal language*

b) IRE dominant, single/limited solutions questions in linked sequences; *for class A some peer-to-peer 'debates'; for class B some multiple responses to initial I and student-initiated turns*

c) Typical satisfactory norms: 'Correct' responses ⇨ follow-on questions (1/3 superlatives); Some atypical unsatisfactory norms: *For class A 'errors' ⇨ often follow-up questions, some debates; for class B IRE 'errors' ⇨ follow-up, some funnelling; multiple responses ⇨ correct solution acknowledged/ error ignored; student-initiated 'errors' ⇨ bald 'no'* [in other class B lessons, there were few instances of a bald 'no']

d) Revoicing; repeating (occasional rephrasing into more formal/complete phrase)

D) Tasks

a) Focus on rewriting as multiplication (same register different representations); examples used to justify.

b) Model – exercises; limited range of permissible change (Bills et al., 2006)

E) Sequencing

a) Focus on mathematical horizon, (re)direction to 'rewriting' solution strategy

b) Some attention drawn to 'not' rules of indices

c) Alternative ways to consider zero exponents introduced but separately

F) Teacher Cognition

a) Discussion considered important in developing student reasoning.

i) ***Atypical*** *– attempting to adopt a similar approach (discussion) with class B as he often used with class A*

ii) For class B initial belief discussions may be unproductive – off-task behaviour, praise for engagement

iii) Interpretation of classroom activity: students engaged in productive discussions and understood main concepts in a similar manner as class A [whole-class discussions varied in 'success', with some curtailed due to perceptions of engagement]

G) Classroom norms

a) Teacher arbiter of correctness

b) Procedure counts as explanation; examples justify rules

c) For class A errors considered exciting to discuss

H) Cognitive Demand

a) Potential high but range low to high (limited press to move beyond procedural)

5.3.3 Teacher: Rowan

5.3.3.1 Background

Rowan engaged in internal and external CPD opportunities. For example, she had been participating in a year-long CPD course and was drawing on those resources and approaches to introduce more student-led problem-solving activities. In addition, her department meetings regularly focused on resource development and how these supported the learning of mathematics.

She followed her departmental scheme of work and did not make any adjustments to the topic taught to accommodate the recording. Rowan did indicate that the level of planning and development of resources for the lessons reported on here was typical of about one in five of her lessons. So, whilst the lessons reported might not fully represent the students' everyday experience, they were a constituent part of the students' usual experience of school mathematics. The recorded lessons were one hour long and were one of four mathematics lessons held each week.

The school placed students in sets for mathematics. After year 7, their policy was to use internal assessments to make judgments about group composition, but this also involved teacher recommendation and set sizes. Usually most movement occurred once a year, with the two or three students with highest and lowest attainment in each set considered, but the intention was to keep the classes as stable as possible over the two years leading up to GCSE exams, so movement was rare after the first term of year 10. There were two curriculum routes specified for Key Stage 4, namely higher and foundation, linked to the GCSE specifications. In Key Stage 4, two thirds of the sets followed the higher tier route, with the remaining sets following the foundation route. There was some commonality in topics between higher and foundation routes at the start of year 10, but these routes quickly became more distinct. By year 11, the higher route focussed almost exclusively on the content designated as grade 6 and above and the foundation route on grade 5 and below. For the higher route, the content included in the foundation route was considered presumed knowledge, and only met when embedded in other activities and tasks.

5.3.3.2 Teacher's Knowledge, Beliefs and Values

Pre- and post-lesson interviews indicated that Rowan felt she had a relatively free choice about her style of teaching, but some school structures restrained her capacity to act. In particular, she perceived the school's curriculum plans and expectations for exam preparation as constraints. This was mitigated to some extent by resource innovations introduced by the mathematics department, which provided some leeway to deviate from the prescribed curriculum. So, while she felt that the use of more open tasks would be supported by school leaders, the content heavy departmental plans that specified the material to be covered lesson by lesson restricted her movement. However, the level of freedom was related to the attainment of the class, where she perceived greater 'exam pressure' with higher attaining sets but felt she had more freedom of movement with lower attaining sets. Rowan stated she invested time in developing resources that would allow students to move away from following algorithms modelled by herself and towards problem solving approaches.

5.3.3.3 Rowan: Class A

This was a Key Stage 4 class, composed of about thirty students who had attainment profiles higher than average for that year group in the school.

(a) Lesson Specific: Teacher's Knowledge, Beliefs and Values, incorporating Initial Lesson Image

In the pre-lesson interview, Rowan stated that she wanted the students to improve their algebraic manipulation skills and make links to other topics, as well as developing their problem-solving skills. As such her articulated lesson goal was considered to include elements of both learning and performance orientations. Within the lesson, there was a similar duality. For example, when reviewing particular questions, Rowan stated, "that it's a very very common exam question you guys slipped up on in the mocks", but framed the task as "can anyone make sense of that picture and those expressions and how they link".

Rowan indicated she was aiming to develop students' ability to problem solve. Whilst this was her aim, a typical lesson for this class was more 'traditional', in that Rowan modelled a solution, after which students completed exercises containing similar questions. She felt this approach was supported by both colleagues and students as an efficient way to cover the required curriculum. However, Rowan had been introducing more open tasks over the course of the year, believing an improvement in problem solving would develop students' ability to answer the non-routine questions that were incorporated into the then new-style GCSE exams. So, whilst this lesson might not represent the students' usual day-to-day experience, this type of lesson approach was part of the students' overall experience of mathematics lessons.

Rowan felt the students currently exhibited more positive attitudes to a 'traditional' approach, as this produced a greater quantity of written work, considered synonymous with good progress and, as yet, they could not see the purpose of open tasks. However, based on her teaching in a different school, she thought this was based on prior experiences. She argued the students needed to experience positive outcomes of a more open approach, in terms of being able to answer the new-style GCSE questions, in order to be persuaded to engage fully in these types of lessons. In effect, Rowan was attributing a performance orientation to her students (Dweck, 1999).

Before the lesson, Rowan indicated that she thought that the students might struggle to see the relationship between the algebraic expressions and the diagrams, but they were familiar with area as a model for multiplications so that should be an appropriate starting point. In addition, she thought the students would struggle when deciding on a course of action with the open-ended nature of some of the tasks.

(b) TOM: Organisation

The lesson was timetabled for one hour and lasted fifty-six minutes due to lesson transitions. Seventeen minutes were spent at a whole-class level, with the remaining time spent on seatwork. The desks, large enough to seat two students, were arranged in rows and the students were asked to work in pairs. Rowan explicitly stated that she wanted to hear discussions, although the starter activity and the worksheet could have been completed independently. The card sort activity was done in pairs or groups of three, with sets of cards shared, although students were expected to write up the results in their own exercise books.

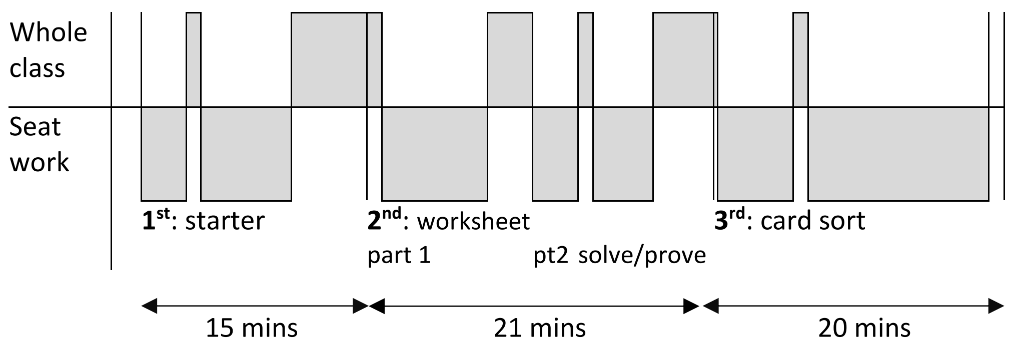


Figure 5.28: Rowan Class A Organisation

(c) TOM: Tasks, Examples and Explanations - Overview

The lesson was entitled ‘The visual representations of algebra’.

Phase 1: Starter

Rowan displayed a question on a PowerPoint slide that the students started as they arrived (figure 5.29). An additional question, entitled ‘bright spark’, was added after a couple of minutes and then there was a whole-class discussion.

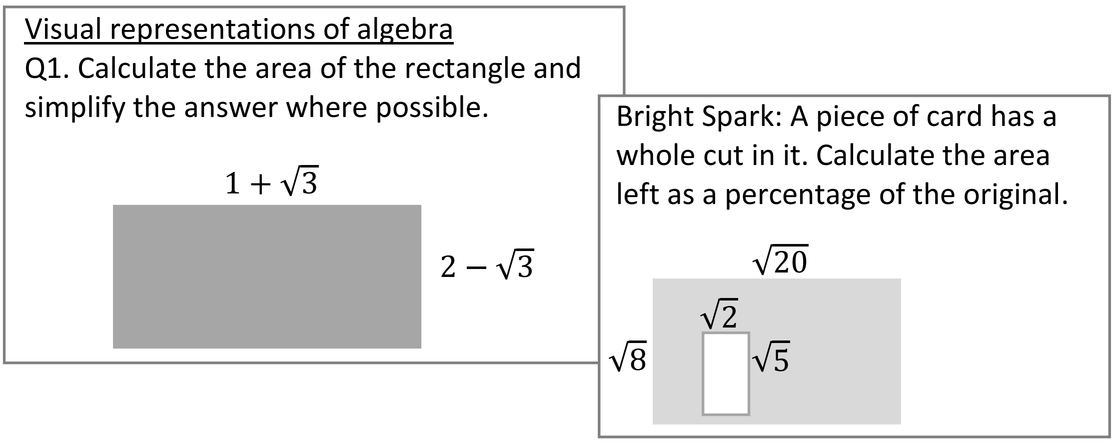


Figure 5.29: Rowan Class A Tasks

Q1 drew on the area of a rectangle as a model of multiplication and required the manipulation of compound surds. For the bright spark, all the surds were simple, but it was a multistage problem. Students took a subtraction approach, initially calculating the area of the two rectangles followed by attempts to subtract the terms; the lack of information as to the exact placement of the smaller rectangle and the naming as a ‘hole’ may have cued this approach. Other dissection strategies would have involved ‘seeing’ the area as equivalent if the removed section was placed elsewhere, with dimensions of the remaining card then calculable. However, it was also solvable without calculating areas if the similarity of the rectangles was recognised.

Phase 2: Worksheet

The students completed tasks relating to a worksheet (figure 5.30).

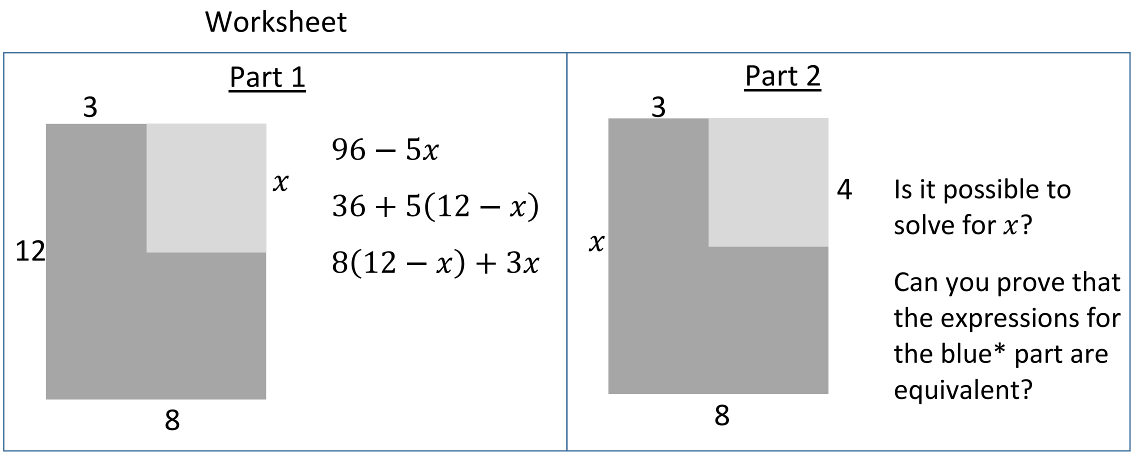


Figure 5.30: Rowan Class A Tasks

*The ‘L’ part of the rectangle was coloured blue and the small rectangle was red

Initially, in reference to part 1, Rowan asked them, “what’s the connection between those expressions on the side and this diagram”. A period of seatwork then whole-class discussion followed. Then Rowan asked the students to move onto part 2; initially students were asked, “how many different expressions can you come up with from that diagram”. After a further period of seatwork, the students were asked to consider the two questions on the worksheet related to solving and proof; the diagram/ expressions from part 1 were used as well as those from part 2.

For the worksheet, and the later card sort, the ‘L’ shape was the common format (figure 5.30). Three dissections were associated with finding the area of the ‘L’ (figure 5.31a-c): subtracting the area of the smaller rectangle from the larger; dissecting the ‘L’ vertically into two rectangles and adding; and dissecting horizontally and adding. Other dissections, such as figure 5.31d, were not included. One dimension of the compound shape was labelled x, with some others given positive integer values; enough information was provided to find all the dimensions of the ‘L’. Conservation of area meant that different but equivalent expressions could be generated for each dissection. Consequently, this had the potential for students to explore the equivalence of algebraic expressions before they could prove this was the case by algebraic manipulation.

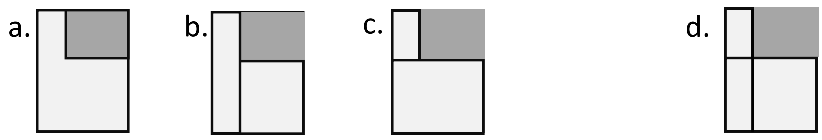


Figure 5.31: Rowan Class A Tasks

The form of the diagrams for part 1 and 2 may have made the critical features harder to discern. They were the same shape and size, and two dimensions, namely 8 and 3, were retained, but 12 was replaced by x and x by 4. The visible likeness and the retention of some dimensions could have cued students to read the diagrams as congruent, and the use of the same letter could have been interpreted as the letter representing the same variable. If the diagrams were read as congruent, then $x = 4$ in the first diagram and $x = 12$ in the second; in this context, if students accepted that x could represent two different lengths then all the dimensions would be known. If x represented the same unknown, then the reader had to ignore the visible 'sameness' and appreciate that if drawn 'to scale' the diagrams would look different. In this situation, equations could be formed and solved for x. Whereas, if x represented different variables then the students would still have to ignore the visible 'sameness' but appreciate that the xs are unrelated and could vary as constrained by the given dimensions. All of which could have taken attention away from the key mathematical features related to equivalence, namely the equivalence of expressions drawn from finding the same area of a compound shape using different dissections.

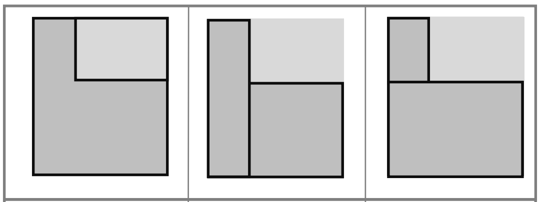
In part 2, two questions were posed, "Is it possible to solve for x?" and "Can you prove that the expressions for the blue square are equivalent?". The first had the potential to draw attention to the meaning of solving, including the conditions under which something is solvable, and the second to the notion of proof. Both questions related to underlying mathematical structures and their juxtaposition could have drawn attention to the assumptions being made and the different conditions under which equivalence and solving apply.

Phase 3: Card Sort

The students worked on a card sort activity, matching equivalent expressions and expressions with diagrams (figure 5.32).

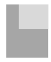
Card sort resource

Printed table:



A		
B		
C		
D		

Cards: 12 cards with algebraic expressions and 'blank' diagrams



$8x - 20$	$63 - 5x$
$2x + 7(9 - x)$	$8x + 5(6 - x)$
$3x + 30$	$5x + 8$
$3x + 5(x - 4)$	$9x - 4(x - 2)$
$18 + 5(9 - x)$	$18 + 5(x - 2)$
$8(x - 4) + 12$	$48 - 3(6 - x)$

Figure 5.32: Rowan Class A Tasks

A range of strategies were possible, including 'guessing' dimensions followed by finding expressions for each dissection, factorising terms in each expression and simplifying expression in order to match the twelve cards.

General

Multiple representations were part of all the tasks, although the level of embeddedness varied. For example, Q1 in the starter used a diagram to present the information, but once the required calculation had been established the problem could be solved without further reference to the diagram. Whereas in later problems there was a greater requirement to shift between diagrams and algebraic expressions. Each activity drew on the area of a rectangle as a model of multiplication. This relationship was not mentioned explicitly, but the students' reactions to the problems indicated that they were familiar with this model. Most tasks also involved compound shapes, which included the possibility of multiple solution strategies.

(d) TOM: Discourse

In terms of the mathematical register, the communication of meaning of key terms was supported by the context. For example, in reference to 'solving', Rowan used the phrase "can you use them to solve and find what x is?". Equivalence of algebraic expressions was a key element of the lesson tasks and Rowan used the term ‘equivalent’ seven times in whole-class discussions; for example:

105 T: ... the version of the expression that we’ve got here to represent the blue area (..) is equivalent to (.) this one that we’ve got for the blue area because they should be equivalent shouldn’t they because they’re both talking about the blue area…

Extract 5.35 Rowan Class A

In line 105, the meaning of ‘equivalence’ could have been inferred from the relationship of the expressions with the same area on the diagram, although the term was not explicitly defined.

There was one occasion when a student appeared to be referring to features related to equivalence:

112 S: work out that they were the same

113 T: prove that they are the same (.) yeah…

Extract 5.36: Rowan Class A

The student used the phrase ‘the same’, which was mirrored by Rowan, although she revoiced part of the student’s contribution, replacing the ‘work out’ with ‘prove’; the term equivalence was not used. In this lesson, students were regularly exposed to the term ‘equivalence’ but there was no press for them to use the term in their own public talk.

In whole-class episodes, approximately 85% of the talk was classified as mathematically related episodes (figure 5.33: subdivision 1). Turn-taking was the most common form of whole-class talk, but monologues were just under half of the mathematically related episodes (subdivision 2). The monologues consisted of explanations or instructions given by Rowan (subdivision 3 T:E). Taken together, IRE exchanges and the variant with an extended evaluative turn were the most common form of turn-taking (subdivision 3 T:IRE). These were often linked to form extended question-and-answer sequences. There was one student-led episode of turn-taking, which occurred when Rowan asked a student to explain her approach (subdivision 3 S:L), although Rowan structured the talk with questions and her own explanations.

This was classified as student-led as the student’s contribution was longer than in typical IRE exchanges and the approach originated with the student.

<table>
<tr><td colspan="4">Whole-class talk</td><td></td></tr>
<tr><td colspan="3">Mathematically Related Episodes</td><td>Other</td><td>Subdivision 1</td></tr>
<tr><td colspan="2">Turn-taking</td><td>Monologues</td><td>Other</td><td>Subdivision 2</td></tr>
<tr><td>T: IRE (inc. variant)</td><td>S:L</td><td>T:E Explains/ Instructs</td><td>Other</td><td>Subdivision 3</td></tr>
</table>

Figure 5.33: Rowan Class A Breakdown of Whole-Class Episodes

Within IRE sequences, Rowan indicated responses were satisfactory in a number of ways. In addition to using explicit affirmative words, such as “yes”, Rowan indicated responses were satisfactory by writing the answer on the class whiteboard, repeating or revoicing the response, drawing on the response in her following turn and moving on. There was only one occasion where Rowan added a superlative to her evaluative turn, in this case “perfect”. All the responses treated as satisfactory could be considered as a mathematically valid response. For example:

88 T: but where’s the five come from (..)
89 S: eight minus three
90 T: OK yes this dimension’s eight this dimension’s three that must be five therefore this must be five x (.) if I subtract that from the whole rectangle I’m left with the blue area (.) yeah
[gesturing at PP]
(.) the thirty six plus (.) five bracket x minus twelve…

Extract 5.37: Rowan Class A

The student’s response in line 89 draws on the appropriate dimensions from the diagram, and line 90 contains an explicit positive evaluation indicating acceptance as a satisfactory response. Rowan extended her turn by adding a further explanation; this occurred in the majority of the IRE interactions. Rowan’s turns were almost always longer than the students’, as extended student contributions were rare.

Rowan indicated that responses were unsatisfactory in less direct ways. Rather than use explicit evaluations, she used follow-up questions related to the original question, usually with some form of simplification. This pattern was also followed for ‘don’t know’ responses. For example, when the ‘bright spark’ starter question was being discussed the following exchange occurred:

31	T:	so what did we get as the area of the large rectangle
32	S1:	four root ten (.) that’s what the calculator said
33	S2:	(root one sixty)
34	T:	root eight multiplied by root twenty is (..)
35	S2:	root one sixty
36	T:	OK (..) …
		[writing $\sqrt{160}$ on the whiteboard]

Extract 5.38: Rowan Class A

After Rowan’s initial question (line 31) two separate answers were distinguished on the recording (lines 32 & 33), although they overlapped and the second was quieter. In line 34, Rowan simplified the question, moving from asking for the area, to specifying the calculation. As “root one sixty” was subsequently positively evaluated by Rowan when offered in line 35, the interpretation made was that in line 34 she was responding to “four root ten” and was treating this as an unsatisfactory response (although $4\sqrt{10} = \sqrt{160}$).

The exchange continued:

36	T:	… what’s the area of the blue section
37	S4:	minus root twenty
38	T:	what’s five 5 times two
39	S5:	ten
40	T:	[adding 10, then $-\sqrt{}$ to show $\sqrt{160} - \sqrt{10}$]

Extract 5.39: Rowan Class A

After asking about the area of the blue section (line 36) and the student’s response (line 37), Rowan asked a simple follow-up question that all the students would be able to answer with ease, and without reference to the previous questions or the tasks undertaken in the lesson (line 38). This indicated “minus root twenty” was treated as an unsatisfactory response. The writing of 10 on the board after line 39 indicated this response was satisfactory, although it was Rowan who completed the original question by writing the square root.

This last extract demonstrated the characteristics of funnelling. This was the simplification of the mathematics being asked to a point where the students could answer with minimal cognitive effort. Whilst there were a relatively small number of student responses treated as unsatisfactory, in four out of seven of these occasions Rowan simplified the mathematics being asked to the point where the funnelling label could be applied.

The precursors for responses to be treated as unsatisfactory did vary. There were a couple of occasions when students' responses would normally be considered a mathematically valid answer. For example, "four root ten" (extract 5.38, line 32) was the requested area but was treated as an 'error'. There were other occasions, as in extract 5.39, line 37, when the response was not the result of the requested calculation and would be considered a mathematical error. Finally, there were a couple of times when the response did not contain a mathematical error *per se*, but Rowan thought was insufficient in some way. For example, when asked why the area was $5x$, a response of "because it's five times x" was subject to a follow-up question.

In overall terms, IRE exchanges were the most frequent form of interaction, where Rowan asked questions with a limited range of mathematically valid responses. The most common exchange was Rowan asking these relatively closed questions, followed by a mathematically valid response, which Rowan treated as satisfactory whilst extending her turn by adding an explanation of her own. Responses that were treated as unsatisfactory were less common and had more diverse precursors. In her evaluative turn, Rowan kept the same focus but tended to follow-up by simplifying the question. IRE exchanges were usually contained in extended sequences related to an overarching process.

Extended periods of talk by Rowan, classified as monologues due to their length and level of self-containment, occurred throughout the lesson. These predominantly focussed on mathematical explanations but there were also some task instructions. No student talk was long enough to be similarly classified. Rowan's monologues, taken in conjunction with explanations in her evaluative IRE turns, meant that she contributed the majority of class talk.

(e) TOM: Sequencing

Rowan controlled the overall trajectory of the lesson, shaped by the use of prepared resources and her selection of students' work as the focus for whole-class discussions. She regulated the mathematical focus through questions asked and explanations given.

In the first phase, Rowan drew attention to a particular method of multiplying surds. Referring to multiplication of brackets done in previous lessons, five times she said, "don't abandon the methods". Stating, "I was really scared that someone was going to do this", Rowan modelled an error on the board for $(1+\sqrt{3})(2-\sqrt{3})$ (figure 5.34).

$$\begin{array}{r} 1 \times 2 = 2 \\ \sqrt{3} \times -\sqrt{3} = -3 \\ -1 \end{array}$$

Figure 5.34: Rowan Class A Tasks

When the 'bright spark' was discussed, Rowan focused on the generation of unsimplified versions of surds that represented the area of rectangles, namely $\sqrt{160}$ and $\sqrt{160}-\sqrt{10}$. This one solution strategy was privileged over other possible approaches. In addition, the focus on unsimplified surds may have made subsequent calculations more difficult and obscured the proportional relationship in 'bright spark'. Rowan did not complete the 'bright spark' problem in the whole-class discussion, leaving the final resolution of the problem with the students.

In phase 2, Rowan drew attention to finding and labelling missing dimensions as the first stage of the task (extract 5.37). She also highlighted relationships between particular terms or expressions and the areas of parts of the diagrams. Whilst some IRE exchanges led to the labelling of dimensions, much of this relationship work was undertaken by Rowan in extended explanations. For example, she outlined how two out of the three expressions in part 1 were derived:

92 T: so if we do (.) if we say five (.) which is this

[pointing at a on figure 5.35]

multiplied by twelve minus x (.) that tells you this dimension (.)

[pointing at b]

and then (.) three multiplied by twelve which gives you this strip here (.)

[point at c]

is thirty-six (.) so that's another expression that represents the blue area (..) what about the last one (.) another way of expressing the blue area if we cut it (.) here

[making a horizontal line with an arm across the top of d]

(.) then this part is three x (.) yeah (.) and this part (.) is twelve minus x (.) multiplied by eight (.)

[pointing at the red rectangle then d]

so three different ways of representing exactly the same area but if you look at those expressions they don't look the same do they (..) how many different expressions can you come up with from that diagram

[pointing at diagram labelled part 2]

Extract 5.40: Rowan Class A

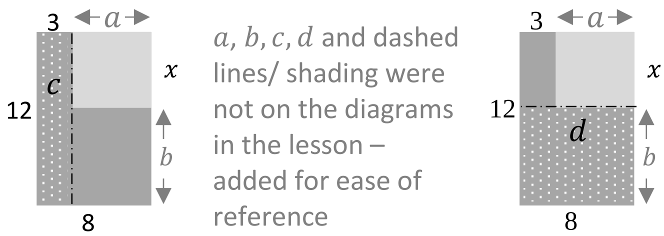


Figure 5.35: Rowan Class A Tasks

Rowan concluded by drawing attention to the notion that different expressions represented the same area but looked different. As this was followed by an immediate transition to finding expressions for the diagram in part 2, there was no explicit discussion about the notion that the expressions were equivalent and would rearrange into the same form.

Phase 2 continued with students generating expressions for the second diagram and exploring the two questions relating to 'proving equivalence' and 'solving'. When Rowan drew the class back together for the final whole-class discussion in this phase, it was more difficult to discern which mathematical concept was the focus of attention; Rowan opened the discussion with:

109	T:	… so did you find making (.) this equivalent to this
		[point at: $8(12-x)+3x$ and $36+5(12-x)$]
		or this equivalent to this (.) did that enable you to solve that equation
110	Ss:	no
		[chorused by a number of students]
111	T:	what did it enable you to do though
112	S:	work out that they were the same
113	T:	prove that they are the same (.) yeah…

Extract 5.41: Rowan Class A

In line 109, although Rowan used the term 'equivalent', her reference to 'making', when the expressions were already equivalent, and her reference to 'solving' made it unclear if she was attending to the equivalence of the expressions. Moreover, as there were no equations written on the board, it was not clear as to which equation she was referring to. The students' responses indicated they thought that solving was not possible in the circumstances prescribed, but their reasoning was not interrogated.

Still referring to part 1, Rowan continued:

113	T:	… so probably end up if you simplified it all d'both of them
		[pointing to: $36+5(12-x)$ and $8(12-x)+3x$]
		you'd have ninety-six minus five x is equal to ninety-six minus five x (..)
		[writing: $96-5x=96-5x$]
		x's on both sides (.) we'd (.) add five x to get rid of it here so ninety-six
		[writing: $96=96$]
114	S:	equals ninety-six
115	T:	oh I've just cancelled out my x's on both sides (.) which proves that 96 equals 96 well we knew that anyway (.) but really interestingly what Beth did (.) erm (.) was…

Extract 5.42: Rowan Class A

Rowan had a surprised tone to her voice at the beginning of line 115; it appeared that she had not anticipated the result of her action. In the second part of her turn, Rowan shifted attention to Beth's approach of setting up and solving an equation. Similar to discussions about part one, there were no explicit discussions about the manipulation of expressions, and in particular, the question about proving equivalence had not been explicitly addressed.

Rowan then asked Beth to describe her approach. Beth had formed an equation from the two simplest expressions for the two blue areas from part 1 and part 2 and solved the resulting equation, namely $8x - 20 = 96 - 5x$. Rowan added a longer explanation that included an explicit articulation that the assumption had been made that the blue areas were the same. However, there was no explicit discussion about the role of x, and in particular that this approach contained the inherent assumption that the xs represented the same unknown in both diagrams.

In the last phase of the lesson, the students completed the card-sort with relatively little whole-class interaction. Apart from organisational instructions, Rowan drew attention to strategies for completion; for example:

143	T:	but if you start by looking at the diagram (.) and writing the same expressions that were written for the previous sheet with the blue areas (.) you doing the exact same thing (.) you'll just know if you've got the answer right cause it will be on- appear on the other cards (.)

Extract 5.43: Rowan Class A

However, the shared strategy did not include discussions as to how the dimensions of the blank diagrams could be worked out or alternative approaches that could be used to match up expressions.

Phase 2 of the lesson contained the majority of whole-class talk related to Rowan's lesson goals. She directed attention towards the generation of expressions and how these linked to the area of the compound 'L' shape. The "same area" was used on a couple of occasions to argue that the expressions were equivalent. The students were not exposed to a sequence of examples. Instead, the structure of the problem drew attention to equivalence, albeit with the previously discussed issues regarding the visual 'sameness' of the two diagrams. In the last phase, the card sort maintained the connection to the principle that different expressions representing the same area were equivalent. As such, the students had the opportunity to develop an understanding that different forms of algebraic expressions can represent the same area, hence build on the notion of equivalence. However, with a variety of solution strategies possible, the task could have been completed without manipulating algebraic expressions to show equivalence.

One key consequence of this management of this lesson trajectory was that the actual algebraic manipulation to prove two expressions were equivalent was not undertaken at a whole-class level. In particular, the whole-class discussions in phase 2 appeared to be side-tracked when Rowan wrote equivalent expressions equal to one another. Consequently, students could have completed the tasks as set without having experienced algebraic expressions are equivalent iff they can be rearranged into the same form.

(f) Interpretation of Classroom Activities

In the whole-class elements of the lesson, Rowan determined the mathematical focus of discussions. In particular, she redirected students to approaches she structured without interrogating student contributions, which included treating as unsatisfactory mathematically valid responses. There was an episode about generating and solving an equation that was more student led, in so far as it appeared Rowan had not considered that approach before the lesson. She discovered this approach during a period of seatwork and chose to share this in the subsequent whole-class discussion, but she controlled this with IRE exchanges. The inference made was that Rowan was attending to her mathematical horizon during whole-class interactions.

(g) Cognitive Demand

The tasks required algebraic expressions to be linked to diagrammatical representations, thereby providing the opportunity for students to consider links between representations and between representations and underlying concepts. In particular, the notion that expressions of different forms can represent the same area, and thereby be ‘the same’ was met repeatedly throughout the lesson. There was the possibility to transition from the principle of expressions being equivalent based on the shared representation of an area to the principle that equivalence is demonstrable through algebraic manipulation. However, this was dependent on individual student activity as this was not an explicit part of whole-class discussions or a requirement of the tasks.

Students' 'talk as mathematics' was almost always classified as procedural or process. These student contributions predominantly occurred in response to questions posed by Rowan. For example:

90	T:	… what dimension is (.) twelve minus x (..)
91	S:	you go from the red line you like draw a line through the whole thing and the bottom half it would be twelve minus x
92	T:	so this bit [pointing at the PowerPoint] the twelve minus x (.) the whole of this side is twelve and its bin- (.) had x subtracted (.)…

Extract 5.44: Rowan Class A

Whilst this student's response was related to this particular example, the process of dissection was applicable to other examples. As such, this was classified as process.

On the other hand, Rowan's contributions had a greater variety of classifications. Whilst many comments were focussed on particular calculations, and hence classified as procedural, there were more examples where comments could be interpreted in a wider context and thereby classified as process or mathematical concept (e.g. extract 5.40). An 'effective student of mathematics' would thereby have been exposed to these ideas, but without the requirement to engage in this dialogue themselves.

The questions in phase 2 about solving and proving expressions are equivalent provided an opportunity for students to consider underlying mathematical concepts, such as proof and the difference between an equation and an identity, albeit without being exposed to all the associated language. However, as the whole-class talk did not explicitly discuss these issues, the engagement with these broader concepts was reliant on independent student activity.

It appears, therefore, that the lesson provided opportunities for an 'effective student of mathematics' to engage in tasks that had the potential to elicit higher level thinking. Whilst, to some extent, all the students were involved in the generation of expression representing area, they could have completed all the required tasks without moving beyond this one procedure. Moreover, much of the 'talk as mathematics' was procedural or process and most high-level talk was undertaken by Rowan. In other words, the lesson had the potential to elicit higher level thinking, but students could have completed the tasks working in a procedural manner, thereby not engaging in these higher levels of thinking.

(h) Classroom Norms

Rowan directed student activities within the lesson and the authority to decide what was done, and how, resided with her. For example, Rowan initiated all the whole-class discussion and decided when to move onto the next task. As such, she could be seen as having high levels of agency within her classroom.

Within the lesson there were a number of events that contributed to a narrative that an 'effective student of mathematics' should avoid errors. Rowan expressed pleasure when an anticipated error in the multiplication of surds was not reported, and when errors did occur, Rowan tended to simplify the mathematics until an acceptable response was uttered. In addition, Rowan signalled that bravery was required to tackle more challenging mathematics.

Rowan regulated the mathematical direction of travel, determining the focus for the whole-class discussions. Rowan's choice to focus on single solution strategies, when alternatives were possible and occasionally voiced by students, contributed to the narrative that mathematics was about finding 'the' answer quickly and efficiently. The IRE/exposition pattern reinforced the norm that Rowan was the arbiter of correctness and responsible for explanations, whereas students' responsibility was contributing correct results or procedural explanations. There was one occasion where Rowan drew on a student's approach for a substantive whole-class discussion. This had the potential to communicate that students' explanations are legitimate and valued; however, Rowan revoiced Beth's contribution and extended the explanation, reinforcing the notion that Rowan was the arbiter of correctness and ultimately responsible for explaining the mathematics.

5.3.3.4 Rowan: Summary for Class A

Two lessons were recorded and analysed with Rowan's class A, with lesson-specific summary OMFs produced as part of the analysis process. These were the key point of comparison across the two lessons with class A and the cross-class comparisons with Rowan's class B. As with Joe and Sam, the OMFs were working documents, with descriptions in note form and bespoke abbreviations used (table 5.1). So, in order to communicate the key themes from the lessons, written summaries are provided before the summary OMFs are presented.

First, there is the written summary for the equivalence lesson, which was reported in detail in the preceding section (5.3.3.3). This is followed by the lesson-specific summary OMF (figure 5.36). Then, the written summary for the second recorded lesson is provided, which is referred to as the algebra lesson. Finally, the lesson-specific summary OMF for the algebra lesson is presented (figure 5.37).

(a) Rowan Class A: Written Summary – Equivalence Lesson

The following outline draws together the key themes from the equivalence lesson reported in detail in the preceding section (5.3.3.3).

A) Curriculum
   a) From a higher tier curriculum route
B) Organisation
   a) Seatwork: individual, peer discussion encouraged and some group work
   b) Interleaved seatwork with whole class
C) Discourse patterns: aligned with patterns previously reported
   a) No press for the use of mathematical terms
   b) IRE dominant, limited solution questions in linked sequences but extended teacher exposition was also a regular feature
   c) Typical satisfactory and unsatisfactory norms: 'Correct' responses ⇨ follow-on questions; 'errors' ⇨ follow-up questions (funnelling)
   d) Revoicing; repeating rephrasing and explanations extended
D) Tasks

    a) Multiple representations used; mathematical structure (equivalence/ conservation of area) embedded in tasks but not an explicit element
    b) Some ambiguous features (scale, visual similarity, letter representation)

E) Sequencing
    a) Focus on mathematical horizon, (re)direction to strategies introduced
    b) Occasional reference to mathematical structure, but this included 'erroneously' equating equivalent expressions and bypassing equivalent ⇔ rearrangeable
    c) One student's solution was made the focus of a whole-class discussion

F) Teacher Cognition
    a) Espoused priority was to develop students' problem solving, but felt pressured to teach more 'traditionally'
    b) Privileged her mathematical horizon when interpreting student responses

G) Classroom norms
    a) Teacher arbiter of correctness
    b) Procedure counts as explanation

H) Cognitive Demand
    a) Potential high but range low to high (possible to complete tasks using modelled single solution strategy)

(b) Rowan Class A: Summary OMF – Equivalence Lesson

As before, analysis was an iterative process and an OMF was populated, with the lesson-specific summary OMF (figure 5.36) the final working document. Descriptions in note form and bespoke abbreviations were again used (table 5.1).

Rowan Class A Summary OMF – Equivalence Lesson

Figure 5.36: Rowan Class A Summary OMF – Equivalence Lesson

(c) Rowan Class A: Written Summary – Algebra Lesson

In addition to the equivalence lesson reported in detail, one other lesson was recorded and analysed with class A. The lesson focussed on the manipulation of algebraic expressions involving brackets and fractions (substitution and rearranging). The main resource used was a set of worksheets with 'graded' questions. This lesson was analysed in the same manner, with a lesson-specific summary OMF produced as part of that process (figure 5.37). The two recorded lessons for Rowan's class A were compared, with the lesson-specific summary OMFs structuring this part of the analysis. Many of the dimensions of the OMF for Rowan's algebra lesson were similar to her equivalence lesson, although the task features were dissimilar. The written summary below provides an overview of the pedagogical features of the algebra lesson. Features seen in this second lesson that were different from the equivalence lesson (5.3.3.3) are indicated by *italics*.

A) Curriculum
   a) From a higher tier curriculum route
B) Organisation
   a) Seatwork: *all* individual, peer discussion encouraged
   b) Interleaved seatwork with whole class
C) Discourse patterns: aligned with patterns previously reported
   a) No press for the use of mathematical terms
   b) IRE dominant, limited solution questions in linked sequences but extended teacher exposition were a regular feature
   c) Typical satisfactory and unsatisfactory norms: 'Correct' responses ⇨ follow-on questions; 'errors' ⇨ follow-up questions
   d) Revoicing; repeating rephrasing and explanations extended
D) Tasks
   a) *Same register (rearrangements);* multiples solution strategies possible but not integrated into tasks – single approach privileged.
   b) *Pre-categorised questions, limited implicit exposure to RoPC e.g. not explicit links, boundary/not questions not met - unsystematic variation*.
E) Sequencing

a) Focus on mathematical horizon, (re)direction to strategies introduced, including treating alternative rearrangement/ solution strategy as an ‘error’.
b) Occasional reference to mathematical structure *through links between substitution and rearranging algebraic expressions*

F) Teacher Cognition
a) Espoused priority was to develop students’ problem solving, but felt pressured to teach more ‘traditionally’
b) Privileged her mathematical horizon when interpreting student responses

G) Classroom norms
a) Teacher arbiter of correctness
b) Procedure counts as explanation – *BODMAS treated as a mathematical principal*

H) Cognitive Demand
a) Potential high but range low to high (possible to complete tasks using modelled single solution strategy)

(d) Rowan Class A: Summary OMF – Algebra Lesson

As an integral part of the analysis of the algebra lesson, a lesson-specific summary OMF was produced (figure 5.37). The same note form for descriptions and bespoke abbreviations were used (table 5.1).

Rowan Class A Summary OMF– Lesson 2: Algebra Lesson

Teacher Cognition

Lesson Image: Teacher Class

Lesson Goals: *Performance; some implied learning (accurate rearranging of formulae; links to substitution)*

Plan Rationale: *Students were familiar substitution – used as a model for algebraic formula*

Hypotheses: *Barriers: misapplication of BODMAS, manipulation of fractions*

Knowledge, MCK : *[rearrangement]*

Beliefs and Values

Values: to shift away from teacher led to student problem solving

Beliefs: 'traditional' teaching seen as the way to cover content, especially with higher attaining sets.

Teacher's Interpretation of Classroom Activity: *Reference point Mostly mathematical horizon (single solution strategy).*

Heuristic

Concepts — Ways of working

Organisation: *Individual, (peer discussion encouraged); interleaved WC/SW*

Discourse Patterns

Registers: *No press for formal terminology*
Interaction: *IRE dominant, extended teacher turns, linked longer sequence.*
Monologues: *Teacher expositions common*
Questioning: *Mostly limited soln – part of step-by-step procedure.* Responses: *Typical satisfactory & unsat.* Errors: *few (treated alternate MSS as error).* Revoicing: *repeat, rephrase, extend*

Tasks, Examples, Explanations *(model-exercises).* Features: Context-*none*
General^n: *no explicit reference to order of operations (beyond examples)*
MSS: *possible but not integral part of tasks (not discussed/ acknowledge @ whole-class).* MR: *same register (rearrangements).*
Example variation: *different types of examples, pre-categorised, appeared unsystematic – no explicit links between examples and RoPC DoV*

Sequencing *Lesson trajectory*
Focus math^{cal} horizon, (re)direction to introduced strategies. Math^{cal} features: Focussed on one way of rearranging – did not acknowledge others (treated as error). Occasional attention drawn to math^{cal} structure (substitution ⇨ algebraic rearrangement) but at example level.
Students' work: *Verbal contributions (IRE)*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act) Some student contributions to explanations but part of procedure. Errors: avoid/ correct*
SMN: *Procedure as explanation, (one procedure to soln).*
ESM: Efficient production of correct answer
MP: *BODMAS accepted as a mathematical principle*

Cognitive Demand

Type of thinking:
Potential: *High*

Range: *Low to high Procedural approach possible, with single solution strategy @WC; students could have taken alternative routes in individual work.*

Individual Student Activity

Figure 5.37: Rowan Class A Summary OMF – Lesson 2: Algebra Lesson

5.3.3.5 Rowan: Class B

This was a Key Stage 4 class, composed of about twenty students who had attainment profiles lower than average for their year group in the school.

(a) Lesson Specific: Teacher's Knowledge, Beliefs and Values, incorporating Initial Lesson Image

In the pre-lesson interview, when asked about lesson goals, Rowan stated "I want students to be fluent in using this visual representation… to rearrange by inspection of the diagram as opposed to following an algorithm". She also made reference to performance in exams, stating "I feel that they are more likely to be able to use this in an exam". As such, Rowan's articulated lesson goals included elements of both learning and performance orientations. In addition to using more diagrammatic representations, she wanted to use more student-led activities to build the independence that may help the transition to exam conditions. There was a similar duality in the lesson. Rowan made reference to the aim being for students to understand rearranging equations, whilst the tasks were presented as hierarchical skills to be successfully completed, with practice the required first step.

Rowan thought this class required a different approach from higher attaining classes taught the same topic. She ascribed poor retention as an issue for these students, so more time was required to revisit and practise basic skills. Rowan stated she used more short tasks with interactive or competitive elements to keep students engaged and thought they needed more encouragement to stay on task. Her use of superlatives and 'reward points' for engaging in tasks and answering particular questions appeared to support a belief that extrinsic motivation was of benefit. Rowan also said the diagrammatic approach with multiple equations would be useful because "this could get them doing more maths without realising it"; it appeared she thought the students would not be intrinsically motivated by the mathematics.

Rowan planned for the numerical and algebraic examples to be modelled by the students with strips of paper. She thought being able to change the physical arrangement of the bars would allow students to build links between bar models and rearranging sums. Whilst the students had used bar diagrams in the past, they were not in regular use. Her aim was to develop students' use of bar models, and as the students could already confidently rearrange integer sums, she thought bar models with integer questions would be an effective entry point. Rowan anticipated the students would be able to use bar models to represent numerical equations involving addition but might struggle with subtraction. Rowan considered the transition to algebra as more complex. For example, she recognised the use of scale in the first example might not help the student to understand scale "did not matter" in the algebraic model.

(b) TOM: Organisation

The lesson lasted one hour, and about twenty-five minutes were spent at a whole-class level, with the remaining time spent on individual seatwork. The desks were arranged in rows. Whilst Rowan asked the students to complete the seatwork in pairs, and moved students to ensure that all had partners, the tasks set could have been completed independently. During seatwork, Rowan circulated and interacted with all pairs of students.

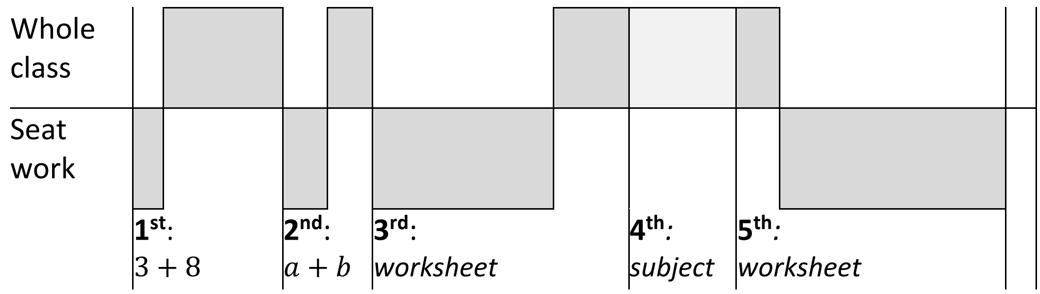


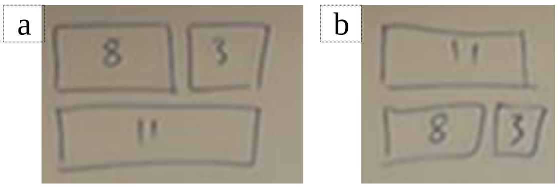
Figure 5.38: Rowan Class B Organisation

(c) TOM: Tasks, Examples and Explanations - Overview

The lesson was entitled 'Rearranging using the bar method'.

Phase 1: Integers $3, 8, 11$ bars

The students were asked to cut strips of paper into bars $3, 8$ and 11cm in length and to arrange them to "show a sum". A whole-class discussion concluded with two diagrams and three sums written on the whiteboard by Rowan (figure 5.39).

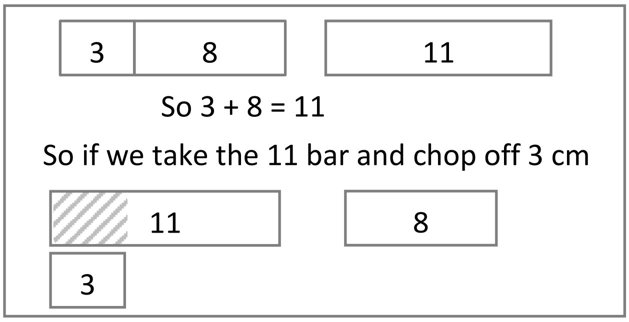


c

$$8 + 3 = 11$$
$$11 - 8 = 3$$
$$11 - 3 = 8$$

Figure 5.39: Rowan Class B Tasks

When subtraction was discussed, a prepared PowerPoint slide was also referred to:

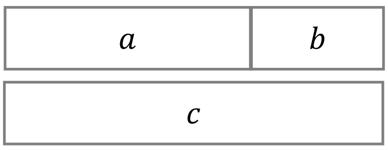


The top half, related to addition, was not discussed.

Figure 5.40: Rowan Class B Tasks

Phase 2: Algebra $a + b = c$

Students were given three new strips of paper and Rowan asked them to “cut them up and arrange them to show that $a + b = c$”. After some individual seatwork, a whole-class discussion ensued. Rowan presented a bar diagram on a PowerPoint slide (figure 5.41) and wrote $a + b = c$ on the board. In response, students offered three more equations, which Rowan added to the whiteboard.



$$a + b = c$$
$$b + a = c$$
$$c - a = b$$
$$c - b = a$$

Figure 5.41: Rowan Class B Tasks

Phase 3: Worksheet

A worksheet was completed by the students (figure 5.42).

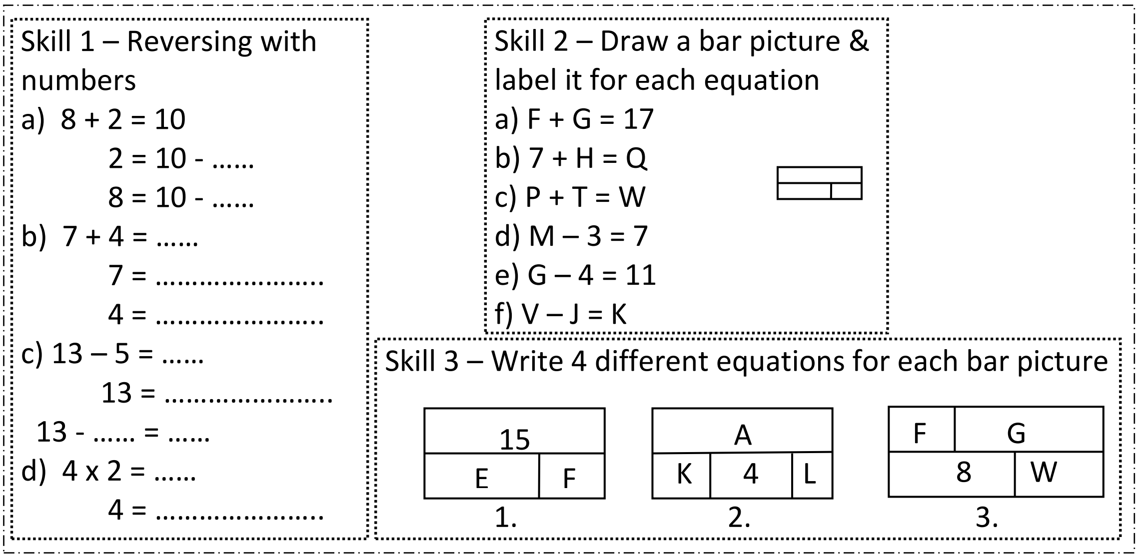


Figure 5.42: Rowan Class B Tasks

After a period of seatwork there was a whole-class discussion focussed on skill 3.

Phase 4: Subject of the formula

The whole-class discussion shifted focus to 'changing the subject of the equation', and in particular the identification of the subject.

Phase 5: Worksheet

The focus returned to the worksheet. A whole-class discussion of skill 3 question 2 was followed by a further period of seatwork.

General

The use of multiple representations was an integral part of the tasks. Apart from the first section of the worksheet, all the examples and tasks required the construction of a diagram to represent an equation or the writing of equations to match a diagram. After the initial $3, 8\ ,11cm$ example, an important mathematical feature of the diagrams was the relative length of the sets of bars. Adjacent bars represented addition, with equivalence indicated when sets of bars had the same overall length, with this feature mapping to the equal sign. A comparison of the 'to scale' first example and later 'not to scale' examples was not an explicit part of the tasks.

Two different layouts of bars were used during the lesson. The majority of the time a vertical layout was used (figures 5.39 & 5.41), where equivalence was shown by the bars end points being vertically inline. In phase 1, two vertical arrangements representing one set of rearranged sums were shared, whereas later one diagram was used to represent four versions of a rearranged equation. As such, the latter diagram could be seen as an object representing the relationships between the letters and numbers, with the absolute positioning of the individual bars not carrying mathematical meaning. A horizontal layout was used in the prepared numerical PowerPoint slides (figure 5.40). Here the bars were more closely associated with the calculation procedure; the order of the bars was the same as the written sum, and in the case of subtraction, the longer bar was manipulated (had a section removed) to represent the process of subtraction. Equations were linked to diagrams throughout the lesson and attention was drawn to the significance of bars being the same length. However, a consideration of how critical features of diagrams varied between examples or the relevance of other features, such as bar position, was not an explicit part of the tasks.

There was a mixture of numbers and letters used, and letters took different roles, both fixed unknowns, such as $M - 3 = 7$, and variables, such as $F + G = 17$. For the majority of the lesson, students were working on questions with three letters/ numbers in the form $A \pm B = C$. The prevalence of this format may have made it harder for students to appreciate alternatives were possible, such as $C = A \pm B$, especially for any students with an operational understanding of the equal sign (Knuth et al., 2006). In 'skill 3', more letters/numbers were introduced through the bars being further subdivided by the addition of extra vertical lines.

(d) TOM: Discourse

In terms of the mathematical register, 'takeaway' was the standard term used when referring to subtraction, although Rowan introduced the term "chop off", initially in relation to the physical strips of paper, but also later when discussing the subtraction model. Both these terms were heard in student talk. Rowan introduced the term 'subject', with the definition of "a number on its own" (which is different from the usual definition). Students responded in appropriate ways to questions framed by

Rowan about ‘changing the subject’, but they were not heard using ‘subject’ at a whole-class level. In this respect, students were introduced to a mathematical term that they were expected to understand but not expected to use in their own public talk.

In whole-class episodes, approximately 80% of talk was classified as mathematically related episodes (figure 5.43: subdivision 1). Turn-taking was the most common form of whole-class talk, with the remaining time classified as monologues (subdivision 2). The monologues consisted of explanations or instructions given by Rowan (subdivision 3 T:E). Taken together, IRE exchanges and the variant with an extended evaluative turn were the most common form of turn-taking, and were often linked to form extended question-and-answer sequences (subdivision 3 T:IRE). There were a few occasions when students-initiated turn-taking exchanges (subdivision 3 S:I). These general discourse patterns were similar to her other class, although less time was classified as monologues.

<table>
<tr><td colspan="4">Whole-class talk</td><td></td></tr>
<tr><td colspan="3">Mathematically Related Episodes</td><td>Other</td><td>Subdivision 1</td></tr>
<tr><td colspan="2">Turn-taking</td><td>Monologues</td><td>Other</td><td>Subdivision 2</td></tr>
<tr><td>T: IRE (inc. variant)</td><td>S: I</td><td>T:E Explains/ Instructs</td><td>Other</td><td>Subdivision 3</td></tr>
</table>

Figure 5.43: Rowan Class B Breakdown of Whole-Class Episodes

There were two distinct patterns of IRE exchanges within whole-class talk. When compared to the rest of the lesson, the episode about ‘the subject of the equation’ in phase 4 had distinctive characteristics, both in terms of the nature of the questions asked and the subsequent interactional patterns.

Outside of the ‘subject’ discussion in phase 4, most questions posed by Rowan had a single or a limited number of mathematically valid replies. Satisfactory responses were indicated in similar ways to her other class, namely by an affirmative word, an immediate repetition or a combination of both. Superlatives, such as “fantastic”, were used in over half these positive evaluations. All responses treated as satisfactory were mathematically valid responses to the question posed. For example:

213	T:	… can you tell me another equation (.) based on that diagram
214	S1:	fifteen takeaway f equals e
215	T:	fifteen takeaway f equals e (.) fantastic
216	S1:	or you could do fifteen takeaway e equals f
217	T:	fantastic (..) what we're actually doing here (.) and what I'm trying to get you to understand (.) is rearranging equations … (..) it's all about (.) doing the inverse on the other side its similar (.) to when we're solving equations

Extract 5.45: Rowan Class B

Lines 213 to 215 formed an IRE sequence, where the response described a legitimate equation and the evaluative turn contained repetition and an explicit evaluation. Rowan does not explicitly ask for another equation in her evaluative turn (line 215) so the student's next comment was classified as self-initiated even though they were responding to the same question (line 216). In line 217, Rowan treated this contribution as satisfactory and extended her turn by including a short explanation.

In a similar manner to class A, satisfactory responses were indicated by positive evaluations provided immediately and often explicitly. Rowan indicated she found responses unsatisfactory using signals that were less overt and often included pauses and follow-up questions. In whole-class discussions, there were no instances of a student's verbal response including a mathematical error. That is to say there were no responses that contained an invalid mathematical statement such as an inaccurate computation or procedure. On one occasion a student did say a when f should have been used; Rowan attributed the error to the poor clarity of the whiteboard image and as soon as she named the actual label the student self-corrected. Instead, the few student responses that Rowan treated as unsatisfactory related to the construction of the bar models. For example, after Rowan asked "show me that a plus b equals c" the following exchange occurred:

100	T:	so have we got a plus b equal c with our strips
101	S:	[student showing their arrangement of bars]

102 T: (.) but are you showing that that one's equal to these two

103 S: (.) no

[moving the bars]

Extract 5.46: Rowan Class B

In line 102, the "but" and the restatement of the question in a more precise form indicated that the student's diagrammatical layout was not in the required form.

There were also a few occasions where students did not respond or indicated a non-response through "don't know" or similar. Twice Rowan asked another student to reply and twice she offered an explanation by outlining what other students had done. Once Rowan modelled the reasoning from the previous question:

138 S: the last one would be a (…)

139 T: what do you think (.) so we've said (.) this (.) takeaway that part is equal to that part (.)

[modelling the previous subtraction on the bar diagram]

is there another one that is there another equation that we could write

140 S: c takeaway a equals b

Extract 5.47: Rowan Class B

In line 138, the student was unable to name the last equation, after which the teacher re-explained the previous equation indicating each bar with arm gestures in conjunction with the verbal commentary; this was considered to be an example of scaffolding (Anghileri, 2006).

Periodically, Rowan used students' work in her explanations. For example, Rowan opened the discussion of $8 + 3$ with:

33 T: let's see what you've done Max (.) what sum are you showing me (.)

34 Max: doneno don't get it

35 T: so this is what Iva showed me (.) this is what Iva showed me initially (..)

[showing the 3 bar moved next to the 8 bar]

and then she started saying because (.) shhh are you watching (.) she started saying because that plus that equals that she moved it over there

Extract 5.48: Rowan Class B

This was one of the occasions when Rowan drew on another student's work after a 'don't know' response. Rowan named the student then explained their actions in her own words, in effect 'broadcasting' that activity to a wider audience. In other parts of the lesson, Rowan 'broadcasted' students' work in her standalone explanations.

There were also times where Rowan paused and then reduced the complexity of the questions within her own turn. Students were usually nominated by Rowan to respond, by either name or gesture, but there were regular occurrences of students self-nominating by calling out answers. Therefore, the pauses may have been an opportunity for students to contribute, but with no overt nominations, this may not have been Rowan's intention nor the interpretation by the students. For example:

106	T:	do you need to actually measure them (..) how could you show that that's equal to that (..) why isn't it equal at the minute maybe (..)

Extract 5.49: Rowan Class B

In line 106, Rowan starts with a question that challenges the role of measuring in the task and moves to a specific question about the bars the students have cut out. This reduction in complexity was considered funnelling as the final question was self-contained and required minimal mathematical thinking.

Phase 4 started when Rowan asked, "what do you think is the subject in this equation", whilst pointing at $E + F = 15$. There followed an extended sequence of IRE exchanges in which the student responses did not contain computation or procedural errors in mathematics *per se*, but Rowan signalled that all the responses were insufficient or unsatisfactory in some way. For example, when $15 - F = E$ was being discussed the following exchange occurred:

228	T:	why do you think e is now the subject
229	S:	because it's the answer
230	T:	because it's the answer (.) what do you mean it's the answer
231	S:	it's what goes from (.) fifteen and f equal to (..)
232	T:	(.) yes sort of (.) but when you say it's the answer

Extract 5.50: Rowan Class B

The question in line 228 is somewhat ambiguous and, depending on the student's understanding of the word 'answer', he could have been attending to 'the letter on its own' in his response. As such, his response could represent some valid mathematical thinking. Rowan repeated the student's contribution, which was a common indicator of a satisfactory response, but she also asked a follow-up question (line 230). Thereby indicating that while the response was not wrong, it was not sufficient either. Again, in line 231, the student's response does not contain a mathematically invalid statement; although the meaning of "and" is unclear. Again, in line 232, Rowan did not completely dismiss the student's response but indicated it was still not sufficient. These types of exchanges with ambiguous evaluative turns continued until Rowan told the students her definition of 'the subject'.

In overall terms, other than phase 4, the majority of Rowan's questions had a limited range of mathematically valid responses, such as naming of an equivalent equation or one-step in a procedure. A minority were simple and self-contained questions of a level where all students could be expected to answer with little cognitive effort. Most common was an IRE pattern with a limited solution question, followed by a valid student response that was treated as satisfactory, and concluded with an immediate transition. On the relatively few occasions when students did not respond or responses were treated as unsatisfactory, Rowan kept the focus on the original question by asking follow-up questions or by offering an explanation. IRE exchanges were usually contained in extended sequences related to an overarching process. As discussed, phase 4 had different characteristics. In particular, the questions focused on a mathematical definition and there were a large number of responses that Rowan treated as being deficient in some manner.

Whilst the IRE was the most common form of whole-class interaction, there were also a few instances of student-initiated turn-taking. These occurred after Rowan had taken a turn that concluded without an explicit question (e.g. extract 5.45, line 216). The reactions of all parties indicated that, whilst student-initiated turns were less frequent than teacher-initiated turns, they were an accepted interactional norm.

Student turns were almost always shorter than Rowan's; questions posed were usually longer than students' responses and Rowan regularly included an element of explanation in her evaluative turn. Extended periods of talk by Rowan, classified as monologues, occurred periodically throughout the lesson and included extended explanations and task instructions; no students talk was long enough to be similarly classified. Consequently, Rowan contributed the majority of class talk.

(e) TOM: Sequencing

Rowan controlled the overall trajectory of the lesson, shaped by the use of prepared resources and her instigation of whole-class discussions. She regulated the mathematical focus through questions asked and explanations given. Throughout the majority of the lesson, Rowan maintained a focus on the deriving of equations from diagrams and the construction of diagrams from equations. When attention was drawn to mathematical features of the diagrams, specifically the role of bar lengths, this was predominantly accomplished through teacher exposition or gesture. As such, the inference made was that Rowan attended to her mathematical horizon when managing most classroom activities.

In the first phase of the lesson, when $3, 8, 11cm$ bars and addition was discussed, Rowan drew attention to a vertical layout of bars through the public endorsement of one student's work (figure 5.39); the horizontal layout for addition on the prepared PowerPoint slide (figure 5.40) was not discussed. As such, it appeared that Rowan had adapted her lesson trajectory, at least in part, in response to student activity. As the discussions continued, Rowan drew an inverted version (figure 5.39b), which she said showed "11 is equal to 8 add 3". This action could have communicated that the position of the bars had meaning. However, this was the only point in the lesson when two vertical diagrams were used to represent one set of equivalent equations.

A student then introduced the notion of subtraction by offering $11 - 3 = 8$. Rowan redirected attention to her prepared horizontal layout (figure 5.40) without responding to the student's use of the single diagram for both addition and subtraction. This shifted the focus to subtraction as a process of removal; however, this model was not referred to after this one example had been presented. After this point, both Rowan and the students associated equivalent equations, involving either addition or subtraction operations, with one vertical diagram. For example, in phase 2, one vertical representation of bars was associated with four equivalent equations, namely $a + b = c$, $b + a = c$, $c - a = b$, $c - b = a$. As the later prepared PowerPoint slides used this vertical layout, this shift appeared to be the intention. However, Rowan did not make an explicit comparison of different layouts nor did she draw attention to why particular formats were chosen.

During the atypical interactional pattern when the 'subject of the equation' was discussed in phase 4, Rowan's goal appeared to be a student articulated definition of 'the subject'. For example, four equations had been generated from a worksheet question, $E + F = 15, F + E = 15, 15 - F = E, 15 - E = F$, when the following exchanges occurred:

220	T:	what do you think is the subject in this equation [indicating $E + F = 15$]
221	S1:	the answer
222	T:	the <u>ans</u>wer this bit here (.) the bit that's on its own (.) so here this is the subject (.) [circling 15] now have changed the subject in this one (.) what's the subject now [pointing at E in $15 - F = E$]
223	S2:	E
224	T:	E (.) why [circling E and writing 'subject']
225	S2:	because 15 takeaway F equals E
226	T:	yeah (..) why does that mean E is the subject

Extract 5.51: Rowan Class B

In line 222, Rowan indicated that "the answer" was accepted by repetition and intonation, but when "the answer" was re-offered by a student in a later interaction, Rowan followed-up by asking for meaning (extract 5.50, line 230). It appeared there were issues with how terms were being understood by different participants and by participants at different points in time. In lines 224 and 226, Rowan indicated partial acceptance of the student responses with repeats but asked "why". This type of exchange continued, and Rowan circled solitary letters or numbers in equations and asked, "why is that the subject". The students replied using a variety of language, such as "at the end" and "outside the equals sign", none of which were treated by Rowan as fully satisfactory responses. On face value, Rowan repeatedly asking students "why" could have been interpreted as exploring student reasoning, but she continued to rearrange and annotate equations. These actions were interpreted as attempts to redirect the focus of the talk towards a particular form of language for the meaning of 'the subject'. After six minutes Rowan concluded with a statement "that this subject of an equation is the thing that is on its own", which mirrored the opening line of the discussion (line 222). (Rowan used 'the subject' label for a solitary letter/number on both the left and right side of any equation.)

The number of student responses treated as deficient, combined with Rowan stating a definition in her concluding remarks, indicated that a student definition of 'the subject' was not achieved. Eight equations had the 'subject' circled, so it may have been possible to identify some characteristics of that position from the information provided. However, the "why" drew student responses that focussed on a range of features, from the relationship to the rest of the equation (e.g. line 225) to position descriptors. For those unfamiliar with the definition of the subject "as the thing that is on its own" there may have been no way to determine which feature Rowan was attending to. Unaligned attention (3.3.7.5), combined with a lack of clarity about the meaning of other terms used, could explain the particular characteristics of this episode.

(f) Interpretation of Classroom Activities

Throughout the whole-class elements of the lesson, the talk almost exclusively remained focussed on mathematical features determined by Rowan, with less evidence of the exploration of student reasoning or exchanges with student-led talk. Although student contributions were drawn on to adapt the lesson trajectory, particularly in terms of the use of diagrams, the inference made was that Rowan was attending to her mathematical horizon during most whole-class interactions.

(g) Cognitive Demand

The multiple representations use in examples and the worksheet provided an opportunity for students to make links between representations and between representations and underlying concepts. Most teacher-posed questions asked students to name equations or construct one diagram. These were linked to particular examples, but as there was a requirement to make links between representations this 'talk as mathematics' was classified as procedural or process. There were a few instances of talk that focussed on wider concepts or mathematical structures, but this was far less common and when it occurred it was undertaken by Rowan. For instance, the lengths of the bars and their equivalence were highlighted by Rowan through exposition:

195 T: … so yours should look something like that (.)
[indicating a bar diagram on the PowerPoint slide]
they could be the other way round (.) there could be different lengths between f and g cause we don't know (.) which one is bigger that which (.) we just know that when you put them together (.) they add to make the same length (.) as seventeen

Extract 5.52: Rowan Class B

Some students appeared to have perceived the bars as placeholders for numbers, rather than the length representing the letter/value and placement representing an equivalence relationship. For example, one student placed the letters and numbers into boxes in the order they appeared in the question and added operations, with the longest bar reserved for the single letter/number on the right-hand side of the equals sign:

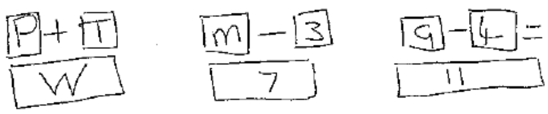


Figure 5.44: Rowan Class B Artefact

As such, there was evidence that some students may not have interpreted length and/or position of the bars as having mathematical meaning.

An inherent part of the tasks was for rearranged equations to be generated. This provided some opportunities for students to experience the equal sign in terms of equivalence, although the prevalence of $A \pm B = C$ format may have made this less visible. Also, as with other features of the representations, the tasks did not include a mechanism to draw attention to equivalence in an explicit manner, and as such may not have been noticed by students. As discussed above, there were constraints as to the type and form of equations and diagrams met by the students, but when the whole lesson is considered, an 'effective student of mathematics' could have engaged with a range of representations. As with the equals sign, this variation could have provided opportunities for students to see patterns that could move them towards understanding generalities, but as a discussion of the range of permissible change was not an explicit part of the tasks, this was more reliant on independent recognition by students.

In overall terms, the majority of 'talk as mathematics' was procedural or process, with attention only occasionally being drawn to mathematically significant features of diagrams in an explicit manner. There were opportunities for students to engage with underpinning mathematical concepts, but students could also have worked in a procedural way to complete the tasks. Consequently, the lesson provided opportunities for students to engage with mathematics that had the potential to elicit higher level thinking, but it was also possible to complete the tasks without engaging at these higher levels.

(h) Classroom Norms

As with her other class, Rowan directed student activities and the authority to decide what was done and how resided with her. As such, Rowan could be seen as having high levels of agency within her classroom.

The positive evaluations associated with 'correct' answers contributed to the narrative that an 'effective student of mathematics' should be able to answer questions quickly and accurately. The focus on procedural talk and the lack of press for explanations beyond procedures also contributed to this narrative. The references to practice and skill levels contributed to the narrative that mathematics is a hierarchy and the learning of mathematics requires the practise of 'lower level' skills first; once a skill is mastered, progression onto the next level is warranted and is a measure of success.

Within the lesson, Rowan regulated the mathematical direction of travel, determining the mathematical approach to be taken almost all of the time. Moreover, the dominance of IRE reinforced the norm that Rowan was the arbiter of correctness, that is to say she judged the legitimacy of mathematical contributions. However, there were also a few occasions where the students took the initiative or offered alternatives. For example, one student unprompted asked “miss, can I explain what we are doing”. Furthermore, on two occasions students did not agree with Rowan’s interpretation of their thinking. For example:

66	T:	Alex the way you’ve laid them out does that show me (.) that this one and this one is equal to this one
67	Alex:	naw I was doing that takeaway that equals that
68	T:	oooh (.) let’s have a look at how we might show takeaway …

Extract 5.53: Rowan Class B

As such, there was evidence that students made some choices about how to act mathematically, displaying aspects of agency.

On a few occasions, Rowan used students’ work to draw attention to a particular mathematical approach (e.g. extract 5.48, line 35). Whilst Rowan chose what to share, and thereby contributed to the notion that judgment about mathematical legitimacy resided with her, the public acknowledgment shifted the status of the contribution from one in need of evaluation to one valuable enough to share. The attribution to particular students also had the potential to communicate that students’ mathematical decisions have a legitimate part to play in mathematics classrooms.

5.3.3.6 Rowan: Summary for Class B

As with Rowan's class A, two lessons with class B were recorded and analysed, which included lesson-specific summary OMFs being produced. As before, the OMFs were working documents, with descriptions in note form and bespoke abbreviations. So, in order to outline the key themes from the lessons, written summaries are provided before the OMFs are presented.

First, there is the written summary for the equations lesson, which was reported in detail in the preceding section (5.3.3.5). This is followed by the lesson-specific summary OMF (figure 5.45). Then, there is a written summary for the directions lesson, which was the second recorded lesson. In this summary, differences between the equations and directions lessons are indicated by italics. Finally, the lesson-specific summary OMF for the directions lesson is presented (figure 5.46).

(a) Rowan Class B: Written Summary – Equations Lesson

The following outline draws together the key themes from the equations lesson, reported in detail in the preceding section (5.3.3.5).

A) Curriculum
   a) From a foundation tier curriculum route
B) Organisation
   a) Seatwork: individual, peer discussion encouraged but not required
   b) Longer blocks of seatwork and whole class
C) Discourse patterns: aligned with patterns previously reported
   a) No press for the use of mathematical terms ('unsuccessful' effort to get students to derive the definition of 'subject'; non-standard meaning)
   b) IRE dominant, limited solution questions in linked sequences; some 'broadcasting' of other students' responses in explanations
   c) Typical satisfactory and unsatisfactory norms: 'Correct' responses ⇨ follow-on questions (superlatives); 'errors' ⇨ follow-up questions, some funnelling, after 'don't know' some broadcasting
   d) Revoicing; repeating, rephrasing and explanations extended
D) Tasks

a) Multiple representations used, including physical models; mathematical structure (equivalence) embedded in tasks but not an explicit element
b) Some ambiguous features (scale, arrangement of diagrams)

E) Sequencing
   a) Focus on mathematical horizon, usually (re)direction to strategies introduced, but shifted to a vertical layout earlier based on student contribution
   b) Occasional reference to mathematical structure, but this included steer in 'subject' discussion
   c) Students' work 'broadcasted' as part of explanations

F) Teacher Cognition
   a) Espoused priority was to develop students' problem solving, but had different expectations (more practice required and more extrinsic motivation)
   b) Privileged her mathematical horizon when interpreting student responses

G) Classroom norms
   a) Teacher arbiter of correctness; student contributions valued through 'broadcasting'
   b) Procedure counts as explanation

H) Cognitive Demand
   a) Potential high but range low to high

(b) Rowan Class B: Summary OMF – Equations Lesson

As before, the analysis was an iterative process and an OMF was populated, with the lesson-specific summary OMF (figure 5.45) the final working document. Descriptions in note form and bespoke abbreviations were again used (table 5.1).

Rowan Class B Summary OMF – Equations Lesson

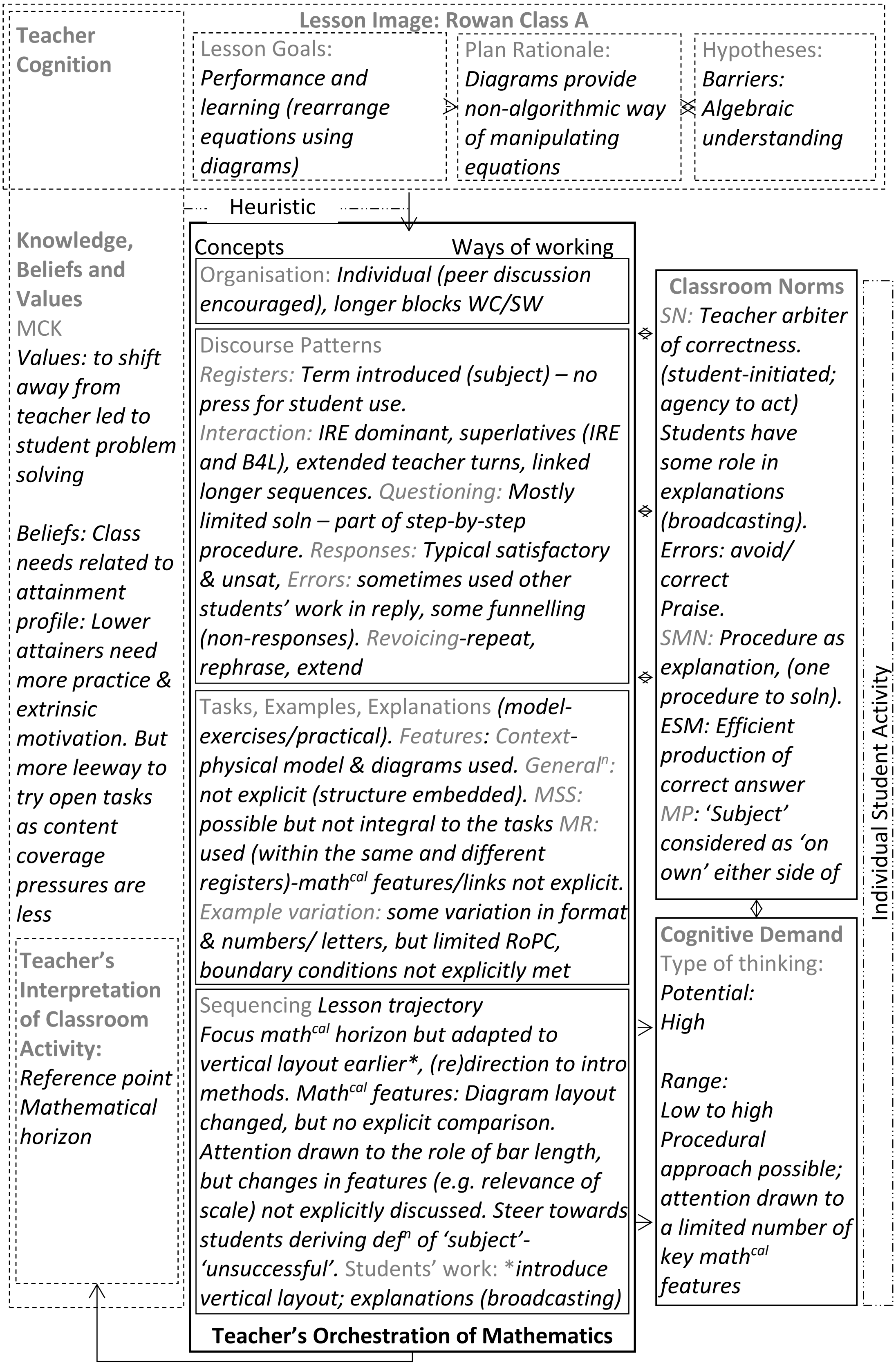


Figure 5.45: Rowan Class B Summary OMF – Equations Lesson

(c) Rowan Class B: Written Summary – Directions Lesson

In addition to the equations lesson reported in detail, one other lesson was recorded and analysed with class B. The lesson focussed on direction, and in particular the use of compass headings and bearings. This lesson was analysed in the same manner, with a lesson-specific summary OMF produced as part of that process (figure 5.46). As with class A, many of the dimensions of the OMF were similar across both Rowan's class B lessons. The following written summary outlines the key themes from this second lesson. Features seen in this directions lesson that were different from the equations lesson are indicated by *italics*.

A) Curriculum
   a) From a foundation tier curriculum route
B) Organisation
   a) Seatwork: individual, peer discussion encouraged *and some paired tasks*
   b) Longer blocks of seatwork and whole class
C) Discourse patterns: aligned with patterns previously reported
   a) Mathematical terms introduced; *use encouraged* but not required
   b) IRE dominant, limited solution questions in linked sequences
   c) Typical satisfactory and unsatisfactory norms: 'Correct' responses ⇨ follow-on questions (superlatives); 'errors' (few) ⇨ follow-up questions, some funnelling, after 'don't know' occasional broadcasting
   d) Revoicing; repeating, rephrasing and explanations extended
D) Tasks
   a) Context and pseudocontext (maps but classroom oriented activities)
   b) Multiple representations used – different registers, but mathematical features and links between representations not an explicit part of the tasks
   c) Some example variation *(angles, lengths, scales)* but no explicit reference to range of permissible change or dimensions of variation
E) Sequencing
   a) Focus on mathematical horizon, usually (re)direction to strategies introduced, but shifted to a vertical layout earlier based on student contribution

b) Occasional reference to mathematical features – *attention drawn to the process of estimating but not to how suitability determined*

c) Students' work 'broadcasted' as one follow-up to nil responses

F) Teacher Cognition

a) Espoused priority was to develop students' problem solving, but had different expectations (more practice required and more extrinsic motivation)

b) Privileged her mathematical horizon when interpreting student responses

G) Classroom norms

a) Teacher arbiter of correctness; student contributions *occasionally* valued through 'broadcasting'

b) Procedure counts as explanation

H) Cognitive Demand

a) Potential high but range low to high as the work could have been completed by following modelled procedures

(d) Rowan Class B: Summary OMF – Directions Lesson

As an integral part of the analysis, a lesson-specific summary OMF was produced (figure 5.46). The same note form for descriptions and bespoke abbreviations were used (table 5.1).

Rowan Class B Summary OMF – Lesson 2: Directions Lesson

Teacher Cognition

Lesson Image: Rowan Class A

Lesson Goals: *Performance; some learning (measuring and estimation skills, language; spatial awareness, application)*

Plan Rationale: *Use of language encouraged through paired tasks (in context)*

Hypotheses: *Barriers: reading off double-scale protractors*

Heuristic

Knowledge, Beliefs and Values

MCK

Values: to shift away from teacher led to student problem solving

Beliefs: Class needs related to attainment profile: Lower attainers need shorter activities and games (competition) for motivation. Real-life connections mentioned

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon

Concepts — Ways of working

Organisation: *Individual, paired tasks (collaboration), longer blocks WC/SW*

Discourse Patterns

Registers: *Terms introduced (directions) – task completable without use, use encouraged but not required in whole-class.* Interaction: *IRE dominant, superlatives (IRE and B4L), extended teacher turns, linked longer sequences.* Questioning: *Limited soln soln – part of step-by-step procedure, or description of activity.* Responses: *Typical satisfactory & unsat,* Errors: *few (broadcasting for 'don't know').* Revoicing- *repeat, rephrase, extend*

Tasks, Examples, Explanations *(model-exercises/activities).* Features: Context- *maps and diagrams used, psuedocontex (map tasks completable without engaging with math^{cal} stated aim – use of language).* General^{n}: *not explicit.* MSS: *possible (T focus on one) but not integral to the tasks* MR: *used (different registers)-math^{cal} features/links not explicit.* Example variation: *some variation in angles /length/ scale, but no explicit reference to RoPC, DoV*

Sequencing *Lesson trajectory Focus math^{cal} horizon, (re)direction to introduced methods, Math^{cal} features: Used direct explanation when task didn't produce intended use of language. Attention drawn to the range of suitable estimates in particular examples, and how estimates could be made, but not how suitable was determined.* Students' work: *Verbal contributions (IRE) (broadcasting)*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act) Some students contributions to explanations. Errors: avoid/ correct. Praise.*

SMN: *Procedure as explanation, (one procedure to soln)*

ESM: Efficient production of correct answer

MP: *Estimation separated from appropriateness*

Cognitive Demand

Type of thinking:

Potential: Low-High

Range: Low to high Procedural approach possible; linking estimations to calculations, and comparing alternative angle calculation strategies (links to parallel lines) was possible

Individual Student Activity

Figure 5.46: Rowan Class B Summary OMF – Lesson 2: Directions Lesson

5.3.3.7 Rowan: Class Comparisons

Two lessons for each class were recorded and analysed with lesson-specific summary OMFs produced (figures 5.36, 5.37. 5.45 & 5.46). These summary OMFs structured the comparison of the two lessons with the same class and the cross-class comparisons. As with Joe and Sam, when the two classes were compared, there were many similarities in Rowan's pedagogical approaches, but differences were also noted.

The comparison process from the perspective of the summary OMFs can be found in appendices 3.5.7, 3.5.8, 3.5.9 & 3.5.10, where underlining has been used to highlight common and differential features between the two classes. Those comparisons, cross-referenced with the more detailed lesson narratives, informed the written summary given below. This written summary provides an overview of the similarities and differences between class A and class B, with *differences* indicated by *italics*. This focusses on the two lessons reported on in detail in the previous sections (5.3.3.3 & 5.3.3.5) so the comparisons can be related to the detailed lesson narratives. However, complimentary or contradictory features from the two other recorded lessons are indicated with square brackets [ ].

A) Curriculum
   a) Class A from *higher tier* and class B from *foundation tier* curriculum route
B) Organisation
   a) Seatwork: individual, peer discussion encouraged and some group work
   b) Class A interleaved seatwork and whole-class; class B had longer blocks of each
   c) *Class B contained fewer students*
C) Discourse patterns: aligned with patterns previously reported
   a) No press for the use of mathematical terms [*Class B – use of terms encourage but not required]*
   b) IRE dominant, limited solution questions in linked sequences. *Class A extended teacher exposition, class B some 'broadcasting' of students' responses*
   c) Typical satisfactory and unsatisfactory norms: 'Correct' responses ⇨ follow-on questions (in *class B superlative used)*; 'errors' ⇨ follow-up questions (some funnelling *and in class B after 'don't know some broadcasting)*
   d) Revoicing; repeating rephrasing and explanations extended.

D) Tasks

   a) Multiple representations used *(physical models for class B)*; mathematical structure embedded in tasks but not an explicit element

   b) Some ambiguous features

   c) [Unsystematic variation of questions in exercises formed of sets of questions on the same topic]

E) Sequencing

   a) Focus on mathematical horizon, *usually* (re)direction to strategies introduced; *class B earlier shift to a vertical layout in response to student contribution*

   b) Occasional reference to mathematical structure, but steer bypassed many key features and sometimes focussed on 'erroneous' features

   c) In class A one students work was the focus of a whole-class discussion; for class B snippets of students' work was 'broadcast' to other students in explanations.

F) Teacher Cognition

   a) Espoused priority was to develop students' problem solving*; for class A felt pressured to teach more 'traditionally', for class B felt more practice and extrinsic motivation was needed.*

   b) Privileged her mathematical horizon when interpreting student responses

G) Classroom norms

   a) Teacher arbiter of correctness; student contributions valued *in class A through use in a whole-class discussion and in class B through 'broadcasting'.*

   b) Procedure counts as explanation

H) Cognitive Demand

   a) Potential high but range low to high

6. Discussion

6.1 Introduction

The previous chapter presented the detailed findings for six lessons, one for each class in the study. For each teacher, this has allowed lesson profiles to be built for their two classes with different attainment profiles, which formed the three pairs of nested cases. This chapter offers a discussion of those findings, drawing on the evidence from the other lessons in the study when this provided confirmatory, complimentary or contradictory evidence not available from the six lessons reported on in detail. The purpose here is twofold; first, to offer a response to the research questions, and second, to position the study to show how the findings relate and contribute to the existing body of knowledge.

6.2 Research Questions

The core motivation for this study arose from the prevalence of setting for mathematics in England and the reported inequities that result from different pedagogical approaches being associated with low attaining sets. Whilst previous studies have provided evidence of trends at cohort levels, there appeared to be less evidence as to how individual teachers change their practice. This led to the research questions to focus on an explication of how teachers' pedagogy may change with different sets.

Research questions:

> RQ: How does a teacher orchestrate mathematics for different groups of students?
>
> > RQa: How does a teacher shift their pedagogical approaches when teaching different groups of students?
> >
> > RQb: How does the character of the mathematics made available to students vary when a teacher teaches different groups of students?

In order to respond to these research questions, the Orchestration of Mathematics Framework (OMF) was developed in this study as an analytical tool. In the following discussions, the viability of the OMF as an analytical tool in this study will be addressed first. After which, the research questions will be considered. This will focus on individual elements of the OMF, where sets for the same teachers will be compared, and comparisons across teachers will be made, including differential features identified in the comparisons of sets. In other words, comparisons will be made within each nested case for each teacher and across the parallel cases (figure 4.1).

6.2.1 The Scope of the Response to the Research Questions

At each stage of this study, the notion that classrooms are complex, dynamic environments with many interdependent factors has been recognised. Of particular relevance is the acknowledgement by Schoenfeld (2013a), amongst others, of the complexities involved in constructing observational schemes. Moreover, as previously discussed (4.4.2), the limitations of developing and implementing an analytical framework by a lone researcher have to be acknowledged (Hollingsworth and Clarke, 2017).

In these circumstances, only a partial confirmation of the analytical power of the OMF can be offered, but the claim made is that the OMF has provided a viable structure for this researcher to chart substantive parts of participating teachers' pedagogical moves. Moreover, inferences were made about the mathematics made available, although this required an additional layer of analysis, with activities interpreted through particular theoretical lenses of how mathematics may be learnt. The efficacy of the OMF for use beyond this study, in terms of both external validity and reliability, would need to be tested with wider use. As discussed in section 6.3, connections to other theoretical perspectives have provided potential starting points for this process.

As discussed in more detail in subsequent sections, the claims made in response to the research questions are that:

> There is sufficient internal validity and reliability in the study to build coherent pedagogical profiles for the teachers that allow cross-class comparisons. Specifically, the orientation provided by the OMF has produced profiles that describe how the teachers orchestrated mathematics for classes with different attainment profiles. As such, commonalties and shifts in pedagogical

approaches have been identified, with some inferences made about the differences in mathematics made available to students.

The evidence for these claims has been presented in the previous chapter. The detailed claims with regard to the teachers' shifts in practice and the mathematics made available to students, along with the warrants for these claims (Toulmin, 2003), are discussed in section 6.4.

6.3 Viability of OMF

6.3.1 The OMF in this Study

In studies that have sought to measure the quality of instruction, there have been some debates about the minimum number of lessons that need to be observed in order to make measures sufficiently reliable (Derry, 2007; Hill et al., 2012; Schlesinger and Jentsch, 2016). One point of consensus appears to be that it depends on the purpose of the study and the facet of instruction under consideration. For example, Schlesinger and Jentsch (2016) argued elements of classroom management were more stable and measurable over single lessons, whereas those elements related to students' mathematical thinking, such as cognitive demand, required nine lessons. This study did not seek to measure the quality of instruction, but rather to build a profile of teachers' pedagogical practice. However, as a relatively small number of lessons have been observed, the notion that some aspects of teachers' pedagogical practice have more local stability, whilst others are more varied, has important implications for these discussions.

In this study, a number of aspects from the OMF were, for the most part, stable between phases within lessons and between lessons with the same class. Discourse patterns was the most notable dimension that had local stability, although particular aspects of other dimensions, such as the teachers' focus on their mathematical horizon, had comparable attributes. In particular, the most common whole-class interaction followed an IRE pattern, with the teachers' management of student responses falling into discernible patterns that were maintained both within and across lessons (5.3.1.7, 5.3.2.7 & 5.3.3.7[1]). For example, the treatment of satisfactory and unsatisfactory student responses followed patterns apparent from the analysis of the first whole-class interaction (e.g. extract 5.2).

That is not to say exceptions did not occur. There were individual instances of atypical behaviours, and occasionally there were extended episodes of alternative patterns of participation. However, these were associated with distinctive and atypical teacher activity. For example, as reported, Rowan's discussion of 'the subject' in class B had

[1] references starting 5.3.1 refers to Joe, 5.3.2 refers to Sam & 5.3.3 refers to Rowan

idiosyncratic features when compared to the rest of the lesson (5.3.3.5). It was associated with an atypical pedagogical focus, namely the students articulating a definition that met her requirements. When individual classes were considered, there were also instances of inter-lesson differences. For example, Sam's use of a bald 'no' was a distinctive feature of his class B lesson that is reported on in detail (5.3.2.5) but was less common in other recorded lessons for this class (5.3.2.6). Again, this occurred with an atypical pedagogical focus, specifically a conscious shift by Sam to try to include more whole-class discussions than typical for that class. Whilst causality is not being claimed, a greater level of student-initiated comments occurred with this partial reorientation of his management of student talk. One possible trigger for the increase in bald negative evaluations could have been the increase in unanticipated student contributions (Baldry, 2019).

Previous studies indicated IRE patterns were likely to be the predominant form of whole-class interaction (Lefstein and Snell, 2011; Drageset, 2015) (3.3.6.2). Whilst that was the case for these lessons, and there were commonalties between teachers, differences were also noted. For example, extended teacher-turns and IRE exchanges being linked together to form longer sequences were common patterns for all the teachers. However, the OMF analysis also identified differential features. Specifically, whilst the majority of questions asked by all the teachers had a limited range of mathematically valid responses, the proportion of simple, single solution questions asked by Sam was higher than for the other two teachers. Moreover, differences between classes with the same teachers were also noted. For example, the rate of inclusion of superlatives by Rowan (5.3.3.7) and the response to multiple student contributions by Sam (5.3.2.7) were relatively stable across different lessons with the same set but were a differential feature when their two sets were compared.

It appeared, therefore, the discourse pattern dimension of TOM had discernible features with relative stability both within and across lessons. In comparison to other features analysed, the finer grain size of turn-taking provided a large number of events for analysis. Moreover, classroom norms shape interactions, making stable patterns in those interactions more likely, and the collective reaction of participants did allow atypical and typical responses to be identified from individual occurrences (3.2.2). As such, this frequency and the stabilising influence of norms would have contributed to discourse patterns becoming apparent over a shorter time frame.

Other aspects, such as task features, had larger grain sizes; with fewer cases, the establishment of any regularities would naturally require a greater level of observation. For instance, the inclusion of context varied, but the presence or absence did not appear to be related to the teacher or the class, but rather to the schools' curriculum designs and the relative prevalence of the topics outside of the mathematics classroom. For example, there are relatively few school-level real-world applications of indices and Sam's two lessons on this topic were recorded as 'no context' (5.3.2.4 & 5.3.2.6). There was an expectation in Joe's school for reference to be made to real-world applications (5.3.1.1). Both his lessons contained elements of psuedo-context, but his lesson on percentages contained some more realistic applications, whilst his lesson on multiplying and dividing contained more context free elements. There were too few lessons to draw conclusions about when and how the teachers would choose to use context, but the identification of context/psuedo-context/no context supported the analysis of the mathematics made available.

As a minimum, it appears the OMF allowed data relating to different aspects of classroom activities to be collected, analysed and discussed using similar language. However, the identified features, such as IRE patterns, have been identified in previous studies using different instruments. A such, the analytical power of the OMF still has to be established. Section 6.4 explores the differential pedagogical practices identified in the analysis; the depth of discussion made possible should, at least in part, help address this question.

6.3.2 Relationship of the OMF to the Didactic Triangle

The origins of the OMF resided in the early reviews of literature undertaken at the outset of this study. During the synthesis of a wide range of research findings that could be brought to bear on the interpretation of mathematics classrooms, a network of key features emerged. When the decision was made to develop an analytical framework, those key features formed the first iteration of the OMF, which was developed further as the study unfolded. Having taken a constant comparison approach, the OMF could be viewed as both the theoretical framework brought to bear on the data (Fram, 2013) and a distillation of my understanding of the field. As such, the conceptualisation of the OMF for this researcher has in-built cohesion; the challenge is the assessment of the OMF from other perspectives.

One question is whether the OMF categories are necessary and sufficient for capturing teachers' pedagogical moves, with internal coherence and acceptable levels of overlap. Furthermore, the framework needs to have analytic power. Within the iterative development of the OMF, one key challenge was the relationship between the form of interactions and their function, and how this was captured in the framework. The most notable examples were how discourse patterns related to the mathematical object under discussion, especially when the use of classroom artefacts was also taken into consideration. The approach taken to the scrutiny of the coherence of the framework was to trace the analysis in terms of the didactic triangle as a conceptualisation of teaching and learning (Straesser, 2007). Whilst the triangle might 'offer an overly idealised model of relations between teacher, student and content' (Ruthven, 2012, p.361), it has nevertheless provided the basis for the conceptualisation of teaching and learning in a range of studies (Herbst and Chazan, 2012; Schoenfeld, 2012; Lerman, 2013). Here, it has provided the point of departure for the theorisation of teachers' orchestration of mathematics. In particular, how the teacher can be retained as the focus of the study whilst taking into account the dynamic relationships between the teacher, students and mathematics (figure 6.1).

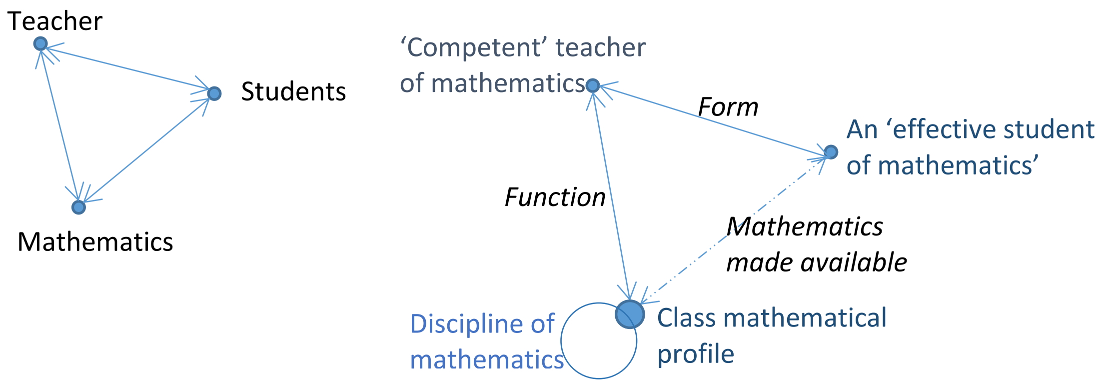


Figure 6.1: Adapted from the didactic triangle (Straesser, 2007, p.165)

The forms of interaction are conceptualised as the relationship between the teacher and students, with the function the mathematics to which the teacher draws attention. The relationship between the students and the mathematics is then their engagement with the mathematics made available to them through the teachers' pedagogical moves.

From this perspective, when the Teacher's Orchestration of Mathematics (TOM) is considered, the discourse dimension focusses on patterns of interaction. As such, this dimension is primarily encompassed by the form of the relationship between the teacher and the students. Whereas, the task dimension focusses on mathematical features and is primarily part of the mathematics element of the didactic triangle. However, classroom activities were selected by the teacher, so there is also a relationship between the teacher and the tasks, starting with the planning process. In addition, the analysis of tasks identified their potential to make mathematics visible to 'an effective student of mathematics'; as such this encompassed the potential relationship between the mathematics and the students. The sequencing dimension is encompassed by the relationships between the teacher and the mathematics and between the teacher and the students, manifest in how the teacher manages the lesson trajectory. In particular, this captures the tension the teacher has to manage between attending to their mathematical horizon and to student reasoning (3.3.7), which is an inherent part of the dynamic triad. As such, the TOM dimensions appear to be interpretable from the perspective of the didactic triangle model of teaching and learning of mathematics.

However, absolute demarcations did not occur, albeit a foreseeable situation from the perspective of the didactic triangle. The dynamic nature of classrooms means there is an ongoing interplay between elements of the didactic triangle (Goodchild and Sriraman, 2012; Ruthven, 2012), though different aspects become foregrounded when the focus is on particular features of the classroom. Here, for example, the discourse patterns in the treatment of student responses appeared dependent on the teachers' interpretation of the mathematical validity of contributions; the norm was for an immediate transition after 'correct' responses and further scrutiny of 'incorrect' contributions (e.g. 5.3.1.2). Moreover, those interactions were usually managed to maintain a focus on the teacher's mathematical horizon, which shaped the mathematics made available to students. As such, all facets of the triangle were 'in play'.

Viewed from the perspective of the didactic triangle, attempts at further decomposition of the OMF to increase discrimination between categories could be seen to compromise the modelling of dynamic interactions. From the perspective of the OMF, these examples would appear to support the view held by Ruthven (2012) that the didactic triangle is 'overly idealised' (p.361). However, the argument made here, and exemplified above, is that the internal validity of the TOM categorisations was sufficient for this author to structure observations and data analysis from which comparisons could be made. Ritchie and Lewis (2013) argued external validity is reliant on the quality of the description being sufficient for 'others to assess their transferability to another setting' (p.268), and as such resides in the eye of the reader.

The wider framework of the OMF is integral to the study. For instance, the learning trajectory cycle enabled the rationale for planning decisions to be captured, along with elements of teachers' beliefs and values. This was considered important due to the range of external classroom factors within which teaching is embedded (Ruthven, 2012), especially as teaching in sets is associated with a wide range of contextual influences (Francis et al., 2017). For example, Rowan's perceptions of 'exam pressure' appeared to have a differential impact on pedagogical choices she made with her higher and lower attaining sets (5.3.1.1). From the perspective of the didactic triangle, the teacher cognition elements of the OMF are encompassed by the 'teacher' node. However, the situation is more complex than just the consideration of the individual perspective of a teacher. For example, Rowan also shared her perspectives on colleagues' and students' expectations of mathematics lessons, with implications for the notion of a shared understanding of what constitutes a 'competent teacher' in particular settings (Gresalfi and Cobb, 2011). The interpretative framework offered by Cobb et al. (2001), holds that social and psychological perspectives are in an interdependent reflexive relationship. The above example from Rowan exposes the same duality in the OMF. Following the lead from Cobb et al. (2001), these intrapersonal and interpersonal perspectives are seen as complementary and co-dependent.

The other two elements of the OMF are cognitive demand and classroom norms. Cognitive demand provides a summary of the 'the level and type of thinking that a task has the potential to elicit' (Boston and Smith, 2009, p.122). As such, this is encompassed by the relationship between the students and the mathematics, providing an evaluation of the level of mathematics made available to them.

The relationship of classroom norms to the didactic triangle is more complex. These norms are a constituent part of a wider interpretative framework from Cobb et al. (2001), which from the outset has been drawn on to argue how different aspects of the didactic triangle can be brought into focus (3.2.6). As such, this has been used in the scrutiny of the OMF above. The social elements of their framework, namely classroom norms, have been privileged in the OMF, as these are more visible in the classroom. As these norms are considered to be recurring patterns of behaviour that fulfils the expectations the teacher and students have for the action of others (Cobb et al., 2009), they are considered to correspond to the relationship between the teacher and students. As the subcategories of classroom norms, specifically social norms, sociomathematical norms and mathematical practices, become more focussed on mathematical ways of working and specific practices, they encompass more aspects of the teachers' and students' relationship with mathematics (figure 6.2).

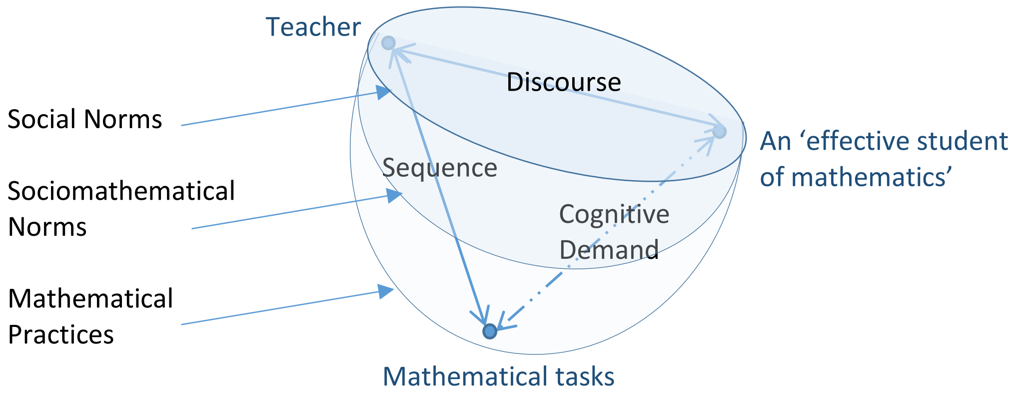


Figure 6.2: The relationship between classroom norms and the didactic triangle

(Straesser, 2007, p.165)

Hollingsworth and Clarke (2017), amongst others, have argued that the complex relationship between classroom activity and learning necessitates research that draws on a ranged of theoretical and methodological frameworks. They called for 'careful parallel analyses of high quality, complex data' by research teams (p.459). Whilst the nature of this study precluded a research team approach, it is hoped that the mapping of the OMF to existing theoretical framework allows the reader to relate this study to the wider field.

6.4 Pedagogical Profiles: Differential Pedagogical Practices

A range of pedagogical practices has been associated with teaching in sets, and low attaining sets in particular. For example, research indicates that low sets tend to access a more limited and fragmented curriculum, with a focus on low-level recall within more tightly controlled whole-class activities and more individual seatwork (Linchevski and Kutscher, 1998; Watson, 2001; Francis et al., 2017) (2.4.1). As demonstrated in chapter 4, for each teacher in this study there were some differential features when pedagogical profiles for their lower attaining set (class B) were compared with their higher attaining set (class A). There were differences between the three teachers' pedagogical profiles, and identified differential features mirrored this variation. The following sections discuss shifts in the teachers' practice and how this relates to other research findings.

6.4.1 Curriculum

One of the most evident differences between the higher and lower attaining sets was the nature of the curriculum to which students were given access. The three teachers all taught from school curriculum plans that contained different routes for different sets.

Rowan's Key Stage 4 classes followed routes aligned with the two tiers of GCSE exams. Class A followed the school's higher curriculum route and class B the foundation route, with any commonality between the routes diminishing as Key Stage 4 progressed (5.3.3.1). Whilst Rowan indicated she had discretion to choose how to teach particular topics, she felt pressure to "cover the content for the exams", especially for the higher attaining class (5.3.3.2). Consequently, her lessons aligned closely with the school's differential curriculum routes and her classes experienced different mathematical topics and material.

For Joe and Sam's Key Stage 3 classes, each school had schemes of work where the same overarching topics were scheduled to be taught at the same time to all classes in the same year, but the specific content identified as appropriate for different sets did vary. Both Joe and Sam said they had the discretion to deviate from the curriculum route suggested for each set, but in practice, their lessons predominantly aligned with these different routes (5.3.1.1 & 5.3.2.1). Whilst there was considerable overlap of content indicated on the different routes, this did result in students in different sets having different mathematical experiences.

For example, the content for Joe's class B lesson was presumed knowledge for class A, and students in class A only met that content when embedded in other tasks. The recorded lessons were part of a percentage topic. When the whole sequence of lessons was considered, it was anticipated that only class A would work on reverse percentage calculations. Over time, the differential routes resulted in Joe's higher attaining set meeting mathematical content, identified in school curriculum plans as 'higher level', which students in the lower attaining set would not meet in that academic year. These stratified curriculum routes, with lower attaining sets experiencing a more restricted curriculum, reflects findings from previous research (e.g. Dunne et al., 2011; Francis et al., 2017) (2.4.1).

The setting policies at both schools were based on measures of prior attainment, starting in year 7 with Key Stage 2 results, after which internal assessments were used. All the teachers reported that some students were moved between sets, but as reported in other studies (e.g. Dunne et al., 2011), the number of students moved was relatively low, at about ten percent of students per year. Decisions were based on teacher recommendation and students being identified as 'outliers', rather than an absolute re-ranking based on attainment measures, and Sam reported a few students were placed because of behavioural issues (5.3.1.1 & 5.3.2.1). Whilst a detailed analysis of the composition of sets and set moves was beyond the scope of this study, the systems in place could have allowed sets to be skewed in relation to issues such as students' socioeconomic status or SEND, as reported in previous studies (Wiliam and Bartholomew, 2004). Moreover, the limited movement of students, with set moves less frequent as the students approached GCSE exams, meant many students in the six classes were in the same or similar sets as they had been in year 7, and this was likely to remain the case. This type of teacher-influenced and restricted set moves mirrors findings in other studies (2.3). Unprompted comments from Sam and Joe indicated that when students were moved into higher attaining sets, gaps in knowledge caused some difficulties, some of which they attributed to the moved students not having previously covered the same material.

Whilst the participating teachers were asked to 'teach as normal', Sam changed his approach to teaching his class B in the recorded lessons (5.3.2.1). In unrecorded lessons he typically used different resources for classes A and B, whereas for the recorded indices lessons, both were based on the same resources, a practice he adopted in the subsequent recorded lessons. When all of Sam's ten lessons are considered, class B tended to take longer to complete tasks and activities, so fewer were completed, but the level of difference varied between lessons. For example, in the indices lessons the content coverage was very similar, with the inclusion of a negative exponent the only additional material used in class A (5.3.2.3 & 5.3.2.5). Whereas, for the three sequential lessons on fractions, class B covered the material from the first two lessons for class A, and did not meet models for multiplication and division that was the focus of class A's third lesson. This did result in class A continuing to meet a wider range of mathematical ideas, but Sam thought the differences were smaller than normal. Moreover, for some lessons, Sam's assessment of students' understanding of concepts was similar for both classes. This change also brought atypical ways of working to class B (5.3.2.5).

As, in general, the three teachers taught their sets in line with the schools' tiered curriculum routes, for many students their curriculum 'diet' was determined by their set allocation and their school's curriculum. However, Sam's departure from his normal practice for his lower attaining set provided some evidence that teachers' expectations unduly reduced the level of mathematics offered (2.4.2); limits that were successfully challenged, albeit partially, in his recorded lessons (5.3.2.5). As the teachers' default positions were to follow the schools' stratified curriculum plans, changing practice would also necessitate changing expectations at a departmental level.

6.4.2 Organisation

For all three teachers, there were fewer students in the lower attaining sets. Across all the recorded lessons, the amount of whole-class work ranged from 15 to 30 minutes, with no discernible patterns regarding which sets spent longer at this level of interaction. For Joe and Rowan, the distribution of time did follow different patterns; for higher attaining sets, whole-class periods were shorter but interspersed throughout the lesson, in effect interleaving whole-class and seatwork, whereas for the lower attaining sets the whole-class talk was contained in fewer longer sections of interaction (figures 5.2, 5.8, 5.28 & 5.38). In Joe's class, whole-class interactions were used to introduce a task that led into longer blocks of seatwork, whereas for Rowan it was the other way around with the whole-class element occurring at the end of a task or activity. Both Sam's classes followed a similar interleaving pattern (figures 5.14 & 5.23); a similarity that could be related to his decision to adopt the same resources and approaches for both classes (5.3.2.3 & 5.3.2.5).

During seatwork undertaken in lower attaining sets, both Joe and Rowan were seen to circulate the room and interact with all the students at least once, either individually, in pairs or small groups. This level of individual interaction did not occur in the higher attaining sets; the larger student numbers and shorter episodes of seatwork may have made this less feasible. One outcome of these different patterns was a greater opportunity for students in the lower attaining sets to interact with their teacher on a one-to-one basis, but reduced opportunities for students to share ideas with their wider peer group. If these patterns persisted, students in the lower attaining sets might have greater individual attention from their teacher but at the cost of a more isolated experience of mathematics, a trait reportedly more common with low attaining sets (Kutnick et al., 2006) (2.4.1).

6.4.3 Discourse

Many of the teachers' discourse patterns were consistent across both of their classes, and indeed, there were many similarities between teachers; the commonalities and exceptions are discussed in more detail below. For instance, in none of the classes was there a requirement for students to adopt the more precise language modelled or introduced by the teachers ([2]5.3.1.3d, -1.5d, -2.3d, -2.5d -3.3d, -3.5d). Moreover, the

[2] For reference to multiple classes, shorthand notation is used. The dash is a placeholder for repeating the underlined. E.g. (5.3.1.3d, -1.5d, -2.3d, -2.5d -3.3d, -3.5d) refers to 5.3.1.3d, 5.3.1.5d, 5.3.2.3d, 5.3.2.5d, 5.3.3.3d & 5.3.3.5d, namely Joe's class A&B, Sam's class A&B and Rowan's class A&B

three teachers often rephrased and extended the students' contributions into more mathematically precise terms. As such it appeared there was no press to induct students into a vertical discourse (Gellert and Straehler-Pohl, 2011) (3.3.6). However, it was noted that Sam used more informal language with class B in comparison to class A (5.3.2.3 & 5.3.2.5).

IRE was the dominant form of whole-class interactions in all the lessons (5.3.1.7, 5.3.2.7 & 5.3.3.7). These exchanges were often linked together to form longer sequences, with the teachers' evaluative turns often extended to include revoiced student contributions or additional levels of explanation. Questions asked usually had a limited range of mathematically valid responses, which were part of a step-by-step procedure structured by the teacher (e.g. extract 5.6). As such, many whole-class interactions could be characterised as 'guided algorithmic reasoning' (Lithner, 2008), with the teacher directing the overall strategy and the students' role often limited to engagement with individual steps. As such, these type of exchanges are often classified as low-level cognitive demand (3.3.3). Watson (2001), amongst others, associated this type of approach, where the students are led through steps in a procedure, with the teaching of low attaining classes (2.4.2). However, in this study this approach was ubiquitous and occurred in all lessons, regardless of the sets' attainment profile, and with all teachers.

Whilst many discourse patterns were common to all lessons, some differences in how satisfactory and unsatisfactory responses were treated were noted. For all, there was a greater use of superlatives with the lower attaining sets (5.3.1.5, 5.3.2.5 & 5.3.3.5). Although for Joe, occurrences were in single figures, so more caution was needed in interpreting this as a recurring pattern of behaviour. Superlatives appeared to be used in different ways. In Joe's class B, these were part of positive evaluations in IRE exchanges that were focused on calculations (e.g. extract 5.10, line 113). While the same types of questions were also regularly asked in Joe's class A, a superlative was only used once in an evaluative IRE turn. The few other instances of superlative use were related to a student's explanation of an alternative solution strategy (extract 5.6, line 166). In Rowan's lessons, there were almost no instances of superlatives being used with class A, whereas in class B these were included in about half of all positive evaluations in IRE exchanges (e.g. extract 5.45). She also used superlatives in relation to behaviour and engagement in class B. Sam, on the other hand, used superlatives at a similar rate and manner in both classes when positively evaluating responses (e.g. extract 5.27), but he only used them in relation to engagement with class B (5.3.2.5d).

As discussed in section 3.3.6.2, if superlatives are interpreted as 'praise' this in of itself is a complex issue (Hattie and Timperley, 2007). Brophy (1981) argued that teachers use praise based on their reading of student need rather than as a reaction to the quality of the response. From this perspective, it is possible that the differential use by the teachers indicated they held different views of the students' social needs related to attainment. Indeed, Sam and Rowan indicated they thought their lower attaining sets needed more encouragement to stay on task (5.3.2.5 & 5.3.3.5). Meyer (1982) argued if the student perceives the activity as easy, praise may be interpreted as inferring they have low 'ability'. In Joe's and Rowan's lower attaining sets and both of Sam's classes, superlatives were usually used in the positive evaluation of a single step in a calculation or a process, which students answered immediately and with mathematically valid responses (5.3.1.5, 5.3.2.7 & 5.3.3.5). This suggested those questions were straightforward for some to answer. There was the potential, therefore, for those questions to be perceived as 'easy' by the students and hence the inclusion of superlatives could have a negative impact on the students' academic self-concept (Ireson and Hallam, 2009). For students in the lower attaining sets, this had the potential to reinforce self-perceptions of 'low ability' that can arise from their set placement (Boaler et al., 2000).

The treatment of errors was a differential feature in Sam's classes (5.3.2.7). In both, Sam's typical action after interpreting a student response as an 'error' was to ask follow-up questions. There were occasions when multiple students responded to an initial question; if these contained one or more errors, in class A he allowed some peer-to-peer 'debates' to run (e.g. extract 5.19). This pattern of participation did not occur in class B; if one of the responses was 'correct' Sam drew immediate attention to that response through direct acknowledgement, otherwise errors were followed-up by questions or explanations. In class B, Sam's reaction to student-initiated comments with errors was the bald 'no'; this had a similar effect of returning the focus of the interaction back to ‘correct’ responses, promptly and in a teacher-controlled manner. Whilst Sam used simple self-contained questions with both his classes, ‘funnelling’ in response to errors was more common in class B (5.3.2.5d). Therefore, it appeared Sam controlled the follow-up to errors more tightly in his lower attaining set.

Kutnick et al. (2006) argued that some teachers controlled whole-class interactions more tightly in lower attaining sets (2.4.2). In addition to Sam’s treatment of errors, there were other indications that the three teachers shifted how they controlled whole-class interactions. For example, in class B, Rowan ‘broadcasted’ some students work to the whole class rather than asking the students to explain (extract 5.48). Whilst this could be seen as a way of valuing the students’ contribution by sharing work with their peers, this was also undertaken in a teacher-controlled manner. In Joe’s classes, there was more individual seatwork and less whole-class discussions with his class B (5.3.1.5b), and students in class A were given time to attempt problems before whole-class discussions, so were more likely to be reporting their solution strategies (5.3.1.3e). Therefore, it did appear that all the teachers controlled whole-class interactions more tightly in the lower attaining sets, albeit in different ways and to differing extents.

6.4.4 Tasks

The curriculum from which tasks were selected varied, dependent on the attainment profile of the sets (6.4.1). This resulted in different activities being undertaken in Joe’s and Rowan’s classes; Sam’s were more closely aligned due to changes he made for the recorded lessons. The type of activities Joe planned, such as questions modelled at a whole-class level and exercises from textbooks, were similar for both his classes. Rowan stated that class B undertook shorter activities that included more practical, interactive or competitive elements (5.3.3.5a) designed to support engagement (6.4.6), whereas class A there had more ‘traditional’ lessons, which consisted of teacher modelling followed by practice (5.3.3.3a). However, as discussed below, when the tasks were compared across the recorded lessons for each pair of classes, the mathematical features were similar. In particular, a clear-cut shift to more ‘drill and practice’ for lower attaining classes, as reported in some studies (e.g. Boaler, 2002), was not in evidence.

Sam and Rowan used multiple representations as an integral part of many tasks in their recorded lessons, which often included visual representations (e.g. figures 5.30 & 5.38). Whilst the tasks required students to use different representations, the focus was on specific examples rather than how features of the representations related to mathematical concepts. For example, the tasks in Rowan's class B required students to use bar models alongside written equations but there was no requirement to comment on critical features of the model and how these changed, such as relevance of absolute or relative length of bars (5.3.3.6). Similarly, in her class A, the tasks required the students to link algebraic expressions with different area decompositions, but equivalence was viewed through those examples. In Sam's indices lessons, expanded layouts were used to justify rules, but without attention being explicitly drawn to how the examples generalised (5.3.2.3c). A key element of Sam's other recorded lessons was the use of multiple representations of fractions (Baldry, 2018). In a similar manner to Rowan, the use of multiple representations was embedded in the tasks but with a focus on examples. For instance, when Sam asked, "you're going to tell me why the shaded area is four fifths" a student responded, "cause it's twenty-five and the number is twenty shaded".

It has been argued that multiple representations can either aid learning or be the source of confusion, dependent on how they are deployed (3.3.5). For example, engaging students in self-explanations about how visual features relate to mathematical concepts is thought to facilitate an understanding of visual representations (Rau et al., 2017). Both Sam and Rowan used multiple representations, and thereby offered students some conceptual variation (3.3.5.2), but with tasks structured around examples; the occasional more generalised comments were predominantly made by the teachers (e.g. extract 5.35). Consequently, students used different representations, but student self-explanations were not an explicit part of tasks and discussions about connections between features of representations, and between representations and concepts, were rare.

In terms of multiple solution strategies, Joe and Rowan predominantly modelled one procedure for each activity, which included problems where more than one approach was possible. Whilst they acknowledged alternative strategies, these were not included in their explanation, and the tasks that were subsequently met could all have been completed using the modelled approach; a comparison of strategies was not a required nor prompted element of the tasks (5.3.1.3c, -1.5c, -3.3c & -3.5c). Sam sometimes included multiple solution strategies, but when used these were introduced sequentially and making links between approaches were not an integral part of the tasks (5.3.2.3c & 5.3.2.5c). As such, the tasks in all classes provided limited access to the multiple solution strategy strand of procedural variation (3.3.5.2), and reinforced the norm that accurate application of a procedure was the expectation for an 'effective student of mathematics'.

All three teachers used exercises containing similar questions in some phases of their lessons (e.g. figures 5.11, 5.18 & 5.42). Rowan's class A lesson was atypical in this regard, as this focussed on the manipulation of equivalent expressions, but her other recorded lesson for this class followed the more typical model/exercise pattern (5.3.3.4). New features, such as transitioning from drawing bar diagrams to interpretation, were introduced as the exercises progressed. Consequently, the tasks exposed students to some variation, although the form and level of control of that variation differed. For example, Joe's percentage questions (5.3.1.3) and Rowan's class A (lesson 2) worksheets (5.3.3.4) had the same method applied to similar problems in slightly different contexts, where an accurately executed procedure for each question was the requirement. With few discernible connections between questions, tasks were usually classified as unsystematic variation (3.3.5.2). When this was coupled with the use of one register and one solution strategy, this limited exposure to variation that could have led to discernment. And while Sam often included multiple representations in his lessons, as with Joe and Rowan, the focus of questions in sequences of examples or exercises was often on the application of a procedure, and questions were usually self-contained with few interconnections (e.g. figure 5.18). As such, in terms of procedural variation, Sam's tasks were similarly classified as unsystematic variation.

Across the lessons, the students experienced a range of questions that could have extended their experience of the range of permissible change (3.3.5.2), but this was often in one dimension with only a small subset of the possible changes met. Moreover, most examples were in a common format; atypical, boundary or non-examples were infrequently seen. So, students may have had a restricted understanding of possible variation, which could have limited their understanding of generalisation. In addition, there was rarely explicit reference to the range of permissible change or dimensions of possible variation.

Considering the lessons from the different perspectives variation theory can offer (3.3.5.2), it appears students could have experienced some variation, but with varying levels of control and generally in an implicit form. This may have limited the students' discernment of critical features and hence generalisation (Mason, 2011a). However, there appeared to be no systemic differences between the higher and lower attaining sets for each teacher in this regard.

When all the recorded lessons are considered, there were occasions when the teachers in this study included problems set in contexts related to the real-world, and aspects of these lessons were coded as 'pseudo-context' as the problems appeared contrived for the classroom (5.3.1.3, 5.3.2.6 & 5.3.3.7). For Sam, there were few discernible differences between how context was used in the higher and lower attaining sets (5.3.2.4 & 5.3.2.6). Rowan used some 'real-world' context with class B, but not with class A. For class B, the second recorded lesson used local maps (5.3.3.6), but the task design allowed the students to complete the work without using compass directions, the stated learning objective. For Joe, there were more links to 'real-world' contexts in class A than class B; the school policy was to include this in all lessons, but Joe indicated he found this harder in some topics, such as his class B lesson. In his class A, there were some instances of students not understanding terminology and others where students made personal judgements about the context rather than attending to the mathematical problem (5.3.1.3). It appears, therefore, that these contexts may have distracted from the learning intentions and produced some conflicts of attention.

Research indicates that students from low socioeconomic backgrounds may be more likely to draw on 'inappropriate' contextual information (Cooper and Harries, 2002) (3.3.5.4), but students' background information was beyond the scope of this study. Here, while some differences in the use of context were noted, the small number of cases meant drawing conclusions about the differential use of context based on attainment was not possible.

6.4.5 Sequencing

In all the recorded lessons, the three teachers predominantly directed or redirected the focus of attention onto approaches they introduced (5.3.1.7, 5.3.2.7 & 5.3.3.7). With IRE the dominant form of interaction, the teachers initiated the focus of attention a significant amount of the time, but they also redirected students back to approaches they introduced far more often than they interrogated student reasoning. This included situations where student contributions could have been part of a mathematically valid solution strategy.

For example, when asked to calculate $3.49 \times 10 \times 3$, a student attempted to multiply by three first; Joe used an IRE sequence to redirect the student back to multiplying by ten first (extract 5.13). In a similar fashion, Sam redirected an exchange back to the previous focus, namely exponents of zero, without exploring the student's reasoning, which appeared to be related to a generalised feature, that of powers of ten always remaining positive with negative exponents (extract 5.21). Rowan often redirected the focus of attention to approaches she introduced (5.3.3.3e), which resulted in students' alternative approaches being treated as errors. For example, in the second lesson with class A, a student offered $x = \frac{g+c}{a} + b$ as a rearrangement of $a(x - b) = g + c$. Rowan redirected exchanges so the bracket was expanded first, as per previously modelled questions, leading to $x = \frac{g+c+ab}{a}$ (figure 5.37: sequencing).

Consequently, it appeared the three teachers predominantly privileged their mathematical horizon over the interrogation of student reasoning in their management of the lesson trajectories (3.3.7.2 & 3.3.7.5). There were some exceptions, such as in Rowan's class B, when she accepted a diagrammatic layout introduced by a student, albeit returning to her prepared layout afterwards (5.3.3.5e). However, these types of episodes were rare and, for the most part, the focus remained on approaches the teachers introduced.

Student contributions were predominantly part of IRE exchanges, with the focus controlled by the teachers' questions. On occasions, the teachers utilised more extensive ideas from students but in a teacher-controlled manner. For example, Rowan, nominated a student to explain their approach, which became central to a whole-class IRE sequence, but Rowan knew about the approach beforehand, and reworded and extended the explanation (5.3.3.3e). In a similar way, Joe integrated a known student's approach into an IRE sequence (extract 5.6 & 5.7). After giving students sixty seconds to prepare, Sam asked three students to explain 'anything to the power zero is one' (extract 5.23), but these explanations were not interrogated, and Sam drew to a conclusion with his own explanation. Consequently, when using students' approaches, the overall lesson trajectories remained aligned with the teachers' mathematical horizons.

Whilst drawing firm conclusions from single episodes in lessons is not feasible, all these instances of intentional shifts of focus to more extensive student explanations occurred in the higher attaining sets. There were no equivalent examples of Joe, Sam or Rowan incorporating more extensive contributions into whole-class talk in their lower attaining sets. Rowan 'broadcasted' students work in her class B, but she selected and worded the contribution rather than the explanation being voiced by the students (e.g. extract 5.42). This could be seen to contribute to the evidence that talk in lower attaining classes was more tightly controlled by the teachers.

Most of the questions in IRE sequences were self-contained and a fair proportion were relatively simple. When combined with the teachers' structuring of procedures, even for students directly engaged in the interaction, the level of cognitive demand could have been low. However, as discussed in 3.3.6.2, the surrounding sequence of events needs to be taken into account when evaluating the significance of particular questions (Lefstein and Snell, 2011). When extended sequences of IRE exchanges were considered, mathematical structures and concepts were involved. For example, as part of the modelled solutions, Joe signalled the relationship between the value and the percentage represented (extract 5.1), Sam structured his IRE turn-taking to include shifts between different representations of indices (e.g. extract 5.17) and Rowan asked how different equations could be formed from a diagram (e.g. extract 5.47). So, extended IRE sequences had the potential to focus attention on aspects of mathematical structure. However, these exchanges predominantly focussed on the particular examples under discussion; explicit links to concepts were rare.

In a similar manner, the sequencing of activities needed to be taken into account. For example, Sam introduced different ways of exploring powers with exponents of zero, Joe presented different ways of multiplying by powers of ten and Rowan had alternative bar-model representations. These juxtaposed alternatives provided some variation, but the teachers usually maintained the focus of attention on the current procedure, with any comparisons left to individual student discretion. It appeared, therefore, that more generalised concepts could have been available if students could see through the specific to the general (3.3.5.2) or if they made links between different classroom episodes. However, the teachers rarely marked critical features explicitly, so relationships and links to concepts often remained implicit. Consequently, students could have engaged in whole-class talk and responded appropriately whilst encountering low levels of cognitive demand, but there was the possibility of higher levels of cognitive engagement as some may have attended to the more mathematically significant features. This, however, would have been dependent on individual students' self-directed discernment.

As with the task dimension, the teachers' management of the lesson trajectory was similar for both their classes, with few systemic differences attributable to attainment. As with the analysis of tasks from a perspective of variation theory, differences in the mathematics made available might lie in how successful the students were at reading the implicit mathematical meaning embedded in sequenced activities. This in turn could be dependent on how they interpreted particular instances of a concept and whether they made self-generated links between sequential episodes. The argument made in section 3.3.5 is that this might occur more often with students in higher attaining sets.

6.4.6 Teacher Cognition

The ambitions all three teachers articulated for students had elements related to developing mathematical understanding. For example, Joe and Rowan discussed problem solving (5.3.1.2 & 5.3.3.2) and Sam referred to developing conceptual understanding (5.3.2.2). However, there were some subtle shifts in lesson goals; while all three teachers had a mixture of learning and performance orientations, learning was given more emphasis in their higher attaining sets (5.3.1.3a, -1.5a -2.3a, -2.5a,

-3.3a, & -3.5a). In addition, Rowan mentioned practice was important for her class B, due to her perception of their poor retention skills, and both Joe and Sam discussed how they planned to adapt lessons because of students' weaker number skills in their respective lower attaining sets.

Kutnick et al. (2006) argued that some teachers perceived student behaviour as being more problematic in lower attaining sets (2.4.2.1). There was some evidence that Sam and Rowan held similar views; student engagement and behaviour were only mentioned in respect to their lower attaining sets (5.3.2.5a & 5.3.3.5a). In particular, Rowan ascribed different motivational states to her class B, with the students requiring greater extrinsic motivation; adaptions she adopted included using shorter, more competitive tasks. Sam stated that students in his class B required more encouragement to stay on task and behaviour could adversely affect whole-class discussions; in class B superlatives were used with reference to engagement, which did not occur in class A (5.3.2.5d). As discussed in 6.4.3, Joe and Rowan used more superlatives with their lower attaining sets within IRE exchanges, which again could indicate they perceived the needs of the students in those classes differently from their higher attaining sets (2.4.2.1 & 3.3.6.2). It appeared, therefore, that all three teachers had elements of their beliefs where they held different expectations for students in different sets.

Sam did change his practice for the recorded lessons (5.3.2.1 & 6.4.1). He acknowledged that he usually used more individual ways of working with his lower attaining set, but planned to adopt a similar discussion-based approach in both classes for the recorded lessons. When he discussed his evaluation of the indices lessons, he was pleasantly surprised at the success of this change, in terms of both engagement and his assessment of student understanding in class B (5.3.2.5f). This offered confirmation that he previously held different expectations for class B and suggested the experience may have challenged aspects of those beliefs. Sam continued to use the same resources and approaches for the remaining recorded lessons. From his perspective there was mixed success; the students could access the resources but engagement in whole-class discussions remained unpredictable, with some discussions curtailed due to a perceived lack of engagement.

The teachers' interpretation of classroom activities and their in-the-moment decision making appeared to be predominantly orientated to their mathematical horizon, with the interpretation of student reasoning at a whole-class level a rare occurrence (e.g. extract 5.10). This was indicated, for example, when teachers used IRE sequences to redirect attention back to processes they had previously introduced (e.g. extract 5.6). However, with IRE patterns of interaction such a consistent part of all the teachers' discourse patterns, many of their in-the-moment decisions may have fallen within algorithmic reasoning rather than being a reflectively conscious act (Watson, 2019) (3.3.7.4). There appeared to be few discernible differences between higher and lower attaining sets in this regard.

In overall terms, the lesson images appeared slightly more performance orientated with the lower attaining groups, with more significant differences in the content covered due to the stratified curriculum routes that the teachers followed (6.4.1). Whilst there were indications that the teachers planned different types of tasks for the different sets, when tasks were analysed there were few differences in their features (6.4.4).

6.4.7 Cognitive Demand

In section 3.3.3, cognitive demand was defined as 'the level and type of thinking that a task has the potential to elicit' (Boston and Smith, 2009, p.122). Here, a common feature of many of the tasks was they could have been completed in a procedural manner based on models the teachers presented to the class. Similarly, in whole-class discussions, there was little requirement to go beyond describing procedures (5.3.1.3g, -1.5g, -2.3g, -2.5g, -3.3g & -3.5g). However, there were also elements that went beyond the purely procedural, such as the use of multiple representations by Sam and Rowan, and the attention Joe paid to structure through the use of place value or percentage/value links. So, as activities had the potential to convey some meaning beyond the particular, the majority of the tasks and 'talk as mathematics' were classified as procedural or process.

As discussed in sections 6.4.4 and 6.4.5, attention was rarely drawn in an explicit manner to the links between examples and concepts or between representations and their mathematically significant features. Consequently, whilst there were opportunities for students to make connections with mathematical concepts, they could have worked in a procedural way when completing tasks or taking part in whole-class discussions. So, the lessons provided opportunities for the students to engage in activities with the potential to elicit higher-level thinking, but the students could have participated, in ways accepted as legitimate, with low-levels of cognitive demand. In this regard, there were no systemic differences discernible between sets with different attainment profiles.

6.4.8 Classroom Norms

In terms of social norms, in all classes the teacher was the arbiter of correctness (5.3.1.7, 5.3.2.7 & 5.3.3.7). The dominance of IRE sequences, with the norms of immediate acceptance of 'correct' responses and follow-up action after 'errors', appeared to be significant contributory factors (e.g. extract 5.2). In this respect, the three teachers' practice mirrored previous research findings (3.3.6.2). In addition, the three teachers tended to redirect students' attention back to a single procedure they introduced, thereby privileging finding the answer through known procedures over an exploration of alternative routes. These repeated patterns of interaction indicated that an 'effective student of mathematics' would be seen as one who could efficiently produce a 'correct' answer and errors were to be avoided or corrected if met (5.3.1.4, -1.6, -2.4, -2.6, -3.4 & -3.6).

Other types of student engagement were valued. In all the classes, students demonstrated some agency to act through student-initiated questions or comments. However, these occurrences were usually limited to a few individual comments within teacher directed whole-class interactions. In terms of differences between classes, as discussed (6.4.3 & 6.4.5), student contributions appeared to be more tightly controlled by the teacher in the lower attaining sets, potentially imposing further limits on those students' agency.

From the perspective of sociomathematical norms, descriptions of procedures were accepted by all the teachers as explanations (e.g. extract 5.2). Moreover, examples were used to justify more general cases without making links explicit or drawing attention to the range of permissible change. For example, Joe accepted one alternative percentage reduction calculation as justification of that process (extract 5.7), Sam used examples to justify the 'rules of indices' (5.3.2.3h) and Rowan discussed equivalence based on particular examples of compound areas (extract 5.35); the resulting message being that examples are sufficient to justify a more general case.

6.4.9 Shifts in Pedagogy

Research indicates when students are taught in sets, classes with lower attainment profiles tend to have access to a more restricted curriculum, be taught by teachers with lower expectations, and have distinctive pedagogical characteristics (2.4.2). If these features occur, they can combine to limit students' access to significant mathematical ideas. The following discussion draws on sections 6.4.1 to 6.4.8 to summarise how aspects of those previous research findings were apparent in the individual teachers' shifts in practice.

In these three pairs of case studies, the stratification of the curricula did place restrictions on the mathematical content students experienced (6.4.1). Whilst the teachers felt they could adapt their school's curriculum plans, in practice they generally followed the prescribed stratified routes (5.3.1.1, 5.3.2.1 & 5.3.3.1). These differences were driven by school level decisions, influenced by England's exam structures (2.4.1). Sam's change to using the same curricula materials for both classes in the recorded lessons (5.3.2.5c) raised the question as to whether teachers' expectations, both in designing the curricula and in selecting content for particular lessons, unduly limits students' access to the curriculum when placed in a low attaining set.

It was noted that the three teachers had similar differential expectations about student engagement related to attainment, albeit manifest in different ways. For example, for the three lower attaining sets, the teachers use of superlatives, Sam's caution about holding discussions and Rowan's use of shorter tasks, indicated they all expected those students to need more support to remain engaged in the mathematics (6.4.6). There were also some subtle differences in the level of control the teachers' exerted in whole-class discussions, with tighter control seen in the lower attaining sets. Specifically, there were differences in the management of ‘errors’ (6.4.3) and the use of student explanations (6.4.5).

However, there were many similar features to discourse patterns, tasks and the sequencing of lessons, and consequently the levels of cognitive demand were also comparable (see 6.4.3 to 6.4.7); key features are highlighted below. All three teachers talked more than the students, frequently using IRE patterns of interaction, where they revoiced student contributions and extended explanations. In general, classroom discourse and tasks focused on the ‘doing’, in other words the efficient execution of procedures, and there was relatively little explicit attention drawn to mathematical relationships, concepts or strategies. Procedures were accepted as explanations and in many whole-class discussions, the teacher directed or redirected attention onto a single procedure with little comparison of multiple solution strategies. Multiple representations were used in similar ways across attainment groups. When attention was drawn to links between representations, and between representations and concepts, this was predominantly undertaken at the level of examples (e.g. extract 5.37), rather than between features of representations and concepts.

Students were exposed to different types of variation, but this varied in type and level of control. For example, the use of multiple representations provided access to an aspect of conceptual variation, but this was likely to remain an implicit part of the task as attention was rarely drawn to these features through the task requirements or whole class talk. Unsystematic variation in exercises was relatively common, as the accurate execution of a procedure was the normal requirement for each question, with few discernible links between them (5.3.1.3c, -1.5c, -2.3c, -2.5c, -3.3c & -3.5c).

In overall terms, there were relatively few systemic differences attributable to attainment when the task, sequencing and cognitive demand dimensions of the OMF were considered. Previous studies have reported common features for lower attaining sets include low cognitively demanding tasks, the breaking down of tasks into simple steps and a shift to ‘drill and practice’ (2.4.2). Here, however, if these features were part of a teacher’s pedagogical approach they tended to occur in both sets, and there was no wholesale shift to ‘drill and practice’. Whereas differences were notable in curriculum access, aspects of discourse and teacher cognition elements of the OMF, specifically in relation to expectations, motivation and level of teacher-control, all of which have been reported in previous studies (2.4.1 & 2.4.2).

7. Conclusion

7.1 Summary of Responses to the Research Questions

In this study, the research questions asked how a teacher orchestrates mathematics for different groups of students (RQ), with a focus on how teachers shift their pedagogical approaches when teaching different groups of students (RQ1a) and how the character of mathematics made available to students varies (RQ1b). Due to the complexity of the classroom, the response to this research question is multifaceted and while there were areas of commonality between the teachers, this was not universal (6.4.9). In overall terms, shifts were noted in the areas of curriculum access, expectations and control of whole class talk, albeit manifest in different ways. The teachers planned lessons from a more restricted curriculum for their lower attaining sets, so those students experienced less content coverage compared to their peers. In relation to engagement and motivation, lower attaining classes appeared to be seen as requiring higher levels of extrinsic motivation, manifest in the use of competitive tasks or a greater use of praise. Whilst many of the teachers discourse patterns were stable across both their respective classes, there was evidence that the teachers controlled whole-class talk more tightly in their lower attaining classes.

However, there were other aspects of the teachers' practice where there appeared to be no systemic differences between the different sets. For instance, how the teachers talked about mathematics and to what they drew attention were similar across their classes. In general, classroom discourse and tasks focussed on 'doing', with attention drawn to procedures. The use of multiple representations and structured solution strategies offered students opportunities to consider some underpinning mathematical concepts and relationships, but examples tended to be used without explicit attention being drawn to how the specific and general relate. For example, exercises tended to have unsystematic variation and when variation was controlled more tightly this was an implicit part of the task; drawing explicit attention to variation or relationships at a conceptual level was rare. Consequently, it is argued that the mathematics made available to students would have been dependent on how successful individual students were at attending to the implicit mathematically significant features.

In terms of the two parts of the research question (RQ), class pedagogical profiles have been built up that have allowed cross class comparisons, which have allowed a detailed response to the first part (RQa). Some inferences have been made about the mathematics made available to students, summarised in relation to cognitive demand, which has allowed a partial response to the second part (RQb).

It should be noted that this study focussed on teachers' pedagogical moves and the mathematics made available to an 'effective student of mathematics'. Only episodes categorised as mathematically related were coded and analysed. As discussed in section 3.2.3.1, students can identify with, merely comply or resist the classroom obligations of an 'effective student of mathematics'. Whilst the normative identity for all classes was predominately related to the efficient production of 'correct' answers, it was beyond the scope of this study to consider if there were differing levels of identification with these obligations.

In order to answer the research questions the Orchestration of Mathematics Framework (OMF) was developed. Consequently, consideration needs to be given to the viability of the OMF as an analytical tool for charting teachers' pedagogical approaches and for characterising the mathematics made available to students. As discussed in 6.3, the OMF provided me with a structure for the coordination of different theoretical lenses when building an understanding of pedagogical moves in 'typical' mathematics lessons. The arguments made in section 6.4 provide the warrants for the claim that the OMF is a viable analytical tool for charting substantive parts of participating teachers' pedagogical moves, with inferences made about the mathematics made available to students. Sections 3.4.3 and 6.3.2 considered the relationship of the OMF to other theoretical perspectives in order to site this work in the wider field.

7.2 Limitations and Ethical Considerations

This thesis is composed of a classroom-based video-study, structured as a case study of three teachers and six classes. To a large extent, this study relies on video transcripts of the lessons and it could be argued that this reliance raises a question as to the appropriateness and robustness of a case study approach. To have meaning beyond this study requires the work to have sufficient trustworthiness and transferability, which can be achieved through providing sufficient descriptions of the study and transparency in the analysis for the reader to relate these cases to settings with which they are familiar (4.4.2).

The study explored the teachers' pedagogical moves. Whilst the complexity of the classroom necessitated selecting a perspective through which to view classroom interactions, foregrounding teachers' moves did place individual students' activities in the background. The argument made was that the notion of classroom norms allowed students' participation to be taken into account without a detailed examination of individual students' activities (3.2.2). However, this did mean that the teachers' activities in relation to supporting equity of access were not considered (3.4.4). In addition, the focus was on whole-class interactions that accounted for about a third to half of the lesson time and other aspects of the lessons were not subject to direct analysis. Technology does exist that would allow students' seatwork to be captured in more detail, both audio and visual, and camera systems can automatically track the teacher. Therefore, it would be worth considering capturing a wider range of data in any future studies.

Another limitation of this study is that it has been undertaken by a sole researcher. The OMF has value if it can contribute to building a shared understanding of what happens in mathematics classrooms. Sections 3.4.3 and 6.2 outlined how the conceptualisation of the OMF had cohesion for this researcher and discussed how it related to other theoretical frameworks, as a starting point for situating the OMF in the wider field. However, the efficacy of the framework can only be tested with wider use. So, while the development of the OMF supported my understanding of previous research and potential relationships between different perspectives, the question remains as to whether a bespoke framework designed by a sole researcher was the most effective way to produce a sufficiently robust study.

In terms of the design and implementation of the study, the difficulties in recruiting participants led to the opportunistic recruitment of three teachers. Whilst this in of itself was not an issue, as I was looking to find a way to explore any typical lessons, it did mean that I had pre-existing and ongoing relationships with the teachers. Two were PGCE alumni and all had been mentors for ITE students on the PGCE course I work on. I did adapt to their preferred ways of participating, for example not challenging Sam when he changed his way of working as he reported this as a positive experience. As the teachers were used to the PGCE model of joint observation of student-teachers, followed by a debrief that included elements of judgment, it proved difficult to steer post-lesson discussions away from that format. Many of the pedagogical approaches that have been challenged in educational research have roots outside of the individual teacher, such as stratified syllabus structures and the prevalence of unsystematic variation in English textbooks. As my analysis progressed, it was difficult to describe and interpret classroom activities in ways that took into account the research literature but without inadvertently implying judgment of the teachers (4.3.3 & 5.2).

7.3 Implications

7.3.1 The Orchestration of Mathematics Framework

Numerous theoretical perspectives can be brought to bear on the interpretation of mathematics classrooms. Previous research studies have offered a range of lesson observation frameworks to interpret classrooms, which have diverse purposes and formats though many have an evaluative stance. Within this field, the OMF provides a holistic pedagogical profile of a teacher's practice that allows different theoretical perspectives to be interpreted in relation to each other. The descriptive and interpretative focus of the OMF could compliment evaluative perspectives by capturing classroom activities that fall outside other studies' criteria for effective instruction. Moreover, the OMF orientation from the perspective of the teacher has the potential to compliment other orientations, such as a focus on the learner more typically found in lesson study (Larssen et al., 2018).

The descriptive and interpretative power of the OMF has the potential to provide teachers with new ways to understand their practice, and thereby support their professional development. The OMF could be used as both a planning and review tool, providing the framework for focussing attention on features that are significant in the learning of mathematics. In particular, the OMF draws attention to elements of practice that are usually less visible, such as classroom norms. As previously discussed (3.4.4.2), the transition from a research instrument to an accessible and effective professional development tool would not be straightforward. Consequently, it is likely that the OMF would initially be used in contexts supported by teacher educators, such as Initial Teacher Education (ITE) or professional development courses.

Within ITE, for example, student-teachers meet a range of theoretical perspectives during their course. Over time, the OMF could provide a structure for relating those perspectives to their classroom practice. Used as a planning tool, different elements of the OMF could be in focus at different stages of their course, so student-teachers could build-up their understanding of the different facets of their practice within a coherent framework. In ITE courses, school-based mentors undertake the majority of lesson observations of student-teachers and usually take the lead in lesson review processes. There can be a tendency for judgments to be made without an explicit and detailed discussion of what happened in the lesson. So, if the mentors could be inducted into the use of the OMF, its descriptive orientation could provide the structure for building a better shared understanding of what happened, and thereby provide the foundations for more productive discussions between the student-teacher and the mentor.

7.3.2 Setting

Prior research has explored the effect of teaching students in sets from a range of perspectives. Studies have sought to explicate the relationships between setting and students' experience of mathematics by analysing attainment, identifying characteristic pedagogical approaches and exploring differential access to teachers and the curriculum. However, the conflation of these factors in the setting process makes it more difficult to determine the relative influence of each. In surveys, teachers have reported how they adapt their practice when teaching sets with different attainment profiles. This study contributes to the field through the detailed study of three teachers' actual classroom activities when they taught different sets, charting shifts in practice and exploring possible antecedents.

Specifically, this study provides insights into the influence of stratified school curriculum plans on individual teachers' lesson choices, and the level of 'in-built' expectations regarding students' capabilities to access the curriculum. As the GCSE exams have a tiered curriculum, stratification appears to be structurally embedded and may therefore be difficult to counter. However, Sam's change in practice for the recorded lessons provided a brief window on the potential of more closely aligning teaching approaches for different sets; a possible avenue for further study.

In this study, some differences were noted in how the teachers controlled the classroom discourse in their higher and lower attaining sets, but there were few discernible differences in patterns of interactions and the management of lesson trajectories. In particular, the steer towards procedures and the low level of explicitness in how mathematical representations, relationships and concepts were discussed were common features across the classes. If the norm is for many aspects of mathematical meaning to be conveyed through implicit means, maybe the question raised is whether students in higher attaining sets are those better placed to discern mathematically significant features and make connections between representations and mathematical concepts themselves. The corollary being that raising levels of explicitness may support all students, but particularly those in lower attaining sets, although that would be a far from straightforward undertaking (Mason, 2011b).

7.4 Learning through Research

This study has been undertaken part-time alongside my work in ITE. Shifting my perspective from tutor to researcher was not always straightforward but brought additional insights. This study highlighted the regularity of patterns of participation in classrooms and the role of normative identities. As a PGCE tutor, I have developed ways of working and there are expectations, from both student-teachers and school-based staff, about that role. Having to consciously shift language and ways of being in classrooms away from evaluation, I believe prompted recognition of some of my own algorithmic reasoning (Watson, 2019). For example, my research diary indicates I moved from summative comments to more detailed lesson observation notes that attempted to capture individual interactions and activities. This was undertaken in response to identifying a need to make the description and analysis stages of observations more explicit. Contending with the tensions between tutor and researcher perspectives has also influenced my professional practice, where, for example, I now have a greater focus on supporting student-teachers to articulate links between tasks and mathematical concepts.

Lesson study is a model of collaborative planning, teaching, observation and review of lessons, undertaken by teams of teachers and framed by pedagogical research. My work in this field focusses on the nature of observation. In the UK, one tradition is to focus the observation on a few case-study students rather than the teacher or the whole class. This challenged my thinking about my argument that lessons can be analysed using the notion of an 'effective student of mathematics'. As previously discussed, the complexity of the classroom cannot be captured with a single model. As such, an area for my future research might be to consider how to bring together the OMF, with its focus on the teacher, and a lesson study approach, with a focus on the students.

7.5 Concluding Remarks

When I moved from teaching mathematics in the state school system to working in the higher education sector I was surprised at the breadth and depth of mathematics education research, and disappointed that so little of this research appeared to be accessed by teachers. This study has sought to integrate the most relevant theoretical perspectives, so these can be drawn on to analyse typical lessons, as it is typical lessons that form the day-to-day experiences of teachers and students. The ultimate goal being that the OMF could contribute to linking theory and practice, and in particular provide a means for teachers to consider ways in which they can ameliorate the impact of setting.

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Appendices

Appendix 1: Ethical Approval

1.1 Confirmation of Ethical Approval

UNIVERSITY OF LEICESTER

University Ethics Sub-Committee for Sociology; Politics and IR; Lifelong Learning; Criminology; Economics and the School of Education

12/06/2016

Ethics Reference: 4511-fb128-education

TO:
Name of Researcher Applicant: Fay Baldry
Department: Education
Research Project Title: Teachers Orchestration of Mathematics

Dear Fay Baldry,
RE: Ethics review of Research Study application

The University Ethics Sub-Committee for Sociology; Politics and IR; Lifelong Learning; Criminology; Economics and the School of Education has reviewed and discussed the above application.

1. Ethical opinion

The Sub-Committee grants ethical approval to the above research project on the basis described in the application form and supporting documentation, subject to the conditions specified below.

2. Summary of ethics review discussion

The Committee noted the following issues:
Please ensure that the opportunity to withdraw without reason appears in letters to parents/children and teachers. We support the gaining of consent from children as a way of showing respect to them in line with their participation being voluntary, even though legally as minors their signatures are not legally binding. Including parent/carer and child signatures on the same form is supported. Good luck with your study.

3. General conditions of the ethical approval

The ethics approval is subject to the following general conditions being met prior to the start of the project:

As the Principal Investigator, you are expected to deliver the research project in accordance with the University's policies and procedures, which includes the University's Research Code of Conduct and the University's Research Ethics Policy.
If relevant, management permission or approval (gate keeper role) must be obtained from host organisation prior to the start of the study at the site concerned.

4. Reporting requirements after ethical approval

You are expected to notify the Sub-Committee about:

- Significant amendments to the project
- Serious breaches of the protocol
- Annual progress reports
- Notifying the end of the study

5. Use of application information

Details from your ethics application will be stored on the University Ethics Online System. With your permission, the Sub-Committee may wish to use parts of the application in an anonymised format for training or sharing best practice. Please let me know if you do not want the application details to be used in this manner.

Best wishes for the success of this research project.

Yours sincerely,
Dr. Laura Brace
Chair

1.2 Parent / Carer Letter of Consent

Dear Parent / Carer,
There is a lot of interest in how mathematics can be best taught, especially in light of the curriculum changes that are currently underway. Observation of lessons is a key tool in understanding how mathematics lessons progress for different learners. Fay Baldry, a researcher from the School of Education, University of Leicester, is undertaking research into how classroom observations can best be used to build a picture of how mathematics unfolds in different settings. She is looking to observe and video a small number of mathematics lessons and ask the class teacher about the work that students completed.

All recordings will be stored securely by Fay Baldry and only viewed by herself and the class teacher to ensure confidentiality. Any reports will be written to ensure that student anonymity is preserved at each stage of the research process (i.e. any reports will not include names or other details that could identify individuals). Any student who would not want to be part of the recorded lessons would not be penalised, and any student could withdraw once the study has commenced without needing to provide a reason.

Fay Baldry taught mathematics in secondary schools for over 15 years before taking on her current role in teacher education; this project forms part of her own PhD studies.

In order to share good practice with others in the education sector, it is hoped that finding from this research could be used for professional and academic publication, in journals or conference proceedings. Any published material will have strict student, teacher and school anonymity.

If you have any further questions concerning this matter, please feel free to contact the school (or Fay Baldry). Please discuss this with your child, and complete the slip below to indicate whether or not you would all be happy for some of their lessons to be recorded and used for this purpose. Thank you for reading this letter and for your co-operation.

Yours faithfully

Fay Baldry

fb128@le.ac.uk
School of Education
21 University Road

✂____

Name of student:____________________________
Parent/Carer Signature:______________________________ Date:___________
Student signature:______________________________ Date:__________
We are happy for lessons for the above to be recorded and used in this study: Yes/No*

*please delete as appropriate

Appendix 2: Pilot Study Stage 1

2.1 Class A (AU3: Box and Whisker Plots)

2.1.1 Transcript Extract: Class A

Downloaded from https://www.timssvideo.com/transcripts.

Annotations are indicated by *[italics].*

Coding:

> Mathematically related; Mathematical organisation; Not mathematically related – only mathematically related was subject to further analysis.

Problem:

> How many choc chips are needed to ensure 6 cookies have at least 3 each?

Transcript:

Teacher: Okay, write down the answers to these please. Yes, Daniel.

Student: Um, what time are we gonna come back from computing?

⋮

Teacher: All right, here's the story. My wife bakes hot chocolate chip cookies, which I like. But lately, the number of cookies- or the number of chocolate chips in the cookie... has been decreasing. We're going to simulate an experiment here whereby we have to find out how many chocolate chips I've got to put into a mixture to create six cookies. Which you have on that sheet that I've given you- so that I can be pretty sure that each cookie is going to end up with at least three, yes? Now, anyone got any ideas as to how many chocolate chips I would have to put into my mixture so that I would end up with at least three in each cookie?

[Discourse: Exposition IRE]

Student: Eighteen?

Student: Eighteen

Teacher: What about if I had 18- now, they're mixed up in this mixture for the six cookies- are you sure that each time you scoop out some of that mixture, you're gonna get three? So you- you're riding on the bare minimum there, aren't you? You're hoping- you're hoping that you're going to get three each scoop.

[IRE: answer treated as error]

What we're going to do is this little simulation exercise and it's just to gather some statistical data so we can carry on with our statistics. You're gonna work in pairs. One person's going to roll the dice. The number of the dice indicates the cookie and your cookies are numbered one to six. For example if I rolled a five that means I've got one chocolate chip for cookie number five. You're going to have to roll the dice sufficient times to end up with a minimum of three chocolate chips in each cookie. Do you understand?

[Context – initially 'real', but link between context and model used not made explicit, and assumptions not considered so psuedocontext as enacted]

Student: Yeah

⋮ [30 minutes of rolling dice until each 'cookie' had 3 'chocolate chips'. Class results collated: Frequency distribution table compiled, median, upper and lower quartile calculated, box plot requested]

Teacher: And just before we go on there... How many chocolate chips do you think you would need to ensure that you ended up with three?

[IRE]

Student 1: Twelve million of 'em.

Student 2: At least 40

Teacher: At least?

Student: Forty

Student 3: There's no way you could tell because if all the cookies- all of the chips (inaudible)

Teacher: Well let's face it, you know... If we- if we put these chocolate chips into the mixture and mixed them around, we would expect them to get mixed a little bit and not sit in one corner, wouldn't we? Right?

[Treated responses as errors – 'ignored'. Interpretation: Student questions related to raw data; Teacher responded with summary statistics]

But using- using the information that we gathered from that little thing-

Student: Twenty-nine?

Teacher: Twenty-nine?

Student: About 30.

Teacher: So where are you getting these figures from?

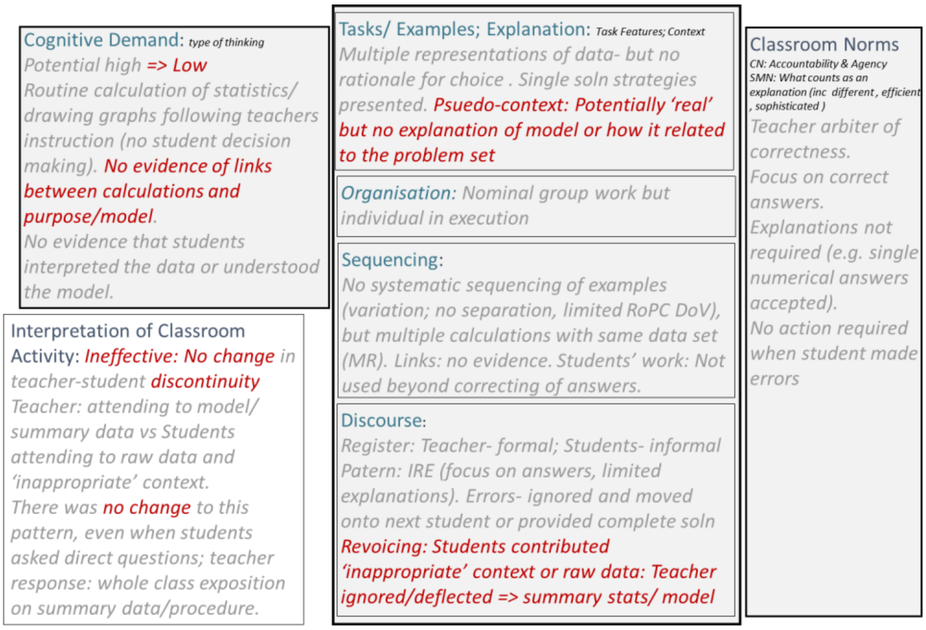
[IRE; treated as errors – 'ignored']

Student: From the mean.

Teacher: The mean! The mean? Okay. Let's just have a look at those three numbers again: a mean of thirty point five seven, a median of twenty-eight point five, and a mode of 25. So you are electing, in this case, to use the mean.

Student: Yeah

2.1.2 OMF: Interim OMF Class A



2.1.3 Researcher's Comments: Class A

Downloaded from:

https://static1.squarespace.com/static/59df81ea18b27ddf3bb4abb5/t/5ca513c0e2c4834be286549e/1554322368430/AU3+Researcher+Comments.pdf

References to national data were removed.

Analysis:

Examples of cross referencing with OMF analysis: Task Features, Cognitive Demand

Researcher's Comments:

The remainder of the problems in this lesson were related to one another, mostly mathematically or thematically. The rest were repetitions. 00:03:38 This problem is situated in a real-life context. The problem also involves the use of physical materials, in this case dice. Students throw the dice to determine which cookie will get a chocolate chip. Here the teacher provides a goal statement, telling the students that they are going to work on a simulation exercise that is related to their study of statistics. 00:06:05 At this point the classroom interaction shifts from public to private. This is the first of 12 shifts between public and private interaction throughout the lesson. Overall, 57% of the lesson is devoted to public interaction and 43% is devoted to private interaction. In this segment, the students are working in pairs. However, during the private interaction segments that occur later in the lesson, the students work individually. 00:24:16 In this problem, in which students are asked to construct a box-and-whisker plot, they are given a choice of how to create the scale. Therefore the problem is considered one in which students are allowed a choice of solution methods. The problem also involves the use of physical materials, in this case rulers 00:34:31 The students are led through an exercise in using their graphing calculators, although they do not have a chance to complete it during this lesson. 5). This time-point also marks the beginning of the introduction of new content in this lesson. Up to this point, the students have been reviewing previously learned content. Therefore, 81% of the lesson time is spent reviewing and 19% is spent introducing new content.

2.1.4 OMF: Summary Class A

Cognitive Demand: ***type of thinking***
Low: Procedural Algorithmic approach, no ambiguity.
Potential High: No overt attempt to link activity with purpose/ concepts of different representations, but the same data set in more than one way may have made links visible to some
High attempted: One 'which is more appropriate' but ineffective as initial question flawed. Checking planned… though procedural

Interpretation:
Teacher-Students discontinuity.
Student questions: raw data
Teacher response: summary data

Concepts Ways of working

Tasks: Examples Explanations
Task Features: Single solution strategy.
Multiple representations of the data (raw, table, summary and box plot); no rationale for choice
Context: Psuedo-context. Initial real, but model flawed and not explained. 'Classic' students drawing on real context in discussion @ end which had to be deflected by the teacher as it did not fit pseudo-model used.

Sequencing:
Variation; No separation, limited range of permissible change (RoPC) or dimensions of variation (DoV). No sequence of examples, but multiple calculations with the same data set
Links; No evidence
Students' work: Not used beyond answers.

Organisation
Individual
~~Group work~~

Discourse: ***Registers; Patterns: IRE, Questioning; Student Responses: Errors, Revoicing***
Register: used formal mathematical register himself, but little or no requirement for explanations off students- they did not use.
Patterns: IRE (focus on answers, no explanations)
Errors: ignored & moved on to correct responses from another student, or provided the complete solution to the whole-class.
Revoicing: DW- responses to students' questions related to correcting procedure

Teacher's Orchestration of

Classroom Norms
Accountability & Agency

Teacher arbiter of correctness.
Focus on correct answers.
Explanations not required. (Single numerical answers accepted)
No action appeared to be required when student made errors.

Individual Student Activity

2.2 Class B (AU4: Ratio)

2.2.1 Transcript Extract: Class B

Downloaded from https://www.timssvideo.com/au4-ratios#tabs-3.

Annotations are indicated by *[italics].*

Coding:

Mathematically related; Mathematical organisation; Not mathematically related

Transcript:

Teacher: Lindsey you can take any spares and put them in the box and bring the box to the front, okay? All right. Now let's just go over what we did yesterday. Yesterday you had to divide your 12 into different ratios. Let's do one of those again. Or two of them. Let's divide the 12 you have in the ratio, uh... five to seven. Divide them into ratio five to seven. Thank you. So that means that you're going to separate them into two groups on your table with how many in the first group?

[Organisation. Explanation (procedure). IRE initiated]

Student: Five.

Teacher: Five and seven in the second one. Okay. What if I say I want them divided into ratio one to two?

[IRE]

Student: (inaudible)

[Error indicated by teacher response]

Teacher: No. Six in each would be?

[Bald no]

Student: Fourteen.

Teacher: Six to six which would be? One to one.

[Ignoring error, answering own question]

Student: Eight in one group and four in the other.

Teacher: Eight in one group. Which group?

[Order error – opportunity to correct]

Student: In the two.

Teacher: Groups, so you really should have told me four to eight because remember with ratio the order is important.

[Ignores student response, corrects but with answer only… the teacher did the explanation].

Now I want you, for what we're moving onto today, to actually make them equal piles. So I want how many piles altogether really?

Student: Two

Teacher: No.

[Bald no. Treated as error although could be considered 'correct']

Student: Three.

Teacher: Three piles all together and I want you to stack them up so you have three piles. One on your left, and two on your right. Stack them up so you have one on your left and two on your right.

[procedural explanation]

One two three four five... six seven eight nine 10 11 12. All right. So what we actually have there- Danny doesn't seem to, but the rest of us should have. Is when we-

We have three piles worth 12 and you have created one pile of four and two piles of eight. Okay?

2.2.2 Researcher's Comments: Class B

Downloaded from:

https://static1.squarespace.com/static/59df81ea18b27ddf3bb4abb5/t/5ca5168271c10b0f2bc1bb0f/1554323074755/AU4+Researcher+Comments.pdf

References to national data were removed.

Analysis:

Examples of cross referencing to OMF: Task Features, Cognitive Demand

Researcher's Comments:

At the beginning of the lesson, the teacher writes the heading for today's class on the board: "Dividing a Given Quantity in a Given Ratio." 00:01:58 Here the teacher is handing out counting blocks that the students will use to help solve various problems dealing with ratios. These blocks are considered to be "special mathematical materials"; that is, mathematical materials used for a mathematical purpose during the lesson. 00:09:56 In this problem, students are asked to divide 200 dollars in the ratio 1 to 3. Problems involving calculations and algebraic manipulations such as this, were considered to have "using procedures" problem statements. During the interval 10:21-13:34 the teacher and students use procedures to complete the problem; specifically, they divide the 200 dollars by four and then assign the appropriate amount to each part of the ratio. 00:21:33 At this point in the lesson, the students are given a problem on which they work privately, in groups of 2 or 3. The lesson switches between public and private work a total of 10 times. Overall, 57% of the lesson is spent in public interaction and 43% is spent in private interaction. During all the other periods of private interaction, the students work individually. Most of the private interaction time (73% per lesson, on average) was devoted to working individually. 00:29:31 This problem, in which students are asked to create a story about why they want to divide Smarties (a type of candy) in a given ratio, is the first one in the lesson that is set up with a real-life connection. (Several of the previous problems had a real-life connection brought in as they were solved, but they were not set up in a real-world context.) 00:43:18 Here the teacher introduces a new method to solve a problem worked on earlier in the lesson (at 14:04). The lesson shifts back and forth between introducing and practicing new content several times. Overall, 10% of the lesson time is devoted to reviewing, 40% is devoted to introducing new content, and 50% is devoted to practicing new content. There are seven shifts in purpose. 00:54:50 A student can be seen here using a calculator for computation purposes. 00:55:54 At this point, the teacher assigns a group of problems for students to begin in class, and then complete at home for homework. Altogether, the class spent approximately 12 minutes working on nine "future homework" problems. These problems all involve Smarties, a type of candy, which are considered both real-world objects used in the lesson and physical materials used to solve problems.

2.2.3 OMF: Summary Class A

Cognitive Demand: ***type of thinking***

Low: Procedural
No links made to underlying concepts
Explanations not sought

Concepts — Ways of working

Tasks: Examples Explanations
Task Features: manipulable but token use
No multiple representations
Preferred solution strategy (ie offered manipulables and allowed alternative procedure, but not critiqued and TWC used one solution strategy)
Context: Psuedo-context , Which did lead to issues with later student work and a 'classic' focus on reality that was not deemed warranted by the teacher.

Sequencing: ***Variation; Links; Students' work.***
Variation: Unsystematic, no separation . Little RoPC or DoV exception: RoPC highlighted 'using easy numbers- factors'.
Links: not made

Organisation
Individual
~~Group work~~

Discourse: ***Registers; Patterns: IRE, Questioning; Student Responses: Errors, Revoicing***
Register: horizontal
Patterns: IRE, short pauses, sometimes quiet individual/chorused answer, teacher answered own questions. Closed down/funnelling without giving students chance to contribute. Errors: ignored/baldly acknowledged, correct answer accepted. No discussions either way.

Teacher's Orchestration of

Classroom Norms
Accountability & Agency

Teacher arbiter of correctness.
Focus on correct answers.
Explanations not expected; any explanations/ justifications provided by the teacher.

2.3 Class C (AU2: Congruent Triangles)

2.3.1 Transcript Extract: Class C

Downloaded from https://www.timssvideo.com/au2-congruence

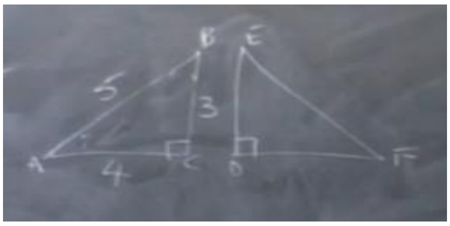
Annotations are indicated by *[italics].*

Coding:

Mathematically related; Mathematical organisation; Not mathematically related

Lesson:

Starter activity: used this as the example of congruent triangles



Main task: activity designed for students to establish the minimum set of rules that would produce congruent triangles.

One student draws a triangle and creates rules, a second student follows the rules and 'tests' congruence.

Transcript:

Teacher : …how many instructions did you actually need for each one?

Student: Yep, I needed four.

['correct' answer: three]

Teacher: You needed four, what were they?

Student: Um, interval AB is vertical and measures five centimetres and AC is horizontal.

Teacher: So if it's saying horizontal it's saying it's at an angle aren't you?

[Reframing: students vertical – horizontal and other no-defining features. Teacher reframes in terms of defining features. Discontinuity]

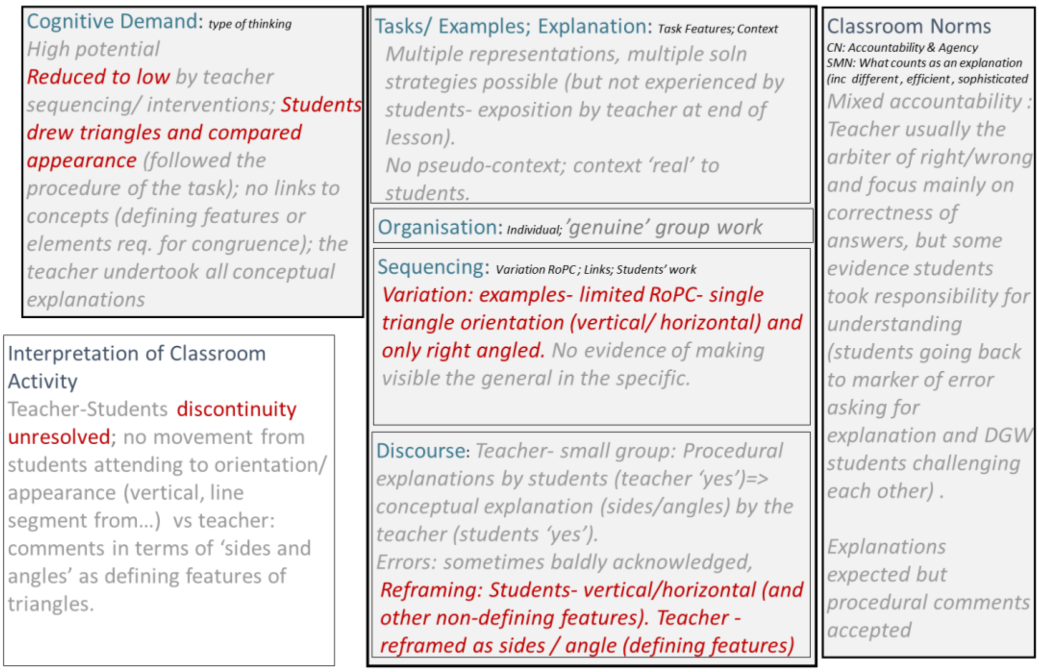
Student: Yeah.

Teacher: So we're saying one's vertical and one's horizontal.

Student: Yeah and you um...

Teacher: Then you're defining the angle between them aren't you?

2.3.2 OMF: Interim OMF Class C



2.3.3 Researcher's Comments: Class C

Downloaded from:

https://static1.squarespace.com/static/59df81ea18b27ddf3bb4abb5/t/5bbe1513a4222f8030f6e571/1539183892086/AU2+Reseacher+Comments.pdf)

References to national data were removed.

Analysis:

Examples of cross referencing: Task Features, Cognitive Demand

Researcher's Comments:

The lesson begins with a review of congruent triangles. The review lasts until 4:51, at which time new content is introduced. Then, near the end of the lesson (at 38:07) there is a brief period during which the students practice and apply what they have just learned. Introducing new content accounts for a relatively large portion of time in this lesson (73%). 00:04:51 This problem is the only one in this lesson where students are explicitly given a choice of solution methods. That is, they are told that there are multiple ways to solve the problem, and they can decide which method they want to use (see 8:56). When the results are presented, several alternative solution methods are shown, and they are explored at length. This problem is mathematically related to the earlier problems in the lesson; more specifically, it is an extension of the mathematical ideas and operations involved in the previous problems. This problem is also considered to be at a high level of procedural complexity due to the fact that it requires more than four steps and has several embedded sub-problems. The problem is stated in such a way that students are asked to conjecture and reason about the minimum pieces of information needed to determine if two triangles are congruent. After students work on the problem, the teacher goes over the results. Because she focuses on the answers and identifying the corresponding rules, without making explicit the mathematical reasoning behind these rules, the problem is considered to have a "stating concepts" implementation. 00:09:32 At this point the students begin working in groups at their seats. This is the first of two periods of private interaction during the lesson (the second begins at 40:00). Altogether, 38% of the lesson is devoted to public interaction and 62% is devoted to private interaction. All of the private work time is spent in groups. The remainder of the time was spent working individually. 00:38:07. These four problems, assigned as a group, are all applications. That is, they require students to apply procedures they previously learned in one context to a different context. These problems are begun in class, but students are expected to complete them for homework (see 44:51). Altogether about five minutes are spent working on them.

2.3.4 OMF: Summary Class C

Concepts Ways of working

Cognitive Demand: ***type of thinking***
High potential: Task had potential for students to make links between features of triangles and congruence, thereby developing an understanding of the critical features of triangles. Reduced by the amount of information provided (such as the number of different approaches and how they were different). Also, as enacted, some pupils were working with their own unchallenged features of triangle construction rather than triangle features.
Low for some students.

Interpretation:
Teacher-Students discontinuity.
Student comments: related to drawing instruction (vertical, line segment from…)
Teacher response: interpreted in terms of 'sides and angles' as defining features of triangles.

Tasks: Examples Explanations
Task Features: Multiple solution strategies available in task, though students selected a limited range for their own work.
Multiple representation: instructions and diagrams
Context: No context. But task 'real' in a classroom context

Sequencing: ***Variation; Links; Students' work.***
Variation: limitation RoPC- single orientation of triangles in starter and use of horiz/vertical in main tasks. Some RoPC/DoV-congruent triangles with different criteria. No evidence of seeing general in the specific.
Links: Not apparent
Students work; generic examples of students work used @ end of lesson, mainly presented by the teacher- minimal student input.

Organisation
Main task genuine Group work.
Individual @ end

Discourse: ***Registers; Patterns: IRE, Questioning; Student Responses: Errors, Revoicing***
Register: used formal mathematical register and some expectation for students (e.g use of hypotenuse).
Errors: sometimes baldly acknowledged, sometimes ignored, accepted/ correct answers often revoiced to draw attention to side/angle defining features of triangles, sometimes appearing to move beyond student meaning.

Teacher's Orchestration of

Classroom Norms
Accountability & Agency

Mixed accountability.

Focus was mainly on the correct answer, but evidence that students' own expectation that they understood (e.g. going back to explanation of error).

DGW: students challenged each other

Appendix 3: Coding and Analysis

The following provides exemplars of the analysis strategy of the transcription coding and the development of the OMF. Appendix 3.1 provides an overview of the timeline. Appendices 3.2, 3.3 & 3.4 provide exemplars of the coding and how this developed over time. Appendix 3.5 provides the annotated summary OMFs that structured the lesson comparisons with the same class and the cross-class comparisons.

3.1 Timeline

As described in section 3.4 and exemplified in appendix 2, the initial coding was developed from the categories on the OMF.

The development of the OMF and coding were iterative and intertwined. The following summarises the main development points, but there were overlaps between events.

Literature Review

> TOM: Initial Conceptual Model (figure 3.4: model A).
> OMF: Model refined, to include sequencing and organisation dimensions and the mathematical teaching cycle, producing the first iteration of the OMF (figure 3.8: model B).

Pilot Study

> Coding: Lesson transcripts based on OMF categories (appendix 2).
> OMF: Category descriptions were refined through application to empirical data in stage 1&2 (figure 3.11: model C).

Transition to Main Study

> Coding: Lesson transcription protocols revisited and formalised (3.4.2.2).
> OMF: Coding cross reference with OMF, with category descriptions refined leading to the final iteration of the OMF (figure 3.14: model D).
> Interview transcripts annotated based on coding protocols. Data from transcripts, interviews and classroom artefacts mapped to OMF.

3.2 Transcription Annotations

Pauses: (.) short notable pause in speech *(originally without brackets)*

(..) pause of about two seconds duration (not timed)

(…) pause of over two seconds

Visual/ contextual information: [square brackets] contain additional non-verbal information.

Truncated speech: a dash - is used when it appears that the utterance is a truncated word e.g. thirt-

Overlapping speech: wavy underlining is used to indicate overlapping speech.

Emphasis: Vocal emphasis is indicated by underlining the stressed word or syllable.

Volume: notably quieter talk was indicated by (brackets).

Indistinguishable talk: (inaudible) indicates talk during whole-class episodes that cannot be heard. This could, therefore, be private talk not intended to be part of whole-class discussions but picked up by the microphones.

T – teacher talk
S – student talk
S1, S2 etc. to indicate different students were contributing to the same exchange.
Ss – multiple students talking

Class Activity Codes

WC: Whole-class – fully public activity

Implied expectation that the whole-class is attending to a single activity. Typically the teacher or a student talking to the class, either exposition or question-and-answer session.

TWC: Teacher Whole-class – fully public activity

As above but where the teacher is the leading voice, either exposition or asking questions

Semi: Semi-public.

Could be heard by most students but without signals that listening was expected.

DW: Desk Work (Seatwork)

Students working at their desks on individual or group tasks

DWI: Individual Seatwork
DWG: Group Seatwork
TM: Teacher Movement

Teacher circulating the room

ITS: Teacher-Student Interaction (Initiated by the teacher)

Individual Teacher Student interaction, not intended for a wider audience

ITSac: as ITS but across room – so audible to other students

IST: Teacher-Student Interaction One (Initiated by the student)
ITSm: Teacher-Student Interaction Many- semi-public

Interaction with one or a small group of students, which is applicable to other students in the vicinity.

ITSpair: as ITSm but to a pair of students
ISS: Student-Student Interaction

Group interactions

Mathematics Activity Codes

MN: Mathematics-not related

Classroom management/organisational issues unrelated to the mathematics. E.g. handing out of resources, collecting in of homework, disciplinary matters, and social interactions.

MR: Mathematically related

MA: Those activities with mathematical content activities

MO: Mathematics Organisation

E.g. practicalities such as arranging into groups

Type of talk/interaction
Expo: Exposition
Ins: Instructions
Q&A: Question and answer sequence
Ir E: Part of Initiate Response Evaluate sequence with capitalisation indicating what part of the sequence.
Ex: Extended explanation as part of IRE sequence.
Evaluation:

+ve treated as satisfactory
+veQ treated as satisfactory with 'praise' – indication of quality
-ve treated as unsatisfactory

Comp: Computation- result of calculation
Pro: Procedure or process (see below)
Proc: Procedure – what was done
Pros: A more generic comment that could be applied outside that particular question
MC: Mathematical concept- links to wider mathematical structure, concepts or generalisation

CN: Classroom Norms
SN: Social Norm - Agency Accountability Authority Responsibility
SMN: Sociomathematical Norm - explanation; learning of mathematics (e.g. practice; hierarchical skills)

3.3 Early Transition Coding Protocols for Lesson Transcripts

The lesson transcripts were in tables, with the first three columns the time stamp, the speaker (teacher or student(s)) and the transcript. Additional columns were added to include the description, level of interaction, an activity code, the level of mathematics and a line number. The table below indicates which columns in the table contained which type of information. The abbreviations are explained in appendix 3.2, but were refined over time.

Time	Speaker	Transcript	Description *Italics - interpretation*	Level inter-action	Activity		Line no.
			IRE *+ve* *+veQ* *-ve* Questions: closed/open; single/multiple solutions; simple; recall; narrow (limited)-procedural (the answer unlikely to new insights); self-contained/ links beyond individual question; reducing parameters (leading; funnelling) *SN: e.g. repetition of response => indicates acceptance* *SMN: e.g. procedure accepted as explanation*	TWC ITS IST ISS ITSac ITSpair ITSm DWI	Ins Expo Q&A Ir E Ex FS – follow on student	MO Comp Pro Proc Proc	

An example of early coding is given below. Section 3.4.2.2 outlined how the coding was formalised. An exemplar lesson transcript using the final version of the coding protocol is given in appendix 3.4.2.

3.3.1 Exemplar Lesson Transcript – Early Coding

		Transcript (date and group):	Description *Italics - interpretation*	Level	Activity	
0044	T	open your books for me please		Semi		
###		** done <no inaud> fine **^ <inaud>				
0115	T	name are you ok				
			teacher sitting at desk register			
0209	T	if you have finished you can go to see if you can do the next one the next one …	Instruction	TWC		MO
###			TM students appear to be completing starter			
0358	T	1 squared	Q: *closed, recall, single soln. self-contained*	TWC	Q&A I	Pro
0359	SS	1 (fairly quiet)	A: recall single result			
	T	ergh				
	S	1 (loud)	*Student interpreted teacher's response to reply more loudly.*			
	T	2 squared	*+ve No comment on response and moved on => acceptance* Q: *closed, recall, single soln. self-contained*	TWC	Q&A r E I	Pro
	SS	4 (louder)				
	T < Ss >	3 squared <9> 4 squared <16> 5 squared <25> 6 squared <36> 7 squared <49> 8 squared < 64>	*+ve* Q: *closed, recall, single soln. self-contained*		Q&A r E I	Pro
0456	T	2 to the power 4	Q: *closed, single soln, self-contained. Level: recall*	TWC	Q&A I	Pro
	S	16	A: single result			
	SS	16				
	T	16 loverly OK . Tom what does 2 to the power 4 mean	*+veQ* *SN: repetition of response => indicates acceptance* *Q: asked for meaning*		Q&A r E I	Pros
	S 1	[not Tom very quietly] 16	A: single result			

	S 2	[hesitantly: probably Tom] 16				
	T	what does it mean	*-ve: follow up question* *16 not acknowledged and followed up by partial repetition*			
0506	S	oh it means 2x2x2x2	*A: procedural/process explanation*	TWC		Proc /s
0516	T	excellent . good . so we sort have got a rough understanding of that . erm .. I am going to tell you first that . 2 .. squared times 2 cubed is 2 to the power 5 . ok um why .. Clarisa	*+veQ SN: procedural explanation accepted* writing on board $2^2 \times 2^3 = 2^5$ *Q: wider – giving sum and asking for explanation*	TWC	Expo Tell Q&A r EEx I	
0535	S	you add the powers inaud	A: single procedural explanation			
	T	so that's what we can do ... Amber	*-ve Follow up question (or =ve??)* *SN: emphasising 'do'* *Implied Q about why student moved onto a more structural explanation => indicating that explanation is more than 'do'* *Q: indicating more than 'do' required*		Q&A r E I	MC
	S	isn't it because it looks like 2x2.x. 2x2x2	*A: process explanation self-mitigation*			Proc /s
	T	wonderful . ok so let's think about this .. 2 squared is just 2x2 . and we are multiplying that . by 2x2x2 2cubed which is 2 to the power 5.. is that OK . yeh good ..	*+veQ* *revoicing explanation*		Q&A r EEx I-next box	Pro MC
0603	T	erm .. so for instance on a few of these . 2 to the power 5 times 2 to the power 4 . is what .	questions on powerpoint *variation: "note* the same base" *-highlighting boundary to students* *Q: closed, single soln, self-contained*		Q&A r EEx I	
0616	S	(/2 to the power 9\)inaudible	A: single result			

3.4 Coding Protocols

After the pilot study, the coding protocols were revisited and were mapped to the OMF (see section 3.4.2.2). This resulted in revisions of the OMF and updated coding protocols. The coding summary is given below, which is followed by an example of the final transcription process.

3.4.1 Coding Summary

A summary of themes that emerged from coding.

- Structure of Talk
    - Level of interaction:
        - Whole-class; semi-public; local
    - Type of interaction:
        - Turn-taking
            - Teacher: **IRE**; IRE variant (extended teacher turn); multiple R
                - I: Simple; Self-contained; Single/Multiple Solutions
            - Student-initiated: peer-to-peer; questions/comments
        - Monologue Exposition:
            - Teacher: explains/instructs
            - Student: explanation
    - Register (vertical ⇔ horizontal)
- Steer: Regulation of Lesson Trajectory
    - Teacher led:
        - Launch, direction/redirection (simplifying, processing, conceptualising)
        - Focus: Mathematical horizon or student reasoning
        - Feedback: sharing solutions
    - Student led:
        - Student-initiated approaches
- Type of Talk
    - Talk as mathematics:
        - Mathematical focus
        - Level (computation; procedure; process; mathematical concepts)
    - Talk about mathematics:
        - The nature of mathematics
        - The learning of mathematics

- Talk about students:
    - Effective student of mathematics

This summary was tabulated for ease of reference for the coding of transcripts.

<table>
<tr><td>Structure of Teacher Talk</td><td>Sequencing: Steer</td><td>Type of Talk</td></tr>
<tr><td>Level of interaction:
Whole-class
Semi-public
Local</td><td rowspan="3">Teacher:
How
Launch
Direction
Redirection
Whose
Mathematical horizon
Student reasoning
Feedback

Student:
Student initiated (S:I)</td><td>Talk as mathematics (TasM):
Computation
Procedure
Process
Mathematical concept</td></tr>
<tr><td rowspan="2">Type of interaction:
Turn taking
Teacher: IRE, type of question
Student initiated (S:I)
Monologue</td><td>Talk about mathematics (TabM):
The nature of mathematics
The learning of mathematics</td></tr>
<tr><td>Talk about students (TaS):
Effective student of mathematics
Motivation and engagement</td></tr>
</table>

3.4.2 Exemplar Lesson Transcription - Final Transcription Protocol

The final protocols contained the descriptions of classroom activity, such as the content of board work in [ ], as part of the transcription.

IRE exchanges were numbered, with the capitalisation indicating the place in the sequence of that particular exchange. (Where one IRE exchange leads into another IRE then both are indicated – e.g. irE(1) Ire(2) in sequence).

The level of interaction was included in the time stamp box to free up space for analysis. Mathematical organisation continued to be shaded green and not mathematically related grey. The type of talk was listed in the last column at transition points only.

Time		Transcription		Inter-action	
1041 TWC	T	OK well let's move onto that last point then how many ways can we think of of writing two thirds (.) or putting two thirds (.) what is two thirds James	Steer: Launch Q: MS (multiple solutions)	Ire(1)	TaM (continues until change)
1050	S1	It's two thirds of a whole		iRe(1)	
1053 TWC	T	Two thirds of a whole (.) how are you going to represent that	+ve (repeat, Q: follow-on) Q: not closed Steer: direction-processing Focus: representation Level: process	irE(1) Ire(2)	SN: correct, efficient
1056	S1	By make three then shade in two		iRe(2)	
1100 TWC	T	Make three what	+ve (Q: follow-on)	irE(2) Ire(3)	
1101	S1	(inaud)			
1102 TWC	T	Of what	Clarification Level: process/concept	Ire(3)	
1102	S1	Whole		iRe(3)	
TWC	T	What are you calling the whole	+ve (use, Q: follow-on) Steer: direction-processing/concepts Focus: representation Level: process/concept	irE(4)	
1106	S1	circle		iRe(4)	
TWC	T	A circle (.) ok [drawing circle on board, splitting into 3 equal parts, shading 2]	+ve (repeat, use) Level: process	irE(4)	SN: responsibility for explanation
	Ss	(circle . pizza)			
1116	T	Like that yeh	Revoicing: Extending student explanation – one word to diagram	[ire(4)]	SMN: explanation
	S1	Yeh			
1117 TWC	T	Cool (.) any other ways (..)	Q: MS Steer: direction-processing Focus: other representations	Ire(5)	
1118	Ss	(square (…) square)	No T response		
1122	S3	You could really do it with any shape can't you	S:I Level: concept	S:I(1)	SN: student agency
TWC	T	rectangle square any sort of shape yeh (..) it might be quite more difficult with some shapes that others mightn't it	Steer: acknowledged S:I but not pursued (horizon)	S:I(1)E	
		So what would that be Lowis [pointing to circle with 2/3 shaded]	Q: single soln, self-contained	Ire(8)	
		Pardon (.)		iRe(8)	

		[teacher pointing at board] Oh zero point six recurring			
1224 TWC	T	Zero point six recurring [writing: $0.\dot{6}$]	+ve (repeat, use)	irE(8)	SMN: what expected wrt explanation
1227 TWC	T	.. Joe what else	Q: MS Steer: direction-representation	Ire(9)	
	S	percentage		iRe(9)	
1229 TWC	T	Percentage yeh so there's lots of different ways . what does percentage mean^…	+ve (repeat, explicit) Q: not closed Level: process/concept	irE(9) Ire(10)	
	S	Out of a hundred		iRe(10)	
1236 TWC		Out of a hundred yeh (.) youknow so if you split up the word percent you get per and cent (.) it literally means per one hundred because cent means one hundred in French (.) yeh (..) so per cent (.)per one hundred (.) erm anything else we can say about fractions then (..) ok (…) well let's see what I thought about it then let's see what I've come up with then (.) I really like a couple of those things by the way I particularly like this one (.) and this one [point at decimal and part-whole on the board] I particularly like one number divided by another number and part of a whole they're quite nice comments (.) so it's four fifths and I'm telling you that actually all of them represent four fifths can we look at all of the rest of those shapes on there (.) all the rest of the diagrams (..) and see how they represent four fifths	+ve (repeat, explicit) Level: process Multiple representations: but no explicit links between representations and concepts or between features of different representations.	irE(10) extend Mono-Expo	SN: responsibility for explanation SMN: explanation

3.5 Comparison of Lessons and Classes using Summary OMFs

The following are the summary OMFs presented in figures 5.7, 5.13, 5.21, 5.22, 5.26, 5.27, 5.36, 5.37, 5.45 & 5.46. The key points of comparison between lessons for the same class and cross-class comparisons were these summary OMFs. These comparisons, cross-referenced with the more detailed lesson narratives, structured the findings and informed the written summaries.

In the versions below, underlining has been used to indicate the common and differential features when class A was compared to class B.

KEY: Common feature; Differential feature; Lesson specific

As discussed in section 5.3.1.4, the summary OMFs were working documents, which have descriptions in note form and bespoke abbreviations are used (table 5.1).

3.5.1 Joe Class A OMF

Teacher Cognition

Lesson Image: Teacher Class

Lesson Goals: *Performance, learning for some (silver: find % change; gold: find alternative methods)*

Plan Rationale: *Online examples offered relevance and rounding opportunities.*

Hypotheses: *Find % then ± would be students' default approach. Contexts may cause issues.*

Heuristic

Knowledge, Beliefs and Values

MCK

Priority: to develop student's understanding of mathematics (particularly problem solving). Discussion beneficial to enrich understanding.

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon (over-interpretation of student reasoning?)

Organisation: *Individual (peer encouraged); interleaved SW-WC (ISW first); self-selecting tiered tasks*

Discourse Patterns

Registers: *term explained (integer) but no press for students' use*

Interaction: *IRE dominant, extended teacher turns, linked longer sequence. Some multiple responses pre evaluation*

Questions: *Most limited soln – part of step-by-step procedure (T summarised)*

Responses: *Typical satisfactory & unsat*

Revoicing: *repeat, rephrase-formal, extend*

Tasks, Examples, Explanations: *(model-exercises)* Features Context: *some 'real-world' questions constructed for classroom. Textbook psuedocontexts.* General^{n}: *not explicit* MSS: *possible, acknowledged and shown, but not required. One method in teacher explanations (% +/-)* MR: *possible-not used* Example variation: *limited RoPC DoV (% change with all information provided), 'contexts' varied but similar form/ format, no 'not'*

Sequencing: Lesson trajectory

Focus on mathematical horizon, redirection to single procedure (% change, finding % then +/-). Drew attention to the relationship between value and % represented (structure). Focus on comparison with online reductions ➪ excluded exploration of rounding. Variation: *Alternative SS acknowledged (used at end of lesson*):* *Students' work: alternative SS shared**

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act). Errors: avoid/ correct.*

SMN: *Procedure as explanation, (one procedure to soln); students articulate next step a few times (prompted). Mathematics as a hierarchy (content, ways of working).*

ESM: efficient & correct answers

MP: 'one percent' approach for %

Cognitive Demand

Type of thinking:

Potential: *High (discovering alternative strategies 'gold goal)*

Range: *Low to high Procedural completion of tasks following modelled examples was possible (no press for higher).*

Individual Student Activity

KEY: (Compared to B) Common feature; Differential feature; Lesson specific

3.5.2 Joe Class B OMF

Teacher Cognition

Lesson Image: Teacher Class

Lesson Goals: *Performance, learning a possibility for some (silver: fluent x10s 100s, 1000s; gold: student-led 'flipped')*

Plan Rationale: *Weak numeracy skills => starter useful and model $\times 10^n$ first to limit decimal use.*

Hypotheses: *Some- still 'add zeros' for decimals $\times 10^n$ Others- errors with single digit $\times$ decimal w.r.t. decimal point*

Knowledge, Beliefs and Values

MCK

Priority: to develop student's understanding of mathematics (particularly problem solving). Discussion beneficial to enrich understanding.

Heuristic

Organisation: *Individual (peer encouraged); block WC then SW (WC first on problems); self-selecting tiered tasks*

Discourse Patterns

Registers: *no terms introduced*

Interaction: *IRE dominant, extended teacher turns, linked longer sequence, more superlatives* Questioning: *Most limited soln- part of step-by-step procedure (T summarised)* Responses: *Typical satisfactory (1/5 superlatives) & unsat*

Revoicing: *repeat, rephrase, extend*

Tasks, Examples, Explanations: *(model-exercises).* Features Context: *notionally in 'big question'; psuedocontext (pens), mostly context free.* Generalⁿ: *not explicit* MSS: *possible, alternatives acknowledge, but one method in teacher explanations ($\times 10^n$ first)*

MR: *column headings/mental*

Example variation: *limited RoPC DoV (powers of ten ⇨ multiples of powers of ten, some required zero placeholder) similar form/ format, no 'not'*

Sequencing: Lesson trajectory

Focus on math^cal horizon and (re)direction usually to single introduced procedure (factorise then $\times 10^n$). Drew attention to place value through column headings (structure). Variation: *Alternative SS when met were acknowledge, considered separately, not compared.* Students' work: *Calculator check 'correct'=> move on, drew attention to one method.*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act). Errors: avoid/ correct Praise*

SMN: *Procedure as explanation, (one procedure to soln), students articulate next step a few times (prompted). Mathematics as a hierarchy.* ESM: *efficient & correct answers*

MP: *factorisation to aid multiplication with 10s, 100s, 1000s…*

Cognitive Demand

Type of thinking:

Potential: *High*

Range: *Low to high Procedural completion of tasks following modelled examples was possible (no press for higher).*

Individual Student Activity

Teacher's Interpretation of Classroom Activity:

Reference point Usually mathematical horizon (over-interpretation of student reasoning?) (attended to student reasoning on a couple of occasions)

KEY: (Compared to A) Common feature; Differential feature; Lesson specific

3.5.3 Sam Class A OMF – Indices Lesson

Teacher Cognition

Lesson Image: Sam Class A Lesson Indices

Lesson Goals: *Learning* & *Performance*

Plan Rationale: *Expanded form + "why" => reasons for rules. Question-rule- question => support generalisation*

Hypotheses: Issues would be with algebraic exponents and coefficients

Knowledge, Beliefs and Values

MCK *Discussion aids engagement and conceptual reasoning*

Examples: Careful choice important in developing mathematical reasoning CPD: variation theory

Teacher's Interpretation of Classroom Activity: *Reference point Mathematical horizon Some interpretation of student reasoning (x^{ab} in post-lesson)*

Heuristic

Organisation: *Individual (peer-to-peer discussion promoted, written work individual); interleaved WC-SW*

Discourse Patterns *Registers: Power terms used – no press for student use. Interaction: IRE dominant, extended teacher turns, linked longer sequence; some peer-to peer turn-taking (debates). Questioning: Mostly limited solution in step-by-step procedure (1/4 simple, self-contained) some funnelling Responses: Typical satisfactory (1/3 superlatives) Errors often typical unsat-sometimes debate prompted Revoicing: repeat, rephrase, extend*

Tasks, Examples, Explanations: *(model-exercises). Features* Context: *No Context. General^n: math^{cal} structure exposed (expanded form), examples used to justify. MSS: mostly single, alternatives separated. MR: same register - different forms (manipulation principles not explicit part of tasks).* Example variation*: Some variation in examples (operations, coeffs); not explicit, limited RoPC DoV (e.g. integer exponents)*

Sequencing *Focussed on math^{cal} horizon, (re)direction to introduced processes (rearrangement). Plans adapted (zero exponents) – different approaches introduced but separately - links not explicit. Variation: Example set – 'not' the same base example introduced. Other boundary conditions not explicitly met. Students' work: Errors used as starting point for peer-to-peer debates*

Teacher's Orchestration of Mathematics

Classroom Norms *SN: Teacher arbiter of correctness. Errors discussion exciting but to be corrected. (student-initiated; agency to act, debates but IRE/T explanation resolution). ESM: Efficient production of correct answer & ideas/debate; SMN: Procedure as explanation, (one at a time procedure to soln). Examples used to justify rules. MP: Expanded form for manipulation of powers. 'Power' used w.r.t exponent and whole term*

Cognitive Demand Type of thinking: Potential: *High (MR math^{cal} structure; zero exponents)* Range: *Low to high Procedural approach possible (no sustained press for higher)*

Individual Student Activity

KEY: (Compared to B) Common feature; Differential feature; Lesson specific

3.5.4 Sam Class A OMF – All Lessons

Lesson Image: Sam Class A - All Lessons

Teacher Cognition

Lesson Goals: Learning Performance

Plan Rationale: Discussion planned to expose misconceptions

Hypotheses: Articulated generic 'ways of working' (e.g. discussion) rather than topic specific

Heuristic

Knowledge, Beliefs and Values

MCK

Discussion considered important- aids engagement and conceptual reasoning

Examples: Careful choice important in developing mathematical reasoning CPD: variation theory

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon e.g. errors not explored (student reasoning)

Organisation: Individual, group, peer-to-peer discussion promoted, interleaved WC-SW

Discourse Patterns

Registers: no press for vertical, colloquial used and accepted

Interaction: IRE dominant, extended teacher turns, linked longer sequence. Questioning: limited solns often part of step-by-step procedure. Responses: Typical sat (superlatives). Errors: often typical unsat ignored or follow-up/direct explanation, sometimes debate. Revoicing: repeat, rephrase, extend.

Tasks, Examples, Explanations: (model-exercises). Features: Context: No context or context with psuedocontext. General^{n}: some math^{cal} structure exposed, examples used to justify. MSS: single or multiple (separately rather than compared). MR: common, either same register different forms or different registers (links when made focussed on specific examples rather than general features). Example variation: Some variation in examples; limited and not explicit RoPC DoV

Sequencing

Focussed on mathematical horizon, (re)direction to strategies introduced

Variation: attention drawn to specific examples, links between or to math^{cal} sig features not explicit; student errors/ explanations usually redirected towards standard procedure. Students' work: Discussion encouraged but steer towards math^{cal} horizon; some students @ board

Teacher's Orchestration of Mathematics

Classroom Norms

SN: Teacher arbiter of correctness. Accountability predominantly with the teacher (student-initiated; agency to act; class discussion-debates) Errors: IRE avoid-debate interesting / correct

SMN: Procedure as explanation, (one at a time procedure to soln). ESM: efficient & correct answers & ideas/debates

MP:

Cognitive Demand

Type of thinking:

Potential: High (MR)

Range: Low to high Procedural approach possible (no sustained press for higher)

Individual Student Activity

KEY: (Compared to B) Common feature; Differential feature; Lesson specific

3.5.5 Sam Class B OMF – Indices Lesson

Lesson Image: Sam Class B

Teacher Cognition

Lesson Goals: *Performance and some learning*

Plan Rationale: ***Atypical: try same approach/ resources as class A.*** *Question-rule- question ➪ support generalisation. Weak number skill ➪ simple numbers*

Hypotheses: *Discussions may be unproductive – off task behaviour/ more encouragement needed.*

Heuristic

Knowledge, Beliefs and Values MCK *Discussion aids engagement and conceptual reasoning. Examples: Careful choice important in developing mathematical reasoning CPD: variation theory Belief: engagement problematic; praise for engagement*

Teacher's Interpretation of Classroom Activity: *Reference point Mathematical horizon Students had been able to* ***engage in discussions*** *in a positive manner, and had understood the concepts* ***similar class A***

Organisation: *Individual; interleaved WC-SW*

Discourse Patterns: *Registers:* *Informal language* (e.g. little number) the norm, but *unprompted student mirroring.* *Interaction:* *IRE dominant, extended teacher turns and multiple student turns wrt initial I, linked longer sequence; some student-initiated turns, superlatives (IRE and B4L).* *Questioning:* *Mostly limited solution in step-by-step procedure (1/4 simple, self-contained); some funnelling* *Responses:* *Typical sat (1/3 superlatives)* *Errors* *often typical unsat, some 'bald' no's (student led), ignored errors parallel to correct.* *Revoicing:* *repeat (occasional rephrase)*

Tasks, Examples, Explanations *(model-exercises).* *Features:* No context. *General^n:* *math^{cal} structure exposed (expanded form), examples used to justify.* *MSS:* *mostly single, alternatives separated.* *MR:* *same register- different forms (manipulation principals not explicit).* Example variation: *Some variation in examples (operations, coefficients); not explicit, limited RoPC DoV (e.g. positive integer exponents)*

Sequencing: pre-emptive strike? *Focussed on mathematical horizon (rearrangement process). Plans adapted (zero exponents) – different approaches introduced but separately - links not explicit* *Variation:* *Example set – 'not' the same base example introduced. Other boundary conditions not explicitly met* *Students' work:* *Verbal contributions shared*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act; turns acknowledged). Errors: avoid/ correct Praise engagement* *SMN:* *Procedure accepted as explanation (one at a time procedure to soln) Examples used to justify rules* *ESM: Efficient production of correct answer;* *MP:* *Expanded form used for manipulation of powers.*

Cognitive Demand

Type of thinking:

Potential: *High (MR math^{cal} structure; zero exponents)*

Range: *Low to high Procedural approach possible (no sustained press for higher)*

Individual Student Activity

KEY: (Compared to A) Common feature; Differential feature; Lesson specific

3.5.6 Sam Class B OMF – All Lessons

Lesson Image: Sam Class B – All Lessons

Teacher Cognition

Lesson Goals: *Performance Learning for some*

Plan Rationale: *Discussion exposes misconceptions Weak 'basis skills' ⇨ simpler examples*

Hypotheses: *Articulated generic 'ways of working' (e.g. discussion) rather than topic specific. Behaviour– more encouragement needed*

Knowledge, Beliefs and Values MCK *Discussion important- aids engagement and conceptual reasoning Examples: Careful choice important in developing mathematical reasoning CPD: variation theory Belief: engagement problematic – **could engage in productive discussions** but 'unpredictable' POST - Similar approaches: less content coverage than A, sometimes 'worked' (wrt student understanding)*

Teacher's Interpretation of Classroom Activity: *Reference point Mathematical horizon.*

Heuristic

Organisation: *Individual, [discussion promoted (interleaved WC-SW)]*

Discourse Patterns *[more WC interactions]* Registers: *no press for vertical, colloquial /informal used more often and accepted.* Interaction: *IRE dominant, extended teacher turns, linked longer sequence, multiple student turns wrt initial I (calling out), superlatives (IRE&B4L).* Questioning: *limited (simple ¼) solns often part of step-by-step procedure, some funnelling.* Responses: *Typical sat (superlatives) [some student-initiated].* Errors: *often typical unsat, ignored (esp. multiple R-accepted correct) or follow-up/direct explanation **[student-initiated ⇨ bald 'no']**.* Revoicing: *repeat, rephrase, extend.*

Tasks, Examples, Explanations: *(model-exercises).* Same tasks as class A Features: Context: *No context or context with psuedocontext.* General^{n}: *some math^{cal} structure exposed, examples used to justify.* MSS: *single or multiple (separately rather than compared).* MR: *common, either same register different forms or different registers (links when made focussed on specific examples rather than general features).* Example variation: *Some variation in examples; limited and not explicit RoPC DoV*

Sequencing *Focussed on math^{cal} horizon, (re)direction to strategies introduced.* *Variation: attention drawn to specific examples, links between or to math^{cal} sig features not explicit; student errors/ explanations usually redirected towards standard procedure.* Students' work: *[Discussion encouraged but steer towards math^{cal} horizon]; some students @ board*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. Agency and accountability resides predominantly with the teacher (student-initiated; agency to act) Errors avoid/ correct*

SMN: *Procedure as explanation, (one at a time procedure to soln).*

ESM: *Efficient production of correct answer*

MP:

Cognitive Demand

Type of thinking:

Potential: *High (MR)*

Range: *Low to high Procedural approach possible (no sustained press for higher)*

Individual Student Activity

KEY: (Compared to A) Common feature; Differential feature; Lesson specific

bold *linked to a particular lesson.* Underlined *atypical (recorded vs normal lesson)*

3.5.7 Rowan Class A OMF – Equivalence Lesson

Teacher Cognition

Lesson Image: Teacher Class

Lesson Goals: *Performance and learning (improve problem solving & algebraic manipulation)*

Plan Rationale: *Students were familiar with area as multiply providing concrete link to equivalence*

Hypotheses: *Barriers: lack of familiarity with open problems; recognising relationships between diagrams and algebra*

Knowledge,
MCK : *[equivalence]*
Beliefs and Values
Values: to shift away from teacher led to student problem solving
Beliefs: Colleagues and students preferred 'traditional' teaching – coverage of content, especially with higher attaining sets.

Teacher's Interpretation of Classroom Activity:
Reference point Mostly mathematical horizon (included a student's approach to 'solving' in whole-class discussions).

Heuristic

Concepts — Ways of working

Organisation: *Individual (peer discussion encouraged), some group (cooperation), interleaved WC/SW*

Discourse Patterns
Registers: *Term used (equivalence) – no press for student use.* Interaction: *IRE dominant, extended teacher turns, linked longer sequence.* Monologues: *Teacher expositions common* Questioning: *Mostly limited soln – part of step-by-step procedure.* Responses: *Typical satisfactory & unsat* Errors: *(some funnelling)* Revoicing-*repeat, rephrase, extend*

Tasks, Examples, Explanations *(model-exercises*).* Features: Context-*area model for multiply.* General^{n}: *not explicit (structure embedded).* MSS: *possible (T focus - single) others 'avoidable'.* MR: *used (same and different registers), math^{cal} features/ links not explicit (area equivalence/ conservation of), ambiguous features (scale, visual similarity, x, specific /general).* Example variation: *few examples* to general model*

Sequencing *Lesson trajectory Focussed math^{cal} horizon, (re)direction to introduce approaches. Focussed on expressions, then (erroneously) on equating equivalent expressions, then solving a case. Occasional attention drawn to math^{cal} structure (area ⇨ equivalent) but not to 'equivalent ⇔ rearrangeable.* Students' work: *solving example (assumption about area articulated, but not others e.g. x)*

Teacher's Orchestration of Mathematics

Classroom Norms
SN: *Teacher arbiter of correctness. (student-initiated; agency to act) Some student contributions to explanations (one pre-selected), but teacher control Errors: avoid/ correct*
SMN: *Procedure as explanation (one procedure to soln). ESM: Efficient production of correct answer*
MP: *Equivalence sometimes treated as 'equals'*

Cognitive Demand
Type of thinking:
Potential: *High*
Range: *Low to high Procedural approach possible and attention not consistently drawn to mathematical features*

Individual Student Activity

KEY: (Compared to B) Common feature; Differential feature; Lesson specific

3.5.8 Rowan Class A OMF – Algebra Lesson

Teacher Cognition

Lesson Image: Teacher Class

Lesson Goals: *Performance; some implied learning (accurate rearranging of formulae; links to substitution)*

Plan Rationale: *Students were familiar substitution – used as a model for algebraic formula*

Hypotheses: *Barriers: misapplication of BODMAS, manipulation of fractions*

Knowledge,
MCK : *[rearrangement]*

Beliefs and Values
Values: to shift away from teacher led to student problem solving
Beliefs: 'traditional' teaching seen as the way to cover content, especially with higher attaining sets.

Teacher's Interpretation of Classroom Activity:
Reference point Mostly mathematical horizon (single solution strategy).

Heuristic

Concepts — Ways of working

Organisation: *Individual, (peer discussion encouraged); interleaved WC/SW*

Discourse Patterns
Registers: *No press for formal terminology*
Interaction: *IRE dominant, extended teacher turns, linked longer sequence.*
Monologues: *Teacher expositions common*
Questioning: *Mostly limited soln – part of step-by-step procedure.* Responses: *Typical satisfactory & unsat.* Errors: *few (treated alternate MSS as error).* Revoicing: *repeat, rephrase, extend*

Tasks, Examples, Explanations *(model-exercises).* Features: Context-*none*
General^{n}: *no explicit reference to order of operations (beyond examples)*
MSS: *possible but not integral part of tasks (not discussed/ acknowledge @ whole-class).* MR: *same register (rearrangements).*
Example variation: *different types of examples, pre-categorised, appeared unsystematic – no explicit links between examples and RoPC DoV*

Sequencing *Lesson trajectory*
Focus math^{cal} horizon, (re)direction to introduced strategies. Math^{cal} features: Focussed on one way of rearranging – did not acknowledge others (treated as error). Occasional attention drawn to math^{cal} structure (substitution ⇨ algebraic rearrangement) but at example level.
Students' work: *Verbal contributions (IRE)*

Teacher's Orchestration of Mathematics

Classroom Norms
SN: *Teacher arbiter of correctness. (student-initiated; agency to act) Some student contributions to explanations but part of procedure. Errors: avoid/ correct*
SMN: *Procedure as explanation, (one procedure to soln).*
ESM: Efficient production of correct answer
MP: *BODMAS accepted as a mathematical principle*

Cognitive Demand
Type of thinking:
Potential: *High*
Range: *Low to high Procedural approach possible, with single solution strategy @WC; students could have taken alternative routes in individual work.*

Individual Student Activity

KEY: (Compared to B) Common feature; Differential feature; Lesson specific

3.5.9 Rowan Class B OMF – Equations Lesson

Teacher Cognition

Lesson Image: Rowan Class A

Lesson Goals: *Performance and learning (rearrange equations using diagrams)*

Plan Rationale: *Diagrams provide non-algorithmic way of manipulating equations*

Hypotheses: *Barriers: Algebraic understanding*

Heuristic

Knowledge, Beliefs and Values

MCK

Values: to shift away from teacher led to student problem solving

Beliefs: Class needs related to attainment profile: Lower attainers need more practice & extrinsic motivation. But more leeway to try open tasks as content coverage pressures are less

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon

Concepts — Ways of working

Organisation: *Individual (peer discussion encouraged), longer blocks WC/SW*

Discourse Patterns

Registers: *Term introduced (subject) – no press for student use.* Interaction: *IRE dominant, superlatives (IRE and B4L), extended teacher turns, linked longer sequences.* Questioning: *Mostly limited soln – part of step-by-step procedure.* Responses: *Typical satisfactory & unsat,* Errors: *sometimes used other students' work in reply, some funnelling (non-responses).* Revoicing-*repeat, rephrase, extend*

Tasks, Examples, Explanations *(model-exercises/practical).* Features: *Context-physical model & diagrams used.* General^{n}: *not explicit (structure embedded).* MSS: *possible but not integral to the tasks* MR: *used (within the same and different registers)-math^{cal} features/links not explicit.* Example variation: *some variation in format & numbers/ letters, but limited RoPC, boundary conditions not explicitly met*

Sequencing *Lesson trajectory*

Focus math^{cal} horizon but adapted to vertical layout earlier, (re)direction to intro methods. Math^{cal} features: Diagram layout changed, but no explicit comparison. Attention drawn to the role of bar length, but changes in features (e.g. relevance of scale) not explicitly discussed. Steer towards students deriving def^{n} of 'subject'-'unsuccessful'.* Students' work: **introduce vertical layout; explanations (broadcasting)*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act) Students have some role in explanations (broadcasting). Errors: avoid/ correct Praise.*

SMN: *Procedure as explanation, (one procedure to soln). ESM: Efficient production of correct answer*

MP: *'Subject' considered as 'on own' either side of*

Cognitive Demand

Type of thinking:

Potential: High

Range: Low to high Procedural approach possible; attention drawn to a limited number of key math^{cal} features

Individual Student Activity

KEY: (Compared to A) Common feature; Differential feature; Lesson specific

3.5.10 Rowan Class B OMF – Directions Lesson

Lesson Image: Rowan Class A

Teacher Cognition

Lesson Goals: *Performance; some learning (measuring and estimation skills, language; spatial awareness, application)*

Plan Rationale: *Use of language encouraged through paired tasks (in context)*

Hypotheses: *Barriers: reading off double-scale protractors*

Heuristic

Knowledge, Beliefs and Values

MCK

Values: to shift away from teacher led to student problem solving

Beliefs: Class needs related to attainment profile: Lower attainers need shorter activities and games (competition) for motivation. Real-life connections mentioned

Teacher's Interpretation of Classroom Activity:

Reference point Mathematical horizon

Concepts — Ways of working

Organisation: *Individual, paired tasks (collaboration), longer blocks WC/SW*

Discourse Patterns

Registers: *Terms introduced (directions) – task completable without use, use encouraged but not required in whole-class.* Interaction: *IRE dominant, superlatives (IRE and B4L), extended teacher turns, linked longer sequences.* Questioning: *Limited soln soln – part of step-by-step procedure, or description of activity.* Responses: *Typical satisfactory & unsat,* Errors: *few (broadcasting for 'don't know').* Revoicing- *repeat, rephrase, extend*

Tasks, Examples, Explanations *(model-exercises/activities).* Features: *Context- maps and diagrams used, psuedocontex (map tasks completable without engaging with math^{cal} stated aim – use of language).* General^{n}: *not explicit.* MSS: *possible (T focus on one) but not integral to the tasks* MR: *used (different registers)-math^{cal} features/links not explicit.* Example variation: *some variation in angles /length/ scale, but no explicit reference to RoPC, DoV*

Sequencing *Lesson trajectory*

Focus math^{cal} horizon, (re)direction to introduced methods, Math^{cal} features: Used direct explanation when task didn't produce intended use of language. Attention drawn to the range of suitable estimates in particular examples, and how estimates could be made, but not how suitable was determined. Students' work: *Verbal contributions (IRE) (broadcasting)*

Teacher's Orchestration of Mathematics

Classroom Norms

SN: *Teacher arbiter of correctness. (student-initiated; agency to act) Some students contributions to explanations. Errors: avoid/ correct. Praise.*

SMN: *Procedure as explanation, (one procedure to soln)*

ESM: Efficient production of correct answer

MP: *Estimation separated from appropriateness*

Cognitive Demand

Type of thinking:

Potential: Low-High

Range: Low to high

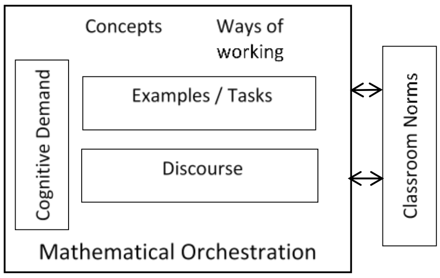
Procedural approach possible; linking estimations to calculations, and comparing alternative angle calculation strategies (links to parallel lines) was possible

Individual Student Activity

KEY: (Compared to A) Common feature; Differential feature; Lesson specific

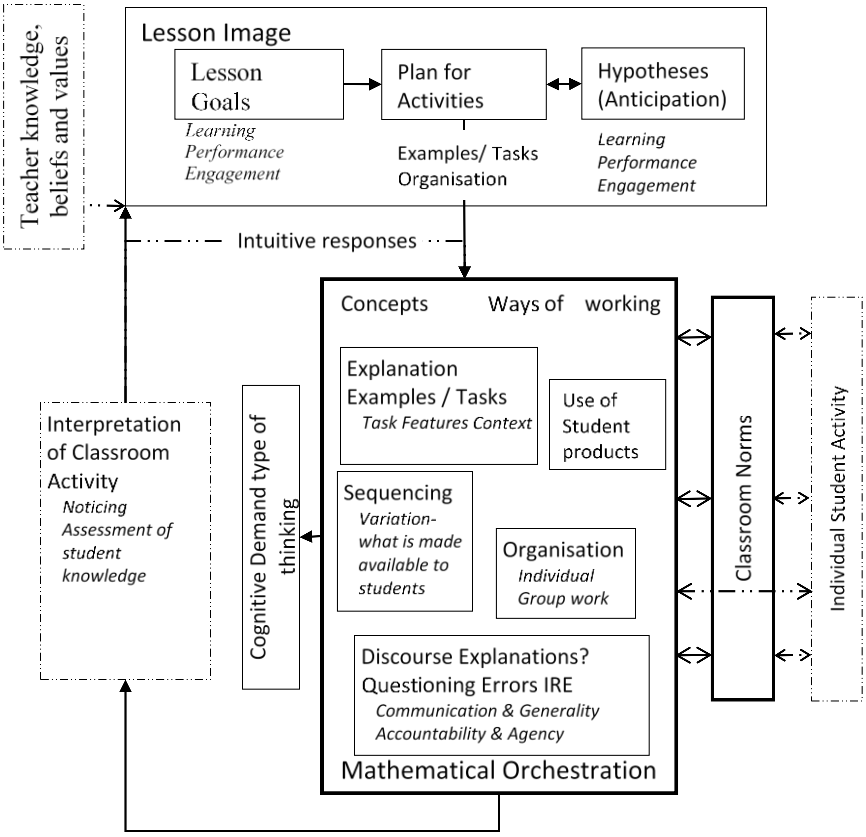
Appendix 4: Early Iterations of Frameworks

4.1 Earliest Conceptual Model



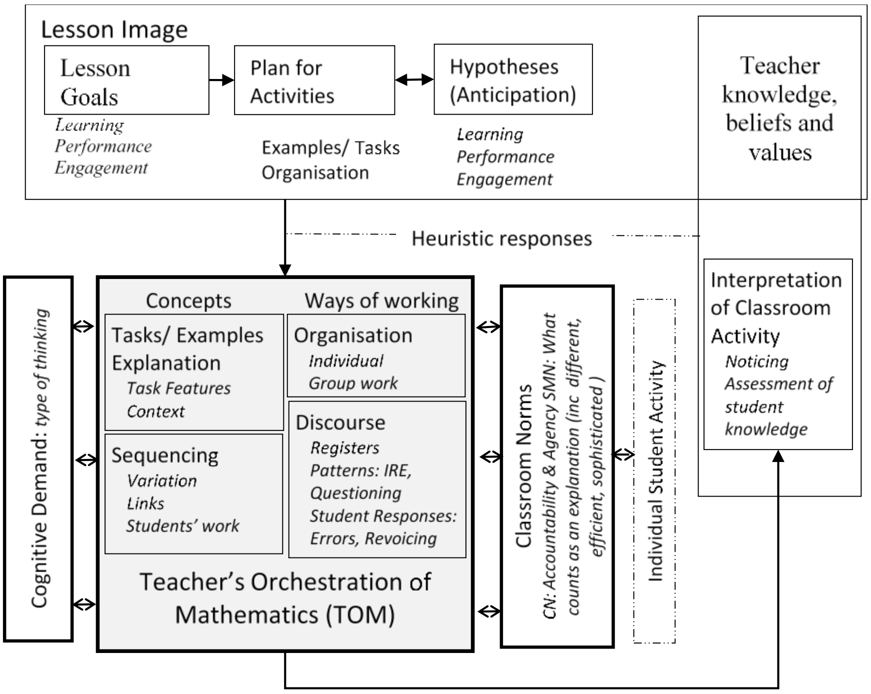
The first key change was to move Cognitive Demand out of the central box, which allowed the focus to shift to the teacher.

4.2 Expansion of TOM



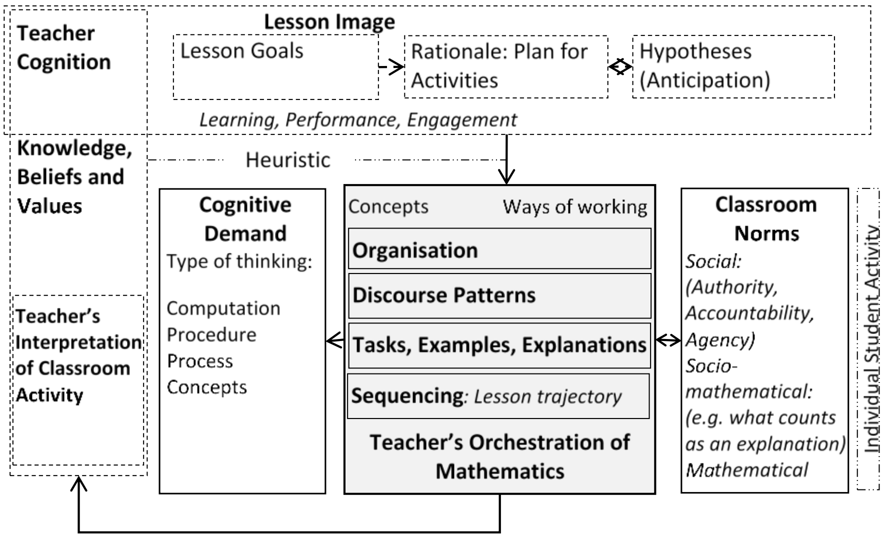
During the literature review, different categorisations of additional dimensions were explored.

4.3 Different Layouts



During use, different layouts were tried for practical reasons and for drawing attention to particular features of the model. E.g. Connecting the lesson image with other aspects of teacher's cognition to recognise the holistic influence of teacher knowledge, beliefs and values.

4.4 Summary Version



Used in publications and presentations.

Appendix 5: Cognitive Demand Criteria

SMITH, M. S. & STEIN, M. K. 1998. REFLECTIONS on Practice: Selecting and Creating mathematical Tasks: From Research to Practice. *Mathematics Teaching in the Middle School,* 3**,** 344-350.

Lower-Level Demands

<u>Memorization Tasks</u>

• Involves either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory.

• Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.

• Are not ambiguous – such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated.

• Have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.

<u>Procedures Without Connections Tasks</u>

• Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task.

• Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it.

• Have no connection to the concepts or meaning that underlie the procedure being used.

• Are focused on producing correct answers rather than developing mathematical understanding.

• Require no explanations, or explanations that focus solely on describing the procedure that was used.

Higher-Level Demands

<u>Procedures With Connections Tasks</u>

• Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.

• Suggest pathways to follow (explicitly or implicitly) that are broad, general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts (Henningsen and Stein, 1997).

• Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.

• Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.

Doing Mathematics Tasks

• Requires complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).

• Requires students to explore and to understand the nature of mathematical concepts, processes, or relationships.

• Demands self-monitoring or self-regulation of one's own cognitive processes.

• Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

• Requires students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.

• Requires considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

(Smith and Stein, 1998, p.348)

Appendix 6: Interview Schedule

Your expectations:

Background

1. Is the material ‘new’ to the pupils or a review lesson?

2. Is the lesson part of a larger unit or sequence of lessons?

Planning

3. What are your lesson goals?

4. How were they decided (i.e. SOW, your choice based on…)

5. What is are the origins of lesson activities (e.g. SOW, text book scheme…)

6. How are the planned activities related to the lesson goals?

7. Is this a ‘typical lesson’?

The lesson ‘expectations’

8. What do you think will happen?

9. What misconceptions might arise?

10. What elements may students struggle with?

Learning

11. Where do you think the learning gains will come from (i.e. what elements of the activities will be particularly useful in meeting your lesson goals).

12. How do you think the lesson will play out for different pupils (or groups of pupils) e.g. are you anticipating that particular students will respond differently to different parts of the lesson?

Organisation

13. How did you decide on the seating arrangements?

14. For paired or group work on what basis were students assigned groups.

Post Lesson Review

What were the main things you wanted the students to learn and are you satisfied that the lesson achieved that purpose?

What moments did you think were particularly important for students’ learning?

Where they anticipated or unexpected?

How did students’ reasoning impact on how the lesson unfolded?

Thinking about how you taught the lesson; how would this compare with how you would ideally like to teach the lesson.

What are the limiting factors; for example to what extent did any of the following limit you from reaching the ideal (National curriculum; School SoW; Student motivation; Class size; Time for planning; Not enough resources (specify); Poor IT resources for the teacher; Poor IT resources for the students; …)